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ABSTRACT

A two-year study was conducted with fourth-grade children in the context of extensive teaching experiments concerned with the learning of rational number concepts. Representational difficulties in using the number line model were investigated. While instruction in the second year attempted to resolve observed learning difficulties, the results of both years showed that children have considerable difficulty with number line representations which show an unreduced form of a given fraction. Explorations of the data suggest difficulty with partitioning and "unpartitioning" number line representations, with translations between various representational modes, and with coordinating symbolic and pictorial information on a number line (Author)

IDENTIFYING FRACTIONS ON NUMBER LINES

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A two-year study was conducted with fourth-grade children in the context of extensive teaching experiments concerned with the learning of rational number concepts. Representational difficulties in using the number line model were investigated. While instruction in the second year attempted to resolve observed learning difficulties, the results of both years showed that children have considerable difficulty with number line representations which show an unreduced form of a given fraction. Explorations of the data suggest difficulty with partitioning and "unpartitioning" number line representations, with translations between various representational modes, and with coordinating symbolic and pictorial information on a number line.

Identifying Fractions on Number Lines

This study was an investigation of (a) the ways students represented, or misrepresented, fractions on number lines and (b) the influence of different instructional strategies on those (mis)representations. The number line model was chosen for study in large part because of its pervasive use in school mathematics instruction.

As a model for representing fractions, the number line differs from other models (e.g., sets, regions) in several important ways. First, a length represents the unit, and this measure construct suggests not only iteration of the unit but also simultaneous subdivisions of all interated units. That is, the number line can conceptually be treated as a ruler. Second, on a number line there is no visual separation between consecutive units. That is, the model is totally continuous. Both sets and regions as models possess visual discreteness. When regions are used, for example, space is typically left between copies of the unit.

Third, the number line requires the use of symbols to convey part of the intended meaning. For example, point A in Figure 1a has no numerical meaning until at least two reference points are labeled. Two possible meanings are given in Figures 1b and 1c. Figures 1d and 1e, however, do convey meaning without any accompanying symbols, though their interpetation requires some standard conventions about the nature of a unit.

INSERT FIGURE 1 ABOUT HERE

The use of symbols to label points on a number line may focus a student's attention on those symbols rather than on the pictorial embodiment of the fractions. This focusing may in turn cue symbolic processes as the predominate mode of manipulation of information. Too, the necessary but not directly used marks on a number line may act as perceptual distractors (Behr, Post, Lesh, & Silver, 1982).

Methods

Two substudies were conducted in successive years with fourth-graders. The instruction in the second year was modified to attempt to overcome the apparent deficiencies in students' performance during the first year.

Substudy 1

Subjects. Subjects were five fourth-graders (three boys and two girls) in an elementary school in northern Illinois. They were selected, through teacher evaluations, to represent a cross-section of facility with arithmetic concepts and were also subjects in an 18-week teaching experiment (Behr, Lesh, & Post, Note 1).

Instruction. Instruction was a four-day lesson concerning association of fraction concepts, relations, and operations with points, comparisons, and transformations on a number line. Specific objectives were (a) to associate whole numbers, fractions, and mixed numbers with points on a number line, (b) to use number lines to help connect "improper" fraction names to "mixed number" names, (c) to use number lines to determine which of two fractions is less or whether they are equivalent, and (d) to use number lines to generate equivalent fractions. The lesson on number line representations was presented near the end of the larger teaching experiment.

Instruction included a variety of activities. The notion of subdivision of the unit was reinforced by use of centimeter rods to develop the analogy with the set/subset fraction concept and by repeated focusing on the 0 (left-hand) endpoint of the first unit. There was considerable attention paid to the equivalence of improper fractions and mixed numbers and to ordering fractions using number lines. Writing equivalent fractions, however, was treated relatively independent of number lines. For example, 2 2/6 and 2 1/3 were compared directly on number lines, rather than by first reducing 2/6 to 1/3.

