DOCUMENT RESUME

ED 257 640 SE 045 576

AUTHOR Verhille, Charles, Ed.
TITLE Proceedings of the 1984 Annual Meeting of the Canadian Mathematics Education Study Group (8th, Waterloo, Ontario, June 2-6, 1984).
INSTITUTION Canadian Mathematics Education Study Group.
PUB DATE Nov 84
NOTE 91p.; Portions may be marginally legible due to small print. For earlier proceedings, see ED 188 872, ED 204 120, ED 234 988-989, and ED 243 653.
PUB TYPE Collected Works - Conference Proceedings (021)

EDRS PRICE MF01/PC04 Plus Postage.
DESCRIPTORS Algebra; Computers; Course Descriptions; Elementary Secondary Education; *Mathematics Curriculum; *Mathematics Education; *Mathematics Instruction; *Microcomputers; Research Utilization
IDENTIFIERS *Canada

ABSTRACT

Papers from the eighth annual meeting of the Canadian Mathematics Education Study Group are presented, beginning with a lecture by Alan Bishop on "The Social Construction of Meaning--A Significant Development for Mathematics Education." Also included are reports of four working groups: "LOGO and the Mathematics Curriculum" (Dale Burnett and William Higginson); "The Impact of Research and Technology on School Algebra Curricula" (Carolyn Kieran and Thomas Kieren); "Epistemology and Mathematics" (Maurice Belanger and David Wheeler); and "Visual Thinking in Mathematics" (Tony Thompson and John Mason). A panel of speakers (David Alexander, Michael Silbert, Dale Drost, and Claude Gaulin) discussed the general trends of current curriculum reforms in school systems in three Canadian provinces. The discussion of a second panel (Peter Taylor, John Poland, and Keith Geddes) on the impact of computers on undergraduate mathematics is briefly summarized. Subjects included conclusions of a report based on earlier Study Group discussions, a college's commitment to the use of computers in first-year courses, and the use of software for the exact manipulation of matrices and functions. A digest of a paper by George Davis, "A Microcomputer for Every Student," is appended. Also included are accounts of two courses with a historical flavor: "Famous Problems in Mathematics: An Outline of a Course" by Israel Kleiner and "Intellectual Respectability--A Historical Approach" by Abe Schenitzer. A list of participants concludes the document. (MNS)

GROUPE CANADIEN D' ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1984 ANNUAL MEETING

UNIVERSITY OF WATERLOO

WATERLOO, ONTARIO

JUNE 2-6, 1984

EDITED

BY

CHARLES VERHILLE

CMESG/GCEDM

NOVEMBER 1984

GROUPE CANADIEN D' ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1984 ANNUAL MEETING

UNIVERSITY OF WATERLOO

WATERLOO, ONTARIO

JUNE 2-6, 1984

EDITED

BY

CHARLES VERHILLE

CMESG/GCEDM NOVEMBER 1984

CMESG/GCEDM
1984 Meeting
PROCEEDINGS

BEST COPY AVAILABLE

EDITOR'S FORWARD

The proceedings of the 1984 Annual Meeting follow the pattern of previous proceedings which reflect the organization of the meeting. How long must one exist to claim traditions? At a young eight years our Annual Meeting has acquired the following traditional ingredients:

a) Major presentations by prominent mathematicians and mathematics educators.

b) Several working groups where topics and issues of interest to this particular small community of scholars are considered.

c) Topic groups permit individuals or groups to make presentations on items of interest to the group.

d) Panels.

These proceedings, in some small way, reflect the above traditions as they unfolded in 1984.

Charles Verhille
Editor

Preface

CANADIAN MATHEMATICS EDUCATION STUDY GROUP
GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUE
1984 MEETING

The eighth annual meeting of the Study Group was held at the University of Waterloo, June 2 to 6, 1984. More than 50 mathematics educators and mathematicians came together to discuss and explore issues in mathematics education. Some issues were new - the impact of computers, recent curriculum reforms, for instance - others were hardy perennials - the place of visualisation and imagery in mathematics, the role of the teacher, for instance. But in the context of the conference, where both kinds were juxtaposed, and where the between-sessions talk cross-fertilised them, it seemed that the "new" always raises some "old" questions, and that "old" questions can be illuminated by deliberately looking at them in "new" ways.

The main invited lectures were given by Leon Henkin (Berkeley) on "Linguistic aspects of mathematics and mathematics instruction" and Alan Bishop (Cambridge) on "The social construction of meaning - a significant development for mathematics education?" Both speakers made spirited and helpful contributions throughout the conference. A panel of speakers discussed the impact of computers on undergraduate mathematics. John Poland (Carleton) outlined the conclusions of a report based on earlier Study Group discussions; George Davis (Clarkson College) communicated the flavour of the College's extensive use of and commitment to computers in first year courses; and Keith Geddes (Waterloo) demonstrated the power of Maple software and its effect on the teaching of integration. A second panel, consisting of David Alexander (Toronto, and Ontario Ministry of Education), Michael Silbert (Hamilton Board of Education), Dale Drost (N.B. Ministry of Education) and Claude Gaulin (Laval), discussed the general trends of current curriculum reforms in the school systems of Ontario, New Brunswick and Quebec. Both panels contained excellent contributions that one would not have wanted to curtail, yet both ran out of discussion time. This was particularly unfortunate in the case of the latter panel as it was scheduled on the last morning, so even informal discussion at subsequent intervals was pre-empted.

The 9-hour (3x3) Working Groups played their usual important part in giving coherence to the conference. Not liked by a few (perhaps because they do not always appear to yield 9 times as much as a 1-hour lecture), they are regarded by most of the participants as characterising a significant feature of the Study Group - an attitude to professional study that includes learning from each other as well as from experts, and that stresses that finding key questions is as important as passing on answers. Group A (Logo and the mathematics curriculum) was led by Dale Burnett and Bill Higginson (Queen's); Group B (The impact of research and technology on school algebra curricula) was led by Carolyn Kieran (UQAM) and Tom Kieren (Alberta); Group C (Epistemology and mathematics) by Maurice Bélanger (UQAM) and David Wheeler (Concordia); Group D (Visual thinking in mathematics) by John Mason (The Open University) and Tony Thompson (Dalhousie).

The programme included information sessions about the Waterloo Mathematics Faculty's involvement in computer-assisted instruction, competitions, and co-operative programmes. A miscellany of topics, including accounts of two courses with a historical flavour given by Israel Kleiner and Abe Shenitzer (York), rounded out the programme.

LECTURE 1

THE SOCIAL CONSTRUCTION OF MEANING - A SIGNIFICANT DEVELOPMENT FOR MATHEMATICS EDUCATION?

BY

ALAN J. BISHOP
DEPARTMENT OF EDUCATION
UNIVERSITY OF CAMBRIDGE

THE SOCIAL CONSTRUCTION OF MEANING - A SIGNIFICANT DEVELOPMENT FOR MATHEMATICS EDUCATION?

BY

ALAN J. BISHOP

I would like today to direct your attention to what I consider to be a significant new research area for us in mathematics education, and the best way I can do this is to explain not only what it is about, but also how I came to see its value. This talk will therefore be a kind of journey through some ideas and will, I hope, convey something of the flavour, and also the substance, of the new area.

However, in order to help you comprehend and evaluate what I have to say you should know that my own working context is in teacher education, at a University Department of Education, and you must also remember that it is within the U.K. system. One consequence is that I start my research from the assumption that the teacher is the most important agent in the whole educational enterprise. Much of the practice of teaching and of teacher education in the U.K. is based on the idea of the 'autonomous teacher'. This idea is a myth, of course, in the sense that every teacher is subject to all kinds of pressures but it is a myth that we value and preserve. I am not concerned today with whether or not this is a good or a bad myth, but I will be happy to agree for now that it has its dangers as well as its blessings!

My research interests have always been concerned with the mysteries and the complexities of the mathematics classroom - the context in which teachers try to acculturate pupils into the mathematician's ways of understanding the world. My research philosophy is that of 'constructive alternativism' (Kelly, 1955) which means that I look for alternative ways of construing and interpreting classroom phenomena in order that the acculturation process can be achieved more successfully than it is at present. One of the first strands of this research to get developed concerned my work on teachers' decision-making. 'The teacher as a decision-maker' was a conception designed to catch the process whereby the teacher deals with the many choices occurring both before and during teaching. I was particularly interested in the decisions made during the classroom interactions, now referred to in the research literature as 'interactive decision making' (Shavelson, 1976). It is a very powerful construct in that it links the work on teachers' knowledge, ideology, attitudes, etc. with the work on teachers' classroom behaviour, methods, language etc. Various aspects of mathematics teachers' decision-making were learnt (Bishop, 1976a) and many more are waiting to be explored. For example, dealing with pupils' misunderstandings and errors constitutes a large part of a teachers activity but the decision-making construct forced me to attend to the fact that, in the classroom situation, what is significant is the teacher's perception of the errors and misunderstandings. This is sometimes forgotten by those researchers who study children's errors in a laboratory-like atmosphere away from the interactive classroom. I therefore looked at perrs (teacher perceived errors) and was particularly interested in the teachers' strategies for dealing with these (Bishop, 1976b). This research developed some very useful activities for teacher education; for example, 'freezing' a moment of decision in a video-tape of a lesson and analysing the choices and criteria open to the teacher. Into such discussion it is possible to inject many constructs from psychological research which would otherwise seem very remote from the classroom.

It is also satisfying to see that this construct has been taken up in a very serious and large-scale manner by the Institute for Research on Teaching at Michigan Stage University. The whole work of the Institute is based on the 'teacher as thinker' model and the decision-making construct is well embedded in that model. This conception recognises the fact that the tasks, constraints and problems of teaching develop certain characteristic ways of thinking in teachers, which clearly has enormous implications for both initial and in-service teacher education (Clark and Yinger, 1979).

The second research strand developed from a long-standing interest in visualisation, and once again was concerned with the classroom situation. My first attempts were with different teaching methods and their interactions with various aspects of spatial ability, but I found both of these constructs (T.M. and S.A.) to be rather remote from the real classroom. I therefore reworked both constructs, and changed 'teaching methods' to 'spatial activities', while 'spatial ability' became 'visual processing'.

Firstly the move away from 'methods' to 'activities' is highly significant. The idea of 'teaching method' creates a distinction between it and mathematical content which I became increasingly uneasy about. 'Teaching method' is also

a researchers' and not a teachers' construct in that no teacher can possibly see the necessary range of teaching that a researcher can, and the teachers I worked with were not happy either about the method/content dichotomy. 'Spatial activities' on the other hand links much better with content and seems to fit more with teachers' ideas of teaching although it is also capable of considerable extension beyond those ideas (Bishop, 1976). It can be embedded in the more general construct of 'mathematical activities' and this is a notion which several researchers are currently exploring. For me, the notion of a mathematical activity relates to both topic and process, and is a unit of both method and curriculum. I particularly value its focus on what the pupils are (supposedly) engaged in and it also enables us to analyse activities by such things as type (open, closed, practice, exploration, analysis etc.) and group size (whole class, small group, individual). I can concern myself with devising relevant, meaningful spatial activities (Bishop, 1982) and can focus my student teachers' attention on the initiation, organisation and control of those activities. 'Spatial activities', as a sub-set of mathematical activities, is I think a very rich and important construct.

The reworking of 'spatial ability' was made possible by analysing the distinction between the ability to Interpret Figural Information i.e. the knowledge, conventions and "vocabulary" of the many figural forms we use in mathematics, and the ability for Visual Processing (Bishop, 1983). Much spatial ability testing only really tests what I call IFI and although that is important in mathematics, I wanted to see what VP could offer in the classroom context. For example, we know that individuals differ markedly in their ability for visual processing, some preferring to do it a lot and some not at all. Krutetskii's (1976) 'geometers' certainly showed extreme preference for it. We know also that there exist differences between individual teacher preferences as well as between those of individual pupils and we can explore how this ability can be developed or how a person can be encouraged not to rely on it. It links with ideas of intuition and imagery, and can also relate to the use of analogy and metaphor.

Of particular interest is how imagery can be shared between teacher and pupil, and this is where the use of diagrams and figures can be an important. Spatial activities can also be studied in terms of their value in helping the externalisation of imagery and the sharing of visual interpretations. Language has a strong part to play here of course because much imagery can be evoked by appropriate language and examples (Kent and Hedger, 1980).

While these two research areas were developing I had become increasingly aware of the gap between much research in mathematics education and the actual classroom situation. In one paper (Bishop, 1980) I concluded that, from the point of view of most theories of learning, the mathematics classroom with its noisy atmosphere, with its multiple objectives, with its fixed-time lessons and with its atmosphere of mutual evaluation, was not a very good place in which to learn mathematics! The problem I could see as a teacher educator was that research on the learning of mathematics was becoming more and more sophisticated while classrooms were becoming more and more of a challenge to most teachers. As a consequence many people were feeling that the quality of learning was declining.

Now I was not the only person to notice this, of course, and I could see different developments which were designed to make the classroom situation more controllable and more "appropriate" for learning as it was thought it should be done. In the USA, and to some extent elsewhere, one development put more effort into the production of the 'ideal' textbook. Much time, money and effort is invested in what some people unfairly call "teacher-proof" texts. These are carefully designed to avoid sex and racial bias, and to build in motivators, reviews, examples, historical quotations, check-tests, spaced practice exercises etc. The student's text and the teacher's text interleave precisely and the teacher is told exactly what must be done. She thereby loses her authority to the text's authors. One can detect in research also a search for lesson components which can be put together to produce the 'ideal' lesson (Good and Grouws, 1979). In the U.K. too we can find our ideas of teacher training dominated by the notion of the 'mathematics lesson'. Lesson planning is stressed, lesson components are analysed, and exercises are given in 'lessonising' the curriculum.

A second move to control classroom learning was also developing in the U.K. and elsewhere. This was the move towards individualised schemes (like SMILE, and KMP) which built to some extent on the earlier research

on programmed instruction. However we have well-documented evidence of the ways in which such schemes totally change the teacher's role from those of teacher, authority, helper, to those of administrator, marker, paper distributor (Morgan, 1977). The danger here is that the more sophisticated the individual materials become, the more they intervene between the teacher and the pupil. The teacher once again loses her authority to the anonymous pieces of paper.

My own response to the challenge of the complexity of the classroom is not to seek salvation in the textbook full of ideal lessons, nor in the loneliness of the individualised material, but to seek better ways to understand the classroom. It is only complex because of our ignorance and if we could understand it better, if we could interpret it more richly, then perhaps we could learn how to handle it better. This brings me to the third research effort which has occupied by mind over recent years.

I refer to it as the 'social construction' frame and like to distinguish it from our more traditional 'mathematics lesson' frame which, as I have already indicated, has tended to dominate our thinking about mathematics education. This 'social construction' conception has grown out of the wider range of research perspectives which have been brought to bear on the phenomenon of life in classrooms. Classroom ethnographers, sociologists, those who study verbal interactions, teachers' decisions and pupil/teacher perceptions have opened our eyes to a rich tapestry of classroom phenomena. We are now, for example, much more aware of aspects like teacher stress, pupils' fear of mathematics, of the effects of interpersonal perceptions, of pupil-pupil interactions, of the powerful position of the teacher in the classroom and of pupils' strategies for coping with their relative powerlessness.

What I have been attempting to do is to pull out from these researches what I feel are the more significant aspects for us in mathematics education.

Fundamental to our understanding of mathematics classrooms is the fact that one is dealing with people. It may seem trivial to say this but the fact can easily be overlooked when discussing details of lesson components, for example, or pupil ability, or motivation, or any other psychological or mathematical construct. It is true, of course, that the classroom, being part of an institution, institutionalises the participants. But each classroom group is still a unique combination of people - it has its own identity, its own atmosphere, its own significant events, its own pleasures and its own crises. As a result, it has its own history created by, shared between, and remembered by the people in the group.

