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ABSTRACT

This study investigated children's representation of addition and subtraction problems with canonical and noncanonical number sentences, to see whether children would directly represent the structure of a problem if both types of sentences were available. Subjects were 22 first graders and 41 second graders randomly assigned either to a Canonical group or a Noncanonical group. Each group received two 30-minute periods of instruction on writing and solving number sentences and writing number sentences to represent word problems. The Noncanonical group was introduced to all six basic open sentence types, while the Canonical group was introduced only to the one for addition and the one for subtraction in which the unknown is to the right of the equals sign. Children were then given two 12-item tests, on one of which they were instructed to write a number sentence for the problem and to solve the problem; on the second test, they only had to write a number sentence for the problem. Results suggest that most first graders are limited to direct symbolic representations of word problems. Both first and second graders can learn to write noncanonical number sentences and use them to represent word problems. Implications for instruction are discussed. (MNS)

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Before they receive formal instruction in addition or subtraction, children can analyze and solve simple addition and subtraction word problems by representing the action of relationships described in a problem with physical objects or fingers or by employing a variety of quite sophisticated counting strategies (Carpenter & Moser, 1983; 1984; Riley, Greeno, & Heller, 1983). At the time children are first introduced to writing mathematical sentences to solve word problems, these informal strategies work and make more sense to the children than the formal strategies being taught. As a consequence, although most children are able to solve the problems using these informal modeling and counting strategies, they see no connection between their solutions and the number sentences they are asked to write by the teacher (Carpenter, Hiebert, & Moser, 1983). The operations represented by the number sentences that children write are often incorrect and inconsistent with the modeling and counting strategies that they actually use to solve the problem.

The difficulties that children experience in writing number sentences to represent word problems occur because the representations they have available (a + b, a - b) do not always correspond to their formal solutions. For children, a - b represents a separating action which corresponds to their solution of separate problems like the following:

John had 8 marbles. He lost 3 of them. How many does he have left?

As a consequence, they have no difficulty writing a number sentence for this type of problem. The following Join Change Unknown problem, however, is generally solved by joining elements to another set and keeping track of the number of elements joined rather than removing elements from a larger set (Carpenter & Moser, 1983).

John had 3 marbles. He won some more. Now he has 8 marbles. How many marbles did he win?

This solution corresponds more closely to the noncanonical number sentence $3 + \square = 8$. This suggests that for young children noncanonical number sentences may be the most natural representation for certain types of addition and subtraction problems. In other words, children may most naturally write number sentences (both canonical and noncanonical) that directly represent the action described in a problem. Further support for this hypothesis is provided by children's solutions to different number sentences ($b - a = \square$ and $a + \square = b$), which are similar to their solutions for the corresponding word problems (Blume, 1981).

Although studies have investigated the relative effectiveness of instruction that includes canonical and noncanonical number sentence and instruction that focuses exclusively on canonical number sentences

(Wilson, 1967), this research has some serious limitations (Zweng, 1968) and does not take into account recent findings regarding the development of children's ability to solve addition and subtraction problems. The study by Carpenter et al. (1983) examined the number sentences that children write to solve different problems and to consider the relation between these number sentences and the informal strategies that children use to solve the problems. This study, however, did not include any of the problems that are most directly represented by noncanonical number sentences, and none of the children in the study had been exposed to noncanonical number sentences.

Purpose

The purpose of this study was to investigate children's representation of a wide range of addition and subtraction problems with canonical and noncanonical number sentences. The primary concern was whether children would directly represent the structure of a problem if both canonical and noncanonical number sentences were available for such a representation.

The study included children who had been exposed to both canonical ($a + b = \square$, $a - b = \square$) and noncanonical ($a + \square = b$, $a - \square = b$, $\square \div a = b$, $\square - a = b$) number sentences as well as children who had only studied canonical number sentences. Two short periods of instruction were provided to make appropriate number sentences available to the children. The study was not designed to investigate the relative effectiveness of different instructional treatments. The brief instruction was much too limited to represent a valid test of instructional treatments. The objective was to identify the number sentences that children in each group used to solve different addition and subtraction problems in order to examine the relation between these number sentences that children wrote and the semantic structure of the problems. Since the semantic structure of word problems plays such a dominant role in children's strategies for solving the problems at this age using modeling and counting techniques, it was hypothesized that the structure would significantly influence the number sentence representations of the children in both groups as well.

