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ABSTRACT

Three models have been proposed to account for the relationship between the principle of commutativity and the development of more economical addition strategies, which disregard addend order. In the first and second models, it has been proposed that either discovery or assumption of commutativity is a necessary condition for the invention of advanced addition strategies. A third model suggests that children may invent labor-saving addition strategies without appreciating the commutativity principle. A study tested these three models by evaluating 36 kindergarteners' responses on two types of commutativity tasks. Both tasks involved predicting whether commuted and noncommuted pairs of problems would produce the same or different answers. Over two sessions, children's addition strategies were also assessed. Strategies noted were spontaneous counting-all with concrete supports; counting-all mentally, starting with the first addend; counting-all mentally, starting with the larger addend; counting-on mentally from the first addend; and counting-on mentally from the larger addend. (Counting-all strategies begin at the number 1; counting-on strategies begin at the value of the first addend selected by the child.) Findings indicated that, as proposed by the second model, commutativity was not naturally assumed by children, but appeared to be discovered. Contrary to the first model and consistent with the third, an understanding of commutativity was not evident in all subjects who invented labor-saving addition strategies. (Author/RH)

The Development of the Commutativity Principle and Addition Strategies in Young Children

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Abstract

Three models have been proposed concerning the relationship between the principle of commutativity and the development of addition strategies which disregard addend order. It has been proposed that the discovery (model 1) or assumption (model 2) of commutativity is a necessary condition for the invention of such advanced addition strategies. A third model suggests that children may invent labor saving addition strategies without appreciating the commutativity principle. This study tested the three models by evaluating 36 kindergarteners on two types of commutativity tasks. Both tasks involved predicting whether commuted and noncommuted pairs of problems would produce the same or different answers. Over two sessions, addition strategies were also determined. Commutativity was not naturally assumed by children (as proposed by model 2), but appeared to be discovered. However, contrary to model 1 and consistent with model 3, an understanding of commutativity was not evident in all those who invented labor saving addition strategies.

The Development of the Commutativity Principle and Addition Strategies in Young Children

Surprisingly little research exists on the development and use of basic mathematical principles such as the commutativity of addition (the order in which terms are added does not affect the sum) (Gelman & Starkey, 1979; Suydam & Weaver, 1975). Some evidence indicates that children appreciate the commutativity principle quite early (Baroody, Berent, & Packman, 1982; Baroody, Ginsburg, & Waxman, in press; Ginsburg, 1982). This study examined children just beginning school for evidence of (a) an implicit knowledge of commutativity and (b) a developmental relationship between this principle and informal addition strategies.

Children appear to invent increasingly sophisticated and economical counting strategies to compute addition (e.g., Carpenter & Moser, 1982; Groen & Resnick, 1977; Ilg & Ames, 1951; and Resnick & Ford, 1981; Starkey & Gelman, 1982). Though it requires more mental effort, some informal mental addition strategies always deal with the first addend first (Baroody, in press). For example, counting-all starting with the first addend (CAF), the most basic mental addition strategy, involves enumerating the first addend and continuing this count as the second addend is enumerated (e.g., 2 + 4: "1, 2; 3[1], 4[2], 5[3], 6[4]--6"). Note that the second stage requires two simultaneous counts (four steps in the case of 2 + 4), which is cognitively demanding. Moreover, the load on working memory increases as the length (numbers of steps) of this double count increases. A somewhat more sophisticated strategy has occasionally been observed (cf. Fuson, 1982). Counting-on from the first addend (COF) involves starting with the cardinal value of the first addend and counting from there as the second addend is enumerated (e.g., 2 + 4: "2; 3 [1], 4[2], 5[3], 6[4]--6"). Note that, while the COF strategy reduces the total count, the number of steps in the cognitively demanding double count is the same as in the CAF procedure (again four steps in the case of 2 + 4).

