DOCUMENT RESUME

ED 264 087 SE 046 234

AUTHOR	Usiskin, Zalman; Bell, Max
TITLE	Applying Arithmetic: A Handbook of Applications of Arithmetic. Part I: Numbers.
INSTITUTION	Chicago Univ., Ill. Dept. of Education.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	83
GRANT	NSF-SED-79-19065
NOTE	155p.; For related documents, see SE 046 235 (part II) and SE 046 236 (part III).
PUB TYPE	Guides - Classroom Use - Guides (For Teachers) (052)
EDRS PRICE	MF01/PC07 Plus Postage.
DESCRIPTORS	*Arithmetic; Elementary Education; *Elementary School Mathematics; *Mathematical Applications; Mathematics Education; *Mathematics Instruction; Number Concepts; *Numbers; *Teaching Methods
IDENTIFIERS	National Science Foundation

ABSTRACT

The first four chapters of a 14-chapter, three-volume work on arithmetical applications are contained in this document. Each chapter details the "use classes" of one broad arithmetical concept. (A "use class" of a concept is a set of examples of real world uses of the concept which share a common structure). Each chapter contains: an introduction and summary; three to six sections, each devoted to one use class and containing a general introduction, questions, and comments; suggestions for teaching or illustrating a given concept; questions which test understanding of the ideas presented; and notes and commentary, with reasons for selecting particular use classes, related research, and short essays on issues related to applying the concepts. Topics of the chapters include: (1) use of single numbers (counts, measures, locations, ratio comparisons, codes, and derived formula constraints); (2) use of ordered pairs, triples, or n-tuples (counts, measures, locations, ratios, codes, and combined uses); (3) uses of collections of numbers (domains, data sets, neighborhoods, and solution sets); and (4) uses of variables (formulas, unknowns, properties, and storage locations). (JN)

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APPLYING ARITHMETIC

A HANDBOOK OF APPLICATIONS OF ARITHMETIC

PART I: NUMBERS

by

ZALMAN USISKIN AND MAX BELL

under the auspices of the
ARITHMETIC AND ITS APPLICATIONS PROJECT

DEPARTMENT OF EDUCATION
THE UNIVERSITY OF CHICAGO

The preparation of this work was partially supported by National Science Foundation Grant SED 79-19065. However, any opinions, conclusions, or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Preface

The writing of this volume has been motivated by two existing gaps in mathematics education. The first gap is between student performance on arithmetic skills and the generally worse performance on realistic problems utilizing those same skills. The second gap is the disparity between oft-stated goals of professional organizations and schools and textbooks (generally supportive of applications of arithmetic) and the classroom reality. After grade 4, realistic applications of arithmetic do not often appear in the classroom, and those that do appear represent only a narrow picture of this broad domain.

The intended audiences are diverse. First, we have designed the book for use by teachers. Each concept is illustrated by a large number of examples, and comments are given following the examples to aid in adaptation for use in classrooms. Each chapter contains a special section entitled "Pedagogical Remarks" to further assist in this task.

Second, we have designed the book for use by those interested in curriculum design or research. Each chapter contains an extended discussion of selected theoretical, pedagogical, philosophical, psychological or semantic issues and research related to the ideas found within the chapter.

Third, because teachers and other professional educators often encounter books like this one only in the context of coursework, we have included a number of questions at the end of each chapter.

Fourth, we hope that the ideas in this book might also be suitable to lay readers interested in understanding the uses of arithmetic. We have tried to make the writing easy to understand and in most places the mathematical prerequisites necessary to comprehend the material are minimal.

Our goal is to improve our society's understanding of the applications of arithmetic. In the past, due to the necessity of having to spend a great deal of time teaching how to get answers, books could not afford to be devoted to teaching when to use particular arithmetic processes. Calculators, in our opinion, allow us to change emphasis from how to when. This book constitutes a first attempt to provide a rather complete categorization of the simpler applications of arithmetic.

The organization of this book is not definitive and in many places may not exhaust the range of applications. Many may disagree with our categorizations. We encourage criticism; we only hope that those who criticize will help us improve the ideas presented here or produce their own improved version.

Zalman Usiskin and Max Bell
June, 1983

Acknowledgements

This book was written as part of the Arithmetic and Its Applications project funded by the National Science Foundation. We are grateful to Ray Hannapel, Ruth von Blum, Harold Stolberg, Andrew Molnar, and others in the foundation for their support and assistance.

The Arithmetic and Its Applications project was assisted in its work by two advisory boards, one consisting of university personnel, the other of junior high school or middle school teachers and supervisors. The advisory board members were: Pamela Ames, Harry Bohan, Sherye Garmony, Alan Hoffer, Jeremy Kilpatrick, James McBride, Kay Nebel, and Jane Swafford. Roberta Dees worked with us on this project for a year. Each of these people assisted in the development of this manuscript in his or her own way (but the authors take full responsibility for the writing).

Early drafts of these materials were tried out in classes at the University of Chicago by us, Sam Houston State University by Harry Bohan, and Ohio State University by Alan Hoffer. We appreciate the willingness of these institutions to support this endeavor and extend our thanks to the students who gave comments to help us improve it.

Our thanks go also to the University of Chicago for providing facilities, colleagues, and students particularly amenable to the kind of thinking this type of writing requires.

Finally, we are each fortunate to have wives who are not only supportive of our work but who also are involved in mathematics

education. They have been responsive sounding boards for most of the ideas presented here and often were the ones who provided an ultimate clarification of an issue. We appreciate their help more than we can put in words.

Zalman Usiskin and Max Bell

APPLYING ARITHMETIC

Table of Contents

Part II: Operations (continued)

Introduction

For most of us, an application of arithmetic begins with an attempt to comprehend numbers we encounter in everyday living. These range from prices of goods and services to interest rates on investments to sports scores to minimum daily nutrient requirements to ID numbers to geographic information found on roads and maps to technical information about objects around the home to results of surveys published in newspapers or magazines. Our society has become increasingly numericized, requiring each of us to process more numbers than many of us thought we would need.

On many occasions, comprehension of numerical information suffices. We only may want to know the protein content of a food, or a sports score, or the time to the airport, or a social security number. At other times we may wish to operate on given numerical information to generate more information. From prices of foods, one may calculate which is more economical and still supply nutritional needs. From interest rates, income can be determined. From temperature data, energy costs can be estimated. From sports data, decisions regarding the quality of teams and participants may be desired. From information about the size of living quarters, wall and floor covering needs can be established. We add, subtract, multiply, divide, take powers, and apply other operations of arithmetic to help us obtain the additional numerical information.

But things are not always so simple. Given numerical information is not always written in a form that makes it easy to operate upon.

We do not always know what to do with such information until we display, scale, or estimate it in some way. We classify rewriting, graphing, scaling, and estimating as maneuvers and recognize that we often maneuver both given numerical information and the results of computations.

These three application skills comprise the subject matter of this book, one part of the book being devoted to each of them.

In	Part I,	Numbers	we ask	To what uses are numbers and number aggregates put?
In	Part II,	Operations	we ask	What are the common uses of the fundamental operations?
In	Part III,	Maneuvers	we ask	For what reasons are the most common types of maneuvers applied?

The three parts are divided into a total of 14 chapters. Each chapter details the use classes of one broad arithmetic concept (e.g., single number, multiplication, or estimation). The notion of use class is at the heart of this book and is roughly defined here.

> A use class of a concept is a set of examples of real world uses of the concept which share a common structure.

The arithmetic concepts in this book have from 3 to 6 use classes each; there are 57 use classes in the 14 chapters. The chapters are organized in the following way.

Introduction

3-6 Sections, one devoted to each use class, each with a general introduction to that class followed by example questions with answers and comments

Summary

Pedagogical Remarks

Questions

Notes and Commentary

Since use classes are defined in terms of examples, the major space in this volume is devoted to the examples, answers, and comments. The purpose of the other components of each chapter is as follows:

The introduction and summary each contain a short synopsis of the types of applications of a particular concept.

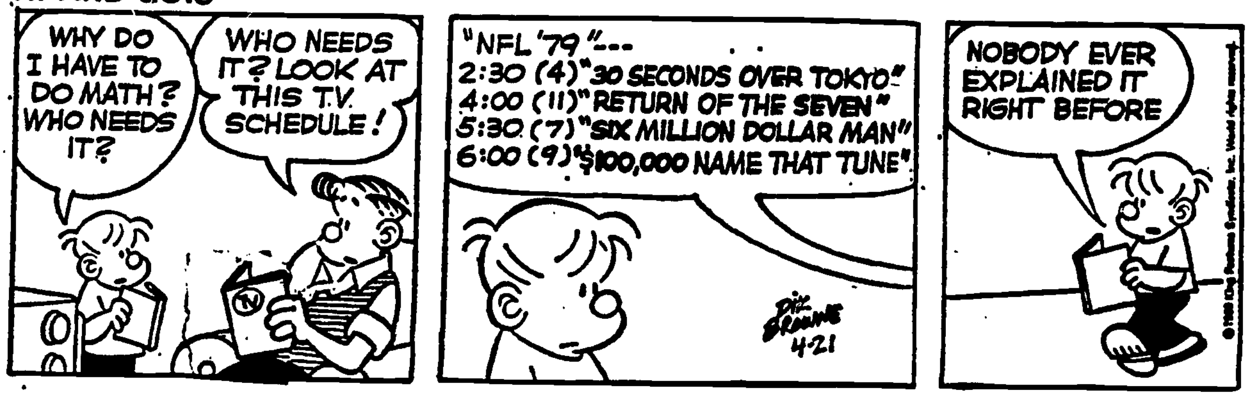
Suggestions for teaching or illustrating a given concept may be found both in the comments following each example and in the pedagogical remarks.

The questions are a test of the reader's understanding of the ideas herein. The notes and commentary include our reasons for the selection of the particular use classes, related research, and short essays on issues related to applying the various concepts.

A calculator is strongly recommended for all sections of this book so that the reader can spend time dealing with the concepts of this book rather than with paper and pencil computation. A calculator with an x^y key is necessary in Chapters 9 and 10.

PART I: NUMBERS

HI AND LOIS


WHY DO I HAVE TO DO MATH? WHO NEEDS IT?
WHO NEEDS IT? LOOK AT THIS T.V. SCHEDULE!
"NFL '79"--- 2:30 (4) "30 SECONDS OVER TOKYO" 4:00 (11) "RETURN OF THE SEVEN" 5:30 (7) "SIX MILLION DOLLAR MAN" 6:00 (9) "$100,000 NAME THAT TUNE"
NOBODY EVER EXPLAINED IT RIGHT BEFORE

Introduction

The basic objects of arithmetic are numbers and aggregates of numbers. The latter include ordered sets of numbers (called n-tuples) and collections of numbers.

Chapter 1 catalogs the uses of single numbers such as 24, 2/5, -3.6, 17, 6%, etc. Each of the uses is important and basic to understanding many of the uses of ideas found later in this book.

From single numbers we move to Chapter 2, n-tuples. These are found to have nearly the same uses as single numbers. For instance, a single mileage marker on an interstate highway can locate an exit, while two numbers--latitude and longitude--serve the same purpose in locating a place on the Earth's surface.

Chapter 3 considers unordered collections of single numbers. These collections are used in ways different from the numbers of which they are constituted. Chapter 4 describes the uses of variables, symbols that can stand for numbers. Variables are important in a variety of ways related to the understanding and use of arithmetic.

In cataloging these uses, we ignore the ways in which the numbers, number aggregates, and variables are represented. A discount of 20% may be advertised as "1/5 off" and calculated by multiplying by .20. To us, the numbers 20%, 1/5, and .20 are being used in the same way. Similarly, we are not concerned whether a set of numbers has braces { }, an n-tuple is expressed with parentheses (, , ... ,), or a variable is represented by a single letter x or a word XMEAN.

Throughout Part I, the topics shift back and forth from those that will be familiar to all readers to other topics familiar only to a few. Yet all of the ideas are important both within mathematics itself and in its application. We find ourselves remarking again and again that one of the reasons many adults have difficulty in processing numerical information is that they have never explicitly been introduced to some of the basic notions in applying arithmetic.

CHAPTER 1

USES OF SINGLE NUMBERS

The myriads of instances in which numbers are used can be sorted into six categories, or use classes. These use classes form the basis on which this chapter is organized.

A. Counts (5 people; 10,200,000 cars)

B. Measures (2.6 kilograms; 35 mph)

C. Locations (36°F; dates)

D. Ratio Comparisons (35, 2/3 as many)

E. Codes (phone number 312-555-2368; zip codes)

F. Derived Formula Constants ($\frac{9}{5}$ and 32 in $F = \frac{9}{5}C + 32$)

As the examples given in parentheses suggest, it is difficult to determine how a number is being used without looking at labels (people, kilograms, zip code, etc.) linked to the numbers. In the examples above, "miles per hour" is a measure unit label, "degrees Fahrenheit" identifies a temperature scale, and "cars" is a counting unit label. These and other types of labels are associated with uses of numbers and are discussed along with these uses.

Single Number Use Class A: Counts

A count is a number that indicates how many things are in a set. The things being counted are usually identified by a counting unit label. For example, in "There are 360 pages in that book," "360" is the count and "pages" is the counting unit. We say that "360 pages" is a quantity.

The process of counting is not the same as the use of a number as a count. Although 360 is a count of pages, few people would begin with 1, 2, 3, . . . to determine how many pages are in a book; most would just go to the last page and use the page number there as an estimate. Counts may be established in many different ways. Sometimes one starts 1, 2, 3, . . . and counts until there is nothing left to count. Sometimes a single one of many presumably identical sets is counted and then multiplication is used to find a total. Sometimes a count reported as a single number is the sum of counts from different places. Very often (more often than conveyed by textbooks) counts are estimated, especially for hard-to-count populations or for counts over 50. Sometimes it is not possible to determine how a count was obtained.

Splitting a counting unit into smaller units changes the meaning of the count. For instance, if there were 400 people at the concert, it would change the meaning to say that there were 800 ears at the concert.

Zero is often used as a count to refer to the absence of any of the things that are being counted. Negative numbers may represent counts if a scale with two directions has been established (see Section C of this chapter).

Examples:

(Here and elsewhere, the reader is encouraged to try to answer the question(s) in each example before reading on to the answers and comments.)

1. In the sentence that follows, (a) identify the numbers being used as counts; (b) identify the counting units associated with these numbers; and (c) indicate how you think each count was determined: Uganda, a country slightly smaller in area than the state of Oregon, has a population of 13.2 million, as estimated in 1979.

   Answers: (a) 13.2 million; (b) people; (c) Probably there was a census within ten or fifteen years before 1979. That census would have estimated the population by actual counts in the subdivisions of Uganda, then adding those counts. Probably a growth rate was then estimated and that rate applied to the census population to get the current population estimate.

   Comment: 13.2 million could be written 13,200,000. Numbers are still numbers regardless of whether they are written out with words or denoted using mathematical symbols. The notation here, using a decimal, is common in discussions involving very large numbers.

2. As in Example 1, identify counts, counting units, and possible sources: Although ridership since 1975 has increased 17.5% to 21.4 million riders, the amount of equipment Amtrak has available has declined from 1981 cars in 1976 to 1607 cars in September, 1979.

   Answers: (a) Of the seven numbers in the sentence, only three are being used as counts: 21.4 million, 1981, and 1607; (b) riders, cars, cars; (c) 21.4 million is an estimate probably based on adding ticket sale figures from various parts of the country; 1981 and 1607 may have been calculated by actual counting and totalling.

Comment: We consider the dates as locations, to be discussed in Section C of this chapter. The number 17.5% is a ratio comparison; such uses are discussed in Section D of this chapter.

3. As in Example 1, identify counts, counting units, and sources:

Hockey score: Hartford 9, Colorado 2.

Answers: (a) 9, 2; (b) goals; (c) by counting.

Comment: In hockey scores, the counting units--"goals"--is understood so is not written. In baseball, the understood label is "runs".

4. As in Example 1, identify counts, counting units, and sources:

In a political convention or primary election, only one person can get the nomination of the party.

Answers: (a) one; (b) person; (c) by regulation.

Comment: The word "one" does not always represent a number, as in the phrase "one can see that . . ." Here the word "a" does not represent a number, though at times it can denote the number one.

5. As in Example 1, identify counts, counting units, and sources:

Go to the store and buy three dozen eggs.

Answers: (a) three dozen; (b) eggs; (c) by estimating how many will be needed based upon past usage.

Comment: "Dozen" is another name for 12 just as "pair" is usually another name for 2.

Single Number Use Class B: Measures

A measure is a number that indicates how much of a particular attribute something has. A measure is always associated with a measure unit label. As with counts, a measure with its unit is called a quantity.

Quantity	Measure	Measure unit	Attribute being measured
30 cm	30	centimeter	length
31.2 mph	31.2	miles per hour	speed
$399.95	399.95	dollar	value or cost
1/2 cup	1/2	cup	volume
8 cc	8	cubic centimeter	volume

It is possible to consider counts as a special kind of measure, a measure of numerosity, but in this book we distinguish counts from measures. Specifically, counts are discrete (not connected) whereas most measures are continuous. That is, there are definite jumps from one count to the next, say from 4 people to 5 people, but there are an infinite number of measure values between, say, 4 centimeters and 5 centimeters.

Generally, a measure has these properties:

(a) A measure of 0 means there is none of the attribute being considered. (An empty milk carton contains 0 quarts of milk.)

(b) The measure unit is arbitrary. (One can measure length in centimeters (cm) or inches (in.); speed in miles per hour (mph or mi/hr) or kilometers per second (km/sec); value in dollars ($), Deutschmarks (Dm), or francs (F); etc.)

(c) Multiplying a measure by n corresponds to having n times as much of the quantity being measured. (A 6 cm length is twice as long as a 3 cm length. On the other hand, 6°F does not represent twice as much of anything as 3°F; such temperatures are locations on a scale--see Section C of this chapter.)

(d) Addition of measures with the same units can be meaningful. (Adding 1/2 cup to 3/4 cup of the same substance yields 1 1/4 cups of that substance.)

(c) Between any two measures of the same attribute there are other possible measures. (Between 1" and 2" is 1.3".)

Many familiar measures can be expressed in terms of just a few base units. In the metric system, the base units are the meter (for the attribute length), second (time), kilogram (mass), kelvin (thermodynamic temperature), ampere (electric current), and the candela (luminous intensity). These measure units are very precisely standardized. For instance, the meter is now defined as "1,650,763.73 wave lengths in a vacuum of the radiation emitted by the transition between the energy levels 2p10 and 5d5 of the krypton-86 atom." Few people need to know the meaning of this definition, but it allows precision to eight or nine significant figures.

From these base units, multiple units are defined. For instance, from the meter, the centimeter ($\frac{1}{100}$ meter), kilometer (1000 meters), millimeter ($\frac{1}{1000}$ meter), and others are defined. Units of length not in the metric system may also be defined from the meter. For example, in the United States, an inch is now officially defined as 2.54 centimeters. Since foot, rod, mile and all other units of length in the English system are multiples of inches, all English units are multiples of units from the metric system.

The base units give rise to many other units. Units for length (e.g., meter) may be elaborated into units for area (square meters or m^2) and volume (cubic meters or m^3). Length (e.g., mile) combined with time (hour) yields velocity (miles per hour). The units used to describe these other attributes are called derived units.

A very common but different sort of attribute is that of value or cost. Just as we may measure length in feet or meters, we may measure value in dollars or francs. These measures of value, or monetary units resemble standardized physical measures in having consistent systems of units and subunits with well-defined conversions. One dollar always equals 100 pennies. But unit conversions from one system to another vary from day to day (and often from hour to hour). In place of a "bureau of standards", brokers in money exchanges set the conversion rates.

Value measures are also used for scores on tests and scores in games. In these cases, the unit may be "points" or some synonymous substitute. Often a definite conversion system exists. For example, in football, one touchdown = 6 points and one field goal = 3 points. On a teacher test, questions may be worth 1/2 point each, a conversion rate that may be known in advance.

Other units used in measures can be rough units (e.g., a pinch of salt) or based upon a temporary or a personal unit (e.g., a pace) which may be fine for occasional purposes but are not always linked by official rules to other measures.

Examples:

1. Name three units for area measure.

   Answer: There are many more than three units that could be named, including square mile, acre, square centimeter (denoted cm^2), and hectare (a metric unit of measurement equal to 10,000 square meters).

   Comment: There are 640 acres in a square mile. A hectare is about 2.5 acres.

2. Give an example of an item whose unit cost is between 4 cents and 5 cents.

   Answer: Any item that sells "2 for 9 cents" or "6 for a quarter" has a unit cost between 4 cents and 5 cents.

   Comment: If foreign currency were used, it is quite likely that one could find coins to indicate a cost between 4 cents and 5 cents. The absence of a unit of money or coins for each fractional cost does not change the fact that cost is a measure.

3. A recipe (Better Homes and Gardens Cooking for Time, Meredith Press, 1968, p. 25) for spicy orange tea contains the following ingredients: 5 whole cloves, 1 inch of stick cinnamon, 3 tablespoons honey, 1/2 cup orange juice, 2 tea bags. Which numbers are counts, which measures?

   Answer: The 1, 3, and 1/2 are measures, the 2 is a count, and the 5 has properties of both. It is possible to think of 4.5 whole cloves, at which point "cloves" is acting as a rough measure unit. It is more likely that one would think of 5 as a count.

   Comment: We do not encourage making a big fuss about the difference between counts and measures; this example is given to make the point that distinctions are not always easy.

4. Consider the following sentence: Our new apartment will have approximately 1400 square feet of space. (a) Identify the attribute being measured. (b) Identify the numbers used as measures. (c) Identify the measure unit for each such number. (d) Classify the measure unit as one of the following: base unit of the metric system, multiple of the base unit, derived unit, monetary unit, rough unit, or personal unit.

   Answers: (a) area; (b) 1400; (c) square feet; (d) derived unit.

   Comment: Square feet is derived from feet, a unit which is a multiple of meter.

5. As in Example 4, identify the attributes, measures, and measure units, and classify the units. If turned on for $3\frac{1}{2}$ hours, a 100-watt bulb will use 350 watt-hours of energy.

   Answers: (a) time, electric power, energy; (b) $3\frac{1}{2}$, 100, 350; (c) hours, watt, watt-hours; (d) hours is a multiple unit (from seconds); the others are derived units.

6. As in Example 4, identify the attributes, measures, and measure units, and classify the units. Some people believe that 5 g per day of vitamin C will reduce the number of colds a person will contract. Sixty to 80 mg per day of Vitamin C are recommended as a minimum by the National Research Council.

   Answers: (a) All are measuring rate of vitamin intake (i.e., average quantity of vitamins taken in over a given period). (b) 5, sixty, 80. (c) Grams per day, milligrams per day, milligrams per day. (d) All are derived units.

   Comment: Though gram is a base unit and milligram is a multiple of a base unit, the "per day" makes these units derived.

7. As in Example 4, identify the attributes, measures, and measure units and classify the units. To buy a $100,000 home with a late 1982 mortgage rate of 17%, 20% down payment, and payment over 29 years it was estimated that an annual income of about $61,000 would be needed.

   Answers: (a) value, time, value. (b) 100,000; 29, 61,700; (c) dollars, years, dollars; (d) monetary unit, multiple unit (from seconds), monetary unit.

   Comment: Both 17% and 20% are ratio comparisons, discussed in Section D of this chapter.

8. As in Example 4, identify the atttributes, measures, and measure units, and classify the units. A 2-ounce candy bar costs a quarter.

