DOCUMENT RESUME

ED 260 904 SE 045 939

AUTHOR	Kirshner, David
TITLE	Spatial Cues in Algebraic Syntax.
INSTITUTION	British Columbia Univ., Vancouver.
PUB DATE	85
NOTE	80p.; Paper presented at the Annual Meeting of the American Educational Research Association (69th, Chicago, IL, March 31-April 4, 1985).
PUB TYPE	Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC04 Plus Postage.
DESCRIPTORS	Adults; *Algebra; *College Mathematics; Educational Research; Higher Education; *Individual Differences; *Mathematical Linguistics; Mathematics Instruction; Secondary Education; *Secondary School Mathematics; Spatial Ability; Symbols (Mathematics); *Syntax
IDENTIFIERS	British Columbia; *Mathematics Education Research

ABSTRACT

Language syntax is discussed, followed by an overview of algbraic syntax. Evaluating algebraic expressions was chosen as the means to investigate the psychological basis of syntactic skills. Subjects' ability to perform evaluative tasks appropriately using standard algebraic notation was ascertained; then similar tasks were presented using a nonce, or artificial, notation designed to display the propositional or deep structure character of algebraic expressions while distorting the surface cues of ordinary notion. Data from three ten-item instruments (included in appendices) were analyzed for 517 subjects in grades 9, 11, first-year calculus, and fourth-year engineering; and professional engineers in Vancouver, British Columbia. Covariates were computing experience and past algebraic achievement. Most subjects (93.6%) demonstrated competence in the regular notation algebra tasks. Both subjects using propositional referents and those using surface cues in syntactic decision making were generally successful (91.0%) at the five simple nonce algebra items, but complex nonce algebra items were more difficult (66.3% was the mean percentage correct). Close nonce notation was significantly more difficult than spaced nonce notation. Variation among respondents was substantial, and individual differences are extensively discussed, as are implications for research. (MNS)

ED260904

Spatial Cues in Algebraic Syntax

David Kirshner

University of British Columbia

SE 045 939

TABLE OF CONTENTS

Spatial Cues in Algebraic Syntax

Every language, be it natural or artificial, employs a system of rules, called a syntax, to govern the arrangement and interrelation of its elements as they occur in "sentences." It is the syntax of English, for example, that determines for the sentence "The man Bill grabbed fell" that it is the man that fell, Bill who did the grabbing, the man who was grabbed, etc. Similarly, it is the syntax of the computer language LOGO that determines that in the command "FORWARD :CHARGE" :CHARGE will be evaluated as a number of units for the turtle to progress. As well, it is the syntax of algebra that determines that in "$3x^2$" it is the x which is squared and the result tripled, rather than the x tripled and the result squared. In general, syntax provides a fixed internal structure for the elements of the "sentences" of a language.

Clearly, syntax must be mastered before any sensible use can be made of a language. In the case of natural languages, syntax is learned unconsciously through informal exposure to a language community. A speaker need not be consciously aware of the "rules" which comprise syntax. For example, the fluent speaker of English, unschooled in linguistic theory, would be hard pressed to identify a rule in which the stress applied to a pronoun determines its reference. Yet every native speaker, will unhesitatingly recognize that pronominal reference for the sentence "Bill hit John and then Frank hit him" is determined by the presence or absence of stress on "him." Indeed, a major project of linguistic research is to discover exactly what the rules of syntax are for natural languages.

For artificial languages, such as computer languages, the situation is reversed. Rules of syntax are deliberately laid out in the original formulation of the language. They remain fixed (or subject to controlled development) throughout the life of the language. For the novice, syntactic rules are learned explicitly from manuals or through structured teaching. The computer insists on explicit formulations for commands, and those who design computer languages and those who "speak" to computers through these languages are well apprised of the fact.

Mathematical languages are usually considered to reside within the artificial camp rather than the natural language camp. Indeed, the very essence of mathematics may be seen as the presentation of rigorously derived rules through a rational and determinate notation. Like computer languages, mathematical languages have been designed for specific technical or scientific purposes. Their rules are completely circumscribed by conscious and rational consideration at the time of their inception. They are communicated to novice users from textbooks or through structured pedagogy.

As compelling as this view of mathematical language may seem, it is assailable on several grounds. First of all, unlike computer languages, mathematical languages such as the system of algebraic notation are not the creation of a single integrated effort. Algebraic results originally were coded in careful natural language, rather like legal language is today. The following translation from the great Mohammed ibn-Musa al-Khowarizmi in his ninth century book Al-jabr wa'l muqābalah (from which "algebra" got its name) illustrates this point:

> You ought to understand also that when you take the half of the roots in this form of equation [quadratic] and then multiply the half by itself; if that which proceeds or results from the multiplication is less than the units above-mentioned as accompanying the square, you have an equation. (Boyer, p. 253).

(This statement expresses the fact that in a quadratic equation the discriminant must be positive in order for there to exist a real solution). Specialized mathematical symbols were only gradually introduced, and as Cajori (1928) observed, the systematization of notations was halting and evolutionary rather than decisive and final. Thus the assumption that mathematical languages are like computer languages in having a rationally accessible syntax should not be accepted unquestioningly.

Careful examination of pedagogical methods and materials for algebraic syntax proves particularly damning to this thesis. A survey of several textbooks (Brown, Snader & Simon, 1970; Vannatta, Goodwin & Crosswhite, 1970; Johnson, Lendsey & Slesnick, 1971; Sobel & Maletsky, 1974; Dolciani & Wooton, 1975; Johnson & Johnson, 1975) found only 1 to 6 pages devoted to instruction in algebraic syntax. More importantly, the rules presented were, in most cases, inadequate to the actual requirements of syntactic skill. Often, exponentiation and radical were entirely omitted from the discussion (which typically is near the beginning of the text) since these operations had not yet been introduced. Occasionally the left-to-right precedence of related operations was unstated [(5 - 3 + 1 = (5 - 3) + 1, not 5 - (3 + 1)].

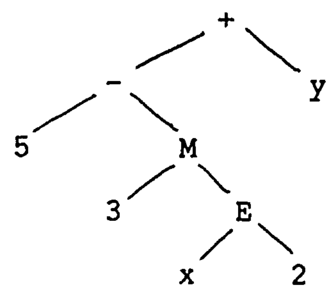
Similar deficiencies can be discerned in classroom based syntactic strategies. In the present study many grade 9 and grade 11 students tested reported the use of the acronym BOMDAS which stands for Brackets, Of, Multiply, Divide, Add, Subtract. This indicates that operations within parentheses are precedent to those outside of parentheses and that multiplication and division have precedence over addition and subtraction. "Of" is a relic of older notations as illustrated in "3/8 of 24." A similar technique uses the mnemonic My Dear Aunt Sally for which the initials correspond to the MDAS of BOMDAS. Such methods are inadequate even to the parsing of such simple examples as $3x^2$, and $x - y + 1$.

It is apparent from these observations that the assumption that syntactic knowledge is transmitted to the neophyte in an explicit form through the propositions espoused by text and teacher is highly suspect. Thus one may ask what is the nature of syntactic skill, if it is not based on textbook rules? How are these skills actually acquired by students if pedagogical methods are incomplete? Are all students equally successful in acquiring this syntactic knowledge which is fundamental to any sensible use of algebraic symbolism?

