DOCUMENT RESUME

ED 230 415		SE 041 909
AUTHOR	Hiebert, James; Wearne, Diana	
TITLE	Students' Conceptions of Decimal Numbers.	
INSTITUTION	Delaware Univ., Newark.	
SPONS AGENCY	National Science Foundation, Washington, D.C.	
PUB DATE	Apr 83	
GRANT	SPE-8218387	
NOTE	60p.; Paper presented at the Annual Meeting of the American Educational Research Association (Montreal, Quebec, Canada, April 11-15, 1983).	
PUB TYPE	Reports - Research/Technical (143)	
EDRS PRICE	MF01/PC03 Plus Postage.	
DESCRIPTORS	*Cognitive Processes; *Decimal Fractions; Educational Research; Elementary Secondary Education; Interviews; *Mathematical Concepts; *Mathematics Education; *Mathematics Instruction; Number Concepts; Testing	
IDENTIFIERS	*Mathematics Education Research	

ABSTRACT

Decimal numbers have become an increasingly important topic of the elementary and junior high school mathematics curriculum. However, national and state education assessments indicate that students have incomplete and distorted conceptions of decimal numbers. This paper reports initial data from a two-year project designed to elicit and describe students' understanding of decimals. Students in grades 3, 5, 7, and 9 were given written tests and interviewed individually on a variety of decimal tasks. Of primary interest here are tasks that considered decimals as (1) quantities that have value; (2) extensions of whole numbers; and (3) equivalents of common fractions. Results indicate that students perceive decimals primarily as symbols upon which to perform syntactic maneuvers. Although many students have significant hidden understandings, they rarely connect these with the procedural rules they have memorized. (JN)

Students' Conceptions of Decimal Numbers

James Hiebert Diana Wearne

University of Delaware

Paper presented at the annual meeting of the American Educational Research Association, Montreal, April 1983

The project reported herein was performed with the support of National Science Foundation Grant No. SPE-8218387. Any opinions, findings, conclusions, or recommendations expressed in the report are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Abstract

Decimal numbers have become an increasingly important topic of the elementary and junior high school mathematics curriculum. However, national and state education assessments indicate that students have incomplete and distorted conceptions of decimal numbers. This paper reports initial data from a two-year project designed to elicit and describe students' understanding of decimals. Students in grades 3, 5, 7, and 9 were given written tests and interviewed individually on a variety of decimal tasks. Of primary interest here are tasks that considered decimals as (1) quantities that have value; (2) extensions of whole numbers; and (3) equivalents of common fractions. Results indicate that students perceive decimals primarily as symbols upon which to perform syntactic maneuvers. Although many students have significant hidden understandings, they rarely connect these with the procedural rules they have memorized.

Students' Conceptions of Decimal Numbers

Two fundamental topics in the elementary school mathematics curriculum are whole numbers and fractions. A great percentage of time is devoted to developing students' skill in working with these two types of numbers. For most of elementary school, whole numbers and fractions are treated separately, relegated to different problem situations and taught as different symbol systems requiring the application of different sets of rules. However, as students move into upper elementary school and junior high school they are asked to integrate the concepts of whole numbers and fractions to form the system of decimal numbers.

Although adults may view the construction of decimal numbers as a simple extension of whole number and fraction concepts, the genuine integration of these two major ideas to form a mature notion of decimal represents a major intellectual advance. In fact, a number of cognitive development theories (Case, 1978; Piaget, 1960; Werner, 1948) regard the integration of two fundamental and previously independent ideas as a hallmark of intellectual development. Thus, it is likely that the introduction of decimal numbers places heavy demands on student learning. If students enter instruction on decimals without a full understanding of the whole number system and/or common fractions, if their cognitive capacities are exceeded by the new integration demands, or if instruction fails to present decimals in appropriate contexts, students may acquire only a partial understanding of decimals. Consequently, it is reasonable to assume that the topic of decimals would present special difficulties for many elementary and junior high school students.

Previous research suggests that many students do, in fact, experience great difficulty learning about decimals. Since the research base is quite limited, and is mostly comprised of general surveys of written performance, the origins and the full nature of students' difficulties are not entirely clear. But the surveys leave little doubt that the difficulties exist. Results from the mathematics assessment of NAEP (Carpenter, Corbitt, Kepner, Lindquist, and Reys, 1981) indicate that nine-year olds have little familiarity with decimals and about 50% of 13-year olds lack even basic understandings of decimals. Several items asked students to order decimals by recognizing the value of the decimal positions. While most 13-year olds realized that a number greater than one is larger than a number less than one, they had substantial difficulty ordering two decimals less than one. Almost 50% ignored the decimal points and treated the numbers as whole numbers; they did not recognize that, for example, .3 is equivalent to .30. The relationship between decimals and common fractions appeared to be equally cloudy for students. About 50% of the 13-year olds could change common fractions expressed in tenths and hundredths to decimal fractions, but less than 40% changed 1/5 to its decimal equivalent. Almost no 9-year-olds could change decimal fractions to common fractions, or vice versa. Apparently, a large number of elementary school students do not relate whole number place value and common fraction concepts to decimal representations.

Further evidence of the lack of relationship between previous knowledge and decimals is provided by Erlwanger (1975) and Ekenstam (1977). These reports suggest that often students see work with decimals as unrelated to previous work, either inside or outside of school. It seems that they construct separate systems of rules to deal with decimals, and see little connection between the meaning of a decimal and a whole number or a fraction.

The problems students encounter as they are learning about decimals are of special concern because of the importance of decimals, both from a mathematical and a practical point of view. Mathematically, the decimal system represents a more powerful representation system than those the student has worked with previously. In addition to culminating previous work with whole numbers and fractions, the decimal system provides new ways of representing quantitative situations and encourages new insights into the properties of number systems themselves. Practically speaking, decimals are becoming an increasingly important part of the mathematics curriculum due the recent influx of calculators and computers into the schools and the growing emphasis on the metric measurement system. Decimal numbers are a central part of the mathematics language in this technological age, and it is imperative that students acquire a firm grasp of this language.

The purpose of this paper is to report on one aspect of a two-year project that is studying the acquisition of decimal concepts and skills. The project is designed to elicit the conceptions and misconceptions that students have of decimal numbers, and to map out the rule systems that students use to manipulate decimal symbols. The goal of the project is to construct a complete description of how students view decimals: what concepts and skills they have acquired, how these are linked to previous knowledge, and how this knowledge is expressed in the rules they use to solve decimal problems. The focus of this paper is on the conceptions and misconceptions that students acquire about decimals, especially those that emerge when viewing decimals as representations of rational numbers that integrate whole number and fraction concepts.

What is a Decimal Number?

The notion of a decimal number is quite complex. Many subconcepts and skills contribute to the development of a complete decimal number construct. One way of analyzing the construct into its component parts is to consider what a competent student knows about the concept of a decimal and about solving conventional school problems that involve decimal numbers. Figure 1 depicts one possible outcome of this kind of analysis.