Method

The subjects for the study were 22 first-grade children and 41 second-grade children from an elementary school that draws from a predominantly middle-class area of Madison, Wisconsin. Prior to the study, the subjects had only limited exposure to solving word problems which focused primarily on simple joining and separating situations (cf. problems 1 and 4 in Table 1). No instruction had been given on writing noncanonical number sentences or on writing complete open sentences with a box to represent the unknown.

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Insert Table 1 about here

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Children were randomly assigned to one of two groups: a Canonical group and a Noncanonical group. Each group received two 30-minute periods of instruction on writing and solving number sentences and writing number sentences to represent word problems. On the first day, the Noncanonical group was introduced to all six basic open sentence types ($a + b = \square$, $a - b = \square$, $a + \square = b$, $a - \square = b$, $\square + a = b$,

$\square - a = b$). For the noncanonical number sentences they were told only how to interpret the symbols. For example, $6 + \square = 8$ was read as "six plus what number equals eight?" or "what number can I add to 6 to equal eight?" No specific instructions were given for solving the number sentences. Instruction lasted approximately 15 minutes. For the duration of the period, the children solved a variety of number sentences working individually. On the second day, the Noncanonical group was taught to write number sentences to represent word problems. The children were told to use the number sentences they had learned about the proceeding day to represent the action in the problems. Instruction lasted approximately 15 minutes. For the duration of the period, the children worked individually writing number sentences to represent problems and solving the number sentences. All the word problems included in instruction and individual seat work were of three types: simple Join, simple Separate, and Join/Change Unknown (Table 1, problems 1, 4, and 2, respectively). Thus, although children were shown the complete range of number sentences on the first day of instruction, the only noncanonical number sentences that they used to represent word problems during instruction were of the form $a + \square = b$.

Instruction for the Canonical group followed the same pattern except that only canonical number sentences were included. On the first day, children solved open number sentences of the form $a + b = \square$ and $a - b = \square$. On the second day, they represented word problems using one of these two forms. The problems covered during instruction were exactly the same as these used with the Noncanonical group. For the Join/Change Unknown problem (Table 1, problem 2), a part-whole analysis was employed (Kouba & Moser, 1979; Riley et al., 1983; Wilson, 1967).

Based on this analysis, if both parts are given and the whole is the unknown, the operation is addition; if one part and the whole is given, the operation is subtraction. Formal procedures for analyzing part-whole relationships were not specified, and the technique was simply illustrated in the context of solving Join/Change Unknown problems.

The first and second graders were instructed in separate groups. The number domain of the problems for the first graders consisted of number facts with sums less than 10; for second graders, number facts with sums greater than 10 were used. Instruction was provided by two of the experimenters. The experimenter who taught the first-grade Canonical group taught the second-grade Noncanonical group and the experimenter who taught the first-grade Noncanonical group taught the second-grade Canonical group. One first-grade class and two second-grade classes participated in the study. Half of each class, identified using random selection procedures, stayed in their regular classroom, and the other half went to a separate classroom. All groups were instructed as a group with the last half of the class devoted to individual seat work. For each group, instruction occurred on consecutive days.

Following instruction, children were tested on their ability to write number sentences to represent addition and subtraction word problems. The test was divided into two parts, which were administered on consecutive days. The first part consisted of 12 addition and subtraction problems (Table 1) selected from the categories of problem types identified by Carpenter and Moser (1983) and Riley et al. (1983). The problems include all six Join and Separate problems, which correspond to each basic type of open sentence. The clear action of these six problems is directly modeled by a specific number sentence (see Table 1). For the other problems, the relation between problems and number sentence is more ambiguous. The matching strategy used to solve Compare problems is not clearly represented by any particular number sentence, and the Combine subtraction problem could reasonably be represented by several different number sentences. Only problems similar to problems 1, 2, and 4 had been included in instruction.

