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ABSTRACT
This document primarily consists of papers scheduled for presentation at the third annual meeting of the North American chapter of the International Group for Psychology in Mathematics Education (NA-PME), held in September 1981, at the University of Minnesota. A total of 27 papers are arranged alphabetically by author. An additional three late arrivals are included in the back. It is noted that the North American chapter was founded in 1979 at Northwestern University. The existence of this chapter is viewed as evidence of a need to communicate, collaborate, critique, and lend support to current and future research efforts. The materials presented here are seen as indicative of the degree of diversity and of the high quality of research currently underway. (MP)

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PSYCHOLOGY OF MATHEMATICS EDUCATION

Proceedings of the Third Annual Meeting
of the North American Chapter
of the International Group
for the Psychology of Mathematics Education

MINNEAPOLIS, MINNESOTA
SEPTEMBER 10-12, 1981

Edited by Thomas R. POST
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PREFACE

Recent progress in the field of research in mathematics education has been quite impressive. There is a clear trend toward cooperation, collaboration and the development of research support systems among researchers. The early successes of the Georgia Center, the ERIC Research monographs, and the recent efforts of NIE and NSF to fund collaborative efforts all attest to the emergence of a new perspective toward the field in the United States. As a result significant progress has been made in several important areas. The development of early number concepts, problem solving and in the area of rational number learning, to name but three.

Such collaboration has not been solely confined to the United States. The formation in 1976, of the International Group for the Psychology of Mathematics Education (PME) as an affiliate of the International Congress for Mathematics Education can be viewed as evidence that these trends are indeed international in scope. The accomplishments of PME during its brief life span have in our opinion, been no less than outstanding. In six years a core of researchers in mathematics education has evolved, representing diverse yet complementary fields of interest. The potential for future collaboration remains very promising. In 1979 at Northwestern University, the North American Chapter of the PME (NA PME) was founded. The existence of the North American Chapter is yet further evidence of a perceived need to communicate, collaborate, critique and lend support to present and future research effort in our field.

We are pleased to welcome you to the University of Minnesota for the third annual meeting of NA-PME and trust that your time will prove to be well spent. We are grateful to the authors of the papers contained in these Proceedings for submitting their manuscripts promptly and in the form requested. They [illegible] alphabetically by first author for easy reference. The conference program will consist mainly of presentations of coordinated groups of papers all having a common or closely related theme, along with ample time for discussion and the planning of future research. These papers are indicative of the degree of diversity and of the high quality of research currently underway.

The editors are indebted to the Department of Curriculum and Instruction and to the Dean's Office of the College of Education, for providing a portion of the financial, clerical and moral support so necessary to the planning of a conference of this type.

We are also indebted to the executive board/program committee of NA-PME for their wise counsel and encouragement during these last two years.

Thomas R. Post - Editor
Mary Patricia Roberts - Editor

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TABLE OF CONTENTS

A COMPUTATIONAL ERROR CLASSIFICATION SYSTEM AND ITS VALIDATION

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As far back as 1909 (Buswell and Judd, 1925), efforts have been made to identify and classify students' computational errors. These studies show that such errors are systematic (Brueckner, 1932; Buswell and John, 1926; Cox, 1975; Graeber and Wallace, 1977; Grossnickle, 1935; Meyers, 1924); can be reliably established (Brueckner and Elwell, 1932; Grossnickle and Snyder, 1939); can be used for diagnosis (Ashlock, 1976; Brown and Burton, 1978); and that without instructional intervention are evident a year later (Cox, 1975).

Many classification systems have been proposed (Buswell and John, 1926; Englehardt, 1977; Gist, 1917; Osburn, 1924; Roberts, 1968; Smith, 1968). Although these systems describe the computational errors made by students, they lack one desirable property of a classification system: that of being mutually exclusive. Thus an error could be classified in more than one category. Such classification systems, also, limit their usefulness in the teaching-learning process. Some of the error categories evoke similar prescriptions. Or, if the category system is too broad, no particular prescription is forthcoming.

One purpose of this paper is to introduce a classification system which has mutually exclusive categories based upon thinking strategies, rather than on error types or error clusters, directed toward the addition and subtraction algortihm. The categories of this classification system are:

I. NOT DISTINGUISHING PLACE-VALUE

The student does not distinguish place-value. For example,

$$\begin{array}{r} 23 \\ +\ 14 \\ \hline 10 \end{array} \quad \text{or} \quad \begin{array}{r} 76 \\ -\ 4 \\ \hline 32 \end{array} \quad \text{or} \quad 76 - 4 = 36$$

II. DISTINGUISHING PLACE-VALUE, BUT ISOLATING COLUMNS

The student consistently isolates the columns within each exercise which requires renaming and treats each column as if the other columns did not exist as part of the same exercise. For example,

$$\begin{array}{r} 402 \\ -\ 288 \\ \hline 286 \end{array} \quad \text{or} \quad \begin{array}{r} 4\cancel{0}2 \\ -\ 288 \\ \hline 224 \end{array}$$

III. INAPPROPRIATE INTERACTION

The student consistently has inappropriate interaction between (among) the columns for exercises which require renaming. For example,

$$\begin{array}{r} 402 \\ -\ 288 \\ \hline 24 \end{array} \quad \text{or} \quad \begin{array}{r} 402 \\ -\ 288 \\ \hline 124 \end{array}$$

IV. APPROPRIATE INTERACTION.

The student correctly performs at least three of the five exercises which require multiple renaming.

The validation of the classification system, as part of a larger research study, involved 164 third grade students from two Southwest Chicago Catholic Schools. These schools draw students from white, middle-class families. All third graders from these schools were tested on an author-constructed criterion referenced subtraction computation test (SCT) to select students for the research study. The SCT consisted of fifteen skill levels for subtraction, with at least five exercises for each skill level.

Student responses on the SCT were reviewed by the author. All incorrect responses were examined to determine what systematic error, if any, was made. (An error is defined as systematic for a specific algorithmic computation when a pattern can be established and occurs in at least three of the five exercises for a given skill level.)

Students were selected to be in the sample if they demonstrated a (symbolic) knowledge of the basic addition and subtraction facts and fell into one of the categories in this system.

The classification system was modified for this study for the following two reasons: 1. Not distinguishing place-value was an empty category for these third grade students, and 2. As a result of the study, category III (Inappropriate Interaction) was divided into two groups. Group IIIa consisted of errors involving the multiple renaming process. That is, students incorrectly performed subtraction exercises requiring multiple renaming. Group IIIb also consisted of errors involving the multiple renaming process but was only evident when there were zeroes in the minuend. Students who committed the second error did not also commit the first error. It was thought that the thinking processes for these errors might be different.

The validation, then, was with these four groups: II, IIIa, IIIb, IV. Within each group a random sample of twenty-six members, stratified by school, was chosen to be interviewed on an author-constructed, structured, but flexible, Place-Value Interview based upon a task analysis of the subtraction algorithm.

The interviews took place over a three-month period. During this time the teachers reported that a major portion of the time for mathematics was spent on multiplication. A small portion of the time was devoted to reviewing the addition and subtraction algorithm with some of the time spent on renaming with zeroes in the minuend. What was being taught during this period is pertinent in that some of the students' performances on the subtraction algorithm during the interviews were different than their performances on the SCT. Although these changes had not been anticipated, especially in light of Cox's study (1975) where she found that the systematic errors persisted a year later when there was no instructional intervention, it was necessary to recategorize the students based upon their responses during the interview as it is those responses which will help elucidate the thinking processes used in the performance of the subtraction algorithm. A statistically significant association exists between the original classification of the students and the reclassification ($\chi^2 = 46.89$, $p < .001$). The reclassification yielded thirteen students in group II, twenty-one students in group IIIa, twenty-six students in group IIIb, and forty-three students in group IV. For all but six students, the shift was into higher numbered categories.

Tables 1 and 2 display the results for each group and the entire sample for the concept percentile scores and the computation percentile scores of the Iowa Test of Basic Skills (ITBS). The ITBS was administered by the schools as part of the schools' evaluation program. The results from the ITBS for each of the students in the study were obtained from the schools. T-tests performed on the differences for the mean percentile scores for each of the groups indicate that for computation these groups are statistically different. For the concept percentile scores, only groups IIIa and IIIb are not significantly different. These results confirm the differences among the groups for computation and confirm that conceptually groups IIIa and IIIb are similar. A correlation between the groups and each of the percentile scores indicate a significant relation. The regression equations yield results close to the mean percentile scores for each group.

TABLE 1

ITBS CONCEPT PERCENTILE SCORES

GROUP	RANGE	N	MEAN	VARIANCE
II	11 - 67	12	31.75	254.02
IIIa	10 - 89	20	54.15	558.73
IIIb	12 - 89	25	57.20	527.12
IV	26 - 97	40	72.13	257.91
SAMPLE	10 - 97	97	59.89	549.21

$y = 11.59\ x + 25.6$

$r = 0.5213\ (p < .0005)$

TABLE 2

ITBS COMPUTATION PERCENTILE SCORES

GROUP	RANGE	N	MEAN	VARIANCE
II	10 - 79	12	29.25	366.69
IIIa	8 - 91	20	41.85	614.43
IIIb	5 - 92	25	52.00	596.88
IV	22 - 99	40	71.25	373.89
SAMPLE	5 - 99	97	55.03	708.96

$y = 14.85\ x + 11.25$

$r = 0.5851\ (p < .0005)$

One of the tasks presented during the interview tried to ascertain if the students saw a relation between a pencil and paper task of finding the difference between forty-two and twenty-eight and finding that same difference using popsicle sticks grouped by tens and ones. Group II students, in general, could not see the relation of the two tasks (only five of the thirteen students saw such a relation), whereas only eight of the ninety remaining students could not see the relation. Two of these eight students said it "shouldn't match." The other students replied "I don't know." The results from this task help point out the differences between group II students and the students in the other three groups.

A task to find the difference between three hundred and one, first "in their heads" and then with pencil and paper yielded interesting differences between group IV students and students in the other three groups. Table 3 displays the results for each of the groups.

TABLE 3

TASK: DIFFERENCE BETWEEN THREE HUNDRED AND ONE

GROUP	NUMBER CORRECT	NUMBER IN GROUP	PERCENTAGE CORRECT
II	1	13	7
IIIa	9	21	43
IIIb	6	26	23
IV	39	43	91

Ninety-one percent of group IV students could perform this task correctly, whereas only twenty-seven percent of the rest of the students could do so.

The interview tasks ranged widely from one-to-one correspondence between an object and a number name to finding the difference between two four-digit numbers with multiple renaming in a paper and pencil task. A grouping and a place-value manipulative was used to ascertain the student's ability to display numbers and subtract without and with regrouping (trading). Students were asked to create smallest (largest) numbers using digit cards, write and read numbers, and explain the processes they used in six subtraction exercises. Further results from the interview will be shared during the presentation.

The validation of the classification system was limited, because of the thrust of the study, to subtraction errors. However, the ITBS computation scores included results from exercises in addition, subtraction and multiplication. Classification of students with regard to addition errors would yield similar results.

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THE EFFECT OF VISUAL PERCEPTUAL DISTRACTORS ON CHILDREN'S LOGICAL-MATHEMATICAL THINKING IN RATIONAL-NUMBER SITUATIONS[1]

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During the past two years (1979-1981) the National Science Foundation has sponsored efforts at five University sites to develop, field test, and implement instructional and evaluation materials over a broad spectrum of rational number concepts. One question of primary concern to the Rational Number Project has been, "What is the nature of the impact of manipulative materials on the learning of rational number concepts?"

The paradigm used by the project's instructional component has been the teaching experiment. During the 1980-81 school year, 18-20 week teaching experiments were conducted with a) six Grade 4 children in DeKalb, Illinois, b) six Grade 4 children in St. Paul, Minnesota and c) five Grade 5 children in St. Paul, Minnesota. In addition, extensive evaluation materials were developed at Northwestern University under the direction of Richard Lesh and by Ed Silver and Diane Briars of San Diego State and Carnegie-Mellon Universities respectively. Both the instructional and evaluation materials were utilized at all project sites. As a result a fairly substantial body of data has been collected and is currently undergoing analysis.

[1]The research was supported in part by the National Science Foundation under grant number SED 79-20591. Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[2]The authors are indebted to the following people who assisted during the research: Nik Pa Nik Azis, Nadine Bezuk, Kathleen Cramer, Issa Feghali, Leigh McKinlay, Roberta Oblak, Mary Patricia Roberts, Robert Rycek, Constance Sherman and Juanita Squire; special thanks go to Nadine Bezuk, Robert Rycek, and Juanita Squire, who provided valuable contributions in the preparation of this paper. Constructive criticism from Professor Margariete Montague Wheeler, about an earlier draft was invaluable.

Six major data strands have emerged from the teaching experiments conducted in Illinois and Minnesota. They are:

1) The effect of visual/perceptual distractors on children logical-mathematical thinking.
2) Hierarchies in the learning of order and equivalence.
3) The emergence of proportional reasoning.
4) Difficulties involved in applying rational number concepts to problem situations. (This data strand is being pursued in conjunction with the Applied Problem Solving Project at Northwestern.)
5) Children's ability to perform translations within and between various modes of representation.
6) Children's ability to synthesize various rational number sub-constructs, i.e., part-whole, measure, quotient, operator, decimal and ratio.

It is the purpose of this paper to define perceptual distractor and begin to define its role in children's understanding of rational number concepts. It is of particular interest to show how perceptual distractors influence children's thinking. It is hypothesized that perceptual distractors overwhelm logical thought processes and cause children to interpret problems and tasks in extra-ordinary ways.