Insert Figure 1 about here

Several features of the diagram shown in Figure 1 should be highlighted. First, the diagram is visual evidence that the full notion of decimal number is an extremely complex one. Many pieces of information, from a variety of instructional and experiential sources, fit together to form the competent student's knowledge of decimal. Space does not permit a full description of each cell in the diagram, but two examples may help to provide a sense of the meanings represented by the cell labels. The term "ordered sequence" in the box under the base ten (whole) number understanding refers to the fact that adjacent places in a base ten numeral have a clearly defined relationship--they differ in value by a factor of ten. As a second example, the box in the bottom row on the right containing the label "$a \cdot b^{ths}$" represents the elemental intuitive understanding of common fractions read as "$a \cdot b^{ths}$" and most often conceived as part of a region. It provides a common foundation for many later interpretations of fraction such as measure and operator. These two specific examples take on added significance in the context of several distinctions that are emphasized by the diagram in Figure 1.

The first distinction, and the second feature of the diagram to be discussed, is the major distinction between form and understanding. "Form" includes the modes of representation and the rules that are used to operate on the representations. The rules can be carried out with or without knowing why they work. The rule components identified under "Form" can also be thought of as making up the syntax of the system. "Understanding", on the other hand, refers to the conceptual underpinnings of the system. When we say that students understand decimals we usually mean that they have learned the constructs and relationships depicted in the right half of the diagram. These components make up the semantics of the system.

The distinction between form and understanding, or syntax, and semantics, seems to capture the essence of a common phenomenon often expressed by teachers: "some students can get the right answer but they don't understand what they're doing." Stated in more general terms, the same sentiment can be expressed by saying that form and understanding can be learned independent from each other. In fact, some researchers note that this is a rather common occurrence. Davis and McKnight (1980) conclude that, with respect to borrowing in whole number subtraction, "...the effect of semantic knowledge on algorithmic behavior is easily described; it has no effect" (p. 75). Resnick (1982) presents data on beginning arithmetic performance and concludes that "...even when the basic concepts are well understood, they may remain unrelated to computational procedure" (p. 136).

The relationship between conceptual knowledge and procedural correctness represents a persistent and critical issue in the study of human learning in general, and mathematics learning in particular. It is reminiscent of the understanding versus skill debate of the past. However, it now seems clear that this debate will not be resolved by arguing for the advantage of one over the other, but by explicating the important relationships between these two domains (Glaser, 1979). The topic of decimal numbers appears to be an especially rich arena for studying the relationships between form and understanding, or syntax and semantics. A primary objective of this project is to describe these relationships for decimal numbers, and ultimately to speculate about the nature of these relationships for mathematics learning in other domains.

A further distinction that will help to set the stage for this report has been proposed by Van Engen (1953), and later, in somewhat different form, by Greeno (1980). Van Engen distinguishes between understanding and meaning, Greeno between implicit understanding and explicit understanding. Although the labels are different, the distinction is essentially the same. Understanding, or implicit understanding, refers to an intuitive knowledge of the conceptual ideas, knowledge of what the ideas are and how they are related, knowledge contained within the understanding side of Figure 1. Meaning, or explicit understanding, refers to a knowledge of the concepts <u>and</u> the formal language used to express them.

This extends implicit understanding to include knowledge of how to represent concepts or operations using appropriate symbols. Actually, this form of understanding occurs anytime the student establishes a link between a component of form and a component of understanding.

An added note of emphasis is in order regarding the importance of making connections between form and understanding. The understandings listed in Figure 1 are best thought of as simple abstractions of concrete experiences or as ideas based directly on previous understanding. Drawing the connections between a set of understandings and an appropriate symbol system is considered by many to be a central feature of intelligent human activity (e.g., Hofstadter, 1979; Werner and Kaplan, 1963). Of specific interest here is the fact that the connecting process plays a critical role in learning mathematics, and has been identified as a primary objective of mathematics instruction (Van Engen, 1949; Skemp, 1971). The current project is investigating children's attempts to make connections between symbols and understandings and this paper will report on some of these.

Two of the links that give meaning to the decimal representation form, and help to define what a decimal is, are shown as dotted lines in Figure 1. One of these links runs between the representation of a decimal and ordered sequence (described earlier). The other link ties the decimal representation with common fraction and part/whole concepts represented by the label $a \cdot b^{ths}$. In this way the decimal representation serves as a focal point for the convergence and integration of the concepts of place value and part/whole. The full meaning, or explicit understanding, of a decimal representation depends upon constructing both of these links.

Procedures

Scope. As implied by Figure 1, there are many aspects of decimal numbers that provide legitimate domains of investigation. The two-year project referenced in this report is focusing on several of the constructs and relationships diagrammed in the figure; this report will be restricted further to a subset of these components. Figure 2 shows the precise nature of these limitations.

Insert Figure 2 about here

To summarize the information contained in Figure 2, this report will consider students' conceptions of decimal fractions and will describe some of the subconcepts upon which they base their notions of decimals, especially those subconcepts derived from previous work with whole numbers and fractions. Of particular interest will be the interpretations students impose on representations of decimal numbers. The question of whether students connect notions of whole numbers and/or fractions to decimal representations will be of primary importance.

The report, however, will not consider a variety of other important issues. For example, decimals are representations of rational numbers, and as such they may be given a variety of interpretations

(Kieren, 1980). This report considers part-whole and measure interpretations but does not cover those of quotient, ratio, or operator. In addition the report does not describe the rule systems students use to compute with decimals. The focus is quite strictly on the conceptions and misconceptions of decimal fractions that students have acquired.

Sample. The sample during the first year of the project, and upon which this report is based, consisted of students in grades 3, 5, 7, and 9. The initial written test was given to 115 fifth-graders, 256 seventh-graders, and 212 ninth-graders. Twenty-five students in each of grades 3, 5, 7 and 9 were then interviewed individually. The 25 students in grades 5, 7, and 9 were selected to ensure a representative sample based on their written test performance. The third-graders were chosen by asking the four third-grade teachers in the target school to identify a group of 25 students that would represent equally the achievement levels of the third-grade population. Seven students in each grade were then selected to receive a second interview. In each case, the subsample of seven students was chosen to represent equally the performance levels exhibited on the first interview. All subjects were drawn from schools located in a moderate size midwestern city and containing a racial and socio-economic mixture of students. All students in participating classrooms who returned parent permission forms received the initial written test.

Assessment techniques. Three different forms of assessment were used in order to obtain a more complete picture of students' conceptions of decimal numbers. The first assessment was a 30-45 minute paper-and-pencil group test that included items on decimal computation, translation between decimal numbers and common fractions, positional value of decimal digits, ordering decimal numbers, using number lines and partitioned regions to represent decimal values, and solving decimal word problems. A pool of items was constructed initially and item sampling was used to build several test forms. This permitted the administration of a greater number of items and helped to ensure that each grade level received items that were of appropriate difficulty.

