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ABSTRACT

This study examined various misconceptions committed by junior high school students in fraction addition and subtraction problems. Almost 600 subjects were administered two tests, addition and subtraction, and their performances were analyzed by several computer programs, including two programs (FBUG and SPBUG) which are so flexible to any items generated by computer that they can be used for any teacher-made tests. A painstaking error analysis and construction of buggy programs were carried out and summary statistics are described. The analysis results indicate that individual differences in applying different strategies and procedural skills varied more among students than expected. Many erroneous rules were committed by students who used them sporadically. These rules are often observed only once per student and never used repeatedly by the same individuals. Various error types (sources of misconceptions) cover almost all the levels of tasks involved in solving fraction problems. A close examination of frequency distributions of erroneous rules revealed that some errors tend to appear among high-score students while others appeared only among low-score students. Systematic investigation of "bug-behaviors" will lead further understanding in human cognition and learning and thus it will bring about further improvement in American education. (JN)

Computer-based Education Research Laboratory **CERL**

ANALYSIS OF ERRORS IN FRACTION ADDITION AND SUBTRACTION PROBLEMS

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Introduction

The purpose of this study is to examine various misconceptions committed by junior high students in fraction addition and subtraction problems. Almost 600 subjects were administered two tests, addition and subtraction, and their performances were analyzed by several computer programs written by Robert Baillie and Doris Shaw. Especially FBUG and SPBUG (Baillie, et.al) are so flexible to any items generated by computer that they can be used for any teacher made tests. SPBUG will provide descriptive statistics, response pattern analysis as well as error diagnosis. This program is also applicable to any fraction addition tests including teacher-made tests.

The rate of diagnosing erroneous rules (algorithms) of operation is as high as 90% of all responses. Most rules are used with less than perfect consistency by many students, and sporadic deviations appeared in one or two items per student. Nevertheless, certain rules are observed in the same individual, where the student applies the rules systematically throughout the test.

It is natural to assume that the erroneous rules resulting from the same type of misconception or incompleteness or lack of knowledge tend to be seen persistently in a variety of different types of items in a test. But as can be seen in Figure 3 in which the performances of students on the test items are conveniently tabulated (S-P table, see Harnisch and Linn, 1981; Tatsuoka, 1978), some rules are seen over all items as well as for many students regardless of their scores. Other errors are observed more frequently in high-score rather than low-score students, while yet other rules are more common in low-score students. Some student performances--bug patterns--support "repair theory" (Brown & VanLehn, 1981), but many others exhibit more complicated bug patterns which are probably affected by complicated mental models based on past knowledge that the students have acquired.

Methods for solving mixed fractions: Addition and Subtraction

Method A. The mixed (or whole) number is converted to an improper fraction and then the strategy for adding (subtracting) two fractions is used. This method is especially noticed in subtraction problems where borrowing is needed. However, since this method involves manipulations with large numbers it is more subject to arithmetic errors. Reducing and converting the result to a mixed number is more frequently needed when this method is used than when other methods for solving mixed number operations are used.

Method B. The fraction part and the whole number part are dealt with separately. The advantage of this method is that once the whole number part is separated the student can manipulate smaller numbers as compared with the ones that would be manipulated had Method A been used. However, when using this method the student has to remember to repeat the operation twice: Once with the whole number and then with the fraction part. When the fraction part involves finding common-denominator equivalent fractions, and reducing and converting an improper fraction to a mixed number, the original whole number part can easily be forgotten. Moreover, in subtraction problems where borrowing is needed it introduces an additional complication which does not exist when Method A is used.

Examples:

Method A	Method B
5 9/22 - 2 9/11	5 9/22 - 2 9/11
= 119/22 - 31/11	= 5 9/22 - 2 18/22
= 119/22 - 62/22	= 4 31/22 - 2 18/22
= 57/22 = 2 13/22	= 2 13/22

Rules for solving fraction addition and subtraction problems.

Adding or subtracting fractions $(\frac{b}{c} \pm \frac{e}{f})$

I. Like Denominators (c=f)

A. Add (or subtract) numerators for the numerator part of the result. (b $\pm$ e)

B. The denominator of the result is one of the two like denominators. (c,f),(c′,f′)

II. Unlike Denominators (c$\neq$f)

A. Check if you can divide each fraction by a common factor. (See flow chart, Klein, et al., 1981, pg. 36.)

B. Find a common denominator using one of the following methods.

1. Prime Factoring Method
2. Multiples Method
3. One of the denominators is a multiple of the other (For a flow chart of 1-3, see Ibid, p. 33-35.)
4. Automatic Method (Multiply the denominators together to get a common denominator $b/c + e/f = bf/cf + ce/cf$)

C. Find equivalent fractions (see chart Ibid, p. 37).

D. Add (or subtract) the same as in step I.

E. Reduce and/or convert improper fraction to mixed number (for chart see Ibid, pp. 40-41)

Adding or subtracting mixed numbers.

Choose one of the following methods:

Method A: Convert the mixed number to an improper fraction. Proceed with steps I. through II.E. (For a chart, see Ibid, p.32)

Method B: Add (or subtract) the whole numbers first then repeat steps I. through II.E. for the fraction part. (In subtraction problems watch for cases where $b' < e'$. You will have to borrow from the whole number part a, as in $a\frac{b}{c}$)

Figure 1 displays a flowchart for solving fraction addition and subtraction problems according to the above mentioned rules.

Insert Figure 1 about here

Classification of observed errors according to the task analysis.

Based on the task analysis the observed errors can be classified according to the following categories. The numbers or letters in the parentheses refer to the list of bugs in Table 1 for addition or subtraction.

Insert Table 1 about here

1. Errors associated with an incorrect use of Method A*

Classified into this category are errors resulting from applying

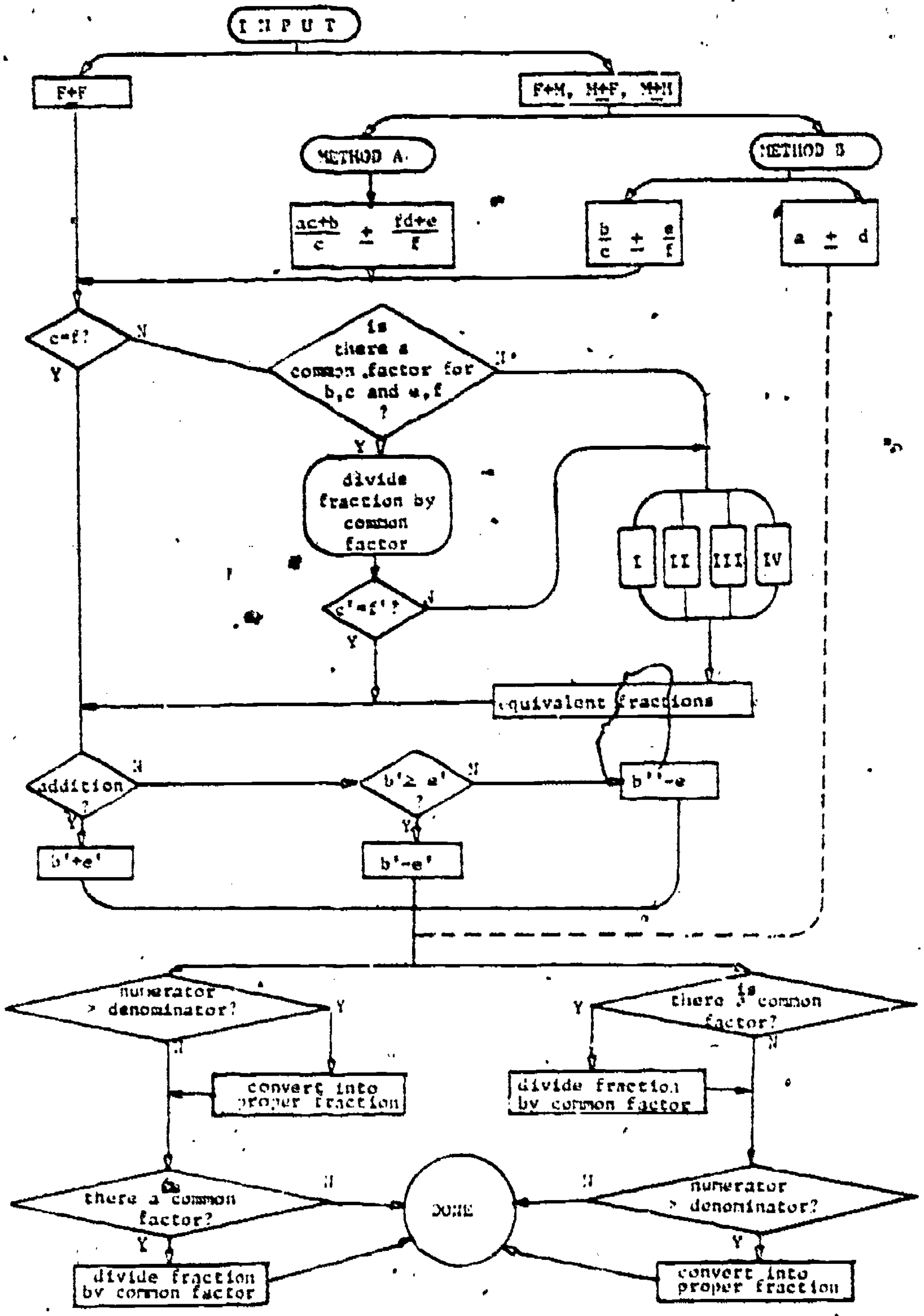

Figure 1: A Rough Sketch of a Flowchart Describing Different Methods for Solving Fraction Addition and Subtraction Problems

Table 1

Classification of Erroneous Rules or Algorithms

According to the Task Analysis

1.	Incorrect Use of Method A	2, 5, 7, 28, 37, 40, 53 C, D, H
2.	Incorrect Use of Method B	32, 33, 39, 43 A, B, E, F, G, J
3.	Incorrectly finding common denominator	12, 14, 59, 62, 64 c, d, h
4.	Inccorectly finding equivalent fractions	10, 15, 17, 18, 19, 21 23, 26, 60, 63, 66 f
5.	Faulty algorithms	11, 25, 39, 46, 47, 48, 50 51, 54, 55, 57, 58, 61, 65, 67 a, b, g, n, x, y
6	Simplification (reducing) (converting)	 1b, 6, 10, 11, 12 1a, 2, 3, 4, 5, 7, 8, 9

The numerals and letters represent the rules (bugs or algorithms) innthe list given in the appendix.

incorrect rules of converting a mixed number to an improper fraction. Examples of these errors are multiplying the whole number by the numerator and adding the denominator (2); keeping the whole number part after conversion to an improper fraction has taken place (5); converting by multiplying the whole number by the numerator (7); converting by adding the whole number part to the numerator (28)(H); or converting the mixed fraction to a whole number by ignoring the denominator(53)(D). In subtraction a mixed fraction is converted by adding all three parts of the fraction for the numerator (C); or in addition adding the three parts of the first fraction for the numerator of the result and the three parts of the second fraction for the denominator (37),or similarly adding the whole number and denominator of each fraction to obtain the numerator and denominator of the result (40).

2. Errors associated with an incorrect use of Method B.

Classified into this category are errors resulting from applying incorrect rules to the whole number or fraction parts such as operating only with the fraction part and omitting the whole number part (33)(G); or operating with the whole number part and omitting the fraction part (J). Adding the numerator and denominator of the first fraction to obtain the numerator of the result and similarly adding the numerator and denominator of the second fraction to obtain the denominator of the result (39); multiplying the whole numbers (43), or adding a one to the whole number part (32) are noted in addition. Errors in borrowing are unique to subtraction problems. Among the incorrect rules that lead to borrowing errors are the following; when borrowing reduce the whole number by 1 but add 10 to the numerator (A); borrow but forget to reduce the whole number from which it was borrowed (B); when borrowing, add 1 to the whole number from which it was borrowed (E); and subtract 1 from the numerator as well as from the whole number (F).

3. Errors associated with finding common denominators.

Classified into this category are errors resulting from applying incorrect rules for finding a common denominator such as choosing one of the two denominators as the common denominator, the first or second denominator, the smaller or larger denominator (12,62,64,c,d,h).

Several errors are associated only with unequal denominators such as (14, 59b).

4. Errors associated with finding equivalent fractions.

The following incorrect rules result in errors classified into this category; the numarator of each equivalent fraction is the sum of the numerator and denominator of that fraction (10); while a common denominator is found by multiplying the denominators or the LCD (least common denominator) is found; the numerators are added (15,17) or subtracted (f) or the denominators are added together (26).

A fraction that has 1 as its numerator is treated as if the entire fraction equals 1 (18). The LCD is correctly found to be less than the denominators multiplied together, but then the numerators are cross multiplied by the original denominators (19); the common denominator is found but the original denominators are used as the numerators of the equivalent fracion (21), or each numerator is multiplied by its denominator (23)(60), or the numerator of the second fraction is added to its denominator (66) or the numerator is cross added to the other denominator (63).

5. Errors associated with the addition/subtraction algorithms.

The following errors result from applying incorrect rules of operation adding or subtracting corresponding parts (11)(a): Converting mixed numbers to improper fractions and adding numerators and denominators (67); inverting the second fraction and adding corresponding parts of the fraction (61); adding or subtracting only fraction parts that have different numbers in the numerators (58)(g); multiplying corresponding parts of the fraction (48); or adding whole numbers and numerators but multiplying denominators (54).

Another incorrect rule of operation is cross cancelling followed by multiplying numerators and adding denominators (47); subtracting smaller from larger corresponding parts (x); adding numerators and denominators (46); or adding numerators and multiplying denominators(57).

After finding a common denominator the smaller numerator is subtracted from the larger for an addition problem (50); or the numerators are multiplied (55).

Other incorrect rules are subtracting the smaller from the larger number or adding both numbers in the first fraction to obtain the numerator and in the second one obtain the denominator (39)(y); adding numerators and subtracting denominators (65) or vise versa (b); using the division algorithm(51)(n); ignoring the numerator when the value is 10 (25).

6. Errors associated with simplifying the result (reducing/converting)

In reducing the fraction part of a mixed number, omit the original whole number part (1b) or divide all three parts, whole number, numerator and denominator by the same divisor (6). In reducing a fraction where the numerator and denominator are equal, the fraction part equals the value of the numerator or denominator (10). In reducing the fraction part of an answer the numerator and denominator are divided by different numbers (12) or the numerator is divided by the greatest common divisor and the gcd becomes the denominator (11).

In simplifying a mixed improper fraction, convert the fraction part but omit the original whole number (1a) or retain the original whole number obtained from converting the improper fraction part (2).

In simplifying the improper fraction part of the answer the new whole number becomes the numerator and any remainder is lost (3). In converting the improper fraction part the original numerator becomes the new denominator (4).

Any conversion of an improper fraction part will equal one (9). This is an error only when the conversion does not equal one. In another conversion error of an improper fraction the whole number is the numerator minus any remainder after the denominator is divided into the numerator (8).

A proper fraction has the smaller numerator divided into the larger denominator (7) but the denominator stays the same as the original unlike when the fraction part is inverted and then converted or reduced (5).

The Test

Two lists, one in addition and the other in subtraction, were carefully designed in order to provide diagnostic information. For a detailed description of the item construction see Klein, et al., 1981.

The addition test consisted of 48 items and the subtraction one consisted of 42 items. Two parallel sets of items were included in each test, comprising the following 8 types of items $F \pm F$; $F+M$; $M \pm F$; $M \pm M$; $W \pm F$; $F+W$; $M \pm W$; $W \pm M$ (where M = mixed numbers, F = fractions and W = whole number). A preliminary analysis indicated that problems including a whole number were clustered on a different dimension than the other types of items. Hence, these items were separated from the rest and will not be discussed in this report. Thus, we will refer to two parallel sets of 19 items each in addition and two parallel sets of 16 items each in subtraction. Figures 2 and 3 display the item numbers according to their type. (The item numbers in the charts refer to the original

Insert Figures 2 & 3 about here

numbers in the test. As can be seen in the charts, the items in each test included like and unlike fractions some of which could be simplified, i.e., reduced and/or converted to a proper fraction, prior to the addition, subtraction operation. The subtraction chart also specifies the items that require borrowing -- a procedure required where Method B is used.