The particular emphasis in this report is to exhibit differences among children's dependence on visual-perceptual information, as compared to their ability to apply logical mathematical thinking. It will also address the transition from dependence on visual information to logical-mathematical thinking.

A series of tasks in which visual-perceptual distractors were deliberately introduced was developed. Emphasized in this report is information which indicates differences among children's ability to "put aside," "overcome," or "ignore" the distractors and deal with the tasks on a logical-mathematical level. The extent to which a child is able to do this -- resolve conflicts between visual information and their logical-mathematical thinking -- is viewed as one of several important indicators of how solid or tenuous is the child's understanding of the rational-number concept in question.

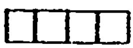
Overview of the Tasks

The term "visual-perceptual distractor" is used in this paper to refer to the introduction of information into a standard school-type rational-number task which is either consistent with the task, irrelevant to the task, or inconsistent with the task. A) Consistent cues are designed specifically to aid in the solution of a task or problem. B) Irrelevant cues contain extraneous but neutral information. Such cues require the solver to ignore certain information. C) Inconsistent cues are those which conflict with the conceptualization of the task or problem and therefore, must be reconciled prior to solution. This is normally accomplished by ignoral followed by reconstruction. This latter category has proven to be the most troublesome for students, perhaps because it involves a multi-faceted solution.

An example will illustrate these distinctions.

Task: To shade three-fourths of the rectangle:

Solution Strategy

A) Consistent Cue:  Subject shades 3 of 4 parts

B) Irrelevant Cue:  Subject ignores every other line; clumps 2-1/8's as 1/4 and shades 3 such clumps.

C) Inconsistent Cue:  Subject ignores all lines, reconstructs diagrams and proceeds as in A).

The rational number tasks, of three general types, were distinguished by the physical embodiment of the unit:

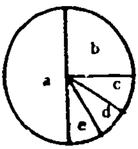
1) A continuous model such as a rectangle or circular region.
2) A set of discrete objects.
3) A line segment on a number line.

The normal order of task presentation involved first the task without the distractor, followed by the same problem with the distractor present. Sometimes the task was physically transformed from a consistent to an inconsistent situation while the subject observed. Such transformation often caused the child to provide not only a different response but also a different rationale when explaining her procedure, a phenomenon reminiscent of pre-operational children's responses to Piaget's conservation tasks.

The theory-based instructional materials developed for the teaching experiment provided a very rich instructional environment which relied heavily on the systematic use of manipulative aids. Manipulative aids used in the instructional program included continuous embodiments for rational number such as cut-out fractional parts, paper folding, and centimeter rods; discrete embodiments, such as chips, and various number lines. The instruction emphasized the part-whole and measure subconstructs of rational number. Concepts taught included the basic fraction concept, order and equivalence relations, addition and subtraction of like fractions and multiplication. Instruction dealt with fractions less than, equal to, and greater than one, as well as mixed numbers.

Continuous Embodiment Tasks

One perceptual distractor concerns children's ability to deal with a part of a whole as a ~~region~~ and as a partitioned region. This ability is an [illegible] ious precursor to dealing with notions of equivalent fractions. [illegible] s the observation that two equivalent parts of a whole can each be [illegible] the same fractions when one part is appropriately partitioned. In the [illegible] ing figure b and c/d/e, equivalent parts, can each be named as 1/4 and 3/12.



Of interest was whether the child could ignore the partition lines in cde in order to consider it one-fourth and imagine partition lines placed in b to consider it as three-twelfths. This was one of several contexts in which we found the existence of subpartitioning lines to be a distractor to children's logical-mathematical understanding of rational-number concepts.

Several of our interviews suggest that for some children a part (or group of parts) can only have one fractional name at a time. Part b is either 1/4 or 3/12 but cannot be both at the same time. The same is true for cde. While the part cannot have two names at the same time, the subject does exhibit flexibility in terms of the part being either 1/4 or 3/12 at any given time. This contrasts with a lower level response where a part has one and only one fractional name at all times.

For example, one child Mk, was not able to give two names for b: according to his thinking it could be 1/4 or 3/12 but not both. This same child was unable to see that another name for cde was 1/4: it was only 3/12.

Results suggest a linear trend in the development of this aspect of fraction identity. A first level of understanding consists of b and cde having each a single label (1/4 and 3/12, respectively). Level two consists of b having two labels (1/4 and 3/12), but not simultaneously, while cde still has only one label. Level three would indicate that both b and cde can have two labels (1/4 or 3/12) but not simultaneously. And level four consists of both b and cde, each having two labels (1/4 and 3/12) simultaneously.

Discrete Embodiment Tasks

To investigate the strength of children's logical-mathemtical thinking about rational number in the context of discrete embodiments, several tasks involving perceptual distractors were developed. The distractor was a transformation of the consistantly arranged set into one which was inconsistent with problem conditions.

Task 1 involved an initial presentation of six paper clips arranged as ||| ||| and transformed to || || || ; task 2 involved an initial presentation of ten paper clips arranged as ||||| ||||| and transformed to ||| |||| |||. For each part of tasks 1 and 2 the subject was asked to produce a set of paper clips equal in number to 3-halves the number of clips in the stimulus set. Task 3 involved a set of twelve paper clips; for the initial presentation they were arranged as |||| |||| |||| and transformed to |||||| ||||||. The problem for the subject in each case in task 3 was to present a set of clips equal in number to 5-thirds the number of clips in the stimulus set. As might be suspected, the second part of every task proved to be much more difficult for students, since the transformation diverted the attention of the solver from the basic concept intended by the problem presenter. Of special note is the fact that after providing an acceptable explanation to a correct solution to the first part of each task some students completely abandoned these "logical" structures and adopted other faulty ones which reflected the physical situation. For example in task #1 one student correctly suggested the 3/2 of ||| ||| was ||| ||| |||, while providing an appropriate explanation. She then concluded that 3/2 of || || || was the same set (i.e., || || ||) because "you already have 3 groups of 2." Another child took one set of 2 away from || || ||, reasoning that "we already have 3/2's." This child apparently reinterpreted the task to be one of reconstructing the unit. In this case the perceptual distractor not only altered the quality of the child's thought process but also caused him to alter the perceived task so as to more closely correspond with the physical setting.

Number Line Tasks

A series of tasks which involved two kinds of perceptual distractors on the number line was developed. One involved variations in the number of subdivisions of the unit, the other variations in the size of the unit. Space here does not permit discussion of these results, except that they were similar in nature to observations in both the continuous and discrete contexts. Children's logical thought processes were unduly interferred with, in the presence of visually distracting elements in the problem conditions.

Other types of distractors also seem to be emerging as we continue to examine our pool of data. These include: language, numerical distractors, and sequencing conditions resulting in an Einstellung or mental set.

These and related issues will be discussed more fully in "Rational Number Concepts," a chapter to appear in Acquisition of Mathematics Concepts and Processes, Lesh and Landau (Eds.), Academic Press, 1982.

Meaningful understanding of mathematical ideas and the mathematical symbolism for these ideas depends in part on an ability to demonstrate interactively the association between the symbolic and manipulative-aid modes of representation. Theoretically, as children deal with mathematical ideas, embodied by manipulative aids, the mathematical ideas are abstracted into logical-mathematical structures. As childrens' logical-mathematical structures expand, it is presumed that their dependence upon the concrete manipulative aids decreases; ultimately, logical-mathematical thought becomes sufficiently strong so that it dominates the visual-perceptual information. The extent to which children's thinking is dominated by visual-perceptual information therefore, seems to be an indication of the relative strength of their logical-mathematical thinking.

The extent to which children can resolve conflicts between visual information and logical-mathematical thought processes might at first be viewed as a simple indicator of how firmly a child has internalized a given concept. However, the issue is probably more complex than that. It is suspected that the ability to resolve such conflicts is differentially related to field-dependent and field-independent learners. By definition the field-dependent child is unable to (or has great difficulty) ignoring or overcoming irrelevant environmental stimuli accompanying problem conditions. Witkin (1977) states that:

> "The person who is relatively field-dependent is likely to have difficulty...with that class or problems where the solution depends on taking some critical element out of the context in which it is presented and restructuring the problem material so that the item is now used in a different context."

> "The relatively field-independent person is likely to overcome the organization of the field, or to restructure it, when presented with a field having a dominant organization, where as the relatively field-dependent person tends to adhere to the organization of the field as given."[1]

Similarly Goodenough (1976) suggests that:

> "Field independence is considered to be the analytical aspect of an articulated (as contrasted to a global—or field-dependent, insert ours) mode of field approach as expressed in perception."

> "If field-dependent subjects accept the organization of the field as given, then they should be dominated by the most salient ones in concept attainment problems. In contrast, the analytical ability of field-independent subjects should make it possible for them to sample more fully from the non-salient features of a stimulus complex in their attempt to learn which attributes are relevant to a concept definition."[2]

It is indeed tempting to discuss the issue of perceptual distractors within the framework of field-dependence theory. It seems clear that the abandonment of previously internalized cognitive structures in the presence of visual stimuli inconsistent with problem conditions and/or requirements is quite similar to the individual who is "...dominated by the most salient cues in the concept attainment problems." (ibid) It may be then that the effect of perceptual distractors on student learning is a function of where the individual appears on the field independence — field dependence continuum. The linkages suggested also imply that the issue of such distractors transcends the learning of rational number concepts per se and is relevant to a much broader spectrum of concepts.

Our data suggests just such a differential impact. Some students were obviously more "bothered" by the visual miscues presented in the problem tasks. It was nevertheless possible in all cases to teach children to overcome the impact of these distractors in specific situations. It should be noted however, that there was a strong tendency for the (some) children to again be influenced when the distractors were presented in a different context (e.g., continuous and then discrete).

Implications

Distractors represent one class of instructional conditions which make some types of problems more difficult for children to solve. Knowledge of their impact will be helpful in the design of more effective instructional sequences for children.

Although performance with rational numbers is affected by the presence of distractors, children can be taught to overcome their influence. It is expected that strategies generated by children to overcome these distractors will result in more stable rational-number concepts.

Our research has raised important questions about the role of such distractors in the learning process. Issues of sequencing, interactions with learning style and ability levels as well as questions related to appropriate procedures for overcoming their influence will need to be addressed.

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COUNTING AND NUMERATION CAPABILITIES OF PRIMARY SCHOOL CHILDREN: A PRELIMINARY REPORT*

Max S. Bell and Jean F. Burns

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ABSTRACT

Counting and other number ideas of preschool children have been extensively studied and so have computation abilities of children in grade 4 and beyond. Our work is directed at better understanding of the transitions from the one to the other. The research with primary school children (K-3) outlined here aims to construct a coherent picture of the development of children's capabilities in verbal counting (forwards and backwards by ones, tens, and other integers), in reading and writing numbers, and in certain other aspects of symbolic arithmetic. We also record the content of each child's arithmetic learning experience. We have developed efficient interview based methodologies for those purposes and results so far have suggested a number of interesting links among the things surveyed. The picture of children's capabilities and experience thus developed may suggest alternatives to what is now common in primary school instruction.

INTRODUCTION

Quite a lot is known now about the counting and numeration ideas of preschool children (e.g., Gelman and Gallistel, 1978; Fuson and Hall, in press). Quite a lot is also known from National Assessment of Education Progress (NAEP) about the symbolic arithmetic performance of children in fourth grade and beyond. Our work begins where the developmental work on early number ideas leaves off and stops just short of the achievement studies of performance of symbolic arithmetic. It seeks to illuminate the development of children's ideas and capabilities with respect to the counting and numeration systems and the links of these to arithmetic. For those purposes we have developed efficient brief interview methodologies that quickly assess the limits of verbal counting and written numeration knowledge of K-3 children and at the same time assess their capabilities in certain basic arithmetic procedures. We have also developed ways of getting a good record of the actual content of the arithmetic instruction of these same children.

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To this point (June 1981) our research program has included October interviews of 75 laboratory school children in second and third grade and October and May interviews of 120 children in grades K-3 in a middle class suburban school near Chicago. Both schools enroll at least 40% non-white students. We have also done limited pilot studies with other group of children over a wide range of socio economic and school backgrounds. Figure 1 indicates the sorts of data we have. It includes about half the tasks surveyed in the initial interview, for just second and third grade children of the surburban school.

Some of the most interesting data come from the forward and backward counting tasks, coded in the first few columns of Figure 1. We ask the children to count by ones. If a child gets past 30, we ask him to pretend he has counted to 68 and to go on from there; then similarly for 98-101, 197-203, and so on. We record the highest completed interval, ask the child to count back from that point, and probe for lower backwards counting capability if need be. . similar task checks counting by tens. The stop and skip feature of the counting tasks prevents mere alphabet-like recitation of a memorized word string and also permits finding efficiently the outer limits of counting understanding for each child. The interview is informal, brief (5-10 minutes), and unintimidating. Each child is pressed to his/her limits for each task, with graceful exits when those limits are apparent. Children respond willingly to the tasks and appear to enjoy the interview sessions.

SOME PRELIMINARY RESULTS

Figure 1 displays some typical results with second and third grade suburban school children. Children are listed there from high to low on their performance with the verbal counting tasks. Footnotes to the table explain the coding system used; for example, an asterisk (*) indicates a correct response and responses to the writing and reading of numerals are coded to reflect actual responses as nearly as possible.