Algebraic Syntax

In order to begin to address these questions, it is necessary to provide a comprehensive account of algebraic syntax. The syntax of algebra assigns a fixed internal structure to the elements of each well formed algebraic expression. A tree notation adapted from linguistic theory (Matz, 1980) is used to display this structure. This notation is illustrated in Figure 1 for the expression $5 - 3x^2 + y$. The operations closer to the bottom of the tree are said to be "precedent" to those above, while the higher ones are "dominant" to those below. In keeping with standard practise, syntactic conventions will be formulated

Figure 1
Syntactic Tree Diagram for $5 - 3x^2 + y$



"M" represents multiplication
"E" represents exponentiation

as precedence rules, although elsewhere, the author has argued (unpublished) that cognitively, such rules are represented in terms of dominance. That discussion, however, is orthogonal to the present concerns.

There are two kinds of syntactic conventions in elementary algebra. The first concerns the physical presence of syntactic markers. Besides the usual parentheses, there are two other types of syntactic markers. The vinculum as it occurs in both fractions and radicals is an indicator of grouping. (Compare, for example, $\sqrt{x + y}$ and $\sqrt{x} + y$). Similarly, the raising of symbols in exponents carries syntactic import (compare x^{5y} and x^5y). Inclusion of an operation within parentheses, above or under a vinculum, or within an exponent gives it precedence which it otherwise would not have.

The second kind of syntactic convention operates in the absence of physical markers. A comprehensive treatment of this syntactic component utilizes a hierarchy of operation levels introduced by Schwartzman (1977). The six operations are grouped into three levels of two inverse operations each. The operations at each successive level are repeated applications of those at the previous level:

Operation Hierarchy

Level 1:	addition	subtraction
Level 2:	multiplication	division
Level 3:	exponentiation	radical

(In this hierarchy, Level 3 is said to be "higher" than Level 2 which is "higher" than Level 1).

Using this hierarchy, a syntactic convention can be stated for instances where physical markers are absent.

Syntactic Convention

1. Higher level operations have precedence over lower level operations, and
2. In case of an equality of levels, the left-most operation has precedence.

Thus an expression such as $5 - 3x^2 + y$ is analysed as in Figure 1 with exponentiation (Level 3) having first precedence and multiplication (Level 2) having next precedence. Addition and subtraction are of the same level (Level 1) therefore subtraction precedes addition because of its leftwards position.

The foregoing provides a brief overview of the syntax of symbolic algebra. The delineation of rigorous and rational rules of syntax, however, does not answer the questions which were posed at the end of the introductory section. On the contrary, the articulation of a relatively complex and intricate syntax merely highlights the inadequacy of standard instructional techniques and materials as a vehicle for syntactic mastery and raises anew the questions of student learning.

Hypothesis

The syntax of natural language is difficult to specify, in part, because of the complex relationship between "deep" and "surface" structures in language. Deep structures carry the meaning of sentences. Surface structures are the result of transformations of the deep structures into expressive form. For example, in the surface representation of the sentence "John is easy to please," John appears to be the subject. The deep structure, however, is more akin to [It [for someone to please John] is easy]. John is the direct object for an embedded clause in the deep structure representation.

Algebraic language also has deep and surface aspects. Multiplication and exponentiation, for example, are positionally marked by juxtapositions, rather than by explicit operation symbols. Thus, the deep structure representation "3 multiply x" has as a surface representation "3 horizontal juxtaposition x."

Recall that "operation levels" (Page 7), which figured in the Syntactic Convention, were defined by an inventory of the operations at each level. (Level 1 is addition and subtraction, etc). This, therefore, is a deep structure definition of "levels." Operation level, however, can be reformulated in surface structure terms:

Alternative Definition of Operation Level

Level 1: wide spacing; $a \quad b$ ("a + b" and "a - b")

Level 2: horizontal or vertical juxtaposition; ab, $\genfrac{}{}{0pt}{}{a}{b}$

Level 3: diagonal juxtaposition; a^b , ${}^a b$

The syntactic convention (page 7), assumes a quite different character under these new definitions. For example, the first rule "Higher level operations have precedence over lower level operations" becomes "diagonally juxtaposed symbols have higher precedence than horizontally or vertically juxtaposed symbols, which have higher precedence than spaced symbols," rather than the more usual "exponents and radicals precede multiplication and division which precede addition and subtraction." One could easily imagine two computer programs for the parsing of algebraic expressions, one involving the assignment of deep structure terms to the expression prior to parsing, the other parsing directly on the basis of surface features. The two programs would produce equivalent syntactic decisions.

The delineation of two well-defined characterizations of "operation level" raises the question as to which version actually underlies the syntactic knowledge of the fluent algebraist. Is syntactic knowledge coded as information about "operations" and other deep structure constructs, or is it coded in terms of the spacing and positioning of symbols? Are the degenerate rules of syntax which are present in our classrooms and in our texbooks somehow the vehicle for a propositional (deep structure) syntactic knowledge, or does the student him/herself respond, untutored, to the positional cues in the notation?

Method

The task of evaluating algebraic expressions was chosen as the means to investigate the psychological basis of syntactic skill. A typical example of such a task is to evaluate $3x^2$ when $x=2$. A result of 12 is taken as an indication that the expression has been analysed appropriately as $3(x^2)$. A result of 36 is taken as an indication that the expression has been analysed inappropriately as $(3x)^2$.

An overview of the basic strategy of investigation in this study is as follows: Firstly, it is necessary to verify that the subjects can perform evaluative tasks appropriately using standard algebraic notation. Then similar tasks are presented using a nonce, or artificial, notation. This notation is specially devised to display the propositional or deep structure character of algebraic expressions while distorting the surface cues of ordinary notation. The ability of subjects to perform appropriate syntactic analyses in the nonce notation is taken as an indication that syntactic knowledge is propositionally encoded in deep structure form. Inability to transfer competent behaviours to the nonce setting indicates a dependence upon the surface cues found in ordinary notation.[1] (Several possibly confounding factors to this simplified inferential scheme are considered throughout this report).

[1]The use of nonce forms is a standard paradigm in psycholinguistics. See, for example, Braine, (1971); Marslen-Wilson & Welsh, (1978).

The Nonce Forms Study

A nonce notation was devised which displays the deep structure relations in algebraic expressions, but distorts the usual spatial arrangement of the symbols:

aAb	$= a + b$	aSb	$= a - b$
aMb	$= ab$	aDb	$= \frac{a}{b}$
aEb	$= a^b$	aRb	$= \sqrt[a]{b}$

In each case a capital letter abbreviation is used to identify the operation. Thus the notation may communicate, in propositional form, the operations within an algebraic expression. The nonce form, however, spaces all of the symbols equally in a horizontal array. This distorts the surface, spatial cues available in standard notation for the determination of operation levels.

Three kinds of tasks were devised using the nonce forms notation: arithmetic; simple algebraic; and complex algebraic. The arithmetic tasks were simple binary combinations such as 5E2 (5^2). These items were included to insure that subjects could indeed retrieve the appropriate propositional information from the nonce notation.

The algebraic items, simple and complex, consisted of algebraic expressions expressed in nonce form. Each expression contained a single occurrence of the variable "x." The task was to evaluate the expression for x=2.

Simple algebraic items were of the form 5MxS2A1 (ie $5x - 2 + 1$), in which the correct precedence of the operations is from left to right. Complex algebraic items were of the form 1A3MxE2 ($1 + 3x^2$) in which the correct order is other than from left to right.

This distinction was made because of the nature of the spatial cueing hypothesis. If subjects do in fact rely upon such cues then they might be expected to transfer their response patterns to the nonce notation. But the crucial indicator according to this hypothesis is the relative spacing of the symbols. Since the nonce notation spaces all symbols equally, this would be construed as indicating that the levels of all of the operations are equal. In this event, syntactic subrule 2 (page 7) would dictate a left-to-right assignment of operation precedence. Thus for the items identified as "simple algebraic," subjects using spatial cues, as well as subjects using propositional information, would be expected to perform correctly. For the complex algebraic items however subjects using spatial cues would tend to continue with the left-to-right assignment of precedence, while those employing propositional strategies would adjust their precedence procedures appropriately.