Test. The fraction test of Novillis (1980) was given immediately prior to and immediately after the instruction. This 16-item, multiple choice test can be partitioned into two 8-item subscales in several ways: (a) fraction given with representation to be chosen versus representation given with fraction to be chosen, (b) number line shows 0 to 1 versus number line shows 0 to 2, and (c) representation on number line shows unreduced fraction versus representation shows reduced fraction. For each item there were five choices, one of which was "Not Given"; this choice was never the correct choice. In all cases, the fraction symbol in the correct fraction/representation pair was reduced even if the representation was for an unreduced equivalent fraction.

Results. Scores on the six possible subscales are given in Table 1. For five of the subscales, performance uniformly increased or remained constant from pretest to posttest. The sole exception was when the representation was unreduced and the fraction symbol was reduced. As a follow-up of this subscale, scores were separated according to the other categories of items. (See Table 2.) With the exception of student 1, students were unable to choose a reduced fraction name when an unreduced equivalent form was represented on a number line.

INSERT TABLES 1 AND 2 ABOUT HERE

To help determine what processes the students might be using, incorrect responses on the unreduced representation subscale were examined. On the pretest, 10 of the 31 incorrect responses were "Not Given"; two were blanks. On the posttest, however, 28 of the 30 incorrect responses were "Not Given"; none were blanks.

Additional information was available from videotaped interviews. In three interview tasks, students were to find equivalent fractions, 5/3 = ?/12, 8/6 = ?/3, and 8/6 = ?/12. Student 1 solved these problems symbolically and used the number line only to verify the solutions. Student 4 used counting strategies but solved all three correctly. Student 2 combined number line and symbolic algorithms and solved only the first and third tasks involving larger denominators. Students 3 and 5 used addition and subtraction strategies and solved the last two tasks correctly, but possibly only because of the 2:1 ratio of the denominators in each case.

Substudy 2

Subjects. Subjects were eight fourth-graders (four boys and four girls) in the same elementary school used in Substudy 1. They were again selected to represent a cross-section of arithmetic facility and were also subjects in an extended teaching experiment (Behr, Lesh, & Post, Note 1).

Instruction. Instruction on use of number lines lasted eight days, September 14-24, 1983. The instruction in Substudy 1 was extended by including more activities on equivalence, on translations between the number line and area models, and on using equivalent fractions to name a single point on a number line.

Tests. A variety of tests were administered to the subjects between March 1982 and January 1983. A list of test dates is given in Table 3. Some items in these tests involved use of the number line model in ways similar either to Novillis' test or to the instruction. Sample items are given in Figure 2.

INSERT TABLE 3 AND FIGURE 2 ABOUT HERE

Results. Performance on Tests I and III is given in Table 4. Although the improvement in average performance was significant ($p < .05$), the change in error patterns was also of considerable interest.

INSERT TABLE 4 ABOUT HERE

Performance on Novillis' test is given in Table 5. There was considerable improvement on the "reduced representation" subscale ($p < .05$) but not on the "unreduced representation" subscale. Further refinement of the scores on this subscale (Table 6) along with analysis of the errors revealed shifts in the students' error patterns. On the pretest, 25 of the 47 incorrect responses (53%) were "Not Given" and 3 more (6%) were consistent with use of the interval from 0 to 2 as the unit. On the posttest, these results were, respectively, 30 of the 38 incorrect responses (79%) and 7 out of 38 (18%). This shift is consistent with the data of Substudy 1.

INSERT TABLES 5 AND 6 ABOUT HERE

Two of the items in Test VII were identical with two items in Novillis' test. The performance on these two items across the three administrations is given in Table 7. Of particular interest is the slight improvement even after number line instruction ended.

INSERT TABLE 7 ABOUT HERE

Performance on the Number Line Test is given in Table 8. The improvement was substantial, but since the items were closely related to the instruction, this may reflect only a practice effect. The errors on this tests were of three primary types. On the pretest, 38 of the 90 incorrect responses (42%) were consistent with the student's having used the wrong unit, 14 (16%) were consistent with the student's having counted marks instead of intervals, and 12 (13%) were consistent with the student's having represented the inverse of the given fraction; 14 responses (16%) were "I don't know." On the posttest the corresponding data were 11 of the 34 incorrect responses (32%), 0 (0%), and 4 (12%); 8 responses (24%) were "I don't know." The apparently complete lack of counting marks instead of intervals was expected since the instruction explicitly dealt with the number line from a measurement rather than a counting interpretation.