Many things about the links of counting to other tasks are suggested by simple tallies and displays. For example Figure 1 makes it obvious that (for these children) being good at counting is linked to being good at many other things. "eyeball analysis" then confirmed by statistical analyses permits some

Figure 1: Counting and Numeration: Beginning Second and Third Grade Children

Grade	ID	Counting[2] Ones F	Ones B	Tens F	Tens B	Total	Writing Numbers[3] ("ten more")[4] 57	67	77	498	5004	1/2	Reading Numbers[5] 498	5004	1/3	Operations[6] +	-	x	÷
1	012	7	7	6	6	26	•	•I		•	•	105	•	•	DK	•	•	•	I
2	026	7	7	4	6	24	•	•I		•	50004	•	•	504	•	•	•	DK	I
2	027	7	7	6	5	25	•	•I	•I	•	•	•	•	•	•	•	•	•	I
3	022	7	7	4	5	25	•	•I		•	•	141/1	•	•	•	•	•	•	•
3	008	7	7	6	5	25	•	•I		•	•	•	•	•	C	•	•	•	•
2	031	7	7	6	6	26	•	•I	•I	•	504	50	•	•	12	•	•	•	DK
3	010	7	6	4	5	14	•	•I		•	•	•	•	•	•	•	•	•	I
3	062	7	7	5	5	24	•	•I		•	•	•	•	•	•	•	•	•	DK
2	053	7	7	5	5	25	•	•I		•	•	•	•	•	•	•	•	•	•
2	055	7	7	6	1	24	•	•I		•	•	•	•	•C	141/2	•	•	•	DK
3	020	6	5	6	6	21	•	•C	•I	•	•	2/1	•	•	•	•	•	•	I
2	054	6	5	5	5	21	•	•I		•	5,0000,4	DK	•	•	1o12	•	•	•	I
3	015	7	6	5	[illegible]	20	•	•I		•	DK	•	•	DK	•	•	•	•	I
3	011	7	2	5	[illegible]	20	•	•I		•	DK	•	498	•5Tzero4	1of1	•	•	•	I
3	001	7	6	5	2	10	•	•I		•	5,4	•	•	504	•	•	•	I	I
3	019	6	6	6	2	20	•	87C	97C	•	•	141/2	•	•	•	•	•	•	•
3	049	6	5	6	6	19	•	•C	•I	4101	DK	10	•	504	1,2	•	•	•	DK
3	002	6	6	5	8	27	•	•C	•C	•	50004	161/2	•	504	•	•	•	•	2
2	012	3	3	4	1	26	•	•C	•I	40098	50004	50	•	5004	•	•	•	DK	I
3	025	3	3	6	1	16	•	•I	•I	•	5,4	•	•	•	•	•	•	•	I
2	044	3	3	1	1	16	•	•C	•I	•	514	1/	894	485	3	•	•	DK	DK
3	003	7	6	1	1	15	•	•C	•I	•	•	DK	•	•	2-2	•	•	I	DK
3	037	6	2	5	2	15	•	•I	•I	•	•	DK	•	•	DK	•	•	•	DK
2	030	5	5	4	1	15	•	63C	72C	•	50004	DK	•	•		•	•	DK	DK
1	063	5	5	4	1	15	•	60C		•	50004	DK	4-9-8	504	1,1	•	•	•	I
3	006	4	1	4	1	14	•	•I		•	5000,4	•	•	504	•	•	•	•	DK
2	038	6	2	5	1	16	•	•C	•I	•	50004	DK	•	•		•	•	•	•
2	028	4	3	5	2	14	•	•I	•I	410098	5^04	DK	4-9-8	5-0-0-4	1,2	•	•	•	DK
3	004	5	5	1	1	14	•	•C		•	5,4	1/8	•	•	241/2	•	•	•	I
2	045	5	1	4	1	11	•	•C	•I	•	•	•	•	•	3	•	•	I	DK
2	040	6	2	6	1	13	•	•C	•I	•	•	1/,2		•		•	4	I	DK
3	018	4	2	2	1	12	•	•I		•	•	DK	•	•	141/2of2	•	•	•	DK
3	064	5	6	1	1	12	•	•C	•I	•	5,4600	1/4	4-9-8	•	2HALF	•	•	•	•
3	041	4	4	2	1	11	•	•I		•	•	DK	•	•	12	•	•	•	•
2	026	4	4	2	1	12	•	•I	•I	•	•	DK	•	•	1,2	•	•	DK	DK
2	034	4	2	1	1	11	•	•C	•C	•	504	05	•	•	16DK	•	I	•	DK
2	029	5	1	1	1	11	•	•I	•I	40098	50004	DK	•	504	DK	•	•	DK	DK
1	017	4	4	2	1	11	•	•I	•I	•	5400	DK	•	•	1/2of1	•	•	•	DK
3	014	4	4	1	1	11	•	•C	•C	400098	50004	DK	49-8	504	141/2	•	•	DK	DK
2	040	6	1	2	1	10	•	•C	•I	•	•	I	•	•	141/2	•	•	•	•
2	049	4	1	1	1	10	•	•C	•I	4098	•	DK	•	•	142	•	•	I	DK
3	056	3	1	1	1	10	837	DK		•	504	DK	•	504	3	•	•	DK	DK
2	039	3	1	3	1	10	•	•I		4098	DK	•	40898	•	141/1	•	•	I	DK
2	061	3	3	1	1	10	•	•C	•I	•	504	DK	•	DK	DK	•	•	DK	DK
1	068	4	4	1	1	10	•	DK		400098	850004	05	4898	504	3	I	I	DK	DK
2	060	4	4	1	1	10	•	•C	•C	149800	1504	DK	•	•	DK	•	•	I	DK
1	005	3	3	1	0	9	•	69C		40098	50,04	•	•	504	•	•	I	I	DK
2	047	4	1	1	1	9	•	•I	•I	40098	50004	•	•	504	2•1	•	•	I	DK
3	009	4	1	1	1	9	•	•I		4,98	5,4	1 1/1	•	•	1of2	•	•	•	DK
2	059	4	6	1	0	9	•	DK		4098	50004	DK	•	504	DK	•	•	DK	DK
1	031	4	4	1	0	9	•	•C	•C	40098	460004	DK	49-8	504	3	•	•	I	DK
3	016	4	4	1	0	9	•	107C		418	514	DK	•	504	141/2-1	•	•	I	DK
2	042	4	1	2	1	8	•	68C		40098	5004	817	198	1854	1TA2	•	•	I	DK
2	041	1	3	1	1	8	•	66C		40098	•	DK	49-8	504	1=2	•	•	DK	DK
2	052	4	11	1	1	7	837	869C		84100098	5100004	DK	DK	504	1T41	•	•	•	I
3	007	4	1	1	1	7	•	•I	•	•	540	DK	•	504	1o2=)	•	•	I	•
2	057	4	1	1	0	7	•	70C		40098	50004	10/20	4-9-8	504	1,1	•	I	DK	DK
2	015	4	1	1	0	6	•	DK		EX	DK	•	DK	DK	142	•	•	I	DK
3	021	4	1	1	0	6	•	•C	•C	4098	DK	DK	•	516	1,2	•	•	DK	DK
2	021	4	2	0	0	6	•	DK		40098	50004	10	4948	546	DK	•	•	DK	DK
2	058	3	1	1	0	5	•	58C		•	DK	10	4-98	5854	142	•	•	I	DK
3	011	3	1	1	0	5	•	DK		4100900	DK	DK	40-9-8	5074	1,2	I	•	DK	DK
2	033	1	2	1	0	5	•	DK		DK	DK	DK	4-9-8	5-0-0-4	142	•	•	DK	DK
2	050	2	0	1	0	1	87	DK		1810	DK	DK	400	500	1,2	DK	DK	DK	DK
2	054	2	1	0	0	1	DK			DK	51004	DK	4-9-8	5-0-0-4	1,2	DK	DK	DK	DK

Notes:

1. An notation "•" means a correct response; DK means "don't know"; blank means question not asked.

2. The number recorded indicates best correct sequence in the count-skip-count task coded as at right. Backward probing began at end of best forward sequence.

Code	0	1	2	3	4	5	6	7
Ones forward	DK	<30	to 30	68-71	98-101	197-103	997-1003	3148-3151
Ones backward	DK	10-1	30-20	77-61	101-91	203-191	1003-991	3151-3141
Tens forward	DK	<150	to 150	180-110	480-520	980-1020	2970-3010	
Tens backward	DK	50-10	150-90	210-150	520-450	1010-950	3010 2950	

3. Actual responses are recorded. "B" indicates a backward digit, e.g., Ƨ for 5. All but two wrote "100" correctly.

4. Task. "Write fifty-seven for me. Now write the number that is ten more than fifty-seven. ...ten more ..." "C" indicates a counted response and "I" an instant response.

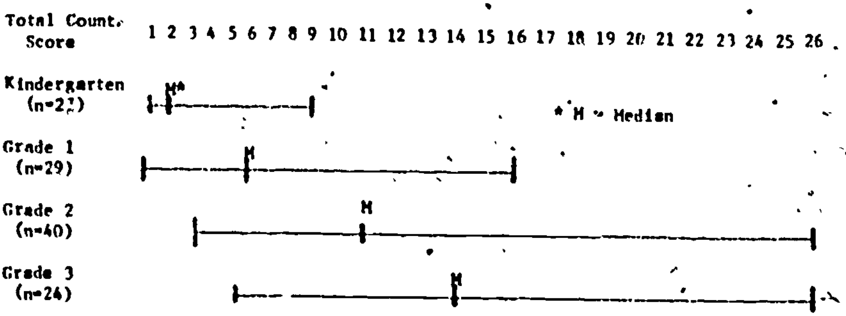
5. Coded versions of verbatim responses to numerals written on cards, e.g., "5H4" means "five hundred (and) four" and "49-8" means "forty-nine, eight." Nearly all responded correctly to "57" and "100."

6. "I" indicates the child gave the correct operation name but not the correct answer.

tentative conclusions from our Autumn 1980 interview. Detailed analyses of the rich data available from this group of children is continuing. Here are some of the preliminary conclusions:

1. Among children at each grade level, there is a wide range of counting skills. The range and medians of the total count score in October 1980 of K-3 children at the suburban school is indicated in Figure 2.

Figure 2: Range and Medians of Total Counting Scores of K-3 Suburban School Children (Max scores = 26)



Note that maximum scores move by larger year-to-year jumps than do minimums or medians. This result is also common in our pilot longitudinal studies, and reflects the existence of a group of children who remain poor counters year after year--a potentially serious handicap.

2. Verbal counting skills are associated with a variety of symbolic arithmetic skills; for example, with reading and writing of large numbers; skipping by ten; notation for operations, fractions, and money; and solving $\square - 4 = 16$.
3. For grades 1-3, backward counting is as easy as forward counting for the best forward counters; but beyond about the middle of the distribution of total counting scores, backward counting becomes a substantially more difficult task. The latter difficulties confirm many results with younger children working with small numbers but we and others find surprising the ease of backward counting for many of the more capable children.
4. Only 5 of the 27 beginning kindergarten children had any difficulty pointing to and counting fourteen objects; one of those children could nevertheless count on past thirty. Of the 22 children who could easily count fourteen

objects, 9 could continue without objects but not past thirty; 7 could count beyond thirty but not skip to 68-72; one could skip to 68-72; and 6 could skip to 98 and count past 100. Those seem to us rather remarkable performances for five-year-olds; certainly they are well beyond what most kindergarten or 1st grade school books exploit.

5. At all grade levels, reading numbers is a lot easier than writing numbers. For example, essentially all beginning first graders could read 100 from a card, but only seventy percent could write 100 from dictation. In beginning third grade only one of twenty-four children could write $1.47 properly from dictation but twenty-two of the twenty-four could read it.

6. There was frequent confirmation for the clinical findings (e.g., Ginsburg 1977) that children pushed beyond their familiar range of numerals offer such "logical" responses as writing "40098" or "410098" (but almost never "400908") for dictated "four hundred ninety eight," or read a written "5004" as "five hundred (and) four." The frequency with which beginning second or third graders (and even some first graders) respond by writing "50004" for dictated "five thousand four" seems remarkable to us considering that "5000" is beyond numbers with which such children are likely to have had direct experiences and far beyond what most school programs for K-2 admit as possible to learn.

7. Recognition of the arithmetic operation symbols +, −, x, ÷ seems to be school related, except possibly for children ranked at the top on verbal counting ability. (Those children may have learned most of what they know about the arithmetic symbol system independent of their school experience.)

Results of tasks not shown in Figure 1:

8. Our results with this rather large group confirm surprising findings of clinical studies concerning meanings children attach to the "=" (equals) symbol. (Behr, et. al., 1980) Nearly all the children in our studies see this not as an "equals" or "same as" relation but as requiring an operation on the left linked to an answer on the right. Hence, they typically reject 4 = 4 because there is "no problem", 2 + 2 = 3 + 1 because there is "no answer," and 4 = 2 + 2 because it is "backwards."

9. Most second and third grade children and many first graders respond correctly to a task that asks them to make maximum and minimum numbers from three

individual digit cards (4, 3, 7). But well over half of the first grade children who did this correctly were unable to read the (correct) number they had constructed.

10. Only four of the twenty-four beginning third grade children could calculate 65 - 37 correctly with pencil and paper although such problems are solidly in the 2nd grade curriculum.
11. "Mental arithmetic" skills are poor in our 3rd graders; only nine of twenty-four of them could handle 27 - 21 without paper and pencil and only two were able to answer 26 + 19 correctly. We often found the children trying to visualize the problems as written down, then doing some algorithm on their mental blackboard.
12. There are typical errors in backward counting at the decades and hundreds; for example the decade is left out (72, 71, 69 . . .) or the next lower decade is inserted (72, 71, 60, 69, 68 . . .). Some children do better on this task if the responses are written instead of verbal, so they can see the patterns. Since our classroom observations indicate that backward counting tasks are not a part of K-3 work (except possibly "10, 9, 8, . . . 1), the errors are not surprising.