The instrument contained three ten-item subtests. The first subtest consisted of nonce arithmetic items. The second subtest was an arbitrary arrangement of five nonce algebra simple and five nonce algebra complex items. The third subtest featured regular notation algebra items which correspond in syntactic structure to the nonce algebra items. Thus item #7 in subtest 2 was 1A3MxE2[2] and item #7 in subtest 3 was $3 + 2x^2$. These sections of the instrument are included as Appendix A.

Multiple Choice Distractors. Five multiple choice options were presented with each item, but a blank space was also provided for subjects' answers not corresponding to the given choices. The five options for the algebra items included all of the possible parsings of the algebraic expression. For example, the nonce item 1A3MxE2 (x=2) had 13, 16, 37, 49, and 64 provided as possible answers. These answers correspond to the parsings 1A[3M(xE2)], (1A3)M(xE2), 1A[(3Mx)E2], [1A(3Mx)]E2, and [(1A3)Mx]E2 respectively.

Alternate Nonce Notation

It was anticipated that performance using a new and unfamiliar notation might decline somewhat relative to performance using the usual notation just because of the unfamiliarity, and quite apart from any particular hypothesized causes. To guard against the possibility that poor performance on the nonce items might be attributed to

---

[2]The problem of subjects transcribing the nonce notation into regular form (either in writing or mentally) was considered and is discussed below.

the general unfamiliarity of the notation, rather than specifically to the distortion of spatial cues, an alternate form of the nonce notation was devised. This notation differs from the first nonce notation only in that the spacing of the symbols more closely resembles spacing in the regular notation.

$$a \quad A \quad b = a + b \qquad\qquad a \quad S \quad b = a - b$$

$$a\ M\ b = ab \qquad\qquad a\ D\ b = \frac{a}{b}$$

$$aEb = a^{b} \qquad\qquad aRb = \sqrt[a]{b}$$

(The Level 1 operations are more widely spaced than the Level 2 or Level 3 operations). It was reasoned that inferior performance on the first nonce forms relative to performance on the spaced form could not be explained in terms of the general unfamiliarity of the notation, but only in terms of the differential spacing characteristics of the notations.

Two versions of the instrument were devised differing only in the type of nonce notation used. (See Appendix B for the spaced form version of the instrument). The two versions were randomly distributed within each of the groups to which tests were administered.

Subjects

The instruments were administered to 562 subjects.[3] Twenty-two of these subjects failed to provide information on at least one independent or covariate variable (see "Covariates") and their scores could not be included in the analyses. Of the remaining 540 subjects, 23 (4.4%) were eliminated because of failure to obtain at least seven correct answers out of nine nonce arithmetic items.[4] It was reasoned that subjects either lacking basic arithmetic skills, or unable to grasp the representational system of the nonce abbreviations would not be fairly tested by the instrument.

Of the 517 subjects (352 male, 165 female) whose responses were analysed, 133 were students in grade 9 (75 male, 58 female), 159 were students in grade 11 (93 male, 66 female), 81 were students in first year calculus classes (49 male, 32 female), 124 were students in fourth year engineering (115 male, 9 female), and twenty were professional engineers (all male). The grade 9 and 11 students were drawn from two predominantly middle and lower

----

[3]Additionally, 68 subjects in grade 9 remedial mathematics classes were tested. These subjects, in the main, lacked even rudimentary exposure to elementary algebra and their scores were not included in the analysis.

[4]A tenth item dealing with the radical operation was dropped because this operation does not appear in any of the subsequent algebra items, and because many of the grade 9 students had not yet been introduced to computation with radicals.

middle class secondary schools of the public school system in Vancouver, Canada. The calculus and engineering students were enrolled at the University of British Columbia, and the professional engineers were in attendance at a meeting of the Professional Engineers Association of British Columbia. Due to the small number of professional engineers participating, their scores were grouped together with those of the graduating engineering students.

The broad range of "grade" levels was included to determine whether syntactic reasoning skill develops with increased mathematical experience and maturity, as well as to provide a reference group of unquestionably competent algebraists with which to anchor the study. All subjects, however, were expected to be reasonably proficient in the syntactic analysis of the fairly simple, standard notation algebraic expressions presented.

Test Administration

Equal numbers of the two forms of the instrument were randomly distributed within each gender grouping in each of 16 participating classes. Participants were instructed to circle or write down only final answers for the items, since a transcription of the nonce items into regular notation would undermine the intended effect of the notation. Compliance was enforced by the use of pens rather than pencils to prevent erasure, and by prohibition on the use of scrap paper.

Subjects were given adequate time to complete, check and correct the current subtest before proceeding to the next. Correcting of answers in the previous subtest, however, was not permitted. This was monitored by requiring the initial of the test administrator next to each authorized (within subtest) correction.

Covariates

Computing Experience

Several calculator and computer languages employ syntactic rules for arithmetic evaluation which differ from the standard of algebraic notation. For example, in the programming language APL, the expression 3*2+5 is evaluated as 3*(2+5), 21, rather than as (3*2)+5, 11. (The "*" represents multiplication). Also, virtually all computer languages employ explicit symbols for multiplication and exponentiation which, in standard algebraic notation, are only positionally marked. Thus subjects with experience in computer languages have already learned to function with more than one system of surface representations and might be expected to have developed a more flexible syntactic rule. Accordingly, each subject was asked to indicate whether he or she had "none," "some," or "quite a bit" of experience in programming a computer.

Table 1

mean COMPUTING EXPERIENCE from a 3 point scale:
1 = "none"; 2 = "some"; 3 = "quite a bit"
of computing experience.

SEX \ GRADE	GRADE 9	GRADE 11	CALCULUS	ENGINEERS	TOTAL
MALE	1.49 75	1.71 93	1.63 49	2.29 135	1.88 352
FEMALE	1.48 58	1.50 66	1.34 32	2.22 9	1.50 165
TOTAL	1.49 133	1.62 159	1.52 81	2.28 144	1.76 517

This measure provides only a crude indicator of programming experience, since "quite a bit" to a grade 9 student undoubtedly means something different to a professional engineer. Nevertheless, the vastly greater computing experience of the fourth year engineering students is reflected in the responses to this question. (See Table 1).

Because of the likelihood that the measure used tends to inflate the relatively lesser computer exposure of the younger subjects, developmental trends would be difficult to evaluate. A covariate adjustment would only partly compensate for the greater computer experience of the engineering students. The computing experience scores were entered as a covariate in the analyses of nonce algebra scores in order to obtain this partial control.

Past Algebraic Achievement

The tendency for men to outperform women in higher level mathematical tasks such as algebra is well documented. (See for example, NAEP 1981). In a pilot study, the gender of the respondent had appeared to be related to nonce notation performance. Data were collected on the past algebraic achievement of the subjects in order to insure that apparent gender differences were not merely an artifact of general algebraic aptitude. Post secondary school subjects were asked to record the final mark in their final high school algebra course. The most recent (November 1983) report card marks in algebra were obtained directly from the schools for the secondary student participants. The data were entered as a three point covariate measure against nonce algebra scores.

Results

Regular Algebra Subtest

As anticipated, most of the subjects demonstrated competence in the regular notation algebra tasks. The mean score on this subtest was 93.6% across all grades. There was a significant grade effect ($\alpha = 0.001$). Cell means ranged from 86.4% for the grade 9 subjects to 96.8% for the engineers and graduating engineers. Men and women performed about equally well on the regular notation items (93.8% and

Table 2

NONCE ALGEBRA-SIMPLE PERCENTAGE
BY TEST FORM, SEX and GRADE
WITH ALGEBRA ACHIEVEMENT and COMPUTER EXPERIENCE as covariates

TOTAL POPULATION
91.03% nonce algebra-simple percentage
(517)

TEST FORM

closed	spaced
91.39	90.66
(260)	(257)

SEX

male	female
90.91	91.27
(352)	(165)

GRADE

9	11	1'st yr	4'th yr+	
86.17	92.20	88.64	95.55	***
(133)	(159)	(81)	(144)	

*** significant at 0.001 level

93.3% respectively). The interaction of sex with grade was not statistically significant.