More sophisticated informal addition strategies dispense with the larger addend first. In a recent case study, Baroody (in press) discovered that Felicia, a preschooler, would resort to a counting-all starting with the larger addend (CAL) strategy (e.g., 2 + 4: "1, 2, 3, 4; 5[1], 6[2]--6"). The most advanced informal strategy is counting-on starting with the larger addend (COL) (e.g., 2 + 4: "4; 5[1], 6[2]--6"). Note that both the CAL and the COL procedure reduce the double count to a minimum (two steps in the case of 2 + 4). Thus, in order to save mental labor, it behooves the child to disregard the order of the addends and always start with the larger addend. Does the invention of the cognitively more economical CAL or COL strategy also imply an appreciation of commutativity (cf. Groen & Resnick, 1977; Resnick & Ford, 1981)?

Three models have been proposed concerning the relationship between commutativity and the development of more economical addition strategies. According to the first (Resnick, 1983), the discovery of commutativity allows children to invent more economical addition strategies. For example, a child might compute various pairs of problems such as 5 + 1 and 1 + 5 by counting-on from the first addend (5 + 1: "5; 6[1]--6" and 1 + 5: "1; 2[1], 3[2], 4[3], 5[4], 6[5]--6") and notice that the sums are the same. Since one problem can be substituted for the other to get the sum (the commutativity principle) and since it is easier to start with the larger addend, the child adopts a more economical COL approach. That is, the child always begins with the cardinal value of the larger addend (e.g., 1 + 5: "5; 6[1]--6"). In the second view--a variation of the first--children just naturally assume that addition is commutative. The <u>assumption</u> of commutativity may also eventually result in children inventing counting-on from the larger addend (Resnick, 1983). There is a third and dramatically different possibility. According to this model, the development of more economical addition algorithms is <u>not necessarily related</u> to an appreciation of commutativity. In other words, the child simply searches for ways to save cognitive effort, and starting with the larger addend accomplishes this end (cf. Resnick & Neches, in press). This labor saving maneuver works because addition just happens to be commutative. For instance, a child might use a COL strategy to solve 5 + 1 and 1 + 5, but not realize that these problems will produce the same answer.

This study collected data on kindergarteners' understanding of commutativity and their addition strategies over several sessions in order to accomplish two objectives. The first aim of the study was to determine whether commutativity is assumed or discovered by young children. The second objective of the study was to determine the relationship between commutativity and the development of the more economical CAL and COL addition strategies--i.e., to test the three models described above.

Method

Subjects

A total of 36 children (15 boys and 21 girls) ranging in age from 5 years - 4 months to 6 - 9 (*M* = 5 - 11) participated in the study. An additional five children from the subject pool were not included in the data analyses because a response bias was evident, the child was inattentive, or sums were produced automatically and an informal strategy could not be determined. The participants were drawn from three kindergarten classes in two middle- to upper-class suburban schools. All children participating had the permission of a parent or guardian.

Design

During a familiarization session, both experimenters played math games with small groups of subjects. The addition task ("Car Race" game) was introduced at this time to ensure familiarity with the written addition format used in the study--including addition involving zero. If a child had no organized addition strategy, s/he was shown a counting-all with concrete supports procedure. Half the addition trials were administered in the first experimental session, several days later; the other half were administered in the second experimental session, one week after the first. During these experimental sessions, a counting-all procedure with blocks was retaught as necessary. An understanding of commutativity was gauged in two ways. Half the trials of commutativity task 1 were presented during experimental session 1; half during experimental session 2. Individual subjects received the addition task and commutativity task in the same order across both sessions. The order of the tasks as well as the experimenters were counterbalanced. There were no significant order or tester effects. Commutativity task 2 was administered during experimental session 3, which followed session 2 by several days. This task was administered last in order not to bias the results of commutativity task 1. The design is summarized in Table 1.