A corollary which is of significance to the teacher is that each individual person in the classroom group creates her own unique construction of the rest of the participants, of their goals, of the interactions between herself and the others and of all the events, tasks, mathematical contents which occur in the classroom. Such 'objects' as children's abilities, mathematical meaning, teacher's knowledge, rules of behaviour, do not exist as objective facts but are the individual products of each person's construction.

Recognition of this social construction of phenomena leads me to propose a new orientation for mathematics education. This orientation views mathematics classroom teaching as *controlling the organisation and dynamics of the classroom for the purposes of sharing and developing mathematical meaning.* This orientation has the following features:

1 it puts the teacher in relation to the whole classroom group,
2 it emphasises the dynamic and interactive nature of teaching,
3 it assumes the interpersonal nature of teaching, i.e. that the teacher is working with learners not merely encouraging learning,
4 it recognises the 'shared' idea of knowing and knowledge, reflecting the importance of both content and context,
5 it takes into account the pupil's existing knowledge, abilities and feelings, emphasising a developmental rather than a learning theoretical approach,
6 it emphasises developing mathematical meaning as the general aim of mathematics teaching, including both cognitive and affective goals,
7 it recognises the existence of many methods and classroom organisations, i.e. it does not by definition exclude any methodological techniques already established,
8 it is a conception which permits development of the teacher through initial teacher training and beyond.

Central to this view of classroom teaching is the idea of mathematical meaning - a notion which should perhaps be clarified at this point. What I am seeking to emphasize is the personal nature of the meaning of any new mathematical idea. A new idea is meaningful to the extent that it makes connections with the individual's present knowledge. It can connect with the individual's knowledge of other topics and ideas in mathematics but it can also connect with knowledge of other subjects outside mathematics. It may well relate to imagery, analogy and metaphor, but these connections will be of a different type. The idea can be an example of another mathematical idea (because that is the nature of mathematics) and may well generate examples of its own. Finally, and arguably most importantly, it can connect with the individual's knowledge of real world situations. It is obvious therefore that no two people will have the same sets of connections and meanings, and in particular teacher and learner will have very different meanings associated with mathematics. The teacher will 'know' the ideas she is teaching in terms of the connections they make with the rest of her mathematical knowledge. The learner however is the 'meaning maker' (Postman and Weingartner, 1971) in the educational enterprise and must establish the connections between the new idea and her existing knowledge, if the idea is to be learnt meaningfully. As Thom (1973) says "The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of meaning, of the existence of mathematical objects". The educational goal we are concerned with here then, is that of sharing, and developing, mathematical meaning.

This conception has enabled me to focus my analysis on three fundamental aspects:

mathematical activities	- chosen to emphasise the learner's involvement with mathematics rather than the teacher's presentation of content,
communication	- chosen to emphasise the process and product of shared meanings,
negotiation	- chosen to emphasise the non-symmetry of the teacher/pupil relationship in the development of shared meanings.

I have already illustrated the importance of the idea of mathematical activities but a few more thoughts are necessary here. It is significant for the teacher's pre-class decisions in that the teacher no longer thinks of how she will present content during the class but rather she must make the didactical conversion from mathematical content and knowledge to mathematical activities suitable for the pupils. In the UK we still 'think' too much about content, knowledge and mathematical topics and not enough about what the pupils' activity will be in class. A focus onto mathematical activities for the pupils can improve that situation and can put the pupil's activity at the centre of the teachers' concerns.

Not only does this affect pre-class decisions but it also affects the teacher's interactive decision-making. Teaching is, as a result, more concerned with the initiation, control, organisation and exploitation of the pupils' activity. There is more of a dynamic, organic-growth, feeling in the classroom than of a compartmentalised list of specific knowledge or skills to be taught from nothing, and to be finished at a set time.

Another aspect which "mathematical activity" makes us attend to is collaborative working. Pupils value collaborative working but mathematics teachers in the UK have an ambivalent attitude towards it - most seem to prefer pupils to work on their own but will say things such as "you can work with your friend if you don't make too much noise"! In fact in UK mathematics classrooms there will exist much collaborative learning but most of it will be covert and often 'illegal' instead of being deliberately planned and encouraged by teachers. If only we could develop more small group mathematical activities for pupils the teachers could be encouraged to take a more positive attitude towards collaborative and interdependent working than they do at present.

Communication is not a new construct in education but in my view it has never been well analysed or activated within mathematics education. In general, in the UK, mathematics classrooms are places where you do mathematics not where you communicate or discuss mathematical meanings. Meanings and understanding are about the connections one has between ideas - a new idea will be meaningful for a pupil to the extent to which it connects well

with the pupil's existing ideas and meanings. Communication in a mathematics classroom is therefore concerned with sharing mathematical meanings and connections. We can only share ideas by exposing them, and 'talk' is clearly a most important vehicle for exposing connections. Also important are symbolism, uses of diagrams for conveying images, examples from different contexts, analogies and metaphors, and written accounts and descriptions. Some of these we know relatively more about (symbols, definitions..,) others we know relatively less about (analogies, metaphors, contexts). Moreover if we add into the construct of communication the dimension of sharing, then the three-way process, from the pupil to teacher as well as teacher to pupil and pupil to pupil, shows us how ignorant we are about pupils' analogies, metaphors, contexts, examples, etc. and about ways of enabling these to be exposed and shared. For example, activities can be developed which encourage and legitimise this exposition and sharing, such as investigations which involve creating symbolism, or projects which draw on knowledge of the pupils' environment, or discussions of mathematical ideas and their diagrammed analogies (number lines etc.). Several research studies have shown us that in classrooms it is the teacher who does most talking. What I should like to see are developments which show teachers how pupils can be encouraged to take more part in the sharing of mathematical meaning. I think that exploiting the ideas of two-way and three way communication could be a profitable way forward.

If communication is about sharing meanings then negotiation is about developing meanings. Without wishing to suggest that the teacher is the authority for mathematics in the classroom, it is the case that the teacher is given authority and power by the society for the specific education of her pupils. This authority means that the teacher has certain goals and intentions for the pupils and these will be different from the pupils' goals and intentions in the classroom. Negotiation is goal-directed interaction, in which the participants seek to attain their respective goals. We can include in this idea the working out of a "modus vivendi" in the classroom i.e. the rules of procedure, discipline and behaviour which teachers already know much about. What the construct of negotiation also offers is an idea of "modus sciendi", a way of knowing, which is what the teacher is trying to develop by the use of her own, necessarily richer, mathematical knowledge and understanding. This construct then specifically catches the necessary power imbalance implicit in the teaching/learning relationship but it describes it in such a way that we can see alternatives to the mere imposition of knowledge from the powerful teacher.

What it therefore forces us to do is to consider how to encourage teachers to use their power not to impose their knowledge on the pupils. It makes us think more about how teachers can encourage the negotiation process, how teachers can encourage pupils to play a greater part in the development of their own mathematical meanings, how teachers can recognise more positively the pupils' context and goal structure, and how teachers might evaluate better the development of meanings.

In conclusion then, may I suggest that this 'social construction' conception and the three constructs, 'activities', 'communication' and 'negotiation', offer us many rich avenues to explore in research. Like any good construct, they recast what we know about and direct us to what we need to know about. I would, as a result, particularly urge more attention to the following:

- the development of activities, particularly those which exploit the pupils' context and those suitable for small group work,
- the analysis of the relationships between activities and mathematical topics,
- studies of teachers' interactive decisions with pupils engaged in activities of different types,
- studies of teachers' techniques to encourage sharing of mathematical meanings,
- the analysis, from the 'sharing' perspective, of pupil-pupil discussions,
- studies of the process whereby visual imagery can be shared,
- the analysis of teachers' decision-making concerning mathematical authority,
- the analysis of teachers' strategies which permit negotiation,
- the development of methods of evaluating the development of meaning.

REFERENCES

Bishop, A.J.: 1975, 'Visual Mathematics', Proceedings of the ICMI-IDM Regional Conference on the Teaching of Geometry, IDM University of Bielefeld, F

Bishop, A.J.: 1976(a), 'Decision-making, the intervening variable', Educational Studies in Mathematics, 7, 41-7.

Bishop, A.J.: 1976(b), 'teachers' 'on-the-spot' strategies for dealing with pupils' errors', Psychology of Mathematics Education Workshop, Chelsea College, University of London.

Bishop, A.J.: 1980, 'Classroom conditions for learning mathematics' in R.Karplus (ed.) Proceedings of the Fourth International Conference for the Psychology of Mathematics Education, University of California, Berkeley, California.

Bishop, A.J.: 1982, 'Towards relevance in the teaching of geometry' in G.Noel (ed.) Colloque International sur l'enseignement de la geometrie, Universite de l'Etat a Mons, Belgique.

Bishop, A.J.: 1983, 'Space and Geometry' in R.Lesh and M.Landau (eds) Acquisition of Mathematics Concepts and Processes, Academic Press, New York.

Clark, C. and Yinger, R.: 1979, 'Teachers' thinking' in P.Peterson and H. Walberg (eds) Research on Teaching, McCutchan, Berkeley, California.

Good, T.L. and Grouws, D.A.: 1979, 'The Missouri Mathematics Effectiveness Project an experimental study in fourth grade classrooms' Journal of Educational Psychology, 71,3,355-362.

Kelly, G.A.: 1955, The Psychology of Personal Constructs (Vols 1 and 2), Norton, New York.

Kent, D. and Hedger, K.: 1980, 'Growing tall', Educational Studies in Mathematics, 11,137-179.

Krutetskii, V.A.: 1976, The Psychology of Mathematical Abilities in Schoolchildren, University of Chicago Press, Chicago.

Morgan, J.: 1977, Affective Consequences for the Learning and Teaching of Mathematics of an Individualised Learning Programme, DIME project, Stirling, Scotland.

Shavelson, R.J.: 1976, 'Teachers' Decision Making' in N.L.Gage (ed.) The Psychology of Teaching Methods, National Society for the Study of Education, University of Chicago Press, Chicago.

WORKING GROUP A

LOGO AND THE MATHEMATICS CURRICULUM

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

JUNE 1984 MEETING (WATERLOO)

REPORT ON WORKING GROUP (A)

LOGO AND THE MATHEMATICS CURRICULUM

GROUP LEADERS: DALE BURNETT
WILLIAM HIGGINSON

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

1984 MEETING
UNIVERSITY OF WATERLOO

REPORT OF WORKING GROUP

"LOGO AND THE MATHEMATICS CURRICULUM"

The description of the working group read:

The computer language Logo and its underlying educational philosophy, developed at the Massachusetts Institute of Technology over the past fifteen years shows signs of being one of the most popular pieces of educational software of the 1980's. The developmental history of Logo has been such that the time period between very limited accessibility (1968 - 1982) and widespread accessibility (1983 -) has been quite short. A major implication of this is that mathematics educators have had little time to consider a number of important questions.

It is the purpose of this working group to provide a forum for the discussion of these questions. Issues to be considered will include problem-solving in a Logo context, the influence on school curricula and Logo investigations. It will be assumed that participants will have some rudimentary knowledge of Logo.

Working Group Leaders: J. D. Burnett and W. C. Higginson

Some fourteen members of the Study Group focussed their attention on these and related questions over three, three-hour, periods during the 1984 meeting. The initial session was given over to an examination and discussion of a wide range of resource materials and the sharing of diverse experiences which had arisen out of participants' research and teaching activities with Logo. Following this session one of the participants (J. Clark) summarized some of the questions which had arisen:

1) What do we mean when we say Logo? Is there a distinction between Turtle Geometry and Logo philosophy?

2) How do Logo philosophy and Turtle Geometry support existing elementary and secondary mathematics curricula?

3) Other than that which relates to mathematics, what can students learn from Turtle Geometry?

4) When and how should Logo be introduced into classrooms?

5) What are some models for classroom implementation which deal with time, space student access, etc?

6) As a device for learning mathematics, why is Logo superior to other teaching devices (such as geoboards and pattern blocks)?

7) Is it important for children to learn programming? If so, when and how should it be taught?

8) What are the advantages of Logo as a programming language?

9) What is the role of the teacher in the child's Logo environment?

10) Is there some connection between debugging and how students view errors in general?

11) What are other "objects to think with" like Turtle Geometry? (See Mindstorms page 11 and page 122)?

In the second and third sessions the majority of time was given over to the exploration and discussion of a number of Logo investigations. Two of these were Dale Burnett's "Turtle Billiards" microworld and Gary Flewelling's "Transformation" investigation. The latter was posed in the form:

> Using only the Logo primitives FD, BK, RT and LT, write a small procedure (6 - 10 instructions) that will draw a small 'doodle' on the screen.
> Investigate the effects of carrying out various transformations on this procedure.

The group found that activities of this sort generated significant mathematical questions extremely quickly. It was felt, however, that teachers without substantial mathematical training would probably not recognize all of the potential of these situations. Commenting on the problem stated above, for example, Gary Flewelling wrote:

Here many teachers would have to be given a sample of possible avenues of inquiry, for example, reverse the FD's and BK's, or reverse all the LT's and RT's, or reverse the order of the procedure, or replace all FD's and BK's with the procedure itself, or...

Teachers might also have to be told the task's target audience. In this example, the task is appropriate for Junior, Intermediate, Senior and Post-Secondary students.

Teachers will also have to be shown the links between the various lines of inquiry and the Mathematics curriculum they are trying to implement. In this example, various investigations link to such topics as transformation geometry, algebra, group theory and combinatorics.

Six conclusions arrived at by the working group near the end of its deliberations were:

1) Logo, not linked more strongly to existing school programmes, runs the risk of becoming trivialized.

2) Logo deserves to be integrated into school math programmes.

3) Logo is a tool that can be used by teachers to help them implement their mathematics curriculum.

4) Teachers need proof of the two previous points.

5) Proof is particularly needed in the form of student tasks which illustrate these two contentions.

6) Samples of such tasks need to be developed and delivered to teachers.

Further discussion about the nature of these Logo tasks led to the following consensus:

The tasks should satisfy many of the following criteria:

1) Can be used by students with a variety of interests and abilities.

2) Allow for a number of lines of inquiry.

3) Can be returned to again and again through the school year

4) Can be utilized through the grades at varying levels of sophistication

5) Has links to the mathematics curriculum

6) Contains an element of choice/freedom/modifiability for both the teacher and the student.

7) May require the student to construct various procedures to carry out related investigations.

8) May have to be accompanied by prewritten procedures.

9) Invites the cooperative effort of more than one participant.

Gary Flewelling (Mathematics Consultant, Wellington County Board of Education, 500 Victoria Road North, Guelph, Ontario N1E 6K2) volunteered to coordinate a 'task' force to generate a collection of such tasks.

Participants in the working group: J. Bergeron, D. Burnett, J. Clark, G. Flewelling, W. Higginson, J. Hillel, B. Hodgson, H. Hough, R. McGee, A. McLean, M. Rahim, P. Rogers, P. Taylor, E. Williams.

WORKING GROUP B

THE IMPACT OF RESEARCH AND TECHNOLOGY ON SCHOOL ALGEBRA CURRICULA

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

JUNE 1984 MEETING (WATERLOO)

REPORT OF WORKING GROUP (B)

Global Approaches

A report of the working group on Computers and Algebra

by

Carolyn Kieran

and

Thomas Kieren*

* Reporting for and using ideas contributed by the working group: David Alexander, Dale Drost, Gila Hanna, Nick Herscovics, Lars Jansson, Dave Kirshner, André Ladouceur and John Poland.