The second assessment was an individual interview using a standardized format. The interviewers were the two principal investigators and a graduate assistant who had been trained to administer the interview. Interview tasks primarily consisted of matching concrete representations of decimal numbers using base ten Dienes' blocks with written representations. The goal of the interviews was to assess, in a facilitative context, students' understanding of decimal subconcepts such as partitioning and regrouping (see Figure 2), and to determine what meanings they had assigned to the written decimal symbols. Most items asked students to represent a decimal number with the blocks, perform a specified operation with the blocks, or write a number corresponding to a given block display. Consequently the responses were unambiguous and were recorded on a prepared coding sheet. Interview sessions lasted about 25 minutes.

The third assessment was a flexible, in-depth interview. The primary objective of this interview was to explore more deeply the rules students use to solve decimal problems and to test hypotheses about these rules that had been generated from analyzing responses on the written test. The interviewers were the two principal investigators. Interview tasks involved presenting students with written problems (computation, translating decimals to fractions, etc.) and asking students to explain their work by using a variety of probe questions. The initial items were predetermined, but the follow-up questions were based on the student's preceeding responses and therefore varied somewhat from student to student. The interview sessions, 20-30 minutes in length, were audio-taped. Notes were taken during the sessions and responses indicating confirmation or rejection of a specific hypothesis being tested were coded immediately.

Since the third-graders had not yet received decimal instruction they received different interview tasks during the in-depth interview. A sequence of tasks was constructed that exactly paralleled the standardized interview but dealt with common fractions rather than decimals. These tasks were given to the third-graders in order to collect some information on the foundation knowledge of fractions with which students enter instruction on decimals.

Written test items, interview protocols, and interview scoring sheets are available from the authors.

Assessment schedule. A summary of the times during the year when each assessment was administered, and the number of subjects involved at each stage, is given in Table 1.

Insert Table 1 about here

Tasks. The tasks that are of primary interest in this report are those that focused on the meaning of a decimal number in various contexts. As indicated in Figure 2, the possible meanings of a decimal were not exhausted; for example, the notion of a repeating decimal (derived from the quotient interpretation), was not considered in any of the tasks. However, the tasks did deal with the following meanings: (1) decimal numbers are extensions of whole numbers that have positional value; (2) decimal numbers are rational numbers that have common fraction equivalents, and that can be treated as measures; and (3) decimal numbers are quantities that have value, i.e., that can be ordered.

Results

Tasks that assess the meaning, or explicit understanding, that students attribute to decimal numbers can be of two types. The task can present the standard decimal representation and ask the student to relate the symbol to other knowledge they may already have or may be able to construct in the course of solving the task. It is assumed that the response indicates the presence of one of more understandings (see Figure 1). Alternatively, the task can present a stimulus which is assumed to represent for the student one or more of the understandings, and ask the student to write the associated decimal representation. Each of the tasks to be presented shortly can be classified as one of these two types, or as variations of them (e.g., translating between symbol systems.)

Decimal numbers as extensions of whole numbers. Instruction on decimals in elementary school often is designed to build upon previous work with whole numbers. Decimal representations are treated as symbols that are similar to whole numbers, with the value of the positions decreasing by a factor of ten as you move to the right (just as they did before) and the decimal point standing between the position values of more than one and less than one. Sometimes pictures of base ten Diene's blocks are shown (with the large block as the unit) to reinforce the values of the positions, just as bundles of sticks are used to show the values of the whole number positions.

The first set of tasks to be reported considered students' ability to relate decimal symbolic representations to pictorial and concrete representations, which presumably have a value interpretation that is easy to discern. Items of this kind appeared both on the group tests and in the standardized individual interviews. As a general word of explanation, results of group test items are usually presented in tables showing the percentage of students, by grade level, who responded correctly. The tables also list frequent incorrect responses that were given on each item, with "frequent" being defined as a response that accounted for at least 15% of the errors on that item and having at least four occurrences.

The results shown in Table 2 indicate that by the time stu-

Insert Table 2 about here

dents reach grade five they are able to write symbolic representations of whole numbers that are shown pictorially. However this link between representations for whole numbers does not extend to decimals. The results in Table 3 suggest that students

Insert Table 3 about here

experience new problems in dealing with decimal symbols. Apparently they are not able to simply "do the same thing" with decimals as with whole numbers.

A closer look at the ability of students to relate symbolic representations to other representations that more naturally show the value of the number is provided by tasks from the standardized interview.

Insert Tables 4 and 5 about here

The results in Table 4 show that students did better, probably for a variety of reasons, on the interview tasks than they did on the analogous group test items that showed pictures of blocks. However, about one-third of the seventh graders were still unable to write a decimal number for a block display, even when they knew the values of the blocks and had been shown how to write decimals for other displays. On the other hand, it is clear that most third-graders had not yet been exposed to decimal notation, but after a brief interaction about one-fourth of them could regroup blocks when necessary and write the appropriate decimal number (see Task 4).

The most interesting aspect of the data in Table 4 is the variety of errors. Many of the errors are made by the younger students who have not worked extensively with decimals, and it is the nature of these errors that is most intriguing. In Tasks 1 and 3, the younger students are asked to write a number, the form of which may be nearly novel. It is interesting to analyze students' inventions under these conditions. In general, students do not write nonsense symbols, but rather try quite hard to figure out ways of representing the block values so that the form itself carries some meaning. The invented notation often builds in a meaning that is quite obvious, usually more obvious than the standard notation. It is tempting to conclude from the kinds of errors students make here that students would like to connect form and understanding, if given a chance. More will be said later about this point.

The tasks presented in Table 5 involved the reverse skill of those in Table 4: students were shown a written decimal and asked to put out the blocks that would show that number. They had been through all of the tasks in Table 4 before receiving those in Table 5. Even assuming improved performance due to this brief "learning" session, the striking feature of the results in Table 5 is the high level of success. Students had little trouble laying out the appropriate blocks and most students could regroup the blocks to show alternate representations (see Task 4). It is surprising that third-graders, who had received no classroom instruction on decimals, were as successful as seventh-graders on this task. It appears that the concept of a decimal number and its associated symbolic notation is not beyond the cognitive capability of most third grade students.

The results shown in Tables 6 and 7 provide some evidence of

Insert Tables 6 and 7 about here

students' facility with more conventional problems on the value of whole number and decimal positions. The items were presented in multiple-choice formats on the group tests and consequently are shown using slightly different form in the tables. Because of considerable differences in the amount of instruction on decimals that had been received by fifth-graders as compared to seventh- and ninth-graders, these two groups received different items. As seen in Table 6, fifth-graders did quite well on the straightforward questions about place value, but quite poorly on the less conventional items, even though these dealt only with whole numbers. Entering decimal instruction with the deficiencies in whole number knowledge displayed here is bound to create further problems. Students certainly are unable to extend knowledge of whole numbers to create meaning for decimals if the knowledge does not exist.

The data in Table 7 suggest that about one-half of the seventh-graders and three-fourths of the ninth-graders were able to identify the values of decimal and whole number positions. While some of the errors can be arributed to careless reading (e.g., Item 2), this is still a relatively low success rate on conventional kinds of questions.