   Answers: (a) weight, value; (b) 2, quarter; (c) ounces, dollars; (d) multiple unit (of grams), monetary unit.

   Comment: While the word "quarter" can be considered as denoting a coin, it also is literally a "quarter of a dollar", equal to $.25, and denotes the number 1/4.

9. As in Example 4, identify the attributes, measures, and measure units, and classify the units. In 1980, in a U.S. family in which both parents were about 40 years old, the average number of children was 1.7.

   Answers: (a) time, family size; (b) 40, 1.7; (c) years, children per family; (d) multiple unit (of seconds), derived unit (from counting units).

   Comment: 1980 is a location on a scale, discussed in Section C of this chapter.

   Comment: The 1.7 children per family is a measure that some people think is weird because they view "children" as the unit. Actually, "children per family" is the unit. It is indeed impossible to have 1.7 children, but the concept of 1.7 children per family is quite reasonable, and the accuracy to tenths is required in questions concerning population growth. E.g., the number of children per family decreased from 6.3 to 4.7 during the late industrial period in England.

10. The exchange rates between the British pound and the U.S. dollar changed from $2.80 per pound in 1965 to $1.82 per pound in 1978.

(a) Was this change favorable to U.S. purchasers of British goods?

(b) What would have been the change in dollar cost over this time period of something that cost 2 pounds in 1965?

Answers: (a) favorable, since the dollar could buy more pounds;

(b) from $5.60 to $3.64.

Comment: Usually, when the price of a foreign currency goes down in relation to the dollar, it is a sign of relative strength for the dollar. Similarly, when the price of gold goes down on the overseas money markets, it is a sign of strength for the dollar.

11. Give the value of each of these: (a) in U.S. football, a safety; (b) in college basketball, a field goal; (c) in women's gymnastics, a small error in a routine; (d) in tennis, a let serve.

Answers: (a) 2 points, (b) 2 points; (c) usually -.1 points, that is, a deduction of a tenth of a point; (d) 0 points--a let serve does not affect the score.

Comment: While football and basketball avoid fractions, fractions are the rule in diving, gymnastics, and figure skating.

Number Use Class C: Locations

A location is a number that denotes a place in a rank, scale, or other ordered framework, but does not have all of the properties of counts or measures. Here are some examples of numbers used as locations.

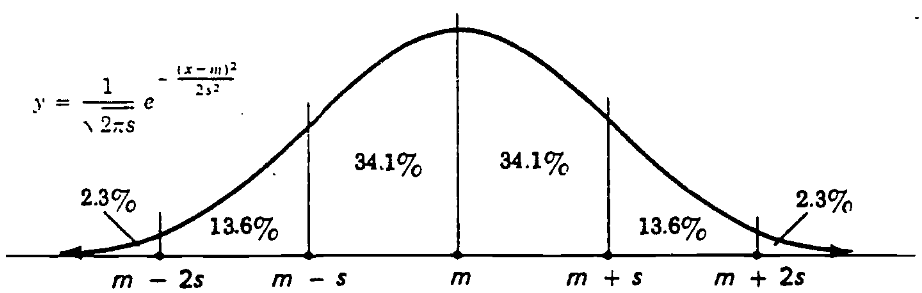
Context	Name or type of framework
Normal room temperature is about 21°C.	Celsius temperature scale
My brother was born on December 21, 1980.	Gregorian calendar
That horse finished in first place.	rank order
Some mathematics books are shelved under 512.3.	Dewey decimal system.

The various scales or frameworks can be categorized by the mathematical operations one can meaningfully perform on the numbers.

Some location frameworks are based upon counts or measures, in that a particular count or measure is taken as zero and from that zero one counts or measures in two directions. These frameworks are called <u>interval</u> scales and include Celsius temperatures (based upon kelvin units but measured from the ice point of water), calendars (based upon time, measured from some important date in history), and altitudes on Earth (based upon length, measured from sea level). With interval scales, there is a unit distance and subtraction is meaningful for telling you how far apart two locations are. For instance, it is meaningful to subtract a day's low temperature from a high temperature to get a spread of temperatures. However, adding temperatures yields a sum that has no physical referent. There is usually no putting together addition (see Chapter 5, Section A) with locations. Addition and then division might, however, be used to estimate <u>average</u> temperature.

Other frameworks have different mathematical properties. The decibel scale for sound intensity, the magnitudes for brightness of stars, and the Richter scale for earthquake intensities are all based on exponential scales. Since $A^n/A^m = A^{n-m}$, ratio comparison of such values is accomplished by subtracting the values. For example, to compare sound intensities of 60 and 75 decibels, subtract 60 from 75 and divide by 10, yielding 1.5, then take 10 to the 1.5 power (use a calculator to determine $10^{1.5}$). The result, approximately 31.6, indicates that the 75-decibel sound is about 31.6 times as intense as the 60-decibel sound.

IQ's, SAT scores, and many other education measurements are normalized scales based on the normal curve, often used, at least in the past, as a grading curve. If a distribution of scores fits such a curve, one can expect to find about 68% of the values within one standard deviation of the mean (34% on each side of the mean) and about 95% within two standard deviations from the mean (leaving a little more than 2% at each tail beyond two standard deviations).



For example, most IQ tests are scaled with 100 as the mean and 15 as standard deviation, so on these tests about 16% of the people will have an IQ above 115. People sometimes subtract to compare such scores but that is seldom meaningful. For example, the 15 point difference between

IQ's of 100 and 115 embraces 34% of the population, but the same difference between 130 and 145 includes only about 2% of the population.

In contrast to the above scales, some location frameworks possess no mathematical property other than order, and are consequently called ordinal scales. Examples include library shelving systems, the Beaufort scale for wind velocities, the Mohs hardness scale for minerals, and the Mercalli scale for earthquakes. These scales possess non-mathematical descriptions of each of their values to make up for the lack of mathematical properties.

Not all frameworks are so easily classified. The Fahrenheit temperature scale is a special kind of interval scale not so simply related to the kelvin measure. Rank orders have the properties of interval scales when only one thing can have a given rank (as in most horse races) but are ordinal scales when many things can be assigned the same rank (as in science fairs). Calendars are very much like interval scales (based upon time and measured from an important date) but all calendar systems have quirks, such as leap years, that make it more difficult to precisely compare dates.

Numbers used in interval scales are identified with the counting or measure units upon which they are based, so they look like counts or measures. Numbers in the other frameworks mentioned above are identified with the name of the framework; we speak of an "SAT score" of 512, 3.6 "on the Richter scale", etc. Locations not associated with counting or measure units are examples of scalars.

Examples:

1. Hardness of minerals is given by the Mohs hardness scale, a scale with the following values:

1	talc	9	topaz
2	gypsum	10	garnet
3	calcite	11	fused zirconia
4	flourite	12	fused alumina
5	apatite	13	silicon carbide
6	orthoclase	14	boron carbide
7	vitreous pure silica	15	diamond
8	quartz		

Are minerals with higher numbers on this scale harder or softer?

Answer: Harder, since diamond, the hardest of minerals, is given the highest number.

Comment: An older version of this scale had only 10 values: 1-6 as above, quartz and topaz as 7 and 8, corundum 9, and diamond 10. Man-made minerals made it necessary to put in more values at the hard end of the scale.

Comment: The Knoop hardness scale is a measure of the identation in a mineral made by a special device dropped on the mineral. It is used when more precise notions of hardness are desired.

2. Name at least one mathematical property of the Richter scale, a framework that is used for measuring earthquakes.

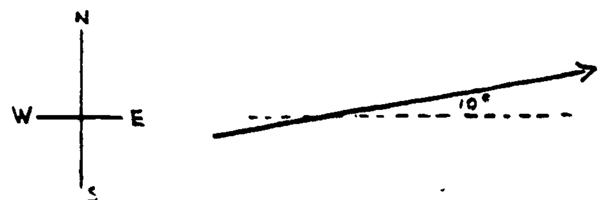
Answer: The Richter scale has the property that if two values differ by 1, then they represent a 10-fold increase in the force of the earthquake. Specifically, a value of 6.3 on this scale represents a quake that is 10 times the force of one that has a value of 5.3. Forces with values over 6 are considered strong enough to cause damage.

Comment: The framework is named after the American scientist, Charles F. Richter (1901-), who invented it in 1932.

Comment: Another scale for measuring earthquakes is the Mercalli scale, with values ranging from I to XII.

3. A tornado is spotted traveling in the direction 10° N of E. What does this mean?

   Answer: It means that, with east taken as 0°, the tornado is travelling in a direction from 10° south of west to 10° north of east, as shown by the arrow in the diagram.



   Comment: The key directions in this framework are the directions north, east, south, and west. By measuring from them, no angle measure greater than 45° is needed. For instance, 50° N of E would be described as 40° E of N.

4. Identify the locations in the following sentence: The news on Channel 2 runs from 6:00 to 7:00 PM and last month had a rating of 11.5.

   Answer: 2, 6:00 PM and 7:00 PM.

   Comment: Channel numbers also serve as identifications, but are located in a framework based upon the frequency of the radio waves over which the signal travels. Both 6:00 PM and 7:00 PM locate a time within a framework that is quite familiar to us; however, in the Army and in Europe these times would be denoted as 1800 hours and 1900 hours, respectively. The 11.5 is a ratio comparison, 11.5% of the audience, a use of numbers discussed in section D of this chapter.

5. Identify the numbers used as locations and name at least one key value in the overall framework discussed in the next sentence:

   In theory, 68% of the population should have IQs between 85 and 115, and similarly, 68% of the population should score between 400 and 600 on SAT exams.

   Answer: 85 and 115 are locations on an IQ scale. Both 400 and 600 are locations on the SAT test score scale. One key IQ value is 100, the theoretical mean. SAT scores were originally designed to have a mean of 500, a key value on that framework.

Comment: Both of these frameworks are called normalized scales, because a constant difference from the mean corresponds to a particular predesigned percentage in the distribution of scores, this percentage based upon the normal (bell-shaped) curve.

6. Identify the numbers used as locations and name at least one key value in the overall framework discussed in the next sentences:

The artificial satellite Sputnik I was launched by the Soviet Union on October 4, 1957. The first United States satellite followed three months later in January 1958, and was quite a bit smaller.

Answers: (a) I; first; October 4, 1957; and January 1958.

(b) "I" and "first" are ranks, based on time order. The two dates are in the Gregorian calendar, where the year 1 was the estimated birth year of Jesus.

Comment: Best estimates now place Jesus' birth as occurring between 4 and 6 B.C. The dates could be denoted numerically (and are often written) 10/4/57 and 1/58. This shows that October is "10th month" and January is "1st month". Etymologically, October means 8th month--octo being a prefix for 8--since in early Roman times the year began with March and not with January. This explains as well September, November, and December.

7. Name the two temperature frameworks in common use in the United States and give two key values in each.

Answer: The frameworks are the Fahrenheit and Celsius (formerly Centigrade) scales. Key values are: 32°F = 0°C = ice point of water, and 212°F = 100°C = steam point of water.

Comment: In the metric system, there is a base unit of thermodynamic temperature called the kelvin. When measured in kelvins, temperatures are measures like those in the previous section. 273.15 kelvins = 0°C and 373.15 kelvins = 100°C.

8. On January 1, 1980, we were in the year 5740 of the Jewish calendar and 1640 of the Japanese calendar. Since years in these calendars are about the same length, one can compare dates with little error. (a) What year will January 1, 2000, in the Gregorian calendar be in these other calendars? (b) To what date in our calendar did the year 1 in these other calendars correspond?

Answers: (a) January 1, 2000 will be in the year 5760 of the Jewish calendar and 1660 of the Japanese calendar. (b) the year 1 of the Jewish calendar occurred about 3760 B.C. (the year in which Jews of ancient times thought the world was created) and the year 1 of the Japanese calendar occurred in 341 A.D.

Comment: These corresponding values point out a difference between conversion of locations from one scale to another and conversion of measures from one unit to another. One can use multiplication to convert from one unit to another, but this seldom works with scales.

Comment: The length of a year in the Muslim calendar is 354 or 355 days, so dates in that calendar cannot be compared as easily to dates in the Gregorian calendar. On January 1, 1980 it was the year 1400 of the Muslim calendar, which began in 622 A.D., the traditional date for the flight of Mohammed from Mecca to Medina.

9. Rand McNally's <u>The International Atlas</u> gives the number 4418 beside Mount Whitney, California, and the number <u>86</u> (note the underline) beside Death Valley, California. What do these numbers mean?

Answer: The numbers represent the altitude (in meters) of the highest and lowest spots in California when compared to sea level. The underline indicates a negative, -86, to represent a spot below sea level.

Comment: Though we speak of the "height" of a mountain, the value given is seldom the vertical distance from the base to the peak. It is almost always the height above sea level.

Mt. Rainier in Washington and Pikes Peak in Colorado have about the same heights above sea level. But Mt. Rainier is more impressive because its base is nearly at sea level, whereas the land around Pikes Peak is already about 1700 meters above sea level. The height of Mt. Rainier from base to peak is almost 50% more than the corresponding height of Pikes Peak.

10. Aristotle lived around the year -300. What does the negative sign signify?

Answer: In some books, -300 signifies 300 B.C.

Comment: The negative sign is used in place of B.C. to make the scale more neutral with respect to religion.

11. On a clear night away from lights, a person can see stars of magnitude 5 or 6. In a city it is difficult to see stars beyond the second magnitude. Which is brighter, a fifth or a second magnitude star?

Answer: The lights of cities make it more difficult to see stars, indicating that lower numbers are identified with brighter stars and greater magnitudes with dimmer stars.

Comment: A difference of 5 magnitudes corresponds to a brightness factor of 100. So a 1st magnitude star is about 100 times brighter than a 6th magnitude star. Stars can be brighter than 1st magnitude; other than the Sun the brightest star in the sky is Sirius, with a magnitude of -1.47 (making it 100 times brighter than a star with magnitude 3.53). The North Star, Polaris, has a variable magnitude averaging about 2.

12. On a test, the highest score was Amanda's, a 97. Next were George and David, with 96. Then came Louise, with 95. Would you rank Louise 3rd or 4th highest in the class?

Answer: Louise is 4th highest, because there are 3 scores above hers. That George and David tied does not affect her rank.

Comment: Occasionally one sees, in this kind of situation, Louise given rank 3 in the class. This incorrect ranking for her is caused by confusing the rank of her score with her rank.

Certainly her score was 3rd best, but she was 4th best.

Comment: George and David are for some purposes said to be tied for second, but for other purposes (such as statistical analysis) assigned the rank 2.5.

Number Use Class D: Ratio Comparisons

The cost of living in 1980 was approximately 4 times what it was in 1946. The number 4, used in this context, represents a ratio comparison of costs from the two years. A ratio comparison is a number which can be thought of as the result of dividing two measures or counts with the same unit (but may not have actually been calculated this way). This use of numbers is quite common.

Context	Ratio	Quantities Compared
20% discount	20%	amount of discount and original price
probability of 1/1024	1/1024	number of ways an event can occur and number of total events possible
specific gravity of mercury is 13.596	13.596	mass of 1 cm^3 of mercury and mass of 1 cm^3 of water

All numbers used as ratio comparisons are scalars, numbers not associated with counting or measure units because, in dividing two measures or counts with the same unit, the units cancel. In place of units, ratio comparisons are labeled by descriptors, such as 40% "discount" and a "probability" of .8. (Recall that many of the numbers used as locations in Section C of this chapter are also scalars; their labels are the names of the frameworks.)

Though "ratio" and "rational" have the same root, numbers used as ratio comparisons need not be rational numbers: Pi, the golden ratio, and most sines, cosines, and tangent ratios found in trigonometry are irrational numbers. Still it is not the derivation of these numbers that causes us to classify them as ratios, but their use. For example, the number pi, though defined as a ratio (of circumference to diameter in a circle), can be a measure (the area of a circle with radius 1 meter is π square meters) and is very commonly found as a derived formula value (see Section F of this chapter).

Examples:

1. When we speak of the scale of a typical state map as being 1:500,000, the number $\frac{1}{500,000}$ has a ratio comparison use. What is being compared?

   Answer: The two quantities being compared are distances on the map and the corresponding distances on the Earth.

   Comment: One can also think of a map scale as an ordered pair. See Section A of Chapter 2.

2. One week in 1980 a six-month certificate of deposit yielded interest at an annual rate of 14.806%. Of the four numbers in the preceding sentence, which is a ratio comparison and what is being compared?

   Answer: The ratio comparison is 14.806%. The two quantities being compared are the interest given on this deposit and the amount of money deposited.

   Comment: The percent sign (%) is part of the number, acting like 1/100. The computation done to arrive at the percentage is not simple division but a more complicated process, taking yearly rates into account.

3. The cost of living in 1979 was about 3.6 times what it was in 1946. Briefly describe how the number 3.6 might be arrived at.

   Answer: The number 3.6 might be found by dividing the cost of a "market basket" of goods and services in 1979 by the corresponding cost in 1946. However, changes in life style and products often force adjustments to be made in values used in "market basket" formulas.

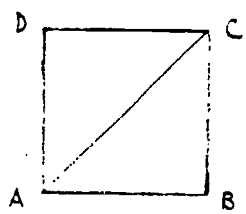
Comment: Salaries of many workers are modified based on changes in the cost of living, so how such a ratio is calculated is significant.

4. The average woman weighs about 4/5 as much as the average man. How might the number 4/5 in this context be used in calculations?

Answer: If you know the weight of the average man, multiply by 4/5 to get the weight of the average woman.

Comment: One advantage of using fractions is that the comparison "the other way" is also easy to make. The average man weighs about 5/4 as much as the average woman.

5. Measure $\overline{AB}$ and $\overline{AC}$. Then divide the length of $\overline{AC}$ by the length of $\overline{AB}$ to arrive at a number that compares the length of a diagonal of a square with the length of a side. Give your answer in a sentence of the same form as Example 4.



Answer: A sentence such as "The diagonal is about 1.4 times as long as the side" would be correct. Any answer must, of course, approximate the results from the Pythagorean theorem, which in this case is that the diagonal is $\sqrt{2}$ times the length of a side. (Or, equivalently, the length of a side is the length of the diagonal divided by $\sqrt{2}$).

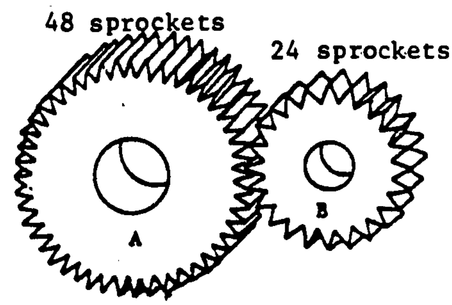
Comment: For most computations, one approximates $\sqrt{2}$ by a decimal, perhaps 1.4 or 1.414, with more decimals used as greater accuracy is needed. The sequence of fractions 7/5, 17/12, 41/29, 99/70, 239/169, . . . (Can you see the way each term is formed from the previous?) also provides better and better approximations to $\sqrt{2}$. But if one needs to physically construct a length involving a square root, it is possible to do so without recourse to measuring by using diagonals of squares and rectangles.

6. That microscope can have as much as 512 power. What does the 512 compare?

Answer: Being compared are the actual size of an object and the perceived size as viewed through the microscope, 512 times as great.

Comment: We could also say that the object was only 1/512 as big as its microscope image.

7. Indicate how negative numbers could be used to describe the ratio of revolutions of the two gears.



Answer: The gear at right will turn twice as fast as the gear at the left and in the opposite direction. So we could say that the right gear turns -2 times as fast as the left gear.

Comment: If the gears were separated and turned in the same direction, then the ratio would be positive. This kind of analysis can be used to simplify very complicated gear systems.

8. In sentences (a) to (e), what number represents each probability? (a) The probability of an event as likely not to happen as to happen. (b) The probability of an event that is sure to happen. (c) The probability of an event almost sure to happen. (d) The probability of an event that is possible but quite unlikely. (e) The probability of an event that is impossible (such as getting 11 questions right on a 10-question test).

Answers: (a) 1/2; (b) 1; (c) some number close to 1 and less than 1, such as .9, or .95, or .983, or .999994; (d) some positive number close to 0, such as .03, .0046, or .00000006; (e) 0.

9. One store advertises a chair on sale at 20% off. A second store has the same chair on sale for 30% off. Which is the better buy?

Answer: Not enough information is given to determine which is better. If the chairs originally sold for the same price, then the second store has the cheaper sale price, of course.

Comment: The larger the discount, the cheaper the product, but only if the original price is the same. Stores have been known to utilize this lack of information and advertise "1/2 off" sales (a good deal, it seems) without indicating either the original or the sale price.

Number Use Class E: Codes

Every person is identified by many numbers: credit card numbers, social security number, driver's license number, employee number, phone number, etc. These numbers exemplify a use class we call <u>codes</u>. Codes identify things and, as a result, can be used to transmit information. The authors' zip code 60637 contains three pieces of information:

6 refers to the Midwest: Illinois, Missouri, Nebraska or Kansas

06 refers to Chicago

37 refers to the neighborhood of Chicago, including the University of Chicago

Many codes are <u>alphanumeric</u>, that is, they use letters as well as numbers. License plates are alphanumeric because the ability to use any one of 26 letters in a space gives more choices than the use of any one of 10 digits. Model numbers of airplanes and automobiles are often alphanumeric codes because letters help to break up long strings of numbers and make the code easier to read.

It is seldom meaningful to use an entire code in any sort of computation. But particular digits in a code may signify measures or counts or locations or comparisons and thus be amenable to computation. In fact, identification numbers often hold more information than simple identification.

<u>Examples</u>:

1. What is the meaning of the 1 and the 800 in the phone number 1-800-555-2368?

   Answer: The 1 signifies a long distance call; the 800 indicates that this is a call for which the caller is not charged.

Comment: One does not see 18005552368 written without spaces or dashes. Spaces and dashes make numbers easier to remember.

Comment: 800 is one of the few three-digit beginnings to long distance phone numbers that do not signify geographic areas. The area codes were originally selected so that the most telephoned areas would be easiest to dial and require the fewest clicks. For this reason, New York is 212, Los Angeles 213 and Chicago 312.

2. More DC-10's have been sold than L1011's. What do DC and L mean in these alphanumeric codes?

   Answer: The L in L1011 stands for Lockheed and the DC for Douglas Corporation (now McDonnell-Douglas), the manufacturers.

   Comment: The 10 in DC-10 is roughly a tenth model-type in the Douglas line.

