DOCUMENT RESUME

ED 271 322		SE 046 678

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TITLE	The Construction of Logic in School Subjects: The Case of Multidigit Subtraction.
INSTITUTION	Virginia Commonwealth Univ., Richmond.
PUB DATE	Jun 86
NOTE	27p.; Paper presented at the Annual Symposium of the Jean Piaget Society (Philadelphia, PA, June, 1986).
PUB TYPE	Reports - Research/Technical (143)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	*Algorithms; Concept Formation; *Elementary School Mathematics; Fundamental Concepts; Grade 2; Grade 3; *Mathematical Logic; Mathematics Education; *Mathematics Skills; Primary Education; Problem Solving; *Subtraction

ABSTRACT

This paper presents an examination of the construction of logic in multidigit subtraction. Interviews were conducted with 90 grade 2 and grade 3 students to determine whether they understood the logic of borrowing and whether the construction of the logic was related to procedural expertise or corresponding conceptual knowledge. Of 34 students identified as proficient, only 12 recognized that the value of the first number in the subtraction problem is conserved during the borrowing procedure. Responses to questions about the values exchanged during regrouping suggest that conceptual knowledge does not distinguish those who understand the logic of borrowing from those who do not. Students who demonstrated procedural proficiency with the subtraction algorithm also demonstrated higher levels of conceptual and logical knowledge about borrowing than those who were less proficient. (Author/JM)

The Construction of Logic in School Subjects: The Case of Multidigit Subtraction

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Paper presented at the Annual Symposium of the Jean Piaget Society, Philadelphia, PA, June, 1986.

Abstract

Ninety second and third graders were interviewed about multidigit subtraction to determine whether they understood the logic of borrowing and whether its construction was related to procedural expertise or corresponding conceptual knowledge. Of the 34 students identified as procedurally proficient, only 12 recognized that the value of the first number in the subtraction problem is conserved during the borrowing procedure. Responses to questions about the values exchanged during regrouping suggest that conceptual knowledge does not distinguish those who understand the logic of borrowing from those who do not. Children need more opportunities to construct the part-whole logic of number.

The Construction of Logical Knowledge: A Study of Borrowing in Subtraction

It is increasingly clear that although students acquire procedural proficiency in mathematics, they often do not demonstrate the corresponding conceptual knowledge that defines understanding (Hiebert, 1984; National Assessment of Educational Progress, 1983). A related concern is whether children also fail to construct the logic of number. This logic is important for two reasons: (1) it reflects an understanding of number and its logical properties, not merely the particular algorithm; (2) it provides a basis with which to reason about more advanced mathematics.

This paper will explore the extent to which children construct the logic of multidigit subtraction.

Logical knowledge is distinguished from two other aspects of knowing, procedural knowledge and conceptual knowledge (Cauley, 1985). Logical knowledge refers to the logical structures proposed by Piaget (such as conservation, class inclusion and transitivity) that organize thinking across domains. Logical knowledge undergoes qualitative or stage changes as it is constructed. More importantly, the pattern of these qualitative changes is common to many conceptual domains. Also, contrary to traditional Piagetian theory, each stage in logical knowledge is treated as a description of the modal level of logical ability (Snyder & Feldman, 1977, 1984; Feldman, 1980a, 1980b). In this view, an individual may demonstrate logical reasoning at both higher and lower levels with different tasks because the logical structures are constructed as the individual experiences task contents and demands.

In contrast to logical knowledge, procedural knowledge is defined as the task specific skills, rules, strategies and procedures that are employed to perform tasks. In this analysis, procedures are regarded as tools of knowing because they themselves are not ordinarily used to construct knowledge, but rather to engage action. The principal difference with logical knowledge is that procedures do not show stage development and are task specific.

Conceptual knowledge ranges from knowledge of specific facts or word definitions that are relatively isolated, to the more complex, elaborated bodies of knowledge of the expert. While conceptual knowledge may be constructed by the individual, it does not show shage development. Evidence of qualitative change in conceptual knowledge are considered to be domain specific. The part-whole logic of number, for example, is relevant to all arithmetical skills. In contrast, conceptual knowledge for subtraction, while overlapping in part with other arithmetical skills, also has features like borrowing that are unique to it.

Each type of knowledge can be identified for the subtraction algorithm as it is taught in most schools. Procedural knowledge refers to the series of steps employed to solve subtraction problems. Adequate procedural knowledge is confirmed by the correct solution of the problem. Conceptual knowledge of the traditional subtraction algorithm, adapted from Omanson, Peled, and Resnick (1982), is: (a) knowing that the goal of subtraction is to take the whole bottom number away from the whole top number; (b) knowing the values that are exchanged during borrowing; and (c) knowing the compensation rule, that the decrease in one column equals the increase in the other column(s). Logical knowledge of the traditional subtraction algorithm is knowing that the value of the whole top number is conserved during borrowing or regrouping. More specifically, the logic of the borrowing algorithm is based on the part-whole relationship of number. That is, a whole number can be composed and recomposed into parts in various ways, each composition being equivalent to the whole. The borrowing or regrouping algorithm conserves the whole while rearranging its parts.