THE CONTENT OF PRIMARY SCHOOL MATHEMATICS INSTRUCTION

We have not completed our analyses of that content covered by the children we have interviewed but certain tentative conclusions are possible. Over 80 percent of the time children work by themselves with pencil and textbooks, workbooks, or teacher worksheets, going to the teacher for individual instruction when needed. Virtually all the materials are heavily oriented to paper and pencil work; few alternative ways of proceeding are offered. There are wide child to child variations in the actual amount of material covered even within the same classroom. The base ten place value counting and numeration systems (beyond small numbers) are not themselves explicitly linked to operations and other arithmetic procedures, even though in theory the links are fairly transparent. "Borrowing" and "carrying" uses of base ten notation are exceptions but even those are taught more as memorized procedures than as aspects of the counting and numeration systems. It is rare for school work to indicate any uses of numbers or of arithmetic procedures. Calculators and other computation aids in universal use in the world outside of school are essentially never found in the K-3 classrooms we have seen.

SUMMARY

Our results indicate that many children have counting and numeration capabilities that do not come from primary school work and are not exploited in primary school work. Those findings suggest to us that some rather simple things could be done that might markedly improve the results of early school arithmetic instruction.

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PROBLEMS RELATED TO THE APPLICATION OF A MODEL OF UNDERSTANDING TO ELEMENTARY SCHOOL MATHEMATICS

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Several models describing various modes of understanding mathematics have been suggested (Bruner,1960; Skemp,1976; Byers & Herscovics,1977; Skemp,1979). A version integrating the first three models was used in an experiment whose main objective was to determine if such a model could be assimilated by elementary school teachers (Bergeron et al.,1981). This research, involving a group of 28 teachers, has shown that they could apply the model, that is, they could identify various modes of understanding associated with notions such as number, the four operations of arithmetic, place-value notation, as well as the addition and subtraction algorithms.

These conceptual analyses have had some important psycho-pedagogical effects on teachers (Herscovics et al.,1981). They seem to have changed their perception of mathematics and also that of their own mathematical competence. Equally important in terms of didactics, the teachers have developed a constructivist approach towards learning: they now de-emphasize the importance of the written answer and focus on the thinking processes. They have become aware that only through an appropriate form of questioning can the child's reasoning be uncovered. This evolution in the teachers' attitude was achieved despite the weaknesses of the model used.

These results have been so encouraging that they warrant the construction of a new model better suited to the analysis of mathematical concepts and which is discussed in a companion paper (Herscovics,Bergeron,1981). However, it is essential first to study the problems encountered by teachers in our last experiment and to identify the internal contradictions of the model used in order to prevent their recurrence.

MODEL USED IN THE EXPERIMENT

As mentioned earlier, the model used in our experiment was a synthesis of previous models developed by Bruner, Skemp, Byers and Herscovics. A brief summary of these models is necessary in order to understand how they were adapted.

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Bruner (1960) described two complementary modes of thinking, namely intuitive thinking (global and implicit perception of a problem, unawareness of the processes used in getting the answer) and analytic thinking (steps are explicit, full awareness of the relevant information and operations). Skemp (1976) distinguished between instrumental understanding ("rules without reason") and relational understanding ("knowing what to do and why"). Basing themselves on the Bruner and Skemp classifications, Byers and Herscovics (1977) combined them in a model and, moreover, discriminated between content (mathematical ideas) and mathematical form (their representations). They described the following four modes of understanding:

instrumental understanding is evidenced by the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works

relational understanding is evidenced by the ability to deduce specific rules from more general mathematical relationships

intuitive understanding is evidenced by the ability to solve a problem without prior analysis of the problem

formal understanding is evidenced by the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning

As can be seen from these descriptions, the three models dealt primarily with rules and problem solving. In order to use them for the analysis of concepts, they had to be somewhat modified. Thus, in order to characterize intuitive understanding, we have added to Bruner's "global perception" criteria, such as visual perception and estimation (cf. comparison of quantities) and primitive unquantified action (eg, to add to, to bring together).

Similarly, in the case of the instrumental and relational modes, we had to adapt Skemp's definitions to concept formation. In this context, rules and procedures are not an end in themselves but rather become the means used in the construction of new mathematical notions. Thus, the instrumental mode was assigned the double meaning of "memorization" and that of "initial construction" (cf, in the case of addition: bringing together and counting from one). The relational mode also was assigned a double meaning, that of "justification" given to it by Skemp, and that of "links and relations leading to notions of invariance and reversibility" (for addition: bringing together and counting on from the first term may reflect some invariance of number · to perceive subtraction as the inverse operation of addition is an example of reversibility).

Finally, insofar as formal understanding is concerned, the first part of the Byers and Herscovics description was used without however specifying what was meant by "relevant mathematical ideas" since these seemed quite obvious. The second part was interpreted more in the sense of "logical justification" than in the sense of "formal proof" which lies beyond the elementary school curriculum.

DIFFICULTIES ENCOUNTERED BY TEACHERS

As can be seen, the model we used was rather complex. Not only was it a synthesis of the three preceeding models, but moreover, the description of a given mode could now involve several new criteria. Thus it is not too surprising that certain teachers experienced some difficulties in applying it to specific concepts.

For example, in a test verifying the transfer from addition to subtraction, teachers were asked to identify four modes of understanding for each operation. Out of 28 teachers, 13 were mistaken in their identifications, the greatest confusion occurring between the intuitive and instrumental modes on one hand, and between the relational and formal modes on the other (Herscovics et al., 1981). In the first instance they called "intuitive" the initial construction of addition (cf. bringing together and counting from one) and in the second case they interpreted "relational" to mean the relation between the symbolic expression and its enactive and iconic representations. Nonetheless, in spite of these difficulties, a pedagogically important result remains: teachers were able to perceive several ways of understanding a given concept.

While agreeing that there were many ways of understanding the same concept, they repeatedly raised the question "How can one be sure that the understanding is not instrumental?" For instance, how can one decide if the procedure indicating a relational understanding of addition (bringing together and counting on) was not the product of rote learning? As a matter of fact, nothing can guarantee that the processes used are not resulting from pure memorization. This problem brings out one of the difficulties experienced by teachers in their transition from the evaluation of skills to the evaluation of understanding. As far as skills are concerned, the student's success can be easily verified by the right or wrong answer and his results are proof of his mastery. In contrast, there are no "proofs" in the evaluation of understanding. The processes used as criteria can at best be construed as indications and these may lead to some inferences regarding the nature of the pupil's understanding. It took our teachers five weeks to achieve this transition and to realize that only through a questioning of the child could they verify the validity of their inferences about his thinking processes and reasoning.

INTERNAL CONTRADICTIONS OF THE MODEL

The teachers' equivocal reaction about instrumental understanding raises questions about the internal consistency of the model we used. Its analysis reveals several contradictions.

The first one of these relates to the instrumental mode. As mentioned earlier, we used this terminology to describe a specific process (the initial construction of a concept) and also to qualify anything learned by rote. Of course, by rote learning we are not referring to any "automatisms" one develops following a process of assimilation (as in the memorization of number facts following the construction of the concepts of addition and multiplication). We mean specifically those processes which are memorized without the intervention of any reasoning. Now, since understanding necessarily involves thinking processes, any memorization devoid of reasoning can hardly be qualified as comprehension. Thus, it is not surprising that even at the level of definition, the instrumental mode as defined by Skemp should have conflicted with the usual meaning of the word "understanding" (Collis, 1981).

The second contradiction involves both the instrumental and relational modes. Indeed, by accepting Skemp's definitions and by adding other criteria based on procedures, different but contradicting evaluations of understanding become possible. For instance, the understanding of addition in a child counting all (counting from one) can be called instrumental since it corresponds to the "initial construction" criterion. On the other hand, it could also be qualified as relational if the pupil can justify it. How then can one characterize the student who can justify the more evolved procedure of "counting on"? As shown by this example, a model of understanding cannot use simultaneously a criterion based on a procedure and also a criterion based on its justification. In fact, just as procedures can be learned by rote, they can also be justified. Since justification bears witness to thinking processes and reasoning, it is bound up with the global notion of understanding and thus cannot be considered the attribute of any particular mode.

A third contradiction is linked to the relational and formal modes. In the model we have used, justification served as a criterion for relational understanding. However, as mentioned above, any justification summons up the child's logical thinking and could even reveal "the combination of mathematical ideas in a chain of logical reasoning". It thus follows that a given evidence of understanding could be viewed both as relational and formal.

CONCLUSIONS

The contradictions we have just described arose from our attempt to graft onto the three preceding models additional criteria needed to describe concept formation. Thus we ended up with a "hybrid" model which, as Skemp (1981) has pointed out, was trying to describe both various states of understanding as well as the construction of understanding. In summary, we have tried to include "initial construction" in the sense of "operationalization" as a criterion of instrumental understanding; we have associated with the relational mode criteria involving notions of invariance and reversibility born out of "reflective abstraction"; and we have interpreted the formal mode as a "formalization" of relational understanding. In fact, these new criteria which we added to the older models provide us with the means to describe the various stages in the construction of a concept, a construction which must be based on the child's intuitive knowledge. Consequently, it will be necessary to construct a new model which will provide a proper framework for these new criteria while avoiding the past contradictions.

Of course, the new model will have to meet other requirements. First, it will have to answer the psycho-pedagogical needs of the teachers and attune them to the children's thinking. This is essential in a constructivist approach to learning wherein the teacher's role is to actively guide the child in the construction of his knowledge. Moreover, the new model will have to be applicable to the analysis of concepts, of arithmetic operations, their properties and their algorithms, while distinguishing between content and form. Some psychological questions regarding the new model are discussed in a companion [illegible] Herscovics and Bergeron, Minnesota, 1981).

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Young Children's Best Efforts in Solving Word Problems: Strategies and Performance of Individuals[1]

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What native understanding do children bring as a basis for study of word problems in elementary school? We address this question through detailed observation of individual children of kindergarten and pre-school age (4 and 5 years), with the aim of assessing their understanding of situations involving addition and subtraction and their ability to use this understanding to solve such problems.

Our theory of performance for problem solving in elementary mathematics and science (Briars and Larkin, 1981) is that problem solutions involve either a *naive* problem representation that corresponds very directly to the familiar real-world situation described by the problem statement, or a *mathematical* representation that reflects the abstract mathematical structure of the problem. For the simple addition and subtraction problems considered here, naive representations involve either combining two groups of objects, separating one group into two, or matching one group against another followed by an appropriate combination or separation (see Table 1). The mathematical structure of each problem consists of a part-whole structure with appropriate subsets and supersets mapped onto its components. Then any missing part is found by appropriate combination or separation.

Our initial expectation was that young children would most readily solve those problems for which the answer simply appears when the problem situation is acted out, problems readily solvable with a naive representation (c.f. problems 1 and 2 in Table 1). Conversely, they should find most difficult problems like 3 through 6 in Table 1 for which acting out the situation yields an answer only through further manipulation. In these more difficult problems, the action (combination, seperation, or matching of sets) does not necessarily correspond to the mathematical operation (addition, subtraction) required to find the answer directly.

Previous research does support this expectation (Carpenter & Moser, 1979, Nesher, 1979, Riley, 1979). However, past work has not addressed two important aspects of children's problem-solving: performance following some minimal explanation and consistent ability to solve a number of problems of a particular type. These factors are important in examining children's skill because word problems are often unfamiliar to children. Consequently failure to solve any given problem could result from simply not understanding what the problem was asking. Correspondingly, a single correct solution could be a lucky guess.

In the study reported here the goal was to assess the abilities of young children to solve simple addition and subtraction problems. More specifically we assessed their use of naive strategies (acting out the problem) and mathematical strategies (transforming the problem into a standard form and flexibly applying the appropriate combination or separation operator).

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Table 1: Example of problems used in the study.
The descriptions in parens indicate the naive representation suggested by the verb (join, separate, match), the mathematical operation (+, -) and a description of the unknown (e.g., result, change, large set, small set).

Part-Whole, Action Verb Same as Mathematical Operation

1. Wally has 5 pennies. His father gives him 4 more pennies. How many pennies does Wally have now? (Join/Addition/Result Unknown)

2. There are 7 cars in a parking lot. 3 of the cars drive out. How many cars are in the parking lot now? (Separate/Subtraction/Result Unknown)

Part-Whole, Action Verb Different From Mathematical Operation

3. Tim has 6 marbles in his pouch. Then he finds some more marbles. Now Tim has 10 marbles. How many marbles did he find? (Join/Subtraction/Change Unknown)

4. 10 people are riding on a bus. Then some of the people get off the bus. Now there are 7 people riding on the bus. How many people got off of the bus? (Separate/Subtraction/Change Unknown)

5. Terry has some sea shells. Then her friend gives her 5 more sea shells. Now she has 11 sea shells. How many sea shells did she have to start out with? (Join/Subtraction/Start Unknown)

6. There are some birds sitting in a tree. 3 of the birds fly away. Now there are 6 birds sitting in the tree. How many birds were sitting in the tree in the beginning? (Separate/Addition/Start Unknown)

Comparison Matching Language

7. 5 children are wearing hats. There are 4 extra hats. How many hats are there? (Match/Addition/Large-Set Unknown)

8. There are 7 cups. There are 10 saucers. How many saucers won't have a cup? (Match/Subtraction/Difference Unknown)

9. There are 9 children and some swings. 3 children don't get a swing. How many swings are there? (Match/Subtraction/Small Set Unknown)

METHODOLOGY

Procedure

We used the following procedure in an effort to maximize the children's opportunity to show us the extent (and limitations) of this understanding:

1. We worked with the children individually, constructing detailed records of their behavior.

2. Counters (poker chips in two colors) were made available. These materials were intended to aid children in showing us their models of the situations described, and could also act as computational devices for children largely without knowledge of number facts.