Insert Table 1 about here

Measures

Commutativity Task #1. One commutativity task took the form of a "Quick Look" game. Pairs of horizontal addition problems were typed in large print

(.4 cm high) on separate 3 x 5 cards. The second problem of a pair was typed 2.5 cm below the first. Four types of problem pairs were used: (1) commuted pairs (12 trials; correct answer: "same"), (2) identical problems (4 trials; correct answer: "same"), (3) problems with the same sum as N + 0 problems (2 trials; correct answer: "same"), and (4) problems with different sums (12 trials; correct answer: "different"). The noncommuted trials were included to prevent or detect a response bias or inattentive responding. The 30 trials and the session they were presented are delineated in Table 1. During each session, trials were presented in random order.

The child was given the following instructions. "We're going to play the 'Quick Look' game. I'll show you a card with two adding problems on them like this one [example trial: 2 + 1 & 2 + 8]. I'll only show you the card for a short time--so you will only have a quick look. Now you won't have enough time to figure out the answers--just try to see if the adding problems would give the same answer. How about 2 + 1 and 2 + 8? Do you think they would add up to the same or different answers?" Before beginning the task, children were shown a second practice trial [1 + 0 and 1 + 1], which the experimenter, if necessary, helped answer. The experimenter then showed the child a stimulus card, read the problems, and asked if the problems would add up to the same or different answers (the order of "same" and "different" was counterbalanced). The total exposure time of each stimulus card was about 4 seconds. Children were encouraged to respond quickly without computing the problems. If a child did insist on computing (counting) to determine the answer for a commuted pair, the trial was scored as incorrect. A child was scored as consistently correct on the commuted trials if s/he was correct on 10 to 12 of the trials ($p < .02$, Sign test) (if a response bias could be discounted), inconsistent if correct on 3 to 9 trials, and consistently incorrect if correct on 0 to 2 trials. A subject was scored as consistently correct on identical trials if s/he was correct on all 4 trials and inconsistent if correct on 3 trials (no subject got less than 3 identical trials correct). A child was scored as consistently correct on same sum as N + 0 trials if s/he got both trials correct, inconsistent if one trial was correct, and consistently incorrect if incorrect on both trials. (The results indicated a systematic--though usually incorrect basis--rather than a random approach for responding to these same sum as N + 0 trials.) Finally, a participant was scored as consistently correct on different sum trials if s/he got 11 or all 12 trials correct. Only one child was inconsistent on these different sum trials (7 correct).

Commutativity Task #2. An understanding of commutativity was also gauged in a brief, structured interview (task 2). A child was presented with a problem (usually 6 + 4), asked to figure it out, and then asked if a second problem would add up to, for example, ten--the same thing--or something different. A second commuted problem (e.g., 4 + 6) was then written down, and the child's reaction was recorded. As a check, some subjects were given a second commuted pair and/or a pair with different sums. A child was scored as correct on task 2 if s/he responded automatically with "the same" to the commuted trial(s); incorrect if s/he said "different" or resorted to computing; and other if s/he hesitated more than several seconds before responding or was inconsistent in responding to the commuted trial(s).

Overall Scoring for Commutativity. In addition to the scores for each task, performance across both commutativity task was determined by the following criteria.

(a) Overall, "success" was defined as being consistently correct on task 1 and being automatically correct on task 2.

(b) "Mixed success" was defined (1) as being consistently correct on task 1, but hesitant/inconsistent or incorrect on task 2 or (2) as being automatically correct on task 2 but inconsistent or consistently incorrect on task 1.

(c) An "unsuccessful" commutativity performance was defined as being inconsistent or consistently incorrect on task 1 and hesitant/inconsistent or incorrect on task 2.