INSERT TABLE 8 ABOUT HERE

Discussion

In many ways the instruction seems to have been effective. The shifts in error patterns, in particular, suggest that the instruction at least sensitized students to the need to attend to some characteristics of the number line model. For example, in Substudy 1, the increase in "Not Given" responses may have resulted from learning to look for a representation with a unit that is subdivided as indicated by the denominator of the fraction. Failure to recognize unreduced representations, however, may indicate either an inability to "unpartition" (Behr, Post, Lesh, & Silver, 1982), a lack of skill at reducing fractions, or an inflexibility in translating between modes of representation. In Substudy 2, the shift in errors on the Number Line Test seems to support this. The decrease in very inappropriate responses (e.g., counting marks instead of intervals and representing the inverse of the given fraction) and the concurrent rise in the percent of "I don't know." responses suggest that these students at least learned the major characteristics of the model that needed to be attended to.

The instruction of the second year also seems to have been marginally more successful at helping students deal with unreduced representations. This may have been due to the added attention given to translations between part-whole displays and number lines, to finding units on number lines, or to greater emphasis on the measure construct. In Substudy 1, only student 1 spontaneously used symbolic algorithms for finding equivalent fractions in interview tasks. In Substudy 2, however, six of the eight students were able in interview tasks to use symbolic algorithms for generating equivalent fractions. However, only four of the eight seemed to have any success at coordinating the symbols with the number line model. Clear and easy access to these symbolic algorithms may indicate a well-developed concept of fractions.

The data of this study, notably those in Table 6, also indicate that unpartitioning of a given representation is possible; that is, if a reduced fraction is given and a correct representation is to be chosen, students can sometimes identify the proper representation, even when it is of an unreduced equivalent fraction. However, when the representation is given in unreduced form, the students are almost universally unable to choose the correct reduced symbolic fraction. They apparently do not look back at the given representation and try to make each of the symbolic fraction choices fit that number line. Perhaps the symbol takes on an identity of its own once it is generated from a given representation and the connection to its perceptual base is lost.

Implications

Number line instruction is difficult. During the instruction, the students seemed to be able to perform adequately; the improved performance on the Number Line Test supports this observation. However, transfer of knowledge to slightly different situations (e.g., Novillis' test) was not particularly successful, especially when the representations of the fractions were in unreduced form.

One primary feature of the number line model may help explain this observation. Since the model consists of pictorial information with accompanying symbols, there may be a cognitive difficulty in connecting the information contained in the two modes of representation. The data from Tests I and III support this contention. With items like 2a (Figure 2) students could generally identify the proper unit. However, with items like

2b, students didn't do as well. Possibly this is because there are more symbols to coordinate in the representation of the item information; there may be an overload on the capacity of the students to coordinate the two modes of information. An hypothesis arising out of this analysis is that THE NEED TO COORDINATE SYMBOLIC AND PICTORIAL INFORMATION WITH THE NUMBER LINE MODEL POSES DIFFICULTY IN MATCHING FRACTION NAMES WITH NUMBER LINE REPRESENTATIONS.

A similar situation was observed by Gerace and Mestre (Note 2). In their study, Hispanic high school students in a beginning algebra class were initially very rigid in labeling number lines; that is, the first tick to the right of 0 was always supposed to be labeled with a "1." Later, the number lines frequently seemed to be labeled with (mis)conceptual labels rather than with mathematically correct labels. For example in Figure 3a, the first "1/3" denotes the first third, while the second "1/3" denotes the third closest to 1. In Figure 3b, the labeling illustrates a common mistake in ordering common fractions. In more concrete situations (e.g., in problems in which the number line represented distances or in which there was use of East/West designations) the students were noticeably more successful.