3. What common code is often used for room numbers in a school or other large building?

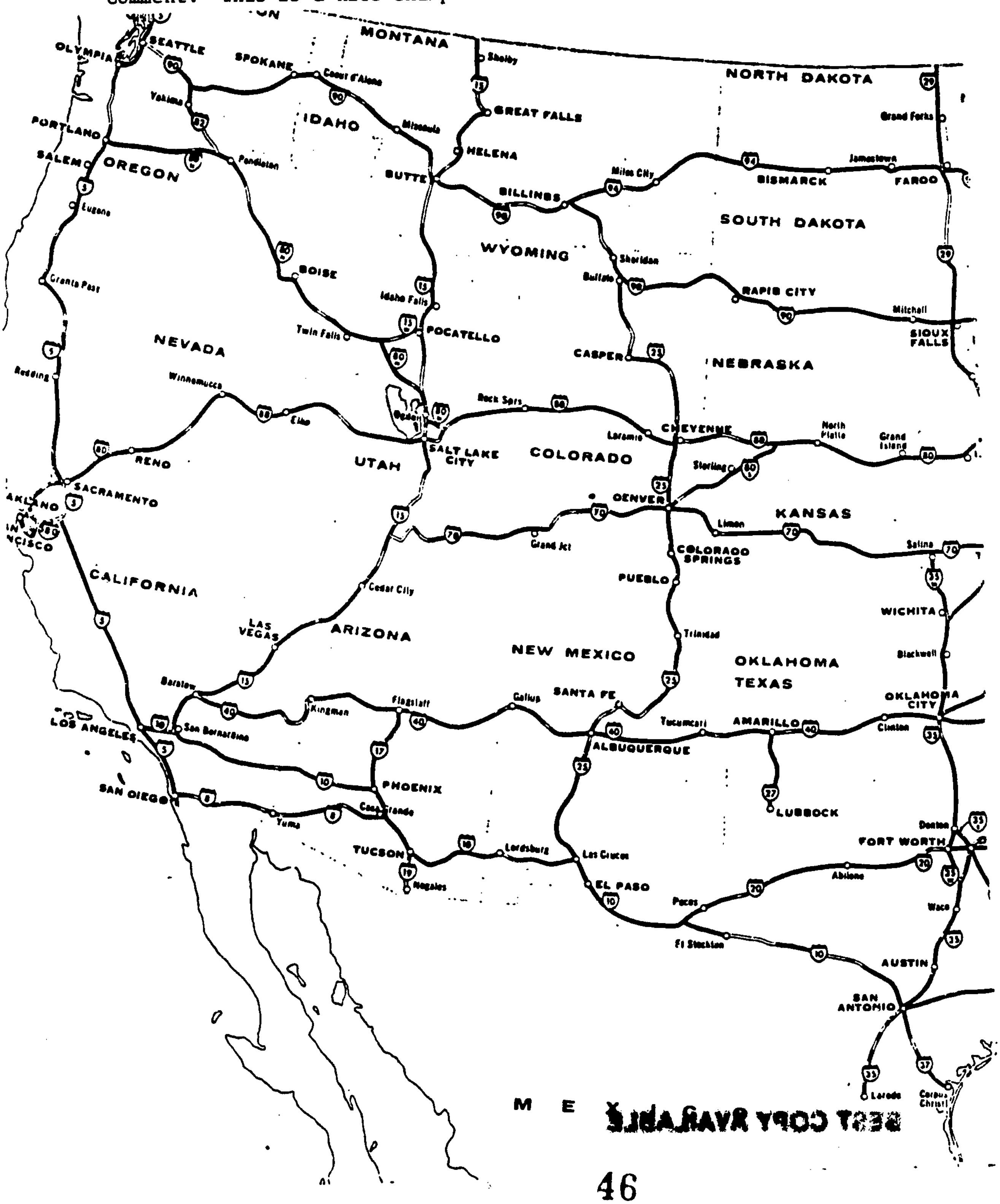
   Answer: Rooms are identified by 3- or 4-digit numbers. The hundreds place (or both hundreds and thousands, if over the 9th floor) denotes the floor, the units and tens digits the room. Thus 312 will usually be room 12 on the 3rd floor. (The 12 may have little meaning.)

   Comment: When there are over 100 rooms on a floor, as in a large hotel, either an ordered pair such as 3-112 or an alphanumeric system is needed. In Europe, rooms are coded in the same way but the floors are numbered to be 1 less than our floors. So 312 would be on what people in the U.S. would call the 4th floor, but what Europeans call the 3rd floor.

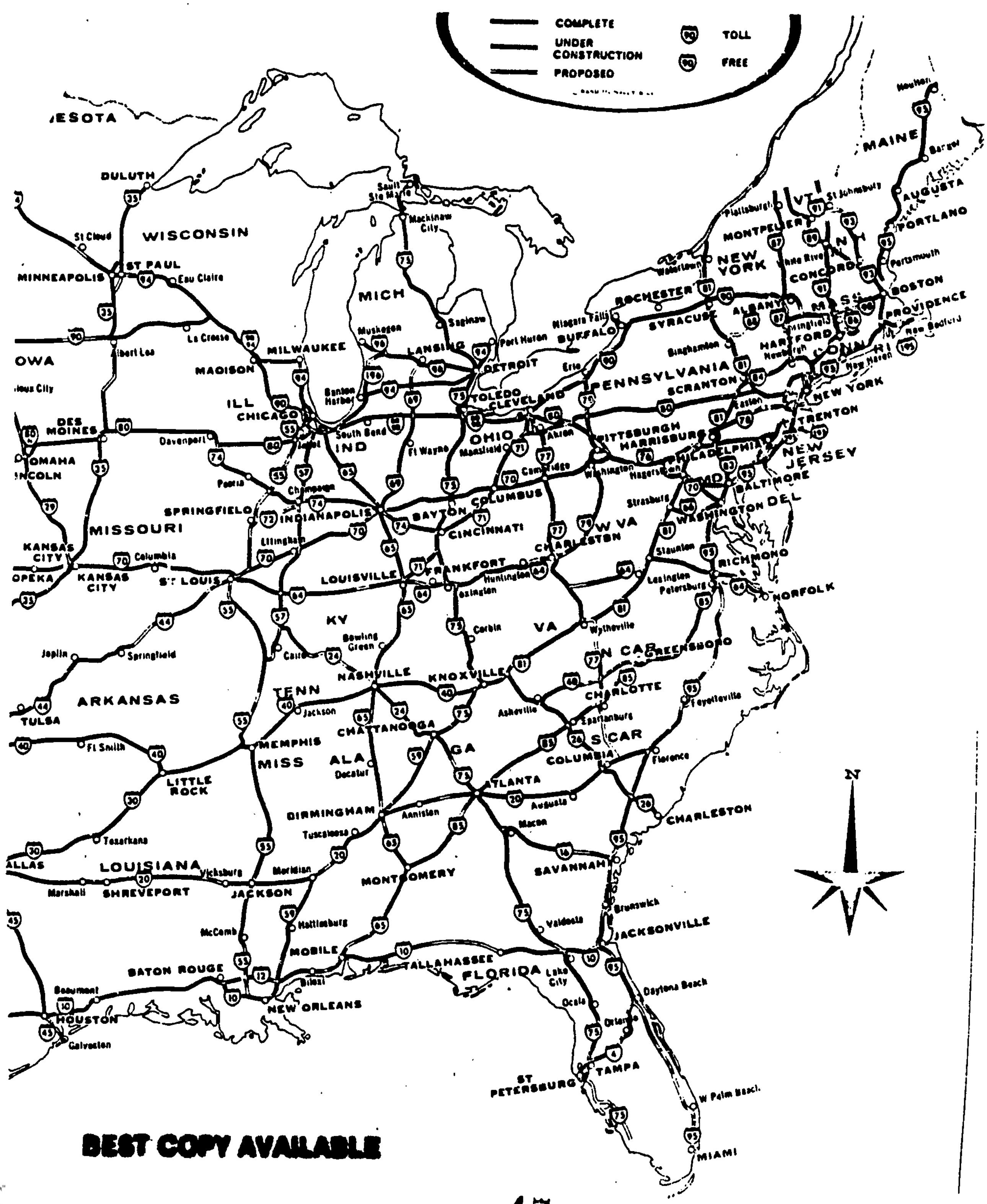
4. Use the map of interstate highways on the next pages to figure out how numbers were assigned to interstate highways.

   Answer: Interstate highways in the U.S. are identified by two-digit codes so that odd numbers generally represent north-south routes and even numbers represent east-west routes. Also, the numbers are coordinated nationwide so that north-south highway numbers increase from west to east and east-west highway numbers increase from south to north. Three digit numbers are used around cities.

Comment: This is a nice example of a code that has no secrecy to it.



Interstate System Map



Number Use Class F: Derived Formula Constants

A formula is a generalization of a relationship among numbers and quantities. Some common formulas are:

$P = 2\ell + 2w$	for the perimeter of a rectangle,
$A = \pi r^2$	for the area of a circle, and
$F = \frac{9}{5}C + 32$	for converting Celsius to Farhenheit temperatures,

but these are only a few of literally tens of thousands of formulas that have uses.

In the above formulas, P, A, ℓ, w, r, F and C are called <u>variables</u> because (and this is the whole point of formulas) their values may change from situation to situation. (See Chapter 4, Part A for additional discussion of this use of variables.) The numbers 2, π, $\frac{9}{5}$, and 32 are <u>constants</u>. When number constants in formulas are analyzed (see Chapter 10 for examples of such analyses), we find that each number has a meaning, which can often be traced back to one of the uses of numbers discussed in earlier sections. Sometimes, however, the particular way that a number is used in a formula is difficult to link to these uses. Whether the linkage is obvious or obscure we call these numbers <u>derived formula constants</u>; they are derived from more basic uses.

For example, the number π is usually defined (as was discussed in Section C) as a ratio. But the use of π in the formula $A = \pi r^2$ is not as a ratio but as a number derived in a not-so-simple way from the earlier definition. On the surface, the number 2 in this same formula seems to indicate only the number of times r is used as a factor so it seems to be a count. But analysis of the derivation of the formula shows it to be related to the way in which we calculate area, beginning with the area of a square, and a more general analysis shows it to represent area as a two-dimensional measure (see Chapter 9, Section A).

An analysis of the temperature conversion formula $F = \frac{9}{5}C + 32$ is done in Chapter 10, Section C, and shows that the 32 comes from the equality of 32°F and 0°C (as ice point of H_2O) while the $\frac{9}{5}$ is a ratio of unit length in these scales. So the numbers have meaning, and their use in the formula is derived from these meanings. In the equivalent more complicated formula, $F = \frac{9}{5}(C + \frac{160}{9})$; the number 160/9 is derived using mathematical considerations and has no particular significance in either of these temperature scales. Still another equivalent formula is $F = C + 4(.2C + 8)$. This formula is easier to fit on a typewritten line and exhibits no division but its derivation is more obscure. The numbers 4, .2, and 8 in this last formula have no significance for temperature. They are outstanding examples of derived formula constants.

Examples:

1. The Earth is approximately spherical with a radius of about 3960 miles. The surface area S of a sphere with radius r can be found by the formula $S = 4\pi r^2$. What is the surface area of the Earth?

   Answer: $S \approx 4 \times 3.14 \times (3960)^2$ square miles

   $\approx 197{,}000{,}000$ square miles

   Comment: The formula indicates that the surface area of a sphere is 4 times the area of a circle with the same radius, but there is no very simple way to show this. The derivation of this formula requires notions of limits not usually studied until calculus. Thus we identify the 4 as a derived formula value use of numbers.

   Comment: This question also exemplifies estimation (3.14 for π) for ease of use (Chapter 12, Section B). We use 3 significant figures because the given radius has that and because the earth itself is not an exact sphere.

   Comment: The seven largest countries in the world, by area (millions of square miles), are the U.S.S.R. (8.6), Canada (3.9), China (3.7), U.S. (3.6), Brazil (3.3), Australia (3.0), and India (1.3).

2. The "rule of 72" can be applied to determine when money will double. If money is invested at 12% annual interest, compounded, it will double in approximately 6 years, because $72 \div 12 = 6$. (a) Use the rule of 72 to determine how long it takes money to double if invested at 10% annual interest compounded. (b) In the first two sentences of this example, which number is the derived formula constant?

   Answers: (a) $72 \div 10 = 7.2$, so it takes 7.2 years for money to double at 10%. (b) The number 72.

   Comment: Keeping track of the units in this computation makes no sense; normally 12% x 6 years = .72 years, but the rule of 72 does not mention years. In derived formula constant uses of numbers, the unit is often absent or obscure.

Comment: If money is invested at an annual rate of r% compounded, then it will double in (log 2)/log (1 + r) years. For small values of r as are typically found in interest rates, the fraction is quite close to 72/r, close enough to give this approximation so useful to bankers and savers.

Summary

We recognize six use classes of single numbers: counts, measures, locations, ratio comparisons, codes, and derived formula constants.

A <u>count</u> is a number that indicates how many things are in a set. The things counted are usually identified by a counting unit.

A <u>measure</u> is a number that indicates how much of a particular attribute some thing has. A measure always is associated with a measure unit; that unit may be a base unit (with very precise standardization), a multiple of a base unit, a derived unit, a monetary unit, a rough unit, or a unit based upon a temporary or personal definition. Zero measure signifies absence of the attribute in the thing under consideration.

A <u>location</u> is a number that denotes a place in an ordered framework but is not a measure or count. Such frameworks include ordinal scales, which indicate nothing but order, interval scales; and other scales possessing sophisticated mathematical properties.

A <u>ratio comparison</u> is a number which can be the result of comparing by division two measures or counts with the same units. Ratio comparisons do not have a measure or counting unit.

A <u>code</u> is a number that is used to identify an object. Codes are often alphanumeric; that is, they involve both numbers and letters. It is seldom meaningful to use an entire code in any sort of computation.

A <u>derived formula constant</u> is a number that is found in formulas or rules, often derived utilizing mathematical manipulations or properties of numbers. At times the origins of such numbers can be traced back to the other uses; at other times the origins may be obscure.

Pedagogical Remarks

A person normally encounters all of the uses of numbers given in this chapter, but the school curriculum does not accurately represent the relative importance of these uses. In particular, counting is emphasized so much that some students believe all other uses of numbers to be artificial. In recent years, books have given greater attention to measures that in the past. We encourage this trend and particularly suggest that a wide variety of measure units be used in teaching.

In preparing this chapter, we were surprised at the number of examples of locations which we could easily find; the present curriculum had led us to expect a narrower selection. We encourage teachers to give students experience with all kinds of orders and scales. This is also the natural place to lead students into the uses of negative numbers.

Ratio comparisons are already in the curriculum. In this chapter, ratios are considered as single numbers, a notion often missed by students. Uses of ratios considered as ordered pairs are discussed in Chapter 2 and a section of Chapter 8 is devoted to ratios as a use class of division.

Since codes are rarely used in computation, they are not and perhaps need not be as prominent as the other uses. Still, students should be made aware of them in manners such as those discussed below.

Derived formula constants have always been standard in junior high school work and beyond, though seldom treated explicitly as a use of numbers.

Getting started. To introduce middle or junior high school children to the uses of numbers, we like to begin with an exercise involving a daily newspaper. All of the uses of numbers can be found in most newspapers quite easily, so this is a rich source.

We ask the students to circle all numbers they find on a page. With some pages, such as that including stock market reports, there are so many numbers that locating different types becomes the task. Then we ask them to count or estimate the number of numbers on the page. Most pages contain well over a hundred numbers. Students are usually surprised at how many numbers there are. That is the first purpose of the newspaper exercise--to emphasize the ubiquity and variety of numbers.

This exercise always provokes interesting questions. Is "three" a number? (Yes, a number does not have to be written in Arabic numerals.) Is 9:30, the time of a store opening, one number or two? (One, because the 30 represents 30/60 of an hour. Had our method of telling time not originated with the Babylonians, who preferred to use 60 as a number base, 9:30 might be represented as 9.5, i.e., as nine and a half hours.) Is a date, "October 13, 1982" or "10/13/82" one number, two, or three? (One number, for it represents one location in a time frame.) In the phrase "front row in an auto race", is "front" a number because it could mean the same thing as "first"? (We prefer to think not. If something is not stated mathematically, let us not give mathematical emphasis to it. We have to draw the line somewhere.) Do not be pedantic by insisting upon one interpretation; many distinctions are a matter of opinion.

For the next phase of the exercise, list the use classes on the chalkboard. Ask the students to find numbers (with labels, when appropriate) for each use class.

All students need discussion to clarify the differences between the uses. Such discussion is necessary to the success of the exercise and should be done with everyone participating. For example, page numbers may be both counts and locations. "It takes ten hours to get to Hawaii" and "ten o'clock" both signify time, but "ten hours" is a measure and "ten o'clock" is a location in a time framework.

The point is not to restrict each instance of a number to one class, but to establish each use as deserving of mention. This exercise will help the students to begin to think about the uses of numbers and provides a basis for more detailed discussion of certain use classes, such as measure units. (You might even keep a list of weird or borderline examples which do not seem to fit the classes perfectly.)

Counts. Counts are covered well in schoolbooks. Still, there are certain points which need to be emphasized. First, there is a difference between a count as a use and the action called counting. They do not necessarily occur together. To find the number of seats in a classroom, we may multiply the number of rows by the number of seats in a row. So we have not counted every seat even though we are getting a count. Conversely, we may count to help us determine, with a yardstick, the measure of something like the length of a room.

Second, counts are not necessarily small numbers. They may be very large, as in population or inventory counts.

Third, counts are not necessarily exact. They may be estimates, for example, of the number of people watching a parade, the number of animals that will be sighted on a safari, or the number of hamburgers a store sells in a year.

Measures. We cannot overemphasize the importance of measure units when considering measures. (At one time we considered devoting an entire chapter of this book to measure units and other kinds of labels.) Yet, in general, measure unit labels are ignored in mathematics classes. One even sees problems in books such as the following: "Find a triangle with sides 3, 4, and x whose area and perimeter are equal." Yet it is impossible for the area and the perimeter to be equal since one is measured in linear units, and the other in area units. (It is permissible to get around this difficulty by asking for an area and a perimeter that are numerically equal.)

Because students are taught (unwisely, we believe) to count to get measures, they often have a hard time distinguishing measures from counts. The key is the measure unit. Unlike a counting unit, a measure unit can be split into smaller units. Even a penny can theoretically be split into small units. (In fact, the mil is one-tenth of a penny.) Splitting a measure unit does not change the attribute being measured, but it will change the number called the measure.

Because there are so many measures, the teacher may wish to discuss four types of measures separately: monetary measures, measures of value (points, etc.), geometric measures (length, area, angle measure), other physical measures. All of these types of measures are important.

We encourage teachers to use the metric system, but we feel that to ignore the present customary units is a disservice to our students. For example, it may take a long time before carpenters use the metric system.

Emphasize those things that are already metric when teaching students the prefixes: vitamins and minerals are measured in grams, micrograms, and milligrams; camera film and some cigarettes in millimeters; blood and other liquids used in medicine in cubic centimeters (cc's) or milliliters; energy usage in kilowatt-hours; computer speed in nanoseconds; energy force of bombs in megatons; our monetary system in cents and mils. Stress the prefixes that are in common use, and be aware that even experienced scientists refer to tables when they encounter an unusual prefix.

Locations. This use class is exceedingly rich, and examples range from the simple to the complex.

Begin with addresses. Use a map of the school neighborhood and help students locate where they live. Is there any regularity to the addresses? Are odd numbers on one side of the street and even numbers on the other? Could you predict what the addresses will be on a block you have never seen?

Another good example is time. It is easy to order dates. You may wish to draw a time line, but remember that years are not points on a line but intervals. After you have discussed our usual (Gregorian) calendar and put in some key dates, superimpose some of the other calendars used in today's world (Jewish, Muslim, Chinese).

Temperature is a particularly good example to motivate negative numbers. One degree on the Fahrenheit scale is only 5/9 as big as a degree on the Celsius scale, something like the Muslim year being about 97/100 as long as the Gregorian year.

Ordinal scales are easy to understand. Put descriptions of the Mohs hardness scale, or the Beaufort scale, and perhaps a grading scale, on the bulletin board. When there is a day with a very strong wind, ask students to determine what this wind would be described as on the Beaufort scale. Bring in a stone and see if its hardness can be determined on the Mohs scale.

Ranks are particularly important locations. Students need to know the ordinal names of the numbers: first, second, third, etc.

More complicated scales, such as the Richter scale for measuring earthquakes, should be mentioned but discussed only as time permits (or on the day following a local or newsworthy earthquake).

Ratio comparison. As a use, stress the notion that a ratio may be considered as a single number. This is hard for students to grasp if they conceive of all fractions as two numbers separated by a bar.

One way to get the notion of ratio as a single number is to shift back and forth from fraction to decimal to percent notation. For instance, a copy machine makes copies 3/4, 2/3, and 1/2 original size. Which gives the smallest copy? The question could also be stated: A copy machine makes copies 75%, $66\frac{2}{3}$%, and 50% original size. Which button should be pushed to make the smallest copy?

Ratios can be, and often are, numbers larger than one. Stress the notion that a fraction can be larger than one. This may be correcting something which was wrongly taught (or wrongly learned) in earlier grades.

Codes. Students should be given at least one experience in trying to make and break a code. See Section A of Chapter 2 for some examples. It is instructive to make a list of all identification numbers that people can think of. It is interesting and perhaps depressing to note the size of this list. Students may wish to discuss if having a single identification number would be better.

Derived Formula Constants. It is important to stress that formulas do not merely appear; they have origins in basic ideas from which they are derived. Many of the derivations utilize ideas found throughout Part II of this book.

Connecting use classes with types of numbers and with operations. Today's schoolbooks are not organized by use class; rather, it is common to organize by types of numbers or by operations. Examining the tables on the next page can help in deciding when to study a particular use. For example, negative numbers are not appropriate with counts or measures (unless these are extended into a scale) or codes, but are appropriate with some locations and comparisons.

The tables clearly show why some students have difficulties in learning to apply new types of numbers. A student who sees numbers only as counts cannot hope to understand how to apply fractions. A student who sees numbers only as counts or measures will not understand how negative numbers are applied.

Part II of this book discusses uses of operations, and these tables provide a nice beginning for that kind of discussion. For instance, by asking students to come up with subtraction situations for counts, measures, locations, and ratio comparisons, a variety of different uses of subtraction may appear. For example, with locations, take-away subtraction is inappropriate, comparison subtraction is quite common. (How much warmer is 35°F than 30°F?)

Finally, we caution against the teaching of sorting into use classes as if this were the important skill. Knowing the use classes of numbers is a means to better understanding of how arithmetic is used and is meant to be a help in organizing thinking about these uses. Being aware of the different use classes and being able to examplify each in various ways are the important ends; being able to catalog numbers into use classes is not itself a skill upon which to drill students. (A similar comment applies to all chapters in this book.)

Amenability of Uses to Types of Numbers and Operations

Is the operation appropriate?

Use Class	+	-	x	÷	Ordering	Associated Label
Counts	Y	Y	Y	Y	Y	counting unit
Measures	Y	Y	Y	Y	Y	measure unit
Locations	S	S	N	N	Y	framework name
Ratio Comparisons	Y	Y	Y	Y	Y	descriptor
Codes	N	N	N	N	N	code name

What kind of numbers are typically involved?

Use Class	Whole	Zero	Decimals or Fractions	Negative
Counts	Y	Y	N	N
Measures	Y	Y	Y	N
Locations	Y	S	Y	S
Ratio Comparisons	Y	S	Y	S
Codes	Y	S	N	N
Formula constants	Y	N	Y	Y

Y = yes or almost always, S = sometimes, N = no or rarely.

Questions

1. Consider the following quote from an actual news story.

   Free-Way, a one-passenger, three-wheel economy car, gives you 100 mpg or refunds its $2800 price, says H.M. Vehicles, 1116 E. Highway 13, Burnsville, MN 55337. A one-cylinder, 345-cc engine with automatic transmission, it squeezes 900 miles from a tank, says the maker.

   (a) Identify each number being used as a count. Identify the counting unit associated with each number. (b) Identify each number being used as a measure. Give the measure in full for each such number. Classify the measure unit as one of the following: base unit of the metric system, multiple of the base unit, derived unit, monetary unit, rough unit, or personal unit. Identify each attribute being measured. (c) Identify each number which is a location in a framework and name the kind of framework. (d) Identify each number which is used as a code.