Datasets

The tests described in the previous section were administered three times to junior high students. The first group was comprised of 148 8th and 9th graders who took the tests in the spring of 1982. The second and third group obtained in the fall of 1982 were comprised of 171 and 273 7th and 8th graders. The responses to the paper and pencil 38- and 32-item tests were typed on the PLATO system. The answers before reducing to the simplest form of fractions as well as those after reducing were coded separately so error analysis could be divided into algorithmic and reducing parts.

Computer Program

An algorithmic computer program for diagnosing erroneous rules used by students in fraction addition problems was written in FORTRAN on the PLATO system. A similar program was developed on an Apple computer, but this version was not algorithmic and diagnosed errors by checking two numbers--one being a student's response to an item and the other, the numbers (stored on a disk) obtained by applying each erroneous rule to the item.

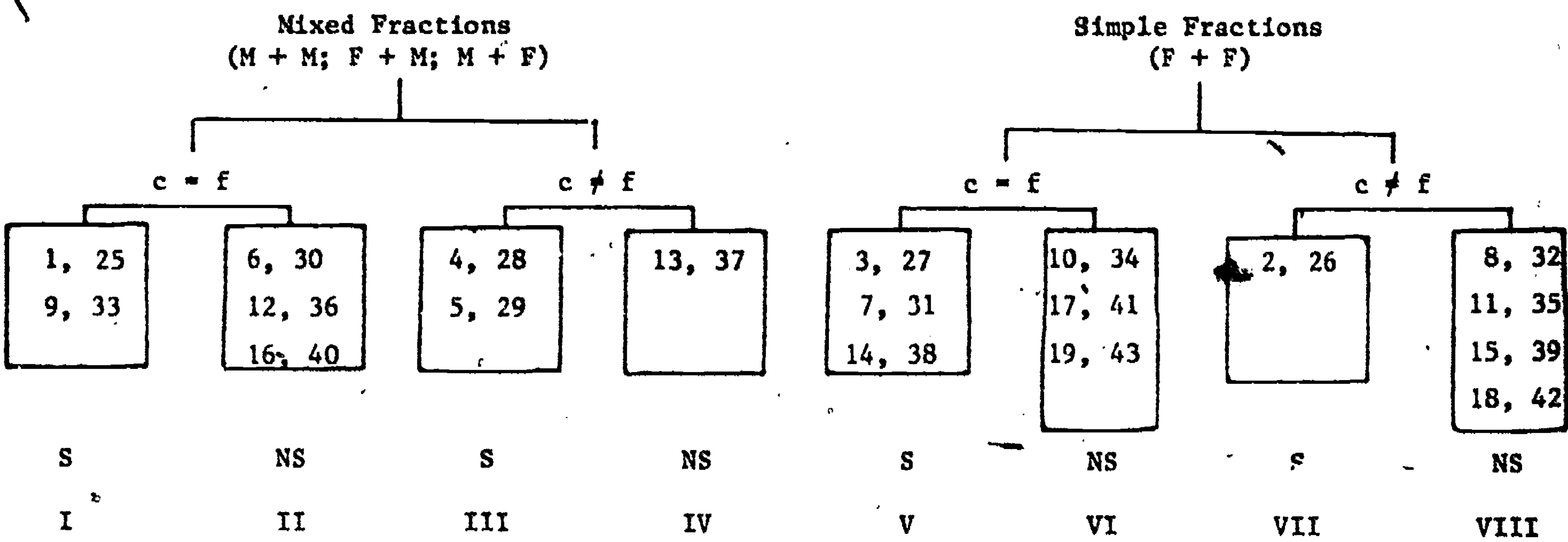

Figure 2. Fraction Addition Item Classification Chart (38 Items).

S = Simplifiable (Item Before Operation)

NS = Non Simplifiable (Item Before Operation)

M = Mixed Number

F = Fraction

Note: $a\frac{b}{c} + d\frac{e}{f}$. The numbers in the boxes are item numbers.

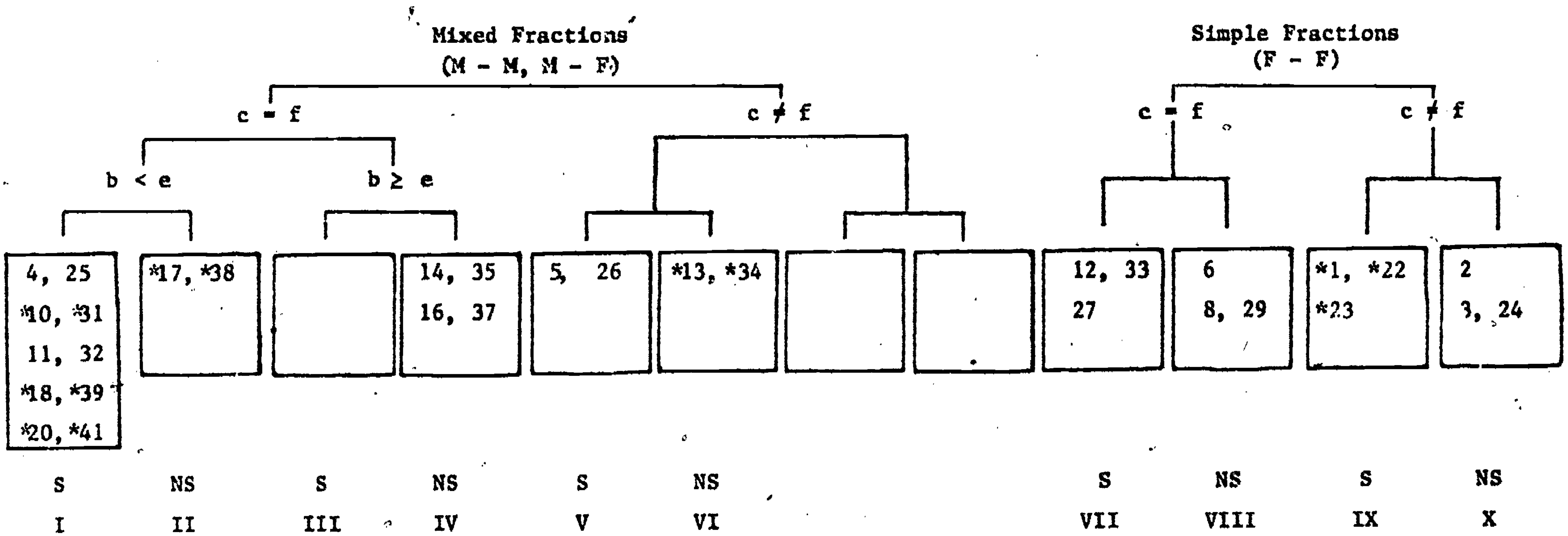

Figure 3. Fraction Subtraction Item Classification Chart (32 Items).

S = Simplifiable (Item Before Operation) M = Mixed Number

NS = Non Simplifiable (Item Before Operation) F = Fraction

Note: $a\frac{b}{c} - d\frac{e}{f}$. The numbers in the boxes are item numbers.

* needs borrowing

The program generates an S-P table by sorting the rows and columns of the original datamatrix by the orders of item difficulties and total scores. Then it calculates a correlation matrix, means and standard deviations of items and students, caution index for each student and item. The second S-P table is obtained by replacing 1-0 scores by the number and letters designating seventy erroneous rules of operation in fraction addition and subtraction problems. An example of the second S-P bug table is given in Figure 4.

Insert Figure 4 about here

Analysis results: What rules are frequently seen and what rules are observed together

In the S-P bug table, missing data is replaced by 99, the use of the right rule is 1 and unidentified wrong rules or typographical, careless errors are marked by 0. The right-most column contains the total scores arranged in descending order from the top to the bottom. Later in this report, the three groups divided by the two lines shown in Figure 4, which are drawn at the score of 29 and 10 will be designated as high-, middle- and low-score groups and a summary of each groups' descriptive statistics and error analyses will be given and compared.

Insert Figure 5 about here

As can be seen in Figure 5, the use of erroneous rule 11, adding the corresponding parts of two fraction numbers, for the lower-score group is somewhat comparable to the use of the right rule in the high-score group.

The order of item difficulties clearly reflects the difficulties predicted in task analysis. Table 2 gives the summary statistics of the dataset including the S-P bug table as a part shown in Figure 4. The common factor in Item types VI, II, V, and I is that the items in these

Insert Table 2 about here

groups have equal denominators. The 16 items from the right-most column have unequal denominators. Figure 4 clearly shows that students in the middle group don't know how to obtain the largest common denominator of

ITEMS

I	S.NUM	17	10	41	34	7	12	16	31	36	40	6	19	27	38	43	14	33	25	9	30	3	1	11	15	32	8	4	35	39	28	13	18	2	37	42	29	5	26	TOTSC(I)
1	1	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	-1	38
2	124	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	0	37
3	148	1	1	1	1	-1	1	1	1	1	42	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	-1	37
4	25	1	1	1	1	-1	1	-1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	0	-1	36
5	33	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	-1	-1	-1	-1	-1	1	1	-1	1	-1	0	1	1	1	1	-1	36
6	36	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	11	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	99	36
7	77	1	1	1	1	-1	1	1	1	1	32	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	0	1	-1	1	1	1	1	1	-1	36
8	95	1	1	1	1	-1	1	-1	1	1	1	[illegible]	1	1	1	1	1	1	1	1	1	11	1	0	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	-1	36
9	101	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	0	[illegible]	36
10	123	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	33	1	-1	36
11	145	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	0	59	1	1	1	1	[illegible]	36
12	3	1	1	1	1	-1	1	-1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	[illegible]	35
13	6	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	33	1	[illegible]	35
14	18	1	1	1	1	-1	1	-1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	0	-1	-1	-1	-1	-1	1	1	-1	1	[illegible]	1	1	1	1	1	-1	35
15	39	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	0	33	1	0	35
16	59	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	12	1	1	21	35
17	72	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	-1	-1	1	12	-1	1	1	-1	1	-1	1	32	1	1	1	-1	35
18	87	1	1	1	1	-1	1	-1	1	1	1	1	0	1	1	1	1	11	1	1	1	1	11	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	-1	35
19	113	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	99	1	1	33	1	1	1	7	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	1	-1	35
20	141	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	21	1	1	1	0	99	35
21	142	1	1	1	1	-1	1	-1	1	1	1	1	1	1	25	1	-25	1	1	1	1	1	1	-1	-1	15	1	-1	1	1	-1	1	-1	1	1	1	1	1	-1	35
22	143	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1	-1	-1	-1	-1	-1	1	1	-1	0	-1	1	1	1	1	1	-1	35
23	7	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	11	1	1	1	1	-1	-1	1	12	-1	1	1	-1	1	-1	0	1	1	1	1	0	34
24	15	1	1	1	1	-1	1	-1	1	1	1	0	1	1	1	1	1	43	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	1	7	99	34
25	16	1	1	1	1	-1	1	-1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	1	-1	-1	-1	-1	-1	1	1	-1	12	-1	1	1	1	1	7	-1	34
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27	20	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	18	-1	-1	-1	19	1	1	0	1	-1	1	1	1	1	1	-1	34
28	21	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	60	1	1	0	0	99	34
29	22	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	54	59	1	1	99	34
30	70	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	-1	-1	-1	-1	1	1	1	99	1	-1	1	1	1	21	1	99	34
31	94	1	1	1	1	-1	1	-1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	-1	-1	58	1	-1	1	1	-1	1	-1	0	1	1	1	1	62	34
32	116	1	1	1	1	-1	1	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	0	0	1	1	-1	34
33	140	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	0	1	1	0	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	0	1	0	1	1	1	34
34	146	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	-1	-1	-1	-1	0	1	-1	1	-1	1	1	1	1	1	0	34
35	4	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	99	1	1	1	1	1	1	0	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	99	1	1	1	99	33
36	19	1	1	1	1	-1	1	-1	1	33	1	1	1	1	1	1	1	1	1	1	11	1	1	-1	-1	-1	-1	-1	1	1	-1	0	-1	1	1	1	1	[illegible]	99	33
37	34	1	1	1	1	-1	1	-1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	1	-1	-1	-1	-1	0	1	-1	1	15	1	1	1	1	0	-1	33
38	58	1	1	1	1	-1	1	-1	1	1	1	1	1	0	1	1	1	1	1	43	1	1	0	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	0	1	99	33
39	86	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	0	-1	-1	-1	-1	1	1	-1	1	1	1	1	0	1	1	0	33
40	88	1	1	1	1	-1	1	-1	1	1	1	11	1	1	99	1	1	1	1	11	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	0	1	99	1	1	1	-1	33
41	96	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	49	-1	-1	-1	1	59	-1	33	-1	1	1	1	1	1	0	33
42	104	1	1	1	1	-1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	-1	-1	0	1	1	1	1	0	1	-1	0	1	1	1	1	0	33
43	105	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	1	-1	-1	1	1	12	12	1	-1	1	33	1	33	1	0	33
44	120	1	1	1	1	-1	1	-1	1	1	1	32	1	1	1	1	0	1	1	5	0	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	59	1	1	-1	33
45	121	1	1	1	1	-1	1	32	1	1	1	1	1	1	1	[illegible]	1	1	1	1	1	1	5	-1	-1	-1	-1	-1	1	0	-1	1	-1	1	1	1	1	1	0	33
46	125	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	11	-1	1	1	-1	1	0	1	1	1	0	7	0	33
47	147	1	1	1	1	-1	1	-1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	0	1	0	33
48	149	1	1	1	1	-1	1	-1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	0	-1	-1	-1	-1	-1	1	1	-1	1	-1	[illegible]	1	1	54	1	0	32
49	60	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	0	0	7	99	32
50	90	1	1	1	1	-1	0	-1	1	1	1	0	1	1	0	1	1	1	1	0	1	1	1	-1	-1	-1	-1	1	1	1	0	1	-1	1	1	1	1	1	99	32
51	100	1	1	1	1	-1	0	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	1	15	-1	1	1	-1	11	-1	1	11	1	1	0	11	32
52	123	1	1	1	1	11	0	-1	99	1	1	1	1	1	1	1	1	1	1	1	1	0	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	99	1	1	1	0	32
53	48	1	1	42	1	-1	1	-1	1	1	1	1	1	0	1	1	1	1	1	1	1	10	0	-1	-1	-1	-1	-1	1	1	-1	0	-1	1	1	1	1	21	99	31
54	61	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	-1	-1	1	12	-1	1	1	-1	1	0	1	1	59	0	45	0	31
55	130	1	1	1	1	0	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	-1	0	-1	-1	-1	1	1	-1	1	1	0	0	59	1	1	0	31
56	133	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	0	1	7	1	7	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	33	0	26	7	-1	31
57	28	1	1	1	1	-1	33	7	1	1	1	1	1	1	1	1	1	1	1	1	0	11	33	66	-1	-1	-1	12	1	1	1	1	-1	1	1	1	1	1	0	30
58	63	1	1	1	1	-1	1	-1	1	1	1	1	0	1	1	0	1	1	1	1	1	1	1	99	1	-1	-1	-1	0	99	-1	1	-1	12	33	1	1	1	0	30
59	79	1	1	1	1	-1	1	-1	1	1	1	0	1	1	0	1	0	1	1	1	1	0	1	-1	-1	-1	-1	-1	1	1	-1	1	17	0	1	1	1	0	99	30
60	93	1	1	1	94	-1	1	-1	1	99	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	0	-1	-1	1	1	1	99	1	1	1	99	99	99	0	99	29
61	43	1	1	1	1	-1	1	-1	1	1	1	0	1	1	25	1	1	43	1	1	1	1	0	-1	-1	0	1	0	1	1	1	1	0	1	1	1	33	7	0	28
62	80	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	-1	1	48	11	12	1	11	1	1	11	11	11	0	1	1	11	28
63	38	1	1	1	1	11	0	-1	1	1	1	1	11	0	1	1	1	1	1	0	1	1	0	-1	-63	-1	-1	-1	1	63	-1	1	0	1	1	0	0	1	0	27
64	46	1	1	42	1	-1	1	39	1	1	1	32	1	1	1	42	1	1	33	1	0	1	5	1	-1	-1	-1	-1	10	1	-1	32	-1	1	1	1	1	7	99	27
65	5	1	1	1	1	-1	1	-1	1	1	1	0	1	1	1	1	1	43	0	1	32	0	0	-1	1	-1	-1	-1	1	0	-1	19	19	12	-1	1	1	99	0	26
66	49	1	1	1	1	-1	1	-1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	-1	-1	58	58	11	1	1	0	11	-11	11	0	11	1	99	11	26
67	106	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	-11	-11	11	22
68	127	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	15	0	1	15	54	11	0	0	0	0	21	0	0	21	99	0	22
69	8	1	23	1	1	0	0	-1	1	1	1	1	23	1	0	1	0	0	1	53	1	23	0	1	-1	-1	-1	-1	0	1	-1	0	0	57	1	0	12	0	1	21
70	110	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	11	11	11	11	-11	-11	11	11	11	11	11	11	0	12	10	11	21
71	136	1	1	1	1	1	1	1	0	0	99	1	1	1	1	11	1	1	7	1	0	1	0	1	-1	-1	-1	11	17	1	7	0	1	0	0	0	0	0	0	21
72	40	1	1	1	1	11	1	1	1	1	11	1	1	1	1	1	1	1	1	1	1	1	1	11	11	11	11	-11	0	11	11	11	-11	11	11	11	-11	-11	11	20
73	47	1	1	42	1	-1	1	-1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	11	11	11	58	-11	-11	11	11	11	0	11	11	11	-11	-11	11	20
74	51	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	7	1	1	33	-17	-12	15	11	12	-17	-12	12	-17	-17	0	17	17	11	0	99	20
75	117	1	1	1	1	-1	0	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	0	0	56	11	51	12	0	0	40	0	51	0	65	0	0	0	20
76	119	1	42	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	-11	-11	11	20

Figure 4: S-P Bug Table: The Data Matrix Filled by Student Bugs.