INSERT FIGURE 3 ABOUT HERE

From a slightly different perspective, the data of this study and other studies indicate that students' difficulty at adding partitioning points to generate higher term fractions or at mentally removing partitioning points to generate lower term fractions is not unique to the number line model (Behr, Post, Lesh, & Silver, 1982; Payne, 1976). Moreover, the greater difficulty children have generating lower term fractions by unpartitioning pervades work with both continuous and discrete models and with symbolic equivalent-fractions tasks. Generating higher term fractions symbolically seems to be easier than generating lower term fractions. At the symbolic level, this difference in difficulty may be due to children's greater facility with multiplication than division.

With manipulative tasks the children seem to rely heavily on the visual representation of a fraction; flexibility in the perception of equivalent fractions, independent of the given representations, is not yet achieved. Children not only seem distracted by "extra" points but also seem to question "the rules of the game." That is, some children have been observed to add partitioning points, but when faced with the removal of points, these same children hesitate and may query the teacher or interviewer about whether it is "alright" to take out points. Other children have been found totally unable to perceive lower term fractions in the presence of extra points. More generally, the partitioning/unpartitioning phenomenon seems to pervade many children's work with most models for rational numbers. An hypothesis arising out of this analysis is that AS LONG AS PARTITIONING/UNPARTITIONING IS DIFFICULT FOR CHILDREN, NUMBER LINE REPRESENTATIONS OF FRACTIONS MAY NOT BE EASILY TAUGHT.

The instruction of Substudy 2 seemed more effective at helping students deal with representations on number lines from

0 to 1 than on number lines from 0 to 2. In Substudy 1, the instruction seemed to be ineffective at helping students deal with representations on both kinds of number lines. Perhaps the increased emphasis on identifying the unit during the instruction phase of Substudy 2 was responsible for the different pattern of effectiveness. If so, then increased emphasis on identifying the points on a number line which represent 2, 3, 4, etc. might help students further generalize their concepts of fraction representations.

A major hypothesis of the research project of which this study is one part is that translations between and within modes of representation facilitate learning (Behr, Post, & Lesh, Note 1). As noted earlier, the instruction provided models of translations of three types: (a) symbol --> number line, (b) symbol --> number line --> different number line, and (c) number line --> symbol --> different symbol. Inclusion of translations such as symbol --> number line --> different number line --> different symbol might have helped children make the symbol --> different symbol translations in generating equivalent fractions. (See Figure 4.) The symbol --> different symbol translation is a condensed version of the longer string of translations. Until students are able to collapse this sequence, it may be helpful to provide settings in which all parts of the sequence are explicit. (See Bernard & Bright, in press, for further discussion of the notion of collapsing of processes and operations.)

INSERT FIGURE 4 ABOUT HERE

More translations between different kinds of models might also have been helpful. The inclusion in Substudy 2 of translations between part-whole models and number lines may have been partially responsible for the apparent improvement in performance. If so, more instruction of this type might have enhanced the improvement.

Too, interconnecting the symbolic generation of equivalent fractions to the number line models needs more attention. Some children were quite successful at symbolic tasks but did not spontaneously use their symbolic skills in number line situations. Perhaps they did not see the connection between these two kinds of tasks (or, modes of presentation of information).

Two possible teaching techniques seem to arise from these considerations. First, multiple number line representations of a single fraction might be presented. At most, one of these representations would be the reduced representation of the fraction, while all others would be unreduced representations of the fraction. Some of these unreduced representations would be labeled with the unreduced fraction name, while others would be labeled with the reduced fraction name. Illustrations of appropriate number lines are given in Figure 5.

INSERT FIGURE 5 ABOUT HERE

Second, number lines with different subdivisions might be matched and then labeled. Symbolic expressions of the equivalence of the fraction represented in the several ways could then be presented. Illustrations of sample number lines are given in Figure 6.

INSERT FIGURE 6 ABOUT HERE

Knowledge of equivalent fractions seems to be important to the full utilization of number line representations. Knowledge that is developed only through symbolic algorithms may be isolated and not called upon in the context of manipulative tasks. Work with the number line model during instruction on equivalent fractions would probably be needed. For example, partitioning units of a number line first into halves, then fourths, etc., would illustrate the notion that to every point on the number line there are associated many equivalent fractions. Before using the number line for more complex tasks (e.g., to model addition and subtraction, especially of unlike fractions) more skill with equivalent fractions in the context of the number line is essential. Automatic generation of equivalent fraction representations, through further partitioning or unpartitioning of the number line "in the mind's eye," could facilitate flexibility in perception. Such flexibility might significantly enhance students' performance.