The work of this group can be characterized by what we considered,* but perhaps best by the proposed ideas and followup work which came as a reaction to the material considered. Finally the work of the group can be focussed by extensions which we did not have time to consider and by material and directions which we deliberately chose not to follow.

Background

As background to our work members of the group brought the following kinds of concerns:

- What is it we want in school Algebra in 5 - 10 years? What distortions will the use of micros cause in algebra?
- What will be the effect of new technologies in algebraic symbol manipulation and its meaning?
- What implications do computer related thinking, learning and knowledge building theories (eg. AI) have for algebra?
- What effects will the computer experience backgrounds of students have for teaching algebra to them?
- What kinds of software and computer uses will help in algebraic concept development? How can this be assessed?
- Can computers be used to ameliorate student learning difficulties? How can this be studied in algebra?
- There is high powered software which "does algebra".
  How does one learn to use it?
  How does one teach its use?

* The group specially commended Carolyn Kieran whose extra efforts in providing the technical set-up made many of our considerations possible.

How does it impinge on what is currently taught in algebra?

What ought to be taught?

- How does one teach teachers about computers for teaching algebra?
- What is the interaction among human algebraic knowledge building tools, algebraic games on a computer and programming? How does one study this phenomena? Measure its effect?

Materials Considered

The working group considered four kinds of materials: Documents, software, oral reports and site visits.

1. The group considered a number of documents. Two studied and discussed by all were:

   - Fey, J. T. (ed.) Computing and Mathematics, a Report of a 1982 conference. NCTM, 1984. (especially the Algebra chapter)

   - Blume, G. W. A review of research on the effects of computer programming in mathematical problem solving. A paper from the New Orleans AERA meeting, 1984.

   We also had available a small library of text materials using computers in school algebra from 1969 - 1984.

2. The group together reviewed some pieces of relevant software, in particular Algebra Arcade and Mu Math.

3. Group members provided oral reports, particularly one by Alexander on the changes in algebra curriculum in Ontario and its relationship to computer applications and one by Kieren reviewing his and Hatfield's research of the late 1960's.

4. The group visited Project MAPLE in the Computing Science Department of the University of Waterloo.

   Included was a demonstration of the algebraic capabilities of MAPLE.

Group deliberations

As suggested by the title of this report, consideration of the above materials seemed to stimulate discussion of more global approaches to school algebra. These global approaches seemed mediated by whole-graphic approaches to the study of families of algebraic objects, by the use of the language of powerful algebraic computing programs and through the use of structured programming languages.

Algebraic Games

As an opening exercise the group looked at the game Algebroids from the software Algebra Arcade. In this game a person is presented with a number closed figures called "algebroids" on a coordinate plane. The object of the game is to use algebraic expressions to generate graphs which "hit" the algebroids. Since higher point values are awarded for hitting more than one algebroid with one graph (without hitting a "monster" or designated negative region), the player is encouraged to use other than linear graphs. A graph length limitation keeps one from using curves which would more or less cover the screen (eg. $f(x) = 100 \sin 100x$). There was a uniform positive reaction to this game by group members both in terms of playing and in terms of potential for use in secondary school mathematics. This game and its uses tended to dominate the group's discussion throughout its sessions.

A first question arising questioned whether this game focussed on analysis rather than algebra. Although it has analytic features, the group consensus was that it aided in the teaching of the properties of families of curves and particularly on the effects of algebraic transformations on notation (related to graphical transformations on the screen).

The group noted two immediate learning benefits. One was the speed at which one could produce (and hopefully hold on a practice screen for comparisons) various members of a family (eg. linear functions $y = ax + b$), related families ($y = ax + b$, $y = m|x| + n$, $y = ax^2 + p$) or transformations within families ($y = ax^2 + p$, $y = a(x - k)^2 + p$). Thus one's imagination in symbol use would be supported by rapidly developed related images. The second benefit relates to the "games" phenomena. The work of Malone on qualities of video games and of Loftus and Loftus on the psychology of playing such games was discussed. Thus there seemed to be two impacts - an emphasis on <u>families</u> of algebraic expressions as opposed to single manipulations and the motivational effect of relating images to algebraic expressions (perhaps making their manipulation and study more intuitive and object related).

The most significant thesis that arose from the group deliberation can best be seen in curriculum/research questions.

> Is it necessary to have an atomic understanding of the functions to play the game? Should we not take this more global approach to teaching about algebraic objects such as linear, quadratic or exponential expressions? Does one need to learn ordered pair ideas at all to study this material? Is a different sequence of instruction in school mathematics (say in the algebraic aspects of grades 7 - 11) suggested?

These questions and reactions to them dominated the group's discussions. Some interesting points were as follows.

> The global approach through the manipulation of expressions and generation of related graphs seems more <u>intuitive</u> than the ordered pair first emphasis.

It was questioned whether one could look at operators or transformations without knowing about states or objects transformed.

Alternative curriculum sequences were discussed. One such experimental sequence constructed was as follows.

I. In grades K - 8 graphing experiences using points and interpreting graphs would be introduced and developed.

II. "Algebroids" or a like game could be used to introduce a family of functions and the manipulation of related expressions and equations; eg. for what values of b does an expression of the form $3x + b$ hit an algebroid centred at (2,7)?

II. a) An alternate "algebroid" sequence would consider the family $y = a|x| + b$ because in such families it is easier to see the effects of all parameter manipulation and such graphs can hit more "algebroids" at once - hence are motivating.

III. The study of graphs as point sets would follow the study of families of expressions/graphs. In the game context this would be done by reducing the "algebroids" to single point locations necessitating the consideration of the graphs as sets of points.

The group considered the teacher education necessary for such a changed sequence and emphasis. It was noted that such an approach would aim at many of the current objectives of high school algebra, but the objectives and activities would be dramatically different from a teacher and classroom organization point of view.

<u>Algebraic Computers</u>

The group saw a demonstration of the very powerful MAPLE manipulation system which is aimed at being usable in the next generation of micro computers. The group also used the currently available Mu Math system. This latter system is slow in doing certain kinds of recursive algebraic computations because of its list processing character. Still the group saw a number of implications and questions for school algebra arising from algebraic calculator use.

1. Is a call for renewed emphasis on algebraic manipulation in our curriculum obsolete? What is the role of such learning now?
2. With the use of algebraic calculators comes the need to introduce a new notation system for algebraic expressions; will this notation become standardized?
3. Since the algebraic calculator can generate correct sentences, there will be an increased need to study sets of sentences and look for patterns or properties. Thus as with graphic capabilities, the algebraic calculator sponsors the study of the global.
4. Might more new algebraic topics (eg. matrices) be introduced or introduced earlier into the curriculum?

Follow up activities

The group, because of its interest in the effects of algebraic games, set some goals for study during the year. With respect to a game like Algebroids the following research was proposed:

1. Can we develop and test a sequence of instruction incorporating "Algebroids"? This question will be studied in a large curriculum sense (where does such activity fit into the algebra curriculum?) and the specific sense (what is an appropriate sequence using Algebroids to study a particular family of expressions, say quadratics?).
2. How would one modify the "Algebroid" program to enhance learning? There is currently a practice screen. Could its use be improved by allowing graphs to stay on? be in different colours? Could more information about available expressions and manipulations be built into the game?

3. Can we use "Algebroids" as a testing device?
   - Given a graph give expression.
   - Given expression give graph.
   - Can we use shape recognition program in this effort (eg. GMATCH at the Univ. of Alberta)?
4. In what ways is the use of a global approach to algebraic expressions through "Algebroids" intuitive. How can such use be related to a theory of mathematical knowledge building?

   Can one generate evidence of knowledge-building from intuition in algebraic novices (eg. Gr. 9 students) and more expert students (eg. Gr. 12 students studying conics) as they use tools and games such as "Algebroids"?

Our group hopes to be able to have a continuation session at next year's meeting to share results of this work with one another and with CMESG/GCEDM at large.

NOTE: 1. The group does not see much use for computer-aided practice in algebraic manipulation. Thus, there was limited study of any such curriculum materials.

2. The group did not consider image-related games which might be related to teaching algebraic structures such as groups. This is an obvious update of the work of Dienes in the 1960's. Such computer graphical/symbolic use could also be used at a more advanced level to provide background for the study of various types of groups or other algebraic structures. Such study would be an extension of the work proposed above.

29

WORKING GROUP C

EPISTEMOLOGY AND MATHEMATICS

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

JUNE 1984 MEETING (WATERLOO)

REPORT OF WORKING GROUP (C)

MATHEMATICS AND EPISTEMOLOGY

LEADERS: MAURICE BELANGER
DAVID WHEELER

Working Group C: Mathematics and Epistemology

In the first session the group attempted various partial descriptions of epistemology, and then worked for a while on the epistemological foundations of subtraction (in order to compare their analysis with one by Gérard Vergnaud in his article, "Cognitive and developmental psychology and research in mathematics education: some theoretical and methodological issues"). Whereas in his article Vergnaud makes his analysis on the basis of the problems that students at school have to solve, the group began with a model: "Given some objects, remove some, how many are left?" The limitations of this knowledge model were soon apparent: it is clear that to "know subtraction" involves knowing more than a single model offers. After the coffee break the group looked at some research results concerned with students' arithmetical errors - the connection being that the researcher (or the teacher) must try to reconstruct what is in the child's head, i.e. the child's knowledge. A rather loose excursion through the landscape of bugs, debugging and feedback followed.

The discussion in the second session was launched by reading two articles - Alan Schoenfeld, "Metacognitive and epistemological issues in mathematical understanding", and Caleb Gattegno, "Curriculum and epistemology". Schoenfeld's paper raised sharply the question of "relative epistemologies: do we all have the same epistemology? do students and teachers share the same epistemology? The Piaget story suggests that epistemology is evolutionary in the individual. Gattegno's paper, while not shedding much light on the nature of epistemology (in a holistic sense), gave some instructive insights into the various sources of epistemological information that can be tapped.

In the third session the group attempted to formulate some research questions.

1. What are teachers' and students' implicit epistemological beliefs about mathematics? How do they compare with each other and with mathematicians' epistemological beliefs?
2. Make a didatical-epistemological study of various elementary mathematical topics, using as entry point the distinction between signifier and signified.
3. Compare the epistemological questions raised by the history of mathematics with those raised by the development of mathematics in the individual.
4. Generate a taxonomy of epistemological assumptions that could be applied to the analysis of mathematical textbooks.
5. Study the epistemology of the individual, e.g. how does anyone derive general knowledge from particular instances, or particular knowledge from general principles?

6. What is the epistemological significance of a currently fashionable topic: the representation of knowledge?

The group became aware that their discussions were difficult and not making obviously significant discoveries. The feeling was expressed that much more work in this field has been done by e.g. French mathematics educators, influenced by Bachelard, and that one obvious step would be to find out more of what they have achieved and are working on.

The appended statements were volunteered by members of the group.

Raffaella Borasi

Although the title of the working group focussed on the word "epistemology", we desperately tried in the course of all three session to "define" such a term. Looking back, I realize now that this specific objective could have made us overlook some very interesting outcomes of our discussion. We may not have reached an agreement over a "rigorous" or even satisfactory definition of epistemology (what a disappointment for a group of mathematicians!), but I think we contributed to identify some important elements and questions concerning mathematics education research.

I will try briefly to list some of them:
- what is the student's/the child's/the mathematician's/the math. ed. researcher's/ conception of mathematics?
- what does it mean "to know" something?
- how do we "get to know", what is the "knowing process" in the child/ the mathematician/the math ed researcher?
- how can we evaluate our methodologies and results in mathematics education?
- what is the child's/the mathematician's/the math ed researcher's system of beliefs about how we get to know in mathematics, how we "do" mathematics?
- for each of the previous points we might have to distinguish whether we are speaking about mathematics in general or a specific math topic or concept
- in, what sense can we talk of "the mathematician's"/"the child's" epistemology (or more specifically, conceptions, system of beliefs, etc.), rather than a specific person's epistemology?

Dieter Lunkenbein

Three domains of epistemology seem to be of importance in the discussion of the theme: mathematical epistemology, genetic epistemology and epistemology of mathematics education. It is crucial to separate these three domains in order to avoid fundamental misunderstandings.

Mathematical epistemology studies mathematical science in order to determine or to elucidate its logical origins, its value and its importance. It is a bridge between mathematics and philosophy and of great relevance to the mathematics educator, since it informs him about the nature of mathematical knowledge, the content of mathematics teaching and learning.

Genetic epistemology, created by Jean Piaget, who is also its most important representative up to now, studies the emergence and the evolution of mathematical notions and conceptual contexts in the individual (in particular in the child) and it is intended to explain and to rationalize the individual's behaviour in situations where such notions emerge or can be applied. This domain is of particular relevance to the mathematics educator since it informs him about developmental processes and their most important variables, processes he wants to stimulate, accelerate or direct.

Epistemology of mathematics education studies processes and methods to obtain and to validate knowledge in mathematics education. Its results are of importance to the mathematics educator since they inform him about the validity of his methods and the relevance of his results.

While the distinction between these three domains is crucial for a clear and systematic discussion of the theme, only the synthesis of all three aspects indicates the importance and the scope of the epistemological approach in mathematics education.

Brock Rachar

As individuals, we seem to have some sense of the point at which we know that something is the case. We may even be able to identify how we know.

The how may be based on inductive experience or a kind of gestalt perception, or it may be derived deductively from previous knowns by an acquired system of logic.

We are also aware that different individuals may come to the state of knowing that something is the case without the how of the knowing coinciding.

This latter observation has decided implications for curriculum. As mathematics educators we should have available different strategies for enabling the learning of the knowing of something. As teacher educators we should be looking at those approaches that are most likely to be successful and most likely to be free of setting up erroneous conclusions. The settings we use for investigating mathematics are important to the kinds of learning or epistemological framework that children (learners) acquire.

If we believe that mathematics is the development of skills in performing algorithms in order to answer standard sets of problems, then we impart one view of mathematics. If mathematics enables you to know "what to do when you don't know what to do" in solving a problem, then that is a whole other world view of mathematics.

As Alan Bishop pointed out in his lecture, what we do and the connections we make in mathematics learning are largely determined by the way we think about the purposes of mathematics and the ways it is learned.

David Wheeler

Before making any attempt to "report" on the group's activities, I present some examples in order to ask whether the insights embedded in them are epistemological - or something else.

Ex. 1. (This is a quotation from a review of an article about mathematical induction.)

> "We must not overlook the conceptual/technical difficulty of handling the vital step $P(n) \rightarrow P(n+1)$. Most encounters of students with things like $P(n+1)$ have been straight substitutions - substitutions of $n+1$ for some variable in a known expression (function). But in the induction proof the student has to handle almost the reverse of this. For example, he takes $P(n)$ and adds something to it, then has to arrange the new expression to show that it is in fact $P(n+1)$. This requires getting it into the form $P(x)$ while simultaneously "thinking" of $n+1$, and not x as the variable. This is really hard, in many cases, because one is not using the alebra to simplify but to force a correspondence to a certain model. Where else are students required to do this?"

Ex. 2. A "classical" problem runs: show how to detect the false coin, which is too light, in a set of 9 coins using an equal-arm balance twice.
The correct solution requires splitting the 9 coins into 3 sets of 3 and weighing two of them, then weighing two single coins from one set of 3. Most solvers begin with sets of 4 or sets of 2, perhaps because of the strong "binary" flavour of the setting - two arms to the balance, two weighings, two kinds of coin. A "classical" puzzle often has this quality of temporarily deflecting the approach to a solution - this may be what makes it a good puzzle.