3. When a child miscounted a set of poker chips, we asked for a recount, and assessed the solution as correct if ultimately the correct answer was given. Our experience is that children often give

answers incorrect by 1 or 2, due either to miscounting or to losing poker chips.

4. Most importantly, when a child worked a problem incorrectly, or showed no knowledge of how to proceed, we showed the child briefly, "Here is how I would do it." Our final assessment of the child's ability was then based on three or more independent correct solutions for subsequently presented problems of the same type.

Problems

The problems used, of which a sample are shown in Table 1, are similar to those used in classrooms and studied by other investigators (c.f., Carpenter & Moser, 1979, Nesher, 1979, Riley, 1979, Hudson, in press). The first problems involve either two parts combined to make a whole or a whole separated into two parts. All involve action verbs (e.g., fly away, give) that directly specify combination or separation. For these problems the appropriate mathematical operation (combination or separation) can be either the *same* as the action described by the verb in the problem or *different* from that action (see Table 1)[2].

The remaining problems involved comparison between two sets. As shown in Table 1, there are basically three configurations for such problems, two that require the mathematical operation of subtraction (separation) to find the smaller or the difference set and one that requires addition (combination) to find the larger set. Each problem was presented in a language that made explicit the naive matching between sets (through the use of naturally paired objects, cups and saucers, hens and nests), and through phrases like "How many won't get".

Thus, all the problems have an apparent naive representation suggested by the action verb or by the naturally paired objects. The action in the problem can be directly represented by combining, separating, or matching poker chips. The mathematical operation needed to solve the problem may be the same or different from this naive strategy. We consider separately (and do not report here) problems with alternative language less directly connected to a naive representation; although these problems were generally more difficult, the difficulty patterns were similar to those discussed below.

Subjects

The subjects were three kindergarteners (mean age 5 y. 5 mo.) and six pre-kindergarteners (mean age 4 y. 10 mo.) from a university laboratory school, and three pre-kindergarteners (mean age 5 y. 3 mo.) from a local day care center. This sample reflects a range of educational experience and age. The kindergarteners had received some initial formal instruction in adding and subtracting, while the pre-schoolers from the day care center had received little. Only children who could correctly count and make a set of 11 objects were included in this study.

Coding of Problem Solutions

The children's solutions were categorized by the following actions used to relate the sets in the problem representation.

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[2] In problem 4 it initially appears that the mathematical operation matches the verb, however, the decrement cue suggests an action different from that of the mathematical operation of subtraction required to find the answer. The cue corresponds to 10 − a = 7, while the subtraction corresponds to 10 − 7 = a.

- Combine and Combine-To. Combine two sets, either by making both and then combining (C), or by making one and combining to it one-by-one enough counters to make a given larger set (CT).

- Separate and Separate-To. Make a set representing the larger number and remove a set representing the smaller number (S); or remove a set so that the smaller number is left (ST).

Both of the preceding actions can be used with either a naive representation (if the action corresponds to the action described in the problem) or with a mathematical representation (if the action is the same as the mathematical operation required to find the answer). Consequently, whether these actions reflect naive or mathematical representations depends on the problem solved.

- Estimate. Make an arbitrary set to represent "some". Combine with it or separate from it a set representing a known change. Check that the final result corresponds to the known result set. If not, adjust the arbitrary set and repeat the cycle. This action always corresponds to a naive problem representation, and is always used with a combine or separate action.

- Dual. Use a single set of counters in a dual role representing either of two sets in one-to-one correspondence. This action always reflects the mathematical representation of two matched sets as equivalent and is used with a combine or separate action to solve comparison problems.

- Match. Represent corresponding sets by two sets of counters placed in explicit one-to-one correspondence with each other. This action always reflects a naive representation of correspondence; it is used with a combine or separate action to solve comparison problems.

For each problem type, we determined the action(s) based on the mathematical representation of that problem and based on the naive representation. For the mathematical representation of part-whole problems, the associated action is simply combine or separate, depending on whether the operation required to find the answer is addition or subtraction. For comparison problems, a mathematical representation involves using a single set of counters in a dual role (D). This action is executed with an appropriate combine or separate action (DC for problem 7; DS for problems 8 and 9).

For part-whole problems the actions based on the naive representation are basically combine or separate depending on whether the verb in the problem describes combination or separation. These actions are all that are needed for problem 1 (C) and 2 (S). If the change is unknown, then a naive-based action must be used in a different way: (CT) combining elements up to a set of given size (problem type 3); or (ST) separating elements off until a set of given size is left (problem type 4). If the starting set is unknown (as in problems 5 and 6), this set must be estimated before any combination or separation can be done, and the naive-based actions are EC and ES. For comparison problems, naive-based actions always include explicit matching (M), together with the necessary combine or separate (MC for problem 7, MS for problem 8, SM for problem 9).

RESULTS

Table 2 summarizes performance on seven of the nine problem types described in Table 1[3]. These problems

[3]Performance on the two part-whole result unknown problems (1 and 2) are not included in Table 2; all children solved all of these problems at criterion with help given to only one child. In addition, these problems provide no data about the underlying representations because the same action is associated with both naive and mathematical representations of these problems.

are labeled by their number from Table 1, by the nature of the unknown (e.g., result, change), and by the nature of the cues to a naive representation (join or separate verbs, matching phrases). Letters indicated the main actions to relating sets as described earlier.

Table 2: Children's performance on problems 3-9 from Table 1.
Letters refer to the main action used to relate sets in the problem: Combine, Combine-To, Separate, Separate To, Estimate, Dual, Match.

	Part-Whole Problems				Comparison Problems		
Unknown Set:	Change		Start		Large	Difference	Small
Cue:	Join	Separate	Join	Separate	Match	Match	Match
Problem #:	3	4	5	6	7	8	9
Performance							
At Criterion:	11	12	8	10	11	11	11
Given Help:	7	4	7	9	2	5	5
One Error:	2	2	5	4	2	1	4
Actions							
Naive:	CT	ST	EC	ES	MC	MS	SM
Mathematical:	S	S	S	C	DC	DS	SD
Dominant:	CT	-	CT	C*	-	MS	SM
Action Use							
Naive Only:	11	6	0	0	4	8	5
Math. Only:	0	0	0	7	4	0	4
Both:	0	6	3	3	3	3	2

Two aspects of children's performance are of interest: (1) children's problem-solving abilities under conditions designed to maximize performance; (2) the problem representations (naive or mathematical) used in these successful solutions.

Problem Solving Ability

As the top of Table 2 indicates, children have a good ability to solve these problems. All of the kindergarteners and five of the pre kindergarteners ultimately produced consistent correct solutions for all of the problems. Consistent with previous research, the most difficult problems involved an unknown starting set. Other problems were solved consistently by 75% to 100% of the children.

Other measures of problem difficulty, the number of children requiring a demonstration and the number incorrectly solving one problem of a particular type, indicate the following pattern of problem difficulty which is similar to those found by other researchers (see Table 2) (Carpenter & Moser, 1979, Nesher, 1979, Riley, 1979, Hudson, in press). Of the part-whole problems, result unknown problems were the easiest and those with unknown starting quantities the most difficult. Thus under less than optimal conditions (no demonstration, presenting only one problem of a type), our study probably would have replicated previous results. Previously observed difficulties with some of these problems may thus have been due to a lack of familiarity with the problems, rather than a lack of understanding of the problem situations.

Problem Representation

Given that children are capable of solving these word problems, what kind of problem representations are used in their solutions? The actions implemented with the poker-chip counters reflect these representations. The bottom of Table 2 summarizes the actions described earlier as associated with the naive and mathematical representations for each problem, and the actions of the subjects.

Children were remarkably consistent in their approaches to problems. With two exceptions (problems 4 and 7) discussed below, each problem was characterized by a single dominant action used in the solutions of over 75% of the children. In some cases, this consistence was maintained even when problems were presented after an interval of several days.

Part-Whole Problems

For three of the four problems no child used only actions reflecting a mathematical representation. For the change-unknown problems (3 and 4) all and half the children respectively used actions associated with naive representations. Of the six children using both types of actions for problem 4, only two used the mathematical S action more often that the naive ST.

For the start-unknown problems (5 and 6) the predicted naive-based actions were not used exclusively by any child, although in both cases three children used these strategies along with others. In problem 5, five of the children used exclusively the CT action, reversing the roles of the start and increment sets to make this problem type identical to the increment-unknown problem 3. While this action does not reflect the mathematical representation allowing direct use of separating to undo the combination described in the problem, it is clearly a more sophisticated pattern of reasoning than simple acting out of the problem situation with estimation.

In the separation start-unknown problems (like problem 6), the dominant action was combination, superficially corresponding to a mathematical representation. However, most children employed this action in the context of acting out the problem. For example, a typical solution was:

> Here are the 3 birds that flew away (push set away). Here are the 6 that stayed home. Now bring the 3 birds back. At the beginning there were 9.

Thus the C* in Table 2 reflects our belief that for some children the correspondence to the mathematically based combine action is an artifact. In fact these children had an unanticipated way of acting out their naive representation that happened to result in a combine action. Other children, however, used a combining action that did seem to reflect a mathematical representation.

Comparison Problems

In addition to consistency of action used for each problem type, most children used a consistent approach for all of the comparison problems. Four children used actions based on a naive representation on all three problem types, three used consistently actions based on a mathematical representation for two of the three problems.

In two of the three comparison problem types (7 and 9), nearly half the children are indicated as using the mathematically based "dual" action in which one set of counters stands for two matched sets. However, as was the case in problem type 6, we suspect this result is in part an artifact in which an unanticipated form of acting out the problem resulted in an action corresponding to the mathematical representation. Specifically, in solving problems of type 7, children tended to speak of matched items (e.g., hats on children) and unmatched items (extra hats). The desired total number of hats is then found by combining. But the use of a single set of counters to represent "children with hats" may less reflect understanding of equivalent [illegible] tched sets, but rather the simple single set of hats worn by children.

DISCUSSION

If such young children can demonstrate such superb performance on a comprehensive set of word problems, why are word problems so universally acknowledged a difficulty for older children? To answer this question, let us look at how the young children solved a problem like problem 2 in Table 1, and ask whether a similar procedure might be used by an older child on a comparable problem.

The young child hears that Steve has some marbles and then gives away 3. Mimicking the action of the problem, he puts out an estimated group of marbles, takes away 3, and adjusts the previously held counters in order to obtain the specified number 5. At no time need he explicitly recognize that sets of 5 and 3 poker chips must be combined into a set including 8.

Now let us imagine an older child confronted with the problem, Steve has some marbles, he gives away 34 more and now had 96. How many did he have in the beginning? Unless the child has the patience, accuracy, and time to act out the situation (e.g., with pencil marks), she can not use a strategy analogous to that used by the younger child. She must, in order to solve the problem, have the insight that this story involving addition of marbles must in fact be solved by subtraction.

Thus, future research on word problems must focus on how children make or fail to make the transition between acting out problems (what we have called using a naive representation) and re-representing these real-world situations in terms of their mathematical structure so as to determine the appropriate mathematical procedure.

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MICROCOMPUTERS AND MATHEMATICS EDUCATION: IMPLICATIONS FROM AND FOR RESEARCH

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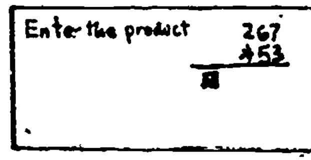
While computer-assisted instruction has been a part of mathematics education for many years, the availability of the inexpensive stand-alone microcomputer, coupled with other developments in educational technology, heighten the importance of computers in mathematics education today. In this paper it will be argued that rese..h in mathematics education, and especially that research which has focused on psychological variables, can and should have a considerable impact on the directions for development of educational software.

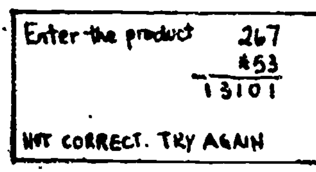
Recent advances in computer-related technologies have the potential for providing educators in general, and mathematics educators in particular, with a dazzling array of new instructional tools. Moreover, industry projections for the development of new hardware, the expansion of multi-purpose computer networks, and the decrease in cost of each of these over the next few years indicate that the immediate future will put an even greater array of attractive tools at our disposal. We find ourselves faced with an abundance of new gadgets which can release us from a world of textbooks and dittos to a land of graphic displays, individualized musical feedback, instantaneous hints for reluctant learners, and voice synthesis for non-readers. It is tempting to reach into this array of hard- and software, grab some bit of microcomputer capability, and adapt it without thinking (though possibly with a great deal of programming skill) to a teaching objective that predates the turn of the century.

Unfortunately, this temptation seems to have guided the developers of much of the software which is commercially advertized as enhancing mathematics education. An all too frequent outcome of this indulgence is an inappropriate application of the technological tool to instruction. A single example can illustrate the nature, though not the scope, of the potential mismatches* between technology and curriculum. More than one educational software package includes programs which combine the following elements.