Too, further investigation of the ways students translate between different representations of knowledge is needed. Experts (e.g., teachers) seem to make these translations easily, and frequently they seem not to be consciously aware that translations are used. In some sense, experts seem to view all modes of presentation of information as equivalent. Novices (e.g., students) on the other hand need explicit help in learning how to make these translations. Much more needs to be known about processes that students use in translating before instruction can be effectively modified to help students learn to make translations between the modes of representation.

Reference Notes

1. Behr, M. J., Post, T. R., & Lesh, R. Construct analysis, manipulative aids, representational systems, and learning of rational numbers. Proposal submitted to National Science Foundation, 1981. (Funded as Grant Number SED-20591)

2. Gerace, W. J., & Mestre, J. P. A study of the cognitive development of hispanic adolescents learning algebra using clinical interview techniques (Final Report, NIE contract #400-81-0027). Amherst, MA: University of Massachusetts, 1982.

References

Behr, M. J., Post, T. R., Lesh, R., & Silver, E. Rational number concepts. In R. Lesh & M. Landau (Eds.), Acquisition of mathematical concepts and processes. New York: Academic Press, 1982.

Bernard, J. E, & Bright, G. W. Student performance in solving linear equations. International Journal of Mathematical Education in Science and Technology, in press.

Novillis, C. Locating proper fractions on number lines: Effect of length and equivalence. School Science and Mathematics, 1980, 80, 423-428.

Payne, J. N. Review of research on fractions. In R. Lesh & D. Bradbard (Eds.), Number and measurement: Papers from a research workshop. Columbus, OH: ERIC/SMEAC, 1976.

Table 1

Subtest Scores[a]

Subtest	Student 1	Student 2	Student 3	Student 4	Student 5	Number of Times Posttest-Pretest >0	=0	<0
fraction given	4(8)	4(4)	2(4)	1(4)	2(4)	4	1	0
representation given	4(8)	2(4)	2(2)	1(3)	0(3)	4	1	0
number line 0-1	8(8)	5(5)	2(3)	2(4)	1(4)	3	2	0
number line 0-2	0(8)	1(3)	2(3)	0(3)	1(3)	5	0	0
reduced representation	4(8)	5(7)	3(6)	1(6)	0(7)	5	0	0
unreduced representation	4(8)	1(1)	1(0)	1(1)	2(0)	1	2	2

[a] x(y) means x = pretest score, y = posttest score. Each student had a maximum possible socre of 8.

Table 2

Refined Scores for Unreduced Representation Subscale[a]

Subcategory	Student 1	2	3	4	5
fraction given with					
0 to 1 number line	2(2)	1(1)	1(0)	1(0)	1(0)
0 to 2 number line	0(2)	0(0)	0(0)	0(1)	1(0)
representation given with					
0 to 1 number line	2(2)	0(0)	0(0)	0(0)	0(0)
0 to 2 number line	0(2)	0(0)	0(0)	0(0)	0(0)

[a]x(y) means x = pretest score, y = posttest score. Each subcategory had a maximum possible score of 2.

Table 3

Test Dates, Substudy 2

Test	Administration Date
Written Test I	1982 Mar 10
Written Test II	1982 Apr 06
Written Test III	1982 May 03
Written Test IV	1982 May 25
Novillis Pretest	1982 Sep 13
Number Line Pretest	1982 Sep 13
Novillis Posttest	1982 Sep 27
Number Line Posttest	1982 Sep 27
Written Test V	1982 Oct 18
Written Test VI	1982 Dec 03
Written Test VII	1983 Jan 20

Table 4

Scores and Error Types, Tests I and III, Substudy 2

Student	Test I		Test III	
	Score[a]	Error Type[b]	Score	Error Type[c]
1	0	M	0	M6
2	0	M	2	D
3	0	M	0	D
4	0	M	2	D
5	1	D	0	blanks
6	0	D	2	D
7	0	M	2	D
8	5	-	6	-

[a]Maximum score = 12

[b]Error M(ultiplcation): response = numerator × denominator for regions like 1/6 ¦ 1/6 ¦ 1/6

Error D(enominator): response = denominator for regions like 1/6 ¦ 1/6 ¦ 1/6

[c]Error M6: response is consistent with error M, but only for sixths and not fourths.