Nonce Algebra-Simple Subtest

The expectation was that both subjects using propositional referents and those using surface cues in syntactic decision making would be generally successful at the five simple nonce algebra items. This expectation was borne out by the analysis, with overall mean score being 91.0%. Furthermore there were no significant deviations in this performance between men and women or between those who received the spaced and the unspaced form of the instrument. There was however a statistically significant grade effect. None of the interaction effects was significant. (See Table 2).

Nonce Algebra-Complex Subtest

The complex nonce algebra items proved more diffcult for most subjects than any of the other items. The mean percentage correct for all subjects was 66.3%. The analysis of covariance also indicated that these items were significantly more dif cult when presented with the unspaced form of the notation than when presented with the spaced form (α = 0.001). Also, men scored significantly better than women on this subtest (α = 0.05), and there was a significant interaction between gender and form (α = 0.05). Women using the unspaced form of the instrument experienced greater difficulty than other subjects. There were no significant interactions other than gender and form. See Table 3.

Distractor Selection. There had been an expectation that subjects unable to apply propositional rules of syntax to the nonce notation would process the symbols from left to right (page 13). In each case, the left-to-right solution was the most frequently chosen distractor. For the item 1A3MxE2 ($1 + 3x^2$), for example, 140 subjects (53.8%) chose the solution corresponding to the correct parse,

Table 3

NONCE ALGEBRA-COMPLEX PERCENT
BY TEST FORM, SEX and GRADE
WITH ALGEBRA ACHIEVEMENT and COMPUTER EXPERIENCE as covariates

TOTAL POPULATION

66.27%	nonce algebra-complex percentage
(517)	

TEST FORM

closed	spaced	
59.54	73.07	***
(260)	(257)	

SEX

male	female	
70.06	58.18	*
(352)	(165)	

GRADE

9	11	1'st yr	4'th yr+
55.79	62.77	65.93	80.00
(133)	(159)	(81)	(144)

FORM / SEX	male	female	
closed	65.78	45.50	
	(180)	(80)	*
spaced	74.53	70.12	
	(172)	(85)	

* significant at 0.05 level
** significant at 0.01 level
*** significant at 0.001 level

1A[3M(xE2)], and two subjects offered no response. Of the 118 subjects who chose an incorrect response, 51, (43.3%) selected the response corresponding to the left-to-right parse, [(1A3)Mx]E2. The pattern of distractor selecton for this item is shown in Table 4.

Table 4

DISTRACTOR SELECTION for
1A3MxE2 $(1 + 3x^2)$ with x=2

RESPONSE	PARSE	N	%
16	(1A3)M(xE2)	19	16.1
37	1A[(3Mx)E2]	29	24.6
49	[1A(3Mx)]E2	16	13.6
64(expected error)	[(1A3)Mx]E2	51	43.3
other		3	2.5
		118	

This propensity towards left-to-right processing, however, is difficult to interpret. It could result from the assumption (page 13) that the uniform spacing provided by the nonce notation was construed as indicating a single level for all operations. Alternatively, subjects may have "read" the symbols from left to right as English text. But more than one half of the respondents selecting an incorrect response rejected the left-to-right option indicating that they actively grappled with the syntactic question though without adequate tools.

Conclusions

Almost all of the subjects participating in the study were well able to evaluate expressions such as $1 + 3x^2$ (x=2) when presented in standard notation. It proved, however, to be significantly more difficult to transfer this ability to the closed nonce notation, 1A3MxE2, than to the spaced nonce notation, 1 A 3 M xE2. These two notations differ only in the spacing of the symbols. The latter notation was devised specifically to mimic spacing features of ordinary notation. Thus it seems necessary to conclude that for some students, at least, surface features of ordinary notation provide a necessary cue to successful syntactic decisions.

Individual Differences in Nonce Notation Processes

Although a general tendency for subjects to score less well on the complex nonce items than on similar standard notation items was observed, variation amongst respondents was substantial. Many subjects were able consistently to render correct syntactic decisions using the nonce notation, while others were consistently unable to do so.

What do these results indicate about differences in the cognitive strategies employed by the successful and unsuccessful participants? Can the conclusion be drawn that the more successful subjects had access to correct propositional rules of syntax which were unavailable to their cohorts?

An alternative explanation of these differences is that the successful subjects created a mental picture, in standard notation, of the algebraic expression presented in nonce notation. They could then "read" the correct syntax from the spatial cues in their mental image of the items. The prohibition against physically transcribing the nonce items into regular notation (page 17) could not be extended to the imagination of the subjects. Subjects employing such

Table 5

SELF REPORT of
VISUALIZATION STRATEGY versus RULE BASED STRATEGY

VISUALIZATION USED?	RULE USED? YES	RULE USED? NO	ROW TOTAL
YES	120 (36.0%)	15 (4.5%)	135 (40.5%)
NO	184 (55.3%)	14 (4.2%)	198 (59.5%)
COLUMN TOTAL	304 (91.3%)	29 (8.7%)	333 (100%)

a strategy might achieve success because of a superior ability to visualize and manipulate mental images rather than through access to correct propositional rules of syntax.

Post Hoc Analysis. In an attempt to discover whether success was due to such mental gymnastics, or if successful subjects were truly using explicit rules to guide their responses, each subject was asked to provide an introspective account of his or her mental processes for one of the complex nonce items with which he or she had been successful (if any).

Subjects were asked to record "yes," "no," or "not sure" to a question such as "For the problem 1A3MxE2, did you imagine or visualize or picture in your mind $1 + 3x^2$?" Similarly, subjects were asked if they used rules of order of operations in solving the problem and if so, which rules. Space was provided for students to elaborate on their response. The portions of the instruments dealing with this part of the study are included as Appendices C & D.

Ideally, this kind of datum should be acquired through careful interview to assure that subjects fully understand the intent of the questions. Of the 449 subjects who had correctly solved a complex nonce algebra item without guessing or using a flawed procedure, 116 were unable to provide a definitive "yes" or "no" response for each of these questions. The majority (55.3%) of the remaining 333 subjects claimed to have used precedence rules, and not to have used the visualization technique (Table 5). Only 15 subjects (4.5%) claimed to have visualized the problem in ordinary notation and *not* applied rules of operation precedence. Thirty six per cent of the subjects claimed to have used both strategies.

Verbal corroboration for subjects in this last category was skewed. None of these subjects provided a verbal account of the visualization strategy, however, many did report the use of propositional rules. For example, one such grade 9 student wrote "exponentiation (a^b) is first in order of operations, then multiplication and addition last," and said nothing about forming a mental image in standard notation.

In fact, only one or two subjects in the study made a clear statement of using the visualization technique. One of these, a grade 9 student, wrote "I tried to visualize the capital letters as being math symbols and worked on the exponent part of the question which was the hardest part." It does not appear from these self reports, however, that success on the nonce item tasks was often accomplished through the visualization of nonce expressions in standard notation. This seems to confirm that access to adequate propositional information about operation precedence is truly variable from one individual to the next.