Ex. 3. It seems intuitively "clear" that infinite sets can have different numerosities. It is "obvious" that there are more natural numbers than even numbers, more points in a unit square than on a line segment. Yet by adopting the 1-1 correspondence rule for comparing numerosities we can show that "in fact" there are the same number of natural numbers as even numbers, and the same number of points in a unit square as points on a line segment.
So now it is intuitively "clear" that what were thought to be different infinities are not. So, intuitively, all infinities are equivalent.
But no! for Cantor's diagonal procedure shows that there are different orders of infinity, non-equivalent to each other.
And so it goes.

Ex. 4. When we see an algebraic expression like

$$\frac{x-1}{x^2-x-6} - \frac{2}{x-3} + \frac{1}{x+2}$$

we see 5 minus signs, but do they all mean the same thing? Well, yes and no. No because, if we set about simplifying the expression, we treat them in 3 different ways. The minuses in $x-1$ and $x-3$ are like letters in a word. The minuses in x^2-x-6 are signals or "controls" which guide us in factoring it as $(x-3)(x+2)$. Finally, the minus between the first two fractions almost means subtraction - although, provided we get our "rules of signs" right, we needn't be aware of this when we collect all the numerators together.
And now, if we add one almost insignificant stroke to our writing,

$$\frac{x-1}{x^2-x-6} = \frac{2}{x-3} + \frac{1}{x+2}$$

we "see" the whole system totally differently.

If there is a common characteristic of these examples, it is that although they each relate to some familiar mathematics, they focus more on "awareness" than on knowledge. The points they make are overlooked in the usual mathematical accounts. Yet a student involved with mathematics needs these awarenesses as much as he/she needs to know the appropriate mathematical content. Here is potentially an epistemology - an epistemology that studies awareness rather than knowledge - which is just as important for pedagogy as epistemology of the usual sort. I venture to suggest that epistemology is the right word because of my confidence that such a study can be objective. That is, although focused on awareness (which sounds personal, subjective), what can be discovered has the quality that can make us say, "Yes, these awarenesses are part of what everyone (who is involved with this particular mathematics) must know."

This "epistemology of mathematical awareness" is not the whole of "didactical epistemology", which must contain empirical and observational ingredients, as well as "knowledge of mathematical knowledge", but its significance for me at the moment is that it is a part that is overlooked by most who have discussed the epistemological foundations of mathematics teaching.

Bruce Rigopolos

Let me remind you that the Oxford English Dictionary, a generally reliable arbiter where matters of meaning are concerned, defines epistemology as follows: "The theory or science of the method or grounds of knowledge." If we embrace this definition, then I feel we in mathematics education are bound to address certain issues. "Method" and "grounds," it seems to me, both deal with coming to know more often than knowing, and with either of these more often than the content that is or is to be known. It may well be that the nature of what we seek to know shapes to some extent the methods we use to search for it and the standards by which we judge whether or not we have found it, much as the design of a tool is influenced by the nature of the materials to which it is to be applied. I think it would be a mistake, however, to focus on the goal or the finished product at the expense of the process.

Now reflect for a moment on the question of what methods and grounds of knowledge students are expected to operate with in the mathematics classroom. All too frequently, I suspect, the method is absorption and the grounds lie in external authority. Students 'soak up' what the teacher, who of course can not be wrong, inundates them with. That such a situation is inappropriate and unacceptable I take as a truism. How can intellectual growth take place in soil of this sort? I believe that absorption should give way to inquiry and external authority to internal validation. Perhaps it is not realistic to expect young children to begin their learning careers with inquiry and internal validation, though I would argue that it often is realistic. Perhaps this 'giving way' is something that students should actually experience as their education progresses. In any case, I feel certain of the direction in which I would like to see things go.

Thoroughly consistent with all of this is a strong sense of the sanctity of the individual in the quest for knowledge. Not surprisingly, I find a preoccupation with common epistemological assumptions more than a little unsettling. While I will grant that common ground is a nice place to begin inquiry, I do not think that it is the only place. Even if it were, is it not that the differences which set the working of one mind apart from the workings of all others so often command our attention when we try to teach mathematics? Do these differences constitute obstacles to learning, or are they invitations to explore the richness of the individual? Are they something to be worked around and smoothed out, or are they to be relished and revelled in? I, for one, have always preferred revelry to work.

WORKING GROUP D

VISUAL THINKING IN MATHEMATICS

CANADIAN MATHEMATICS EDUCATION STUDY GROUP
JUNE 1984 MEETING (WATERLOO)
REPORT OF WORKING GROUP (D)
Working Group on Visualizing Mathematics

Tony Thompson & John Mason

Our (Thompson and Mason) intentions were to make a catalogue of standard topics in upper-secondary/first-year, which students find difficult, and to explore ways in which the explicit use of imagery might assist students. Our background assumption is that many students believe that mathematics takes place on paper; that they have no supportive imagery or connections and consequently they fail to participate in an essential aspect of mathematics.

We soon found that the instruction "make a record of what comes to you when I say the following words", followed by sin(x); $(1+x)^n$; $1, r, r^2, r^3$, ... produces a wide variety of responses. It emerged that the word "image", which I used in preference to visualizing in order to admit acoustic and muscular responses as well as pictorial images, means many things to many people. For example, on examining images connected with absolute value, the idea of a dog on a leash was suggested, to describe $|x-a|<b$. On the other hand one might hope that students would have a geometric picture of an interval of length 2b centred on a, and an accompanying sense of being on a and reaching out b on either side. This highlights a distinction between what I would call 'metaphoric associations' (dog on a leash) and 'mathematical images' (like points on a line). Even though I could probably not give a definitive distinction between these I feel that I want students to contact and be friendly with the purer mathematical images. It may be that metaphor or simile can help, but I am not too sure. Most participants seemed content to employ a shot gun approach in class, trying out a variety of metaphors so that students can respond to ones that appeal particularly to them., I, on the other hand, still believe in, and seek, core mathematical images which lie at the heart of the hard mathematical ideas. When I spoke of appropriate and powerful images, it was pointed out that these must be relative terms - relative to individual propensities and to intentions. I nevertheless feel that some images are more valuable than others. Here are two examples:

1. Multiplication by -1 is notoriously difficult for many students. All sorts of metaphoric models have been proposed involving profit and loss, temperature and so on, as attempts to justify (-1) × (-1) = 1. I believe that this is a structural fact arising from the wish to retain the 'laws of arithmetic' when extending to negatives. It also has a very potent geometric interpretation as a rotation of the numberline about 0, through 180°. I call it potent, because rotations through other angles (not just 90°) can lead to useful investigations as well as supporting de Moivre's theorem and Argand diagrams in the 90° case. Thus it extends consistently to more general situations while providing a geometric image to accompany arithmetic calculation, an image which will not have to be modified later.

2. Continuous functions are usually thought of as functions which have no breaks, and no infinitely wiggly portions (à la sin(1/x)). For most students only the former image is available, and the formal definition of continuity seems like a lot of unnecessary symbolism. One of the features of mathematics is the modification of images (intuition) as a result of experience. Perhaps time devoted to the necessary modifications of intuition would assist students, in conjunction with making it clear to students <u>when</u> the formalising of intuition is taking place (so eloquently spoken of at the conference by Professor Henkin) and <u>when</u> intuition is being worked on and modified.

We kept returning to the distinction between particular and general - I imagine a square; if I draw it, it is particular yet I see it as general; in my imagination it is sort of particular, yet sort of general in its fuzziness. Am I always sure in a class that the students are seeing the generality that I am seeing when looking at or working on a particular example?

Some time was spent seeking primitive images connected with various mathematical topics, but in my view not enough progress was made to report on. Participants tentatively agreed to work on imagery in their classes, and to report some examples next year. I tried to draw what is for me an important distinction between talking about an image, and talking directly from an image, but it requires considerable practice. To talk from an image is to consciously form and enter a mental image, and to describe it to someone who is not present. To talk about an image is to talk in generalities as if you expect others already to 'have' your image. We all found it difficult to form and then speak from an image, yet it is essential to talk from if students are to be assisted in forming images. I recommend that reports of trying imagery in classes consist of

(i) explicit instructions to form an image and work with it, as illustrated in the appendix,

(ii) an account of what students made of it and lessons learned in working with imagery.

We finished by watching and working on a Nicolet film on conics. I was able to demonstrate what I mean by 'reconstruction', which is a necessary but overlooked aspect of studying that deserves more attention. The last event was a request to participants to jot down one or two salient moments from the workshop, and some of these follow.

1. The earliest event that made an impression on me was during the first 'exercise', when we did a 'free association' of images for sin x, $(1+x)^n$; at some point, I became vividly aware of how disparate are the possible ways of 'imaging' such objects, and what a huge variety of things are going on in my students' minds when I pick on one such image to teach a given concept.

2. The second items that sticks out in my memory stems from the discussions at the end of day 1 and most of day 2: namely, the realization of how difficult it is to create a specific set of instructions to convey a particular image (like $|x-a|$ and its possible images), even once you've chosen a specific image. Also to what a large extent I talk about things rather than trying to get my students to think about them, and

3. I learned how to be more passive while inviting my students to become more active, when I am lecturing.

4. Images that illustrate an example of a phenomenon vs. images that create an initial awareness of a phenomenon. We frequently blurred the distinction (eg. imagery for π).

5. A difference between metaphor and image which I had not thought about before struck me forcibly.

6. I suddenly realized that the inability for me to demonstrate motion on the blackboard, in the text book, on the overhead in situations where I had been waving my hands could be possibly be overcome by deliberately evoking motion in mental images. I feel I always personally "saw" motion in my mind and hoped my students did.

7. To share imagery 'both' participants must be allowed to expose their own versions of the image. On more than one occasion during the seminars I became conscious that I was trying to impose my image on the other person - to know that I was doing that embarrassed me. I was in the 'teacher-trap' feeling that 'my' image was better than 'yours' and mine was the right one.

Reassuringly I wasn't the only person in the group making this mistake!

8. John was asking us to visualize a square moving through two fixed points and introduced a third point (which I visualized as being not the square already) and then suggested we move the square through the three points and I was stuck, unable to see that a square would go through three points!

Images: Multiplication by -1

* Imagine a numberline marked off with 0, 1, 2 , ..., -1,-2, ...
Make your numberline go from left to right.

* Rotate it about the 0, through 180 degrees.
Where does 2 end up? ...

This is one way of thinking about multiplication by -1.

Exploration

* Go back to your original number line.
Rotate it about 2, through 180 degrees.
How does its position relate to its starting position?
Where are 2, 3, 4, .. now?

Consider the advantage of showing a brief dynamic image (film/tape) of a line rotating about a succession of points, then inviting inner images. The film should provide a common basis for images, as well as a memory trace on which to impose particular questions.

Generalize: rotate about two points in succession; where will the line be then?

Resist algebra until you feel ready to express a generality which is as substantial and precise as you feel is possible, before using symbols.

Go back to your original number line. Rotate it about 2 through 180 degrees.
Now rotate it about the old position of 3 through 180 degrees. Where is the line now copared to its starting position?

* Generalize: rotate about two points in succession; where will the line be then?

* Generalize: rotate through several points in succession, all identified by their original names; where will the line be then?

Game: A move consists of instruction to rotate your numberline about some point (identified either by its new name, or its original name, but be consistent in each game). After a predetermined number of moves, or perhaps on a move selected by some random event such as rolling a 1 or 6 with a die, the next player MUST return 0 to its original position. Players may then challenge whether it has been done

successfully. Avoid using symbols to 'work it out'. Try to see it directly in your head.

Generalize: rotate about the point 2, through 90 degrees; now rotate about the current point 3 throuh 90 degrees; where is the line now relative to its starting position?

* Generalize to several successive rotations; probably best to agree that the line should always come back to the horizontal before trying to describe its position.

* Generalize to different angles.

Possible notaton for describing effects of rotations:
rotations through 180 degrees correspond to multiplication by -1; rotations through 90 degrees correspond to multiplying by a symbol, say r (which can be used for other agreed angles as well), or say i, which is traditional for 90 degrees.

Rotation through 180 degrees is a potentiely potent image for multiplication by -1 because it extends in such a rich manner.

PANELS

P 1 - CURRICULUM GUIDELINES - A HOPE IN PRINT - BY: DAVID ALEXANDER
MICHAEL SILBERT
DALE DROST
CLAUDE GAULIN

P 2 - THE IMPACT OF COMPUTERS ON UNDERGRADUATE MATHEMATICS BY:

PETER TAYLOR
GEORGE DAVIS
JOHN POLAND
KEITH GEDDES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

JUNE 1984 MEETING (WATERLOO)

REPORT OF PANEL ON CURRICULUM GUIDELINES - A HOPE IN PRINT

Statement for CMESG panel, Wednesday, June 6, 1984

<u>Curriculum Guidelines - A Hope in Print</u>

- D. W. Alexander

In asking me to participate in this session, David Wheeler suggested that the focus should be, "What are the problems of making curriculum change <u>effective</u> in the schools".

The SIMS project has given us a clarified model for the process of curriculum implementation in emphasizing the distinctions among

the intended curriculum
the implemented curriculum
and the attained curriculum.

The implementation of curriculum involves 'negotiations' between developer and teacher, teacher and student (or more completely: between developer and author, author and teacher, and author and student). Each 'negotiation' involves the exploration of and reinterpretation of 'meaning'.

This model re-emphasizes the fact, which we all recognise, that curriculum change supposedly mandated in documents such as Ministry guidelines, is nothing more than "A Hope in Print". Before it becomes implemented, it must be understood and believed in by teachers. A commitment must be made upon the part of teachers to implement the change and finally implementation must be engineered in such a way as to change the learning of students.

Fullan and Park in the resource booklet <u>Curriculum Implementation</u>, Ministry of Education, Ontario, 1981 identify the overall problem:

> "most efforts at curriculum and policy change have concentrated on curriculum development and "on paper" changes. ... the implementation process has frequently overlooked people (behaviour, beliefs, skills) in favour of things (e.g. regulations, materials) ... While people are much more difficult to deal with than things, they are also much more necessary for success".

They also state that crucial to implementation is that teachers:

see the need for the change;
are clear about the change, and do not perceive the change as too complex;

have available or are able to generate materials that incorporate the change and are practical.

While "on paper" changes are clearly only a start, they can contribute to implementation by making the change intended clear. It is this clarity I have been concentrating on over the last two years.

The easiest example to refer to is the issue of calculator usage. For a number of years curriculum leaders in Ontario, as elsewhere, have been advocating increased use of the calculator in mathematics classrooms. In 1980, the Intermediate Guideline Committee suggested that teachers explore methods of using calculators in a variety of ways, but they did not mandate the use of calculators, and they did not clarify the relationship between traditional arithmetic computation and computation using calculators.

A survey of students in Grades 7 to 10 inclusive, made in 1981, showed that 80% of the students owned or could easily borrow a calculator but less than 60% were permitted to use them in class and only 20% of those in Basic Level classes were permitted to use them on tests. (Classroom Processes in Teaching Mathematics, Ministry of Education, Ontario, 1983).

We decided to clarify the change advocated and include the following statement (which some of you may recognize as being an offspring of Mathematics Counts.)

Use of Calculators

> The calculator has become an integral part of our way of life. In recognition of this fact, schools should ensure that students become proficient and discerning in the use of calculators.
>
> Students should possess some reliable methods of carrying out calculations without the use of a calculator when a small number of digits are involved, and with a calculator when a large number of digits are involved.
>
> Calculators shall be used when the primary purpose of a given activity is the development of problem solving or other skills in which computation is of secondary importance.

It should be noted that the "shall" in the last sentence makes this a policy statement in Ontario. Thus we are

clarifying the change - the choice is no longer there for the teacher (on paper) - the calculator shall be used.

In support of this change we have also identified for Grades 7, 8, 9, 10 and for Grades 11 and 12 General Level Mathematics (for which the interface with the Colleges of Applied Arts and Technology imposes some terminal objectives) clarification of what "small number of digits" means in terms of calculations without a calculator. This clarity has an additional purpose, to encourage the change, on the part of Grade 7 and 8 teachers in particular, from spending so much time on computation that they have little time left for geometry and measurement.