The tasks shown in Table 8 probe more deeply into the underly-

Insert Table 8 about here

ing notions that produce surface similarities between whole numbers and decimals. Namely, the tasks assess students' knowledge of the relative size of numbers with the decimal point in different positions. It is the type of errors that students make that again provides the most interesting data. For example, the most frequent error on a task asking students to write a number 10 times as big as 437.56 resulted from the misconception that adding a zero to the end of decimal number increases its value 10 times. Apparently, some students extend their whole number rules to decimals in toto, without recognizing the important distinctions. Thus, it appears that in some cases students are unable to extend their whole number understanding to decimals because it doesn't exist; in other cases they overextend the rules for manipulating whole number symbols, possibly because the links between form and understanding are absent.

Decimal numbers as rational numbers. The tasks presented in this report considered two aspects of decimals viewed as rational numbers: (1) decimal numbers are measures that can be represented by a point (or segment) on the number line and by a part of a region; (2) decimal numbers have common fraction equivalents. Many of the group test items used to assess this knowledge were specially constructed as a nested series of items. Based on a logical analysis of the task and some preliminary information from pilot-testing on how students solve the tasks, the ultimate goal task was pared down into a series of precursor tasks, with each paring reducing the amount of knowledge needed to solve the task. This generated a nested series of tasks; each task involving more information than the one before it. In an obvious way, this procedure also produced a predicted order of difficulty associated with each series. Since in some sense each item is embedded in the next, the items theoretically should form a perfect scale. However, there are other factors that affect item difficulty, such as the salience or appeal of an inappropriate solution. Since this factor appears to be quite influential but is not yet well understood, it is difficult to construct a series of items that is scalable. Nevertheless, the nested series of itmes were expected to be monotonically increasing in difficulty for the sample as a whole. The order of the items was randomized within the series for presentation in the group tests; the items are listed in their predicted order of difficulty in the tables.

A series of items that asked students to place decimals on a number line is presented in Table 9. Interpreting decimals as

Insert Table 9 about here

points on a number line is one aspect of the measure meaning that can be attributed to decimal representations. The items are listed from simplest to most complex, based on the logical task analysis described above. The predicted order of difficulty is confirmed by the data. The considerable drop in success rate from Item 1 to Item 2 probably is due to a misreading of the scale on the number line in Item 2. Larson (1980) reports a similar phenomenon with common fractions. Often it is assumed that students simply count over the given number of marks, treating each segment as one unit. In their hurry to find an answer, they overlook the fact that each segment is more, or less, than one unit. However, the errors on this item do not just result from a careless misreading: students could not count over 7 units between 2 and 3 (there are only 4 marks), and there is no real convergence to an appealing erroneous solution. It appears that even when students are forced to reconsider the scale (after a simple count does not work), a large number of them are unable to interpret the scale appropriately.

The items shown in Table 10 asked students to shade a (decimal)

Insert Table 10 about here

fraction part of a region. The predicted order of difficulty for this series of items was confirmed with the exception of one item for the ninth-grade sample. One of the interesting features of these data is the high rate of performance of the fifth-graders relative to their older counterparts. There are several possible explanations for this. A salient one is that the fifth-graders in this study had not yet studied hundredths to any substantial degree, and consequently ~~they~~ did not have this added "knowledge" to confuse them. Evidence for this interpretation is contained in the error columns. No fifth-grader shaded hundredths rather than tenths in Item 1, and relatively few (compared to seventh- and ninth-graders) did so in Item 3. It is assumed, of course, that fifth-graders would not have done as well as the older students on the hundredth items, but it remains a disconcerting fact that instruction introduces some errors that would not otherwise appear. This same instructional-interference phenomenon appears in later items.

The items in Table 11 reverse the process of those in Table 10,

Insert Table 11 about here

with markedly different results, at least for seventh-graders. Seventh-grade students essentially were unable to write the decimal representation for a shaded region that is not divided into tenths or hundredths. Some of them gave a correct common fraction, and a portion of this group may have misread the instructions. But a larger group used the numerals from an appropriate common fraction to create a decimal (1.5 for 1/5 and 1.4 for 1/4). This is a rather nonsensical response since the shaded region is clearly less than one. It indicates that, in this measure context, the decimal number symbols have little meaning for the students.

Items that considered decimal numbers as common fraction equivalents are shown in Tables 12 and 13. Two nested series of items

Insert Tables 12 and 13 about here

assessed students' ability to write decimals as fractions (Table 12) and fractions as decimals (Table 13). The predicted order of difficulty for both series is confirmed by the data. This validates, in part, the analysis of the task in terms of the knowledge that students use to solve the task. Taken together with an analysis of the errors that students make on these items, it is possible to speculate about the processes students use to solve the problems.

It is clear from the data in Tables 12 and 13 that the processes students used to convert between decimals and fractions change as they move from relative novice to relative expert status with regard to decimals. On each task, nearly one-fourth of the fifth-graders simply converted the numerals of the given number to the alternate form (e.g., .37 = <u>3/7</u>). This error all but disappears by grade nine. The most frequent error in grade nine was to confuse the values of the decimal positions (e.g., 83/100 = <u>.083</u>). At first glance this error seems to carry more understanding of the decimal/fraction relationship than the error most popular in fifth grade. It probably does. But apparently the error also can be explained by an increased awareness of proper form, independent of understanding. Evidence for students' increased sensitivity to appropriate form, in this context, comes from several tasks administered during the in-depth interviews. Students were asked to change decimals to fractions, and vice versa, and were asked to explain their procedures. Most of the seventh- and ninth-graders indicated that a decimal can be written as a fraction by using a denominator of 10 or 100 or 1000, etc., and that a fraction with a denominator of 10 or 100 or 1000 can be written as a fraction by writing the numerator and placing the decimal point. But fewer of them could explain why this rule worked. And more than one-half of the seventh-graders and one-third of the ninth-graders were unable to write 1/4 as a decimal. One seventh-grader summarized a number of students' perceptions of decimal/fraction relationships by saying, ".6 is 'six-tenths', and 6/10 you say the same way, but it's different."

Based on these observations, it is reasonable to hypothesize that most students do not come to decimals through factions; they do not initially attribute meaning to decimal symbols based on their understanding of fractions. If they do acquire an early meaning for decimal number symbols, it does not appear to be a meaning that links the symbols to fraction concepts. Apparently, this linking comes much later, after students have developed some high level skills in manipulating both fraction and decimal symbols.

Decimal numbers as quantities. One way to assess whether students attach a notion of magnitude or size to decimal symbols is to ask them to order decimal numbers, to deal with relative magnitudes. Like most items, these are not independent of other understandings, such as positional value; but the focus is on decimals as quantities, as things that have a magnitude and therefore as things than can be ordered.

The items appearing in Table 14 are multiple choice items.