Explaining Individual and Group Differences

To some degree, differences in nonce item performance can be accounted for by differential instruction. Students in some grade 11 classes had been taught the BOMDAS rule (page 4) in the more complete form of EBOMDAS, where "E" stands for exponentiation. These students tended to apply their rule of operation precedence successfully to a greater range of nonce tasks than did other subjects. Additionally, the propensity for better nonce task performance at higher grade levels might be attributable to a cumulative exposure to more adequate instructional fragments over time.

Such explanations however do not easily account for the fact that within each class, the range of performance was great. In grade nine classes, for example, the likelihood is that the students have experienced the same instructional situation for their entire algebra careers. Thus differential instruction does not likely account for the entirety of individual differences. A more plausible explanation concerns cognitive style differences of the subjects. See "Implications For Research."

Individual Differences in Regular Notation Processes

The study has pointed to differences in the rule structures which subjects have available to them. Some subjects have adequate propositional rules available and others do not. Those who do not must be relying upon the spatial cues in order to perform correct syntactic analyses with ordinary notation. But what of the subjects who do have adequate propositional rules available to them? Do they may make use of their propositional knowledge, or, like their cohorts, do they make use of surface cues in ordinary notation syntactic decisions? This study has provided no direct evidence about these students' regular notation processes.

It is this researcher's bias that all (or nearly all) people proficient in the manipulation of algebraic symbols do normally make use of the spatial cues in ordinary notation to assist in syntactic decisions. There are two main reasons for this view. Firstly, the presence of surface rules for which there are no usual propositional counterparts is suggestive. For example, x^{5y} is interpreted, syntactically, as $x^{(5y)}$ despite the fact that normally exponentiation has precedence over multiplicaton. A propositional formulation of such a rule might be "operations within an exponent have precedence over the associated exponentiation." The presence of such a formulation, even if only in limited usage, would suggest that some portion of the population requires propositional constructs in order to function syntactically. Its absence (at least in the limited experience of this author) opens the possibility that some aspects of surface processing of syntactic cues may be universal.

The second reason is that spatial cues in standard algebraic notation tend to mimic syntactic cues in natural language. Consider, for example, the interpretation of the morphemic string *light house keeping*. Its ambiguity ({light + house keeping} versus {light house + keeping}) is resolved on the basis of several factors including temporal spacing of its lexical units. (In written form, of course, the parse is indicated directly by a physical gap). Thus the untutored predisposition to interpret physical spaces in algebraic notation as syntactic markers may result from an adaptation of a learned linguistic response.[5]

This final observation provides an answer for one question which has hitherto remained unasked in this report. Why, if students do not develop the fragmentary propositional rules provided by instruction into a viable deep structure system, would they be predisposed to discover a complex surface structure syntax completely unaided by pedagogic assistance? It may be the case that virtually any surface pattern is more easily apprehended than an incompletely specified propositional system. This explanantion should be considered, but it seems to this author that a truly arbitrary surface system would provide little prospect for spontaneous discovery. It seems more likely that students are predisposed to "discover" the system of surface notation cues because of patterns of syntactic analysis which have already been established in natural language. The hypothesis that natural language competencies underlie algebraic syntax skills calls for additional theoretical and empirical investigation.

----

[5]In fact the temporal spacing between words in oral language is not achieved by an actual break in the flow of speech, but rather by a slight lengthening of word initial and/or word final speech sounds in accordance with intricate but unconscious rules of speech. Additionally, phonetic features such as word initial aspiration for voiceless stops, voicing for glides, and syllabic stress contribute to the identification of internal juncture (Stageberg, 1966, pp. 69-71).

In its long and gradual evolution our system of algebraic notation has incorporated into itself regular surface patterns perhaps unbeknownst even to the innovators themselves. The presence of such surface regularities enables the unconscious and automatic parsing of expressions without the necessity of semantic processing. This must be counted as a significant source of the strength of algebraic symbolism. As with any powerful technology, however, there is a need to carefully monitor its implementation and to assure that no harmful side effects result from its introduction.

Implications for Research

Constructivist Perspectives

The constructivist epistemology in mathematics education, as advocated in recent works (eg. Houlihan & Ginsburg, 1981; Cobb & Steffe, 1983; and von Glasersfeld, 1983), holds that the student is to be seen as an active agent in his/her own learning, rather than as a passive recipient of predigested knowledge. The present study falls within the purview of constructivist research in mathematics education, since the student has been shown to take an independent (if unconscious) initiative in the development of syntactic skill.

Support for a constructivist epistemology has been provided in such focal mathematics education studies as

Erlwanger (1974) and Brown and VanLehn (1980). In such studies, constructive involvement is evidenced in the persistent misconceptions or errors of the neophyte. But a constructivist epistemology goes beyond merely a model of learning. Cobb and Steffe (1983) observe,

> the adult cannot cause the child to have experience _qua_ experience. Further, as the construction of knowledge is based on experience, the adult cannot cause the child to construct knowledge. In a very real sense, children determine not only _how_ but also _what_ mathematics they construct. (p. 88)

In the present study, students were successful in their standard notation syntactic decisions. The differences in nonce performance reflected differences in the nature of the mathematical competence which was acquired, rather than just in the way that it was acquired. Thus, in the limited sphere of evaluating algebraic expressions, a measure of support has been found for a more radical constructivism.

Algebraic Symbol Competence

Beyond the limited sphere of expression evaluation tasks, what are the consequences to mathematical performance of surface versus deep representations of algebraic syntax? The hypothesis presented in this study was elaborated by the author in the construction of a generative transformational syntactic theory of algebraic symbol manipulation (unpublished). In an associated semantic theory (so far only outlined) a theoretical evaluation of this question is underway. Indeed, the "Implications For Practise" section of this paper presents a firm opinion as to the outcome of that evaluation. Nevertheless, independent empirical verification is required. Correlations between mode of syntactic representation and performance on a variety of algebraic symbol manipulation tasks should be sought.

Spatial Cues in Real Number Properties

Having shown that spatial cues are used in syntactic decisions, it is worthwhile to question whether other aspects of algebra symbol skill are also cued by the spatial representations of symbols. In this connection, it is germane to note that Schwartzman (1977) introduced the notion of "operation levels" for a topic of elementary algebra other than algebraic syntax. He observed that several real number properties share certain characteristics related to distributivity:

(1)

$$(a + b)c = ac + bc, \qquad (a - b)c = ac - bc,$$

$$\frac{a + b}{c} = \frac{a}{b} + \frac{b}{c} \qquad \frac{a - b}{c} = \frac{a}{b} - \frac{b}{c}$$

(2)

$$(ab)^c = a^c b^c, \qquad \left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$

$$\sqrt[c]{ab} = \sqrt[c]{a}\sqrt[c]{b} \qquad \sqrt[c]{\frac{a}{b}} = \frac{\sqrt[c]{a}}{\sqrt[c]{b}}$$

He also noted that the operation being distributed was, in each case, exactly one level higher than the operation over which it was being distributed. [Properties in (1) above are Level 2 over Level 1, and those in (2) above are Level 3 over Level 2]. Since then, the author (unpublished) has argued that the Generalized Distributive Law which states that "any operation distributes over any one lesser level operation" is manifest as a cognitive structure in the successful manipulation of algebraic symbols.

Having proposed a formal theory of the cognitive representation of real number properties which incorporates the notion of "levels," it is again maintained that this rule can accept both the deep structure and surface structure characterizations already delineated. As in the present case, a means for selection on the basis of psychological considerations should be sought.

In the present study, the hypothesis of spatial cueing in syntactic decisions has been supported. The system of syntax however is an essentially arbitrary agreement among the users of the notation. Any other well-formed system could suffice. It is therefore intriguing to speculate that the immutable properties of the real numbers themselves may be psychologically represented as artifacts of the positions of symbols rather than in a propositional form such as mathematicians are wont to display them.