The purpose of this paper is to investigate whether logical knowledge is constructed in the context of learning the subtraction algorithm. If logical structures assimilate mathematics instruction, one might assume that children would construct the logic of mathematics. On the other hand, it may be that children need to experience and reflect on the results of subtraction procedures before they construct logical knowledge about the subtraction algorithm. A secondary question is how logical knowledge is related to conceptual or procedural knowledge given that research suggests that students' conceptual knowledge of multidigit subtraction lags behind procedural skill or at least is not connected to it (Omanson, Peled, & Resnick, 1982; Resnick, 1982). The data presented here describe the knowledge of procedurally proficient children about the logic underlying the conventional multi-digit subtraction algorithm. Of particular interest are the characteristics of students who seem to have integrated procedural expertise with conceptual and logical knowledge of borrowing in multidigit subtraction.

Method

Subjects

Forty-two (21 female) second graders and 48 (27 female) third graders (age range = 7 years 3 months to

10 years 3 months) completed the interview about multidigit subtraction. Twenty-two second graders and 29 third graders were enrolled at a racially integrated, suburban public elementary school. The remaining subjects were enrolled at an urban Catholic grade school. National Percentile Scores for the math subtest of the California Test of Basic Skills (CTBS) ranged from 1 to 99 with a median of 56 for the second graders and ranged from 3 to 99 with a median of 52 for the third graders at the public school. The CTBS scores were higher for the parochial school students, ranging from 14 to 99 with a median of 89 for the second graders and ranging from 50 to 99 with a median of 91 for the third graders.

Materials

A set of plastic coffee stirrers, 4 3/4 in. (15 cm) long, consisting of 30 single sticks, 20 bundles of ten sticks, and 10 bundles of 100 sticks (10 sets of 10) were available for the child to use when answering questions that required the manipulation of materials. In addition, a hand puppet was employed as a confederate to relax the children and to elicit information from them. In conjunction with a verbal presentation of the questions, all problems were presented on 5 in. X 8 in.

(16 cm X 28 cm) sheets of paper, which the subject wrote on as necessary.

Procedure

Students were interviewed individually in a quiet room according to the format of the Subtraction Interview (available from the author). Before the interview began, each child was asked to complete a pretest consisting of ten subtraction problems of varying difficulty (e.g., no borrowing, two digit, three digit, borrow-across-zero, and double borrow problems). When the child finished the problems, the puppet was introduced and the 20 to 30 minute interview followed.

The initial part of the interview asked subjects to represent numbers with sticks, to assess the effect of regrouping the sticks, and to compare the magnitude of numbers. The portion of the interview data reported here involved paper and pencil subtraction. The hand puppet solved the following problems incorrectly: "56-38," "635-241," and "802-455." The student was to determine if the puppet was right and if not, to show her how to do the problem correctly. After students finished correcting the puppet, they were asked questions to assess conceptual knowledge. The questions differed slightly for each problem to vary the questions and reduce the length of the interview. Students were asked the following questions to assess conceptual knowledge for Problem 1 (56-38): (a) why did you cross out this number (5); (b) what did you do with it (what you borrowed); (c) how much did you put there (ones column)? For Problem 2 (635-241) they were asked, (a) how much did you borrow; (b) what did you do with it; and (c) how much did you put there (tens column)? For Problem 3 (802-455) they were asked, (a) how much did you borrow; (b) what did you do with it? Students were also asked the following logic question after each problem: Before you borrowed you had ___ and after you borrowed you had this much (the top number and all borrowing marks were circled), did you have more before you borrowed, or after, or was it the same?

Scoring. Students earned one point for each correct response to the conceptual knowledge questions for a maximum score of eight. For logical knowledge, students scored one point for each of the three problems if they said the value of the minuend remained the same after borrowing and logically justified their response.

Results

The mean scores on the subtraction pretest (0 - 10) for males (M=7.6) and females (M=6.9) of both schools combined did not differ significantly, t (88) = 1.11. Therefore, gender was not included in subsequent analyses.

The sample was divided into three groups based on the pretest scores. The first group included those students who were correct only on the problems not requiring borrowing; the second group included those who were correct on four to nine problems and the students in the third group correctly solved all 10 problems. As Table 1 indicates, these groups differed significantly on both the Conceptual Knowledge of borrowing and the Logical Knowledge of borrowing. For Conceptual Knowledge $\underline{M}$ = 0.16, 2.27, 3.07 respectively, $\underline{F}(2,87) = 21.43$, $\underline{p}<.001$. For Logical Knowledge $\underline{M}$ = 0.00, 0.22, 1.18 respectively, $\underline{F}(2,87) = 11.84$, $\underline{p}<.001$. Since as a group the procedurally proficient students did not approach the maximum possible scores, subsequent analyses were performed to identify their particular strengths and weaknesses.