Another area we have addressed is that of problem solving. The 1980 Guideline Committee had emphasized problem solving, but in very general terms. Curriculum committees trying to incorporate problem solving in the course outlines and resource document of their respective boards had wrestled with the issues of what should be done and at what grades. Recommendations were made to us to clarify the intent.

I endorse the view that in this area we must take what Fullan refers to as the Adaptive approach (as opposed to the Fidelity approach) to curriculum change. That is, I believe that the nature of the change we want in this area must be clarified through the process of implementing it. On the other hand, I also believe that the work such as that of Charles and Lester (JRME, January 1984) gives us a basis for being clearer in our expectations.

Thus, while we do not include specific content objectives for problem solving we do make statements which provide a framework for development of problem solving and further, we have tried to identify a sequence for the teaching of heuristics throughout the grades.

In the introduction we state:

Problem Solving

> Developing the ability to solve problems is a major goal of mathematics education. Problems are solved by drawing on past experiences - sometimes in a systematic manner, but often in flashes of creativity and intuition. Problem solving is not exclusively the domain of mathematics; it is an integral part of all subjects and of everyday life.

Systematic problem solving involves the following stages:

I The awareness of a situation in which there is given information and a goal
II The consideration of possible strategies (models)
III The choice of a strategy (model)
IV The carrying out of the strategy
V The verification of the solution in the situation

There are two major types of problems:

1. A strategy is evident immediately. Difficulties in solution are related to ability to carry out the strategy correctly.
2. A strategy is not immediately evident. Difficulties in solution are initially related to choosing an appropriate strategy.

A given problem may be of the first type for one individual, but of the second for another.

Word problems assigned after a mathematical concept or skill has been taught are usually of the first type, for most students, requiring only the application of a known algorithm to process a solution. It is essential that students also have experiences throughout each grade with problems of the second type. Generally, these problems should be solvable by a variety of strategies or by models and techniques that have not been recently taught or practised.

The following should be stressed in connection with the stages of systematic problem solving:

I Identify relevant and irrelevant information:

Read
Understand
Paraphrase
Summarize
List

II Identify possible strategies:

Classify information (insufficient, conflicting, extraneous, redundant)
Search for a pattern
Draw a diagram or flow chart
Construct a table

Estimate (guess and check; improve the guess)
Choose operations and sequence them
Assume a solution and work backwards
Use a mathematical operation, a formula, or write an equation
Solve a simpler problem (part of the problem)
Account for all possibilities
Check for hidden assumptions
Make an assumption and draw a conclusion

III Consider reasons for the choice of strategy:

Familiarity
Ease of implementation
Efficiency (elegance)

IV Carry out the strategy:

Work with care
Check work
Present ideas clearly
Persist (try, rest, try again, try another strategy)

V Determine how good the solution is:

Verify in the problem situation (reasonableness of result)
Generalize the solution to similar problems
Search for a better solution

Throughout the Intermediate and Senior Divisions, courses must include planned experiences based on Type 2 problems, which will strengthen the students' problem-solving skills summarized above.

While for Grade 7 and 8 we identify the following strategies for particular attention:

Problem Solving

Emphasis should be placed on the following aspects of problem solving:

I Identify relevant and irrelevant information:

Read
Understand
Paraphrase

II Identify possible strategies:

Classify information (insufficient, conflicting, extraneous, redundant)
Search for a pattern
Draw a diagram or flow chart
Construct a table
Estimate (guess and check, improve the guess)
Choose operations and sequence them
Assume a solution and work backwards
Solve a simpler problem
Make an assumption and draw a conclusion

III Consider reasons for choice of strategy:

Familiarity
Ease of implementation

IV Carry out the strategy:

Work with care
Check work
Present ideas clearly
Persist

V Determine how good the solution is:

Verify in the problem situation (reasonableness of results)

Hopefully this will provide sufficient clarity to give a basis for growth on the part of teachers. The danger here is that materials (and textbooks) will produce "false clarity" and translate this material into work with types of problems to illustrate each strategy, rather than emphasizing the critical aspect of conscious selection from a set of possible strategies.

The hope inherent in my last illustration may be the least attainable, since its implementation will be influenced more by economic and social concerns than by actions of teachers.

Some background is necessary:

Since 1972 there have been two major programs in Grades 11 and 12 in mathematics in Ontario. One, Foundations of Mathematics, was developed essentially as a pre-calculus program. The other, Applications of Mathematics, was developed to provide a general program for students not intending to take mathematics (or mathematics related subjects) at university.

One of the major destinations of students enrolled in the Applications of Mathematics courses should be Technology or Business programs of the Colleges of Applied Arts and Technology, however, the perception of teachers and students, strengthened by the level of difficulty of many CAAT courses, is that the program is inadequate preparation for Technology programs, in terms of the algebraic skills developed. The result has been an increasing pressure on the Foundations courses to accommodate students who in understanding and motivation are better suited for the General Level program.

Taking as a basis, a statement of desirable prerequisites which was endorsed by the Deans of Technology, we have constructed a program which should provide students with those prerequisites. The problem is that as long as there is a pool of Advanced Level graduates to draw on, it seems unlikely that the graduates of our Mathematics for Technology program will get preferential treatment for entry into Technology programs. For implementation of the change we desire, we must see a dramatic drop in students taking Advanced courses. This seems to be asking too much of a change in parental and student expectations in terms of "keeping the door to university open".

My dream is that the Deans of Technology will find the graduates of our new program so superior in the prerequisites they require that they will create admission policies that will enhance the appeal of the program. My fear is that we are in a chicken/egg situation and many students will continue with the new Advanced programs to the detriment of both themselves and the program.

I have tried to examine the problem of effective change in curriculum from the perspective of the development of Ministry guidelines. Mike Silbert will look at the same issue from a perspective a little closer to the classroom, that of a mathematics coordinator for a large school board.

In my view, intended change can only be achieved through the modified teacher belief and behaviour which comes from appreciation for the need for change and commitment to it. In a rational world, this would precede the development of the "on paper" change, but our systems of education do not seem to be adapted to such a process and it is quite clear our world is far from rational!

REACTION TO CURRICULUM GUIDELINES

A HOPE IN PRINT

By
Dale R. Drost

Before reacting to the presentations of David Alexander and Mike Silbert, I would like to describe briefly the context in which which I work in New Brunswick. New Brunswick is a bilingual province with less than 150 000 students enrolled in grades 1 through 12, approximately two thirds of whom are anglophone. The provincial department of education has two program development and implementation branches, one for curriculum work in each of the two official languages.

I am the mathematics and science consultant for the anglophone section of the department. Along with other tasks to which I am assigned, I am responsible for overseeing the mathematics and science curricula from grades 1 to 12. I attempt to monitor the situation and to keep teachers content. Silbert's notion of a consultant as a navigator providing information and charting a course based on the best information and technology available describes my position; however, due to the wide range of curriculum offerings in mathematics and science, more time is spent providing information than charting courses. Ontario is criticized for not having a permanent staff member responsible for looking after the interests of Mathematics on a full time basis. Although New Brunswick has such a person in theory, the wide range of responsibilities makes it difficult, if not impossible to do all that needs to be done in mathematics education.

In New Brunswick, we have three mathematics curriculum development advisory committees to work with the consultant to formulate and make recommendations regarding mathematics curricula. There is a committee responsible for each of the elementary mathematics curriculum, the junior high mathematics curriculum, and the senior high mathematics curriculum. Each committee contains members from the university community, as well as from the public school teaching community. Teachers form a majority on each committee.

Whenever necessary, a committee conducts an assessment of the current program and reviews the aims and objectives. Input is received from classroom teachers as much as possible. The statement of aims and objectives is transformed by the committee into a set of criteria for evaluating textbooks and other instructional material. Following a review of available materials, a selection is

made from those which rate highest upon evaluation. Pilot classes are established in several schools to assess further the materials and to identify inservice needs and other tasks attached to implementing the program.

Based on the results of the pilot projects, the committee makes a recommendation to a provincial curriculum advisory committee (PCAC). The PCAC is composed of representatives of such groups as the school superintendents, the teachers' association, trustees, and universities. If the recommended material is approved, steps are then taken towards implementation throughout the schools of the Province. If not approved, the recommendations are returned to the committee for further study.

In theory the process described above provides input into the curriculum development process from all concerned bodies. It also allows for curriculum materials of any nature, either those produced by commercial publishers or those developed locally. In reality, the various programs usually are defined eventually by the textbooks which are adopted.

To a large degree in Canada, the content of textbooks is defined by guidelines developed in the more populous provinces, particularly Ontario. Provinces such as New Brunswick, because of their smaller enrollments and therefore smaller market, have limited impact on textbook content and availability.

Recently, after evaluating our general mathematics programs at the grades 10 and 11 level, a decision was made to search for new textual materials. Two Canadian publishers had series available which appeared to meet our needs for the grade 10 course and had plans to develop an additional book which possibly could meet our grade 11 needs. However, because of delays in the production of guidelines in Ontario, each company has postponed publication of its additional book until it is more certain of Ontario's direction. In the meantime, we in New Brunswick have piloted and selected one of the programs for grade 10 and wait anxiously for the production of the additional book.

To be fair to publishers, they do allow us some input into their products. Sometimes they are prepared to add a chapter for us, or provide us with a booklet containing material not in the main textbook. Recently with respect to chemistry programs, publishers agreed to make changes to their programs to make them more Canadian. Although such changes are often minimal, they are of importance to the success of the program.

Silbert comments that curriculum guides with a content listing as a central feature create difficulties since they often do not provide a primary focus on the rationale for the curriculum. Most curriculum guides already provide at least a short statement of rationale - but do teachers read it, and if they do, do they abide by it? Most of us here would agree that statements on rationale for the curriculum, statements on issues such as calculators and problem solving, and statements as to specific elements of content that are to be included in the curriculum need to be included in provincial curriculum documents. Insurance that such intents are actually implemented and attained must come from extensive inservice programs, rather than from the documents themselves.

Alexander refers to the problem of preparing students for post secondary programs in technology in Ontario. We have experienced similar problems in New Brunswick. Students who have traditionally enrolled in some technology programs have studied general mathematics courses in high school. Community colleges complain that these students have an inadequate background in algebra and trigonometry. As a result many more students are now opting for the high school academic program to satisfy entrance requirements for technology programs. To some extent this lowers the average ability of students in the academic classes and to minimize failure rates, standards are sometimes lowered. Then universities complain because students are not prepared. A catch twenty-two situation results. In mathematics in most provinces we have several levels within our high school programs--levels which allow most students to study mathematics at a level commensurate with their background and ability. If our programs are to retain credibility, standards within each level must be set and maintained.

Many problems associated with implementing change in mathematics are mentioned by Silbert. These include political constraints, human constraints, school administrators who lack curriculum training, and teachers other than those trained in mathematics teaching mathematics. In New Brunswick, the provincial government currently is investigating the possibility of extensive reform in the educational system. Concern has been expressed that at the secondary level, a single core program for all students is being considered. Such a decision would be at the political level and not at the educational level. The other concerns mentioned by Silbert are also real and unlikely to change in the near future. Yet, he is optimistic about curricular change.

It could be argued that if publishers are unable to provide us with material congruent to our needs, then we should take more initiatives in developing our own material. For several reasons, this is a difficult alternative in New Brunswick. Our population is not only small, it is also largely rural and spread over a large area. To bring teachers together is costly and hence not feasible on a regular basis in the province. Finally, New Brunswick has tried to maintain the same program for all of its students with the same basic material used in all schools. In New Brunswick, development of materials at the local school or board level to supplement a core provincial curriculum is a feasible task; development of materials for that core is much more difficult, if not impossible given current circumstances.

Silbert asserts that 'curriculum guides are overrated documents in terms of impact'. Alexander adds that 'while on-paper changes are clearly only a start, they can contribute to implementation by making the change intended clear'. In New Brunswick our curriculum documents are modest ones which provide a brief rationale of the program and in most cases outline the sections of a textbook which constitute the content in the curriculum.

My informal observation of what is done in the classrooms of New Brunswick is that most of the mathematics taught is, in fact, recommended in the curriculum outline. However, everything in the outline is seldom done. Notable examples of this are geometry topics in the elementary and junior high grades, statistics and probability in our academic eleventh grade program, and an emphasis on 'real' problem solving throughout the curriculum. In other words, in New Brunswick the intended curriculum is not congruent to the implemented curriculum. I suspect the same is true in other provinces as well. The document may contribute to implementation but the impact is limited by the teacher's perception of what is important.

Alexander elaborates on statements on calculators and problem solving that are to be recommended for inclusion in Ontario's curriculum documents. Will such statements make any difference in the classroom? Will they have an impact? I suspect that, although the statements provide general direction, most teachers will be unable or unwilling to translate them into specific activities. The number of teachers who read the statements and implement the corresponding intended curriculum may be inversely proportional to the length of the statement.

Silbert advocates an ongoing rather than a shot-gun approach to change. None of us would disagree with this position, yet in New Brunswick, and I suspect in other provinces as well, the approach tends to be shot-gun. We tend to react to situations rather than keep them under constant review. Can it be otherwise with a single consultant responsible for all mathematics and science programs for grades 4 through 12.

Both Alexander and Silbert emphasize the importance of the involvement of teachers in all aspects of the change process. The system must be open, and must also be perceived to be open, to all of those involved - teachers, pupils, parents, trustees, administrators. In New Brunswick we attempt to keep our system open through our curriculum committees and by close cooperation with teachers' subject councils and district professional development committees. Yet, ironically, those whose views do not prevail see the system as not involving them. It is difficult, if not impossible, to keep all people happy with respect to the curriculum.

How can the intended, the implemented, and the attained curricula be made more congruent. In New Brunswick beginning in June 1985, there will be compulsory provincial examinations in mathematics for all students at the end of grade 11. Will this place teachers in a position where they must at least cover the intended curriculum? Already some teachers complain because of too much material in the curriculum and too little time. At the same time, institutions of higher learning complain because students do not know enough. It is possible to make the intended curriculum congruent to the attained curriculum. I suggest yes, but it will not be easy.

The teacher is the key and teachers for the most part need direction - specific direction. This can be done through curriculum guides, making teachers aware of the intended curriculum and the rationale for it. Teachers also need inservice - specific inservice directed at particular problem areas. The inservice needs to be more extensive than is presently often the case to ensure that the intended curriculum becomes implemented. And finally teachers need feedback - specific feedback on how they and their students are doing. Given proper use of this feedback the implemented curriculum can become the attained curriculum. And, of course, to do all of this requires funding - specific funding directed at particular problem areas. That funding must come from provincial governments.

Silbert claims 'the real challenge is to prepare curriculum for teachers learning, not student learning'. To a large extent I agree, however we must not lose sight that the teacher learning must result in student learning. At that time the 'hope in print' will become a reality.