2. Answer Questions 1a-1d for the following quote, from the Guinness Book of World Records some years ago.

   The tallest living woman is Sandy Allen (born June 18, 1955 in Chicago) who lives in Shelbyville, Indiana, and works in Indianapolis. In September, 1974, she measured 7 feet 5-5/16 inches and is [was] still growing. A 6 1/2 pound baby, her abnormal growth began soon after birth. She now weighs 421 lbs. and takes a size 16EE shoe. She uses 6 yards of material to make a dress.

3. These following words represent multiples of units. For each, state the prefix, give its value and tell what the unit measures: (a) millisecond; (b) microgram; (c) decibel; (d) kilohertz; (e) centimeter.

4. Many derived units in the metric system are named after famous scientists. Use a dictionary or other source to determine what each of the following units measures and how it is derived from the base units. Then name one thing for which the scientist is known. (a) newton; (b) gauss; (c) faraday; (d) watt.

5. Newspapers carry the exchange rates between the dollar and foreign currencies at least once a week. Using the most recent information available find the value of the dollar in terms of the yen, franc, British pound, and Deutschmark.

6. The Beaufort scale indicates wind velocities on a scale from 0 (calm) to 17 (extremely violent hurricane). Find the wind velocities that correspond to the Beaufort scale numbers 0 through 17 and give the Beaufort description of these velocities.

7. Give the whole number that indicates where you would look for each of the following in the Dewey decimal system. (a) a translation into English of a novel by Dostoevsky; (b) an algebra book; (c) a book on the history of the United States; (d) a book on vitamins.

8. All numbers in the statements that follow are ratio comparisons. For each indicate the two quantities being compared.

   (a) Doll houses are generally built to be 1/12 actual size.

   (b) The precipitation probability for this area is 20% tonight and 30% tomorrow.

(c) If you invest money at a yearly interest rate of 8%, then each year you can think of your money as having been multiplied by 1.08.

9. A business lists its net profit for the year as -$32,050. What does the (-) sign before the dollar sign indicate?

10. Give a context in which 0 is used as (a) a count, (b) a measure; (c) a location; (d) a ratio comparison.

11. Name ten codes that are used by various organizations or businesses to identify you. (You do not have to give the actual numbers.)

12. Some numbers identify ideas or feelings in the minds of many people. What do these numbers signify? (a) 13 (b) 7 (c) 3, to some Christians, (d) 18, to Jews (e) 666, to some Catholics and Fundamentalist Christians.

13. A formula for the approximate distance d it takes a car to stop from a speed of V miles per hour is $d = \frac{11}{200} V^2 + \frac{12}{11}V - \frac{11}{8}$. (a) According to this formula, what is the stopping distance at 50 mph? (b) Give two reasons why the formula must yield an approximation. (c) Name all derived formula constants in the formula.

Notes and Commentary

(At the beginning of each "Notes" section, the topics of the notes are listed for convenient reference.)

Note	Topic
1.	The structure of the book related to the modelling process
2.	Use classes of others
3.	Our previous work
4.	Counts vs. measures
5.	Non-treatment of locations and codes by others
6.	Ratio comparisons
7.	Derived formula constants
8.	Other use classes

1. The structure of this book related to the modelling process. The three parts of this book roughly parallel steps in what has been termed the modelling process. The steps in that process are often characterized as the following:

1. confrontation of the real situation
2. simplification of the real situation (so as to prepare for step 3)
3. translation of the simplified situation into a mathematical counterpart (the mathematical model)
4. solution or other treatment within the mathematical model
5. translation of the mathematical results back to the original situation
6. check of the feasibility of the solution in the real situation

Seldom does one go through all six steps, and very often one shifts back and forth among the steps. The mathematical subject matter in this book is restricted to models from arithmetic. Part I of the book relates to step 1 of the modelling process, Part II (Operations) relates to steps 3 and 5, and Part III (Maneuvers) relates to step 2. Step 4 is generally considered to be adequately covered in elementary school textbooks, and step 6 depends completely on the particular situation being analyzed. Thus we have covered virtually all the areas we can with respect to the treatment of applications of arithmetic.

Yet there is one major area not treated here, that of the construction of numerical information in the first place. We are not concerned in this book with how one arrives at suitable counts, measures, scales, codes, sets, ordered pairs, or n-tuples. Such tasks may fall to anyone but often are the work of statisticians, scientists, economists, cryptographers, computer programmers, and other specialists. In this book, as in many situations in which we use arithmetic, we are consumers rather than creators of data. For example, a consumer may deal with an interest rate on a mortgage, savings account, or investment, not create those rates. For all of us in most situations in which we confront arithmetic, the numbers are already there.

2. Use classes of others. The terminology use class was invented for the purposes of this volume; the corresponding ideas are often found labelled meanings of number by others. In the 1890's, reflecting the emphasis on precision and rigor that preoccupied mathematicians, psychologists tended to adopt one meaning of number. McLellan and Dewey (1895) distinguish number from quantity:

> "Number ... as distinct from the magnitude which is the unit of reference ["unit" in this volume], and from the magnitude [attribute] which is the unity or limited quality to be measured is:
>
> The repetition of a certain magnitude used as the unit of measurement to equal or express the comparative value of a magnitude of the same kind. It always answers the question 'How many?'" (p. 71)

They go on to say,

> "... the number and the measuring unit together give the absolute magnitude of the quantity. The number by itself indicates its relative value. It always expresses ratio--i.e., the relation which the magnitude to be measured bears to the unit of reference. (pp. 71-72, emphasis theirs)"

We find this definition to be too restrictive. The 30 in 30 F has only a distant relation to the absolute magnitude of anything. No ratio is expressed by one's driver's license number, and there is no unit. SAT scores express no ratios. And so on.

McLellan and Dewey find agreement with their one-faceted view of number in the views of two eminent mathematicians.

> Newton: "Number is the abstract ratio of one quantity to another quantity of the same kind."
>
> Euler: "Number is the ratio of one quantity to another quantity taken as a unit." (p.72)

D. E. Phillips (1897) selects the series meaning of number as most basic. In this conception, seven means the number in the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9,..., the number preceding 8 and following 6. This meaning is even more restrictive than that of Dewey and McLellan.

Thorndike (1922) criticized those who concentrated too much on one meaning. He identified four meanings: the series meaning, the collection meaning (7 means the number of single things in a colection of 7 dolls), the ratio meaning of McLellan and Dewey, and the relational meaning (i.e., determining the meaning of 7 from its relations with other numbers).

> Ordinary school practice has commonly accepted the second (collection) meaning as that which it is the task of the school to teach beginners, but each of the other meanings has been alleged to be the essential one,... (p. 2)

If we now realize that, to these psychologists, number almost always means "whole number", then Thorndike's meanings are quite close to those found here. Only codes are missing.

Thorndike et al.	Ours
series	locations
collection	counts
ratio	measures (but not ratio comparisons)
relational	comparisons

The catholic view of Thorndike seems to have been lost on later mathematics educators. Morton, author of an extensive analysis of arithmetic (1937), asserts (p. 57) that "Counting is the fundamental number experience.". He dismisses measurement, citing Smith's finding (1924) that first-graders count much more than they measure in out-of-school experiences with numbers. We see two fundamental problems in Morton's conclusion. First, the processes of counting and measurement are being equated with the concepts of how numbers are used in experience. To us, counts (as a use of numbers) are not counting (a process) and to speak of measures is not the same as speaking of measuring. Second, the experiences of very young children do not necessarily reflect how concepts "naturally" develop. The experiences of young chilren are affected by the ways they have been exposed to number concepts.

The methods books of recent times (since 1960) have tended to concentrate on structural aspects of arithmetic or concrete embodiments of these aspects. In some books, one finds many activities involving counting or measuring, but these activities tend to be focussed on the concretizations rather than on actual uses, and on the processes rather than on the results.

In all of these eras, the dependence of the school curriculum on computational algorithms, which are different for fractions than for decimals, and different still for whole numbers, has led educators to consider uses of number in conjunction with particular sets of numbers. Counting is generally considered to be the province of whole numbers, measurement and ratio the province of decimals and fractions. Locations are ignored except for the primitive notions of cardinal and ordinal numbers.

Thus sixty years after Thorndike, we find his thinking to be closest to ours. We expect that, with people from so many diverse fields analyzing various aspects of numbers, others have done analyses like ours, and urge those who are aware of such analyses to write us.

3. Our previous work. The ideas of Chapter 1 began in Max Bell's Mathematical Uses and Models in Our Everyday World, (1972). In that work there is specific discussion of single numbers for descriptions, ordering, identification, and coding. The first of these is elaborated upon there as follows:

> "Usually our first and most frequent encounter with mathematical models is with numbers used to describe or quantify some situation. Often the numbers are whole numbers obtained by some counting process; frequently they express measures of something; sometimes a single number description is obtained from combining several other numbers." (p. 5)

The notion of numbers being used for indexing appears in a section title but the term is not illustrated with problems given in that section.

In 1974, Bell identified the "main uses" of numbers among a list of "What Does 'Everyman' Really Need from School Mathematics?". These main uses were said to be counting, measuring, coordinate systems, ordering, indexing, identification numbers and codes, and ratios.

In Usiskin's Algebra Through Applications (1976), Bell's ideas were adapted in a scheme which related uses of numbers to types of numbers. Natural numbers were related to counting, identification and coding, and ordering. Rational numbers were associated with measurement, scoring, locating, and comparison. Negative numbers were applied to situations that have two opposite directions.

Compared to our previous works, in the present work ideas are more carefully defined, more amply illustrated, and primary attention is directed to the use of rather than to the type of number.

4. Counts vs. measures. This distinction is not always made, and indeed both counts and measures satisfy the mathematical properties of outer measure functions. Specifically, M is a finitely additive measure on a collection of sets, if for all sets A and B in the collection, the measure M(A) is never negative, is zero only for the null set, and is additive in the sense that whenever A and B are disjoint, M(AUB) = M(A) + M(B). Thus there is mathematical backing for grouping counts and measures together. We considered doing so, but felt that the special nature and importance of counting and the enormous variety of measures made it wiser to separate these two crucial ideas.

The mathematical commonalities between counts and measures do not support the notion that all measuring is counting, as is sometimes taught. The common phrases "count how many feet long the room is" or "count your change before you leave the store" confuse the issue. Counts may be considered as measures, but not vice-versa.

5. Non-treatment of locations and codes by others. All methods books in arithmetic treat counting as important, and all mention measurement and ratio. Locations and codes are generally ignored. For example, a recent article that analyzes newspapers, in a way similar to our analyses of everyday experience identifies eight categories of uses of mathematics (not just arithmetic).

> "The first category is termed small number concept. This category contains articles that use whole numbers between 1 and 999, primarily for the purposes of measurement or comparison. For example, articles that mention 3 weeks, 45 children, $947,or 19 inches fall into this category. Articles that contain numbers as labels (e.g., Proposition 13), dates (e.g., 10 September 1957), and time (e.g., 9:57 A.M.) are not put in this or any other category." (Czepiel and Esty, 1980)

In this way the authors consciously dismiss two of our use classes of number. (By the way, The New York Times, the newspaper analyzed, would probably not write September 10, 1957 as 10 September 1957. Studies of uses should report actual uses, not wishful alterations of uses.)

Locations are particularly important because only in them do we observe uses of negative numbers. We would like to see more use of negatives but most people and essentially all elementary school books delay them as long as possible. This is a vestige from the late entry of negatives into the world of numbers; only in the Middle Ages were negatives first considered to exist.

Outside of school, negatives are being seen in more uses. Accountants no longer use red ink because it is impossible to differentiate red from black ink in duplicated copies; instead, they either circle numbers to indicate negatives or use the mathematical (−) notation. Negatives for situations like ahead-behind in bowling and under and over par in golf are seen on TV sports programs. Changes in stock prices are represented as positive and negative numbers. Overdrafts in many checking accounts are denoted with negative numbers. The prospect is for these uses to increase because calculators work as easily with negative as with positive numbers.

6. Ratio comparisons. Ratios are treated in three different ways in this book. The uses of ratios considered as single numbers fall in this chapter. Ratios considered as ordered pairs are treated in Chapter 2. Ratios considered as an application of division are found in Chapter 8. Detailed commentary is given in

Note 10 of Chapter 2 and Note 9 of Chapter 8.

7. Derived formula constants. Uses of numbers interplay with mathematical operations (Part II of this book), maneuvers (Part III), and mathematical theory. Often the referent for a number is obscured by the number of steps or the sophistication of these interplays. We find this particularly true in various formulas, rules, or rules of thumb. For instance, there is a "rule of 78" used in figuring out how much of a loan has been repaid. The 78 comes from the sum of the whole numbers 1 through 12. It is not a count, nor a measure, nor a location, nor a ratio comparison, nor code. Yet is is a number with a use. Thus we found ourselves forced into this category, which might be considered a "catch-all" for those things we could not classify, but more accurately represents the realization that relationships among numbers and operations on numbers give rise to other numbers whose most immediate referents are the relationships and operations themselves.

8. Other use classes. At one time we had a use class entitled nominal uses, including derived formula constants, lucky numbers, random numbers, and numbers used for their own sake (such as when one memorizes places in the decimal expansion of pi). Ultimately, derived formula constants prevailed as the use class. These include random numbers, which are themselves derived. Lucky numbers are categorized under codes, since, e.g., the number 13 does or does not signify various things to various people in much the same way as do ID numbers and other codes. We decided to delete numbers used for their own sake. By this we mean situations of which the following are examples.

1. Add 35 and 22.
2. 67.3 x 43.5 =
3. 3 feet 2 inches = ____ inches
4. What is the L.C.M. of 24 and 30?
5. What is the third angle of a triangle with two angles of measures 45° and 32°?
6. Find a number that is 5 times the sum of its digits.

This is not to say that there are not situations in which all of the above might be useful (though we can't think of one for the last example). But to us, uses require a real context, a real situation beyond even that given for examples 3 and 5. All these examples, do, however, fall on the border between real and contrived use. This border is not narrow, for if one needs to know how to compute in order to pass a test for getting a job, say as a postal employee, then any of the above examples might prove to be most useful indeed for that person. Yet classifying something as useful only because it may later be found on a test is not classifying at all. Anything may later be found on a test.

The border is wider than this, however. Many mathematicians are not concerned with the external applicability of their subject yet make their living dealing with numbers and symbols. Here are some of those uses.

1. The only even prime is 2.
2. The numbers 220 and 284 are the smallest pair of numbers with the property that each equals the sum of the others divisors.
3. The limit of the ratio of consecutive terms of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... is $.5(\sqrt{5} - 1)$.
4. The fundamental period of the sine function is 2π.
5. The area between the graph of $y = 1/x$ and the x-axis and the line $x = 1$ and the line $x = 7$ is $\ln 7$.

For these mathematicians, the nominal uses of number are real world uses. Yet, because all of the ideas in this book have uses within mathematics, and because other books, including most mathematics texts, are almost wholly devoted to these uses, here we purposely exclude the intra-mathematical uses of numbers and all other topics. For us, for this entire volume, uses refer to situations outside of mathmatics itself.

CHAPTER 2

USES OF ORDERED PAIRS, ORDERED TRIPLES, OR n-TUPLES

Single numbers do not suffice for all uses. It takes two numbers to identify a particular location on Earth, and three numbers if one wishes to consider altitude as well (perhaps to identify the location of an airplane). Ratios usually involve two numbers, but extended ratios may involve three or more numbers. It takes nine numbers to describe a team score in most baseball games (one number per inning). Dialing a long distance call requires ten or eleven numbers. An ice cream store with 32 flavors of ice cream requires 32 numbers just to keep track of its ice cream inventory; large stores and companies may require thousands of numbers to describe sales.

The situations of interest in this chapter are not those in which there are merely lots of numbers, but situations in which two, three, or more numbers <u>in a specified order</u> represent a single idea. Here are some examples.

Situation	Idea
New York is located at latitude 41°N, longitude 74°W.	location
The proportion of adult males to adult females to children attending the movie is 5:4:12.	ratio
Keeping track of the inventory requires knowing how many of each of 2430 items are in stock.	count

The location 41°N, 74°W, is an ordered pair. The ratio 5:4:12 is an ordered triple. The inventory will require an ordered sequence of 2430 numbers.

An n-tuple is an ordered sequence of n numbers, called components. If there are two components, the n-tuple is called an ordered pair. If there are three components, the n-tuple is called an ordered triple. Mathematicians denote n-tuples by using parentheses, with the n components separated by commas. The n-tuples above could be written as (41°N, 74°W), (5, 4, 12), and as an n-tuple with 2430 components. The number n is the dimension of the n-tuple.

The use classes of n-tuples parallel those for single numbers. We call them counts, measures, locations, ratios, codes, and combined uses.

n-tuple Use Class A: Counts

A multi-dimensional count appears when one is attempting to keep track of more than one count at a time. The number of dimensions is the number of item-types one is counting. The examples are self-explanatory.

Examples:

1. Inventory. Tickets for a particular concert are in three price ranges, which we will call C (cheap), M (moderate), and E (expensive). The total number of tickets available is 4700, of which there are 2000 C, 2500 M, and 200 E. (a) Name an advantage of keeping the information as a 3-tuple (2000, 2500, 200) rather than as a single number. (b) If all tickets are sold except for 124 tickets in the moderate range and 5 in the expensive category, what 3-tuple represents the present inventory?

   Answers: (a) Having the detail makes it easier to respond to inquiries regarding seats available and simplifies computation of total sales. (b) (0, 124, 5).

   Comment: Store inventories are much more complicated and may contain thousands of components. These would normally be separated into n-tuples of fewer components, perhaps by section of the store or by type of item, to make the inventory manageable. Computers are in common use for the task of helping keep track of inventory. Cash registers in some stores are linked directly to the computer to provide updating of inventory after each transaction. This requires coding of all items.

2. Box scores. The "box score" from a baseball game consists of the number of runs by both teams scored in each inning, the total runs, hits, and errors. Since there are nine innings, the box score is usually a 12-tuple in each of two rows, one for each team.

	Inning											
	1	2	3	4	5	6	7	8	9	R	H	E
Team A	0	0	1	0	2	0	0	1	0	4	9	2
Team B	3	0 . . .										

The partial box score given here indicates that team A scored 1 run in the 3rd inning, 2 runs in the 5th inning, and 1 in the 8th inning for a total of 4 runs. Also team A had 9 hits and 2 errors. Complete the box score for the opposing team B, if B scored 3 runs in the first inning, 2 in the 6th, had 12 hits and no errors, Who won?

Answer: Team B's 12-tuple is: 3 0 0 0 0 2 0 0 0 5 12 0. The 10th component, the total number of runs, determines who wins. So team B, with 5 runs, won.

Comment: Baseball, hockey, and soccer use counts in determining scores. Football and basketball give different values for different kinds of scoring and we would classify their box scores as measures.

Comment: A box score for a nine-inning baseball game is a 2 x 12 matrix.

n-tuple Use Class B: Measures

There are two types of uses of n-tuples for measures. In the first type, the n-tuple (usually an ordered pair or ordered triple) stands for a measure that cannot be described by a single number. An example is force, which in scientific discussions involves both a magnitude and a direction, and a description must contain enough information to determine both. The second type is the inventory application, analogous to the previous section, with measures kept rather than counts. In uses of the second type, each n-tuple may contain thousands of components.

Examples:

1. Weather. A hurricane is described as moving at 30 knots in a north-westerly direction. This velocity can be written as an ordered pair (30, NW) or (30, 315°), the 315° being measured clockwise from due north. (A knot is a unit of speed equal to about 1.15 mph.) Describe the movement of a storm that is moving at (25, 90°).

   Answer: The storm is moving due east at 25 knots.

   Comment: Like physicists do, we distinguish velocity from speed. Velocity combines speed with direction; thus to measure velocity ordered pairs are natural objects to use. The ordered pair is used as a single measure, with its components being a measure (25 knots) and a location (90°).

   Comment: These kinds of ordered pairs are akin to polar coordinates in advanced high school mathematics courses and though used here to describe the movements of weather systems, are used as well to locate weather, ships at sea, and planes in the air.

2. Inventory. What is an efficient way for the manager of a 35-flavors ice cream store to keep information regarding the amount of ice cream in stock and the amount sold?

   Answer: The manager could use a 35-tuple, with each component of the 35-tuple standing for a specific flavor. Inventory could be kept as one 35-tuple, sales each day as another. So, for example, if the components were (vanilla, chocolate, . . .) and the day began with 40 gallons of vanilla and 35 gallons of chocolate, and 20.3 gallons of vanilla and 13.7 gallons of chocolate were left at the end of the day, then

(40, 35, . . .) would be inventory at beginning of day, and

(20.3, 13.7, . . .) is inventory at end of day.

Subtracting the corresponding components gives (19.7, 21.3 . . .), the sales for the day.

3. Diet. For a 128-lb, 5'4" female age 25, the recommended daily dietary allowances are 2100 calories, 58 g protein, 0.8 g calcium, 18 mg iron, 4000 I.U. Vitamin A, 1.0 mg thiamine, 1.2 mg riboflavin, 13 mg niacin, and 70 mg Vitamin C. This information can be stored as an n-tuple. What would that n-tuple be and to what use could it be put?

Answer: (2100, 58, 0.8, 18, 4000, 1.0, 1.2, 13, 5). Each component of the n-tuple stands for a different component of the diet. Units would be recorded for each but not stored. In planning a diet, a dietitian can make a corresponding n-tuple for each food in the diet. Then, by adding the numbers in the respective components of all foods in the diet for a given day, the totals can be compared with this recommended amount to see if there are any deficiencies.

Comment: This type of analysis seems to have been first done toward the end of World War II, when some food rationing did exist in the United States and there was the possibility that more rationing might be required. There was then a desire to see if recommended dietary allowances could be achieved through judicious balance of available foods.

n-tuple Use Class C: Locations

The latitude 19°24'N does not locate any single point on Earth, nor does longitude 99°5'W. However, together as the ordered pair (19°24'N, 99°5'W), they stand for the approximate center of Mexico City. This ordered pair exemplifies the use of an n-tuple as identifying a single location, with the same intent as single number locations.

The notion of dimension of an n-tuple comes from this use; the ordered pair gives a two-dimensional location on the surface of the Earth. Were we to take into account Mexico City's altitude above sea level (about 2240 meters), an ordered triple would be needed.

As with the inventory use of n-tuples to represent multi-dimensional counts and measures, there also exist multi-dimensional scales or profiles. These are common in psychological testing, economic analyses, and other mathematical analyses of complex phenomena.

Examples:

1. Location on Earth. The latitudes and longitudes of the centers of five largest metropolitan areas in the world are:

Tokyo	39°45'N, 135°45'E
New York	40°45'N, 74°00'W
Paris	48°50'N, 2°20'E
Shanghai	31° 1'N, 121°25'E
Mexico City	19°24'N, 99° 5'W

Which of these cities is farthest north?

Answer: Paris, since it has the largest latitude.

Comment: The use of negative numbers to represent west and south would make it possible to avoid the use of letters, but this is not customary.

Comment: It makes little sense to speak of farthest east or west.

2. <u>Astronomy</u>. A star in the sky can be located by giving two coordinates, its altitude and its azimuth. The altitude measures the angle of the star from the nearest point on the horizon. The azimuth measures how far clockwise that point on the horizon is from North. Give the altitude and azimuth of a star that is $\frac{1}{3}$ of the way up in the sky in a southwesterly direction.

Answer: The altitude is $\frac{1}{3}$ of $90°$ or $30°$. The azimuth is $225°$.

Comment: Because the locations of stars in the sky change as the Earth rotates, astronomers more often use two other coordinates to locate stars: the right ascension and the declination. The right ascension is measured in hours and minutes after the Earth rotates past a particular point in space. The declination is measured like latitude--north (positive) or south (negative) of the plane containing the Earth's equator. Sirius, the brightest star in the evening sky, has right ascension 6h 42.9m (h and m stand for hours and minutes) and declination $-16°39'$.