Insert Table 1 about here

The data in Table 2 suggest that on the average only 40% of the proficient students knew that the minuend was the same amount after borrowing because either (a) the amount taken from one column was the same as that added to another column; or (b) if the amounts in each column after the borrow were added they would equal the original number (e.g., 40 + 16 = 56); or (c) the borrow could be undone (e.g., what was borrowed could be put back and it would be the same). Across the problems, 60% of the proficient students did not recognize that the minuend was the same amount after borrowing as before. Thirty-two percent to 42% of the proficient students thought they had more _before_ the borrow, because either (a) what was taken from one column is gone (e.g., 46 is left instead of 56 as in Figure 1a); or (b) the first number was higher before (e.g., 5 is more than 4 as in Figure 1a or b). Eighteen percent to 24% of the proficient students thought they had more _after_ the borrow because either (a) the borrowed number changed the place value (e.g., it is 416 instead of 56 or 40+16 as in Figure 1b) or (b) the magnitude of the column changed when the amount was added to it and the decrement is ignored (e.g., it is 16 instead of 6 as in Figure 1b).

Insert Figure 1 about here

Insert Table 2 about here

On the Logical Knowledge questions, 12 of the procedurally proficient students had a maximum score of

3, 19 had a score of 0, and the remaining 3 scored 1 or 2, nearly an all-or-none pattern. The 12 who scored the maximum could not be distinguished by gender (seven males and five females) nor by grade (four second graders and eight third graders). Further, while their scores on the CTBS ranged from the 79th to the 99th percentile, this does not distinguish them either because the scores of many others in the sample were also in that range. They are distinct, however, in that those students who attained maximum scores on the Logical Knowledge questions also attained significantly higher scores on Conceptual Knowledge ($\underline{M}$ = 4.5) than those students having lower scores on Logical Knowledge ($\underline{M}$ = 2.3), $\underline{t}$ (32) = 4.29, $\underline{p}<.05$.

Yet, as Table 3 shows only one half to one third of the high logic subsample knew the value that they borrowed. Further, Table 4 indicates that while all of the high logic subsample knew the value added to the ones column, only half of them knew that they added 100 to the columns to the right in the three digit case.

Insert Table 3 about here

Insert Table 4 about here

Table 5 suggests that while almost all (82%) of the high logic subsample were at least consistent in the values that they claimed were exchanged during borrowing, less than half of them knew the correct value.

Insert Table 5 about here

Finally, that most children (75%) did not conserve the value of the minuend during the borrowing transformation does <u>not</u> mean that these children are preoperational in other domains. Evidence from another portion of the interview with concrete materials suggests that the majority (85%) of the procedurally proficient students understood the logic of regrouping in the concrete context.

Discussion

Students who demonstrate procedural proficiency with the subtraction algorithm also demonstrate higher levels of conceptual and logical knowledge about borrowing than those who are less proficient. Yet, the conceptual and logical knowledge of the subtraction algorithm possessed by these students is often incomplete and inaccurate. This is consistent with previous research which also suggests that procedural owledge does not guarantee conceptual understanding.

It is striking that few children grasp the logic of the borrowing algorithm even though data from a parallel task using concrete materials indicated that the majority of the total sample have the logical competence to form and conserve new part-whole relationships in that context. It is interesting that the twelve students who did construct the logic of borrowing could not be distinguished by gender, grade, mathematical ability, or scores on the conceptual knowledge of borrowing. Further, the data of individual students indicated that only five of the twelve students in the high logic subsample were both correct and consistent in the values they said were exchanged. Five others were consistent, but incorrect, in the values they said were exchanged. In other words, some children logically justified borrowing without knowing the exact values that were exchanged. Thus, while procedural, conceptual, and logical knowledge appear to be related when groups are considered, an examination of individual students revealed that the connections among procedural, conceptual and logical knowledge were tenuous. Most children clearly did not acquire a complete, integrated body of knowledge about subtraction in the second and third grade.

Children's responses to the logical and the conceptual questions about the values exchanged during borrowing suggest that the problem may be a poor grounding in place value conventions. As in preoperational reasoning, students appear to center on particular features of the symbolic representation of borrowing and try to explain it with their limited knowledge of place value conventions. So, if the minuend looks like it has four digits with the borrowing symbols instead of the original three, they conclude that the number now is thousands instead of hundreds. This is not an unreasonable assumption because when they add numbers they must carry rather than leave more than 9 ones or 9 tens in a column. Students who argue that they have less in the minuend after borrowing center on the column they have decremented and they ignore symbols representing that value in another column, and so assume that the borrow left them with less. Thus, the incorrect rationales that students give for their judgments that borrowing increased or decreased the size of the minuend indicates a weakness in their knowledge of place value.