Table 5

Novillis' Test Scores, Substudy 2

Subscale	Student 1	2	3	4	5	6	7	8
reduced representations	5(4)[a]	3(4)	1(6)	4(4)	1(5)	6(8)	2(8)	7(8)
unreduced representations	0(0)	3(5)	1(4)	4(4)	3(0)	0(0)	4(6)	2(7)

[a]x(y) means x = pretest, y = posttest. Maximum score on each subscale is 8.

Table 6

Refined Scores, Unreduced Representation Subscale, Substudy 2

Subscale	Student							
	1	2	3	4	5	6	7	8
fraction given with								
0 to 1 number line	0(0)[a]	1(2)	0(2)	2(2)	1(0)	0(0)	1(2)	2(2)
0 to 2 number line	0(0)	1(1)	0(2)	0(0)	1(0)	0(0)	1(2)	0(2)
representation given with								
0 to 1 number line	0(0)	1(2)	1(0)	2(2)	0(0)	0(0)	1(2)	0(2)
0 to 2 number line	0(0)	0(0)	0(0)	0(0)	1(0)	0(0)	1(0)	0(1)

[a] x(y) means x = pretest score, y = posttest score.
Maximum score on each subscale is 2.

Table 7

Scores on Two Common Items, Novillis' Test and Written Test VII, Substudy 2

Test	Student							
	1	2	3	4	5	6	7	8
Novillis Pretest	0	1	1	2	0	0	1	1
Novillis Posttest	0	2	1	2	0	0	2	2
Written Test VII	1	1	2	2	0	2	2	2