Field Dependence/Independence

Why should some students respond to their instructional situation by constructing a propositional knowledge of syntactic rules which allows transfer of syntactic skill to the nonce notations, while their cohorts are reliant upon surface forms? Individual differences in the ability to translate skills from a familiar context to a new one have been studied by psychologists involved in cognitive style research. Witkin and Goodenough (1981), pioneers in such research, note that:

> Subjects identified as field dependent in perception of the upright were found to have greater difficulty in solving that particular class of problems in which the solution depends on taking an element critical for solution out of the context in which it is presented and restructuring the problem material so that the element is now used in a different context. (p. 17)

The task of transfering syntactic skill to the new nonce environment appears to fall within the "class of problems" which Witkin and Goodenough describe.

The terms field dependence/independence originally derived from variations in judgements of vertical self-orientation on the basis of the visual field (field dependence), or alternatively, on the basis of the vestibular stimulation caused by the action of the gravitational force on the human body (field independence). In ordinary circumstances, external and internal stimuli are mutually reinforcing since objects in the visual field are usually aligned to gravity. In many ingenious experimental settings, however, the visual and vestibular cues were displaced from each other and subjects were measured as to which stimulus predominates in decisions of vertical orientation. Eventually, other cognitive indices such as spatial/visualization ability and the ability to find hidden figures embedded within a compelling visual format were correlationally associated with field independence (Maccoby & Jacklin, 1974).

The nonce forms experiment has some apparent similarities to field dependence studies. In ordinary syntactic processing, spatial cues and propositional frameworks are mutually reinforcing and usually lead to adequate decisions. The nonce notation however has displaced these cues from one another and the effect upon subjects decisions has been observed. In this light, the ability to transfer syntactic skill successfully to a new notational environment may be part of a general tendency towards field independent stimuli restructuring. The embedded figures correlations and similar studies have already shown that intellectual processes can mediate in the process of field independent self-orientation. Therefore, the propensity to structure notational stimuli so as to make them transportable to the new context of the nonce notation seems, almost by definition, to be a measure of field independence. The sex differences found in nonce notation performance tend to support this hypothesis, since cognitive style and gender are well documented correlates (Fennema, 1975; Witkin & Goodenough, 1981). The hypothesis could be empirically tested by appending a test such as the Embedded Figures Test in a replication of the nonce forms study.

Sex Differences in Mathematical Achievement

Sex differences in mathematical achievement (especially for more abstract content domains) have consistently been observed in mathematics education studies (eg. NAEP, 1975; NAEP, 1981). Observed correlations between cognitive style and mathematical inclination (Witkin, Moore, Oltman, et al, 1977), as well as an apparent relationship between mathematical processes and cognitive style correlates, have provided a source for intriguing conjecture about the nature of these gender differences. However, verifiable explanations have not been forthcoming. For example, Fenemma (1975) speaking with regards to spatial visualization skill notes:

> It appears reasonable, therefore, to hypothesize that since there is a concurrent developmental trend and since tests of spatial visualization ability contain many of the same elements contained in mathematics, the two might be related. Perhaps less adequate spatial visualization ability may partially explain girls inferior performance in mathematics. However, there are no data available which enables one to accept or to reject this hypothesis. (p. 37)

What is lacking is a firm demonstration that specific mathematical functions are actually dependent upon the presence of these generally operative cognitive structures.

If, as hypothesized above, the propensity to formulate propositional, deep structure syntactic rules from surface notational cues is a function of cognitive style, then a link is established between cognitive style and the specific mathematical task of syntactic analysis. Of course, such a link provides an explanation of sex differences in mathematical achievement only to the extent that the presence or absence of such propositional rules can be shown to be a determinant of algebraic competence. To some degree, this result is contraindicated by the present study since even subjects dependent upon surface cues were shown to be reasonably proficient at the standard notation expression evaluation tasks. In the concluding section, however, this question is considered in more detail.

Implications for Practise

The study has suggested that many students develop syntactic skill on the basis of their informal experience of notation, and quite apart from the propositional content of instruction. As an educator, how does one evaluate this situation? Should one be content that students seem able to master the syntax of algebra without explicit instruction, or is there cause for serious concern?

Instruction in algebraic syntax (such as it is) usually focusses on evaluative tasks similar to those that have been employed in this study (eg. evaluate $5x^3 + 2y$ when $x = 2$ and $y = -4$). Evaluative skill is essential for the important functions of checking solutions to equations, and for checking identities between open algebraic sentences.

Matz (1980) has noted that these tasks cause difficulty for some students. There is, however, no systematic evidence available which suggests that these problems persist beyond an initial phase of algebra experience. Indeed, for the fairly simple expression presented in this study, performance across grades was excellent (93.6%). Thus for the evaluation of specific expressions, students' mastery levels are probably sufficiently high so that no major concern need be expressed.

Less well recognized, however, is that syntactic skill also plays a crucial role in the representation of real number properties. Consider, for example, the cancellation law for fractional algebraic expressions, $\frac{a\cancel{x}}{b\cancel{x}} = \frac{a}{b}$. What constitutes knowledge of this property adequate to its successful application in the simplification of algebraic expressions? Clearly, understanding that elimination of like subexpressions in the numerator and denominator constitutes a legal move is part of that knowledge. However, as the frequent occurence of the error-type $\frac{a + \cancel{x}}{b + \cancel{x}} = \frac{a}{b}$ (Matz, 1980) indicates, this is only part of the required knowledge. Equally important is a thorough grasp of the syntactic contexts in which the rule can be applied.[6] Similarly, every real number property, or

---

[6]Also, the author has argued (<u>unpublished</u>) that common error types $(x + y)^2 = x^2 + y^2$ and $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$ are instances

algorithm of elementary algebra specifies its own syntactic setting of application. An abstract knowledge of syntactic context is, therefore, essential to the application of real number properties. But as Davis and McKnight (1979) have documented, the whole notion of application of real number properties may be entirely absent.

> R, an adult studying college algebra at a community college, was confronted by the need to compute $(\sqrt{x+a} - \sqrt{x-a})^2$ by carrying out the squaring. When the instructor referred to $(A + B)^2$, R considered this an irrelevant digression, and did not see how to make use of it. This was a typical response by R. Apparently R has not recognized that one has certain key patterns in one's mind - call them <u>semantic templates</u> - and that mathematics sometimes requires an effort to match up some notation on paper with an appropriate semantic template. R feels that something explicitly present in $(\sqrt{x+a} - \sqrt{x-a})^2$ should determine the next step. (pp. 106-107)

They observe:

> The <u>syntax</u> of algebraic expressions may be a key or milestone kind of knowledge in algebraic learning. The degree of syntactic security seems to be a crucial element in a student's predisposition to regression under strain. (p. 56)

For educators, fluent in the manipulation of algebraic symbols, the role of syntactic skill may be difficult to discern. As in natural language, the focus of conscious consideration is the intention of expression rather than the mechanics of expression. In particular, the presence of a tacit system of spatial markers which enables superficial

---

[6](cont'd) of the application of a distributivity principle to an inappropriate syntactic context.

demonstration of syntactic mastery may serve to divert attention away from this fundamental competence and towards "higher level" topics such as real number properties, notions of variable reference, etc. Thus the focus of remediation, instruction and research may be skewed towards the system of real number properties, or the notions of variable reference while the underlying syntactic deficit goes undiagnosed.

The study has shown that important individual differences in students' cognitive structures result from the abdication of serious, rigorous syntactic instruction. Some students, perhaps due to a more field independent cognitive style, develop propositional notions of operation hierarchies which permit the abstract conceptualization of syntactic contexts. Other students, including proportionately more women than men, develop syntactic rules which are inextricably bound to the physical representation of symbols on a page. They do not obtain flexible cognitive structures underlying transferable syntactic knowledge. As long as educators continue to accept, as evidence of syntactic mastery, students' ability on expression evaluation tasks, there will be no way to distinguish between these two groups except possibly as the successful and the unsuccessful, the mathematically inclined and the mathematically disinclined, the able and the disabled.