This weakness is also evident in their discussion of the values that were exchanged during borrowing. For example, those who claimed that they borrowed "10" when in fact they borrowed "100" apparently overgeneralized from the 2-digit case with which they learned subtraction, suggesting limited knowledge of place value. Further, student's largely correct responses to the values exchanged in the two-digit problem suggests that they learned that particular case well. Those who claimed that they borrowed or added "1" probably read the digits veridically.

Kamii (1986) suggests that children's difficulty with place value may result from a more fundamental problem with constructing the logic of number. She argues that the place value conventions will become meaningful for the children only as they construct number as a system of ones, a system of tens, a system of hundreds, etc., all at the same time. Under traditional instruction, when experiences to construct logic are minimal, the student is forced to learn conceptual knowledge by rote, and logic may be constructed more slowly. Thus, Kamii (1985) suggests that early instruction in mathematics should focus on providing opportunities for children to construct the part-whole logic of number. Her data suggest that children not only learn to add and subtract, but they also learn to reason about number. Children may learn the algorithms without such experiences, but their conceptual understanding will be limited to rote regurgitation. This approach is also consistent with other constructivist approaches to mathematics instruction (Confrey, 1985a, 1985b).

Additional research is needed to determine whether children eventually construct the logic of borrowing on their own under traditional instruction, and at what point they can easily grasp the relationships when taught. Informal discussions with randomly selected students after the interview was completed suggest that with brief instruction some students could understand that borrowing conserves the value of the minuend, and it was an exciting discovery for them. Other students clearly thought it was absurd to even entertain the notion that the minuend could be the same after borrowing and not more or less. If some students appear competent with minimal instruction, and if what distinguishes these students from those who are not easily helped could be identified, we could begin to organize instruction so children use their logical competence to understand the mathematical algorithms they employ.

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Table 1

Means (Standard Deviations) of Composite Variables for Students at Each Level of Procedural Skill (A, B, C)

		Knowledge Types	
Level	n	Conceptual Borrowing (max=8)	Logical Borrowing (max=3)
A (0-3 on pretest)	19	0.16 (0.34)	0.00 (0.00)
B (4-9 on pretest)	37	2.27 (1.68)	0.22 (0.71)
C (10 on pretest)	34	3.07 (1.82)	1.18 (1.42)
F(2,87)		21.43**	11.84**
Contrasts		A vs. B* A vs. C*	A vs. C* B vs. C*

* $p < .05$. ** $p < .001$.

Table 2

Percent of Responses to the Question: "Did You Have More Before You Borrowed, or After, or Was it the Same?"

		Responses to the Question			
Group	n	C[a]	MB	MA	NJ
Problem 1 (56-38)					
Proficient	34	41	32	24	3
Problem 2 (635-241)					
Proficient	34	39	42	18	0
Problem 3 (802-455)					
Proficient	34	39	36	18	6

Note. C = conservation; MB = more before; MA = more after; NJ = no justification.

[a] correct response

Table 3

Percent of Subjects' Responses to the Question: "How Much Did You Borrow?"

		Value Borrowed		
Group	n	100 or 10 tens[a]	10	1
Problem 2 (635-241)				
Proficient	34	24	44	26
High Logic	12	50	50	0
Problem 3 (802-455)				
Proficient	34	38	26	32
High Logic	12	67	33	0

Note. To vary the questions asked and to reduce the length of the interview, this question was asked only for Problems 2 and 3.

[a] correct response

Table 4

Percent of Subjects' Responses to the Question:

"How Much Did You Put There (in That Column)?"

		Value Added			
Group	n	100 or 10 tens	10	1	other
Problem 1 (56-38)					
Proficient	34	NA	88[a]	12	0
High Logic	12	NA	100[a]	0	0
Problem 2 (635-241)					
Proficient	34	21[a]	67	9	3
High Logic	12	50[a]	50	0	0

Note. To vary the questions asked and to reduce the length of the interview, this question was asked only for Problems 1 and 2.

[a] correct response

Table 5

Consistency of Subject's Responses to the Values Exchanged in Problem 2 (635-241)

		Consistent Responses			
Group	n	100[a]	10	1	Inconsistent
Proficient	34	15	41	9	35
High Logic	12	42	42	0	17

[a] correct response

Figure Caption

Figure 1. Sample problem solutions.

$$\begin{array}{r} \overset{4}{\cancel{5}}\,{}^{1}6 \\ -\;38 \\ \hline 18 \end{array}$$

a

$$\begin{array}{r} \overset{4}{\cancel{5}}\overset{16}{\cancel{6}} \\ -\;38 \\ \hline 18 \end{array}$$

b