For the first set of 12 problems, children were instructed to write a number sentence for the problem and to solve the problem. The numbers for these problems were within the range of basic number facts so that canonical and noncanonical sentences could be readily solved by using counting strategies that matched the structure of the open sentence or by recalling the appropriate number fact. The second part of the test included 12 problems that were similar to those in Table 1 with minor modifications in the context of the problem. For these problems children were instructed to only write an appropriate number sentence and not to solve the problem. The numbers in these problems were two-digit numbers so the answers were not readily calculated. No experience writing or solving number sentences with two-digit numbers had been provided during the two days of instruction. Carpenter et al. (1983) found that a number of children solved some problems before writing the number sentence. This part of the test was designed to force children to write open number sentences with a box to represent the unknown. The numbers for the first set of problems were selected from basic facts with sums less than 10 for the first graders and from basic facts with sums greater than 10 for the second graders. In the second set of problems, numbers were selected with sums in the 20s for the first graders and sums greater than 50 for the second graders. In each problem set, the order of the problems was randomized. Except for the choice of numbers, the problems and administration procedures were identical for all subjects.

The test was administered to children from Canonical and Non-canonical groups combined in their regular classroom. The test was administered by a third experimenter who had not participated in instruction. Each problem was read by the experimenter to the entire group. Children were instructed to listen to the problem all the way through before writing a number sentence. Each problem was read twice, and children could raise their hands to request an additional repetition. The problems were also printed in the test books. Each problem was printed on a separate page, and children were instructed to not turn the page until they were told to do so.

Results

For the first set of problems, both answers and number sentences were categorized for each problem. An answer was scored as Correct without regard to whether the number sentence was correct, where the answer appeared in the number sentence, or whether the answer was designated by a box. For example, if an appropriate number sentence for a given problem was $5 + \square = 7$, all of the following would have been categorized as correct answers: $5 + 2 = 7$, $5 + 2 = \boxed{7}$, $2 + 5 = 7$, $5 + 7 = 2$. This procedure was adopted to distinguish between the process used to calculate the answer and the analysis involved in writing the number sentence, as previous research had found that children often do not relate the two (Carpenter et al., 1983). Similarly, answers were categorized as based on the Wrong Operation strictly on the basis of the inclusion in the number sentence of a number which would result from an inappropriate computation. For the problem given above, a response that included a 12 in the number sentence would be classified as having an answer based on a wrong operation. Other errors included computational or counting errors, responding one of the numbers given in the problem or failing to generate any answer.

Four primary categories of number sentences were identified. Number sentences were categorized as Canonical if they were of the form $a + b = \square$ or $a - b = \square$ and were appropriate number sentences that would produce a correct answer for the given problem. If the wrong operation was chosen ($a + b = \square$ when the correct sentence was $a - b = \square$ or $b + \square = a$), the sentence was classified as wrong operation. This category is distinct from the wrong operation category for the answer to the problem and the use of the wrong operation was not consistent over the two categories.

Two major categories of noncanonical sentences were identified. The first (Noncanonical) included only complete noncanonical sentences that when solved would result in a correct answer to the problem. Incorrect noncanonical sentences were infrequent and are noted in the text. A number of children attempted to reflect the structure of the problem in their number sentences but did not write appropriate number sentences with a box to represent the unknown. For example, rather than writing $5 + \square = 7$ or $5 + \boxed{2} = 7$, they wrote $5 + 2 = 7$ or $5 + 2 = \boxed{7}$ These responses were labeled Incomplete Noncanonical.

The results reported below are for the first set of 12 problems in which children wrote sentences and calculated the answers. Responses for the second problem set followed the same general pattern with respect to sentence writing. Since the numbers were larger and children were instructed to not compute answers, it was more difficult to write a noncanonical sentence unless a box was used to represent the unknown. As a consequence, there were fewer incomplete nonconical responses. But contrary to instructions, a number children attempted to calculate the answer and write number sentences modeling the action in the problem.