References

Baxter, W. H. & Newton, K. E. (1975). Mathematics for a modern world (Revised edition). Toronto: Gage Educational Publishing Limited.

Boyer, C. B. (1968). A history of mathematics. New York: John Wiley & Sons Inc.

Braine, M. D. S. (1971). On two types of models of the Internalizaton of Grammars. In Slobin, D. I. (Ed.), The Ontogenesis of Grammar. New York: Academic Press, 153-188.

Brown, J. S. & VanLehn, K. (1980). Repair theory: A generative theory of bugs in procedural skills. XEROX, Palo Alto Research Center Cognitive and Instructional Sciences Series. CIS-4 (SSL-80-8) August.

Brown, K. F., Snader, D. W. & Simon, L. (1970). Introduction to high School Mathematics. River Forest, Ill: Laidlaw Brothers.

Cajori, F. (1928). A history of mathematical notations. Chicago: Open Court Publishing.

Cobb, P. & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14, 83-94.

Davis, R. B. & McKnight, C. C. (1979). The conceptualization of mathematics learning as a foundation of improved measurement (Development report number 4. Final report). Washington, D. C.: University of Illinois, Urbana/Champaign. (ERIC Document Reproduction Service No. ED 180 786)

Dolciani, M. P. & Wooton, W. (1975). Modern algebra: Structure and method (Module 1), (Revised edition), Markham, Ontario: Houghton Mifflin Canada Limited.

Erlwanger, S. H. (1974). Case studies of children's conceptions of mathematics. Doctoral Dissertation, Univestiy of Illinois, Urbana, Illinois.

Fennema, E. (1975). Mathematics, spatial ability and the sexes. In E. Fennema (Ed.) Mathematics learning: What research says about sex differences. Mathematics Education Information Report. Columbus, Ohio: ERIC Information Analysis Center for Science, Mathematics, and Environmental Education.

Houlihan, D. M. & Ginsburg, H. P. (1981). The addition methods of first- and second-grade children. Journal for Research in Mathematics Education, 12, 95-106.

von Glasersfeld, E. (1983). Learning as a constructive activity. Proceedings of the Fifth Annuual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 1, 41-69.

Johnson, R. E. & Johnson, C. G. (1975). Algebra the language of mathematics. Menlo Park, Addison-Wesley Publishing Company.

Johnson, R. E., Lendsey, L. L. & Slesnick, W. E. (1971). Algebra (Second Edition; Teachers' Edition), Philippines, Addison-Wesley Publishing Company.

Maccoby, E. E. & Jacklin C. N. (1974). The psychology of sex differences. Stanford: Stanford University Press.

Marslen-Wilson, W. D. & Welsh, A. (1978). Processing interaction and lexical access during word recognition in continuous speech. Cognitive Psychology, 10, 29-63.

Matz, M. (1980). Towards a computational theory of algebraic competence. The Journal of Mathematical Behavior, 3, 93-166.

NAEP. (1981). Results from the second mathematics assessment of the national assessment of educational progress. National Council of Teachers of Mathematics.

NAEP. (1975). The first national assessment of mathematics: An overview. Education Commission of the States, Denver, Colorado.

Sobel, M. A. & Maletsky, E. M. (1974). Mathematics II. Canada: Ginn and Company.

Schwartzman, S. (1977). Helping students understand the distributive property. The Mathematics Teacher, 594-595.

Stageberg, N. C. (1971). An introductory english grammar, (Fourth edition). New York: Holt, Rinehart & Winston, inc.

Travers, K. J., Dalton, L. C., Brunner, V. F. & Taylor, A. R. (1976). Using advanced algebra, Toronto: Doubleday Canada Ltd.

Vannatta, G. D., Goodwin, A. W. & Crosswhite, F. J. (Consultant). (1970). Algebra 1. Columbus: Charles E. Merrill Publishing Company, 1970.

Witkin, H. A. & Goodenough, D. R. (1981). Cognitive Styles: Essence and Origins. New York: International Universities Press, (Psychological Issues, Vol. 51).

Witkin, H. A., Moore, C. A., Oltman, P. K., Goodenough, D. R., Friedman, F., Owen, D. & Raskin, E. (1977). The role of field-dependent and field-independent cognitive styles in academic evolution: A longitudinal study. Journal of Educational Psychology, 69, 197-211.

A P P E N D I X   A

NONCE FORMS INSTRUMENT (closed)

Sections 1, 2, and 3

Section 1 CAPITAL LETTER Arithmetic

In each of the following ten examples an arithmetic problem has been translated using the following CAPITAL LETTER notation:

aAb	= $a + b$	aSb	= $a - b$
aMb	= ab	aDb	= $\frac{a}{b}$
aEb	= a^b	aRb	= $\sqrt[a]{b}$

eg. 3M5 means 3 is multiplied by 5

Calculate the answer to each of these problems in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

1) 3E4 = 7, 12, 64, 81, __

2) 5A16 = -11, 11, 21, 80, __

3) 3R8 = 2, 3, 5, 11, __

4) 7M12 = -5, 5, 19, 84, __

5) 22S17 = 5, 13, 14, 39, __

6) 18D6 = -12, 3, 12, 24, __

7) 4E3 = 7, 12, 64, 81, __

8) 26A14 = -40, -12, 12, 40, __

9) 12M4 = 3, 8, 48, 60, __

10) 7E1 = 1, 6, 7, 8, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

In each of the following ten examples an algebraic expression has been translated using the CAPITAL LETTER notation:

aAb = $a + b$ aSb = $a - b$

aMb = ab aDb = $\frac{a}{b}$

aEb = a^b aRb = $\sqrt[a]{b}$

Evaluate each of the algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions x = 2.

1) 3MxA4 = 10, 13, 18, 48, __

2) 2MxE3 = 10, 16, 27, 64, __

3) 5M(2Ax) = 12, 20, 100, 625, __

4) 5A3Mx = 4, 11, 14, 16, __

5) xE4S2 = 3, 4, 6, 14, __

6) 3E4SxA1 = 3,10,27,78,80, __

7) 1A3MxE2 = 13,16,37,49,64, __

8) 10S3MxA1 = 1,3,5,15,21, __

9) 6A(3MxS2) = 0,6, 9, 10, 16,__

10) 6SxE2A1 = -2,1,3,17,64,__

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 3 Algebra(usual notation)

Evaluate each of the following algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions $x = 2$.