Insert Table 14 about here

The choices are listed along with the percent of students selecting each choice. In general, it appears that the majority of fifth-graders ignored the decimal andtreated the numbers as whole numbers, while most of the ninth-graders read the decimals correctly. Some of the seventh-graders responded like fifth-graders and some responded like ninth-graders.

The items shown in Table 15 asked students to write a decimal

Insert Table 15 about here

number that "comes between" two given decimals. The problem was relatively easy if the decimal essentially could be ignored (e.g., 1.56 and 1.72) but more difficult if the response required a familiarity with the decimal system (e.g., .2 and .3). The response of .2 1/2 for a number between .2 and .3 is especially interesting, because this response violates the rules for form, but it is loaded with meaning. Responses like this, and those shown in Table 4, are somewhat unique. They display an understanding, but represent the understanding with nonconventional, invented notation. Semantics before syntax, if you will. In contrast, most errors reported here have been responses that adhere to proper form, but expose a genuine lack in understanding. These are responses that are written in standard form, and may have some surface or form similarity to the problem, but at the deeper, understanding level show little relationship to the question. Syntax before semantics, if you will.

Discussion

The overriding objective of this report is to provide some insight into students' conceptions of decimal numbers. What do students think decimals are? In the interests of dealing with this question, a distinction was made between form and understanding, and it was argued that useful mathematics knowledge results from linking these domains. Meaning, in mathematics, occurs when an understanding or concept or idea is linked with the symbols that express and communicate the idea. This is a central issue in mathematics learning, and it was this issue that provided the focus for much of the research effort on decimals reported here.

Based on the results, it appears that students have created few links between decimal form and decimal understanding. Many students appear to operate within one or the other of these domains, dependent on the context, but fail to see the critical connections between them. Consequently, they are unable to draw on one source of knowledge to assist with the other. For example, even though fifth- and seventh-graders have worked with tenths and hundredths written as decimal fractions most are unable to use this information about form to write a number between .2 and .3. Some students invent a new form to express their understanding. They do not recognize that the standard form, with which they are familiar, will represent their idea equally well or better. Connecting form and understanding is a late, rather than early, development in learning about decimals.

A second observation is that instruction seems to be doing a better job of teaching form than understanding. Very few of the students' errors (in fifth grade and beyond) violated form conventions. Most responses were written in standard decimal notation. The error usually resulted from the fact that the specific response represented a number that was unrelated to the question. At times it appeared that the form was floating on its own, not tied to understandings that would provide some consistency and prevent many of the unreasonable responses.

A third observation, related to the first two, is that students' perceptions of decimal fractions are influenced more by their knowledge of form than by their understanding. Many students view a decimal number as a special form, as a new way of writing a number. This special representation comes with a unique set of rules that tell how to manipulate the symbols. Although many students demonstrated some fundamental understandings of the decimal system (on the standardized interview tasks), it is their knowledge of form that seems to determine their behavior.

The implications are clear. Instruction must be designed to encourage and facilitate the construction of bridges between form and understanding. Whereas now students seem to build these links after they are reasonably competent in both domains, it is reasonable to believe that learning would be increased by helping students make the connections from the outset. One possible approach would be to take advantage of students inventive powers, and have them create an informal, transitional symbol system that has meaning for them. This approach has been advocated elsewhere (James & Mason, 1982; Resnick, 1980; Skemp, 1982; Woodrow, 1982) and warrents further careful study.

References

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. Decimals: Results and implications from the second NAEP mathematics assessment. Arithmetic Teacher, 1981, 21(8), 34-37.

Case, R. Intellectual development from birth to adulthood: A neo-Piagetian interpretation. In R. S. Siegler (Ed.), Children's thinking: What develops? Hillsdale, N. J.: Erlbaum, 1978.

Davis, R. B., & McKnight, C. The influence of semantic content on algorithmic behavior. The Journal of Mathematical Behavior, 1980, 3, 39-87.

Ekenstam, A. On children's quantitative understanding of numbers. Educational Studies in Mathematics, 1977, 8, 317-32.

Erlwanger, S. H. Case studies of children's conceptions of mathematics--Part I. Journal of Children's Mathematical Behavior, 1975, 1(3), 157-183.

Glaser, R. Trends and research questions in psychological research on learning and schooling. Educational Researcher, 1979, 8(10), 6-13.

Greeno, J. G. Analysis of understanding in problem solving. In R. J. Kluwe & H. Spada (Eds.), Developmental models of thinking. New York: Academic Press, 1980.

Hofstadter, D. R. Gödel, Escher, Bach: An eternal golden braid. New York: Vintage Books, 1979.

James, N., & Mason, J. Towards recording. Visible Language, 1982, 16, 249-258.

Kieren, T. E. The rational number construct--Its elements and mechanisms. In T. E. Kieren (Ed.), Recent research on number learning. Columbus, OH: ERIC/SMEAC, 1980.

Larson, C. N. Seventh-grade students' ability to associate proper fractions with points on the number line. In T. E. Kieren (Ed.), Recent research on number learning. Columbus, OH: ERIC/SMEAC, 1980.

Piaget, J. The psychology of intelligence. Totowa, N. J.: Littlefield, Adams & Co., 1960.

Resnick, L. B. The role of invention in the development of mathematical competence. In R. H. Kluwe & H. Spada (Eds.), Developmental models of thinking. New York: Academic Press, 1980.

Resnick, L. B. Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: A cognitive perspective. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1982.

Skemp, R. R. Communicating mathematics: Surface structures and deep structures. Visible Language, 1982, 16, 281-288.

Van Engen, H. An analysis of meaning in arithmetic. Elementary School Journal, 1949, 49, 321-329; 395-400.

Van Engen, H. The formation of concepts. In H. F. Fehr (Ed.), The learning of mathematics. Twenty-first Yearbook of the National Council of Teachers of Mathematics. Washington, D. C.: NCTM, 1953.

Werner, H. Comparative psychology of mental development (Rev. ed.). Chicago: Follett, 1948.

Werner, H., & Kaplan, B. Symbol formation. New York: John Wiley & Sons, 1963.

Woodrow, D. Mathematical symbolism. Visible Language, 1982, 16, 289-302.