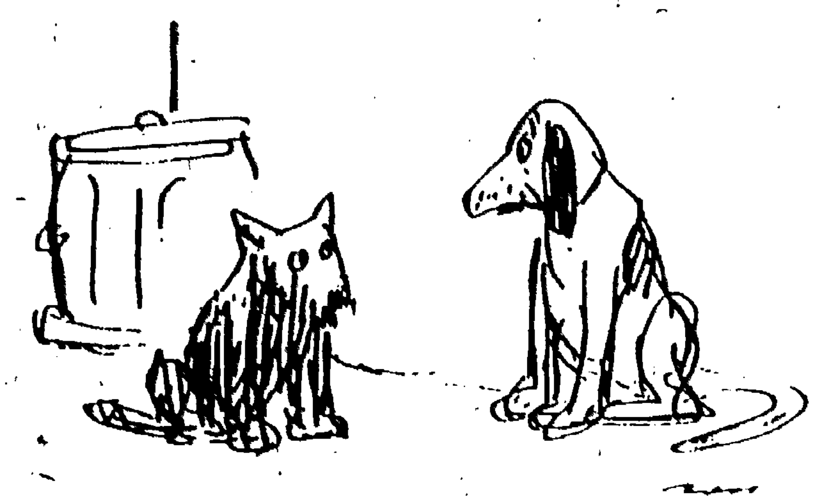
CURRICULUM GUIDELINES: A HOPE IN PRINT†

Panelist: Michael R. Silbert

In order to put my comments in perspective, I feel that I should share with you my mandate. As Supervisor of Mathematics for a reasonably large urban school board, I am officially responsible for all aspects of the quality of program and instruction in Mathematics from Kindergarten through Grade 13 throughout our jurisdiction. This includes staffing our Mathematics programs as well. While my comments will focus primarily on the consultative aspects of my mandate -- those aspects which relate to effecting curricular and instructional change -- I will be touching, albeit briefly, on issues related to staff selection.

What is a consultant? Well, I use to have a tom cat. He'd sleep all day and be out all night carousing and causing a lot of trouble in the neighbourhood. So I took him to the vet and got him fixed. Now he's only a consultant!

The following cartoon puts it another way:



"I don't know. My bark isn't worth a damn and my bite isn't worth a damn."

† I owe a tremendous debt of gratitude to my colleague, Gail Rappolt, for the discussions we had prior to the preparation of these remarks and for permission to paraphrase and excerpt from her recent article, "Effecting Educational Change", in Contact 61 published by the Canadian Studies Foundation.

An ongoing theme in the literature on teaching and learning is that teachers tend to project their own learning style through the same content and methodology employed in their own education. If this is so, and my own observations tend to corroborate this, then one of the most difficult dilemmas facing a consultant trying to implement new curriculum is how to intervene effectively in this cycle and help people change.

Often, teachers appear unwilling or unable to change. They seem determined to hang on to classroom strategies and evaluation practices which result in high failure and drop out rates. They hold on to these techniques in spite of a host of theoretical and empirical evidence indicating that learning will take place using more diverse and exciting techniques. The solution to this dilemma is part of the problem -- our own education. As Bertrand Russell noted in his Skeptical Essays, "We are faced with a paradoxical fact that education has become one of the chief obstacles to intelligence and freedom of thought."[1] Although it is difficult to break out of our own thinking patterns, we must try to focus on changing our curriculum and teaching strategies based on the findings in two areas: curriculum and learning theory. Instead of enthusiastic support for these new ideas, I encounter resistance and disbelief.

Unfortunately, rather than facing this problem -- and it is certainly one of considerable magnitude -- it is both easier (and safer) to lash out at an impersonal, distant curriculum which appears to deny the *reality* of the teacher that we *do* have and *will continue* to have in our classrooms. One thing that has gradually become evident to me is that you cannot depend on your eyes when your imagination is out of focus, and my own focus, given my mandate, must be on intervening in an effective way with teachers to help them recognize and alter those behaviours and beliefs which impede curricular and instructional change.

The problem we face is a tremendous one and the resources are clearly limited. But constructive change, I believe, is indeed possible.

Thinking of the following historical analogy brings the problem into sharper focus.

Picture an inn in Palos, Spain, August 3, 1492. At the table beside you a group of sailors cum philosophers are talking after several jugs of wine. It is the evening after Columbus has sailed. The conversation you overhear may have gone something like this:

First Sailor: "Any fool can see the world is flat!"

Second Sailor: "The world may well be round, but around here it's flat."

Third Sailor: "I have read, listened to the experts and looked at the maps and charts -- there is no other logical explanation -- the world must be round."

[1] Bertrand Russell, Skeptical Essays. (London, 1928), p. 160

Fourth Sailor: "Round, flat or triangular -- finding out won't put food on my table -- we are all still out of work!"

Over the past ten years I have come to accept that my attempts to convince teachers to change their practice, other than superficially, are about as likely to succeed as attempts to get consensus at that table in Palos. Note that even sailor three, who was convinced, was not on board one of those ships!

How does the analogy fit? Like Columbus, I begin with an assumption: that ideal education is process oriented, addresses several different learning styles, recognizes right/left brain dominance and is individualized to the degree that learners believe their own needs are being met. The following figure presents my image of the IDEAL -- analogous to Columbus' ROUND WORLD! I haven't seen the ideal on a large scale yet, but I believe strongly enough that it is both desirable and possible that at least I'm on board one of the ships!

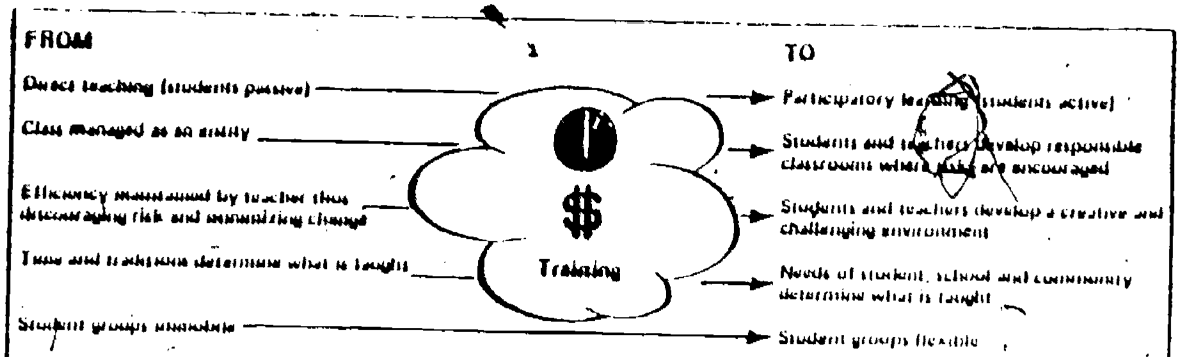


Figure 1
Goals for Educational Change

permission to reprint granted by G. Happold & Associates

However, as a consultant, I am aware that many teachers are still in Palos, and I might add, feel the same way about my educational ideal as sailor one felt about the round world. Given this, I must now be prepared to accept dismissal of my view of the world and see the responses of the four sailors as merely points along the continua of knowledge, skills and attitudes. Realistically, what possible evidence would the average sailor encounter that would suggest the world was round?

Consider how foolish it would have been for sailor one to have been on board the Pinta. He was thoroughly convinced that the world was flat -- he had no other experience. Likewise, teachers' views reflect their experience -- be it from their own education, their own reading, or from in-service or upgrading.

The key, as a consultant, is to see the teacher as merely having certain beliefs at a given point in time (not forever) and to see the consultant as a navigator providing information and charting a course based on the best information and technology at his/her disposal. However, the consultant is not the captain, and thus is critically dependent on information, credibility and marketing to make her/his case.

But enough of this talk of sailors from a bygone era! This panel is about curriculum -- "Curriculum Guidelines: A Hope In Print".

What is meant by curriculum? Is it, as the Ontario Ministry of Education suggests, "all those experiences of the child for which the school is responsible"?[2] This definition causes me some personal difficulty as it suggests that curriculum is a consequence of authority rather than a result of planned action that arises from a carefully thought out, well-researched plan. While this 'after the fact' definition certainly suggests that accountability coexists with authority, it focuses primarily on the implemented and attained curriculum and not on what was intended but perhaps not experienced.

Curriculum guidelines, in the Province of Ontario at least, are vastly overrated documents in terms of their direct impact on classroom instruction. These general documents, vague in intent and distant from the reality of the teacher and the classroom, have little direct impact when compared to learning materials such as texts and workbooks. Admittedly, the content listing in the guideline makes itself felt in the classroom indirectly through these learning materials but, in terms of impacting on the process component of learning, the present guidelines are abysmally unsuccessful.

One of the profound difficulties with a curriculum which has a content listing as a central feature, be this intentional or otherwise, is a lack of a primary focus on the rationale for the curriculum as a framework for interpretation. This lack of focus can lead to significant misinterpretations of the intent of the statements in the content listing and to a course development that is diametrically opposed to that intended by the guideline developers. This point was forcefully brought home to me in recent weeks by two mathematics leaders who have played an active role in the current process and who are preceived as 'innovators'. It was their belief that the 'caps' (restrictions) on some of the more traditional arithmetic and algebraic operations meant that more time was to be spent on these topics -- not less! They had equated increased specificity in the core content listing with increased proficiency, or greater mastery, if you like, and greater time on task.

Documents do not effect change; people effect change. As Cooper and Petrosky reported on a study made almost a decade ago, "Students' attitudes and learnings are directly influenced by the personal qualities of teachers and the classroom climate they create."[3]

[2] Ministry of Education, Ontario, Education in the Primary and Junior Divisions. (Toronto, 1975), p. 3.

[3] Charles R. Cooper and Anthony Petrosky, "Secondary School Students' Perceptions of Math Teachers and Math Classes", in Mathematics Teacher, (March, 1976), p. 227.

The minimal impact of our curriculum documents on even the implemented curriculum is clearly evident in the findings of the recent provincial review of Mathematics in the senior division and the early 70's study of "Current Ontario Elementary School Mathematics Programs". Clearly something in the process is not working; what was intended is not being implemented.

Unfortunately, in Ontario, politics has been the significant driving force behind the current set of curricular revisions although I would be the first to agree that some of the areas of change are long overdue. This conclusion is only too easily formed. The guidelines for Grades 7 to 10 which were developed in the early 60's took over 15 years to be superseded and yet revisions on their successor commenced less than 2 years after they had been disseminated to the schools. One of the early criticisms that we, as a writing team had to face, was how we could possibly be changing courses that had not yet been fully implemented. Such an action, while being consistent with other non-curricular educational thrusts, left some curriculum support personnel at the local level shaking their heads in disbelief of the Ministry's apparent lack of respect for the integrity of the implementation process.

The curriculum model that, for all intents and purposes, is operational in Ontario today is the "add water and stir" or "snapshot" (some might argue "slapshot") approach to curriculum design, development, implementation and review. This discrete model treats the phases of the curriculum cycle as "events" rather than as a series of complex, interrelated processes.

The curriculum process, as a whole, is admittedly operating under a number of environmental constraints, noteably:

- Political constraints
- Human constraints (time available, rate of learning)

There is a serious under-resourcing of the process at both the local and provincial levels; for example, there is no one presently on permanent staff in the Curriculum Branch of the Ministry whose primary responsibility is to look after the interests of Mathematics on an ongoing basis.

Another matter of considerable personal concern is the partitioning of assessment, curriculum and instruction. At a time when there is a strong cry for "accountability" we should be attempting to develop a symbiotic relationship between curriculum, instruction and assessment. It is my personal belief that assessment should reflect instruction, and instruction should reflect curriculum, both the intent and content, to the greatest extent possible. Is it any wonder that our teachers have not developed a coherent, integrated approach to these three aspects of the teaching/learning process given the model that we have provided? When "push comes to pull" between curriculum, which is general, and assessment, which is specific, it is the latter which rules the day with curriculum taking a back seat! When we come across "exemplary strategies", do we build them equally into both our instructional and assessment repertoires? The answer, generally, is no.

At the local level, there are two additional problems that have significant impact on effecting curricular change.

At the administrative level, many administrators lack the curriculum training and experience and hence do not have either the tools or the credibility to convince teachers that anything but a teacher-centred, rule-oriented mode of operation will provide them with the necessary security and stability in the classroom.

At the classroom level, many educational jurisdictions in Ontario have responded to declining enrolment patterns by establishing a subject-independent seniority system. As a result, an increasing number of teachers are teaching mathematics without adequate background in the discipline and with no formal mathematics education preparation. In our jurisdiction, 10-15% of our instructional lines are being taught by these "cross-over" teachers while at the same time, of the 22 probationary teachers in our secondary schools who recently had their contracts terminated due to declining enrolment, 7 (31.8%) were teaching mathematics on a full or part-time basis.

Given the present constraints we face, is change possible? My answer is an optimistic, enthusiastic YES!

Clearly what is needed is a dynamic, responsive curriculum process with an appropriate allocation of resources to support all phases, noteably the implementation and assessment phases.

How might the "add water and stir" process we now engage in be modified to optimize the possibility of effective curricular change? Certainly, a more rational procedure for effecting change would provide for an ongoing rather than a shot gun process of curriculum development and implementation with appropriate mechanisms built into the process for monitoring, maintaining and updating the curriculum.

Two notes of caution apply here:

1. As with other "sequential systems", the process is bound by its least effective phase. If inadequate resources are applied in the implementation phase, for example, additional resources in other phases will neither speed up nor improve the quality of the overall process;

2. As with virtually all human endeavours, there is a "law of conservation" which governs the curriculum process.

Change must be manageable in amount and pace and should be instituted as an accepted part of the teaching/learning process. When we are assessing teacher performance we must go beyond looking at what the teacher knows and should be asking, "What have you done recently? What have you learned?"

Experience has demonstrated that most individuals are reluctant, in general, to change their present practices unless they are forced to do so or believe that they have something to gain as a result of their change in behaviour. Two of the strongest non coercive motivators for change are a reduction in work load to achieve the same level of output or an increase in

perceived productive output for the same level of energy expenditure. With these factors in mind the incentives and rewards for the users of new curriculum innovations are very dependent on the design and extent of the support provided on implementation. We must increasingly shift our focus to developing and nuturing intrinsic motivation to change on the part of teachers as evidence suggests that "appeal to authority" alone will not work.

In order for students to derive full benefits from a new curriculum, teachers need to be prepared through the identification of needs, an awareness of the potential of the curriculum to meet those needs and a familiarity and comfort with both the conception and actualization of the curriculum. The development of these environmental factors is as important as the development of the curriculum itself. Unless the clients perceive both needs on their part and significant benefits resulting from their use of the curriculum, the cost of the project will not be justified by its benefits.

In order to encourage full participation in new curricular developments, individuals and groups must, in addition, perceive that their participation will not be directed against their own interests. The openness of the current process has done much to encourage an active rather than reactive participation on the part of members of the educational community across Ontario.

Motivation to change, alone, is a necessary but not a sufficient condition. Even with a high level of intrinsic motivation there is still a need for training.

If curriculum is to be more than just "a hope in print", time and money must now be invested in quality teacher education programs in order to implement the curriculum that has been developed.

For those who wish to design better curriculum, the real challenge is to prepare curriculum for teacher learning, not student learning.

That this will be done effectively is my hope!

CANADIAN MATHEMATICS EDUCATION STUDY GROUP
JUNE 1984 MEETING (WATERLOO)
REPORT TO PANEL ON
THE IMPACT OF COMPUTERS ON UNDERGRADUATE MATHEMATICS

Panel on the impact of computers on undergraduate mathematics
Chairman: Peter D. Taylor (Queen's)

At the June 1982 meeting of the CMESG one of the working groups, led by Tony Thompson and Bernard Hodgson, explored the impact of computers on undergraduate mathematics education. The group felt its deliberations to be rather successful, and at the following meeting, in June 1983, Bernard and John Poland sat down and prepared a paper which they published in the November 1983 edition of the Notes of the Canadian Math Society.

At the June 1984 meeting of the CMESG, the topic was again pursued with a panel chaired by Peter Taylor. The three panelists were John Poland (Carleton), George Davis (Clarkson) and Keith Geddes (Waterloo). John spent 15 minutes highlighting the paper referred to above, in particular, addressing the question of how the current undergraduate curriculum might be modified to take advantage of (or be enriched by) new developments in computer hardware and software.