Addition Task. The children's addition strategies were assessed in the context of the "Car Race" game. A subject was presented addition problems typed horizontally in large print on a 4 x 6 card. The sum indicated how many spaces the subject or experimenter could advance their race cars around a track. A total of 12 smaller addend first (SAF) problems and 12 larger addend first (LAF) problems were presented over two sessions in random order (see Table 1). Scoring focused on the SAF problems, since only this type of problem permits differentiation between strategies which deal with the first addend first or those which start with the larger addend. If a child had no organized strategy for adding, s/he was taught to count-all with concrete supports. Other strategies noted were spontaneous counting-all with concrete supports, counting-all mentally starting with the first addend (e.g., 2 + 3: "1, 2; 3[1], 4[2], 5[3],--5") (CAF); counting-all mentally starting with the larger addend (e.g., 2 + 3: "1, 2, 3; 4[1], 5[2]--5") (CAL); counting-on mentally from the first addend (e.g., 2 + 3: "2; 3[1], 4[2], 5[3]--5") (COF); and counting-on mentally from the larger addend (e.g., 2 + 3: "3; 4[1], 5[2]--5"). Each subject was scored on the predominate and most advanced strategy (when used more than once) for the SAF problems for each session (93% interrater agreement for 12 or 33% of the subjects) and across sessions.

Results and Discussion

The Development of Commutativity

The results indicated that the commutativity principle is not simply assumed by children, but abstracted from their addition experience. Only about half of the subjects were successful on each of the commutativity tasks. On task 1, specifically, 18 (50%) were consistently correct, 4 (11%) were inconsistent, and 14 (39%) were consistently incorrect (see Table 2). In contrast, nearly all (33 or 92%) of the children were consistently correct on the identity trials. Only one subject was not consistently correct on the different sum pairs. Because this subject was inconsistent across all types of trials, his data are excluded from further analyses. With the exception of this one subject, then, responses were discriminate and hence a response bias or carelessness could be discounted. Only five subjects (14%) got both same sum as N + 0 trials correct. However, this can be explained by the fact that the N + 0 rule ("adding zero leaves a number unchanged") and/or recall of the basic doubles involved (1 + 1 and 2 + 2) were not sufficiently automatic for most subjects to make a quick and accurate comparison of the sums in these trials. On task 2, 18 (51%) were automatically correct, 3 (9%) were hesitant/inconsistent, and 14 (40%) were incorrect or counted. Overall, only 14 (40%) of the kindergarteners were successful on both commutativity tasks, 8 (23%) had mixed success, and 13 (37%) were unsuccessful on both tasks (see Table 3).

Insert Tables 2 and 3 about here

Moreover, among the ten subjects that had to be retaught the counting-all procedure during the experimental session(s), none were successful on both commutativity tasks. Five had mixed success, and five were unsuccessful on both tasks. Thus it seems that, among children just learning an addition strategy, commutativity is either only tentatively understood or not appreciated at all.

Finally, of the eight children having mixed success, four were "successful" on commutativity task 1, but not on task 2. The behavior of two of these subjects on task 1 as well as on task 2 suggested that the commutativity principle was not yet firmly established (see Table 3, footnotes b & d). The other four having mixed success were unsuccessful on commutativity task 1, but successful on task 2. Behavioral evidence suggests that, for some of these children, computing the sums for a few inverted problems may have been sufficient experience to abstract the commutativity principle (see Table 3, footnotes a & e). Kate (S #25), for instance, clearly abstracted the principle during the course of the study. She was presented 2 + 4 & 4 + 2 as her first trial of commutativity task 1. After a pause, she indicated that they would give the same answers. On trial #2 (5 + 3 & 3 + 5), she responded to the experimenter's question with: "I can't tell." After a pause (in which she probably counted), she responded,"The same." On the third trial (4 + 6 & 6 + 4), she commented, "That's the one [type of problem] I got so much trouble over. The same? Later she also paused for 8 + 2 and 2 + 8. When task #1 was readministered a week later, Kate responded quickly to the commuted trials. In the follow-up interview, she explained that, yes, 4 + 6 would produce the same answer as 6 + 4 because, "I figured it out when I counted when we played the other game."