77	126	1	1	1	1	-1	1	1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	-17	-12	15	15	11	-17	-12	12	-17	0	11	17	17	54	0	11	20
78	139	1	1	1	1	-1	1	42	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	11	11	11	15	-11	-11	11	11	0	-11	12	11	11	-11	-11	11	20
79	144	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	0	1	11	1	11	15	11	58	-11	-11	0	11	11	-11	11	11	11	-11	-11	0	20
80	37	1	11	1	1	-1	42	0	1	1	1	1	1	1	1	1	1	1	33	42	1	11	0	99	11	1	11	11	0	1	50	0	11	1	0	1	0	48	95	19
81	76	1	1	1	1	1	1	-1	11	1	1	1	1	1	1	1	1	1	0	1	11	1	1	11	11	11	11	-11	-11	0	11	11	-11	11	0	11	-11	-11	11	19
82	78	1	1	1	1	1	1	1	0	1	42	1	1	11	1	1	1	1	1	1	1	1	1	11	11	11	58	-11	-11	11	11	11	-11	0	11	11	-11	-11	0	19
83	52	23	1	99	1	1	1	1	99	99	99	1	23	1	99	99	1	23	1	0	99	1	47	1	18	-1	-1	-1	15	99	-1	1	13	1	99	99	99	1	94	[illegible]
84	2	11	1	1	1	-1	1	-1	1	1	1	1	11	1	1	42	11	1	23	1	1	1	1	11	41	11	48	0	-11	11	2	67	-11	11	0	11	67	12	0	[illegible]
85	35	1	1	1	1	-1	1	-1	1	1	1	1	1	1	0	1	0	11	0	1	1	1	0	11	11	58	11	-11	-11	11	11	11	-11	0	11	11	-11	-11	11	17
86	29	1	1	1	1	11	1	-1	1	1	1	0	1	1	25	1	-25	1	0	0	1	11	0	11	11	11	11	11	-11	11	0	11	64	0	11	11	-11	-11	0	14
87	41	1	1	99	1	1	1	1	99	1	99	32	1	1	99	99	1	0	1	1	11	1	5	1	11	11	11	-11	11	99	11	11	11	11	99	99	11	0	11	14
88	42	11	1	99	99	1	1	10	93	99	99	1	99	99	99	99	11	99	93	1	99	1	11	1	11	99	1	1	99	99	99	99	1	1	99	99	99	1	99	12
89	45	1	1	1	11	11	1	-1	11	1	1	11	11	11	1	11	1	43	1	1	0	0	1	11	15	11	11	-11	-11	11	11	11	-11	48	11	11	-11	-11	11	12
90	49	99	1	1	1	11	1	-1	11	1	1	0	1	1	0	1	0	43	1	0	1	0	0	11	0	58	58	54	-11	11	11	12	0	11	55	11	11	0	11	12
91	74	1	1	1	1	11	99	1	1	99	93	99	1	1	1	1	1	99	99	99	99	1	0	49	11	11	15	2	11	11	99	59	-11	11	59	11	99	0	11	12
92	44	1	1	48	0	11	1	1	48	0	0	1	1	0	42	1	1	0	67	1	0	11	1	11	11	43	11	11	48	48	67	11	11	0	0	0	0	10	11	10
93	71	0	11	0	1	11	11	1	1	0	0	1	11	1	1	0	1	1	11	11	1	11	1	15	11	11	58	11	11	0	0	11	1	12	11	0	-11	-11	11	10
94	57	1	1	1	0	-1	28	-28	1	28	28	28	1	1	1	0	0	40	28	28	0	1	0	0	0	59	-59	0	21	59	0	0	0	0	0	12	0	49	59	9
95	122	0	11	1	11	11	0	7	11	67	1	1	11	11	1	1	25	1	67	1	1	11	1	11	11	11	15	14	-11	11	-26	67	46	0	0	11	67	0	46	9
96	13	1	1	1	23	-1	42	1	1	0	0	11	42	0	0	0	1	0	0	11	0	1	0	11	11	0	11	0	-11	0	40	11	11	11	11	59	0	10	0	8
97	92	99	95	99	99	48	1	99	95	1	99	1	99	99	99	99	99	1	0	1	1	99	0	99	99	99	99	56	99	99	99	1	99	48	1	99	99	99	99	8
98	132	11	11	1	0	0	0	10	1	1	1	1	11	11	1	1	11	0	1	11	11	11	11	11	11	58	11	-11	-11	11	11	17	-11	11	11	11	-11	-11	11	8
99	54	42	1	42	42	-1	0	-7	1	28	7	7	11	1	1	1	1	28	7	0	28	0	7	48	0	58	15	0	0	0	0	0	0	11	37	0	0	0	0	7
100	62	0	1	99	99	1	28	0	99	99	95	1	0	99	99	99	0	99	99	1	99	1	1	12	0	99	11	67	99	99	99	0	0	1	99	99	99	67	99	7
101	85	1	11	11	1	11	53	0	11	0	0	53	11	1	1	1	1	0	0	53	53	11	0	11	8	99	11	55	99	99	99	0	12	15	0	99	0	0	99	6
102	24	11	11	11	11	11	67	-67	11	67	67	0	11	11	11	1	11	1	1	1	67	11	1	11	11	11	15	67	-11	11	67	67	-11	11	67	11	67	15	11	5
103	115	11	0	1	11	11	11	0	11	11	1	11	11	11	1	1	11	11	1	11	11	11	25	11	11	11	11	-11	-11	11	11	11	-11	11	0	11	-11	-11	11	5
104	134	1	11	11	11	99	33	1	11	11	0	0	11	11	0	0	1	1	1	11	11	0	0	11	0	11	11	37	-11	11	11	11	-11	0	11	11	11	0	11	5
105	50	99	11	99	99	11	1	99	99	99	93	0	11	99	99	98	11	99	99	67	99	11	1	26	99	99	1	0	99	99	99	0	1	0	99	59	99	26	99	4
106	26	1	11	1	11	11	0	10	11	11	1	38	11	0	11	11	0	11	51	51	11	11	38	11	11	11	11	0	-11	11	0	57	-11	0	11	11	-11	-11	0	3
107	82	11	11	11	1	11	11	-11	11	11	11	11	11	11	11	1	11	11	1	11	11	11	11	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	11	0	11	3
108	103	1	11	1	11	11	11	-11	11	11	11	11	11	11	11	11	11	11	1	0	11	11	11	11	11	11	11	11	-11	11	0	11	-11	11	11	11	-11	-11	11	3
109	107	1	[illegible]	11	11	48	1	10	15	11	99	5	0	11	11	11	42	11	[illegible]	0	0	48	54	15	48	48	58	48	11	11	48	47	15	48	11	11	11	54	11	3
110	11	0	0	0	11	11	0	28	1	0	1	11	11	11	0	11	11	11	0	0	11	11	0	15	48	11	58	54	0	0	0	0	0	47	0	0	0	0	0	2
111	53	11	11	11	11	11	1	7	11	67	0	0	11	11	0	11	11	7	67	99	67	11	1	11	11	99	11	5	-11	11	67	67	11	0	67	0	67	67	11	2
112	89	11	11	11	0	11	11	-11	11	11	11	1	11	11	11	11	11	11	11	11	11	11	1	11	11	11	11	54	0	11	0	11	-11	11	11	11	-11	-11	11	2
113	96	99	1	99	99	11	11	99	99	99	99	11	11	99	99	99	99	99	99	1	99	11	11	11	99	99	11	11	99	99	99	99	17	0	99	99	99	10	99	2
114	131	10	10	-10	10	-10	67	-10	10	0	-10	67	10	10	10	10	0	67	67	67	67	0	0	1	10	10	10	14	10	10	67	67	10	10	0	10	10	-10	10	2
115	137	11	11	11	11	11	11	-11	11	11	11	1	11	11	11	11	11	11	11	11	11	11	1	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	-11	-11	11	2
116	32	1	11	11	11	11	93	99	11	99	93	99	11	11	99	11	99	99	99	0	99	11	99	99	11	11	0	99	11	11	99	99	-11	11	99	11	99	99	11	1
117	64	45	45	-45	45	45	99	99	45	99	94	99	45	45	97	45	99	99	99	99	99	45	99	-45	45	0	37	99	45	45	99	99	45	45	99	99	99	45	-45	1
118	66	11	11	11	11	11	11	-11	11	99	11	11	0	11	11	11	11	11	0	1	11	11	0	11	15	15	15	-11	-11	11	11	47	0	11	11	11	54	0	11	1
119	68	0	11	11	11	0	11	-11	11	11	11	1	0	11	11	11	11	11	11	11	11	11	0	11	11	12	11	11	-11	11	0	17	0	11	11	11	-11	-11	0	1
120	69	41	0	11	11	11	0	-11	0	11	11	11	11	11	11	11	0	11	11	11	11	11	1	11	11	66	11	11	-11	11	0	11	-11	0	11	11	0	0	11	1
121	75	11	1	11	11	11	0	-11	11	11	11	11	11	11	11	11	11	11	-11	11	11	11	41	26	26	11	26	11	11	11	0	0	-11	26	11	11	1	0	11	1
122	61	11	11	11	11	11	11	0	11	0	11	1	11	11	11	11	11	11	11	67	11	11	0	11	11	11	11	-11	-11	11	11	11	-11	11	0	11	-11	-11	0	1
123	97	11	11	11	11	11	11	10	11	11	1	11	11	11	0	11	11	11	-11	11	11	11	11	11	11	11	11	-11	-11	11	11	11	11	0	11	0	11	0	11	1
124	107	11	11	0	0	11	11	0	11	11	11	1	0	0	11	11	0	0	0	11	11	11	11	11	11	18	58	-11	-11	0	11	11	64	11	11	0	11	0	0	1
125	108	15	11	11	11	11	11	-11	11	11	11	1	11	11	1	11	11	11	0	11	11	11	0	11	15	11	-11	54	-11	11	11	11	-11	0	11	11	11	0	11	1
126	109	99	11	11	11	11	99	39	0	0	11	0	99	11	0	11	99	11	-11	1	11	11	11	99	99	66	11	0	11	11	11	59	99	11	11	11	-11	-11	11	1
127	4	11	45	11	11	45	11	-11	11	0	11	45	11	11	11	11	0	11	11	45	0	45	0	17	11	11	37	45	11	11	11	11	0	45	0	11	11	0	11	0
128	10	11	11	11	11	11	0	7	11	37	15	37	11	11	0	11	11	37	37	0	37	0	37	15	11	11	11	37	11	11	0	0	-11	11	37	11	37	0	11	0
129	12	0	11	11	11	11	11	-11	11	0	11	11	11	11	11	11	11	11	0	11	11	11	0	11	11	11	11	-11	-11	11	11	0	64	11	11	11	-11	-11	11	0
130	24	37	37	11	11	37	40	28	11	0	0	40	37	11	11	11	0	0	0	40	0	37	40	37	37	11	37	37	11	11	0	0	37	15	0	11	0	41	11	0
131	23	11	11	11	11	11	11	-11	11	11	11	0	0	11	11	0	11	11	11	11	11	11	0	11	15	11	11	-11	-11	11	11	11	-11	0	11	11	11	59	11	0
132	27	11	11	11	11	11	99	99	11	99	99	11	11	11	99	11	0	99	99	11	99	11	99	11	11	11	11	11	-11	11	99	99	11	11	99	0	99	0	11	0
133	30	11	11	11	11	11	11	-11	11	11	11	11	11	11	11	11	11	11	-11	11	11	11	11	11	11	11	11	11	-11	11	0	11	-11	11	11	11	-11	-11	11	0
134	31	11	11	11	11	11	11	-11	11	11	11	11	11	11	11	11	0	11	-11	11	11	11	11	11	11	11	11	18	-11	11	0	11	-11	0	11	11	-11	-11	11	0
135	55	48	11	11	11	11	11	0	15	11	0	11	11	11	11	11	11	11	0	11	0	11	11	15	15	11	11	54	11	11	11	11	0	11	11	11	11	54	11	0
136	56	11	11	11	11	11	39	-11	11	11	11	27	0	11	11	11	23	11	11	45	11	11	0	11	15	11	18	45	-11	11	11	45	-11	46	11	11	0	45	11	0
137	65	11	11	11	11	11	67	-67	11	67	67	67	11	11	11	11	11	67	67	67	15	11	0	11	11	11	11	67	-11	0	67	67	-11	11	67	11	67	67	11	0
138	67	11	0	11	11	11	11	-11	11	11	11	11	11	0	99	11	99	11	11	11	11	11	0	15	11	11	11	-11	11	11	11	11	-11	11	11	11	-11	-11	11	0
139	73	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	0	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	0
140	83	11	42	11	11	11	11	-11	11	11	11	11	11	11	0	11	11	11	-11	11	11	11	11	11	11	11	15	-11	-11	11	11	11	-11	11	11	11	-11	-11	11	0
141	84	11	11	11	11	11	11	-11	11	11	11	0	11	11	11	11	11	11	0	11	11	11	11	11	11	11	11	54	-11	11	11	11	-11	11	11	11	-11	-11	0	0
142	91	11	11	11	11	11	11	10	11	11	0	11	11	11	11	11	11	11	-11	11	11	11	11	11	11	11	11	54	-11	11	11	11	-11	0	11	0	0	10	11	0
143	111	14	11	11	11	11	11	-11	11	11	11	11	11	11	0	11	11	11	-11	11	11	11	11	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	0	10	0	0
144	112	11	11	11	11	11	11	0	11	11	11	11	11	11	0	11	0	11	11	11	11	11	0	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	-11	-11	11	0
145	114	99	49	49	99	11	93	99	99	99	99	99	99	99	99	99	99	99	99	99	99	0	0	99	99	99	99	99	99	99	99	99	99	0	99	99	99	99	99	0
146	118	11	11	11	11	11	11	-11	93	11	11	11	11	11	0	11	0	11	2	11	99	11	25	11	11	11	11	-11	-11	11	11	11	64	11	11	11	11	11	11	0
147	128	11	11	11	11	11	11	-11	11	11	11	99	99	11	11	11	11	11	0	11	11	11	99	11	11	11	11	-48	-11	11	11	17	-11	99	11	11	11	48	46	0
148	135	11	11	11	11	11	11	-11	11	0	11	11	11	11	11	11	11	11	-11	11	11	11	11	11	11	11	11	-11	-11	11	11	11	-11	11	11	11	11	0	11	0
149	138	11	0	0	11	11	67	-67	11	67	67	67	11	11	11	11	11	67	0	0	67	11	67	48	11	11	58	67	0	11	67	67	-11	11	67	11	67	0	0	0
		96	95	92	91	87	87	87	87	87	87	86	85	84	84	84	83	83	82	80	79	73	69	66	65	65	64	63	62	62	60	59	58	55	53	52	49	48	27	