George Davis is a mathematics teacher and Professor of Educational Development at Clarkson University. He is instigator, architect and administrator of the much talked about Clarkson scheme which puts a microcomputer into the hands of every entering freshman. He presented some of the philosophy behind and details of the scheme and then gave a number of pedagogical examples from his lab course in mathematical modelling. He has kindly prepared a digest of his presentation, which is appended.

Keith Geddes is Associate Professor in the Computer Science Department at Waterloo and works with algebraic algorithms for symbolic computation. He is currently involved in the implementation of MAPLE, a language suited to exact manipulation of matrices and functions. In his presentation he described the central features of this language. It aims to provide software which can be easily used by undergraduate students on microcomputers, and which will manipulate and factor polynomials, differentiate and integrate functions, and manipulate matrices, all with rational arithmetic. He hopes to have a version available in 1-2 years.

Appendix A Microcomputer for Every Student

A. George Davis - Clarkson University

Clarkson University has an undergraduate enrollment of approximately 3300 students and a graduate enrollment of nearly 300. Of the undergraduates 63% are engineering students, 20% are science students and 17% are in management studies.

In the spring of 1982 the President asked the faculty to submit proposals for "peaks of excellence" in undergraduate programs. In the broad computer area three proposals were submitted:

(1) hardware for computer labs for majors
(2) a retraining institute for solving the shortage of college level computer science instructors.
(3) a microcomputer for every undergraduate.

Given the nature of our institution these were all natural proposals and by October 1982 the Board of Trustees was persuaded that the funding of these proposals would create a "peak of excellence" in education.

I will concentrate on the microcomputer proposal since it was mine. The heart of the proposal was to require each student to have a microcomputer (one specific model to be chosen). The student would upon entry pay a fee for maintenance of the computer for the 8 semesters in attendance. At the conclusion of the four years the student would be given the computer but until that time the University would maintain ownership. The program was to be phased in over 4 years by providing computers for each first year class. Thus, the institutional investment in dollars would level off after 4 years. The program began in the fall of 1983 so we have completed one year of experience. The faculty who were to teach freshmen courses in the fall of 1983 were provided with a computer in December of 1982 so that they could have lead time to develop material for their courses that relied on the computer. Since every student at Clarkson is required to take a course in computer programming some use was guaranteed. The biggest use outside of these courses was in the humanities course where word processing was widely used. Broad use of computer-based tutorials was made in the basic physics course where the professor produced some 30 tutorials and students produced approximately 200 tutorial lessons. These tutorials were distributed through the curriculum support area of the Educational Resource Center. The major question facing us now as we prepare for sophomore courses and beyond is "How do we make certain that we don't have horseless-carriage courses?". By that I mean the micro should not only influence how we teach but also what we teach. Real thought needs to be given to what should course content be now.

To that end let me describe my experiment with a mathematics laboratory. This course was started 4 years ago with one micro and 10 programmable calculators. Its basic thesis is that interesting problems are investigated now - not the traditional approach that you can only look at them after you have taken courses A, B, C, etc.

Programming is not the emphasis - but the development of mathematical models is the focus. Simple programs that are easy to alter are used to investigate the implications of the models and the predicted outcomes are checked with the real world pheonomena.

This past year among the projects investigated were:

(1) A practical non-euclidean geometry with implications for city traffic, robotics and information storage. Here at first a rectangular pattern of "streets" was assumed and all points were considered, not just "street-corners" as in "taxi cab geometry". Questions addressed the nature of "straight lines", perpendicular bisectors, etc. Some of the better students investigated similar questions for a radial pattern of streets.

(2) Exponential decay or growth. Here one of the goals was to get students to guess at a closed form of the solution.

(3) Pendulum. This was investigated after the answer for the period of a simple pendulum had been developed in their physics course assuming small oscillations. Questions addressed include "How small is small", how long must arm be in order to just detect a 5% change in gravity - different answer for different initial angle?, interaction of precision of time measuring device with the above question. My question is, given the tools now available why do we do the "small oscillation" work at all?

(4) Mechanics. (a) The flight of a baseball. Questions include how should the batter behave differently in Fenway Park, air is heavy - impact?, the complex job of getting under a fly ball, how to throw balls to home on the fly or bounce? (b) Advantage of being tall in basketball - aside from rebounding.

(5) Length of Arc. (a) Length of first arc of the sine curve. Interesting, here students don't believe the answer because when we draw sine curve we rarely use the same unit of distance on each axis. A chance to investigate mathematical theory when outcome conflicts with intuition. (b) Length of an ellipse

(6) Monte Carlo Methods.
- (a) Buffon Needle problem
- (b) arc length of an ellipse
- (c) logarithm
- (d) why did using average radius of ellipse work well for arc length but fail for area? (a model question)

(7) Random numbers (central limit theorem)
- (a) shuffling a deck of cards
- (b) energy states in atomic chemistry and physics

I will spend some of my summer working out new questions to illustrate these same concepts so I won't be tempted to short cut investigations but be truly as involved as the students in the model work. In some way I will, get my questions anywhere - Scientific American, todays newspaper, the gambling table, etc. ...

TOPIC GROUP L

FAMOUS PROBLEMS IN MATHEMATICS
AN OUTLINE OF A COURSE

BY: ISREAL KLEINER
DEPARTMENT OF MATHEMATICS
YORK UNIVERSITY

CANADIAN MATHEMATICS EDUCATION STUDY GROUP
JUNE 1984 MEETING (WATERLOO)
REPORT OF TOPIC GROUP (L)

Famous Problems in Mathematics: An Outline of a Course

Israel Kleiner, York University

This is a one-semester course at the 3rd year level offered in the department of mathematics at York University. (Some technical details about the course are given at the end.) It is not a course in the history of mathematics, but it has a significant historical component. In fact, one of the objectives of the course is to make students aware that mathematics has a history, and that it may be interesting, useful, and important to bring history to bear on the study of mathematics.

The course tries to legitimize in the eyes of students the point that it makes sense to talk about mathematics in addition to doing mathematics; that it makes sense to deal with ideas in mathematics in addition to dealing with "mathematical technology". In brief, the course attempts to make students more "mathematically civilized". (The phrase in quotes is the title of a "letter to the editor" written by Professor O. Shisha; it appeared in the A.M.S. Notices, v.30 (1983), p.603).

Before dealing with the so called "famous problems", let me list some ideas or themes which I try to pursue in the course, with brief indications of intent:

(a) The origin of concepts, results, and theories in mathematics.

A relevant major theme of the course is that "concrete" problems often give rise to abstract concepts and theories.

(b) The roles of intuition and logic in the creation of mathematics

Students often see only the logical side of the mathematical enterprise. But, in the view of Hadamard, "logic merely sanctions the conquests of the intuition". History often bears him out. On the other hand, there were times in the evolution of mathematics when logical rather than intuitive thinking was the creative force. (The creation

of non-Euclidean geometry and set theory are prime examples.) For the working mathematician there is, of course, an ongoing interplay between intuition and logic.

(c) Changing standards of rigor in the evolution of mathematics.

The concepts of "proof" and "rigor" in mathematics are not absolute but change with time. Moreover, the change is not necessarily from the less to the more rigorous - there are fluctuations in standards of rigor. In fact, I think that what we are witnessing nowadays (both pedagogically and professionally) is a reaction against the strict rigor and abstraction which have dominated mathematics for much of this century.

(d) The role of the individual in the creation of mathematics.

The sociological theory concerning the development of mathematics can be summarized succinctly and poetically by the following statement of J. Bolyai: "Mathematical discoveries, like springtime violets in the woods, have their season which no human can hasten or retard." At the same time, the discoveries are made by humans - humans with personalities, passions and prejudices which sometimes have a bearing on the mathematics they create. (Cantor is a case in point.)

More generally, the intent is to pay attention to the creators as well as the creations of mathematics (i.e., the human drama in the creation of mathematics).

(e) Mathematics and the physical world.

The relationship between mathematics and the physical world is a longstanding one. It has enriched both mathematics and our understanding of the physical world. Moreover, our view of this relationship has changed over time (especially in the 19th century). Witness the following words of Whitehead: "The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."

(f) The "relativity of mathematics".

By this I mean that mathematical truths are not absolute but depend on the context. For example, the statement "If $a + b = a + c$ then $b = c$" is true in the domain of, say, real or complex numbers but false in the domain of transfinite numbers. Again, the equation $x^2 + 1 = 0$ has no solutions in the domain of real numbers, [illegible] solutions in the domain of complex numbers, and infinitely many solutions in the domain of quaternions.

(g) Mathematics — discovery or invention?

This question arises more or less naturally in connection with various mathematical developments in the 19th century which are dealt with in the course. Moreover, one need not opt for one characterization or the other. Davis and Hersh (in their book The Mathematical Experience) suggest that the typical working mathematician is a Platonist on weekdays and a formalist on weekends (thus viewing mathematics at one time as a discovery and at another as an invention).

Remark

The above themes are, of course, of major importance in the history and philosophy of mathematics, and one can not treat them exhaustively in a one semester course. They are, however, central to the course. Moreover, they are not dealt with one by one (as listed above), but rather are discussed in the course of dealing with the "famous problems". So much

for the underlying themes. Now to the "problems".

The content of the course is flexible and one can choose the problems more or less as one pleases. Here are some of my choices. They are dictated by personal taste, by the level of the course, by the fact that the subject matter of the problems is usually not dealt with in the standard courses, and, most importantly, by the relevance of the problems to the themes which I am trying to expound. (You will note that "problems" are interpreted quite broadly.)

1. Diophantine equations

e.g., $x^2 + y^2 = z^2$; $x^2 + y^2 = z$; $x^2 + 2 = y^3$; $x^n + y^n = z^n$, $n > 2$.

The common thread in the four equations is that one embeds concrete questions about integers in a theoretical (algebraic) framework of unique factorization domains (e.g., $x^2 + y^2 = z^2$ yields $(x + yi)(x - yi) = z^2$, an equation in Gaussian integers). Once developed, the theory bears on the solution of the concrete problems one started with. (The story of Fermat's equation is more complex.)

Such questions gave rise to a new branch of mathematics, namely algebraic number theory and, in particular, to such concepts as unique factorization domain, ring, field, ideal.

2. Distribution of primes among the integers

Here one embeds "practical" questions about integers in the theoretical framework of analysis. The starting point is Euler's identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ \text{prime}} (1 - p^{-s})^{-1},$$ s any real number > 1, which

Euler used to give an alternate proof of the infinitude of primes and to show that the primes are (in some sense) "dense" among the integers (i.e., that $\sum_{p \text{ prime}} 1/p$ diverges with $\sum_{n \text{ integer}} 1/n$; cf. the convergence of $\sum_{p \text{ twin prime}} 1/p$). This was the first instance of the use of analysis in number theory and led to analytic number theory.

Other problems considered are prime-producing formulas (e.g., Fermat & Mersenne primes, the Matijasevich polynomial, Dirichlet's theorem on primes in an arithmetic progression), the Prime Number Theorem, and the Riemann Hypothesis.

Remark on Problems 1 and 2

In addition to providing illustrations of some of the themes mentioned above (e.g., (a), (b), (e)), the study of Number Theory, as exemplified in the first two problems, sheds light on the following points:

(i) "Simplicity" in mathematics is complex (there is an abundance of "simple" questions to which there are, as yet, no answers).

(ii) To study problems in a given system (in this case, the integers) it is often very helpful to enlarge the system (a recurrent theme in mathematics).

(iii) The role and limitations of the computer in mathematics.

3. Polynomial equations

The cubic, the quartic, and higher degree equations.

Students think that it was the quadratic equation (e.g., $x^2 + 1 = 0$) which led to the introduction of complex numbers. This is not the case. In fact, it was the cubic which gave rise to complex numbers. The "why"

and "how" of this "story" are explored.

The complex numbers are an interesting case study of the genesis, evolution, and acceptance of a "mathematical system".

Some indication is given of how the study of permutations of the roots of a polynomial equation aids in the study of solutions of the equation - an important source of the rise of the group concept.

This problem illustrates themes (a), (d), (e), and (g).

4. Are there numbers beyond the complex numbers?

The answer depends on what we mean by "numbers". We explore the historical evolution of the various number systems (indicating gains and losses at each stage of the evolutionary process), and then introduce the quaternions and the octonions (Cayley numbers), indicating how these, in turn, led to the study of non-commutative algebra.

This problem illustrates themes (a), (b), (d), (e), (f), and (g).

5. Why is (-1)(-1)=1?

This is an instance of the general problem of the (logical) justification of the laws of operation with negative numbers. It became a pressing problem (for both pedagogical and professional reasons) at Cambridge University around 1830. Peacock and others set themselves the task of resolving this problem by codifying the laws of operation with numbers. This was perhaps the earliest instance of axiomatics in algebra. The seeds of "abstract algebra" that emerge here are:

(i) The manipulation of symbols for their own sake (so called symbolical algebra); interpretation comes later.

(ii) Some freedom to choose the laws obeyed by the symbols.

The problem illustrates themes (a), (b), and (c).

Remark on Problems 3, 4, and 5.

Problems 3, 4, and 5 come from algebra and are an indication of the transition from "classical" algebra (the study of polynomial equations and laws of operation with "numbers") to "modern" algebra (the study of axiomatic systems).

6. Euclid's parallel postulate.

This problem gave rise to the creation of non-Euclidean geometry, the re-evaluation of the foundations of Euclidean geometry, and the study of axiomatics. It is an excellent topic for raising interesting issues (e.g., what is mathematics?) and, in particular, addressing all the themes (a) to (g) mentioned above.

7. Uniqueness of representation of a function in a Fourier series.

The study of Fourier series had a great impact on subsequent developments in mathematics. The problem of unique representation was addressed by Cantor and this led him to the creation of set theory and the clarification of the concept of the (actual) infinite.

In this connection, we study cardinal arithmetic, and algebraic and transcendental numbers.

This is an excellent topic for illustrating themes (a), (b), (d), (f), and (g).

8. Paradoxes in set theory.

Various approaches to resolving Russell's paradox concerning the set $N = \{x : x \notin x\}$ led to different axiomatizations of set theory in the

early 20th century. (e.g., Russell's theory of types forbids asking if $N \in N$; the Zermelo-Fraenkel theory forbids the formation of N; the von-Neumann-Gödel-Bernays theory classifies N as a class but not as a set.)

Among other causes, these axiomatizations led to various philosophies of mathematics (logicism, formalism, intuitionism).

The problem helps illustrate themes (a), (b), (c), (d), and (f).

9. Consistency, Completeness, Independence.

Here we study the continuum hypothesis and, especially, Gödel's theorems and their impact.

These matters illustrate themes (b), (c), (f), and (g).

Remark on Problems 6, 7, 8, and 9.

In addition to illustrating the various themes as indicated, these problems relate to questions in the philosophy of mathematics, and especially to the fundamental question about the nature of mathematics.

Notes

(i) In a one-semester course one can deal with only some (at most six) of the above nine problems.

(ii) No textbook is used. However, many references are given and students are expected to go to the library and read some of them!

(iii) The prerequisites for the course are any two mathematics courses. Students with only this minimum prerequisite are asked to take concurrently at least one or two more mathematics courses. (One is looking for the elusive quality of "mathematical maturity".)

(iv) The technical aspects of the course (which constitute about 1/3 to 1/2 of the course) are not very demanding. Many students, however, find the intellectual aspects demanding. To deal with ideas in mathematics, to be asked to read independently in the mathematical literature, to write "mini-essays", are tasks which mathematics students are not - but should become - accustomed to.