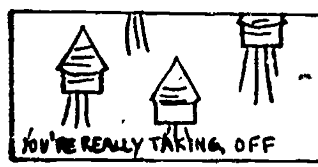
A problem is displayed. The cursor flashes in the appropriate position for the correct number of decimal places.



Should an erroneous answer be entered, "non-threatening" negative feedback is given. (The wrong answer then disappears and the cursor reappears).



Should the correct answer be given, the problem and solution are immediately removed from display and replaced by a clever but irrelevant graphic and musical "reward."



One can imagine the student weak in computation, and therefore "needing" this program, trying to juggle pencil and paper, copying the problem, multiplying, adding; all the while the cursor is flashing "hurry up." A wrong answer and there is no escape from the "non-threatening" sentence to trying again. Insofar as this example is typical of computer use in mathematics instruction, and it is closer to the norm of available software than we would like to believe, the microcomputer will not enhance students' learning of mathematics, but will shortly find a home in classroom closets around the country.

If the microcomputer is to become a useful teaching tool we must first remember that, unlike the textbook and cuisennaire rods, it was not designed as a teaching tool, but rather for other purposes. It will become a good tool for teaching mathematics only if, on the one hand, we recognize these

* It is interesting that one of the most common error messages a novice BASIC programmer receives is "Type mismatch". The computer protects itself from internal mismatches of data types but does nothing to prevent us from mismatching input demands and the nature of the task display.

purposes as they effect society's needs for mathematics instruction and, on the other hand, we seek to apply the computer to instructional tasks and problems for which it is uniquely well suited. Our primary guides in this latter task must be the nature of mathematics and the research literature in mathematics education and cognitive psychology. The rema .der of this paper will sample the potential for development of computer-based curricula; and will raise related research questions.

INDIVIDUALIZING INSTRUCTION

Over the years computer-assisted-instruction has had a major impact on the individualization of instruction. Not only has the computer been used to monitor and administer individualized treatments as in IPI but also the availability of computer models for instruction has motivated the development of countless other individualization models which rely on the actual availability of a computer terminal in degrees ranging from total dependence to complete independence. However, the individualization of instruction has, in general, beer effected primarily by the pacing and branching of students through rather traditional linearly ordered print materials (Mitzel, 1981). Little attention, has been paid in these programs to individual differences variables such as field-dependence/independence, spatial visualization, learning style, symbol comprehension, or related variables , nor to factors related to the learning environment. In Kilpatrick's (1975] terms, the individualization is based upon task variables, but not subject variables or situation variables. There is a clear need to individualize instruction with regard to the latter two types of variables as well. The microcomputer affords us the opportunity to individualize with respect to each type of variable independently or jointly. The potential directions for such individualization are almost infinite and suggest numerous areas for both development and research.

Aptitude-treatment-interactions have been studied extensively (Cronback and Snow 1977, Snow 1978). Although Begle (1979) has argued that ATI has not yielded promising results, McLeod and others (e.g. 1980 a,b, 1978) have found evidence for ATIs between field independence and performance on a variety of mathematical tasks; several writers (e.g. Snow) have argued that failure to find ATIs results from failure to contrast treatments that are essentially different, rather than from non-existence of interactions. The computer can provide the opportunity to teach students with instructional treatments that are radically different from the "regular classroom." Combinations of graphics and text, - real-time simulation vs. sequential description of processes, availability of hints and second chances, use of sound and visual cues to aid in internalizing concepts, number and diversity of examples and non-examples available, amount of redundancy, and numerous other variables can be manipulated for both instructional and research purposes. Moreover, computer storage and analysis of selected data on students' performance on various sequences and types of tasks, could be used in individualized construction of tasks and task sequences for later instructional components.

While the foregoing discussion relates to use of variables such as field independence and spatial visualization to assign students to instructional treatments, it is interesting to speculate whether these aptitudes might, in fact, be teachable using interactive computer graphics. While there is no evidence for this hypothesis, it does make some a priori sense; the computer can be programmed to allow students to remove parts of figures (and replace them), rotate, translate, and reflect figures, color parts of figures, and so on. These are actions which (presumably) must be performed mentally in tests of spatial visualization or field independence, but can be performed "actively" on the computer.

MATHEMATICS LABORATORIES

Manipulation of figures, whether for this purpose, or for other instructional purposes, is but one of many ways in which children can "take charge" and experiment with mathematical concepts using microcomputers. Much of the research support for use of mathematics laboratories (Lesh 1974, Fitzgerald and Higgins 1975) supports, either by direct use in treatments (e.g. Davis 1981) or by implication, the value of computer-based manipulation of objects and symbols. One view of the computer in the mathematics laboratory is as a generator of objects lying between concrete materials and pictorial representation on the concrete to abstract continuum. In this view, computer generated objects can serve as a bridge from three-dimensional objects to the two dimensional representations on the printed page. Research is needed to determine the extent to which this view is useful.

LEARNING STYLES

There are several models of the brain and brain functioning which provide direction for exploration of mathematics learning (Languis et al, 1980); the information processing and bicameral models are both foci of current research on mathematics learning. Greeno and others working with the information processing model are discovering how children develop, construct, or learn numerical, geometric, and measurement concepts. This line of research can have tremendous implications for development of computer-based-education; the view of the "student with computer" as an information processing dyad with a single problem solving structure should be a powerful research tool.

The bi-cameral or hemispheric model has led to a good deal of research on "learning styles;" and the identification of two distinct styles, one characterized by a holistic approach to learning tasks and the other a more analytic approach. Of particular interest in this ragard is the clinical research of Pat Davidson (1979) on mathematics learning styles through which she has identified approaches to mathematical tasks and concepts associated with each of these global learning styles. The implications of this line of research for computer use and research in mathematics education have not begun to be explored. There are potential implications for both the individualization of initial approaches to concepts and the sequencing of instruction to facilitate assimilation of new knowledge to established schemas.

LEARNING ENVIRONMENTS

Many writers have addressed the role of the computer in creating new learning environments. Papert (1980) has argued that the computer can afford children the opportunity to mathematize and to construct mathematical structure rather than study about mathematics. Goodman (1981) addresses a related effect of the computer on the child's learning environment in his discussion of the opportunity the computer can provide children to send powerful messages, that is, to send messages that cause things to happen.

At the moment there are many debates concerning the global effects of computers on school curricula, the nature of computer literacy, the "best" computer language to teach children, and related issues. While these debates continue into the future reflecting new developments in computer technology, mathematics educators can make many advances by exploring the interface between our considerable existing research base and the capabilities of modern technology.

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COGNITIVE SCIENCE AND MATHEMATICS EDUCATION

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For the past seven years a program has been in progress at the Curriculum Laboratory of the University of Illinois to develop a conceptualization for discussing human thinking about mathematical tasks, and to obtain observational data to enable this conceptualization to be extended and refined. Observational data comes both from published reports of work done elsewhere, and from our own Long Term Study, which follows individual students for five years, or even longer. Students included in the study range from third graders to adults in community college courses, with the greatest emphasis on grades seven through twelve. (Davis, 1976; Hannibal, 1976; Davis and Douglas, 1976; Davis, Jockusch, and McKnight, 1978; Davis (in preparation)).

Basic concepts are adapted from recent research in cognitive science (or "artificial intelligence"), especially from Minsky (1975; 1980), Papert (1980), Minsky and Papert (1972), and Schank and Kolodner (1979), and deal with structures capable of representing knowledge in such forms as to make possible various kinds of information processing that are known to occur in human thought. Two fundamental concepts reported previously are *procedures* and *frames*, the latter introduced by Minsky (Minsky, 1975). (See also Davis, 1980.)

Recent work has dealt with the following: (1) identifying certain frames that are commonly shared by most students at certain approximate ages (Davis, 1980); (2) control, and the transfer of control from one structure to another (Davis, in preparation, Matz, 1980; Kissland, Note 1); (3) methods for storing information in memory so as to make possible certain forms of retrieval that are known to occur.

By way of illustration, we present one episode, and a storage-retrieval mechanism that would make it possible: B., a mathematics teacher, was talking with a mathematics educator who lived in Oklahoma, and told him of some relevant work being done in Texas. B. thought the Texas and Oklahoma researchers might get together. Then, recalling that Texas is rather large, B. wondered if he had just said something silly. He suddenly recalled hearing that there are places in Texas that are nearer to Chicago than they are to certain other places in Texas. That instantly reminded him that he had also heard that there are places in Africa that are closer to Alaska than they are to certain other places in Africa. How might these associations have occurred?

'Sending.' One could imagine that when the Texas story was heard, a pointer was inserted, saying: 'Refer also to Africa-Alaska-Africa comparison.' There are at least two things wrong with this assumption. First, in order to realize that such a pointer ought to be inserted, some mechanism has to look at this story, compare it with the Africa-Alaska story, and recognize that both stories have an identical abstract skeletal structure. In short, a powerful 'pattern-matcher' would be required, which goes beyond any mechanisms commonly postulated at present.

The second difficulty is that B. heard the Texas-Chicago story years before he heard the Africa-Alaska story. When the Texas-Chicago story was coded and stored in memory, there was no Africa-Alaska story to list as a reference! Are we to postulate that whenever new data is acquired, every existing memory entry must be up-dated with a reference to the new data whenever they share a pattern, character, scene, or what-have-you in common?

This second objection is a fundamental one. The twenty-third time that a pattern appears, one may conceivably be prepared to recognize it. But how do you ever get started? The *first* time that a pattern appears it is NOT a pattern, since 'pattern' is defined in terms of aspects that are common to several instances. (On its first appearance, what was the 'pattern' of the Texas-Chicago story? That it dealt with a state whose name began with 'T'? That it dealt with the largest state in the United States (which, at the time, it did)? or what?

Mechanisms to deal with this question have been proposed in Minsky (1980), Lawler (1981), and Schank and Kolodner (1979). The key idea is the postulation of a main collection of open frames. When the 'Texas-Chicago' story first appears, it involves *inequalities in distances*. It calls for the retrieval of the basic *distances* frame, which includes something like distance measurement, the triangle inequality, the definition of the 'distance from a point to a set', and so on. (Perhaps this is really a kind of mega-frame, and we are dealing with only a small sub-frame contained within it.) In the course of the Texas-Chicago story, a copy of the basic distance frame has its slots filled with relevant information, and becomes an *instantiated* frame.

Now--what is stored in memory? *After* hearing the Texas-Chicago story, B. still has in mind the original basic distance frame. (We never destroy our collection of basic frames.) But, as a result of hearing the story, he now also has in memory:

(i) the instantiated frame, its variables filled with 'T ', 'Chicago', etc.

(ii) a record of *uses* of the basic frame, that refers us to all those places in memory where various instantiations of this frame have been stored.

Notice that this solves several of the basic problems:

(i) How is 'pattern' defined the *first* time it appears?
Answer: By the portion of the basic frame that was used. (Since the *distance* frame was used to compare the distance from some point A to some set B, as against the distance from point A to some point A', *this* is the pattern. The fact that 'T' is the initial letter of 'Texas' is NOT part of the pattern.)

(ii) How can the Texas-Chicago story send us to the Africa-Alaska story, which did not exist when the Texas-Chicago story was entered into memory?
Answer: It doesn't--at least not directly. What happens is this: when *either* of these stories arises--or even when B. wonders how close a city in Oklahoma is to a city in Texas--what is retrieved is the *basic distance frame*. This is an 'open frame'--its slots are not filled with specific data from any single event. It is there to help us 'make sense' out of this new input data.

But--whenever the basic 'open' distance frame is retrieved, it shows us a list of previous uses, and where the results are stored in memory. If any of these sound helpful or interesting, we know where to go to find them in the form of instantiated frames. It does not matter which use comes first on the list of previous users.

NOTES

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SELECTED PIAGETIAN TASKS AND THE ACQUISITION OF THE FRACTION CONCEPT IN REMEDIAL STUDENTS

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Students who enter secondary school have all received some instruction in basic mathematical concepts and skills. Competency testing in many states has made educators aware of the large number of students who have not mastered minimum skills in spite of having received instruction. In particular, many students have not mastered the concept of fraction or operations with fractions. Even when correct answers are produced, a lack of correspondence between the understanding of basic mathematical concepts and the possession of symbolic algorithms has been observed, both in elementary school children (James, 1980) and in upper elementary and secondary school students (Noelting and Gagné, 1980).

Those studying the structure of a mathematics concept often propose a learning hierarchy on the basis of logical relationships among subconcepts. Various aspects of the fraction concept and operations with fractions have been considered by Greeno (1976), Kieren (1975, 1981), Novillis (1976), and Uprichard and Phillips (1976, 1977). Some mathematics educators have suggested examination of the underlying cognitive structures of mathematics concepts in general (Lovell, 1975; Carpenter, 1979) and of fractions in particular (Hiebert and Tonnessen, 1978; Kieren, 1975; Lesh, 1975; Steffe and Parr, 1968). A current project reported by Lesh, Landau and Hamilton (1980) seeks to describe for children in grades 2 through 8 the acquisition of rational number concepts and the role played by various modes of representation. According to a companion paper (Behr, Post, Silver, and Mierkiewicz, 1980), the study makes use of the following theoretical models of learning: the perceptual model of Dienes as modified by Post; an extension of Bruner's instructional model, given by

Lesh, and an information processing model. This paper reports an attempt to determine whether the cognitive development theory of Piaget could add to our understanding of the student's concept of fraction.