Sirius is 8.7 light years from Earth, so its position in space 8.7 years ago could be given by the ordered triple (6h 42.9m, $-16°39'$, 8.7). Three more coordinates would be needed to describe its motion. Thus in general a computer would store a 6-tuple to describe where a heavenly body is and how it is moving.

3. <u>Profiles</u>. Scores on the Scholastic Aptitude Test of the College Board form a two-dimensional profile of the examinee, the first component V being the verbal ability, the second component Q standing for quantitative ability. Each component ranges from a low score of 200 to a highest possible score of 800, with mean scores at present being about 430 for V and 470 for Q. How would you describe a student with each of the following (V,Q) profiles? (a) (250, 600); (b) (470, 500); (c) (720, 740); (d) (430, 300); (e) (460, 430).

Answers: (a) Very weak verbally, strong mathematically. (This might possibly be a student for whom English is a second language.)

(b) An above average student;

(c) A student with high ability;

(d) An average student with poor mathematics preparation;

(e) An average student.

Comment: Colleges use more than SATs in making judgments regarding admission. Their profiles may contain many more components, some of them non-numerical. We would classify such a profile under combined uses, as in Section F of this chapter.

n-tuple Use Class D: Ratios

If 7 A's and 14 B's are given in a class, then we say that the ratio of A's to B's is 7:14, and this ratio can be reduced (as if it were a fraction) to 1:2. If there are 14 C's as well, then the extended ratio 7:14:14 can be formed, and this extended ratio can be reduced to 1:2:2. Once the ratio is reduced, the original data is irretrievable, so many people prefer to keep the original data. We view the ratio 7:14 as an ordered pair, simply another way of writing (7,14), and the extended ratio 7:14:14 as an ordered triple. Though a ratio of two numbers can be reduced as if it were a fraction, extended ratios serve as a demonstration that, in general, ratios used in this way are not always fractions.

Examples:

1. Ratios. Suppose there are 15 girls and 10 boys in a class. What is the ratio of girls to boys?

   Answer: 15:10, or reduced form, 3:2.

   Comment: Since 15 divided by 10 is equal to 3 divided by 2, one often divides and says that the ratio is 1.5. This use of division is found in Chapter 8, Section A, and the resulting single numbers are ratio comparisons as discussed in Chapter 1, Section D.

   Comment: When thinking about ratios as ordered pairs, we recommend using the colon, as in 15:10. Using the fraction notation 15/10 confuses the ordered pair with the quotient.

2. Extended ratios. In the United States, percentages for blood types O, A, B, and AB among Blacks are 47%, 28%, 20%, and 5%, respectively, which can be written as the extended ratio 47:28:20:5. Among whites the extended ratio is 45:40:11:4. What does this second ratio signify?

   Answer: About 45% of whites are type O, 40% are type A, 11% are type B, and 4% are type AB.

Comment: Because the ratios of blood types differ among races, they are used by anthropologists to help trace the origins and movements of societies.

Comment: Having components of extended ratios add to 100 simplifies the conversion to percentages and makes comparing the ratios easier.

3. Map scales. The scale on a map is written as 1:1000000. What does this ratio signify?

Answer: The map is one millionth actual size.

Comment: The scale 1:1000000 is suitable for exhibiting a small country or state on a single page map. With it, one kilometer on Earth is represented by one millimeter on the map. Various nations have cooperated to make a world-wide map with this scale. (See David Greenhood, Mapping, University of Chicago Press, 1965, p. 48.)

Comment: One sometimes sees units written on scale maps, with an equal sign instead of a colon, as in $\frac{1}{4}$" = 1' for a plan (say of a house) where 1 foot in the actual house is $\frac{1}{4}$ inch on the map. For surveys of lots, a common ratio is 1" = 20'. This use of equals can be confusing.

4. Odds. When the odds favoring (or against) something are 3 to 1, the event is expected to occur (or not to occur) 3 times for every times it does not occur (or occurs). What does each of the following mean? (a) Odds of 5-2 that candidate A will win. (b) Odds of 10-1 against a labor strike.

Answers: (a) Candidate A is expected to win with a probability of $\frac{5}{7}$, or about .71. (b) The labor strike is expected to occur with a probability of $\frac{1}{11}$, or about 9%.

Comment: Odds of 3 to 1 may be written 3-1 or 3:1. The expression "odds are 1 in 6" means a ratio of 1:5 of occurrence.

Comment: Odds in betting situations do not signify probabilities or expected chances of winning or losing, but are based upon the amounts bet for and against the event occurring. They are designed so that the race track or bookie or betting establishment will almost always pay out less than it takes in.

n-tuple Use Class E: Codes

The zip code 60637 in section E of Chapter 1 was explained as having three parts: 6 for the region, 06 for the city, and 37 for the neighborhood. Hence, we might conceive of a zip code as a 3-tuple (6,06,37). Many codes have this flexibility; parts can be separated or joined to form distinct or unified bits of alphanumeric information.

Other codes, particularly those used to convey information, may consist of very long strings of numbers (containing thousands or even millions of individual digits). The decoder must first know how many symbols constitute a single bit of information and then treat the information as an n-tuple with many components, each of which is a smaller code.

Examples:

1. Phone numbers. For the phone number 1-900-555-2368, identify:

   (a) the long distance signal; (b) the exchange; (c) the area code.

   Answers: (a) 1; (b) 555; (c) 900.

   Comment: As of 1982, some places in the country do not require the long distance signal. The code 900 is a special area code, signifying a long distance call to the nearest central office where the particular phone number has meaning.

   Comment: Of course, one could think of this phone number as a single 11-digit number 19005552368, but for memory we separate it into the components given above and ignore the long distance components when we are making a local call.

2. Coded messages. Assign to each letter of the alphabet a two-digit number from 01 to 26. For this example, suppose that Z = 01, Y = 02, X = 03, and so on with A = 26. Then a message beginning with "USA, hello . . ." and containing 1000 letters would begin 0608261722151512 . . . , have 2000 digits, but be decoded considering the 1000-tuple (06, 08, 26, 17, 22, 15, 15, 12, . . .). Using this code, decode (24, 12, 23, 22, 08, 26, 09, 22, 21, 06, 13).

Answer: CODESAREFUN.

Comment: By using numbers to represent punctuation marks and spaces, one can make decoded messages look more like actual messages.

Comment: The use of codes to transmit financial information is routine and confidentiality is a must. Thus secret codes are found in many situations, in wartime or among competing world powers, and coding theory has become an increasingly substantial portion of mathematics. The codes used in the real world tend to be quite a bit more complex than the one used in this example.

n-tuple Use Class F: Combined Uses

When information about a particular thing is gathered from a variety of sources, this information is often stored in computers as n-tuples. Earlier use classes have given some examples of storage as in inventory, but here we consider examples where the components may be quite diverse, perhaps including alphanumeric information. Generally one does not go to the trouble of creating these n-tuples unless there is a reasonably large number of n-tuples needed.

Examples:

1. Identification cards. The information on a driver's license may be considered as an n-tuple. Typical components are (name, address, social security number, expiration date, height, weight, birth date, driver's license number). Fill in the 8-tuple as it would apply to you.

   Answer: Answers will vary. Each is of the form (John Doe, 33110 First Avenue, Enid OK, 222-22-2222, 1 June 1990, 5'6", 135 lb, 6 June 1965, Z123456).

   Comment: When n-tuples are used for storage, it is common for some of the information to be alphanumeric.

2. Research data. To study student mathematics performance in a school, data are collected. What kinds of data might be found in the n-tuple for each student?

   Answer: At a minimum, name and responses to each item on all tests given. The responses might be coded numerically as right-wrong or by quality of answer. Other information that might be used: sex, age, grade, teacher, textbook, total scores, grades on other tests, course grades, etc.

Comment: Computers have completely changed the amount of such information one can realistically deal with.

3. Performance. Newspapers contain a great deal of numerical information, particularly in sports and business pages, that can be classified as performance. Here is one line of the New York bond listing for October 20, 1982.

IBM Cr $14\frac{3}{8}$ 86 13.2 52 $108\frac{5}{8}$ 108 $108\frac{5}{8}$ $+\frac{5}{8}$

(a) How many separate pieces of information seem to be displayed?

(b) What numbers stand for the original yield and current yield (i.e., original and current interest rate) of this bond?

Answer: There are 9 pieces of information, in order: Name of bond (IBM Cr), original yield ($14\frac{3}{8}$), year bond matures (1986, represented by 86), current yield (13.2%), sales ($52,000), high for the previous day ($108\frac{5}{8}$), low for the previous day (108), final price ($108\frac{5}{8}$), and change ($+\frac{5}{8}$).

Comment: Note that this one n-tuple of information has ratios, measures, and a location (1986). If it were a listing for common stocks it would have counts (number sold) as well.

Comment: The line presented above is one of hundreds in a typical bond listing. So we think of hundreds of 9-tuples stored and printed in columns. The result is a matrix with hundreds of rows and 9 columns.

Summary

An *n-tuple* is an ordered sequence of n numbers, called components. Ordered pairs and ordered triples, the most common n-tuples, have 2 and 3 components, respectively.

The uses of n-tuples parallel those for single numbers. An n-tuple may represent an n-dimensional *count*, as in inventories. An n-tuple may be a *measure*, as in forces or other quantities that require more than a single number for their description, or an inventory of measures. An n-tuple is often needed for *locations*, as on Earth or in space, or in scale profiles. A common use of n-tuples is as *ratios* or in extended ratios, as when we say that enrollments of boys and girls are in the ratio 5:3. Also, n-tuples are used as *codes*, where a TV picture or hidden message may require an ordered sequence of thousands of numbers. Lastly, there are n-tuples whose main purpose is to store information of a variety of sorts; we call this a *combined use*.

Conceptually, an n-tuple is more than just its components. For example, the components of a ratio are usually counts, but the ratio is a comparison. Thus we distinguish the uses of n-tuples from the uses of the components, though they are often related.

Pedagogical Remarks

Language. The idea of an n-tuple is very easy; the language often is not. The term "n-tuple" is, however, so different and so peculiar that many students find it difficult to forget. Use of the phrase "ordered pair" is universal; 2-tuple is never found. The phrases "ordered triple" and "3-tuple" are used. For more than 3-components, 4-tuple, 5-tuple, etc., are most common.

Locations. We encourage beginning with location uses, as books normally do. Graph using ordered pairs, or look on a map. Two numbers or two letters are needed to locate a point on the graph or on the map. Yet the ordered pair identifies a single location. That's the crucial idea behind n-tuples; a pair or triplet or other ordered sequence of numbers is working to convey a single idea.

A reasonable activity is to take the latitude and longitude of some cities in the U.S. (most almanacs have this data) and graph these pairs to approximate the locations of the cities. Ask questions about the graph: Which city is farthest west? Which two cities are closest? Is city A farther north than city B? Then ask whether this information could be determined from the ordered pair without graphing.

In space, 3 numbers are needed. Connect this idea with dimension, and this gives a natural way to discuss the dimension of an n-tuple.

From this, almost any other use of n-tuples can be discussed next.

Codes. Codes are fun for students and serious business for banks, governments, and others interested in security. For slower or younger students, codes provide a particularly nice way to learn order of numbers and number names.

Room numbers in many schools use a code to identify location. The first digit may stand for the floor or a section of the school; the other digits may locate rooms in order.

Ratios. Many books introduce ratios as ordered pairs. Be sure to conceive of them also as single numbers, as done in Chapter 1. The equivalence of ordered pairs is just like the equivalence of fractions, namely two quantities in the ratio a to b are also in the ratio ka to kb, just as $\frac{a}{b} = \frac{ka}{kb}$. It's good practice for fractions to have students both "reduce" and "enlarge" ratios. Ask: If the ratio of A's to B's is 10:20, what are three equivalent ratios?

Other uses. Inventories, profiles, and the combined use of n-tuples for storage are quite similar conceptually. One way to start is to ask students what information they would want to know about someone in order to describe that person to someone else. Each type of information identifies a component in an n-tuple. For example, one such n-tuple might begin (name, height, weight, sex, glasses?, major interest, second interest, address, . . .). Then ask which components are most important. Suppose a computer had room for only 80 symbols (single digits or letters) for each individual. What information would you choose? These are decisions in which all students can participate and they mirror important decisions that must be made repeatedly in storage of information.

It is not necessary to have large numbers of components in n-tuples to store lots of information. Sometimes many n-tuples, each with only a couple of components, are needed. For instance, one might wish to tabulate how much TV was watched for a month. The data might consist of ordered pairs (day of month, number of hours of TV watched that day). This is a combined use.

<u>Ultimate goals</u>. A first goal in a study of uses of n-tuples is to realize that it is possible to take a large amount of information and order it in a simple, definite way. For students who are overwhelmed by numerical information, the goal is to show them that n-tuples provide a way to deal with data. A second important goal is to work with students to chunk many bits of information into a single idea, for example, to consider scores on 3 tests not as 3 separate numbers but as a profile, to consider counts of 100 items in a score not as 100 separate counts but as a single inventory. Chunking is a most useful type of higher-order thinking. These ideas are understandable to most students starting in primary school and the impact can be powerful.

Questions

1. A motel has rooms of four different prices: \$25, \$28, \$35, and \$40. One evening 17, 8, 12, and 8 rooms are filled at these prices, respectively. Give some advantages of storing this information as two 4-tuples.
2. To record the pets that people have, a pollster uses the n-tuple (number of dogs, number of cats, number of birds, number of rodents, number of other non-people mammals, number of fish, number of other pets). How many components (i.e., what value of n) is the pollster using? If you were being polled, what n-tuple would the pollster wind up with?
3. From an atlas, almanac, or other source, find the latitudes of Rome, Italy, and Chicago, USA. Which city is further north?
4. With a balanced coin, the odds against getting 5 heads in 5 tosses are 31 to 1. What does this mean?
5. Using the code of Example 2, Section E, code the saying: Misery loves company.
6. A shoe store sells 175 different styles of shoes. Indicate how n-tuples could be used to keep track of inventory, prices, numbers of shoes sold, and dollar sales amount.
7. According to the U.S. Bureau of the Census, in 1979 families were of the following sizes with the following numbers: 2 persons, 22.5 million; 3 persons, 13.0 million; 4 persons, 12.0 million; 5 persons, 6.1 million; 6 persons, 2.5 million; 7 persons or more 1.7 million. Place these frequencies into an extended ratio with six components. Then write the

extended ratio using percentages instead of the large numbers, rounding percentages to the nearest percent.

8. As seen from Bermuda, a hurricane is now at the position (30 miles, 225°) and moving at (20 mph, 90°). At this velocity, what ordered pair will indicate the hurricane's position one hour from now?

Notes and Commentary

1. Our previous work
2. Vectors and n-tuples
3. Alternate use-classes for n-tuples
4. Ratios: A survey of conceptions

1. Our previous work. Bell (1972) has a chapter on "Uses of pairs or triples of numbers" which includes uses of coordinate systems, uses of ratios and rates, and "numbers combined by calculation into new information". [Coordinate systems and ratios are part of the present chapter; the others are the subject matter of the entire second part of this book.] However, the sense in which n-tuples of counts or measures play descriptive roles akin to those played by single counts or measures is not dealt with.

2. Vectors and n-tuples. To the mathematician, a vector is one of a set of n-tuples upon which an equivalence relation and two operations, addition and scalar multiplication, are defined and satisfy certain properties. To the computer, any n-tuple, including even those with alphanumeric components, is a vector. The situation roughly parallels the number-numeral distinction, and has conceptual implications. These implications may be easier to see by considering the more familiar number-numeral situation.

There are those who would say that a social security number is not a number (in the mathematical sense of "real number"), because one would not conceive of adding or multiplying it, or applying order. That is, one conception of number is that a number is an element of a system; thus if an object does not have the properties of the system, it is not a number. The alternate conception is that anything that looks like a number should be considered as a number. It is this more inclusive conception which guided us through the fist chapter. (It is quite possible that the more restrictive conception was what caused some earlier authors to ignore the location and code uses of numbers.)

Similarly, we have selected the broad conception of n-tuple in considering their use classes. Anything that looks like an n-tuple is thus considered to be one. This is a particularly adaptable conception for the computer. Extending the idea, we might say that this chapter discuesses vectors. Indeed, we might have titled the chapter "Uses of Vectors" if not for the possibility of scaring off a large number of readers.

3. Alternate use classes for n-tuples. Early versions of these materials organized the uses of n-tuples into four use classes, none of which remains as is in the present scheme. They were: number-like uses, into which the entire chapter has now been conceptualized; storage, for inventory and combined uses; inputs for later operation, for those orderings in which components will later be combined together to yield single numbers or other n-tuples; and sequences for patterns, for orderings of

numbers in which each component is related to the number of its position in the n-tuple.

We were led into this early scheme because we asked the questions "What are n-tuples used for?", motivated by the related question "Why have n-tuples in the first place?" That is, we first sought to respond to the need for the conceptualization of an idea quite different from that of single numbers.

We changed into the present scheme, in which the use classes for n-tuples parallel those for single numbers, because we made a subtle change in the above questions. We decided to ask instead "What do n-tuples represent?" in order to have uniformity between this and the previous chapter.

To some extent, these different categorizations represent different slices of the same pie. Multi-dimensional counts and the other current use classes can contain examples where the n-tuples are used for storage, as inputs for later operation, or as sequences. For instance, in a class of 35 students with 20 girls and 15 boys, the ratio 20:15 might be said to store the given information, but one might use 20:15 as an input for division, with the result that there are 33 1/3% more girls than boys, or one might find ratios from other years and see if there is a pattern in the sequence.

Indeed, many mathematical objects, not just n-tuples, can be said to be used for storage of data, for inputs to later operation, or for the finding of patterns. So we came to believe that what we had started with was not particularly focussed on n-tuples, but rather on more general notions of why one does anything mathematically.

4. Ratios: A survey of conceptions. Textbooks employ two distinct yet related conceptions of ratio. Under the ordered pair conception, the ratio of 2 to 3 is different from the ratio of 4 to 6, these two being called equivalent. Under the division or single number conception, the ratio of 2 to 3 is the number 2/3 (always readable as 2 divided by 3) and is equal to the ratio of 4 to 6.

The differences in these conceptions are more than semantic, for the conception one prefers dictates how one feels ratios should be introduced, and the conceptions chosen may be related to how one feels about fractions and percent as well. Furthermore, we will attest that these conceptions determine how one feels about the uses of ratio, for we found ourselves in disagreement regarding the placement of ratios in this book.

As a way to seek assistance on this issue, one of us (Usiskin) surveyed 11 leaders and 16 doctoral students in mathematics education at the Universities of Georgia and Chicago in February, 1980. The survey form is given below. The results of this survey were a bit surprising to us, and you the reader may wish to respond to the seven items before reading on.

EQUALITY vs. EQUIVALENCE OF RATIOS

(Z. Usiskin, at University of Georgia, 2/15/80)

For my Ed 899 seminar, I would like your opinions on the following questions dealing with ratios. In the left column below are seven wordings of symbolizations associated with ratios. In the right column are the same wordings or symbolizations with numbers ten times as large. Everyone in the seminar believes that, in the language of ratio, each of these pairs is <u>equivalent</u>. But are they <u>equal</u> in the sense of mathematical equality? Put a Y in the blank if you think they are equal; place an N in the blank if you think they are equivalent but not equal. If you think they are equal sometimes and only equivalent others, write S.

1.	2 out of 3	___	20 out of 30
2.	2:3	___	20:30
3.	2/3	___	20/30
4.	2 of these for 3 of those	___	20 of these for 30 of those
5.	2 divided by 3	___	20 divided by 30
6.	the ratio of 2 to 3	___	the ratio of 20 to 30
7.	2:3:7	___	20:30:70

Do not read on before you indicate a response to each question. Now consider the following. By mathematical equality, we mean that numbers may be equal even when their numerals may not look equal. That is, above if 2+3 had been at the left and 5 at the right, we would have wanted you to answer Y. Does this change any answers for you to Y? If so, indicate which ones. ____________

The results of the survey indicated that our disagreements reflected a wide disparity of views that exists in the mathematics education community at large. For clarity, we reorder the items in terms of the number of "Yes" responses.

Item No.	Are these equal?	Yes	No	Not sure
5	2 divided by 3, 20 divided by 30	22	4	1
3	2/3, 20/30	22	5	
6	the ratio of 2 to 3, the ratio of 20 to 30	17	10	
7	2:3:7, 20:30:70	13	13	1
2	2:3, 20:30	13	14	
1	2 out of 3, 20 out of 30	6	21	
4	2 of these for 3 of those, 20 of these for 30 of those	4	23	

The results indicate no agreement on conception and a balance between those whose thinking seems to reflect the ordered pair notion (indicated by "No" answers) and those whose thinking reflects the single number notion (indicated by "Yes" answers). Significantly, the views of 22 of the 27 respondents can be completely described by indicating where in the above list they began to put "No" responses. The response of six educators was to answer "Yes" only to the first two items in the above tally of responses, while the response of five others was to answer "Yes" to the first five items. Three answered all items "Yes"; three answered all items "No".

The phrase "2 divided by 3" and the fraction 2/3 signal a single number to most of these mathematics educators, while the English phrases "2 out of 3" and "2 of these for 3 of those" signal ordered pairs for most of these mathematics educators. However, there is a rather even split when the word "ratio" or the ratio symbol "2:3" is used. Indeed, we were quite surprised that some would answer items 2 and 6 differently; we had thought that one said the same thing as the other, but just in a different way. Is an idea different when it is given in symbols from what it is when it is stated in words?

The meaning of "equality" versus "equivalence" is critical here. When two structures are isomorphic, are they different structures or are they the same structure expressed in two dfferent ways? To some extent these questions are philosophical, but they underlie some of the differences in response. We further questioned some of the respondees. One of those who answered all No's noted that, to him, 2/3 and 20/30 could never be equal. To him the equal sign is misleading when we use it in 2/3 = 20/30. To him 2/3 and 20/30 could at best be equivalent. This view can be related to the way one conceives of the origin of the set of rational numbers. If the rationals are rigorously constructed from the integers, then there is a time when 2/3 and 20/30 are the ordered pairs (2,3) and (20,30) and not equal, but equivalent. Only later in this construction is the isomorphism noted and we write the fraction instead of the ordered pair. If, on the other hand, one conceives of the rationals as a subset of the real numbers--the reals having been postulated as a complete ordered field--then 2/3 and 20/30 represent the same number (or the same point on the real number line) from the beginning and there never is a stage when they are merely equivalent and not equal. Both methods of constructing the rationals are of equal validity, so the issue cannot be resolved by invoking mathematical rigor.

Nor can the problem be resolved by appealing to conventional practice. As division problems, 2/3 and 20/30 are different. But the answers are the same. Should one consider 2 3 as a division problem or as a single number? (This is the reason for the last pargraph on the survey form, a paragraph that affected no one's answers.) It's clear that sometimes we want one way, sometimes the other. A teacher asks a child: "What is the result when 2 is divided by 3?" Some teachers will want the answer 2/3. Other teachers or even the first teacher at other times, will say that 2/3 is incorrect, that the question implies that the child should "figure it out" and come up with something akin to .666...

Surveys of this type are seldom done in mathematics education. This one made us more cognizant of the viewpoints of others and made us more tolerant as well. We decided to consider ratios both as single numbers (as in Chapter 1, Section D) and as ordered pairs (as in Section D of this chapter). We distinguish 2/3 as a single number from 2/3 as indicating division by considering ratio in a third place, as a use of division (Chapter 8, Section A).