Other Problems

Here are a few more problems (technically somewhat more demanding) which may by considered in such a course.

a) The Königsberg bridge problem; the Euler-Descartes theorem for polyhedra; the Four-colour theorem; (motivated the development of graph theory, topology).

b) Measurement - length, area, volume; (motivated the development of the integral).

c) "Exotic" functions; space-filling curves; (motivated the rigorization and arithmetization of analysis).

d) Isoperimetric problems; other maxima and minima problems; (motivated the creation of the calculus of variations).

e) Aspects of Fourier series; (led to a re-evaluation of a number of fundamental concepts of analysis such as function, integral, convergence).

TOPIC GROUP II

INTELLECTUAL RESPONSIBILITY - A HISTORICAL APPROACH

BY: ABE SHENITZER
DEPARTMENT OF MATHEMATICS
YORK UNIVERSITY

CANADIAN MATHEMATICS EDUCATION STUDY GROUP
JUNE 1984 MEETING (WATERLOO)
REPORT OF TOPIC GROUP (H)

INTELLECTUAL RESPECTABILITY - A HISTORICAL APPROACH

ABE SHENITZER, YORK UNIVERSITY

In my talk I described three courses* that I have taught in recent years at York University. Each of these courses has considerable and historical components. I also mentioned a number of recent books that could help teachers devise "intellectually respectable" courses.

Course 1 A topics course that centered on a number of major mathematical issues.

Course 2 Largely a course on the history of the calculus stressing the thematic and genetic approaches. (Main text: C.H. Edwards, The Historical Development of the Calculus, Springer-Verlag, 1979.)

Course 3 Issue (1). Klein's view of Geometry Illustrated by the Study of Euclidean, similarity, affine and projective geometries.

Issue (2). Hyperbolic geometry and the impact of its discovery on mathematics.

COURSE 1

A topics course in Mathematics*

I introduced this course into a program offered by the department of mathematics at York University and intended primarily for in-service teachers.

The following major topics were covered in 1982-83.

1. The evolution of the number system. (The discussion included a comparison of the contributions of Eudoxus and Dedekind, and consideration of major shifts of viewpoint.)

2. The Method of Archimedes.

3. Some Greek construction problems and their modern algebraic solutions.

4. Kepler's law and Newton's law of Universal Gravitation.

5. Huyghens' cycloidal clock. (This was intended as a simple demonstration of the power of the calculus and of the importance of the idea of curvature.)

6. Minima and maxima. From the isoperimetric problem to the calculus of variations.

7. The discovery of hyperbolic geometry and its intellectual implications.

8. Geometry, geometries, and Klein's Erlangen program.

9. Fourier's series - its genesis and impact of mathematics.

10. Hilbert's third problem. (The impossibility of an elementary theory of volume.)

11. Uncertainty as progress. (Comments inspired by some of the work of Gödel.)

---

* A detailed description of the course and of the topics listed above has appeared in volume 30, issue 1/1984 of the German journal Der Mathematikunterricht, published by Friedrich Verlag Velber, 3016 Seelze 6, West Germany. For an English version write to: Abe Shenitzer, Department of Mathematics, Downsview, Ontario, Canada M3J 1P3

Remark

When I teach this course in 1984-85 I shall call the attention of student lecturers to new possibilities. Specifically, topic 4 on Kepler and Newton could profit from the inclusion of relevant material on Einstein along the lines of splendid two-page essay in a first book on physics by A. P. French (see next page). Topic 4 on the evolution of the number system could profit from the inclusion of material on quaternions and octonians. As for new topics, I would like students to discuss:

(a) Infinitesimals from Leibniz to Robinson.

(b) Conic sections in Greek geometry, in astronomy and in 19th century geometry.

(c) Some aspects of the evolution of projective geometry in the 19th century. (More explicity, how projective geometry became an independent discipline.)

(d) Euclid's classification of irrational ratios in Book X of the Elements and the modern classification if irrational numbers.

COURSE 2

A course in the history of mathematics

The Greek roots of the calculus and of geometry and their subsequent evolution.

Last year I taught a course in the history of mathematics that centered on the history of the calculus but included a discussion of some aspects of the evolution of geometry. The approach was thematic and genetic. I relied to a large extent on C. H. Edward's The historical development of the calculus* and on the relevant parts of a recent Russian series of books on the history of mathematics. The main topics discussed in the course were:

a. The roots of the calculus in the works of Archimedes and Eudoxus.

b. Philosophical continuation in the 14th c. (Oresmes, etc.)

c. Technical continuation in the 17th c. (Cavalieri, Descartes, Fermat etc.)

d. Creation of the apparatus of the calculus by Newton and Leibniz. A critical comparison of the approaches of Newton and Leibniz to the calculus.

e. Newton's role in the emergence of differential equations as the core of the calculus and in the growth of mathematical physics. Determinism.

f. Infinitesimals from Leibniz to Robinson.

g. Euler's introduction of the study of functions as an important concern of the calculus.

h. Fourier's series and its impact on mathematics: (1) impact on the function concept, (2) impact on the concept of the integral, (3) impact on mathematical physics, and so on.

i. The contributions of Cauchy, Riemann, Weierstrass and Lebesgue to the development of the central concepts of the calculus.

j. Eudoxus and Dedekind. The arithmetization of analysis.

k. The axiomatic mehtod. Euclidean geometry and geometry in the 19th c.

---

* Those unfamiliar with Edward's book might like to know that it is a unique combination of hard-to-come-by computational material and excellent critical analyses.

EINSTEIN'S THEORY OF GRAVITATION

(A.P. French, Newtonian Mechanics)

We have described earlier how Newton recognized that the proportionality of weight to inertial mass is a fact of fundamental significance; it played a central role in leading him to the conclusion that his law of gravitation must be a general law of nature. For Newton this was a strictly dynamical result, expressing the basic properties of the force law. But Albert Einstein, in 1915, looked at the situation through new eyes. For him the fact that all objects fall toward the earth with the same acceleration g, whatever their size or physical state or composition, implied that this must be in some truly profound way a kinematic or geometrical result, not a dynamical one. He regarded it as being on a par with Galileo's law of inertia, which expressed the tendency of objects to persist in straight-line motion.

Building on these ideas, Einstein developed the theory that a planet (for example) follows its characteristic path around the sun because in so doing it is traveling along what is called a geodesic line—that is to say, the most economical way of getting from one point to another. His proposition was that although in the absence of massive objects the geodesic path is a straight line in the Euclidean sense, the presence of an extremely massive object such as the sun modifies the geometry locally so that the geodesics become curved lines. The state of affairs in the vicinity of a massive object is, in this view, to be interpreted not in terms of a gravitational field of force but in terms of a "curvature of space"—a facile phrase that covers an abstract and mathematically complex description of non-Euclidean geometries.

For the most part the Einstein theory of gravitation gives results indistinguishable from Newton's; the grounds for preferring it might seem to be conceptual rather than practical. But in one celebrated instance of planetary motions there is a real discrepancy that favors Einstein's theory. This is in the so-called "precession of the perihelion" of Mercury. The phenomenon is that the orbit of Mercury, which is distinctly elliptical in shape, very gradually rotates or precesses in its own plane, so that the major axis is along a slightly different direction at the end of each complete revolution. Most of this precession (amounting to about 10 minutes of arc per century) can be understood in terms of the disturbing effects of the other planets according to Newton's law of gravitation.[1] But there remains a tiny, obstinate residual rotation equal to 43 seconds of arc per century. The attempts to explain it on Newtonian theory—for example by postulating an unobserved planet inside Mercury's own orbit—all came to grief by conflicting with other facts of observation concerning the solar system. Einstein's theory, on the other hand, without the use of any adjustable parameters led to a calculated precession rate that agreed exactly with observation. It corresponded, in effect, to the existence of a very small force with a different dependence on distance than the dominant $1/r^2$ force of Newton's theory. The way in which a disturbing effect of this kind causes the orbit to precess is discussed in Chapter 13. Other empirical modifications of the basic law of gravitation—small departures from the inverse square law—had been tried before Einstein developed his theory but apart from their arbitrary character they also led to false predictions for the other planets. In Einstein's theory, however, it emerged automatically that the size of the disturbing term was proportional to the square of the angular velocity of the planet and hence was much more important for Mercury, with its short period, than for any of the other planets.

[1]The apparent amount of precession as viewed from the earth is actually about 1.5° per century, but most of this is due to the continuous change in the direction of the earth's own axis (the precession of the equinoxes — see Chapter 14).

COURSE 3

A Course in Geometry

Last year I taught a geometry course which could be described as an introduction to the geometric ideas of Klein and a discussion of the discovery of hyperbolic geometry and its revolutionary effect on mathematical thought. I regard this material as very well suited for producing the mysterious side effect known as "mathematical maturity". (Groups come to life and so do invariants. History becomes a history of ideas rather than a chronicle of equally insignificant events). The materials I used included a Swiss high school text on transformation geometry (out of print!); * a sketch of the history of the parallel postulate and of the discovery of hyperbolic geometry, ♦ an essay on the significance of the discovery of hyperbolic geometry ✝ and a proof of the consistency of the axioms of plane hyperbolic geometry (the Poincaré model). ▲

* Max Jeger, Transformation geometry, Allen & Unwin Ltd., 1966

♦ Introduction to Kelley & Matthews, The non-Euclidean hyperbolic plane, Springer Verlag.

✝ Guillen, Bridges to infinity, J.P. Tarcher, Inc., pp. 105-115.

▲ Moise, Elementary geometry from an advanced viewpoint, Addison Wesley, 1963, ch. 25.

The contents of the Jeger book are as follows:

CONTENTS

The annotated bibliography of significant and accessible materials is a welcome feature of the Jeger book.

A few books of use in developing similar courses

A few recent books with substantial critical and historical components could be used to develop a course in number theory. The books I have in mind are:

H.M. Edwards, Fermat's Last Theorem.
Weil, Number Theory. An Approach through History.
Scharlau and Opolka, Von Fermat bis Minkowski

The brand new German series of books called Grundwissen Mathematik (published by Springer; volumes 1 and 2 are out and four more volumes are promised for 1984!) may help one develop similar courses in other areas.

ST OF PARTICIPANTS

1984

List of Participants/Liste des participants

CMESG/GCEDM 1984

David W. Alexander
16th Floor, Mowat Block
Queen's Park
Toronto, Ontario M7A 1E2

Ed Barbeau
Dept. of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1

Maurice Bélanger
CIRADE, U.Q.A.M.
C.P. 8888, Succursale A
Montréal, Québec H3C 3P8

Jacques C. Bergeron
Faculté des sciences de l'éducation
Université de Montréal
C.P. 6203, Succ. A
Montréal, Québec H3C 3T3

Tasoula Berggren
435 Donald Street
Coquitlam, B.C. V3K 3Z9

John W. Berry
Dept. of Mathematics
University of Manitoba
Winnipeg, Manitoba R3T 2N2

Alan J. Bishop
Dept. of Education
University of Cambridge
Cambridge, United Kingdom

Raffaella Borasi
Dept. of Learning & Instruction
553 Baldy Hall
S U N Y at Buffalo
Buffalo, NY 14260
U.S.A.

Alberta Boswall
Mathematics Dept.
Concordia University
7141 Sherbrooke Ouest
Sherbrooke, Quebec H4B 1R6

J. Dale Burnett
Faculty of Education
Queen's University
Kingston, Ontario K7L 3N6

Louis Charbonneau
Dépt. de mathématiques
U.Q.A.M.
C.P. 8888, Succ. A
Montréal, Québec H3C 3P8

John Clark
Toronto Board of Education
155 College St. W.
Toronto, Ontario

A. George Davis
Professor of Educational Development
Clarkson College
Potsdam, NY 13676
U.S.A.

Dale Drost
Comp. 17, Site 1, SS #3
Fredericton, N.B. E3B 5W9

Gary Flewelling
The Wellington County Board of Education
500 Victoria Road North
Guelph, Ontario N1E 6K2

Claude Gaulin
Département de didactique
Faculté des sciences de l'éducation
Université Laval
Québec, Québec G1K 7P4

Keith O. Geddes
Dept. of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1

Gila Hanna
O.I.S.E.,
252 Bloor St. W.
Toronto, Ontario M5S 1V6

Leon Henkin
Dept. of Mathematics
University of California
Berkeley, CA. 94720
U.S.A.

Nicolas Herscovics
Department of Mathematics
Concordia University
7141 Sherbrooke St. W.
Montreal, Quebec H4B 1R6

William Higginson
A306 McArthur Hall
Queen's University
Kingston, Ontario K7L 3N6

Joel Hillel
Dept. of Mathematics
Concordia University
7141 Sherbrooke St. W.
Montreal, Quebec H4B 1R6

Bernard R. Hodgson
Département de mathématiques
Université Laval
Québec, Québec G1K 7P4

Martin Hoffman
Department of Mathematics
Queen's College
Flushing, NY
U.S.A.

Lars Jansson
Faculty of Education
University of Manitoba
Winnipeg, Manitoba R3V 1E4

Carolyn Kieran
Département de mathématiques
U.Q.A.M.
C.P. 8888, Succ. A
Montréal, Québec H3C 3P8

Thomas E. Kieren
Dept. of Secondary Education
University of Alberta
Edmonton, Alberta T6G 2G5

David Kirshner
8 - 2975 Oak Street
Vancouver, B.C. V6H 2K7

Israel Kleiner
Dept. of Mathematics
York University
Downsview, Ontario M3J 1P3

Erika Kuendiger
Faculty of Education
University of Windsor
Windsor, Ontario N9E 1A5

Andre Ladouceur
Faculté d'Education
Université d'Ottawa
651 Cumberland
Ottawa, Ontario K1N 6N5

Dieter Lunkenbein
Département d'EMP
Faculté d'Education
Université de Sherbrooke
Sherbrooke, Québec J1K 2R1

Angus Maclean
Board of Education for the City of York
2 Trethewey Drive
Toronto, Ontario M6M 4A8

Peter W. Manuel
Dept. of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B9

Bev Marshman
Dept. of Applied Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1

John Mason
Mathematics Faculty
Open University
Milton Keynes, MK7 6AA
United Kingdom

Robert McGee
Cabrini College
Radnor, PA 19087
U.S.A.

Murray McPherson
740 Lanark Street
Winnipeg, Manitoba

Eric Muller
Mathematics Dept.
Brock University
St. Catharines, Ontario L2S 3A1

Alexander L. Murrie
Peel Board of Education
73 King Street W.
Mississauga, Ontario L5B 1H5

T.L. Pearson
Dept. of Mathematics
Acadia University
Wolfville, N.S. B0P 1X0

John Poland
Dept. of Math. & Statistics
Carleton University
Ottawa, Ontario K1S 5B6

Hal Proppe
Dept. of Mathematics
Concordia University
Montreal, Quebec H3G 1M8

Brock Rachar
c/o C.C. Carrothers Public School
360 Chippendale Crescent
London, Ontario N5Z 3G2

Medhat H. Rahim
Dept. of Stats. & Applied Probability
University of Alberta
Edmonton, Alberta T6G 2G1

Bruce S. Reopolos
20 Brewster St., lower
Buffalo, NY 14214
U.S.A.

Mary Rhodes
Bishops University
Lennoxville, Quebec J1M 1Z1

Pat Rogers
Mathematics Department
York University
4700 Keele Street
Downsview, Ontario M3J 1P3

Abe Shenitzer
Dept. of Mathematics
York University
4700 Keele Street
Downsview, Ontario M3J 1P3

Ralph Steal
Dept. of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1

Peter D. Taylor
Dept. of Math. & Statistics
Queen's University
Kingston, Ontario K7L 3N6

Tony Thompson
Dept. of Math., Stats. & Computer Science
Dalhousie University
Halifax, N.S. B3H 4H8

David Wheeler
Dept. of Mathematics
Concordia University
7141 Sherbrooke West
Montreal, Quebec H4B 1R6

Edgar R. Williams
Dept. of Math. & Statistics
Memorial University
St. John's, Newfoundland A1C 5S7