Table 1

Assessment Schedule

	Written Test (December) N	Standardized Interview (March) N	In-Depth Interview (April) N
Grade 3	---	25	7
Grade 5	115	25	7
Grade 7	256	25	7
Grade 9	212	25	7

Table 2

Pictorial Representations of Whole Numbers (Grade 5)

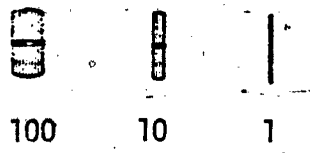
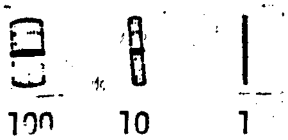
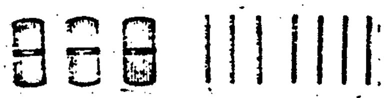
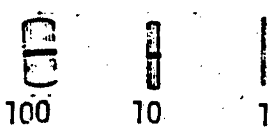
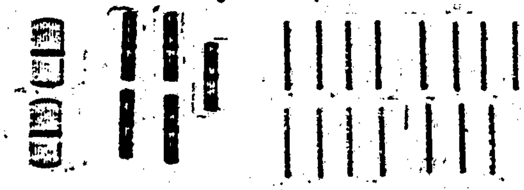
Item	Percent Correct (N=53)	Frequent Errors
1)  Write the number that tells how many are pictured.  (324)	86.8	----
2)  Write the number that tells how many are pictured.  (307)	90.6	----
3)  Write the number that tells how many are pictured.  (265)	75.5	----

Table 3

Pictorial Representations of Decimal Fractions

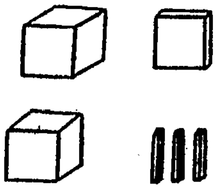
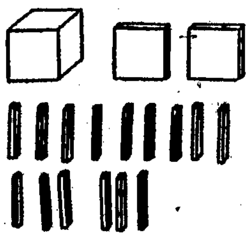
Item	Percent Correct			Frequent Errors	Percent of Responses (Percent of Errors)		
	5th(58)	7th(72)	9th(58)		5th	7th	9th
1)  1 .1 .01 Write the decimal that tells how much is pictured.  (2.13)	15.1	33.3	70.7	.33	0	11(17)	0
				6	13(16)	3(4)	0
2)  1 .1 .01 Write the decimal that tells how much is pictured.  (1.35)	13.2	36.1	53.4	.18	11(13)	13(20)	0
				1.215	4(4)	6(9)	12(26)
				.45	0	11(17)	0

Table 4

Interview Task: Writing Decimal Numbers for Block Displays

	Percent Correct (N=25 each grade)				Interesting Responses	Percent Responding			
	3rd	5th	7th	9th		3rd	5th	7th	9th
1) Write the decimal for 2 whole and 5 tenth blocks. (Students have been shown value of blocks but not how to write decimals.)	16	48	68	88	2.510	4	4		
					2.5/10	4			
					2 5/10		4		
					2 1/2	8			
					don't know	32	16	8	4
2) Write the decimal for 1 whole and 14 tenth blocks. (Students have been shown how to write decimals for non-regrouping situations.)	20	48	56	80	1.14	64	40	40	16
3) Write the decimal for 2 whole, 3 tenth, and 5 hundredth blocks. (Students have been shown value of blocks and how to write decimals with tenths.)	8	36	64	80	2.3.5	28	12		
					2.3.5 100th		4		
					2.3.500	8	8		
					2.3500	4			
					2.3 5/100				4
					2.3 5/10		4		
					2.3 1/2	8			
					2.3/5			4	
					2.8	12		8	4
4) Write the decimal for 2 whole, 4 tenth and 11 hundredth blocks. (Students have been shown how to write decimals for displays like those above.)	28	52	68	80	2.411	52	28	20	12
					2.4.11	4		4	

Table 5

Interview Task: Constructing Block Displays for Written Decimal Numbers

Item	Percent Correct (N=25 each grade)			
	3rd	5th	7th	9th
1) Put out blocks to show "1.32". (Students have been shown how to write decimal numbers to represent block displays.)	96	100	96	96
2) Put out blocks to show ".41".	92	100	92	96
3) Put out blocks to show "2.30".	96	100	88	96
4) Use different blocks to show "2.30". (This instruction was repeated 6 times.)				
Percent of subjects who constructed at least one alternative representation:	80	80	80	96
Percent of subjects who constructed four or more alternative representations:	52	36	56	84

Table 6

Positional Value of Digits (Grade 5)

	Responses	Percent Responding (N=53)
1) Circle the digit that is in the hundreds place of 6734	7*	90.6
	3	5.7
	6	3.8
	4	0
2) Circle the digit in the tenths place of 537.89	8*	75.5
	3	13.2
	7	5.7
	5	1.9
3) Circle all the numbers that have the same value as 574 a. 5 hundreds + 4 ones + 7 tens b. 4 + 70 + 500 c. 50 + 70 + 40 d. 5 hundreds + 6 tens + 14 ones	a,b,d*	5.7
	c	35.8
	a	17.0
	b	13.2
	a,b	11.3
	b,d	5.7
4) If we changed the 5 in 351 to a 7, the number would get bigger. How much bigger?	20*	32.1
	2	24.5
	7	17.0
	70	9.4

*correct response

Table 7

Positional Value of Digits (Grades 7 and 9)

Item	Possible Responses	Percent Responding 7th(72)	9th(58)
1) Circle the digit that is in the tenths place of 67.982	9*	59.7	70.7
	8	23.6	8.6
	6	5.6	6.9
	2	4.2	6.9
	7	4.2	5.2
2) Circle the digit that is in the hundreds place of 7823.456	8*	33.3	22.4
	5	36.1	53.4
	4	16.7	10.3
	2	5.6	8.6
3) Circle the word that tells in which place the 4 is in 23.64	hundredths*	54.2	77.6
	ones	13.9	8.6
	tenths	8.3	5.2
	tens	6.9	5.2
	hundreds	9.7	0
4) Circle the word that tells in which place the 3 is in 645.37	tenths*	58.3	74.1
	tens	15.3	8.6
	oneths	6.9	1.7
	hundredths	8.3	8.6

*correct response

Table 8

Factor of 10 as Basis for Numeration System

Item	Percent Correct 7th(82)	Percent Correct 9th(75)	Frequent Errors	Percent of Responses (Percent of Errors) 7th	Percent of Responses (Percent of Errors) 9th
1) Write a number ten times as big as 437 (4370).	74.4	90.7	43700	1(5)	4(43)
2) Write a number ten times as big as 437.56 (4375.6 or equivalent).	54.9	74.7	437.560	13(30)	7(26)
3) Write a number one-tenth as big as 829 (82.9 or equivalent).	39.0	58.7	829.1 (or equivalent)	11(18)	12(29)
			8290	10(16)	5(13)
4) Write a number one-tenth as big as 62.48 (6.248 or equivalent).	28.0	53.3	62.58	18(25)	12(26)
			624.8	13(19)	11(23)
5) Circle the number in which the value of 4 is ten times as much as the value of 4 in .24 (.423).	52.4	49.3	other possible responses: .894	15(31)	20(40)
			4.368	20(41)	15(29)
			45.67	5(10)	8(16)

Table 9

Placing Decimal Fractions on Number Lines

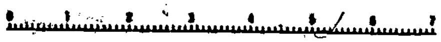
Item	Percent Correct			Frequent Errors	Percent of Responses (Percent of Errors)		
	5th(62)	7th(102)	9th(79)		5th	7th	9th
1) Mark 3.4 on number line. 	35.5	87.3	92.4	3.3	11(18)	3(23)	1(17)
				3.5	0	3(23)	1(17)
2) Mark 2.7 on number line 	8.1	42.2	65.8	2.6	7(7)	8(14)	5(15)
3) Mark .3 on number line 	----	29.4	51.9	3		27(38)	8(16)
				.6		18(25)	14(29)
				.4		11(15)	9(18)
				.1		1(1)	9(18)
4) Mark .42 on number line 	----	16.7	44.3	4.2		13(15)	5(9)
				4.4		21(25)	9(16)