Figure 4: Continuation of S-P BUG Table

BUG	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	
1	375	272	379	288	227	408	400	311	371	434	311	401	273	[illegible]	311	414	432	272	415	367	153	391	276	229	365	394	301	367	421	291	396	243	393	247	386	410	268	395	13044
2	1	0	0	1	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	3	0	0	2	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	11
5	11	0	0	1	1	2	0	0	7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	24
7	4	0	0	2	48	1	0	0	23	0	0	0	1	0	0	20	0	0	0	4	0	0	1	1	0	0	0	14	0	0	0	0	0	0	1	0	0	0	120
10	0	1	2	0	22	0	1	1	0	1	0	0	0	0	1	16	2	2	1	0	1	1	0	1	0	1	1	0	1	9	0	0	1	1	1	1	1	1	71
11	58	141	146	123	68	72	145	142	80	121	156	87	113	87	155	61	112	151	127	68	150	108	106	115	93	110	149	86	108	170	71	116	71	146	72	106	154	107	4251
12	0	6	0	15	4	0	0	14	0	0	5	0	4	0	21	0	0	4	0	0	[illegible]	0	17	5	0	0	3	0	0	3	0	5	0	17	0	0	5	0	134
14	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5
15	0	30	0	5	10	0	0	41	0	1	28	0	4	0	19	2	2	5	0	2	7	0	1	5	1	2	24	0	0	9	0	3	0	6	1	2	5	0	215
17	0	0	0	0	0	0	0	0	0	0	23	0	17	0	0	0	0	19	0	0	0	0	0	0	0	0	0	0	0	20	0	8	0	0	0	0	7	0	94
18	0	0	0	6	1	0	0	1	0	0	5	0	0	0	3	0	1	2	0	0	0	0	4	0	0	0	1	0	0	1	0	0	0	2	0	1	0	0	28
19	0	0	0	4	0	0	0	0	0	0	3	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	11
21	0	2	0	0	1	0	0	0	0	0	13	0	1	0	3	0	0	0	0	0	3	0	0	4	0	0	0	0	0	2	0	0	0	3	0	0	0	0	32
23	0	0	1	2	0	0	0	0	0	1	0	0	1	2	0	2	3	0	3	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	1	20
25	4	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	18
26	0	4	0	0	2	0	0	2	0	0	6	0	2	0	2	0	0	1	0	0	0	0	2	4	0	0	2	0	0	1	0	3	0	1	0	0	3	0	35
27	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
28	0	0	0	0	0	1	0	0	2	0	0	2	0	0	0	3	0	0	0	1	0	0	0	0	1	0	0	5	0	0	2	0	0	0	3	0	0	0	20
32	0	0	0	0	6	3	0	0	0	0	0	0	7	0	0	4	0	0	0	1	0	0	3	1	6	0	0	0	0	0	3	3	0	0	5	0	0	0	42
33	4	0	0	3	0	6	0	0	0	0	0	10	8	0	0	0	0	0	0	7	0	0	2	35	4	0	0	0	0	0	3	6	0	0	2	0	0	0	91
37	2	1	2	4	1	2	3	6	1	2	2	0	1	1	2	1	3	2	2	2	1	4	1	2	2	1	3	2	2	1	2	3	0	1	1	2	1	2	71
38	1	0	0	0	0	1	0	1	1	1	0	0	0	0	2	0	0	1	0	6	0	0	0	0	0	0	2	1	0	0	0	0	0	0	0	0	0	0	17
39	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
40	1	0	0	0	0	1	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	7
41	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
42	1	0	2	0	0	0	5	0	2	3	0	6	0	1	0	4	3	0	3	0	0	0	0	0	0	3	0	1	2	0	6	0	1	0	4	8	0	5	60
43	1	0	0	0	0	2	0	0	13	0	0	3	0	0	0	0	0	0	0	2	0	0	0	0	2	0	0	11	0	0	2	0	0	0	0	0	0	0	36
45	0	5	3	4	4	1	3	0	2	4	2	1	2	1	3	1	3	2	3	2	3	1	2	1	1	2	0	0	2	4	2	1	1	3	1	1	2	1	74
46	0	2	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	3	0	0	1	0	0	0	0	0	0	0	2	0	0	0	0	0	0	12
47	2	6	0	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	21
48	0	9	2	5	4	0	3	7	0	0	3	0	2	0	4	0	2	0	0	0	5	0	2	2	0	1	7	0	0	2	0	0	0	3	0	1	1	0	65
49	0	1	0	0	1	0	0	0	0	0	0	0	0	0	2	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7
50	0	0	0	3	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	1	0	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	17
51	0	1	0	1	0	0	0	2	1	0	0	1	0	0	1	0	0	1	0	1	1	0	0	0	0	0	2	0	0	1	0	0	0	1	0	0	1	0	15
53	0	0	0	2	0	2	0	0	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	8
54	5	0	0	19	21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	3	13	0	0	0	0	0	0	0	6	0	0	5	0	0	0	74
55	0	0	0	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	13
56	0	1	0	3	1	0	0	3	0	0	0	0	1	0	0	0	0	1	0	0	2	0	2	1	0	0	3	0	0	0	0	1	0	0	0	0	0	0	19
57	1	1	0	0	2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7
58	0	0	0	0	0	0	0	36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	23	0	0	0	0	0	0	0	0	0	0	0	59
59	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	2	0	0	0	0	0	0	2	0	0	11	0	19
60	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
62	0	2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	7
63	0	4	0	0	0	0	0	3	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	5	0	0	2	0	0	0	1	0	0	1	0	17
64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	2	0	37
65	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	5	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	2	0	11
66	0	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	3	0	9
67	9	0	0	15	13	12	0	0	14	0	0	16	21	0	0	13	0	0	0	14	0	0	21	24	14	0	0	17	0	0	15	18	0	0	15	0	0	0	251
	480	491	537	517	439	516	560	573	520	568	559	530	474	510	534	544	563	507	554	485	343	505	466	446	490	514	535	506	537	516	504	417	465	467	501	533	466	512	19206

Figure 5: Frequency Table of Bugs x Items

Table 2

Item Difficulties, Values of the Caution Index and Item Types of 38 Fraction Addition Problems

	Item	Item Type	p_i	C_i
1	34	VI	.80	.10
2	41	VI	.80	.05
3	10	VI	.79	.06
4	43	VI	.79	.19
5	6	II	.78	.17
6	17	VI	.78	.06
7	16	II	.78	.10
8	19	VI	.77	.10
9	31	VI	.77	.07
10	36	II	.77	.15
11	14	V	.76	.18
12	27	V	.76	.14
13	40	II	.75	.13
14	7	VI	.73	.11
15	38	V	.73	.25
16	12	II	.72	.17
17	30	II	.72	.14
18	1	I	.71	.20
19	33	I	.71	.18
20	9	I	.70	.23
21	3	V	.69	.16
22	25	I	.67	.26
23	32	VIII	.58	.01
24	39	VIII	.58	.04
25	4	III	.58	.05
26	8	VIII	.57	.05
27	11	VIII	.56	.22
28	15	VIII	.55	.03
29	35	VIII	.53	.05
30	28	III	.53	.04
31	13	IV	.52	.03
32	42	VIII	.51	.06
33	5	III	.51	.05
34	2	VII	.49	.07
35	29	III	.46	.03
36	37	IV	.46	.04
37	18	VIII	.45	.07
38	26	VII	.29	.08

the two numbers, while the students in the low-score group don't know how to add two fractions. The numbers shown in Figure 3 represent erroneous rules.

The caution index (C_i) shows that the items in the fraction addition test were reasonably well constructed because the values of C_i are all nearly zero.

Table 3 shows the percentages of descriptive statistics obtained by error analyses performed on the three datasets. The average percentage of identified rules by "S-P bug" is 90.8% including the right rule, 76.1% excluding the right rule. This rate of error diagnosis is satisfactorily high.

Insert Table 3 about here

The use of erroneous rules of operation varies among individuals. As can be seen in Figure 4, the most popular rule is (fortunately!) the right rule and the second most popular is Rule 11. The 48 different rules in fraction addition were observed and the percentages of their frequencies of use by 592 students were summarized in Table 4. The next

Insert Table 4 about here

rule to Rule 11 in use-frequency is Rule 67. Then Rules 14, 12, 7 and 17. Rule 12 is associated with the task of finding the least common denominator (LCD). Rules 15 and 17 originate from finding equivalent fractions; first the common denominator is obtained by multiplying the denominators or finding the LCD and then the original numerators are added without being multiplied by the appropriate number. Rules 11 and 67 result from misconceptions in adding two fraction numbers. Rule 11 is the addition of corresponding parts of two numbers while Rule 67 applies the same rule as 11 after converting mixed number to improper fractions. Rule 7 occurs when mixed numbers are converted into improper fractions; new numerators are obtained by multiplying whole number parts by numerators, not denominators. Then the right procedure is used to get the final answer.

Rules 60 and 41 are observed only once each in 592 students. Rule 60 is "the numerator times the denominator of the first fraction plus

Table 3

Percentages of Various Descriptive Statistics of

Three Datasets

	Dataset I	Dataset II	Dataset III	Total N=592
Correct	48.68	64.67	57.89	57.55
Incorrect	45.39	31.78	33.59	36.02
Missing	5.92	3.55	8.51	6.43
Identified with Correct Rule	89.51	92.18	90.62	90.81
Identified without Correct Rule	78.26	76.27	74.46	76.12

Table 4

Frequencies of the Use of Erroneous Rules of Operation

in 38 Fraction Addition Problems

(N=592)

Rule	Freq	Rule	Freq	Rule	Freq	Rule	Freq
2	13	23	20	41	1	55	7
5	120	25	18	42	60	56	16
7	112	26	33	43	35	57	7
10	38	27	8	45	66	58	59
11	4234	28	20	46	12	59	18
12	138	31	34	47	21	60	1
14	5	32	46	48	65	62	7
15	216	33	71	49	7	63	16
17	96	37	71	50	17	64	35
18	28	38	18	51	4	65	11
19	4	39	4	53	8	66	9
21	32	40	7	54	73	67	258

the numerator times the denominator of the second fraction to obtain the numerator of the answer. Multiplying the two denominators to get the denominator of the answer." Rule 23 is a rule similar to Rule 60, but 20 cases are observed in the data. Rule 23 is the multiplication of the numerator and denominator to get the new numerator, and the denominator is the common denominator. The right addition procedure follows in both the rules. Since Rule 41 is an odd rule, the description will be omitted here.

According to the classification of erroneous rules based on task analyses, Rule 7 belongs to Category 1, Rule 15 to Category 4, Rule 12 is in Category 3, and Rules 11 and 67 belong to Category 5.

Next, further investigation of the difference between the performances of high and low score groups on the fraction addition test is summarized in Table 5 below. The distributions of the frequencies in the two groups are quite different. The high-score group has more

Insert Table 5 about here

students that used Rules 33, 32, 21, 18, 43 than the low-score group has. These rules are classified in C2 and C4 which are designated by "incorrect use of Method B", and "incorrectly finding equivalent fractions." Especially Rule 33 occurs by omitting the whole number part in the answer. Conversation with teachers in a class confirmed that Rule 33 is due to a careless mistake. Rule 32 is associated with the principle that any number plus zero is that number. The outcome of this rule is that when adding mixed numbers, students add a one to the whole number part of the answer. It is interesting to note that the principle, a + 0 = a for any number a, may be a difficult concept to understand for a junior-high aged student. The third rule strongly differentiates good students from poor students. This rule uses the denominators of the original fractions as the numerators of the new fractions while the right LCD is found and the right addition procedure is used to get the answer.

The students in the low-score group tend to use Rules 15, 48 and 54 sporadically and Rules 11, 67, 10, 45, 26 and 37 systematically. It is

Table 5

Frequencies of Erroneous Rules Observed in the 38-Item Fraction Addition Test by High Score (≥ 29) and Low Score (≤ 9) Groups

Rule	High (N=284)	Low (N=158)	Rule	High (N=284)	Low (N=158)
11	61	2970*	38	3	7
33	51	3	45	3	69*
7	50	14	67	3	215*
12	36	19	62	2	0
32	27	1	14	1	4
21	21	1	23	1	7
42	19	17	26	1	29*
18	15	6	28	1	14
64	15	5	57	1	2
15	14	66	60	1	0
43	14	1	63	1	1
59	12	5	65	1	2
5	10	2	37	0	69*
48	10	27	46	0	12
54	8	24	2	0	9
19	7	1	53	0	7
10	7	25*	40	0	6
17	5	18	66	0	4
47	5	4	39	0	2
49	5	1	51	0	1
55	5	0	41	0	1
58	5	17	27	0	1
25	4	6			
50	4	4			
56	4	11			

*These rules are observed in at least nine items per person interesting to look at why the former appear sporadically throughout a test and the latter systematically. Rules 15 (described earlier), 48 and 54 are classified in C5, faulty algorithms for addition procedure. Rules 11, 67 and 45 belong to C5 but these bugs are usually applied systematically. Rules 10 and 26 belong to C4, incorrectly finding equivalent fractions.

Probability of Bug Occurrences as a Function of the Total-Score Scale

In the previous section, we have found an interesting difference in the frequencies of certain bugs between high-score and low-score students. Further investigation of bug-distribution is carried out in this section.

The formula used to produce probability distribution of bug y given score x is presented below.

$$P(y,x) = \frac{\dfrac{\text{Frequency of bug y among students who score x}}{[(\text{\# of students who scored x}) + 1] \cdot (38-x)}}{\displaystyle\sum_{i=1}^{37} \frac{\text{Frequency of bug y among students who scored } x_i}{[(\text{\# of students who scored } x_i) + 1] \cdot (38-x_i)}}$$

This will provide information about the relationship between total score and appearance of bugs. If total score does not relate with the occurance of bugs, the probability of bug y given score x would be uniform over x and its value would be 1/37 = .02703. This would imply that a bug y is equally likely to be committed by students who answer any number of items correctly. On the contrary, if P(y,x) systematically deviates from the uniform probability distribution, that indicates the existence of a particular relationship between the incidence of a bug and total score. For example, in Figure 6, bugs 32, 33, 43 are seen to be more likely to be found among students who score above 25 than among students who score lower than 25. These three bugs are classified together in Table 1 as being an "incorrect use of Method B" group. Since there were only four occurances of bug 39, its probability distribution was not calculated. A negatively skewed probability distribution (indicating greater probability of occurrence among students with high score) was not found in any other bugsbut these three. It can be said that Method B is attempted only by students who score high. Figure 7 shows the probability distributions of three

Insert Figures 6, 7 & 8 about here

bugs that are more likely to be found among students whose total score is low. These bugs indicate lack of understanding of the meanings of numerator and denominator. For example, a student with bug 37 adds all numbers in the first fraction to obtain the numerator of the answer and does the same on the second fraction to obtain the denominator. In Figure 8, the probability distributions of bugs 15 and 17 are presented and they show that these bugs are most likely committed by students whose scores are in mid range. Both are bugs associated with faulty methods to find the denominator of the answer. These unique bug probability distributions may provide the reason why bug migration occurs and why the direction may be more orderly than one might expect. For example, one student may exhibit bugs 37, 17 and 33 consecutively, as he/she progresses in learning, but not in the reverse order. It should be noted that only bugs which showed narrow concentrations of probability are presented in three figures. Most of the bugs did not show any semblance of a uniform distribution over total score.