The Relationship Between the Development of Commutativity and CAL or COL

What role, if any, then does the abstracted principle of commutativity play in the invention of more economical count strategies for addition? If strategies which start with the larger addend (CAL and COL) are invented as a result of or imply an understanding of commutativity (Model 1) (Groen & Resnick, 1977; Resnick, 1983; Resnick & Ford, 1981), all subjects who use these strategies should appreciate commutativity. If the invention of CAL and COL strategies are simply labor saving maneuvers (Model 3), some subjects who used these more advanced strategies will not be successful on commutativity tasks. While most (6 or 55%) of 11 subjects whose predominant addition strategy (by session 2) was CAL or COL were successful on both commutativity tasks, three (27%) had mixed success, and two (18%) were unsuccessful on both (see Table 4). For example, Meg (S #04) relied primarily on CAF strategy during session 1, but by session 2 relied exclusively on a COL procedure. Yet, she did not appear to appreciate commutativity. In session 1, she was inconsistent on the commutativity task 1 (correct on 3 of 6 trials). For the 5 + 3 & 3 + 5 trial, specifically, she responded to the experimenter's question with: "I don't know?" Asked what she thought, she indicated that the answers would be different. During session 2, she was still inconsistent on commutativity task 1. She answered "the same" on only two of six trials--on one of which (3 + 2 & 2 + 3) she appeared to have computed the answers. In the follow-up interview, she mentally counted to produce the answer to 6 + 4. Asked if 4 + 6 would produce "10--the same or different answer"--she thought for about 60 seconds, appeared to compute the sum, and finally responded, "The same." In brief, Meg's invention of a COL strategy did not seem to require or imply an appreciation of commutativity. While the principle of commutativity might well play a role in the invention of more economical strategies for some children, such inventions can occur solely for the reason of cognitive economy.

Insert Table 4 about here

Clearly, though, children who invert addend order during addition but who do not appreciate commutativity do not have a well developed understanding of addition (Resnick, 1982). Mathematically, addition is the union of two interchangeable sets--i.e., a binary operation in which commutativity is assumed (Weaver, 1982) (see Figure 1). However, young children appear to treat addition as a unary operation (Weaver, 1982)--i.e., they define addition in terms of "action schemes" or changes of state (cf. Baroody & Ginsburg, in press). For instance, young children tend to interpret 3 + 2 as "three and <u>two more</u>" (a unary conception) rather than "combining the cardinality three and cardinality two" (a binary conception). More importantly, young children appear to (implicitly) hold <u>contradictory</u> notions about the effects of addend order. It appears that, initially, children assume that addition is <u>not</u> commutative. In other words, problems with the same, but inverted addends are <u>psychologically different</u> problems for the young child. For example, while 3 + 2 is interpreted as "three and <u>two more</u>," 2 + 3 as is viewed "two and <u>three more</u>." Since the child cannot foresee the outcomes of these "different" problems, the child naturally assumes that the sums will be different. A child is especially likely to invoke this "misconception" when inverted pairs are juxtaposed as in commutativity tasks 1 & 2. Thus, initially at least, children may assume that addend order makes a difference in the definition of the problem and thus in the sum. On the other hand, some children also appear to have a primitive notion of commutativity ("proto-commutativity") or what might be termed an "order indifferent adding scheme": the order in which addends are dealt with does not make a difference in terms of the correctness of the sum (Baroody, Ginsburg, & Waxman, in press). This is manifested in the variety of ways young children organize concrete objects for counting-all (e.g., not always representing the first addend first, or not always starting the answer count with the first set of objects, etc.) (cf. Carpenter & Moser, 1982). This protoconcept or order indifferent adding scheme may also permit the child to disregard addend order during mental addition (to recast problems so that the larger value is always supplemented by the smaller). In effect, it permits the child--for computational purposes--to treat 2 + 3 as though it were 3 and 2 more. Thus invention of CAL or COL to minimize mental computational effort implies or requires only protocommutativity (an order indifferent adding scheme).[1] These logically inconsistent views about the role of order can be resolved by positing "commutativity" and focusing on the outcome of addition. For instance, the child can adopt the view that it does not matter whether you start with two blocks and add three (more) or start with three blocks and add two (more) because the result is the same: five. In this way, the child's psychological meaning of addition better approaches the mathematician's definition of addition.[2]

Insert Figure 1 about here

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Footnotes

[1]This account of the invention of CAL or COL via "protocommutativity" is analogous to a computer program described by Resnick and Neches (in press). The computer program starts with a counting-all procedure and, via a set of self-modifying, economy-directed processes, converts to a COL procedure. This program does not assume commutativity but does assume an order indifferent counting scheme--that the order in which the elements of a set are enumerated does not matter (Gelman & Gallistel, 1978).