Table 10

Shading Decimal Fractions of Regions

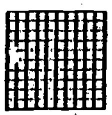
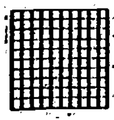
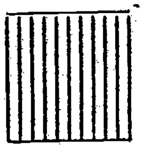
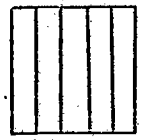
Item	Percent Correct			Frequent Errors	Percent of Responses (Percent of Errors)		
	5th(62)	7th(102)	9th(79)		5th	7th	9th
1) Shade .7 of the figure 	83.9	89.2	77.2	.07	0	5(46)	8(33)
2) Shade .08 of the figure 	----	53.9	69.7	.8		33(72)	11(38)
				.01		2(4)	5(17)
				no response		4(8)	5(17)
3) Shade .4 of the figure 	72.6	50.0	57.0	.04	8(29)	39(78)	29(68)
4) Shade .16 of the figure 	----	25.5	25.3	.6		16(21)	4(5)
				.12		1(1)	18(24)
				no response		17(22)	6(9)
5) Shade .63 of the figure 	----	24.5	35.4	.60		23(30)	10(16)
				no response		21(27)	8(12)

Table 11

Writing a Decimal for a Partially Shaded Region

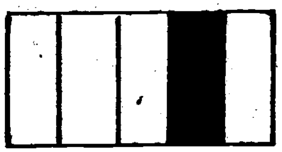
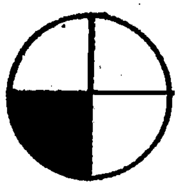
Item	Percent Correct		Frequent Errors	Percent of Responses (Percent of Errors)	
	7th(72)	9th(58)		7th	9th
1) Write the decimal that tells what part of the figure is shaded (.2). 	5.6	56.9	1.5	28(29)	7(16)
			common fraction	22(24)	21(48)
2) Write the decimal that tells what part of the figure is shaded (.25). 	2.8	56.9	1.4	28(29)	7(16)
			common fraction	22(23)	19(44)

Table 12

Converting Decimals to Fractions

Item	Percent Correct			Frequent Errors	Percent of Responses (Percent of Errors)		
	5th(62)	7th(102)	9th(79)		5th	7th	9th
1) .37 = (37/100)	12.9	71.6	84.8	3/7	31(35)	7(24)	4(25)
				no response	24(28)	2(7)	4(25)
2) .09 = (9/100)	8.1	68.6	83.5	0/9	24(26)	5(16)	0
				9/10	15(16)	12(38)	8(46)
				no response	26(28)	1(3)	3(15)
3) 5.02 = (5 2/100) (or equivalent)	4.8	49.0	64.6	5/2	24(25)	10(19)	0
				5/200	3(3)	10(19)	3(7)
				no response	29(31)	7(14)	6(18)

Table 13

Converting Fractions to Decimals

Item	Percent Correct			Frequent Errors	Percent of Responses (Percent of Errors)		
	5th(62)	7th(102)	9th(79)		5th	7th	9th
1) 4/10 = (.4)	30.6	78.4	91.1	4.10	24(35)	8(36)	0
				.04	0	4(18)	3(29)
				common fraction	11(16)	2(9)	3(29)
				no response	18(26)	0	1(14)
2) 83/100 = (.83)	22.6	70.6	91.1	83.100	21(27)	4(13)	0
				83.00	3(4)	8(27)	1(14)
				.083	2(2)	6(20)	4(43)
				no response	27(35)	1(3)	1(14)
3) 3/100 = (.03)	19.3	69.6	82.3	.3	7(8)	7(23)	4(21)
				.003	5(6)	5(16)	6(36)
				3.00	3(4)	5(16)	1(7)
				3.100	21(26)	4(13)	0
				no response	18(24)	0	1(7)
4) 43/10 = (4.3)	9.7	49.0	64.6	43.10	18(20)	6(12)	1(4)
				43.0	18(20)	15(29)	5(14)
				.43	16(18)	19(37)	22(61)
				no response	27(30)	4(8)	1(4)
5) 406/100 = (4.06)	9.7	41.2	59.5	406.100	18(20)	2(3)	1(3)
				406.00	18(20)	11(18)	5(13)
				.406	13(14)	25(42)	23(56)
				no response	34(38)	5(8)	3(6)

Table 14

Ordering Decimal Fractions

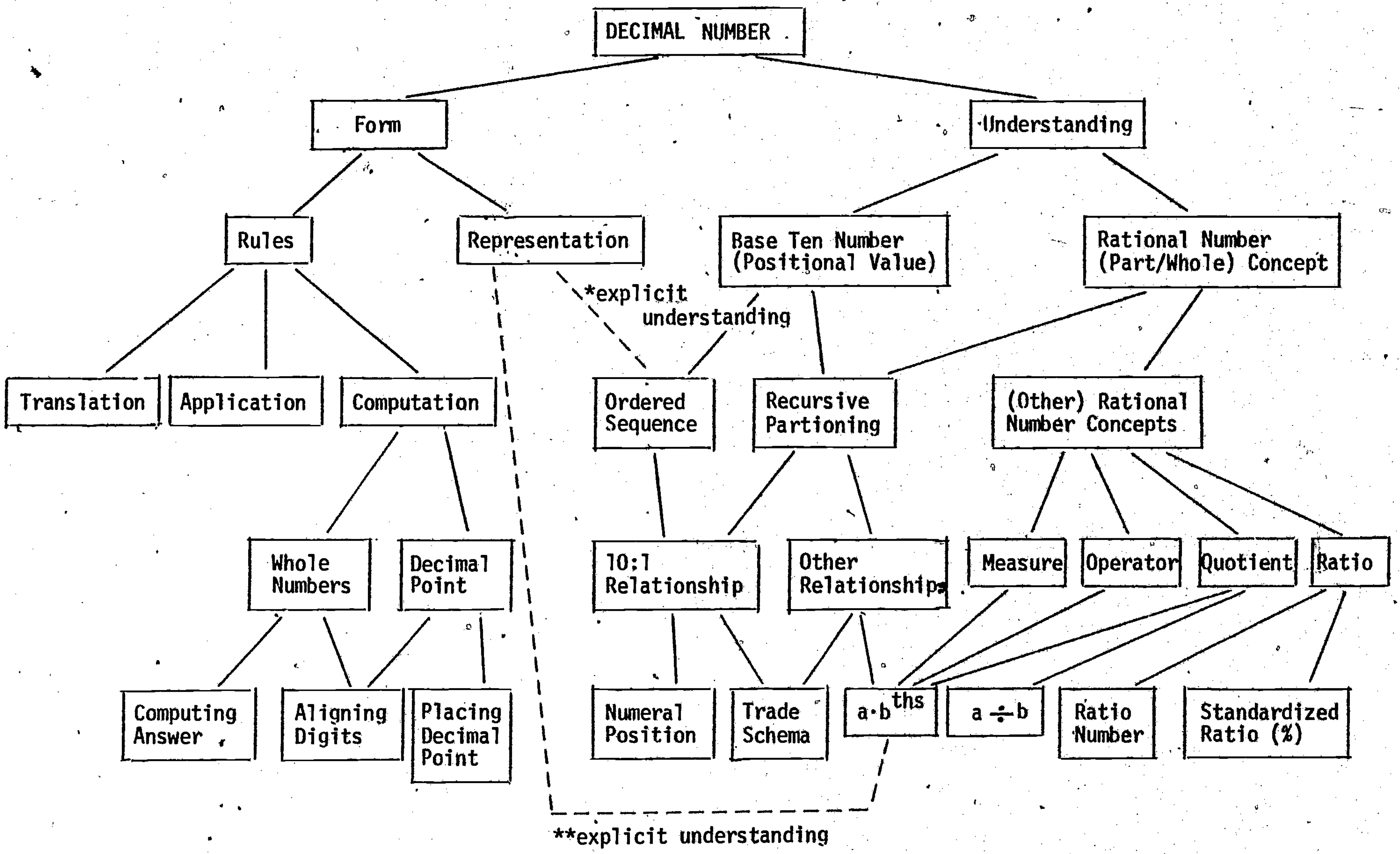
Item	Possible Responses	Percent Responding 5th(53)	7th(82)	9th(75)
1) Circle the number that is greater than .36	.4*	13.2	63.4	82.7
	.360	52.8	13.4	5.3
	.359	5.7	12.2	6.7
	.279	9.4	6.1	5.3
2) Circle the number that is less than .54	.5299*	3.8	50.0	76.0
	1.2	18.9	9.8	4.0
	.6	56.6	29.3	16.0
	.540	9.4	8.5	4.0
3) Circle the number nearest to 7.82	8.0*	----	75.6	92.0
	7.0		19.5	6.7
	9.0		1.2	1.3
	0.8		2.4	0
4) Circle the number nearest to .16	.2*	----	47.6	70.7
	.1		7.3	12.0
	.02		41.4	16.0
	.01		1.2	1.3