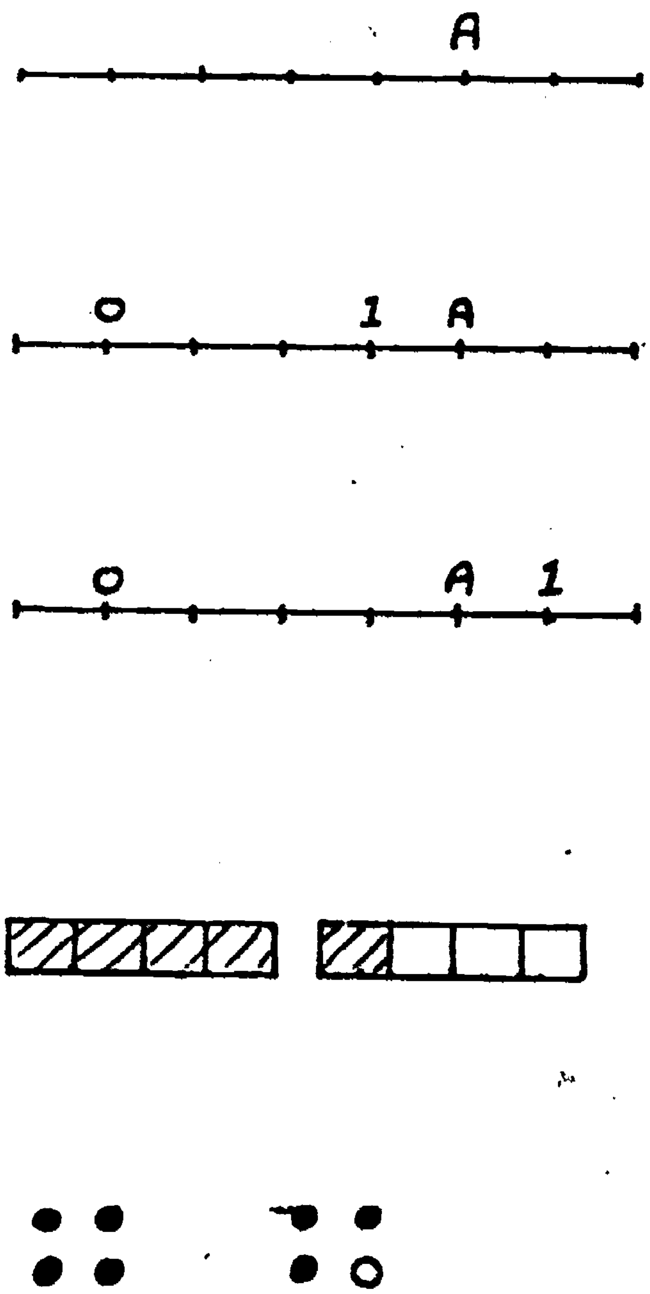
Table 8

Scores on Number Line Test, Substudy 2

Test	Student							
	1	2	3	4	5	6	7	8
Pretest	0	1	1	1	1	0	0	2
Posttest	2	4	6	12	7	12	8	11

Figure 1

Representations of Fractions



a.

b.

c.

d.

e.

Figure 2

Sample Items

a. How many pieces like $\boxed{\frac{3}{6}}$ make one whole? ____

b. How many pieces like $\boxed{\frac{1}{6} \,\vdots\, \frac{1}{6} \,\vdots\, \frac{1}{6}}$ make one whole? ____

c. Mark the point 3/2 on the number line below.