CHAPTER 3

USES OF COLLECTIONS OF NUMBERS

Single numbers do not suffice for all uses. If we ask "How tall is a typical 12-year-old?", probably the best way to answer this question is to gather the heights of as many 12-year-olds as possible. The collection of heights yields information that no single height or average or range can yield. We call this a data set use of collections.

The collection of heights (in cm) of 12-year-olds in a room can be written {125, 146, 179, 156, 153, 137, 136, 145, 166, 162, 169, 159, 149, 149, 156, 150}. Elements in collections may appear more than once. We write the braces { } to emphasize that we are considering the collection of numbers, not the individual elements. For instance, a collection can be infinite. We may generate statistics about the collection, calculating a mean, median, etc. We may look for patterns in the collection. These are things we could not say or do for an individual element.

We recognize three other uses of collections of numbers. Suppose one wants to know the cost of parking in a particular lot for periods of less than 10 hours. The set of times between 0 and 10 hours is being used as a domain. Suppose a weather forecaster estimates tomorrow's high temperature to be between 20°C and 25°C. The interval between 20°C and 25°C is a set of values being used as a neighborhood. Suppose we want to know the dimensions of congruent cards that could be cut without waste from 20" x 26" cover stock. This collection of pairs of numbers we would call a solution set. These four uses constitute the sections of this chapter.

A. Domains

B. Data sets

C. Neighborhoods

D. Solution sets

As with single numbers, a use of a collection of numbers may be classifiable into more than one of these categories.

Collection Use Class A: Domains

In the formula for the area of a rectangle $A = \ell w$, ℓ and w may be any positive number. We say that the domain of definition for ℓ (and also for w) is the set of positive numbers. Values on the Mohs hardness scale range from 0 to 15; this is the domain of the scale. If a class contains students who scored over 75 on an entrance exam, then the set of whole numbers from 75 to the maximum possible score is a domain for that class.

Sets used as domains tend to be reasonably large and usually can be described quite simply. They are often intervals, and often pre-determined.

Examples:

1. Ignoring letters, what is the set of allowable numbers for license plates in your state?

   Answer: Answers will vary. In Illinois, numbers from 0 through 1,999,999 seem to be possible. In some states, all license plates have county letter designations, so no pure numbers are possible. (The set of allowable numbers is the null set in these states.)

   Comment: In sets that are domains, the size of the set is often of critical importance. For license plates, the size must be large enough to handle all of the registered cars. Eight digit places would suffice for any state; to use fewer symbols many states also use letters.

2. Intervals. A thermostat allows settings from 42°F to 88°F. Give some properties of the set of allowable settings.

   Answer: There are infinitely many settings in this interval. Settings recommended for energy savings (55°F in winter and 80°F in summer) are possible. There is a 46° range.

Comment: An interval is the set of all numbers of a given type (whole, rational, real) between two given numbers, called the endpoints. The interval may or may not include its endpoints. Intervals are very commonly found in uses of arithmetic. See notes 4 and 5 for this chapter.

Comment: If only integer settings are allowed, this domain is finite.

3. Discrete vs. continuous domains. Automatic timers for turning lights or appliances on and off are of various types. Some only allow settings each 15 minutes. Others in theory allow any time to be set. Name a time at which the second type but not the first type can be activated.

Answer: 3:20 PM, for example.

Comment: If there was a desire to activate at 3:20, the clock on the first type of timer could be set 5 minutes late.

Comment: We say that the first type of timer is discrete, the second type continuous. Timers with digital displays allow many settings, but are discrete even if one can enter the time to seconds.

4. Inputs. A shoe store advertises an entire collection of shoes on sale for $10.79 a pair. To make things easier for the employees, the store decides to draw up a table of prices with numbers of pairs of shoes being purchased in one column, and total cost in a second column. What numbers would you put in the first column?

Answer: Perhaps 1 through 10; larger numbers could be calculated as sums of smaller ones.

Comment: In this situation, the first column is a set of inputs, corresponding to the technical meaning of "domain" in mathematics as a set of inputs to a function. This use is the counterpart to a use of collections found in Section B, that of outcomes as data sets.

Collection Use Class B: Data sets

A data set is a collection of outcomes or outputs, and is not a "set" in the usual mathematical sense of the word because outcomes can appear more than once. Data sets seldom are in simple patterns; in fact, if the outcomes fall into too regular a pattern, we would often question the data. For example, the distribution of results from tossing a die 600 times usually has each of the numbers 1 to 6 occurring with frequencies between 80 and 120. But if each value came up exactly 100 times, we would think someone tampered with the data. Properties of interest in data sets are such things as symmetry or balance, closeness to an expected data set, range.

Examples:

1. Here is the distribution, by birth sign, of the 38 people (as of 1983) who have been President of the United States.

Capricorn	4	Aquarius	4	Pisces	2
Aries	2	Taurus	6	Gemini	2
Cancer	2	Leo	2	Virgo	2
Libra	5	Scorpio	4	Sagittarius	3

Does this data set suggest a connection between birth sign and the position of President?

Answer: This is not an unusual data set, for one would expect about 3 per sign.

Comment: One way to verify the answer is to statistically test whether this data set is unusual. An appropriate test for this situation is the Chi-square test. We did such a test and found nothing unusual here. A second verification can be done by simulating this situation. Take a coin and a die. There are 12 possible pairs of outcomes for them, just like the 12 signs: H1 (heads on the coin, 1 on the die), H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, and T6. Toss the coin and die 38 times together. See if the frequencies of outcomes you get are similar to the one of the President's birth signs. You might repeat this a few times to see the variety of data sets that are reasonable.

2. When Charles Goren, the bridge expert, writes of a 4-3-3-3 hand distribution, to what is he referring?

   Answer: He is referring to a bridge hand of 13 cards in which 4 are of one suit and 3 are of each of the others. For example, a hand with this distribution might contain 4 hearts, 3 spades, 3 diamonds, and 3 clubs.

   Comment: If the suits were given in the order of the rank of suits in bridge (spades, hearts, diamonds, clubs), we would classify the resulting 3-4-3-3 distribution as a multidimensional count (Chapter 2).

3. In a Scrabble tournament, one of the authors scored 356, 362, 361, and 421 in a four-game qualifying round. What statistics might one be interested in so as to compare this set of scores with those of others?

   Answer: Perhaps the mean of the four numbers (375) or the median (361.5) or the sum (1500). Similar data would be needed for the other contestants.

   Comment: In this tournament, games won counted more than points scored. When people tied in games won, the sum of scores (which for this purpose is equivalent to using an average) was used to break the tie.

4. <u>Frequency distributions</u>. A coin is tossed 20 times to determine how many heads will occur. Here are the results from 428 such experiments. What is the range of number of heads per 20 tosses that occurred in these experiments?

Number of heads in 20 tosses	Frequency	Number of heads in 20 tosses	Frequency
0	0	10	76
1	0	11	64
2	0	12	51
3	1	13	28
4	1	14	16
5	4	15	10
6	11	16	1
7	31	17	0
8	60	18	1
9	73	19	0
		20	0

Answer: The number of heads ranged from 3 to 18.

Comment: The frequency of an event is the number of times it occurs. For this reason, what is graphed above is called a frequency distribution. As the number of experiments is increased, this distribution comes closer and closer to resembling a distribution called a normal distribution. ("Normal" is a technical term here.)

Comment: We can think of the numbers from 0 to 20 as constituting a domain, for they are the possible numbers of heads. The ordered pairs (number of heads in 20 tosses, frequency) are the elements of the data set.

5. During the decade of the 1970s, railroads in the United States carried many passengers. Here is the data by year.

Year	Number of passengers (in millions)
1970	284
1971	272
1972	261
1973	254
1974	274
1975	269
1976	271
1977	275
1978	281
1979	301

Is there any pattern?

Answer: The ridership seems to be on the upswing at the end of the decade after a lull in the years 1972 and 1973. Other that that, there does not seem to be an obvious pattern.

Comment: Because this data set has only 10 ordered pairs, it is rather simple. Even so, not all people would agree with the pattern mentioned above. Because the existence of a pattern might determine policy in buying or preserving equipment, personnel, and so on for future years, the ability to find patterns in data is important. Sometimes a display will help in this regard (see Chapter 14, Section C).

Collection Use Class C: Neighborhoods

We use the term neighborhood to signify a collection of estimates. This kind of collection is used when it is felt that one estimate cannot suffice, or when some sort of tolerance is needed in the estimate. For example, if 63% of the people in a poll favor candidate X, and the poll has an accuracy of 3% almost all the time, then the interval between 60% and 66% is the neighborhood in which candidate A's vote percent is expected to fall.

In advanced mathematics, the word "neighborhood" signifies a set of numbers that are in some way close to each other. This is the idea behind our name for this use class, for estimates (i.e., numbers in neighborhoods) tend to be close to one another in numerical value. But this does not have to be the case. If one would ask fifteen economists to estimate what they think the inflation rate will be next year, the 15 answers constitute a neighborhood (and also a data set) and it is possible that some predictions would be far different from others. Other definitions of "close" are possible; e.g., see Example 6 below.

Examples:

1. Tolerance. The length of a steel bar is given as 2" ± .0002. In what neighborhood does the length lie?

   Answer: The interval with endpoints 1.9998" and 2.0002".

   Comment: The symbol ±, read "plus or minus", is very commonly used in industry. The number .0002 is the tolerance of the measure, generally indicating the accuracy of the machine that cut the bar.

   Comment: When small tolerances are crucial, temperatures will be given with the length, for higher temperatures expand steel bars.

2. Ball-park estimates. According to a newspaper account, attendance at a political rally was in the "neighborhood of 250 persons". This is meant to refer to some interval containing 250 persons. What might be the endpoints of that interval?

   Answer: Perhaps 235 persons and 265 persons.

   Comment: This use of the word "neighborhood" is quite consistent with our use. We derive our answer from common sense, not from any mathematical formula. It seems to us that an attendance of 235 people might fall in the neighborhood, but 230 people might be described differently, perhaps as "between 200 and 250 people". Where you draw the line is a matter of taste.

   Comment: One way to avoid the problem of misinterpretation is to describe the neighborhood using indefinite pronouns or collective adjectives: few, several, many, hundreds, millions, etc.

3. Confidence interval. A poll with 5% accuracy 95% of the time showed that 46% of the residents of a community were willing to pay more than $15 a month for cable TV. Describe the neighborhood of this estimate.

   Answer: The neighborhood is an interval with endpoints 41% and 51%, if you are willing to settle for 95% reliability.

   Comment: Unless everyone who is affected is polled, a poll must have some tolerance and a neighborhood, rather than a single estimate, must be given. In this case, the interval between 41% and 51% is called a 95% confidence interval. If approximately 400 are randomly selected for this kind of poll, 10% is the length and 5% is the tolerance for such a confidence interval; the tolerance can be reduced to 3% by polling about 1000 people. In general, the number n of people needed for a confidence interval of length L with reliability r% is about z^2/L^2, where z is the number of standard deviations so that r% or the normal curve is between -z and z. Tables are needed to find the value of z.

4. Measure accuracy. Many doctor's scales (where a weight comes across a bar to register your weight) have notches only for every quarter

pound. A weight of $130\frac{1}{4}$ pounds on that scale signifies what neighborhood of weights?

Answer: The set of weights between $130\frac{1}{8}$ pounds and $130\frac{3}{8}$ pounds.

Comment: Generally, the tolerance in such a scale is one-half the length of the interval between two adjacent values. Here adjacent values differ by 1/4 pound, so the tolerance is 1/8 pound.

5. Set of guesses. In predicting the result of a football game, 6 sportswriters estimate the following for the result between teams C and D: D by 8 points, D by 3 points, C by 4 points, C by 1 point, C by 6 points, D by 3 points. From this, which team would you call the favorite?

   Answer: Here the collection of estimates is, from the standpoint of team D, {8, 3, -4, -1, -6, 3}. The mean of elements in the set is 1/2, so team D might be called the very slight favorite.

   Comment: This neighborhood could also be classified as a data set. Its elements are close to each other in the sense that they measure the same event.

6. The ISBN number on all books currently published contains 10 digits, the last of which is a "check-digit" designed to detect errors in copying one of the previous nine digits. If an ISBN code number is 0-02-902270-3, give two code numbers that are wrong in one place.

   Answer: 0-02-900270-3, 0-52-902270-3, etc. There are 90 such numbers.

Comment: The set of 90 numbers may be said to constitute a neighborhood of the given code number. If any number of the neighborhood is written by mistake, the check digit detects an error (but doesn't tell where the error is). Here is how it is done for ISBN numbers:

1. Underneath the 9-digit ISBN code, write the numbers 10 through 2 in decreasing order.

0	0	2	9	0	2	2	7	0
10	9	8	7	6	5	4	3	2

2. Multiply the numbers in each column and add the products.

   $0 + 0 + 16 + 63 + 0 + 10 + 8 + 21 + 0 = 118$

3. Find the smallest number that you can add to the sum to make it divisible by 11. That's the check digit.

   $118 + 3 = 121$, which is divisible by 11.

4. Here the check digit is 3. If the check digit is found to be 10 by this process, X is used.

Set Use Class D: Solution Sets

Among how many children can a dozen Easter eggs be evenly divided? There is more than one answer to this question. The eggs could be evenly divided if there were 12 children, 6 children, 4 children, 3 children, 2 children, or obviously if there was only one child. We call the set {12, 6, 4, 3, 2, 1} a solution set. It is a collection of numbers that satisfy given conditions.

Of course, every set or collection of numbers satisfies some condition, the defining condition that got the numbers together in the first place. Solution sets are distinguished from other uses of sets in that it may require some effort to determine their elements. Often one does not even know how many possible solutions there are or whether there is any solution at all. When there are many possible solutions to a real-world problem and only one of these can be picked, the solution chosen is usually selected for non-mathematical reasons such as economy or simplicity.

Examples:

1. Postage. First class mail rates in the U.S. in 1982 were 20¢ for the first ounce and 17¢ for each additional ounce. Post cards cost 13¢ to mail. At these rates, to mail first class letters of any weight and to mail post cards, what set of denominations of stamps are needed?

   Answer: Obviously, having 13¢, 17¢, and 20¢ stamps will do, but so will having only 3¢ and 4¢ stamps.

Comment: There are other sets that will work. One often decides what to do on the basis of efficiency. For instance, {13, 17, 20} (ignoring the units) makes it easier to figure out how many stamps are needed regardless of weight. But {13, 17, 20, 37} might result in using fewer stamps on some letters. An advantage of the {3, 4} solution is that one could keep a smaller total value of stamps on hand and still be ready for any mailing. Which stamps are purchased would probably ultimately depend upon the frequencies with which each denomination is used.

2. Theater capacity. A small theater is to contain between 8 and 10 rows and have a capacity of between 175 and 200. If each row contains the same number of seats, what capacities are possible?

Answer: If 10 rows, 180, 190, and 200 seats are possible; if 9 rows, 180, 189, and 198 seats are possible; if 8 rows, 176, 184, 192, and 200 seats are possible. So there are eight possible capacities ranging from 176 to 200.

Comment: The decision regarding which capacity to have might be based upon the desired width of the seats and hence the number of seats that would be in each row.

Comment: A related problem could involve classrooms with seats that are bolted to the floor. These usually contain 5 or 6 rows with 6 to 8 seats per row and occasionally have 2 seats missing in the front to allow room for a teacher's desk. What numbers of seats are possible in these rooms?

3. Tennis. In tennis, the winner of a set (a non-mathematical use of the word "set") is the first to win six games, but has to win by two games. If both players have won six games, there is a tiebreaker and the winning player is said to have won the set 7-6. What scores are possible in tennis?

Answer: 6-0, 6-1, 6-2, 6-3, 6-4, 7-5, and 7-6.

Comment: Knowing the solution set helps to identify typographical errors, as if a person were reported to have won a set in tennis by the score 6-5.

Comment: In cases where many solutions are possible, as in Examples 2 and 3, it is common to wonder which solutions occur most frequently and why.

4. <u>Telephone cost</u>. The weekday cost (1982) for a phone call on the Bell System from Chicago to Atlanta was $.62 for the first minute call and $.43 for "each additional minute or fraction thereof," plus tax. Give 5 elements of the set of ordered pairs relating time and pre-tax cost.

Answer: (1 minute, $.62), (30 sec, $.62), (2 min 3 sec, $1.48), (10 min, $4.92), (1 minute 59.5 sec, $1.05), etc.

Comment: A formula connecting cost C (in dollars) and time t (in minutes) is C = .62 - .43[-t+1], where [x] is a symbol standing for the greatest integer less than or equal to x. Another formula is C = .62 - .43[-t+1]. Because the formula involves the two variables t and C, each solution is an ordered pair.

Summary

We identify four uses of collections of single numbers. One use is as a domain, a collection from which one picks values. Domains tend to be rather large and easily described. A second use is as a data set, a collection of outcomes or outputs from a search, survey, experiment, or calculation. Data sets are often difficult to describe and statistics are often helpful in dealing with them. A third use is as a neighborhood, a collection of estimates of the same quantity. A fourth use is as a solution set, a collection of numbers satisfying one or more conditions or constraints in which it may be difficult to determine the individual numbers.

Collections thus differ in their uses from single numbers. Accordingly, one seeks information from collections that one does not ask of single numbers. One may wish to know the size of a domain, the mean of a distribution, the width of a neighborhood, or whether or not the solution set has any members. None of these bits of information is appropriate for a single number.

Pedagogical Remarks

Terminology. In a set, as defined in mathematics, something either is or is not an element. No element can be given twice. Thus, in the language of sets, {2, 5, 2} is the same set as {2, 5}. We have used the word collection to describe this chapter because applications often involve collectives in which a single number is repeated. This is very common for data sets (toss dice 20 times and some numbers will come up more than once), but also occurs in neighborhoods (two people may have the same estimate), in domains (certain relationships may require using the same number again and again), and even in solution sets.

The teacher may find it useful to distinguish between sets (where elements cannot appear more than once) and collections (where no such constraint is present). Notes and commentary note 2 for this chapter describe mathematical situations where the distinction is helpful.

Getting started. Data sets provide an easy first use of collections. Begin by collecting one piece of numerical information for each student in reponse to a simple question. For example:

How tall is a typical student in this class?

How far from school do students live?

How much TV did people watch last night?

Tabulate or graph the data. Ask what information the collection provides that an individual datum could not. (The collection displays variability, range, relative frequency.) Ask whether the collection would be a representative sample of all students of this age. Ask how one might describe the collection without giving all of its elements. These kinds of questions move one into the realm of statistics in natural fashion.

Neighborhoods. Some neighborhoods are also data sets. Ask students to estimate some quantity, e.g., the length of the room or the number of students who are absent that day in the school. The set of estimates is the neighborhood. How can this neighborhood be described? (Answer: Its values range from ___, the smallest, to ___, the largest; its mean value is ___; the mode is ___; etc.) What advantage does a set of estimates have that a single estimate does not? (Answer: It might enable a more reliable response to the question.)

A second easy-to-understand type of neighborhood is exemplified by weather forecasts. "A low of from 20 to 25 degrees is expected." "Winds from the south at 12-15 mph." These intervals are neighborhoods.

A subtler type of neighborhood is that inferred in a single measure. An item in the Federal budget is reported as 1.3 billion dollars. Surely it is not exactly that. What neighborhood does the 1.3 billion represent? (Answer: Probably the interval from 1.25 billion to 1.35 billion.) A desk is measured to have a length of 122 cm. Is the length likely to be exactly that? (Answer: No, for no measure of length is exact.) If not, what is an interval in which the length lies? (Answer: The interval from 121.5 to 122.5 cm, which could be written as 122 cm $\pm$ 0.5.)

Domains. Domains are sets of numbers from which one wishes more information. For instance, one might wish to compare the income tax for a single person whose taxable income is $10,000, $20,000, and so on up to $50,000. These five quantities then constitute the domain. Accordingly, one often finds domains as sets of first components of ordered pairs or n-tuples, the other components being the information desired.

As with all collections, a key question concerns how to describe them. Domains usually have simpler descriptions and so provide a nice place to begin with writing out such descriptions. The income tax domain above could be described by listing its elements, since there are only five of them, but also could be described as "the set of multiples of $10,000 up to $50,000".

Solution sets. Be careful not to use solution sets exclusively in situations where there is only one solution and, consequently, only one element in the set. This doesn't convey the important notion, that of a collective of numbers. Begin with situations in which there is more than one answer.

For instance, you might ask all students in a class to write down a list of coins that will guarantee a store being able to give change for any item costing less than $1. Each student's solution becomes a member of a large solution set. What do these solutions have in common? (Possible answers: Each must contain at least four pennies, and at least 99¢ total.) What's the solution with the fewest coins (what might be termed by mathematicians as the "most elegant" solution)? What's a good strategy for a store to have to insure always having enough coins on hand?

Goals. In dealing with the uses of collections, the goals are to make the student more comfortable when he or she encounters a collection of numbers by giving the student language (set, range, mean, relation, etc.) to discuss such collections and raising the questions normally asked about such collections.

Questions

1. Which domains of those given here contain only whole numbers? Which contain only positive numbers?

   (a) The set of reported low temperatures (on the Fahrenheit scale) for Tampa, Florida;

   (b) The set of reported low temperatures (on the Celsius scale) for Tampa;

   (c) The set of body temperatures for a person (on the Celsius scale).

2. Toss a coin 5 times and record the number of heads. Repeat this 10 times to get a set of 10 elements. List your data set and give its mean value and range.

3. Pick a particular size color television set (for example, 19"). Call five stores or look at five catalogs to determine the most inexpensive set of this size that each store sells. Give the mean and the range of the distribution of costs. What do you think would happen to the mean and the range if you called five more stores?

4. In football, a touchdown gives 6 points, an extra point after a touchdown is 1 point, and a field goal is 3 points. If a team scores only touchdowns, extra points, and field goals, list the set of possible total scores under 50.

5. A chart relates heights of men and women to approximate caloric need. What is a reasonable expectation for the domain of this chart, the set of heights? (Answer for each sex.) Give a quantity that you might expect to see in the set of outcomes, that is, in the collection of caloric needs.

6. If 60 seats in a study hall are to be arranged in a rectangular array with the same number of seats in each row, how many rows are possible?

7. Indicate whether each collection of years below is most likely to be a domain, a data set, or a neighborhood?

   (a) The set of all years from 1900 to the present

   (b) {1913, 1922, 1947}

   (c) {1942, 1943, 1940, 1942, 1942, 1943}

8. The population of the People's Republic of China was estimated (1979) as 953,978,000. What interval neighborhood for China's population is suggested by this datum? [As we go to press, China's population (1982) is reported as 1,008,175,288.]

9. All the costs for items in a store are advertised as reduced 20% to 30% in a sale. What might be the sale price of a TV set that before the sale sold for $369.95? What two intervals are involved in this question and what is the use of each?

10. Give the set of legal driving speeds for Interstate highways in the U.S. (under good driving conditions).

11. Name five properties collections can have that individual elements cannot.

Notes and Commentary

1. Origins of this chapter
2. Data sets vs. sets
3. Sets in the curriculum
4. Intervals as mathematical objects
5. Intervals in the curriculum
6. Neighborhoods
7. Alternate use classes for collections

1. Origins of this chapter. A chapter devoted to uses of collections of numbers was not part of our original outline for this book. Rather, in examining uses of numbers, we were surprised at the frequency of occurrence of explicit intervals (those where endpoints are clearly stated, as in "winds between 15 and 20 mph are expected."). Furthermore, there were many occurrences of implicit intervals, those indicated by plurals of number words (dozens, millions, thousandths) or indefinite pronuns (few, several, many). Hence, for an earlier draft we wrote a chapter devoted to uses of intervals.