Analysis of Results: Stability of the Use of a Rule

Both the test for addition and that for subtraction of fraction arithmetic are constructed so as to contain pairs of parallel items. If a student uses his/her own rule consistently, then the answer of the two parallel items should match the responses generated by the rule.

In order to test the stability of a bug, the frequencies of two pairs, (1,0) and (0,1)--a students' use of the bug for solving the first item but not for the counterpart of the first item, and the second binary pair (0,1)stands for that the second item was solved by application of the bug but not the first item--were counted and then McNemar's test was carried out. Table 6 contains the frequencies and the results of a significance test with respect to the right rule.

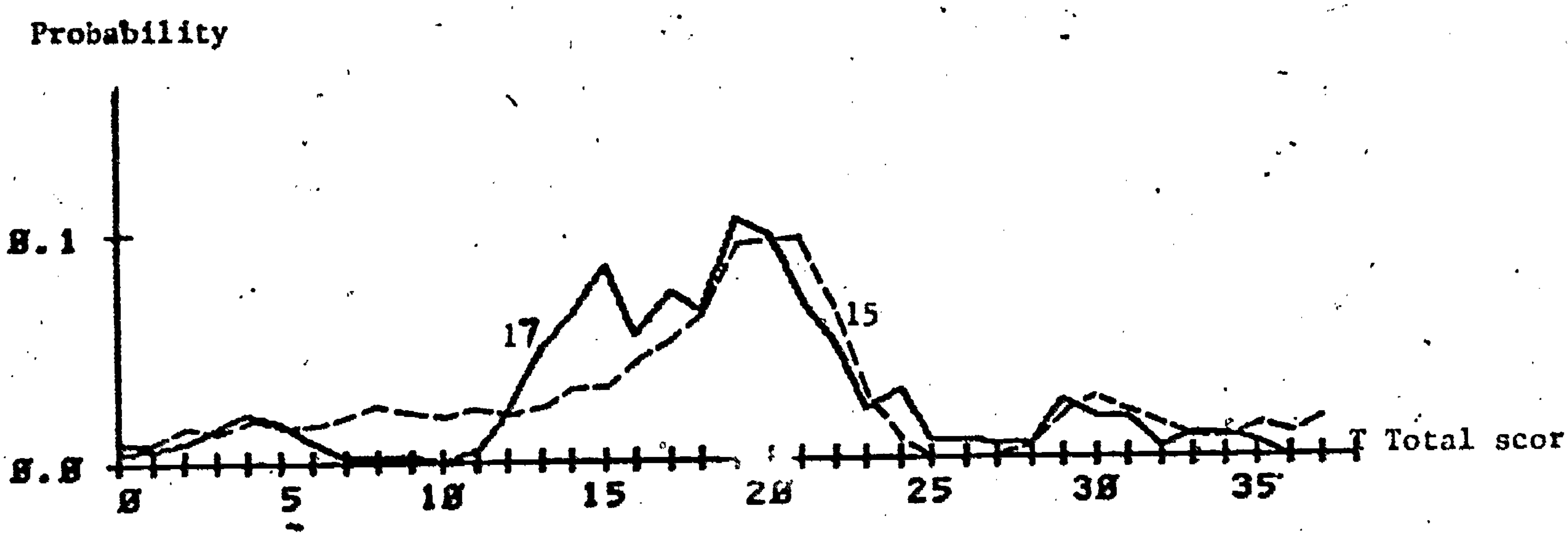

Figure 8 : Probabilities of bug occurrences of 11 and 15

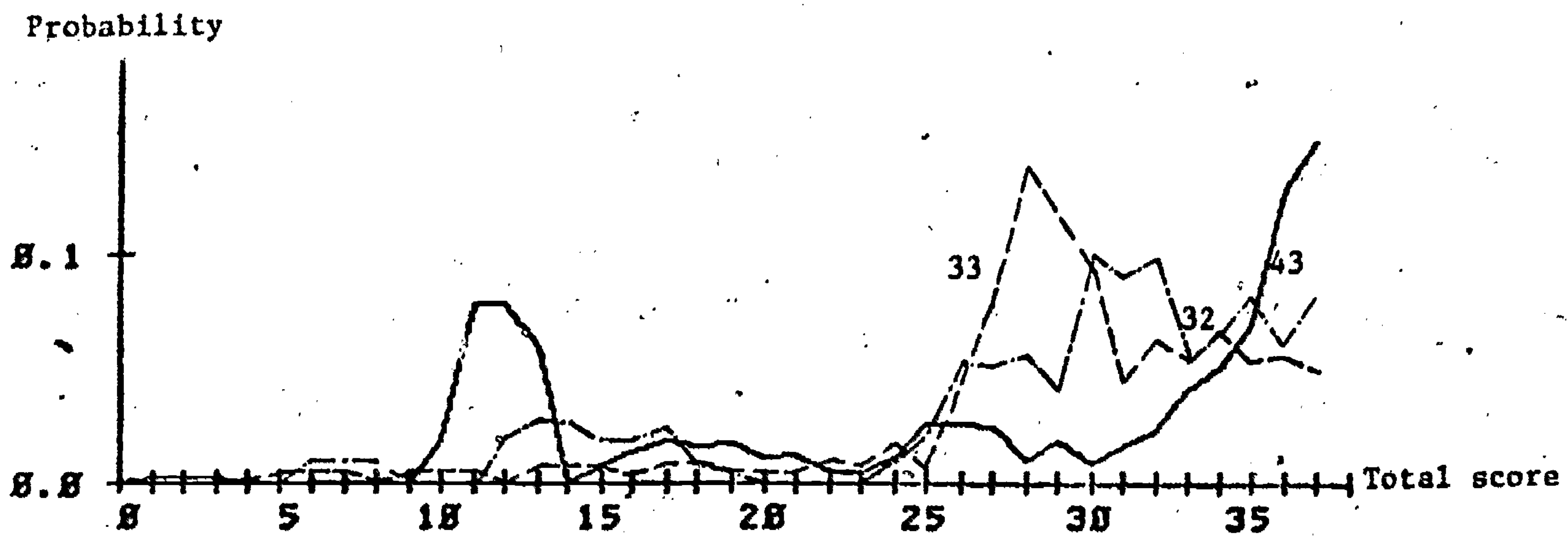

Figure 6 : Probabilities of bug occurrences of 32, 33 and 43

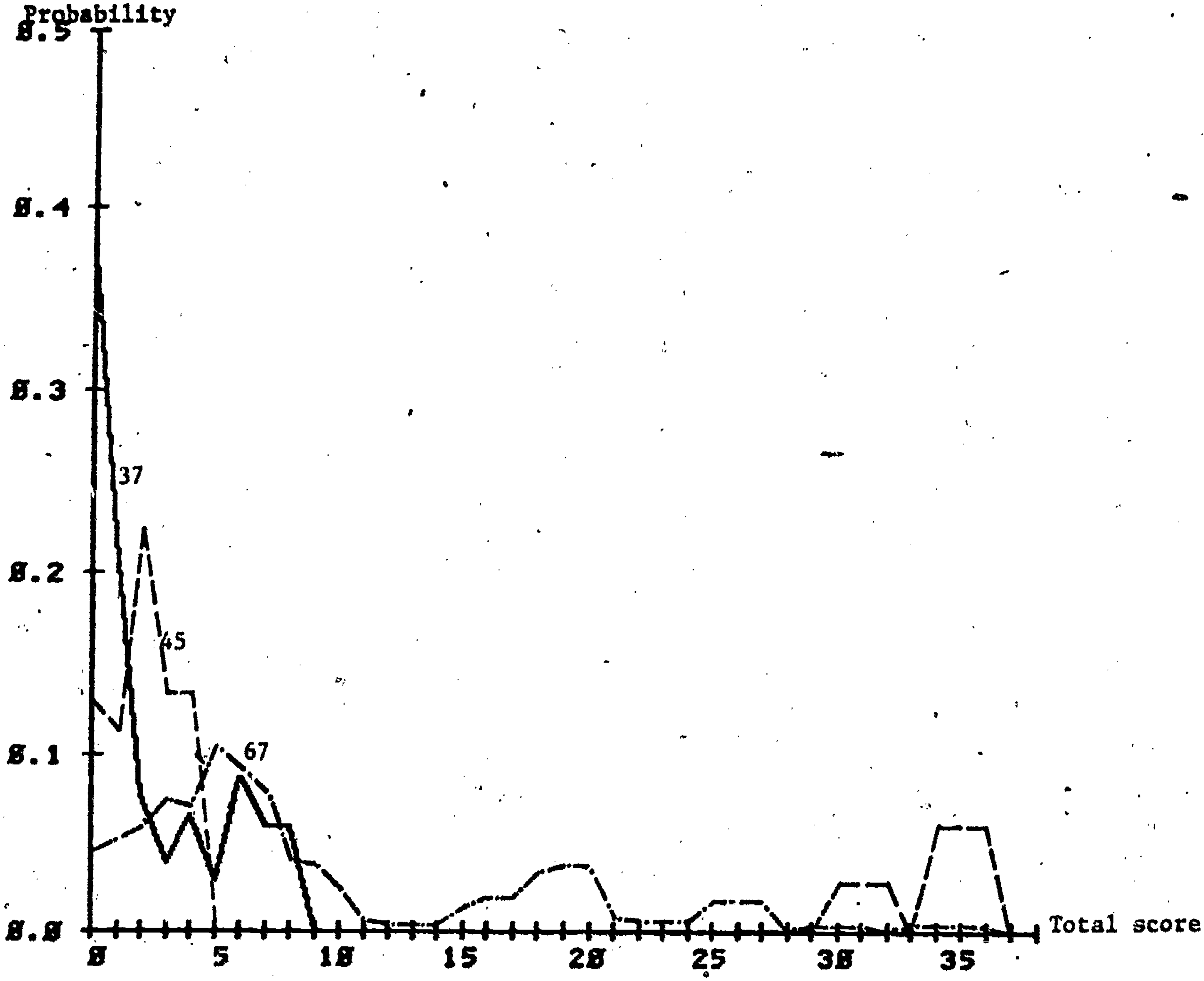

Figure 7 : Probabilities of bug occurrences of 37, 45 and 67

Significantly different pairs of parallel items are listed in Table 7 in terms of the use of various erroneous rules of

Insert Tables 6 & 7 about here

operation. Six pairs of parallel items have the z values of $p > 0.001$ in Table 6. Table 7 shows quite a few bugs are applied inconsistently to the two parallel items. But, we have actually tested 284 pairs for McNemar's test and 31 pairs turned out to be significantly different as can be seen in Table 7. This means that most erroneous rules are used consistently for two parallel items. However, it should be noted that 6 out of 19 pairs in Table 6 showed significant differences in the stability of the use of the right rule. Indeed, all the items in 19 pairs are not necessarily parallel in terms of procedural steps for carrying out all the right rules which were described in the earlier part of the report. More strict, accurate definitions of parallel items referring to each procedural step may be needed.

Table 6

McNemar's Test for Two Parallel Items in Fraction Addition

When the Right Rule was Used

Items	(01)[1]	(10)[2]	z
1, 25	68	76	-.670
2, 26	24	143	-9.29*
3, 27	60	48	1.15
4, 28	35	45	-1.12
5, 29	59	57	.19
6, 30	37	80	-3.98
7, 31	38	44	-.66
8, 32	23	33	-1.34
9, 33	61	65	-.36
10, 34	22	35	-1.72
11, 35	26	46	-2.36
12, 36	46	49	-.31
13, 37	34	64	-3.03*
14, 38	26	54	-3.13*
15, 39	20	34	-1.91
16, 40	31	57	-2.77*
17, 41	23	45	-2.67*
18, 42	50	54	-.39
19, 43	31	51	-2.21

*$p < 0.001$

1 The second item in a pair was answered by the right rule

2 The first item in a pair was answere by the right rule

Table 7

McNemar's Test for Two Parallel Items in Fraction Addition

When Bugs were Used

Item	Bug	(0,1)	(1,0)	(1,1)	z
1, 25	5	0	11	0	-3.32
2, 26	15	0	23	7	-4.80
3, 27	11	17	55	91	-4.48
4, 28	54	1	17	2	-3.77
5, 29	7	1	48	0	-6.71
	10	0	21	1	-4.58
	11	47	0	68	6.86
	33	36	0	0	6.00
	67	17	6	7	2.29
6, 30	11	36	15	57	2.94
7, 31	11	16	51	94	-4.28
8, 32	12	3	14	0	-2.67
	15	9	26	15	-2.87
9, 33	5	0	7	0	-2.65
10, 34	NONE				
11, 35	10	9	0	0	3.00
	15	4	23	5	-3.66
	21	2	13	0	-2.84
12, 36	11	12	28	59	-2.53
13, 37	17	4	13	4	-2.18
	47	0	10	0	-3.16
	54	6	0	0	2.45
14, 38	11	19	35	52	-2.18
15, 39	15	2	15	4	-3.15
16, 40	7	0	19	1	-4.36
	10	0	15	1	-3.87
	11	11	0	61	3.32
	54	5	0	0	2.24
17, 41	NONE				
18, 42	17	2	14	5	-3.00
	59	11	0	0	3.32
	64	0	30	2	-5.48
19, 43	11	14	34	93	-2.89

Simplification Errors

Scoring Procedures

Quite a number of students did not reduce their final answer to the simplest form. Thus, two scoring methods are used in scoring the test items. Since the 36-item test is open ended, each response is viewed as being comprised of three components; the whole number, the numerator and the denominator. With scoring Method 1 student responses are scored 1 only if all the components equaled the corresponding components of the correct answer.

With scoring Method 2, the student responses were converted into a decimal number first, and then they are scored 1 only if the converted decimal number matches the decimal number of the correct answer. By so scoring, the students' answer will be scored correct, regardless of whether or not the answer is reduced to the largest simple form, or converted to a mixed number. As long as the responses are carried out correctly up to the addition or subtraction of equivalent fractions, such responses are credited as correct answers.

The following figure, Figure 9 shows the relationship between scoring Methods 1 and 2. The X-axis stands for the proportion correct for Method

Insert Figure 9 about here

1 while the Y-axis represents the p-values obtained by scoring Method 2. Since Method 1 produced fewer scores of 1 than Method 2, most points in Figure 1 are located above the 45-degree line and only a few students' scores do not change by either method, primarily those having high scores.

Analysis of Simplification Errors: Addition Data

We are looking at simplification errors between the nonsimplified (NS) and simplified (S) answer regardless of the correctness in value of either answer.