[2]According to Weaver (1982), the principle of commutativity describes a property of binary addition--(e.g., combining the cardinality 2 and the cardinality 3 to form a single set: the cardinality 5). The assertion that, for instance, the sum of two and three more equal that of three and two more still involves a unary interpretation of addition. Because it does not describe the property of an operation, Weaver calls such an assertion "pseudocommutativity." While such statements as 2 + 3 = 5 and 3 + 2 = 5 are mathematically equivalent, psychologically they imply different meanings even for adults (cf. Kaput, 1979). It is unclear when (or if) most children also acquire a binary conception of addition and hence an appreciation of commutativity in a mathematical sense.

Table 1: Summary of the Experimental Design and Tasks' Trials

Familiarization Session: Subjects shown a counting-all with blocks procedure for addition and instructed on interpreting written symbols such as "+" and "0."

Session 1 (several days after the familiarization session)[1]:

Commutativity Task 1: Trials 1-15[2]		Addition Task: Trials 1-12[2,3]
2 + 4	4 + 2	2 + 3
3 + 6	6 + 3	2 + 5
4 + 6	6 + 4	2 + 7
5 + 3	3 + 5	3 + 4
6 + 2	2 + 6	3 + 7
8 + 2	2 + 8	4 + 6
3 + 3	3 + 3	4 + 2
6 + 1	6 + 1	6 + 2
1 + 1	2 + 0	8 + 2
0 + 2	10 + 10	5 + 3
2 + 2	5 + 2	6 + 3
2 + 5	2 + 10	5 + 4
3 + 1	0 + 1	
3 + 5	3 + 0	
7 + 2	3 + 2	

Table 1 continued

Session 2 (one week after session 1)[1]:

Commutativity Task: Trials 16-30[2]

2 + 5	5 + 2
2 + 7	7 + 2
3 + 2	2 + 3
3 + 4	4 + 3
5 + 4	4 + 5
7 + 3	3 + 7
1 + 8	1 + 8
4 + 4	4 + 4
4 + 0	2 + 2
2 + 8	10 + 8
4 + 2	0 + 2
4 + 5	1 + 5
5 + 4	5 + 2
5 + 5	1 + 5
6 + 3	1 + 1

Addition Task: Trials 13-24[2,3]

2 + 4
2 + 6
2 + 8
3 + 5
3 + 6
4 + 5
3 + 2
5 + 2
7 + 2
4 + 3
7 + 3
6 + 4

Note 1: Commutativity task 1 and the addition task were presented to individual subjects in sessions 1 and 2 in the same order.

Note 2: Trials were presented in random order.

Note 3: A counting-all strategy with blocks was retaught as necessary.

Table 2: Summary of the Subjects' Performance on Commutativity Task I

Trial type	Consistently correct	Inconsistent	Consistently incorrect	Total
Commuted (12 trials)	18 (50%)	4 (11%)	14 (39%)	36 (100%)
Identity (4 trials)	33 (92%)	3 (8%)	0 (0%)	36 (100%)
Same as N + 0 (2 trials)	5 (14%)	4 (11%)	27 (75%)	36 (100%)
Different sum (12 trials)	35 (97%)	1 (3%)	0 (0%)	36 (100%)

Table 3: Level of Success on Commutativity Tasks 1 and 2

		Commutativity Task 2			
		Automatically indicated commuted trial(s) would produce the same answer as computed trial(s)	Hesitated or responded inconsistently to commuted trial(s)	Indicated that commuted trial(s) would produce a different sum or counted to determine response	Total
Commutativity Task 1	Consistently correct	$^{A}14^{a}$ (40%)	$^{B}2^{b,c}$ (6%)	$^{B}2^{d}$ (6%)	18
	Inconsistent	$^{B}1^{e}$ (3%)	$^{C}0$ (0%)	$^{C}2$ (6%)	3
	Consistently incorrect	$^{B}3^{f}$ (9%)	$^{C}1$ (3%)	$^{C}10$ (29%)	14
	Total	18	3	14	35 (100%)

Table 3 continued

A: Overall commutativity performance scored as "successful."