An early draft also had a chapter entitled "Uses of Sets", to extend Chapter 2 by dealing with unordered collections of numbers. But by definition, a set in mathematics has no repeated elements while there are often repeated elements in useful unordered collections. There is a term "data set" in common use in statistics, but since data sets often have repeated elements, a data set is not a set. Furthermore, intervals are themselves sets, and so we had identified a special kind of set and separated it from others.

The present chapter evolved as we sought to eliminate such confusions. It led us to prefer collection to set. As other notes indicate, this may be a promising conceptualization elsewhere in mathematics.

Our discussion here is not necessarily intended to advocate a renewed emphasis on sets in teaching mathematics. In devoting a chapter to uses of collections of numbers, we wish merely to point out that there are potential uses of sets and other collections beyond their roles in purely formal mathematics.

2. Data sets vs. sets. A particular set is said to be well-defined if there is a finite procedure to determine whether or not a given element is in the set. (For small finite sets the simplest procedure is to list the elements.) For instance, the set of prime numbers is well-defined even though we do not know whether certain very large numbers are prime, because for any given number, there exists such a procedure.

Thus, for any well-defined set, a number is either in it or not in it. No provision is made for a number being in a set more than once; indeed this is generally not permitted. For instance, the set of numbers {1, 2, 3, 3} is usually considered to have three elements, not four.

Not allowing duplicity or multiplicity does create difficulties. The quadratic equation ax^2 + bx + c = 0 has two solutions, judging from the quadratic formula. If they are the same number, is there one solution or still two? With higher degree polynomial equations, such as (x - 2)^2 (x - 1)^5 x = 0, we say that the solution 2 appears "with multiplicity 2", and 1 is a solution with multiplicity 5, while 0 is a solution with multiplicity 1. But are there nine solutions or are there three? Would one want to say that the above polynomial is equivalent to (x - 2)(x - 1) x = 0? The two polynomials have the same solution set but the multiplicity of certain solutions has implications for the way the corresponding polynomial function behaves near the solutions.

It seems that a term is needed for a collection of numbers in which repetitions of the numbers are allowed. There would be use for such a term in both pure and applied mathematical settings. Dataset (with no space between data and set) might be a candidate for such a term.

3. Sets and the curriculum. In the mathematics curriculum reforms of the 1960's, sets were introduced as a unifying concept and, many of us now believe, pretty much got out of hand in various distortions and overemphases of fundamentally sound ideas. For example, some textbooks tried to put in elementary school language the conception of the number 3 as the common property of all sets with three elements. It seems plain that this introduces a big abstraction to get at a much simpler one.

In the junior high school curriculum, sets were (and are) often introduced as solution sets. For example, the equation 3x = 15 is said to have {5} (or worse perhaps {x | x = 5}) as its solution set. But unless equations or inequalities are given with more than one solution, and they often are not, students see little justification for imposing the solution set concept--it seems extra machinery for little purpose.

A third way that sets have appeared is to discuss sets of numbers, such as the set of natural numbers, the set of rational numbers, and so on. These sets are often domains and they provide a better way to introduce students to sets than the first two ways. We speak of properties of the set of rationals (e.g., closure under addition) and this makes it clear that the set has properties not appropriate to the individual numbers in it.

Accordingly, it is predictable that the first two ways have seen less emphasis in recent texts, but the third way remains.

4. Intervals as mathematical objects. Mathematicians distinguish closed intervals (those that contain both endpoints) from open intervals (those that contain neither endpoint). Two

notations are in common use for describing such intervals in terms of the endpoints a and b.

	Ordered Pair Notation	Set Notation
Closed:	[a, b]	$\{x: a \le x \le b\}$
Open	(a, b)	$\{x: a < x < b\}$
Half-Open:	[a, b)	$\{x: a \le x < b\}$
Half-Open:	(a, b]	$\{x: a < x \le b\}$

The variants of ordered pair notation in the left column have the advantages of being amenable to many typewriters and being shorter. The set notation with inequalities has the advantage of exhibiting the definition and using standard notation.

An interval with midpoint m and tolerance t has endpoints m-t and m+t. There are two notations for describing such an interval using these ideas. The first, $m \pm t$, does not allow any distinction between open and closed interval; the second, $\{x: |x - m| \le t\}$ describes a closed interval.

The symbolism $2 \pm .01$ refers to the interval from 1.99 to 2.01, possibly including the endpoints. (Note that the same notation, when used in conjunction with solutions to quadratic equations, refers only to the endpoints.

That is, $x = 2 \pm \sqrt{.0001}$
means $x = 1.99$ or $x = 2.01$
and x does not refer to an interval.)

5. Intervals in the curriculum. The existence of a variety of notations for an idea is an indication of the breadth of application of that idea. Thus Note 4 suggests that intervals are quite a bit more important than the present school curriculum would indicate. This demonstrates that discussion of intervals would be appropriate in teaching about ordered pairs, inequality, sets, absolute value, and possibly quite early as school in connection with addition and subtraction of count and measure estimates. This is in marked contrast to the present situation, where intervals are discussed first usually in conjunction with limits in twelfth grade, even though some students have graphed intervals years earlier as solution sets to certain inequalities.

6. Neighborhoods. Of the four uses identified here for sets the neighborhood use may be least familiar to the reader. The mathematical term "neighborhood", upon which our designation is based, is customarily not encountered until advanced calculus and does not reach its full flowering until topology. The idea, however, is simple: In analysis, a neighborhood of a point P is the set of all points closer than a certain distance to P. In topology, a neighborhood of P is any set containing an open set that contains P. Neighborhoods and open sets are sometimes synonymous, and topology is sometimes characterized as the study

of all the open sets in a space, so the idea of neighborhood is indeed basic.

7. Alternate use classes for collections. We are not aware of any other attempts to categorize uses of sets of numbers, uses of intervals, uses of data sets, or uses of collections.

In a first draft, we classified uses of sets by intent: sets as distributions, sets to show variability, sets to group for later operation. As with n-tuples, we changed this conception to ask what types of objects sets represent. When we did this, we were surprised to find that our conceptualization nicely paralleled ideas already found in textbook discussions of sets. We named our uses domains, data sets, solution sets, and relations. Only the name data sets does not appear in secondary school books, but the idea is famiiar, and we felt that the alternates range, distribution, set of outcomes, or set of outputs were not as descripive.

At the same time, we were quite happpy with the three uses we had identified for intervals: compartments, estimates, and ranges.

Then, as stated in Note 1, we saw problems with our basic organization. (1) Data sets are not sets in the mathematical sense of the term. (2) Data sets are often domains. (3) Intervals are sets. And so we moved to the present organization.

The use of intervals as estimates became subsumed under neighborhoods. The use of intervals as ranges (e.g., students in this class range from 8th to 11th grade) could apply to any of the uss of sets. Intervals as compartments (as in a histogram) is a use of a set of intervals, and extends the idea of a set as a domain. Relations went to the next chapter.

CHAPTER 4

USES OF VARIABLES

A <u>variable</u> is a symbol, usually a single letter such as A or n or x, that stands for a number. The term "variable" derives from the idea that this symbol may stand for any one of a set of numbers or quantities, so its value may change or vary from time to time. For example, in the formula $A = \ell w$ for the area of a rectangle, A, ℓ, and w are variables; their values will change as different rectangles are being considered.

Though we customarily associate variables with the study of algebra, it is also impossible to adequately study arithmetic without implicitly or explicity dealing with them. When a second-grader is asked to find the missing number in $3 + ? = 10$ or to fill in numbers that could make $\square + \triangle = 15$ work, the ?, $\square$, and $\triangle$ are variables. The property "The product of any number and zero is zero" is more succinctly stated with variables: "For any number a, $a \times 0 = 0$." And in the past generation, variables have gained a new use: naming places in a computer's storage.

The uses illustrated above exemplify the sections for this chapter. We call them:

A. Formulas

B. Unknowns

C. Properties

D. Storage locations

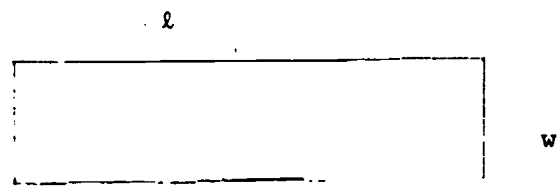
There are other uses of variables connected more with algebra and analysis than with arithmetic. These uses are discussed briefly in the notes following this chapter.

Variable Use Class A: Formulas

We may desire to know the perimeter of a rectangle to find the amount of fence needed for a plot of land, the amount of edging needed for a sign, the distance traveled in a race around a city block, and so on. For these, one adds twice the length of the rectangle to twice its width. A formula states this more succinctly and some would say more clearly.

$$\text{perimeter} = 2 \times \text{length} + 2 \times \text{width}.$$

$$p = 2\ell + 2w$$



Why is English changed into a formula? The reasons are the same as those for other types of rewriting, as detailed in Chapter 11. We may be forced into it because we wish a computer to calculate the value. We may wish to express the relationship differently; for example, if $p = 2\ell + 2w$, then $\ell = \frac{1}{2}p - w$. The formula is clearer to many people, particularly if the letters in the formula are the first letters of the attributes they represent. Thus we call the perimeter p, the width w, and so on. Formulas are easier to deal with in many situations, particularly those that call for the finding of an unknown value (see section B of this chapter).

In Part II of this book, we show that many formulas can be explained by considering the particular operations that are involved in them. For instance, the formula $I = Pr$ (interest on an investment) of Section B of this chapter can be explained as a special case of size change multiplication (Chapter 8, Section A). Another way to derive formulas is by generalizing a pattern. Here is an example.

> In 1982 the cost of a telephone call from Chicago to South America was as low as \$1.66 for the first minute and \$0.71 for each additional minute. Find a formula relating time and cost.

A pattern can be seen by first constructing a table.

minutes	cost
1	\$ 1.66
2	$1.66 + 0.71$
3	$1.66 + 2 \times 0.71$
4	$1.66 + 3 \times 0.71$
5	$1.66 + 4 \times 0.71$

It is important to avoid working out the arithmetic in the cost column, because working out the arithmetic loses the pattern. Now suppose you talked for 40 minutes. What would be the cost?

40 minutes Cost: $1.66 + 39 \times 0.71$

This gives the idea for the formula. The 1.66 and 0.71 will be there and the other number will be one less than the number of minutes, which we'll call m. Thus

m minutes Cost: $1.66 + (m-1) \times 0.71$

The cost C is thus represented by the final expression at right.

$$C = 1.66 + 0.71(m-1)$$

Examples:

1. Speed of sound. According to the 1983 Information Please Almanac, the speed of sound V (in feet per second) through air with (Celsius) temperature t is given by the formula

$$V = \frac{1087 \sqrt{273 + t}}{16.52}$$

State this formula in words.

Answer: The speed of sound is the square root of the sum of the Celsius temperature and 273, that root multiplied by 1087 and then divided by 16.52, OR, equivalently, to find the speed of sound in air, add 273 to the Celsius temperature, take the square root of the sum, then multiply the root by 1087, and finally divide the product by 16.52.

Comment: Thus the formula is briefer and, many would say, clearer. If the numbers 273, 1087, and 16.52 were also written out, the English translation of the formula would be even less intelligible.

Comment: The adding of 273 (more precisely, 273.15) converts the Celsius temperature to kelvins. The other numbers in the formula deal with the conversion of units to feet per second and the fact that air is the medium under discussion. See Chapter 10 for explanations of other formulas.

2. For each of the well-known formulas given here, which letters are variables? For what quantities do they stand? Which letter stands for a definite number and so is not a variable? (a) $C = \pi d$; (b) $E = mc^2$.

Answers: (a) C and d are variables, standing for the circumference and diameter of the same circle; π is a definite number, approximately 3.14. (b) E and m are variables. E stands for the energy in ergs to which a mass m in grams can be converted. The letter c is the speed of light, a quantity presumed to be constant and approximately equal to 300,000 km/sec, and so is not a variable.

Comment: π is the first letter of the Greek word for "around"; it helps to get the distance around a circle.

Comment: The size of c^2 in $E = mc^2$ indicates that just a little mass can produce a greal deal of energy, if it can be converted to energy. That is what happens in nuclear reactions.

3. Bowling. Handicaps are often added to bowling scores to make it possible for bowlers with widely different averages to compete. One handicap H used is given by the formula $H = \frac{4}{5}(180 - A)$, if $A \leq 180$, where A is the average score of the bowler in question. If $A \geq 180$, there is no handicap. Using this formula, what is the handicap of a bowler with an average bowling score of 140?

Answer: When $A = 140$, $H = \frac{4}{5}(180 - 140) = \frac{4}{5}(40) = 32$.

Comment: Thus if this bowler bowled his or her average, he or she would be credited with a score of 140 + 32, or 172.

Comment: In words, the formula states: The handicap is 4/5 of the difference between the bowler's average and 180, if the average is less than 180. If the average is greater than 180, there is no handicap.

4. Postage. The rates for first class mail, as of January 1, 1983, were as follows: 20¢ for the first ounce and 17¢ for each additional ounce or fraction thereof up to 12 ounces. Make a table of postal rates and weight for each whole number weight and give a formula connecting weight and rate.

Answer:

weight (ounces)	rate	
1	.20	
2	.20 + .17	= .37
3	.20 + 2 × .17	= .54
4	.20 + 3 × .17	= .71
...	...	
12	.20 + 11 × .17	= 2.07

In general, when the weight is a whole number between 1 and 12 ounces, $.20 + (w-1) \times .17$.

So a formula is $R = .20 + .17\,(w-1)$.

Comment: This formula does not work for weights less than one ounce. For weights over 12 ounces, a different rate structure is used by the postal service.

Comment: The formula does not work for values of w that are not whole numbers. There is a symbol that can be employed to take care of this problem, called the "greatest integer symbol". [w] is the next integer less than or equal to w. Using the greatest integer symbol, the formula can be adapted to work for all values of w between 1 and 12: $R = .20 - .17[1 - w]$. The reversal of w-1 and 1-w and the change to a minus sign are caused by the need to round up rather than round down.

5. Pendulum. A pendulum is a bar or string that is hanging from a fixed point with a mass on the other end. If a pendulum with length L is swung, then its period, p, the length of time it takes to return to its original position, is approximately $2\pi\sqrt{\frac{L}{g}}$, where π is the usual 3.14... and g on Earth is approximately 9.8 meters/sec^2. (a) Approximate the period for a pendulum whose length is 2.0 meters. (b) One ignores both the mass of the pendulum and the height of the swing in this calculation. Of what significance is that?

Answers: (a) Approximately 2.8 seconds. (b) The duration of the period depends neither on the masses (weights) nor on the height of the swing.

Comment: The variables that are not used in expressions are sometimes as important as those that are used. In this situation, it is surprising to many people to learn that the period of the pendulum does not depend upon the mass of the object swung. This has practical implications in the making of pendulum clocks; in principle, the pendulum length is all that matters. Grandfather clocks use a relatively large pendulum mass because larger masses work more smoothly in mechanically triggering the escapement gear that counts off the seconds.

6. On a particular day, a person weighs 120 lb. Ten days later the person weighs 125 lb. (a) How fast has the person gained weight? (b) Answer the question if the original weight was W_1, there were d days in between, and the final weight was W_2.

Answers: (a) The person has gained 125 - 120 lb, or 5 lb in 10 days, a half a pound a day. (b) We subtracted the weights, then divided by the number of days, so a general expression for the average speed with which the person gained weight is

$$\frac{W_2 - W_1}{d} .$$

Comment: We chose to call the two weights W_2 (for second weight) and W_1 (for first weight). In computers, where more than one letter often is used, we might have used WORIG and WFIN. The use of subscripts $_{1\ 2\ 3\ \text{etc.}}$ is very common for such quantities. The subscripts generally represent locations in a framework.

Variable Use Class B: Unknowns

Example 3 of Section A gives a formula for calculating the handicap H of a bowler from the bowler's average A.

$$H = \frac{4}{5}(180 - A).$$

Suppose a bowler has a handicap of 20. What is the bowler's average?

To answer this question, we first substitute 20 for H.

$$20 = \frac{4}{5}(180 - A).$$

The resulting equation is no longer considered a formula. The variable A has become an unknown, a quantity whose value or values we desire to determine. There are many procedures that can help find A; these are studied in algebra. (Here A is 155.)

Variables as unknowns need not be in equations. Every ten years, the U.S. Census Bureau tries to determine P, where P is the population of the United States. An unknown may have many true values. For example, if a person is driving at a legal speed S on an interstate highway, the value of S may usually be any number between 45 mph and 55 mph, inclusive. We can write $45 \leq S \leq 55$ as the possible values of the unknown speed.

Examples:

1. If a principal P is invested at a rate of r% simple yearly interest, then the income I generated is given by the formula

   $$I = Pr.$$

   A person feels that retirement is possible if the income is $20,000 per year. If it is felt that a rate of 8% can be maintained, what equation can be solved to determine how much money the person must have as an invested nest egg in order to retire?

Answer: Substitute the known values in the formula. This gives

$$\$20{,}000 = P \times 8\%.$$

This would often be rewritten: $\$20{,}000 = .08P$.

Comment: The equation is, in English: Eight per cent of what number is $20,000? The answer is found by dividing $20,000 by 8%. This is a use called size change divisor (see Chapter 8, Section D). The correct value of P is $250,000.

Comment: Because of inflation, the principal in this case would have less buying power each year and so an individual would either have to live on less than $20,000, increase the investments yield to more than 8%, or start with more than $250,000.

2. Guess at the value of the unknown in each case.

   (a) The population of the world in 1750 A.D. was P.

   (b) The population of the world in 1 A.D. was P*.

Answers: The values of P and P* are truly unknown for no one knows for sure nor can either value be estimated with confidence. According to Ansley J. Coale, in "The History of the Human Population" (Scientific American, March, 1974, p. 43), P* is about 300 million, P about 800 million, with these estimates off by at most 1/3.

Comment: We have found that adults generally significantly underestimate P*. In our classes, we have had estimates as low as 50,000 people for P*. Our students seem to ignore the size of Rome and China, Japan and the rest of Asia.

Comment: When two variables' attributes' names begin with the same letter, as in this case where two populations are desired, a variety of techniques have come into being so as not to lose the identification of the variable. One is the use of symbols such as *, ', or " (as in P, P*, P', P"). A second is the use of numbers as subscripts (for the above,

$$P_1 \text{ and } P_2, \text{ or } P_1 \text{ and } P_{1750}$$

would be most appropriate). A third technique is the use of more than one letter to denote the variable (for the above, PEARLY and PLATE might be cute names). The first technique is most common when there are only two or three variables to be treated. The second is used when there are great numbers of variables, as perhaps the populations in all years from 1 A.D. to the present. The third is often used in computer names for variables.

3. F feet of flexible fence can enclose A square feet of area, where

$$A \leq \frac{F^2}{4\pi}.$$

If you have 20 feet of flexible fence, what is the largest whole number square feet of area than can be enclosed?

Answer: Substitute 20 for F and 3.14 for π. This gives

$$A \leq \frac{400}{12.56} \approx 31.7\ldots \text{ sq ft}$$

So the largest whole number square feet is 31.

Comment: The largest area can be attained by shaping the fence in a circle. By turning the fence in on itself, smaller areas can be surrounded.

Comment: This example illustrates two aspects of unknowns, that there can be many values of an unknown that will work in a situation and that sometimes determining the value of an unknown is only a matter of computation.

4. In Section A, a formula relating time m in minutes and cost C of a phone call from Chicago to South America was found to be

$$C = 1.66 + 0.71(m-1)$$

(a) To find out how long a person could talk for a cost of $20, what equation could be employed? (b) To the nearest minute, how long could a person talk for $20?

Answers: (a) $20.00 = 1.66 + 0.71(m-1)$. (b) Up to 26 minutes.

Comment: The equation (a) is found by substituting 20 for C. It is not as difficult to solve as it looks. First subtract 1.66 from each side. That gives

$$18.34 = 0.71(m-1).$$

Now divide both sides by 0.71.

$$25.83 = m - 1$$

Now m can be found in your head. $m = 26.83$. Since talking times are rounded up by the phone company, any call not exceeding 26 minutes will cost under $20. Check by substitution in the original situation. The first minute's $1.66 plus 25 minutes at $0.71 is $19.45, and another minute would cost enough to put the total over $20, so the answer as given is correct.

Variable Use Class C: Properties

If the order of numbers in an addition is switched, the sum is the same. Here are some instances.

$$2 + 347 = 347 + 2$$

$$\frac{1}{3} + -\frac{2}{5} = -\frac{2}{5} + \frac{1}{3}$$

$$7.3 + 46.2 = 46.2 + 7.3$$

The general pattern is described using variables. For any numbers a and b,

$$a + b = b + a.$$

The first sentence that began the paragraph described this property of addition using English words. An advantage of the description with variables is that it looks like the instances, whereas the English does not.

In properties, the same variable or variables may be involved more than once, often on both sides of an equation. This leads to one of the important practical conseqalthough consequences of properties, the ability to substitute one expression for another. Thus properties yield flexibility. For instance, the commutative property of addition illustrated above allows one to learn $2 + 9 = 11$ and $9 + 2 = 11$ as one basic fact rather than two.

Properties have a second major role in arithmetic. They help to demonstrate that shortcut and other procedures are valid. The person who multiplies a number by 19 by first multiplying the number by 20 and then subtracting the number is using an instance of the distributive property. One way of describing that property is: For any numbers a, b, and c:

$$ac - bc = (a - b)c$$

For the case mentioned here, $a = 20$ and $b = 1$:

$$20c - c = (20 - 1)c$$
$$= 19c.$$

Examples:

1. A sales clerk sells many items that cost \$2.98. To calculate the total cost for such items, she multiplies the number of items by 3, then subtracts 2¢ for each item. Describe this process using variables.

   Answer: Let n be the number of items. The total cost is \$2.98n. Instead, the sales clerk calculates \$3n and subtracts \$.02n.

   Comment: That 2.98n gives the same results as 3n - .02n is substantiated mathematically by the distributive property, which can be described by
   $$(a - b)c = ac - bc.$$
   In this case, a = 3, b = .02, and c is the number of items purchased. That is,
   $$(3 - .02)n = 3n - .02n$$
   or
   $$2.98n = 3n - .02n.$$

   Comment: The advantage of using variables does not present itself when there is only one instance to a pattern, and so may not be evident here. But if we realize that a, b, and c in the distributive property can stand for any numbers, then we see that the sales clerk could use the <u>same</u> idea to work with items with other costs. A reasonable question is: What process might the sales person use to calculate the total cost for a bunch of items, each of which costs \$5.89? (Answer: Multiply by 6, then subtract 11¢ for each item.)

2. A supplier of steel announces a 5% increase in prices effective March 1st. To determine the new prices, a buyer multiplies the old prices by 1.05. Describe the general property that is being used here.

   Answer: Suppose the original price is P. A 5% increase is .05P and the total cost will be P + .05P. The buyer is using the expression 1.05P. The general property is

$$P + .05P = 1.05P.$$

That property is true so the buyer's calculations will be correct.

Comment: The buyer has turned a situation that involves two operations, multiplication by .05 and addition of P and .05P, into a situation with just one multiplication. This is an important notion that we apply often in Chapter 7 to explain uses of multiplication.