1) $5x + 7 =$ 3, 17, 32, 45, 70, __

2) $5x^2 =$ 5, 10, 20, 49, 100, __

3) $4(6 + x) =$ 14, 24, 26, 28, 32, __

4) $3 + 4x =$ 9, 11, 13, 14, 24, __

5) $x^3 - 2 =$ 1, 2, 6, 7, 16, __

6) $2^4 - x + 1 =$ 2, 5, 8, 13, 15, __

7) $3 + 2x^2 =$ 11, 19, 20, 49, 100, __

8) $19 - 4x + 2 =$ 3, 9, 13, 32, 60, __

9) $3 + (7x - 2) =$ 0, 3, 15, 18, 21, __

10) $5 - x^2 + 1 =$ -3, 0, 2, 10, 27, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

A P P E N D I X B

NONCE FORMS INSTRUMENT (spaced)

Sections 1, 2, and 3

In each of the following ten examples an arithmetic problem has been translated using the CAPITAL LETTER notation:

a A b = a + b

a S b = a - b

a M b = ab

a D b = $\frac{a}{b}$

aEb = a^b

aRb = $\sqrt[a]{b}$

eg. 3 M 5 means 3 is multiplied by 5

Calculate the answer to each of these problems in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

1) 3E4 = 7, 12, 64, 81, __

2) 5 A 16 = -11,11,21,80,__

3) 3R8 = 2, 3, 5, 11, __

4) 7 M 12 = -5, 5, 19, 84, __

5) 22 S 17 = 5, 13, 14, 39, __

6) 18 D 6 = -12, 3, 12, 24, __

7) 4E3 = 7, 12, 64, 81, __

8) 26 A 14 = -40,-12,12,40,__

9) 12 M 4 = 3, 8, 48, 60, __

10) 7E1 = 1, 6, 7, 8, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 2 CAPITAL LETTER Algebra

In each of the following ten examples an algebraic expression has been translated using the following CAPITAL LETTER notation:

a A b = a + b　　a S b = a - b

a M b = ab　　a D b = $\frac{a}{b}$

aEb = a^b　　aRb = $\sqrt[a]{b}$

Evaluate each of the algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions x = 2.

1) 3 M x A 4 = 10,13,18,48,__

2) 2 M xE3 = 10,16,27,64,__

3) 5 M (2 A x) = 12,20,100,625,__

4) 5 A 3 M x = 4,11,14,16,__

5) xE4 S 2 = 3,4,6,14,__

6) 3E4 S x A 1 = 3,10,27,78,80,__

7) 1 A 3 M xE2 = 13,16,37,49,64,__

8) 10 S 3 M x A 1 = 1,3,5,15,21,__

9) 6 A (3 M x S 2) = 0,6,9,10,16,__

10) 6 S xE2 A 1 = -2,1,3,17,64,_

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 3 Algebra(usual notation)

Evaluate each of the following algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions $x = 2$.

1) $5x + 7 =$ 3, 17, 32, 45, 70, __

2) $5x^2 =$ 5, 10, 20, 49, 100, __

3) $4(6 + x) =$ 14, 24, 26, 28, 32, __

4) $3 + 4x =$ 9, 11, 13, 14, 24, __

5) $x^3 - 2 =$ 1, 2, 6, 7, 16, __

6) $2^4 - x + 1 =$ 2, 5, 8, 13, 15, __

7) $3 + 2x^2 =$ 11, 19, 20, 49, 100, __

8) $19 - 4x + 2 =$ 3, 9, 13, 32, 60, __

9) $3 + (7x - 2) =$ 0, 3, 15, 18, 21, __

10) $5 - x^2 + 1 =$ -3, 0, 2, 10, 27, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

A P P E N D I X   C

POST HOC INSTRUMENT (closed nonce form)

Recall that question 7 in Section 2 was 1A3MxE2 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 13? Yes __ or No __. [Check yes or no].
If your answer was 13, then please proceed to Page 2 now.

If your answer was not 13, then please go on to the next question.

Recall that question 8 in Section 2 was 10S3MxA1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 5? Yes __ or No __. [Check yes or no].
If your answer was 5, then please proceed to Page 4 now.

If your answer was not 5, then please go on to the next question.

Recall that question 10 in Section 2 was 6SxE2A1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 3? Yes __ or No __. [Check yes or no].
If your answer was 3, then please proceed to Page 6 now.

If your answer was not 3, then please go on to the next question.

Recall that question 2 in Section 2 was 2MxE3 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 16? Yes __ or No __. [Check yes or no].
If your answer was 16, then please proceed to Page 8 now.

If your answer was not 16, then please go on to the next question.

Recall that question 4 in Section 2 was 5A3Mx (x = 2).
Without changing it, please go back and check your answer
1) Was your answer 11? Yes __ or No __. [Check yes or no'
If your answer was 11, then please proceed to Page 10 now.

If your answer was not 11, then please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 13 as your answer for.

1A3MxE2.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 3M4 = 12, 1A12 = 13.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 3M4 = 12, 1A12 = 13.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 1A3MxE2, did you imagine or visualize or picture in your mind $1 + 3x^2$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 5 as your answer for

10S3MxA1

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3Mx = 6, 10S6 = 4, 4A1 = 5.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ____________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations 3Mx = 6, 10S6 = 4, 4A1 = 5.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 10S3MxA1, did you imagine or visualize or picture in your mind 10 - 3x + 1 ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 3 as your answer for

6SxE2A1.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 6S4 = 2, 2A1 = 3.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 6S4 = 2, 2A1 = 3.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 6SxE2A1, did you imagine or visualize or picture in your mind $6 - x^2 + 1$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 16 as your answer for

2MxE3.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE3 = 8, 2M8 = 16.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ____________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations xE3 = 8, 2M8 = 16.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 2MxE3, did you imagine or visualize or picture in your mind $2x^3$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 11 as your answer for

5A3Mx.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3Mx = 6, 5A6 = 11.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations 3Mx = 6, 5A6 = 11.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 5A3Mx, did you imagine or visualize or picture in your mind 5 + 3x ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

A P P E N D I X   D

POST HOC INSTRUMENT (spaced nonce form)

Recall that question 7 in Section 2 was 1 A 3 M xE2 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 13? Yes __ or No __. [Check yes or no].
If your answer was 13, then please proceed to Page 2 now.

If your answer was not 13, then please go on to the next question.

Recall that question 8 in Section 2 was 10 S 3 M x A 1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 5? Yes __ or No __. [Check yes or no].
If your answer was 5, then please proceed to Page 4 now.

If your answer was not 5, then please go on to the next question.

Recall that question 10 in Section 2 was 6 S xE2 A 1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 3? Yes __ or No __. [Check yes or no].
If your answer was 3, then please proceed to Page 6 now.

If your answer was not 3, then please go on to the next question.

Recall that question 2 in Section 2 was 2 M xE3 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 16? Yes __ or No __. [Check yes or no].
If your answer was 16, then please proceed to Page 8 now.

If your answer was not 16, then please go on to the next question.

Recall that question 4 in Section 2 was 5 A 3 M x (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 11? Yes __ or No __. [Check yes or no].
If your answer was 11, then please proceed to Page 10 now.

If your answer was not 11, then please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 13 as your answer for

1 A 3 M xE2.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 3 M 4 = 12, 1 A 12 = 13.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 3 M 4 = 12, 1 A 12 = 13.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 1 A 3 M xE2, did you imagine or visualize or picture in your mind $1 + 3x^2$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 5 as your answer for

10 S 3 M x A 1

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3 M x = 6, 10 S 6 = 4, 4 A 1 = 5.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what was in your mind as you figured out the order of operations 3 M x = 6, 10 S 6 = 4, 4 A 1 = 5.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 10 S 3 M x A 1, did you imagine or visualize or picture in your mind 10 - 3x + 1 ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 3 as your answer for

6 S xE2 A 1.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 6 S 4 = 2, 2 A 1 = 3.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 6 S 4 = 2, 2 A 1 = 3.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 6 S xE2 A 1, did you imagine or visualize or picture in your mind $6 - x^2 + 1$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 16 as your answer for

2 M xE3.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE3 = 8, 2 M 8 = 16.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations xE3 = 8, 2 M 8 = 16.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 2 M xE3, did you imagine or visualize or picture in your mind $2x^3$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ____________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 11 as your answer for

5 A 3 M x.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now. Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3 M x = 6, 5 A 6 = 11.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain) ______________________________

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

I would like to know what went through your mind as you figured out the order of operations 3 M x = 6, 5 A 6 = 11.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 5 A 3 M x, did you imagine or visualize or picture in your mind 5 + 3x ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? ______________________________

____

____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

____

____

____

____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.