*correct response

Table 15

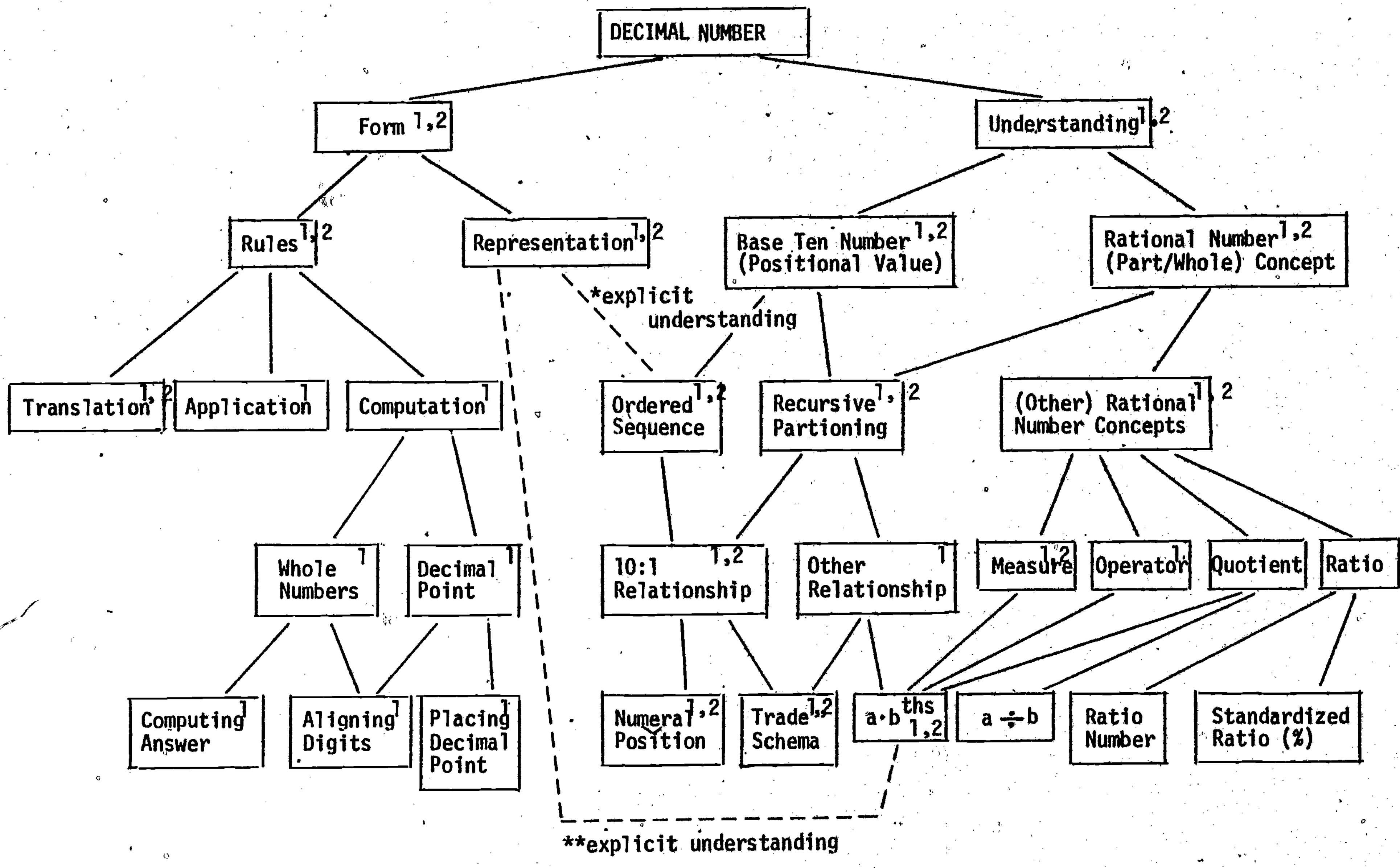
Ordering Decimal Fractions

Item	Percent Correct			Frequent Errors	Percent of Responses (Percent of Errors)		
	5th(53)	7th(82)	9th(75)		5th	7th	9th
1) Write a number that comes between .2 and .3	9.4	41.5	68.0	.2 1/2	4(4)	17(29)	7(21)
				2 1/2	13(15)	5(8)	0
				no response	34(38)	10(17)	5(17)
2) Write a number that comes between 1.56 and 1.72	---	92.7	88.0	(none)			



*connecting decimal form with whole number understanding
**connecting decimal form with fraction understanding

Figure 1 Analysis of decimal fraction



*connecting decimal form with whole number understanding

**connecting decimal form with fraction understanding

[1]Investigated as part of decimal project

luded in this report

Figure 2 Scope of decimal project