Comment: Once again the distributive property is being used. More generally:

$$ax + bx = (a+b)x.$$

3. Strips of paper $\frac{3}{8}$" wide are needed to cover the identifications from some papers before reproducing them. Fifteen of these strips will have a total width of $\frac{45}{8}$". What general property is being employed to do this multiplication?

Answer: The specific instance is: $15 \times \frac{3}{8} = \frac{(15 \times 3)}{8}$.

The general property is: $a \times \frac{b}{c} = \frac{a \times b}{c}$.

Comment: The sign × for multiplication is avoided in algebra because of the obvious potential confusion with the letters x or X. A dot, parentheses, or juxtaposition (placing numbers next to each other) are used instead. For instance:

$$a \cdot \frac{b}{c} = \frac{ab}{c}$$

$$a(\frac{b}{c}) = \frac{ab}{c}$$

4. Each of the following arithmetic statements is true. However, the general pattern is not true. Give the general pattern and an instance when it is not true.

(i) $3 - .75 = 3 \times .75$

(ii) $2 - \frac{2}{3} = 2 \times \frac{2}{3}$

(iii) $99 - .99 = 99 \times .99$

Answer: A general pattern is

$$a - b = a \times b$$

If this pattern were always true, subtraction and multiplication would be the same operations! It's easy to find instances that are not true.

$$6 - 5 \neq 6 \times 5.$$

Comment: We picked the three given true statements very carefully. They are all instances of a less general and more hidden pattern,

$$a - \frac{a}{a+1} = a \times \frac{a}{a+1}.$$

When the expression is defined, i.e., when $a \neq -1$, this pattern is true, as you can verify by substituting any number (except -1) for a. Algebra shows that each side equals $\frac{a^2}{a+1}$.

Variable Use Class D: Storage Locations

In computer programs, when a variable is identified, it does not stand for a number but names a place in the computer's memory where a number can be stored. For instance, in the BASIC language, one may write

```
INPUT N
```

and this means that a place in the computer's memory is reserved for what you call N.

If you write   N = 3

then the computer will put 3 in that place. So far, it looks just like normal work with variables. But the equal sign above is deceptive; it means "<u>replace</u> what's in a certain place called N in my memory with what's on the right side of the = sign". That is, the number 3 replaces whatever was in N. If you now write

```
N = N + 1
```

the computer will replace that 3 by 3 + 1. The number 4 is now in the N-cell in the memory. The statment N = N + 1 thus means "add one to the number in storage location named N".

These kinds of commands are very common in computer languages. Adding one to a number already in a memory location is used to keep counts. Adding 3.98 to what's in a location can give you the price of one more item that sells at 3.98. This can be accomplished by writing

```
N = N + 3.98.
```

Multiplying what's in a location by 5 can be accomplished by

```
N = 5 * N.
```

(In BASIC, the * stands for multiplication.)

If one wishes to name a new variable, that can be done as well. Suppose you write NCOST = 75.00 * N. This says: Reserve a place in memory for a new variable called NCOST. Put in that place 75 times what is in the memory place called N. Now wherever the number in N is changed, the number in NCOST changes as well.

Examples:

1. A computer program is given here. What will be printed? (The left column consists of program line identification numbers.)

```
100  N = 1
110  PRICE = 3.47 * N
120  PRINT N, PRICE
130  N = N + 1
140  IF N < 15 GOTO 110
```

Answer:

1	3.47
2	6.94
3	10.41
4	13.88
5	17.35
6	20.82
7	24.29
8	27.76
9	31.23
10	34.70
11	38.17
12	41.64
13	45.11
14	48.58

Comment: These are the costs of 1 to 14 items that cost $3.47 apiece. Notice that changing one number on line 140 could result in a list as long as you wish. Also notice that the price 3.47 only appears once in the program, so the program is easy to adapt for other prices.

2. A computer program is given here. What does this program do?

```
200  YEAR = 1980
210  PEOPLE = 4500000000
220  PRINT YEAR, PEOPLE
230  YEAR = YEAR + 1
240  PEOPLE = PEOPLE * 1.02
250  IF YEAR ≤ 2000 GOTO 210
```

Answer: This program gives the approximate world population for the years 1980 - 2000 assuming a 1980 population of 4.5 billion and an annual percentage increase of 2%.

Comment: On our Apple, the following was printed

```
1980  4.5E+09
1981  4.59E+09
1982  4.6818E+09
```

and 18 more lines, ending with

```
2000  6.68676327E+09.
```

The E+09 is a variant of scientific notation, meaning "times 10 to the 9th". It is used to avoid superscripts (as in 10^9), an example of rewriting due to a hardware constraint. (See Chapter 11, Section A for other examples.)

3. If H and B stand for lengths of segments, what might A and P stand for in the following program? (∧ means take to a power.)

```
300  INPUT H, B
310  A = .5 * H * B
320  P = H + B + (H∧2 + B∧2)∧.5
330  PRINT H, B, A, P
```

Answer: The area and perimeter of a right triangle with legs of lengths H and B.

Comment: Using more appropriate names for the variables would make the lines longer but would aid understanding. Possibilities are LEG1 and LEG2 for H and B, and RTAREA and RTPERI for A and P.

4. In the computer language FORTRAN, a variable is identified by a 1- to 6-digit alphanumeric code in which the first digit must be a letter. Furthermore, if the first digit is a letter from I to N in the alphabet, then the variable can take on only integer values. If the first digit is any other letter, then the variable can take on any values. For each quantity below, indicate whether it takes on integer values only and give a possible variable name in FORTRAN. (a) The temperature in a room; (b) the number of cylinders in a car's engine; (c) the number of ounces of orange juice drunk at breakfast; (d) the rank of finish in a foot race.

Answers: (a) not an integer, RMTEMP; (b) integer, NCYL; (c) not an integer, OJ: (d) integer, JFIN.

Comment: FORTRAN compilers do different arithmetic operations with integer variables than with other variables, so the distinction is important.

Comment: It is a mark of the unity of mathematics that this chapter finishes with an example requiring thinking similar to that needed in Chapter 1. Knowing the uses of numbers helps in thinking about this question.

Summary

A variable is a symbol that stands for a number. Four major uses of variables are discussed in this chapter. Variables are used to abbreviate names for quantities in relationships such as those given by formulas, they are used as unknowns, they help describe properties, and they identify locations in computer storage.

Most commonly, variables are represented by letters. In formulas, the first letter of the attribute is usually the letter chosen. As unknowns, x and y are the most often used letters. In describing properties, n and letters at the beginning of the alphabet are commonly found, though x and y are used in this way also. For computers, longer names that are more descriptive, such as PRICE or YEAR, are normally used. Generally, if <u>we</u> are working with variables, we like single letters. If there are very many variables, then we use subscripts. We use longer names to help us keep track when computers are doing the work.

Pedagogical Remarks

Our placement of a chapter on variables in this book is based on a belief that work with variables is important in the study of arithmetic and should not be delayed until a formal algebra course.

Goals. The goal for teaching uses of variables is to see this algebraic language as something that makes things clearer and easier. The reasons for rewriting given in Chapter 11 of this book apply to the present chapter as well, for often going to variables is a way of translating an English sentence into mathematics.

The ease with which students can handle formulas and deal with the requirements that computers place on variables makes it clear that work with variables need not be hard nor contrived.

One thing that is done in books that gives a wrong impression is the forcing of variables into situations where they are not needed. For instance, some books will consider the following problem:

> There were 26 people in a room. An unknown number came in but there were 43 in all. How many came in?

and suggest answering it by setting up an equation

$$26 + x = 43.$$

We view the original problem as a subtraction. An equation is not needed. Using an unknown may be a way of dealing with this problem but it should not be forced. The student learns wrongly that algebra is a hard way of doing arithmetic (actually, algebra exists because it provides an easier way of handling many problems).

The different uses of variables feel qualitatively different. The teacher should take advantage of this aspect to utilize those uses first that are easiest. Few students have difficulty understanding formulas, for example. And the language of computer programs seems also to be quite simple for students to learn. On the other hand, the use of variables as unknowns carries a quality of mystery (as in "exploring the unknown") that can create semantic if not conceptual blocks to understanding the idea if the pedagogy is inappropriate. Variables for properties lie somewhere in the middle. (We know of no research that has compared different introductions to variables, so can only go by our own experiences.)

Formulas. Formulas are abbreviations, and most everyone likes abbreviations, but only when they understand what is being abbreviated. Begin with experiments, perhaps with the area of a rectangle or with the cost of many items at the same price (leading either to the formula $A = \ell w$ or $C = np$). Write the relationship in words: Area equals length times width, or Cost equals number of items times price per item. After students have had to write the relationship a few times, they may ask for abbreviations. What abbreviations would be clearest?

Once you have $A = lw$ or $C = np$, reverse the process. For what numbers do these formulas make sense? In the case of $C = np$, does it make sense for n to equal 6.3? In $A = lw$, can $w = 0$? Find values of the variables that work. Find other values. The formula is a general rule as well as an abbreviation.

To derive formulas, start with a simple situation. If a car averages 50 mph, how far will it go:

in one hour? 50 miles

in two hours?	100 miles (but rewrite this as 2×50 miles)
in 3 hours?	3×50 miles
i., 4 hours	4×50 miles

Now the jump. How can all of this be described in one line? Answer: In h hours, the car will go 50h miles.

As a bonus, a formula may work more often than you anticipated. For instance, the value of h can be $1\frac{1}{2}$, for a car can go for $1\frac{1}{2}$ hours. And the general description rightly predicts that the car will go $1\frac{1}{2} \times 50$ miles or 75 miles in this time.

Unknowns. The letter "x" conveys an intrigue that seems to hypnotize many students (and some elementary school teachers as well). The following question below may be easy for a third grader to answer. "What number is missing in

$$3 \times \square + 40 = 52?"$$

This question may be difficult for the ninth grader when it is in the form

"Solve $3x + 40 = 52$."

It's easy for the third grader because it is approached as a puzzle. It's hard for the ninth grader who sees it as symbolic manipulation. In either case, we suggest making the first use of variables as unknowns a puzzle. What number am I thinking of? When I multiply it by 3 and then add 40, I get 52. What number is it? Use boxes and question marks instead of letters if that helps.

Use unknowns in conversing about other topics. This is particularly appropriate for estimates. For instance, in a discussion of the unemployed: How many people are unemployed? Let's let U be the number of unemployed. Do we know anything about U? For starters, there are some people unemployed, so $U > 0$. And there are some people who are working, so in the United States, $U < 250{,}000{,}000$. Can we get closer to U? Think of U as a target which you are trying to find.

Properties. The simplest properties come out of the arithmetic basic facts. When a number is multiplied by 1, the product is the number itself. How can this general statement be described with variables? Answer: For any number n, $n \times 1 = n$. Describing a pattern using variables is usually shorter and, with practice, becomes much clearer than the English language description of the same pattern.

We cannot emphasize enough that teachers should encourage students to look for patterns in the arithmetic they do and look for simple ways to describe these patterns. The attraction mathematics holds for many people is due in part to the seemingly countless number of patterns that are continually being discovered by professionals and often rediscovered by amateurs.

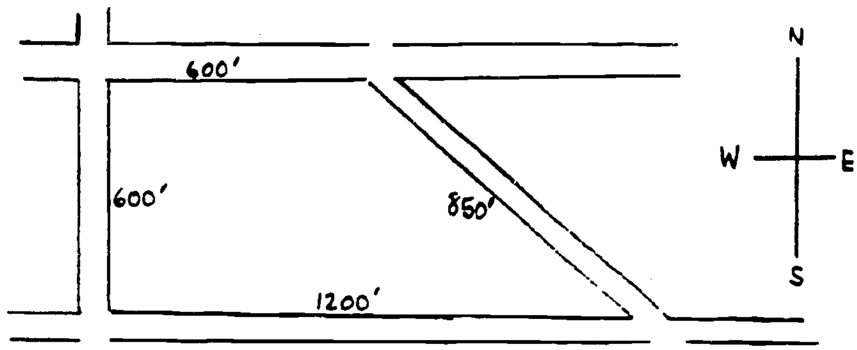
Storage locations. During the $3\frac{1}{2}$ years that it took to write this book, a revolution in the availability of computers occurred in schools. Computer programming, once a province of the later high school years, became available to young students both through specially constructed languages like LOGO and the languages like BASIC and PASCAL used with the Apples, TRS-80s, Pets, Ataris, and other computers in use in schools. The small bit of attention given computer uses in this book, mainly via the storage location section in this chapter, is not as much as they deserve.

Experience with LOGO indicates that young children, certainly as early as fifth grade, can learn to program computers. Work with computers may lead to the typical first introduction to variables for many children.

The difference between the equation $x = x + 1$ (which has no solution in algebra) and the computer statement $X = X + 1$ (which is very common) is bound to be confusing. The equal sign does not mean "is equal to" in the computer statement. A better meaning is "is replaced by". Then in algebra, $x = x + 1$ means "Can the number x be replaced by $x + 1$?" and the answer is No. For computers, $X = X = 1$ means "Can $X + 1$ be put in the slot now occupied by X?" and the answer is Yes. We cannot foresee what the consequences of having both ideas in mind will be, but adults seem to be able to move back and forth between these different uses of variable with relative ease.

Questions

1. The area of this plot of land can be found by using the formula for the area of a trapezoid. What is that formula and what is the area?



2. Describe a general formula of which the following are instances.

   One dozen eggs contains 12 eggs.

   Two dozen eggs contains 24 eggs.

   Three dozen eggs contains 36 eggs.

   Five-and-a-half dozen eggs contains 66 eggs.

3. If the area of a circle equals the area of a rectangle, then

   $$\pi r^2 = \ell \times w.$$

   What equation can be solved to determine the radius of a circular pizza whose surface area is equal to that of a 9" by 13" baking sheet?

4. Write a computer program that will print out the total cost before tax of any number from 1 to 10 of records costing $7.99 each.

5. What does this program do?

```
500  HOURS = 1
510  PRKCST = 2
520  PRINT HOURS, PRKCST
530  HOURS = HOURS + 1
540  PRKCST = PRKCST + .75
550  IF HOURS < 9 GOTO 520
```

6. Describe each general property using variables.

   (a) Instead of multiplying by 25, a person multiplies by 100 and then divides the product by 4.

   (b) To calculate the total cost of 12 cans of pork and beans at a given cost a can, a clerk calculates the prices of 10 cans and 2 cans and adds them.

   (c) Instead of multiplying $\begin{array}{r} 4444 \\ \times\ 789 \\ \hline \end{array}$, a person does the easier $\begin{array}{r} 789 \\ \times 4444 \\ \hline \end{array}$.

7. By combining the Pythagorean theorem with formulas for perimeter, the distance saved d by going diagonally from one corner of a rectangle to the opposite corner rather than "going around" the rectangle can be shown to be given by the formula

$$d = \ell + w - \sqrt{\ell^2 + w^2},$$

where ℓ and w are the length and width of the rectangle. How much distance is saved by walking diagonally across a field 800 meters long and 600 meters wide? How much distance is saved by walking across a field 800 meters long and 100 meters wide? Which is a greater percentage savings?

8. A person makes 6% commission on the first \$100,000 in sales and 5% thereafter. What formula describes the commission on sales of d dollars, if $d \geq \$100,000$?

9. If a principal P is continuously compounded at a yearly rate of r%, then it will grow to $Pe^{r/100}$ in one year, where e is not a variable but an irrational number approximately equal to 2.71828. (a) If $P = 10,000$ and $r = 8$, how much will there be in 1 year? (b) Give another instance of this general statement.

10. When s is the speed of sound, v is your velocity towards a source, and f is the frequency of the sound emitted by the source, you will hear the frequency f', where $f' = f(1 + \frac{v}{s})$. The change in frequency from f to f' is known as the Doppler effect. If the speed of sound is 760 mph and you hear a middle C (frequency 512 cycles per second) as C# (frequency 542 cycles per second), how fast are you travelling toward the sound source?

Notes and Commentary

1. Origins of this chapter
2. Other uses of variables
3. Variables as symbols without referents
4. Work of others

1. Origins of this chapter. This chapter was the last of the fourteen chapters in this book to be written. It arose from our desires to explain the appearance of variables in later parts of the book, to demonstrate that even simple arithmetic naturally flows in various ways into what is traditionally called algebra, and to show that the notion of use classes can extend beyond the bounds of arithmetic.

The content of this chapter is rooted in our previous work. Bell (1971) identifies three uses of variables: placeholders for temporary unknowns, relationships (usually formulas), and universal rules. These correspond to sections A, B, and C in this chapter. In Bell (1972), one of five chapters is devoted to formulas and another chapter is devoted to functional relationships, an application of variables that we discuss in note 2 below. In Usiskin (1976), variables are introduced as pattern descriptors, followed immediately by a discussion of variables in formulas and variables as unknowns.

2. Other uses of variables. The use classes of this chapter characterize "variable" in its rather static mode as a symbol to stand for any one of a set of numbers. Variables have two uses not mentioned in this chapter which are more suggestive of the name.

The first of these is function argument, by which we mean function domain value. Here the replacement set of the variable is the domain of a function, and the domain is conceived of as having order (the normal ordering of the real numbers). We ask: What happens to the value of the function as the variable ranges over the domain. For example, the amount of time (in hours) it takes to travel 100 miles at a rate of miles per hour is 100/r. That is, when

$$f(r) = 100/r$$

the function maps rate into time. We ask: What happens as the rate increases? What happens if the rate is doubled? Here the variable r is the argument of a function and we think of it as changing. As anyone knows who has taught algebra, this is a more difficult use of variable than any of those given in this chapter.

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A related use can be described as <u>changing parameter</u>. Consider the variable t and conceive of its increasing from 0. Now let

$$x = \cos t$$

$$\text{and} \quad y = \sin t.$$

What is happening to (x,y)? The answer is well-known: (x,y) rotates at distance 1 around the origin (0,0); that is, (x,y) sweeps out a circle. This accounts for the name "circular functions" being given to the sine and cosine functions and accounts for the natural applications of these functions to circular motion. The variable t is called a <u>parameter</u> because both domain and range value of the function are in terms of t, and in almost all discussions of parameters of this type we think of the parameter as changing or varying.

Still another static use of variable is as <u>constant parameter</u>, a use somewhat akin to formula abbreviation. An example is in a situation like the following. A library has N books now and adds to its collection at a rate of R books per month. At the end of x months it will thus have N + Rx books. We think of N and R as constants, variables which for a given problem have only one value each. (In fact, the letter x was purposely used instead of M to cause one to think of it more as a function argument.) There is a function mapping x onto N + Rx; this is the linear function with slope R and intercept N. Here R and N are variables used as constant parameters.

Each of these variables is common in school mathematics, particularly in later secondary school courses in which functions are involved. We considered them beyond the proper scope of a volume devoted to applying arithmetic.

3. <u>Variables as symbols without referents</u>. In this chapter, we restrict ourselves to those variables that stand for, or are replaced by, numerical values. The values a variable takes on do not have to be numerical; in geometry, variables often represent points (e.g., when we write "if AB = AC", the letters A, B, and C are representing points); in algebra, variables often represent functions (e.g., if f and g are any functions, then f+g = g+f). This view is very common, and is described in the following quote (May and van Engen, 1959).

> "Roughly speaking, a variable is a symbol for which one substitutes names for some objects, usually a number in algebra. A variable is always associated with a set of objects whose names can be substituted for it. These objects are called values of the variable."

However, this is not the only view possible. To the formalist school, variables are merely symbols on paper related to each other by assumed or derived properties that are also themselves marks on paper. No referents are needed.

While we might consider such a view as tenable to philosophers but impractical to users of mathematics, present day computer algebras such as MACSYMA and muMath (see Pavelle, Rothstein and Fitch, 1981) deal with letters without any need to refer to numerical values. That is, today's computers can operate as both experienced and inexperienced users of algebra do operate, blindly manipulating variables without concern for what they represent. We could call this the nominal use of variables, analogous to the nominal use of number (note 8, chapter 1).

4. Work of others. The roles played by variables in algebra have been analyzed by many others. Here is a sampling.

Thorndike et al. (1923) do not specifically isolate uses of variables, but speak of the nature of algebraic abilities, identifying four as needed: ability to understand and frame formulas; ability with equations; ability with problems; ability with graphs. There is much discussion of uses but the uses aren't organized into types other than by context. Still, this book is a worthwhile starting point for the reader interested in analyses of variables.

In the 50s and 60s, the concept of variable was more likely to be analyzed from the mathematical point of view. Bennett (1955) identifies two uses of variable: free and bound (or dummy or umbral).

> "A free variable may easily be required to remain completely unrestricted, save possibly for specification of a universe of discourse, to avoid certain paradoxes incident to ambiguity as to type....
>
> "A bound variable serves merely as a convenient but theoretically avoidable notational device for discussing a fixed set... A bound variable is recognized as being such by virtue of its mention in a quantifier."

Bennett derides the view of some that a variable is "an unspecific member of a specified set" and shows his preference for thinking of every use of variable as being, more or less, a bound use.

> "...in any completely explicit system of mathematics a variable is a quantified (dummy) symbol in a proposition or designation."

May and van Engen (1959) give only one notion for variable (see note 3 above), but allow that variables appear under different names: placeholder, constant, unknown, and parameter. This is a view found in most textbooks today.

Kuchemann (1981) categorizes children's uses of variables in six ways: (1) letter evaluated, where the letter is assigned a numerical value from the outset; (2) letter not used, where children ignore the letter or do not give it meaning;

(3) letter used as an object, or shorthand for an object; (4) letter used as a specific unknown upon which children operate directly; (5) letter used as a generalized number which can take several values rather than one; and (6) letter used as a variable, representing a range of unspecified values and a systematic relationship is seen to exist between such sets of values. Of these, (3) corresponds to our use of formulas, (4) to unknowns, and (6) to properties. Kuchemann identifies four levels of understanding of these notions and suggests that the uses of variable be introduced to the student in the above order.

Davis, Jockusch, and McKnight (1978) identify an information-processing conception of variable akin to that found in Section D of this chapter.

> "Computers give us another view of the basic mathematical concept of variable. From a computer point of view, the name of a variable can be thought of as the address of some specific memory register, and the value of the variable can be thought of as the contents of this memory register."

This conception is utilized throughout their book-length report.

Recent researchers (e.g., Clement (1982), Rosnick (1981), Wagner (1981)) have focussed on misconceptions students have with variables and often relate these misconceptions to an insufficient understanding of the role or the referent for the variable.

Summary of Part I

The objects of arithmetic are single numbers, ordered collections (n-tuples) of numbers, unordered collections of numbers, and variables that stand for numbers. Understanding the uses of these objects requires first an understanding of the uses of the single numbers upon which they are based.

The uses of n-tuples correspond closely to those of single numbers.

single numbers	n-tuples
counts	counts
measures	measures
locations	locations
ratio comparisons	ratios
codes	codes
derived formula constants	combined uses

N-tuples are often used for storage of numerical information for present use or future accessibility.

Any of the first five uses of single numbers named above may be represented by numbers in a collection. Thinking of the collection as an entity, its role is either as a domain from which numbers are to be chosen, a data set, a solution set, or a neighborhood of estimates.

Variables may stand for numbers used in any of the ways given above. One moves to variables rather than single numbers when one wishes to relate numbers by a formula, when the variable is to stand for an unknown, when there is a general pattern or property to be described, or when one wishes to name a location in a computer where a number is to be stored.

The diversity of uses to which numbers have been put far exceeds the diversity as represented in most schoolbooks. Confronting these uses is a necessary step in the acquiring of the ability to apply arithmetic.