B: "Mixed success."

C: "Unsuccessful performance."

[a] S #12 counted once during session 1 to determine her response to a commuted trial of task 1. Thereafter, she responded automatically to all commuted trials.

[b] S #30 was correct on only 10 of 12 task 1 trials and appeared unsure about a third commuted pair. He then responded inconsistently on task 2, indicating that 4 + 6 would produce a different sum than 6 + 4 and that 4 + 3 would produce the same sum as 3 + 4.

[c] S #35 hesitated before responding to task 2. He may have computed the sum to the commuted problems, though counting behavior was not apparent.

[d] S #23 resorted to counting on commuted trials for both tasks 1 and 2.

[e] During session 1 of task 1, S #25 initially indicated that she could not tell if commuted pairs would produce the same or different answers. She was scored as incorrect on three trials because of the long pauses she took--presumably to compute the sums of the problems. She then responded automatically to the remaining commuted trials.

[f] During session 1, S #10 was incorrect on all commuted trials of task 1. During session 2, she again responded incorrectly to the commuted trials. However, she adopted a questioning tone ("Different answer?") with only the commuted trials. Later, during session 3, she appeared to appreciate commutativity.

Table 4: Overall Commutativity Performance of Those Who Used "Invented" (CAL or COL) Addition Strategies

		Performance on Commutativity Tasks 1 and 2			
		Successful	Mixed Success	Unsuccessful	Total
Predominant Addition Strategy by Session 2	CAL	4[a]	1[b]	1[c]	6
	COL	2[d]	2[e]	1[f]	5
	Total	6 (55%)	3 (27%)	2 (18%)	11

[a] Includes S #36 who swithced from using CAF in session 1 to CAL in session 2. For both sessions, S #36 was administered the addition and then the commutativity task (A/C).

[b] Includes S #23 (C/A) who counted on the first trial of commutativity task 1 and again on commutativity task 2 to determine his response.

[c] Includes S #18 (A/C) who switched from using CAF exclusively in session 1 to using CAF and CAL in session 2.

[d] Includes S #03 (C/A) who counted once during session of commutativity task 1 to determine her response.

[e] Includes S #10 (C/A) who was consistently incorrect on commutativity task 1 but adopted a questioning tone during the second session and S #35 who was hesitant on commutativity task 2.

[f] Includes S #04 (C/A) who switched from using CAF in session 1 to COL in session 2.

Figure Caption

Figure 1: The development of addition and commutativity concepts.

Early Concepts (concrete)

Early addition$_1$: supplementing the first set (unary conception).

Noncommutativity: different addend orders imply different problems and (because the outcomes cannot be forseen) different sums (e.g., 3 + 2 represents "three and two more" and 2 + 3 represents "two and three more").

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Early addition$_2$: supplementing the larger set with a smaller amount (unary conception).

Protocommutativity: the order in which addends are dealt with does not affect the correctness of the outcome (e.g., 2 + 3 may be recast as "three and two more" to facilitate computation--even though 2 + 3 and 3 + 2 may not be equivalent problems--i.e., equivalent in outcome).

↓

Functional concepts (semi-abstract)

"Addition": supplementing a set (unary conception).

"Commutativity": while different addend orders imply different processes, the order does not affect the value of the outcome (e.g., 2 + 3 and 3 + 2 are psychologically different problems, but have identical sums).

? ↓

Mathematical conception (abstract)

Addition is the union of (interchangeable) sets (binary conception).

Commutativity: the order of combining two cardinal values (e.g., the cardinality 3 and the cardinality 2) does not affect the totality (the cardinality 6).