For each problem on a student's paper two answers are recorded, a nonsimplified answer and a simplified or final answer. When only one answer appears on the paper it is repeated for the NS and S answer. One answer may mean that no work is shown to indicate a method used or that after a student reached an answer they left it in whatever

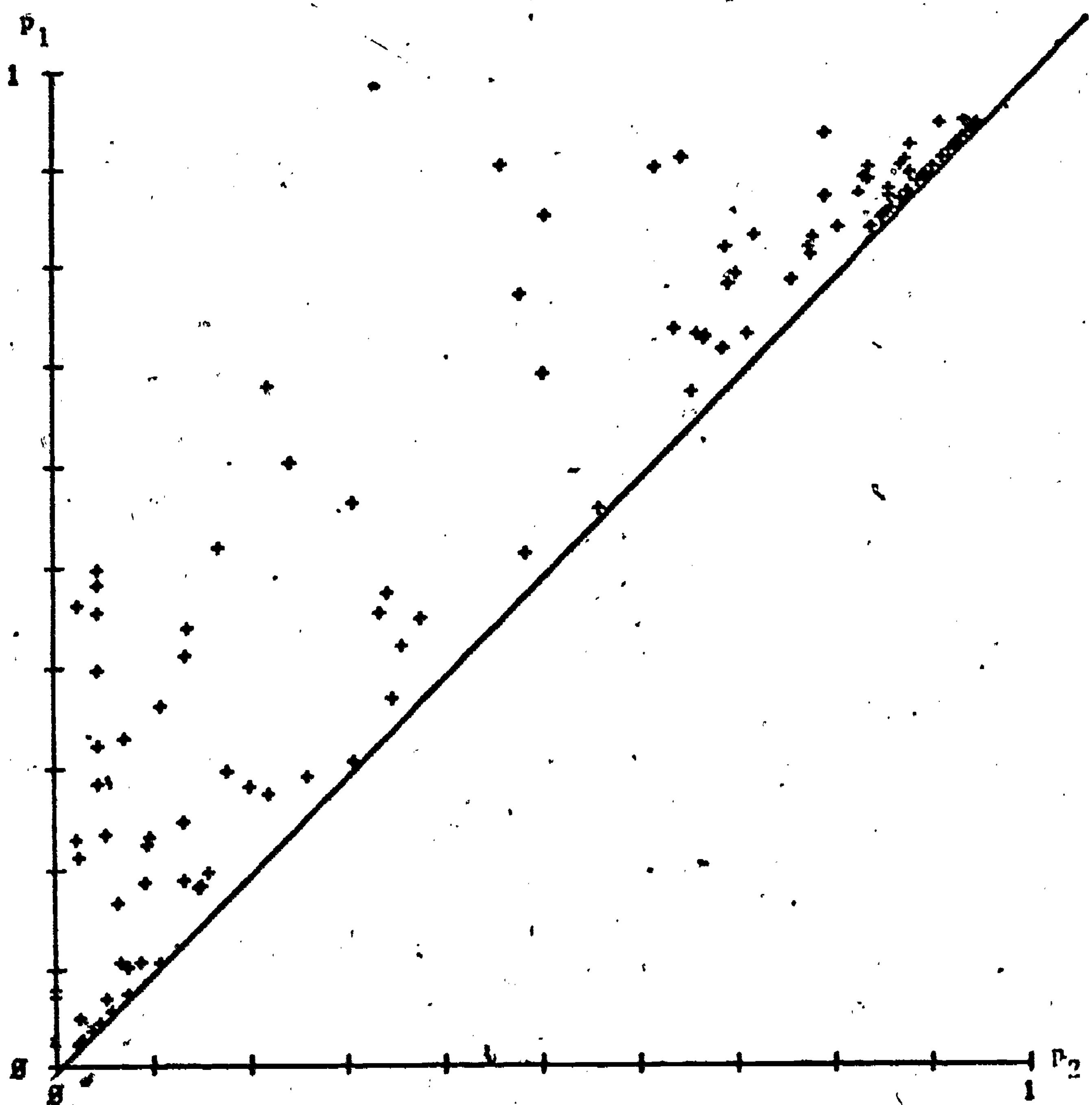

Figure 4: Proportion Correct by Method 1, p_1 against that by Method 2, p_2

form and made no attempt to convert or reduce. In the following tables Ur2 will stand for addition data collected at Urbana Junior High School (UJHS) in the fall of 1982. Ur1 will stand for data collected at UJHS in the spring of 1982 and Su2 will stand for data collected at Sullivan Junior High School in the fall of 1982.

One Answer Problems

	# Students in group (n)	Repeat Answers over Total problems (nx48)
Ur2	171	$\frac{4014}{8208} = 49\%$
Ur1	148	$\frac{4108}{7104} = 58\%$
Su2	273	$\frac{6969}{13104} = 53\%$

Some of the above answers with one answer repeated in the NS and S are blanks. Either the problem was not attempted or an answer was not reached.

	% Students	% Total Problems	Problems for Students with Blanks
Ur2	$\frac{48}{171} = 28\%$	$\frac{333}{8208} = 4\%$	$\frac{333}{2304} = 14\%$
Ur1	$\frac{64}{148} = 43\%$	$\frac{502}{7104} = 7\%$	$\frac{502}{3072} = 16\%$
Su2	$\frac{128}{273} = 47\%$	$\frac{1313}{13104} = 10\%$	$\frac{1313}{6144} = 21\%$

It could be considered a simplification error if the answer was not in a simplified or reduced form. $\frac{48}{6}$, $5\frac{18}{6}$ and 8 are all acceptable answers for problem one because they are numerically equal and correct. Since we don't agree on one final simplified form, we only look at the problems where the student attempted some form of simplification. Many students had changes between the NS and S answer.

Answers with changes between NS and S

Ur1 $\frac{4194}{8208} = 51\%$

Ur1 $\frac{2996}{7104} = 42\%$

Su2 $\frac{6135}{13104} = 47\%$

In looking at answers where there are changes between NS and S we are interested in the ones where the two answers are not numerically equal.

Number of Problems Numerically Unequal

	NS	S	Ur2	Ur1	Su2	
c	incorrect	incorrect	293	470	331	
d	correct	incorrect	291	277	374	
e	incorrect	correct	79	113	148	
			663	860	853	Totals

Number of Students with Numerically Unequal Answers

	Ur2	Ur1	Su2
c,d,e	134	140	216
c,d	129	134	205

Errors between the NS and S answer (NS ≠ S)

	% of students	number of ≠ answers over problems with changes
Ur2	$\frac{134}{171} = 78\%$	$\frac{663}{4194} = 16\%$
Ur1	$\frac{140}{148} = 95\%$	$\frac{860}{2996} = 29\%$
Su2	$\frac{216}{273} = 79\%$	$\frac{853}{6135} = 14\%$

% of Problems with Changes between NS and S that Fit the Following Three Items

	Ur2	Ur1	Su2
c	$\frac{293}{4194} = 7\%$	$\frac{470}{2996} = 16\%$	$\frac{331}{6135} = 5\%$
d	$\frac{291}{4194} = 7\%$	$\frac{277}{2996} = 9\%$	$\frac{374}{6135} = 6\%$
e	$\frac{79}{4194} = 2\%$	$\frac{113}{2996} = 4\%$	$\frac{148}{6135} = 2\%$

The errors in e seem to be caused by the student dealing with the fractions first and adding on the whole number part for the final answer. Although the student was unaware that we were interested in the nonsimplified answer it shows sloppy work on the part of the student and an erroneous concept of an equality.

ex: $\frac{19}{18} = 7\frac{1}{18}$

$\frac{19}{18}$ does not equal $7\frac{1}{18}$ although the latter is a correct answer to the problem number forty eight.

We turn our analysis to the problems with changes that have a numerically incorrect answer in the final form.

Eleven erroneous rules were used to identify simplification errors. These rules are not totally exclusive, one problem may appear in more than one rule type.

R1: In simplifying a mixed number the original whole number is omitted

ex: $4\frac{5}{2} = 2\frac{1}{2}$

$3\frac{6}{9} = \frac{2}{3}$

The student deals only with converting and/or reducing the fraction part and shows an erroneous concept of an equality.

	Ur2	Ur1	Su2
Students	$\frac{27}{129} = 21\%$	$\frac{32}{134} = 24\%$	$\frac{31}{205} = 15\%$
Problem	$\frac{39}{584} = 7\%$	$\frac{64}{747} = 9\%$	$\frac{57}{705} = 8\%$

R2: In simplifying an answer with an improper fraction part the original whole number is recorded and the new whole number from the improper fraction part is omitted.

	Ur2	Ur1	Su2
Students	$\frac{22}{129} = 17\%$	$\frac{20}{134} = 15\%$	$\frac{32}{205} = 16\%$
Problem/ Answers	$\frac{36}{584} = 6\%$	$\frac{21}{747} = 3\%$	$\frac{40}{705} = 6\%$

Some of the problem answers could be examples of R1 or R2. In problem number five the answer could be

ex: $1\frac{27}{14} = 1\frac{13}{14}$

The original whole number and the new whole number from simplifying the fraction are the same number, in this case a one. In problem nine the repeated number is a 2.

ex: $2\frac{16}{7} = 2\frac{2}{7}$

If there are not other clear examples in the students paper, a discussion with the individual student might reveal which simplification error was in use.

R3: In simplifying the improper fraction part of the answer, the new whole number becomes the numerator, ignoring any remainder, with the same denominator. The original whole number part, if any, remains unchanged.

ex: $5\frac{18}{6} = 5\frac{3}{6} = 5\frac{1}{2}$

$\frac{14}{5} = \frac{2}{5}$

	Ur2	Ur1	Su2
Students	$\frac{15}{129} = 9\%$	$\frac{14}{134} = 10\%$	$\frac{28}{205} = 14\%$
Problem/ Answers	$\frac{23}{584} = 4\%$	$\frac{20}{747} = 3\%$	$\frac{49}{705} = 7\%$

There are a few examples where an answer could be R1, R2, or R3.

ex: $2\frac{16}{7} = 2\frac{2}{7}$

The problem leads to an answer involving three 2's, the original whole number, the new whole number from the conversion of the fraction and the new numerator. In this case the nonsimplified answer is correct but the simplified answer is incorrect.

R4: In simplifying an improper fraction divide the denominator by the numerator as usual but use the numerator as the new denominator.

ex: $\frac{7}{5} = 1\frac{2}{7}$

R4 seems to be exclusive in occurence with the problem answers not appearing in more than one bug.

	Ur2	Ur1	Su2
Students	$\frac{11}{129} = 9\%$	$\frac{4}{134} = 3\%$	$\frac{18}{205} = 9\%$

Problem/ Answers	$\frac{18}{584} = 3\%$	$\frac{7}{747} = 1\%$	$\frac{20}{705} = 3\%$

R5: In simplifying a fraction the fraction is first inverted and then usually reduced.

ex: $\frac{10}{5} = \frac{5}{10} = \frac{1}{2}$

$5\frac{18}{6} = 5\frac{6}{18} = 5\frac{1}{3}$

This like R4 seems to be exclusive in occurrence.

	Ur2	Ur1	Su2
Students	$\frac{15}{129} = 12\%$	$\frac{20}{134} = 15\%$	$\frac{34}{205} = 17\%$
Problem/ Answers	$\frac{27}{584} = 5\%$	$\frac{51}{747} = 7\%$	$\frac{65}{705} = 9\%$

R6: In simplifying a mixed number, all parts are divided by the same number.

ex: $2\frac{16}{14} = 1\frac{8}{7}$

	Ur2	Ur1	Su2
Students	$\frac{1}{129} = 1\%$	$\frac{4}{134} = 3\%$	$\frac{9}{205} = 4\%$
Problem/ Answers	$\frac{2}{584} = .3\%$	$\frac{8}{747} = 1\%$	$\frac{14}{705} = 2\%$

c (NS ≠ S): Both answers for this problem are incorrect and numerically unequal.

ex: $5\frac{4}{10} = 5\frac{1}{2}$

$5\frac{4}{10}$ is the wrong answer for problem number thirteen and then the reduction to 5 1/2 is incorrect.

b (Blanks for NS and S): No answer attempted or reached

ex: NS $= 99\frac{99}{99}$

S $= 99\frac{99}{99}$

a (Letter to letter correspondence between the NS and S answer including the blank answers of b.)

ex: NS = $5\frac{18}{6}$

S = $5\frac{18}{6}$

R7: The smaller numerator divided into the larger denominator but the denominator stays the same in the final answer.

ex: $\frac{30}{35} = 1\frac{5}{35}$

$\frac{25}{70} = 2\frac{20}{70}$

	Ur2	Ur1	Su2
Students	$\frac{2}{129}$ = 2%	$\frac{6}{134}$ = 4%	$\frac{6}{205}$ = 4%
Problem/ Answers	$\frac{3}{584}$ = .5%	$\frac{14}{747}$ = 2%	$\frac{10}{705}$ = 1%

R8: In simplifying an improper fraction, the new whole number is the numerator minus any remainder after the denominator is divided into the numerator. The fraction part is handled normally.

ex: $\frac{54}{12} = 48\frac{6}{12}$

$\frac{61}{14} = 56\frac{5}{14}$

	Ur2	Ur1	Su2
Students	$\frac{4}{129}$ = 3%	$\frac{3}{134}$ = 2%	$\frac{6}{205}$ = 3%
Problem/ Answers	$\frac{4}{584}$ = .7%	$\frac{3}{747}$ = .4%	$\frac{7}{705}$ = 1%

R9: Any conversion of an improper fraction is a one.

ex: $2\frac{16}{7} = 3\frac{2}{7}$

$\frac{10}{5} = 1$

$2\frac{11}{5} = 3\frac{1}{5}$

	Ur2	Ur1	Su2
Students	$\frac{30}{129}$ = 23%	$\frac{17}{134}$ = 13%	$\frac{32}{205}$ = 16%

Problem/ Answers	$\frac{40}{584} = 7\%$	$\frac{23}{747} = 3\%$	$\frac{38}{705} = 5\%$

This, of course, is not counted if the conversion should be one.

R10: In simplifying a fraction where the numerator and denominator are equal the resulting whole number is the value of the numerator but not a one.

ex: $\frac{4}{4} = 4$

	Ur2	Ur1	Su2
Students	$\frac{2}{129} = 2\%$	0	$\frac{1}{205} = .5\%$
Problem/ Answers	$\frac{2}{584} = .3\%$	0	$\frac{16}{705} = 2\%$

R11: In reducing a fraction the gcd (greatest common denominator) is found and divided into the numerator as usual but the gcd becomes the denominator.

ex: $1\frac{5}{10} = 1\frac{1}{5}$

$\frac{5}{15} = \frac{1}{5}$

	Ur2	Ur1	Su2
Students	$\frac{13}{129} = 10\%$	$\frac{16}{134} = 12\%$	$\frac{10}{205} = 5\%$
Problem/ Answers	$\frac{17}{584} = 3\%$	$\frac{19}{747} = 3\%$	$\frac{16}{705} = 2\%$

d (NS $\neq$ S): The NS answer was correct but the S was incorrect. The two answers are numerically unequal.

ex: $\frac{26}{36} = \frac{13}{16}$

$\frac{26}{36}$ iss a correct answer to problem number thirty eight but $\frac{13}{16}$ is an incorrect reduction of that answer. It should be $\frac{13}{18}$.

e (NS $\neq$ S) The NS and S answers are numerically unequal but the S answer is correct in numerical terms.

ex: $\frac{19}{18} = 7\frac{1}{18}$

The above example is a correct answer for problem number 48.

ex: $\frac{18}{6} = 8$

This is a correct answer for problem 1 but $\frac{18}{6}$ is not.

f: In either NS or S the fraction has a zero in the denominator of the fraction.

ex: $5\frac{18}{0} = 5$

$5\frac{18}{6} = 8\frac{3}{0}$

Analysis of Simplification Errors: Subtraction Data

We are looking at simplification errors between the nonsimplified (NS) and simplified (S) answer regardless of the correctness in numerical terms of either answer.

In looking at the subtraction fraction problems the problems lead to fewer necessities for simplification in the answer than with the addition problems. The number of one answer problems therefore is greater. Within this smaller group of problems with answers that show a change between the NS and S and are numerically unequal, we see many of the same errors appearing at the same rate in relation to the smaller sample of changes.

For example with the Urbana addition data taken in the Fall of 82, 51% of the problems had changes between the NS and S answer. Some form of simplification was attempted. But with the Urbana subtraction data taken at the same time only 19% of the problems had any changes. This seems reasonable when we look at the difference between addition and subtraction and the particular problems involved in each test.