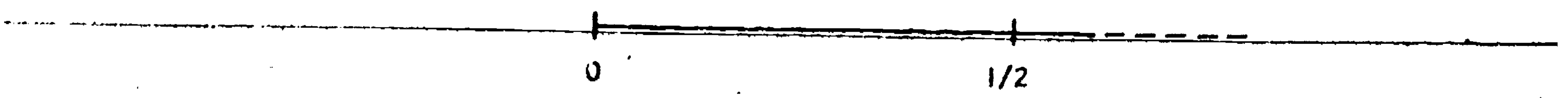


Figure 3

Mislabeling of Number Line

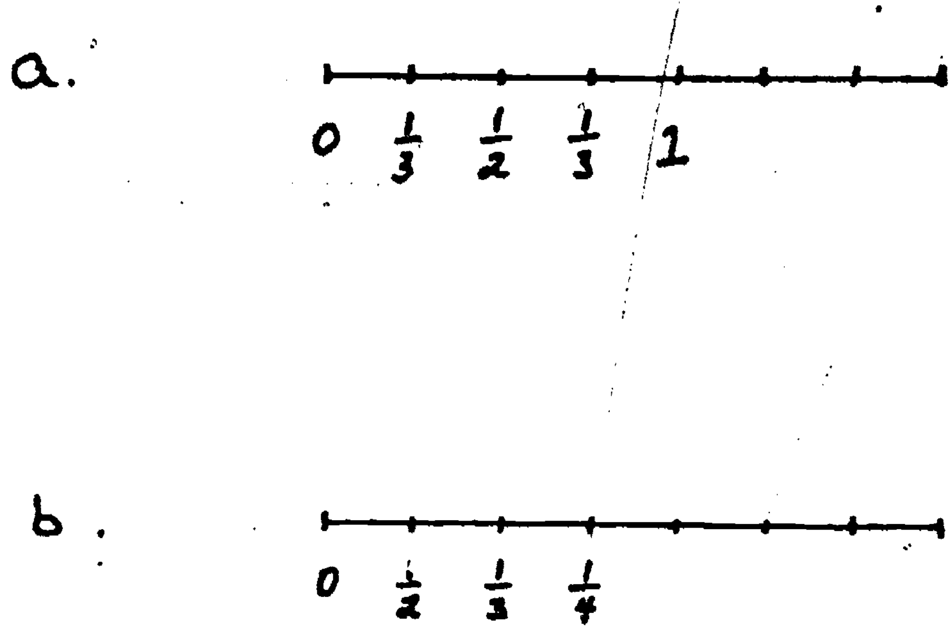


Figure 4

Multiple Representations and Translations

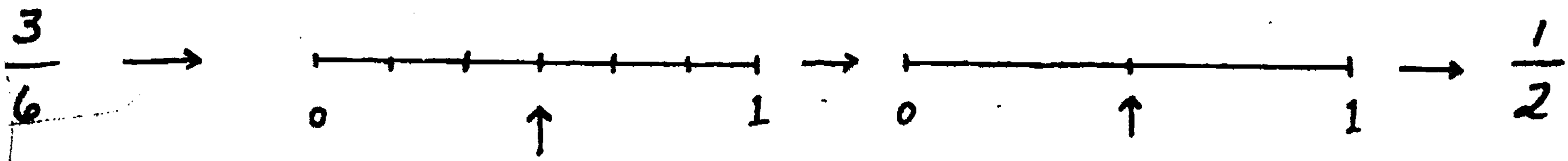


Figure 5

Some Representations of One-half

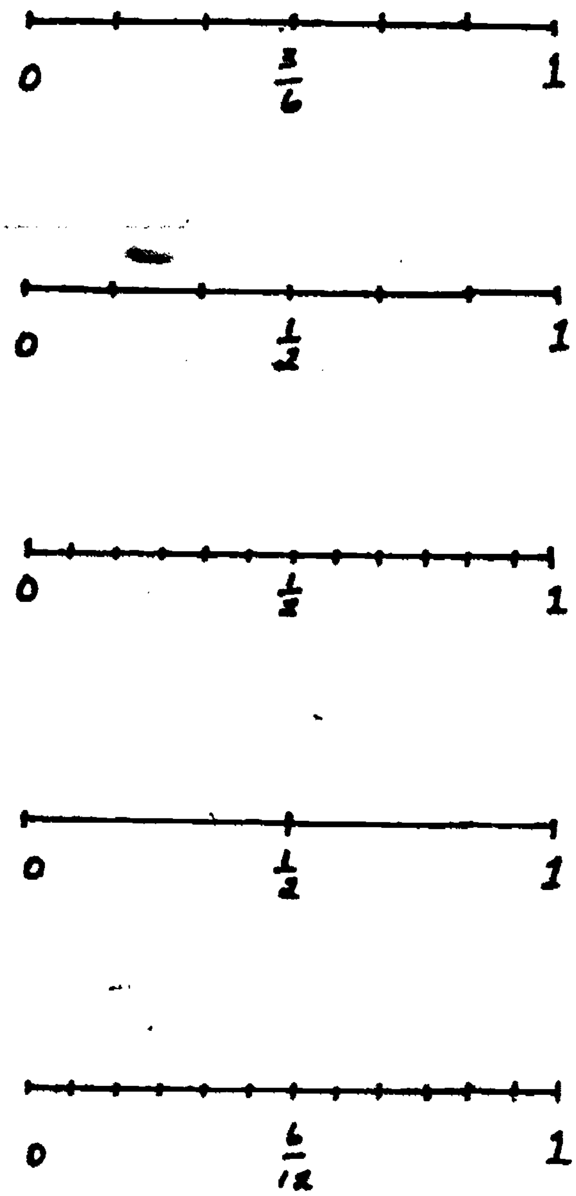
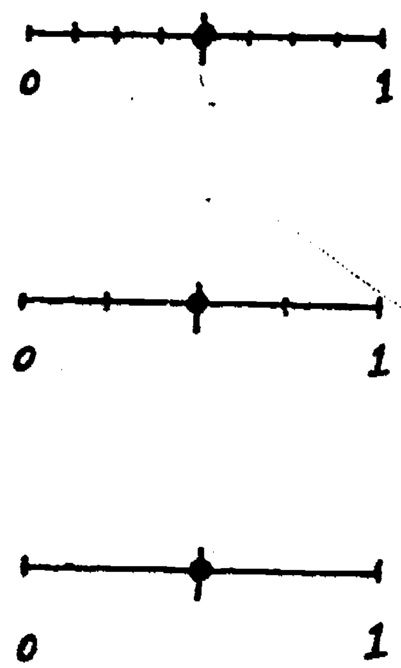


Figure 6

Equivalent Pictorial and Symbolic Representations



$$\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$$