First-grade Results

The results for the two first-grade groups are summarized in Tables 2 and 3. The simple Join and Separate problems (problems 1 and 4) are directly modeled by canonical number sentences and were solved and represented correctly by all children in both groups. The four Join and Separate problems that are best modeled by noncanonical number sentences (problems 2, 3, 5, and 6) provide most interesting insights regarding the children's attempt to write number sentences to represent the action described in problems and the role that noncanonical number sentences may play in these representations. In spite of the fact that children in the Noncanonical group had only been instructed on representing Join missing addend problems (problem 2), over 65% of them attempted to directly represent the action in these four problems. Many of them had not mastered the form of noncanonical number sentences, especially for number sentences involving subtraction which they had no practice writing. As a consequence many of the number sentences did not fit the complete noncanonical sentence format. But all noncanonical sentences, both complete and incomplete, provided a direct representation of the problem (see Table 1). No inappropriate noncanonical sentences were written. Furthermore, for three of the problems (2, 3, and 6), no children in the Noncanonical group wrote a canonical number sentence. For problem 5, 36% of the children in the noncanonical group wrote a canonical subtraction sentence (11 - 8 = $\square$). This may or may not represent a clear attempt to transform the problem to a canonical number sentence. Although 11 - $\square$ = 8 most directly represents the specific action in the problem, both number sentences represent a separating action. Some children may have focused on the action and written the more familiar canonical sentence. This general approach produces a correct number sentence for this problem but may account for the representational errors involving the choice of the wrong operation in problems 2, 3, and 6.

Insert Tables 2 and 3 about here

The Combine and Compare problems are not clearly represented by noncanonical number sentences, and instruction on non-canonical number sentences appeared to little influence on children's representations of these problems. The Equalize problem involves the integration of a comparison and a joining action. Since the joining action could be represented by 5 + $\square$ = 8, this noncanonical sentence might be used to represent this problem, but there were few examples of this response.

Responses from the Canonical group offer further evidence that children of this level most naturally represent problems directly and do not readily transform problems to represent them as canonical number sentences. Although they had been instructed to represent problems as canonical number sentences, the majority of them did not generate a correct canonical representation of the four action problems that would have required transformations (problems 2, 3, 5, and 6). Only 18% represented the Join missing addend problem with a canonical number sentenced in spite of the fact that they had received specific instruction on the canonical representation of that type of problem. Although they had received no instruction in directly representing problems, as many as 45% attempted to write number sentences that represented the action described in these four problems. Because they had not been instructed in writing noncanonical number sentences, they did not represent the unknown with a box, but their number sentences were consistent with the action described in the problems.

Second-grade Results

Results for the two second-grade groups are summarized in Tables 4 and 5. The second graders in both groups were generally more successful than the first graders in representing and solving the more difficult problems. They were much more successful in transforming problems to represent them as canonical sentences. Over 75% of the second graders in the Canonical group wrote correct canonical number sentences for five of the six Join and Separate problems. A few children in the canonical group did attempt to write number sentences that reflected the structure of the problem, but this type of response was less frequent than for the first-grade Canonical group.

Insert Tables 4 and 5 about here

The second graders in the Noncanonical group were generally successful in learning to write noncanonical number sentences, and over 80% used them to represent the two missing addend problems (problems 2 and 3). Responses for two of the Separate problems (problems 5 and 6) were split between canonical and noncanonical representations. Thus, the second graders were generally more successful in learning to write noncanonical number sentences than the first graders in that they almost always wrote complete noncanonical sentences. But many of them were also more flexible and were not limited to directly modeling the action in a given problem. Some second graders were also more flexible in their use of noncanonical sentences and used them to represent Combine, Compare, and Equalize problems. Although no first graders wrote complete or incomplete noncanonical number sentences that were inappropriate for a given problem, there were a total of nine incorrect noncanonical sentences for the second grade group, four of which were for problem 10 ($5 + \square = 7$ rather than $5 + 7 = \square$).

Second graders' flexibility is also illustrated by their representations of the four problems that could be represented by a canonical addition sentence (problems 1, 6, 7, and 10). In the noncanonical group, over half the addition number sentences for these four problems started with the larger number in the problem in spite of the fact that the smaller number was always given first i the word problem. This transformation corresponds to the transformation that older children perform in solving addition problems by counting on from the larger number. None of the first-graders and only about 10% of the second-graders in the noncanonical group made this transformation.