One Answer Problems

	# Students in group (n)	Repeat Answers over Total problems (nx48)
Ur2	167	$\frac{5684}{7014} = 49\%$
Su2	230	$\frac{7616}{9660} = 79\%$
Ur1	139	$\frac{4809}{5838} = 82\%$

Blank Answers

	% students	% total problems	% problems for students with blanks
Ur2	$\frac{49}{167} = 29\%$	$\frac{531}{7014} = 8\%$	$\frac{531}{2058} = 26\%$
Su2	$\frac{94}{230} = 41\%$	$\frac{1223}{9660} = 13\%$	$\frac{1223}{3948} = 31\%$
Ur1	$\frac{81}{139} = 58\%$	$\frac{600}{5838} = 10\%$	$\frac{600}{3402} = 18\%$

Answers with Changes Between NS and SI

Ur2 $\frac{1330}{7014} = 19\%$

Su2 $\frac{2044}{9660} = 21\%$

Ur1 $\frac{1029}{5838} = 18\%$

Number of Students with Numerically Unequal Answers

		Ur2	Su2	Ur1
IICZ	c,d,e,f	85	105	79
II	c,d	77	93	70

Number of Problems with Numerically Unequal Answers

		Ur2	Su2	Ur1	
II	c	149	128	149	
CI	d	60	68	69	
		(209)	(196)	(218)	
IC	e	40	27	23	
ZE	f	78	27	23	
		327	292	260	Totals

Numerically Unequal Answer Between NS and S

	Students	Number of ≠ Answers over problems with changes
Ur2	$\frac{85}{167} = 51\%$	$\frac{327}{1330} = 25\%$
Su2	$\frac{105}{230} = 46\%$	$\frac{292}{2044} = 14\%$
Ur1	$\frac{81}{140} = 58\%$	$\frac{260}{1040} = 25\%$

Percent of Problems with Changes Between NS and S that Fit the Following Three Patterns

	Ur2	Su2	Ur1
NS answer-Si answer			
incorrect ≠ incorrect	149/1330 = 11%	128/2044 = 6%	149/1029 = 14%
correct ≠ incorrect	60/1330 = 5%	68/2044 = 3%	69/1029 = 7%
incorrect ≠ correct	40/1330 = 3%	27/2044 = 1%	23/1029 = 2%
zero denominator in NS or S answer	78/1330 = 6%	69/2044 = 3%	19/1029 = 2%

Summary of Simplification Errors Found in Subtraction Data

	Ur2 Students	Ur2 Problems	Su2 Students	Su2 Problems	Ur1 Students	Ur1 Problems
R1	10/77 = 13%	15/209 = 7%	4/93 = 4%	6/196 = 3%	10/70 = 14%	19/218 = 9%
R2	3/77 = 4%	6/209 = 3%	5/93 = 5%	6/196 = 3%	9/70 = 13%	11/218 = 5%
R3	4/77 = 5%	9/209 = 4%	1/93 = 1%	1/196 = .5%	8/70 = 11%	9/218 = 4%
R4	5/77 = 6%	5/209 = 2%	11/93 = 12%	11/196 = 6%	3/70 = 4%	4/218 = 2%
R5	6/77 = 8%	10/209 = 5%	9/93 = 10%	14/196 = 7%	6/70 = 9%	10/218 = 5%
R6	4/77 = 5%	4/209 = 2%	2/93 = 2%	2/196 = 1%	1/70 = 1%	1/218 = .5%
R7	5/77 = 6%	6/209 = 3%	2/93 = 2%	2/196 = 1%	1/70 = 1%	1/218 = .5
R8	1/77 = 1%	1/209 = .5%	1/93 = 1%	3/196 = 2%	1/70 = 1%	1/218 = .5%

R9	7/77 = 9%	9/209 = 4%	4/93 = 4%	4/196 = 2%	8/70 = 11%	10/218 = .5%
R10	2/77 = 3%	3/209 = 1%	0	0	0	0
R11	3/77 = 4%	3/209 = 1%	4/93 = 4%	4/196 = 2%	3/70 = 4%	3/218 = 1%

Fraction Subtraction Errors : Case Studies

A 42-item test of fraction subtraction problems designed to diagnose erroneous rules as projected in Klein, et al., (1981) was given to 139 junior high school students. The free response test allowed examination of the sudent's wrong answers by several methods of error analysis. The paper and pencil test was scored by teachers and members of the research group. The student responses were then entered into both the PLATO computer at the University of Illinois and a set of analysis programs implemented in Pascal on the Apple II+ computer. A description of these micro-computer programs is presented in another section. Thirty-two of the test items, those not involving whole numbers as either the minuend or the subtrahend, were used for the micro-computer sudy. Various calculational procedures as well as searching techniques were used to select cases which illustrate the error patterns and interpretations, as well as the problem-solving methods used by the students.

The continuing theme that is illustrated in these studies is that obtaining high scores on a test of mathematics problems does not indicate that the student has developed a high degree of understanding of mathematical concepts. A student who uses erroneous rules may reveal knowledge that another student who earns a higher score does not demonstrate. Such information may be obtained from examination of methods of problem-solving and from evidence of the perception level necessary to arrive at certain error patterns. The explanation of these relationships will be discussed as the case studies are presented.

The Meaning of Fractions

Case 1. Some students do not recognize the fraction as an specific quantity. A few try to ignore the fraction part in a mixed fraction. Bea writes that:

$$4\ 4/12 - 2\ 7/12 = 2 \quad \text{and} \quad 3\ 3/8 - 2\ 5/6 = 1$$

She used this rule for 15 items, all mixed fractions. Her misconception concerning the meaning of a fraction was also reflected in her incorrect procedure for subtracting simple fractions. She added the denominators and subtracted the numerators:

5/3 - 3/4 = 2/7 and 6/7 - 4/7 = 2/14 = 1/7

The concept that the written form of a fraction represents a particular quantity requires that the student be able to visualize the partitioning of a unit into any number of equal parts. The unit may be expressed as 5/5, 10/10, or 127/127. The denominator indicates the number of parts that the unit is to be divided into and the numerator represents the number of those parts the quantity includes. The student might say in interpretation of "2/3", "If a unit (one) were divided into three parts, this amount is two of those parts". Erroneous rules often arise when the sudent thinks of the fraction as a "2" and a "3", each to be treated individually according to some procedure. Many of these wrong rules result in wrong answers; however, occasionally they result in correct answers for certain items.

Case 2. Rosemary used the most common erroneous rule to solve fraction subtraction problems. She subtracted the smaller from the larger number in the corresponding parts of the two fractions:

4 4/12 - 2 7/12 = 2 3/0

She had one correct answer on the test:

4 3/5 - 3 4/10 = 1 1/5

That correct answer results from using her erroneous rule on this item. This rule seems to result from a lack of understanding that the denominator of a fraction symbolizes the size parts required by the problem as well as from the smaller from larger rule (Brown and Burton, 1978). This error has been attributed to an "...attempt to commute subtraction problems" (Resnick, 1983).

Case 3. Another common erroneous rule which often results in correct answers is the one in which a student subtracts the smaller from the larger in the numerators, but keeps corresponding equal denominators in the answer. The data shows that Andy used this procedure:

4 4/12 - 2 7/12 = 2 3/12 = 2 1/4 , and

5 3/15 - 3 8/15 = 2 5/15 = 2 1/3

He also quite likely used the same erroneous rule on the 12 items in which the larger numerator was contained in the first fraction:

3/4 - 2/4 = 1/4 and 4 5/7 - 4 3/7 = 2/7

If those correct items had been scored wrong, assuming that he followed an incorrect algorithm, his total score would have been 10 rather than the 22 which was recorded.

Of course, it is possible that Andy does understand that fractions represent quantities, he does calculate common denominators, but that he has just forgotten the procedure that would put the minuend into a form that would allow subtraction of the subtrahend. We cannot tell from his responses.

The Relationship of Method of Problem-Solving and Score

Another type of response is even more difficult to interpret. Some teachers teach students to convert all mixed fractions to improper fractions as the first step in solving fraction subtraction problems. The method is often called by the name "around the world". The student begins with the whole number, multiplies by the denominator, and then adds the numerator. To convert 3 3/5 he might say, "Begin with three, around to five, three times five equals fifteen, go around to three, fifteen plus 3 equals eighteen over five." This procedure sometimes results in large numerators and is time consuming, but if the student works carefully and then uses the procedure of subtracting the smaller numerator from the larger numerator while keeping common denominators, all items with like denominators will be correct. This method of work is called "Method A" by Klein. It is usually possible to tell whether a student has used this method from the scratches on his paper and often from his first, unsimplified, response.

Case 4. Jonathon received the second higest score of the students tested using Method A:

4 3/5 - 3 4/10 = 23/5 - 34/10 = 46/10 - 34/10 = 12/10 = 1 1/5, and

3 3/8 - 2 5/6 = 27/8 - 17/6 = 81/24 - 68/24 = 13/24

He was able to get all but one problem correct, but there is some question about the level of mathematical development that he had attained.

In an attempt to determine whether the selection of a particular method for solving a particular problem could help diagnose student erroneous rules, a comparison was made between the students' use of method A and the students' scores. It was first felt that the selection of the most appropriate method for solving each item, perhaps as a response to some "impasse" (Brown and VanLehn, 1980), might lead to higher scores because the choice of Method A does result in right answers. The data shows that the students in the high scoring third of the sample group had 56% of the uses of Method A. The middle third had 25% and the low scoring third had only 17% of the uses of that method. The use of method A did seem to be associated with obtaining a high score.

Not only did the higest scoring students use method A the most frequently, but they often selected different items on which to use it. At this point it seemed as if a correlation of the frequency of the use c Method A with the number of correct responses for the item might show which items were most appropriate for the use of method A; however, there was no pattern of selection that seemed more appropriate in any of the ability groups. The item with the highest correlation was selected by the lower groups and the item with the lowest (actually sligtly negative) correlation was selected frequently by all groups. It appeared that a student's selection of Method A was unrelated to the characteristics of the item, such as the need for "borrowing".

Borrowing as releted to Problem-Solving Method

Fourteen of the test items were designed to test the ability to "borrow". That ability requires the student to change the form of the minuend so that the subtrahend can be subtracted. In all but two of the items, the two fractions had the same denominators, but the numerators of the second fractions were larger than the numerators of the first. The student should be able to convert as many units as needed from the first fraction to the size parts needed for the problem and then perform the subtraction. In the problem 4 1/3 - 1 5/3 the student should be able to say, "Five thirds is more than one third, so I can't subtract without borrowing. I may change one of the units in the four into three thirds and add it to the one third of the first fraction. That makes three and four thirds and I still cannot subtract five thirds. I must change another unit into thirds and now I have two and seven thirds. Seven minus five equals two; the answer is one and two thirds." A student who can demonstrate the ability to solve borrowing problems in this way has procedural skill as well as an understanding of the meaning of fractions. However a student may get correct answers on those problems by correctly using the Method A procedure and not demonstrate borrowing skills or underlying concepts.

Case 5. One way to find out whether a student who uses that method understands what the fraction means and has chosen that procedure for changing the fraction to allow subtraction might be to find out whether the need to increase the first numerator prompts the student to use Method A. Celeste used Method A when borrowing was needed:

$$7\ 3/5 - 4/5 = 38/5 - 4/5 = 34/5 = 6\ 4/5$$

but when the second numerator was smaller she wrote:

$$4\ 5/7 - 1\ 4/7 = 3\ 1/7$$

She used Method A only when the form of the first fraction required changing.

Upon examination of the data, it was found that of the 17 stdents who used Method A to solve more than 2/3 of the items requiring borrowing, only three of them failed to use Method A on some item not requiring borrowing. Ten of them used Method A on more than two thirds of the items which did not require borrowing.

Case 6. While Ken used Method A in all the problems requiring borrowing:

$$4\ 4/12 - 2\ 7/12 = 52/12 - 31/12 = 21/12 = 1\ 9/12 = 1\ 3/4$$

he also used that method in all the mixed fraction problems that did not require borrowing:

$$3\ 4/5 - 3\ 2/5 = 19/5 - 17/5 = 2/5$$

Since the use of Method A often does not seem to be dependent upon the need for borrowing in the particular item, we do not know from its use whether the student recognizes that need or not.

Case 7. Many of the students who do not use Method A make discriminations which indicate their awareness of the need for borrowing in problems with the smaller numerator in the first fraction. The most common erroneous rule of that type is illustrated by Juanita. She did not convert mixed fractions to improper fractions and has correct answers for the items with the larger numerator in the minuend, however when it is necessary to increase the numerator of the first fraction, she reduces the whole number by one unit and adds 10 to the numerator of the fraction:

4 4/12 - 2 7/12 = 3 14/12 - 2 7/12 = 1 7/12

The use of this rule indicates that the student recognizes the need for borrowing but does not understand the concept that a unit must be divided into the number of fractional parts as determined by the denominator required in the problem. In addition, it indicates familiarity with the procedure used for borrowing in subtraction of multidigit whole numbers. The student may only need to extend his understanding of numbers as compositions of tens and units (Resnick, 1983) to the idea that fractional parts may be other than ten. That erroneous rule does give the correct answer when the fraction is partitioned into ten parts:

4 1/10 - 2 8/10 = 3 11/10 - 2 8/10 = 1 3/10

Several variations of procedural errors were made by students who perceived that borrowing was necessary.

Case 8. When borrowing from a whole number or the whole number part of a fraction, John did not change the whole number even though he correctly increased the numerator:

3 3/15 - 3 8/15 = 5 18/15 - 3 8/15 = 2 2/3

Case 9. Dave added rather than subtracted from the whole number part of a mixed number when borrowing:

4 4/12 - 2 7/12 = 5 16/12 - 2 7/12 = 3 3/4

Case 10. Lisa subtracted from the numerator part of the fraction part as well as from the whole number part when borrowing from a mixed number:

4 4/12 - 2 7/12 = 3 15/12 - 2 7/12 = 1 8/12

Errors of this type result in wrong answers but do give information that the student has reached the level of understanding at which he recognizes that the number of fractional parts in the first fraction must be increased before subtraction is possible.

Common Denominator Errors

The practice of keeping or calculating a common denominator may be a result of following a rote procedure or having knowledge of the concept of needing the same number of parts in each fractional quantity in order to add or subtract. Some indication may be obtained by examining the student's responses on the items which do not have like denominators.

Case 11. Paula kept the denominator when they were the same, but selected the one closer to the equal sign if they were different:

$$5/6 - 1/9 = 4/9 \quad \text{and} \quad 4\ 4/9 - 3\ 5/6 = 1/6$$

She also consistently omitted the whole number part of the answer. There is evidence that she does not understand the meaning of the denominator even though she follows the correct procedure of keeping common denominators in the answer.

Case 12. Brian solved the common denominator problem by using the denominator of the first fraction in the answer:

$$5/3 - 3/4 = 2/3 \quad \text{and} \quad 5/3 - 5/6 = 0/3$$

Case 13. Michelle used the largest denominator of the two fractions as the denominator of the answer:

$$5/6 - 1/9 = 4/9 \quad \text{and} \quad 3\ 3/8 - 2\ 5/6 = 1\ 2/8$$

Case 14. Sometimes a student seems to recognize the need for the same denominator in each fraction but does not recognize the necessity of changing the numerator also in order to keep the quantity property of equivalence in the fractions. Ben had this problem in several test items:

$$5/3 - 3/4 = 2/12 = 1/6 \quad \text{and} \quad 4\ 4/9 - 3\ 5/6 = 1\ 1/18$$

Notice that he subtracts smaller from larger in the numerators.

These solutions show some idea that the denominators should not be added or subtracted, but indicate that the student cannot visualize the denominators as the result of partitioning of units in each fraction into the same size parts.

Errors in Combining Procedures

The use of Method A does not result in a simple solution to problems that involve calculating a common denominator as well as borrowing. Two of the problems on the test required both of those procedures. If the student converts the fraction without regard for the necessity of borrowing, which seems to be the most frequent case, after he has converted both fractions to improper fractions, he must then find a common denominator and the equivalent fractions. Only three then find a common denominator and the equivalent fractions. Only three of the students who used Method A got both of those problems correct.

Case 16. Deren had trouble with the calculations involved, making an error in equivalent fractions:

4 4/9 - 3 5/6 = 72/18 - 58/18 = 14/18 = 7/9

Case 16. Brad subtracted unlike denominators:

3 3/8 - 2 5/6 = 27/8 - 17/6 = 10/2 = 5

Case 17. Louis, who usually used Method A, found the common denominator in this case and then subtracted the smaller from the larger:

3 3/8 - 2 5/6 = 3 9/24 - 2 20/24 = 1 11/24

Case 18. Chandra inverted the second fraction, cancelled, and then subtracted corresponding parts when faced with unlike denominators:

4 4/9 - 3 5/6 = 40/9 - 6/23 = 40/3 - 2/23 = 38/20

At least the last three of these errors result from underlying misconceptions. Students who do not normally use Method A also made errors on those problems.