Piaget has studied important pre-fraction concepts in small children (Piaget, Inhelder and Szeminska, 1960), and other structures he described seem to be logically related to various models of the concept of fraction. The ability to conserve area, for example, would seem to be necessary for the understanding of a fraction as presented in the area model commonly used in textbooks. A clinical study was done to investigate the thinking of secondary school students who were having difficulty with fractions. This paper is a discussion of work on the question: Is there a relationship between a student's understanding of the concept of fraction and the student's level of cognitive development on certain Piaget-type tasks, thought to be logically related to the concept of fraction?

PROCEDURE

Certain Piagetian concepts and certain fraction subconcepts were selected for study. Specifically, this study was designed to investigate possible relationships between:

1) Conservation of number and the discrete model of fraction,

2) Conservation of distance and the number line model of fraction;

3) Conservation of area and the area model of fraction;

4) Class inclusion and the three concepts of fraction; and

5) Conservation of number, seriation, classification, and class inclusion and overall success in the fractions tests.

Three instruments were developed. The first was a set of tasks, similar to those used by Piaget to test for conservation of number, seriation, classification, class inclusion, conservation of distance, and conservation of area. Tasks were prepare in both concrete, or manipulable, and pictorial forms.

The other two instruments were fractions tests, one concrete or manipulable, and one written, containing parallel sections on the concept of fraction (in discrete, number line, and area models) and equivalence and comparison of fractions. The written test also included addition and subtraction of fractions with like denominators.

A pilot study was conducted with four students. Subjects for the main study were 10 girls and 15 boys in the tenth, eleventh, and twelfth grades (median age 16 years) who were enrolled in compensatory classes in a Gainesville, Florida, high school, and whose teachers identified them as "having trouble with fractions." All tests were administered individually; interviews were recorded.

FINDINGS

In general, students scored very low. For example, no students were successful on conservation of area tasks; 8% were successful on classification tasks. The percentage of students successful on each task is given in Table 1.

TABLE 1
Percentage of Students
Successful on Tasks

Task	Percentage of Students
Conservation of numbe	36%
Seriation	44
Classification	8
Class inclusion	8
Conservation of Distance	56
Conservation of area	0

No student passed all sections of either fractions test. Some students could use algorithms to add and subtract fractions with like denominators, but could not answer items related to the concept of fraction. Table 2 lists the percentage of students passing each section of the two fractions tests.

TABLE 2

Percentage of Students Successful on Sections of Fractions tests

Section		Form: Concrete	Form: Written
Fraction Models	Discrete	16%	12%
	Number line	0	0
	Area	12	0
Equivalent fractions		0	4
Comparing fractions		0	0
(Written form only)			
Adding fractions			32
Subtracting fractions			28

In an examination of possible relationships, the data were displayed in Walbesser contingency tables (Walbesser and Eisenberg, 1972). In several cases, the students were not successful at either the task or the fraction subconcept; those cases yielded no information about the relationships. No strong trends were evident, but there were some patterns. Students who could conserve number performed more satisfactorily on the discrete model of fraction. Students performed better on the concrete form of the class inclusion task than on the pictorial (56% to 8%). There seemed to be evidence of learning during the concrete version of the assessment. An inference of the study is that it might be possible to develop a "readiness" test which would indicate students' ability to profit from instruction on the concept of fraction.

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COLLEGE LEVEL DEVELOPMENTAL MATHEMATICS COURSES: WHAT IS DEVELOPED?

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There is a need to learn more about the characteristics of college level students that are deficient in mathematics skills to determine if it is feasible to offer developmental courses with the principle aim of developing abilities to reason abstractly and to solve problems. In addition, such studies of the characteristics of these students might lead to suggestions for improvement of the curriculum for elementary and secondary students.

Many students enter colleges and universities too poorly prepared to succeed in mathematics courses or courses requiring basic mathematics skills. Some of these students come directly from secondary school, but there is a growing population of adults over age twenty-five. They vary in their goals, range of skills and deficiencies, and attitudes. Frequently developmental mathematics courses are offered (with or without academic credit depending upon the particular institution). In a single term or a year students hope to gain the mathematics skills necessary to survive in other college courses.

Concerned about selection of objectives Fey states (Fey, 1980) that "very often the judgement is based on a mixture of tradition and very sketchy or shortsighted estimates of the world in which students will spend their productive lives." Topics for developmental mathematics courses have frequently been selected on the basis of the degree of difficulty students encounter learning specific units in other courses (e.g., expression of large numbers in an astronomy course leads to inclusion of standard scientific notation as a topic in development mathematics courses). If the objectives are simply lists of specific skills needed to pass particular courses, then perhaps these skills ought to be taught in conjunction with the associated courses through additional classes or tutoring sessions or supplementary programmed materials. However, a developmental mathematics class could directly address goals listed in the NIE Conference on Basic Mathematical Skills and Learning Vol II: Reports from the Working Groups (Armstrong, 1975).

Even though students enrolling in such courses have deficiencies, they have made progress toward some of these basic skills. Because of lack of exposure, some students have made little or no progress toward goals such as knowledge of computer uses. However, it appears relatively easy to arrange learning activities directed toward such goals for these students. In contrast, the students encounter great difficulty improving problem solving skills although they may have had considerable previous exposure, and there appears to be great uncertainty of what strategies are most effective in increasing student proficiency in solving problems.

One of the goals listed as a "further desirable goal" by the NIE Conference (Armstrong, 1975) is ability to reason abstractly. Listing this as a further goal seems appropriate when considering students below the college level, but is it possible that for the adult this becomes a basic skill? If so, can it be taught? How? Or are there skills that can be taught that do develop a student's ability to reason abstractly? However, particularly considering the brief time available, one needs to determine characteristics of older students to select appropriate content and methods of instruction to achieve the chosen goals.

If these adult students can be taught to reason abstractly, perhaps they will be able to develop better problem solving skills. In addition, it is hoped this would enhance the reaching of the general goal listed in the report of the NIE Conference (Armstrong, 1975) of "developing level of self confidence necessary to operate effectively in a society that makes use of mathematics and mathematical ideas." College students will be expected to become responsible for their own learning. This is difficult if a student completes specific requirements of a course but returns to a habit of avoiding mathematics because of lack of confidence.

Chiappetta reports studies (Chiappetta, 1976) indicating that 50% of college freshmen have not reached Piaget's level of formal operational thinking, (Piaget's original studies in Switzerland may have utilized a biased sample giving indication of an earlier age of reaching this level.) However, only 1% of mathematics student teachers and none of the calculus students tested in Chiappetta's sample were still at the concrete level. Nevertheless, the level of reasoning acutally used by students is frequently substantially below their level of capacity (Dunlop & Fazio, 1976).

Piaget identifies two types of intelligence - sensori-motor and reflective. Skemp has modified Piaget's definition on reflective intelligence as the functioning of a second order system which:

1 - can perceive and act on the concepts and operations of sensori-motor system

2 - can act on them in ways which take account of their relationships and of other information from memory and from the external environment

3 - can perceive relationships between these concepts and operations

The Skemp Test to measure reflective intelligence includes items designed to test concept formation, reflective activity on concepts, operation formation, and reflective activity on operations. Skemp's study indicated high correlation between reflective activities and mathematics achievement with 10th and 11th grade students in England. Jurdak used the Skemp Test to show better mathematics performance associated with higher reflective intelligence for 12 to 14 year old students (Jurdak, 1980). However, Jurdak's study to investigate whether providing students with experience in playing a game whose rules and strategies reflect the mathematics structure of an operational system, semi group and group would facilitate the learning of these structures found no significant difference between students in the experimental group and control group. But Jurdak does suggest further investigation of the possibility of developing or accelerating the development of reflective intelligence, perhaps by heuristic method of teaching.

Whimbey opposes the position that intelligence is almost entirely inherited and fixed by genetic chemistry and develops the position that, except for a small part of the population having organic brain disorders, intelligence can be taught (Whimbey, 1975, 1976). He supports this position with his own studies of individual adults and other researchers' studies on children and adults. A significant issue is the definition of mental activities that actually constitute intelligence. He also states "Intelligence is paying careful skilled attention to the analysis of relations", and advocates an early practice of Bloom and Broeder in which they found it helpful to point out to college students the close connection between learning a physical skill and learning the thinking patterns of academic reasoning.

Pascual-Leone hypothesizes that the basic intellectual limitation of children is the number of schemes, rules, or ideas they can handle simultaneously - a capacity that increases regularly with age (Carpenter, 1980, p. 182). If adults do have this greater capacity, does this diminish the desirability for a Bruner spiral approach to the curriculum? If a spiral is still desirable, is it possible to shorten the time needed for the spiral approach? Can there be fewer turns in the spiral with a greater amount of learning occurring on each turn?

Assuming adult students do have this increased capacity, perhaps it is possible to teach them generalizations more efficiently than when teaching younger children. Can an adult see the similarities of the following pairs of exercises more easily than a child?

1. Solve for X:
   a. $3x + 5 = 29$ b. $ax + 3 = q$
2. Multiply:
   a. $x^2 \cdot x^5$ b. $x^a \cdot x^{2a} \cdot x$

Texts for developmental courses do not arrange exercises so as to encourage generalizations to any greater degree than those written for secondary school students. Occassionally a starred or optional exercise is included for this purpose. Most texts designed for developmental courses consist of early chapters devoted to arithmetic computations followed by chapters devoted to algebraic skills separated by a few pages to introduce definitions such as 'coefficient', 'monomial', etc. Instead of teaching arithmetic prior to algebra, perhaps integrating the teaching of arithmetic and algebra would enhance understanding (e.g., $\frac{1}{2} + \frac{2}{3}$ could be compared to $\frac{a}{b} + \frac{c}{d}$). Another possible advantage of this integration might be avoiding the student's feeling of "Here I am being taught something that is for 10 year old children, and I'm so stupid that I never did learn it."

Many of these adults do come with a vast collection of rules but seem not to understand them and look for inappropriate cues for selection of rules. Some students have become accustomed to passing tests in short units by memorizing a set of rules. If these rules are not understood, perhaps they are stored in the episodic memory and become associated with irrelevant references, instead of being stored in the semantic memory and able to be retrieved and utilized appropriately in new problem settings. A psychological characteristic related to problem solving ability appears to be degree of field independence (Lester, 1980 and Fennema & Behr, 1980). Field independence is characterized by activities and perceptions which are analytical and an ability to focus upon essential aspects of a problem independent of the surrounding field. Silver, Branca and Adams conclude in a study of 5th and 6th grade students that metacognition (a person's knowledge concerning one's own cognitive processes and products) "does appear to be a necessary condition for attaining expertise in a task domain" (Silver, Branca & Adams, 1980).

Researchers ought to conduct further studies of these characteristics of adults who enroll in developmental mathematics courses and attempt to determine if field independence and metacognition can be increased and by what methods. Can requiring students to verbalize in certain learning situations be beneficial? (Sowder, 1980) The effect of instruction in cue attendance, the requiring of students to describe details that are potentially useful in resolving a particular problem (Wright, 1979), suggests possibilities for increasing problem solving abilities of students. The research of O'Brien & Overton of giving contradictory evidence to subjects giving incorrect conclusions when presented with inferential tasks (O'Brien & Overton, 1980) indicated improvement of college students while it did not help the performance of seventh graders.

Although studies of attitude indicate a low positive correlation between attitudes and achievement (Kulm, 1980), perhaps attitude assessment could indicate environments and methods of instruction more conducive to learning. If these students fear competitive settings, would a cooperative small group activity facilitate learning or be viewed as still competitive?

While some students might prefer individualized programmed materials, others might benefit more by interacting with other people. However, caution is needed in utilizing individualized programmed materials (most are really just self-paced). Although claims for positive results have been made, Schoen came to the conclusion that either the idea of self-paced programmed instruction is wrong or perhaps we do not know how to go about it yet (Begle & Gibb, 1980). It might be more desirable to limit use of such techniques to mastering very specific tasks which later will be integrated with other skills or mathematical ideas using other methods of instruction. Well designed self-paced materials could be used later in conjunction with specific courses.

The mathematics educator must also consider individual differences among college learners even if mathematics achievement tests indicate similar deficiencies. Some of the students enter with severe deficiencies in several academic areas, and some enter with very high verbal abilities and only show lack of mathematical skills Is there are validity to the statement "It's easier to teach algebra to a student who never studied algebra than to the student who studied but failed algebra."?

Since adults do differ from children and adolescents, mathematics educators ought to select studies that have utilized samples from younger populations and replicate them with samples from adult populations taking college level developmental courses. Carpenter (Carpenter, 1980, p. 195) asks "...if a child makes certain errors at a given state will they be resolved as the child requires more mature concepts and skills or will these errors be magnified as new concepts are built on these earlier misconceptions?" Examining errors of college students could help provide the answer.

Studies suggested in this paper could have implications for the curriculum for younger children in addition to the direct objective of providing a richer, more valuable and more efficient learning experience for the college student in the developmental mathematics courses.

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LANGUAGE AND SYMBOLIC FORM IN CHILDREN'S MATHEMATICS

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How children respond to the need to translate from materials arrangements to language descriptions and/or symbolic forms, from language descriptions into materials arrangements and/or symbolic forms and from symbolic forms into language descriptions is of importance when considering instruction in mathematics. It is generally accepted that, in young learners, concepts develop as the result of real experience and that various tokens such as numerals, letters, operation and relation signs and word forms become symbolic only to the extent that the learner successfully relates them to concepts. Whether or not language is a mediating variable in this process is open to question, but the fact remains that language is the major means of communication in classrooms.