Discussion

The results of this study suggest that within the population sampled, most first-graders are limited to direct symbolic representations of word problems. If they are only able to represent word problems with canonical number sentences, they can represent only a limited number of types of problems. It appears, however, that first-graders can learn to write noncanonical number sentences and readily use them to directly represent the action in appropriate problems.

Second-graders also easily learn to write noncanonical number sentences and will use them to represent word problems. Some second-graders are more flexible in writing number sentences and can represent problems with canonical number sentences, even when the canonical sentences do not match the action in the problems.

These results are consistent with the findings of research on children's informal strategies for solving word problems using physical objects and counting strategies. This research has found that children pass through several levels in the development of addition and subtraction concepts and skills (Carpenter & Moser, 1984; Riley et al., 1983). In the initial levels, children solve a variety of addition and subtraction problems by modeling the action in the problems using physical objects and counting strategies. At about the second grade, children begin to pass into more advanced levels in which their counting strategies become more flexible and no longer have to directly correspond to the action or relationships in the problem. The results of this study suggest that these levels apply to symbolic as well as physical representations of problems. In other words, children at the initial levels can only represent problems symbolically with number sentences as long as the sentence corresponds to the action in the problem. However, just as is the case with physical solutions, they can represent a wider range of problems than simple Joining and Separating problems. They solve missing addend problems (Table 1, problem 2) using an adding-on strategy that corresponds to the additive action in the problem, and they can represent the same problem with a noncanonical number sentence ($a + \square = b$) that represents this action. At the more advanced levels, children can transform problems to solve them with the most economical counting strategy or represent them with canonical number sentences.

Implications for Instruction

These results suggest several alternatives for instruction. One is to limit exposure to word problems to types of problems that correspond to canonical number sentences until children attain a level at which they can transform problems. The problem with waiting until children attain the appropriate levels to teach more difficult problems is that children may develop a rather narrow perspective on the types of problems to which addition and subtraction operations apply. Children can solve a much wider variety of problems using manipulative materials and counting strategies, and to restrict the set of problems studied fails to build upon the concepts and skills underlying these solutions.

One possibility is to attempt to accelerate the development of the more advanced levels. One method that has been proposed for accelerating development in this area is by teaching children to analyze problems in terms of part-whole relationships (Kouba & Moser, 1979; Riley et al., 1983).

Another basic alternative is to teach children to represent problems with noncanonical number sentences so that they can represent them while they are still at a direct modeling level. The results of this study suggest that instruction that includes noncanonical as well as canonical number sentences more nearly corresponds to children's natural representations and solutions of word problems than attempting to immediately represent all problem situations in canonical form. Mathematics programs developed in the 1960s included noncanonical number sentences. For the most part, this approach meet with teachers' resistance, because they found that the particular approach to noncanonical sentences that was employed was difficult to teach. At that time, noncanonical sentences were to be solved by relating them to corresponding canonical sentences. Current analyses suggest that such transformations require relatively advanced concepts of addition and subtraction (Briars & Larkin, 1985; Riley et al., 1983). However, young children can solve such sentences directly by using the same modeling and counting processes that they use to solve corresponding word problems (Blume, 1981). The evidence suggest that by the middle of the first grade, most children can solve noncanonical open sentences. (See Carpenter, Blume, Hiebert, Anick, & Pimm [1982] for a review of this research.) The results of the current study suggest that young children will use noncanonical forms to represent and solve appropriate problems.

This study clearly does not resolve the issue of how to teach children to represent and solve simple word problems, and additional research is needed to evaluate instruction based on the different approaches. The results of this study along with the findings of other research on open number sentences (Carpenter et al., 1982) do suggest, however, that instruction involving noncanonical number sentences should not be dismissed out of hand because of presumed past failures. Instruction that includes noncanonical number sentences appears to be a viable approach for building on the informal number concepts and skills that children bring to instruction.