Case 19. Jimmy subtracted denominators when the problem required both borrowing and calculating a common denominator:

3 3/8 - 2 5/6 = 2 11/8 - 2 5/6 = 6/2 = 3

Case 20. Greg, Craig, John, and Theresa all solved problem 34. this way:

4 4/9 - 3 5/6 = 2/18 = 1/9

All of these students had written in the number 17 above the first fraction. Upon questioning, all of them said that the 17 came from adding 8 + 9. They apparently had realized that both borrowing and calulating a common denominator would be required even before they had written the common denominator. After multiplying the 4 by 2 (common denominator calculation), they had added the fraction to 9/9 (borrowing conversion) and had neglected the rest of the calculation. In order to know that the borrowing would be needed before doing the common denominator calculation, the student must have excellent understanding of concepts concerning fractions and also know the procedures involved in fraction manipulation. The application of such complicated processes did result in wrong answers.

Problems with Zero and One

Case 21. Curtis always obtained 1 when he subtracted a number from itself:

2/3 - 2/3 = 1 and 6/7 - 4/7 = 2/1

He also answered 1 when the difference was 1:

3/4 - 2/4 = 1/1

He occasionally kept a common denominator, but he usually subtracted them with the a resulting denominator of 1 in the answer. His procedure as well as his understanding of fractions is faulty.

Some students follow a practice which results in eliminating zero from all answers. This could stem originally from counting errors such as beginning with the wrong number (or finger) when using the decrementation method of subtraction, but with the 13 to 15 year old students in the sample it would more likely be a method to cope with the fact that answers with zero in the denominator or the numerator look unfamiliar or illegal.

Case 22. Nicole followed the practice of omitting a zero whether it occurred in the numerator or the denominator:

3/4 - 3/8 = 0/4 = 4 and 6/7 - 4/7 = 2/0 = 2

The only correct answers that occurred on her paper were found on the items which subtracted whole numbers from mixed fractions and the ones that resulted in zero, items which she apparently solved in her head.

Case 23. Chris arranged not to have denominators of zero by using a cross inversion method:

13/9 - 1/9 = 13/1 - 9/9 = 4/8 and

4 1/10 - 2 8/10 = 4 10/10 - 2 8/1 = 2 2/9

This procedure was only used when the numerator was 1. Otherwise he usually kept a common denominator. When a numerator was calculated as zero, he omitted it and used the denominator as a whole number:

3/4 - 3/8 = 4 and 5/3 - 5/6 = 3

Unusual Erroneous Rules

Case 24. A curiour result of incorrect application of a procedure is found in the work of Michael. He seems to have a goal of making the numerators or the denominators in the fractions agree and then keeping that number in the answer:

5/3 - 3/4 = 5/3 - 4/3 = 1/3 (by inversion) and

5/6 - 1/15 = 1/6 - 1/3 = 1/3 (by cancellation).

The problem 2/3 - 2/3 is easy for him; the answer is 2/3. He knows several procedures and has an objective, but displays a lack of understanding about the meaning of fractions.

Case 25. Rana had no trouble calculating common denominators or converting a whole number to a mixed fraction for the purpose of borrowing, but she was confused when she needed to borrow from the whole number part of a mixed number. She often added all the digits in the mixed number to obtain the numerator of the converted mixed fraction.

4 1/10 = 3 15/10, 4 1/3 = 3 8/3, 4 4/12 = 3 20/12

Case 26. An incorrect procedure which indicates that the student is not thinking of fractions as quantities is the one that Joe uses. He cross subtracts for the numerator and denominator of the answer. Usually, he arranges the answer so the numerator is smaller than the denominator:

4 4/12 - 2 7/12 = 2 5/8 and 11/8 - 1/8 = 3/7

While the same answers would result for problems with like denominators if he subtracted within each fraction, he followed the cross subtraction rule also for those with unlike denominators:

5/6 - 1/15 = 5/10 = 1/2 and 4 4/9 - 3 5/6 = 1 2/4 = 1 1/2

Case 27. Becky used a combination of erroneous rules that illustrate her underlying misconceptions about fractions. In mixed fractions she subtracted the whole numbers for the numerator and added all the fraction parts in both fractions for the denominator:

4 4/12 - 2 7/12 = 2/35

For simple fractions, she subtracted the numerators for the numerator and the denominator was always zero. This treatment of the denominator for like denominator problems gave the same answers as the students who subtracted corresponding parts, but the zero in the case of fractions which had unlike denominators suggests that she used a more general rule. Her procedures seem to be erroneous rules for treating a set of unrelated digits.

Sometimes a student's work may indicate more than a lack of knowledge or concepts.

Case 28. Steve's errors were only 31 percent diagnosed by the computer program and many of those matches seemed to be chance occurrences. He used different rules for almost every item and some of them were unique.

For example:

5/3 - 3/4 = 16/9 = 1 7/9

From the scratches on his paper it was discovered that the 9 was the product of 3 x 3 and the 16 was obtained from 5 x 4 - 4.

48/25 = 1 23/25 and 42/28 = 1 14/28 = 1 1/2

Other examples of Steve's incorrect answers for subtraction of fractions were:

3 1/2 - 2 3/2 = 5 2/1 - 1 1/1 = 6 1/1

4 4/12 - 2 7/12 = 2 8/5 = 1 3/5

Such response patterns may indicate that the student has poor conceptual and procedural knowledge, or that he has some motivational or attitudinal problem. Steve had penciled in the words "Ha Ha" in the space next to one of his answers. Only interviewing students of this type can help explain such performance.

While the total score has long been used as the measure of a student's knowledge, the cases studied for this report indicate that much more important information can result from consideration of students wrong answers. Information about methods of problem-solving and possible underlying misconceptions can lead to more accurate evaluation and more appropriate remediation.

The summary list of frequencies observed in various subtraction erroneous rules is given below. The number of students who used

Inser Table 8 about here

Rule 2 (subtract the smaller number from the larger, corresponding parts) is dominantly large, 558. The second largest number is Rule 8: Subtract the smaller from the larger number in unequal corresponding parts, but keep equal corresponding parts the same. The third-most popular error is a borrowing error: Reduce the whole number of the minuend by 1 and add one to the tens column of the numerator. The first two errors are also often observed in whole number subtraction and signed number addition and subtraction problems. It seems that the idea of a large number minus a smaller number is deeply rooted in many students past knowledge base.

Table 8

Frequencies of Each Subtraction Rule Observed in
138 Students, 32-Item Fraction Subtraction Test

Rule	Frequencies	Rule	Frequencies	Rule	Frequencies
2	558	11	9	30	108
3	29	12	16	31	38
4	31	18	11	33	11
5	92	19	25	34	15
6	63	20	4	35	7
7	9	21	4	36	16
8	287	22	10	37	8
10	7	25	16	38	8
				39	34

Micro-Computer Programs for Error Diagnosis of Fraction Subtraction

The micro-computer programs were implemented because it was felt that computerized assistance for teachers must be available on micro-computers in order to be useful at the present time. The study included analysis programs, user interfaces including both student on-line tests and editors for data entry and modification, and printed output designed to be useful for teachers as well as for research. Several pilot studies were carried out with public school students and teachers in order to explore the potential of the computer.

The programs were written in UCSD Pascal on the Apple II+ computer. Students and teachers from two school districts, Urbana Junior High and Sullivan secondary schools tried the programs and consulted. The test items used were those designed for error analysis of fraction subtraction (Klein, et al., 1981).

The student data was stored in records containing the student ID, grades in English and math, sex, an attitude index, the whole number, numerator, and denominator of two student responses, and up to 64 Boolean variables available for each test item for matching student responses with erroneous responses. The computer program allowed up to 200 records of this structure to be stored on a data disk. Other data files stored over 500 correct and "buggy" student responses.

For the data files there were editors for data entry and merging student records from data collected on different disks. File modification and printout programs for examination of the data both on screen and hardcopy were implemented. The editors were used by teachers and students for recording data and in addition, a test driver was used for collection of data from on-line testing. Programs calculated the correct answers by decimal value as well as fraction forms according to Method A and Method B.

Other programs identified student-items solved by using Method A, marked the student record for that characteristic and compared the use of Method A with correct answers by student and by item. A program sorted the student responses by item, and recorded the frequency of each response, checking both first and second answer. Another program compared all student responses with the classified wrong responses and marked the student record for the different "bugs" as they occurred in the data.

Great effort was made to design printed output that would be helpful for analysis as well as for teacher assistance. The program prints a matrix by student and item with frequencies of occurrence. The student records were sorted according to total score and the items were sorted according to difficulty. A chart, called an S-P table, was made of the sorted files according to student score and difficulty of item. The items were marked with a "+" if the answer was correct, a space if the item was blank, a "?" or "*" if undiagnosed (depending upon the method used) and with a symbol (see descriptions of errors) to indicate erroneous rules. A sample of some student records printed according to this program is included. For the purpose of analysis, several print programs were designed which allowed comparison of student responses over subsets of items as well as calculations such as the caution index for each student and proportions of different response characteristics.

Originally, 31 misconceptions were used to diagnose the 139 cases in the sample. 23 of them were found to occur in more than one percent of the responses, and six of them accounted for 83 percent of the diagnosed erroneous rules in fraction subtraction. 81 percent of the observed incorrect responses have been diagnosed. Some of the underlying misconceptions were comprised of combinations of errors and some error patterns still defy description. The computer programs proved to be very helpful in selection of case studies for illustration.

Some testing was done of programs with special purposes. An item generator was tested with the objective of comparing generated items to those designed for error diagnosis with respect to parallelism. The results were promising and proved to be much faster than expected, which allows for improvement upon the instructions for generation.

Insert Table 9 about here

Table 9

S-P Bug Table for Subtraction Problems

Items 06–11:

Student Number	Total Score	06	12	08	27	33	14	29	37	09	35	16	05	17	18	04	25	30	36	39	01
24	23	+	+	+	+	+	+	a	+	-	+	+	+	?	+	A	A	-	+	+	+
21	22	+	+	+	+	+	+	+	+	+	+	+	+	g	g	g	?	+	+	g	+
18	19	?	+	+	+	+	+	+	+	-	+	+	A	*	+	+	+	-	-	+	t
6	15	+	?	g	+	+	+	g	+	-	*	*	A	+	+	g	g	-	-	+	a
20	14	+	+	+	+	+	+	?	+	-	+	?	?	?	+	A	A	-	+	+	d
15	13	+	+	+	+	+	+	+	+	+	+	+	c	g	g	g	g	-	+	?	f
14	12	+	+	g	?	+	+	+	+	-	+	*		+	+	g	*	-	+	+	t
10	11	a	+	+	a	+	+	+	*	-	*	+	*	+	*	+	*	-	-	*	a
17	11	+	+	a	+	+	a	+	+	+	+	+	+	g	a	a	g	+	-	g	a
25	1	+	+	+	+	+	+	+	+	+	+	?	c	g	?	?	g	+	-	g	a
9	1	+	t	+	+	e	+	+	a	+	D	+	p	+	g	g	+	-	-	p	a
1	0	+		+						+			+			+		-	-		+
4	0	+	+	a	+	+	+	g	+	+	+	+	e	g	g	a	g	-	-	g	a
16	2	e	e	e	e	y	e	?	e	+	e	e	?	e	e	?	e	+	-	?	e
23	2	?	?	?	e	y	e	e	e	+	e	?	?	?	?	?	?	+	-	?	?
5	0	?					J		J	-	J	J	F	J	J	J	B	-	-	J	b
7	0	a	a	a	a	a	a	a	a	-	a	a	a	a	a	a	a	-	-	a	a
19	0	a	a	a	a	a	k	a	k	-	k		?		k	k	k	-	-	k	j

Items 07–34:

Student Number	Total Score	07	11	15	20	38	02	28	03	10	23	24	32	19	22	26	41	31	40	13	34
24	23	+	?	+	A	A	+	+	+	A	+	+	A	-	+	+	A	?	-		?
21	22	-	a	+	g	g	+	-	+	?	+	+	g	-	+	+	?	A	-	I	I
18	19	-	+	-	+	+	g	-	?	+		?	F	-	t	J	+	+	-	*	*
6	15	-	+	-	+	+	g	-	a	+	g	?	+	-	a	a	+	+	-	a	a
20	14	+	?	+	A	A	a	+	d	A	a	a	?	-	d	a	?	?	-	?	?
15	13	-	g	-	g	g	c	+	?	?	c	?	g	-	?	?	g	A	-	?	?
14	12	-	+	-	*	*	?	-	?	*	s	x	*	-	t		*		-	*	a
10	11	-	E	+	+	+	a	-	a	a	a	a	*	-	a	a	G	*	-	a	*
17	11	-	g	-	g	a	a	-	a	?	a	a	a	-	a	a	g	A	-	a	a
25	1	-	?	-	g	g	c	-	c	g	c	?	g	-	c	?	g	A	-	c	?
9	1	-	*	-	*	?	g	-	a	*	?	a	a	-	?	D	p	a	-	?	
1	0	+		-			+	-	+	G				-					-		
4	0	-	a	-	g	g	a	-	a	a	a	a	g	-	a	a	g		-	?	a
16	2	-	e	-	A	?	e	-	a	e	e	e	?	-	e	e	?	e	-	?	e
23	2	-	?	-	?	?	a	-	a	?	?	?	A	-	?	A	?	?	-	?	?
5	0	-	B	-	a	J		-	b	J			B	-		J	J	J	-	J	J
7	0	-	a	-	a	a	a	-	a	*	a	a	a	-	a	a	a	a	-	a	a
19	0	-	k	-	k	?	j	-	j	k	j	j	k	-	j		k		-	k	k

Bug Descriptions

a (2)	Smaller from larger, corresponding parts.
b (3)	Smaller from larger numerator, add denominators.
c (4)	Smaller from larger numerator, use largest denominator.
d (5)	Smaller from larger numerator, use first denominator.
e (6)	Cross subtract.
g (8)	Smaller from larger in unequal parts, keep equal parts in answer.
j (11)	Smaller from larger numerator, denominator = 0.
k (12)	Subtract whole numbers for numerator, add all fraction parts for denominator.
A (30)	Reduce whole number of minuend by 1 and add 1 to the tens column of the numerator when borrowing.
B (31)	Borrow but do not reduce the value of the whole number of the minuend.
D (33)	Calculate improper fractions by Method A and then drop denominators.
J (39)	Subtract whole numbers and omit fraction parts from the answer.
+	Correct answer.
–	Item not used.
?	Undiagnosed, Method B.
*	Undiagnosed, Method A.
blank	No answer

The numerals and letters are from the list of erroneous rules of operation (bugs) given in the Appendix.

Such item generation would be desirable in remediation programs. A student input program which would encourage students to "show their work" much as they do on paper and pencil tests was tested. The purpose was to allow the computer to save their steps in problem-solving. The results have not been satisfactory since the student interface is too complicated for easy use. Probably some additional item type is needed in order to test method of work.

While it was necessary to begin with actual data in order to test the micro-computer capacity, it now appears feasible to generate wrong responses and store them which would make the program useful for different items. The search can be done rapidly enough to make it reasonable to match errors and record them during on-line testing. This makes adaptive-testing a possibility for use with micro-computers and as technology advances it is almost a certainty that it can be done.

Summary

A painstaking error analysis and construction of buggy programs were carried out and summary statistics were described in this report. The analysis results indicate that individual differences in applying different strategies and procedural skills varied more among students than we had expected. Many erroneous rules are committed by students who used them sporadically. These rules are often observed only once per student and never used repeatedly by the same individual. Various error types, i.e., sources of misconceptions, cover almost all the levels of tasks involved in solving fraction problems.

A close examination of frequency distributions of erroneous rules revealed that some errors tend to appear among high-score students while others appeared only among low-score students. Systematic investigation of "bug-behaviors" will lead to further understanding in human cognition and learning and thus it will bring about further improvement in American education.

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