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Table 1

Word Problem Test

Problem	Corresponding Number Sentence
JOIN	
1. Sam had 2 books. Mom gave him 4 more. Now how many books does Sam have?	$2 + 4 = \square$
2. Polly had 5 cookies. Joe gave her some more. Now she has 7 cookies. How many cookies did he give her?	$5 + \square = 7$
3. The boy had some stickers. His aunt gave him 5 more. Now he has 9 stickers. How many stickers did the boy start with?	$\square + 5 = 9$
SEPARATE	
4. 9 dogs were in the park. 7 dogs ran away. How many dogs were left in the park?	$9 - 7 = \square$
5. Kate had 11 jacks. She lost some of them. Now she 8 jacks. How many jacks did Kate lose?	$11 - \square = 8$
6. Joan had some candies. She ate 2 of them. Now she has 6 candies. How many did Joan have to start with?	$\square - 2 = 6$
COMBINE	
7. Tracy has 4 red balls and 6 blue balls. How many balls does Tracy have in all?	
8. The cat has 9 kittens. 6 are brown and the rest are white. How many white kittens are there?	
COMPARE	
9. Ted has 7 toy boats. Sue has 4 toy boats. How many more boats does Ted have than Sue?	
10. Betsy has 4 cards. Her brother has 7 more cards than Betsy. How many cards does her brother have?	
11. Ed has 10 crayons. He has 7 more crayons than Kelly. How many crayons does Kelly have?	
EQUALIZE	
12. Tom has 8 points. Liz has 5 points. How many more points does Liz have to win to have as many as Tom?	

Table 2

First-grade Noncanonical Group Results

Problem	Answer**		Number Sentence**			
	Correct	Wrong Operation	Canonical	Noncanonical	Incomplete Noncanonical	Wrong Operation
Join						
1	100*	0	100	0	0	0
2	100	0	0	45	36	18
3	64	0	0	64	18	9
Separate						
4	100	0	100	0	0	0
5	45	0	36	18	45	0
6	73	0	0	27	45	27
Combine						
7	91	0	91	0	0	9
8	73	0	45	0	0	36
Compare						
9	55	18	27	0	9	45
10	55	9	82	0	0	0
11	27	9	18	0	0	55
Equalize						
12	91	0	0	9	9	64

*percent responding

**Rows do not sum to 100 because some categories of responses have been omitted.

Table 3

First-grade Canonical Group Results

Problem	Answer		Number Sentence			
	Correct	Wrong Operation	Canonical	Noncanonical	Incomplete Noncanonical	Wrong Operation
Join						
1	100*	0	100	0	0	0
2	36	27	18	0	18	55
3	55	27	9	0	45	27
Separate						
4	91	0	91	0	0	9
5	55	0	45	0	36	9
6	55	18	27	0	45	27
Combine						
7	100	0	91	0	0	9
8	27	18	36	0	0	55
Compare						
9	27	55	36	0	0	64
10	27	9	82	0	0	18
11	36	9	55	0	0	36
Equalize						
12	45	9	27	0	9	36

*percent responding

Table 4

Second-grade Noncanonical Group Results

Problem	Answer		Number Sentence			
	Correct	Wrong Operation	Canonical	Noncanonical	Incomplete Noncanonical	Wrong Operation
Join						
1	90*	5	95	0	0	5
2	100	0	5	85	0	5
3	100	0	15	80**	0	0
Separate						
4	90	0	95	0	0	5
5	95	0	55	35	0	
6	55	35	35	35	0	25
Combine						
7	100	0	100	0	0	0
8	85	0	80	15	0	5
Compare						
9	95	0	55	25	0	5
10	50	45	60	0	0	20
11	100	0	70	20	0	5
Equalize						
12	95	0	50	35	0	20

*percent responding

**Includes 30% who responded 8 + $\square$ = 12 rather than $\square$ + 8 = 12.

Table 5

Second-grade Canonical Group Results

Problem	Answer		Number Sentence			
	Correct	Wrong Operation	Canonical	Noncanonical	Incomplete Noncanonical	Wrong Operation
Join						
1	95*	5	95	0	0	5
2	90	5	76	0	14	10
3	90	0	86	0	10	5
Separate						
4	100	0	100	0	0	0
5	90	0	90	0	10	0
6	62	33	57	0	10	33
Combine						
7	95	0	95	0	0	0
8	90	5	86	0	5	10
Compare						
9	95	5	90	0	5	5
10	67	33	62	0	0	29
11	86	5	81	0	5	10
Equalize						
12	81	5	71	0	19	10

*percent responding