During the 1980-81 school year, three of these translation tasks were given to groups of first and second grade students in schools in northeastern Minnesota, and the results tabulated. The purpose of the study were to observe how children handled these translations and the extent to which different instructional practices affected this translating behavior.

Task One

The subjects were forty-seven first grade children, These children were randomly chosen from the 131 who were in a program in which concrete materials are used to develop concepts, numerals and arithmetic signs are slowly connected to concepts, the reading load is minimized, oral language is used more extensively than in textbook based instruction, and in which students are asked to generate symbolic forms before interpreting or 'reading' symbolic forms such as number sentences. The task was administered during the sixth month of the school year. The task was conducted in a one-to-one interview setting. Each child had a piece of paper and a pencil to use in writing a number sentence. The interviewer had a glass cylinder with six marbles of the same color in it, three additional marbles of that color and a recording form to record the order in which the parts of the number sentence were written by the child. The procedure described was used with each child.

Interviewer: "Count the marbles in the tube - how many are there?"
(The child's response was recorded)

Interviewer: "I'm going to do something. When I finish I want you to write a number sentence that shows what happened.

(Interviewer dropped, one at a time, the three additional marbles into the tube to join the six already there.)

Interviewer: "Write a number sentence to show what just happened with the marbles."

(The interviewer recorded the signs in the order in which they are written by the child)

Upon completion of the task, the paper with the child's writing efforts was collected by the interviewer.

Results:

1. 44 of 47 children successfully counted the marbles on the first attempt. 3 others successfully counted on a recount effort.

Table One

2. Signs Written

Order in which the sign were written	6	+	3	=	9
First	40	0	4	0	1
Second	1	46	0	0	0
Third	4	1	34	2	0
Fourth	0	0	0	44	0
Fifth	0	0	3	0	42

3. 38 wrote a correct number sentence. 3 others corrected the first incorrect effort.

4. The incorrect sentences were:
   - 6 + 3 = 8, with the '3' written as 'Ɛ';
   - 7 + 2 = 7, but verbalized as 'six plus three equals nine'
   - 6 + 4 = 9, with the "9" written as "e"

. 6 + 3 = 6

. 5 + 5 = 10

. 6 + 1 + 1 + 1 = 9

. 6 + 3 9, with no ' = " sign

The purpose of the study were to observe how children handled these translations and the extent to which different instructional practices affected this translating behavior.

Task Two:

Twenty one second graders were randomly chosen from the pool of 124 second graders who were in the second year of a textbook-less, materials based instructional program as described under Task One. The task was administered in the sixth month of the school year. Each child had a sheet of paper or pencil so as to be able to write a number sentence, and ten UNIFIX of each of two colors. The interviewer had a form on which to record the child's use of the blocks, and the order in which the different signs were written by the child. The following procedure was used in administering the task to each child.

Part one. Interviewer: "I will read two number story problems. Listen carefully. Use the blocks to show how you are thinking about the problem." The model for the first problem is:

(The interviewer read each problem slowly and paused after each sentence to give the child an opportunity to arrange the blocks)

Interviewer. "This is the first problem. Betty has nine record albums. Gloria has six record albums. How many more albums does Betty have than Gloria?....Now write a number sentence to show what you have done to answer the question."

(Interviewer recorded the child's spontaneous remarks. The way the child used blocks and how the number sentence was written)

Observations

. The children orally answered with the correct number.

. Five children wrote '9 - 6 = 3'.

. Two children wrote '6 + 3 = 9'

. Ten wrote incorrect sentences (see examples)

. Three children did not complete the number sentence

. Twenty children attempted to write a number sentenc

. Nineteen used the blocks, four during the reading of the problem, fifteen after the question was r ad.

Incorrect sentences written were:

3 + 6 + 6 = 9	3 + 1 = 4	9 + 3 = 12	6 + 9 = 16
9 + 6 = 15 (4)	9 + 6 - 3	9 = 3 = 6	
- 2 = 3			

Interviewer: "Johnny has eight toy airplanes. Richard has three more toy airplanes than Johnny. How many airplanes does Richard have?...Write a number sentence to show how you found the answer."

Observations

. Seven offered correct oral answer

. Ten wrote '8 + 3 = 11'

. Eight wrote incorrect sentences

. Nineteen attempted to write a number sentence

. Twenty used blocks to represent the problem, five during the reading and fifteen after the question was asked.

. Two wrote incomplete sentences

The incorrect sentences written are:

3 + 9 + 8 = 11	10 + 1 = 11	8 + 11 = 17
8 - 11 = 3	8 - 3 = 10	7 + 3 = 18
10 - 8 = 3	8 + 3 = 16	

Task Three

A total of 102 second grade students in five different second grade classrooms in four schools were used. These were the children who were present on the day the classroom was visited. The schools used were chosen because of their differences - in socio economic populations and in teacher backgrounds and classroom practices.

School A. This school has first and second grade classrooms using the program described under Tasks One and Two. The school is located in a community of 5000 in which the principal employment is in the taconite mining industry. The teachers of these students had participated in a program of 75 hours of instruction in the psychology of learning and intellectual development and the use of materials to develop concepts in mathematics with children of this age.

School B. The second grades are using a textbook - less, materials based instruction program after the children had had one year in a textbook program while in the first grade. Most of these children live in airbase housing. The teachers had only 20 hours of in-service prior to participation in the program.

School C. This second grade is in the highest socio economic area of a city of 100,000. The school has consistently ranked highest in the city on standard test results. The teacher had 40 hours of instruction in the use of materials in teaching, but uses textbooks as the basis of instruction.

School D. This second grade is in a school that draws from a predominately rural area just outside the same city of 100,000. The teacher is unfamiliar with the use of manipulative materials and uses of form of self-paced individualized instruction approach.

The children were all given the task during the month of the school year. The investigator introduced the four tasks by reviewing with each class (1) a number sentence, and (2) 'story' problems. An example was given of a "story" to go with the open sentence of 5 + 2 = ☐ and opportunity was given for the children to ask questions. Emphasized was the fact that a question had to be part of the story.

The ☐ = 12 - 3 was written on the chalkboard. The children were asked to compose a story problem in their own words so that this number sentence would be used to show what was in the problem and to find the answer to the question. After all children has written a 'story' for the first example, 4 - 3 = ☐ was written on the chalkboard and the instructions repeated. This procedure was followed to present '. - 8 = ☐ and 6 + ☐ = 9. The papers with the 'story' 'problems' as written by the children were collected at the close of the task. Analysis of the children's written 'stories' yielded the following tabulations. No attention was paid to misspellings, sentence structure, etc. The mathematical sense of the stories was the major concern.

Table Two

Stimulus: □ = 12 - 3

	School A	B	C	D
	n = 23	n = 32	n = 28	n = 23
1. correct story	15	16	14	6
2. numbers used correctly but no questions asked	2	7	4	2
3. wrong operation suggested	3	2	4	2
4. closed sentence used in story	0	3	2	6
5. nonsense or no attempt	3	4	4	7

Table Three

Stimulus: 4 + 3 = □

	School A	B	C	D
	n = 23	n = 32	n = 28	n = 23
1. correct story	16	14	15	8
2. numbers used, no correctly	3	5	7	2
3. wrong operation	2	1	4	1
4. wrote closed number sentence	1	5	1	6
5. nonsense no attempt	1	5	1	6

Table Four

Stimulus: 13 - 8 = ☐

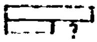
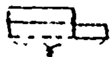
	School A	B	C	D
	n = 23	n = 32	n = 28	n = 23
1. correct story	12	12	13	8
2. numbers used correctly but no questions asked	3	2	2	2
3. wrong operation suggested	3	1	1	2
4. closed sentence used in story	1	10	0	6
5. nonsense or no attempt	4	7	2	5

Table Five

Stimulus: 6 + ☐ = 9

	School A	B	C	D
	n = 23	n = 32	n = 28	n = 23
1. correct story add compare	5 1 6	6 0	2 1 3	2 0 2
2. numbers used correctly but no questions asked	2	8	6	2
3. wrong operation suggested	0	1	6+☐=9 2 other 3 5 total in all	6+ 9= ☐ 4
4. wrote closed number sentence	5	4	1	7
5. nonsense no attempt	6	7	2	8
6. 6 + 3 = ☐ related story	3	7	9	0

General observations:

1. It seems that a slight advantage exists in making these translations for children who have been taught with extended time devoted to concept development at the concrete level, careful connecting of signs to concepts, encoding prior to decoding and experience in all three kinds of translations.
2. Children interpret  both as a "finding the difference" and as a 'missing part', more easily than 
3. Most children will follow a sequence of actions to the end result rather than consider the end result, then how it was obtained, or to think of the part joined, then what it was joined to.
4. Children do surprisingly well at translating a model such as an open sentence into a language description, especially if any emphasis at all is placed on such a translation or its reversal in instruction.

PSYCHOLOGICAL QUESTIONS REGARDING A NEW MODEL OF UNDERSTANDING ELEMENTARY SCHOOL MATHEMATICS

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Following the analysis of problems encountered in applying our hybrid model of understanding to elementary school mathematics (Bergeron, Herscovics, 1981) we have come to the conclusion that some of our criteria were quite useful in describing concept formation. Indeed, we now perceive these criteria as levels of understanding which in fact constitute the backbone of a constructivist model of understanding.

We can perceive four levels of understanding. The first one of these levels, intuitive understanding, takes as a starting point the informal knowledge of the child (pre-concepts, visual perception and estimation, primitive unquantified actions). This knowledge is then coordinated into a procedure leading to a first construction of a concept, a first construction which we consider as a second level of understanding, that of initial conceptualization. We speak of initial conceptualization for at the very beginning the concept in question is blurred and confused with the procedure leading to its construction (for example, at the beginning, the notion of number is confused with the counting procedure). It is only very gradually that the "outline" of a concept gains precision and that it separates from the procedure, making abstraction possible. Abstraction is our third level of comprehension Detaching the concept from its procedure gives it an existence of its own which can be identified as the "content", a content in search of a "form". This requires a process of formalization which we take as a fourth level of understanding.

Since we wish our new model to be constructivist, we must take into account the implications of developmental psychology as well as those of genetic epistemology. Developmental psychology can ascertain that the expectations implied in the new model are reasonable with regards to the intellectual capacity of the elementary grade student. Genetic epistemology, on the other

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hand, reaches the child's construction of knowledge. Of course, we are still far from having examined all the psychological questions raised by the new model. Thus, the present communication is limited to those which have been dealt with, namely: To what extent is the elementary school child capable of abstraction? Is it reasonable to consider abstraction as a third level of understanding? What role can be attributed to written symbolism? Of course, in view of our interest, these psychological considerations will be restricted to the understanding of mathematics.

ABSTRACTIVE CAPACITY AT THE ELEMENTARY LEVEL

The particular nature of mathematics involves two kinds of abstraction that Piaget (1973, pp 81-82) calls "empirical abstraction" (dealing with the physical properties of objects), and "reflective abstraction" (dealing with actions and their coordination). For instance, a child who counts the number of objects in a set but does not "conserve" number has but extracted a pseudo-physical property of the set. On the other hand, the child who is aware that this number is independant of the order in which the objects are counted is at the level of reflective abstraction. It is precisely this last level which characterizes a mathematical activity and this explains why Piaget considers it to be synonymous to "logico-mathematical" abstraction. Whether or not this level of abstraction is accessible to children at the elementary level needs to be examined.

Far from being trivial, this question requires a careful analysis of what is involved in reflective abstraction. Conservation of number lends itself well to such an analysis since it has been one of the most researched topic in arithmetic.

Very briefly, by "conservation of number" we mean the <u>invariance</u> of the cardinality of a set with respect to the layout of its elements. In the well-known Piagetian test, the child who does not conserve number compares two equivalent rows of objects

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* * * * * * *

and believes that the longer row contains more than the shorter one, and this <u>even if he can count</u>. To understand this phenomenon it is necessary to examine and compare some characteristics of the pre-operational thought (2-7 years) with those of the concrete operational level (7-11 years).

At the pre-operational stage, the child focuses essentially on states and not on his actions. Thus it is not surprising that objects are the only things reconstructed internally (empirical abstraction) and not the transformations applied to them, whence his dependence on their physical presence. Even if he admits that by matching the objects, the two rows will have the same number (empirical reversibility), by focusing on the states he is prevented from reaching the reversibility of thought necessary for conservation. Indeed, when examining the two equivalent rows he focuses on their length and this stops him from even considering the transformation which leads to a comparison. It should be noted that his reasoning is quite consistent with the pre-operational child's "transductive" logic (if A causes B, then B causes A) (Mayer, 1977, p. 180). Hence, since experience has taught him "if there is more, then it is longer", he concludes "if it is longer, there must be more". Only later, when he can consider simultaneously the two relevant variables (the length and the density of the row), will he perceive their reciprocal relationship.

This example enables us to better appreciate the intellectual power of a child who has reached the concrete operational stage. By now focusing on his actions he can reconstruct them internally and coordinate them; he can take into account several variables simultaneously and go beyond transductive logic to reach inductive and deductive reasoning, as long as it relates to his concrete world. Finally, his thinking has become reversible. Here are all the elements necessary for the discovery of invariants, which lead to the abstraction of a concept whose source has to lie in the child's reality. Since the concrete operational stage corresponds more or less to the range of the elementary grades, it seems that these children are well endowed to reach the level of reflective abstraction in arithmetic. As a matter of fact, reflective abstraction can in general be applied to arithmetic without requiring formal thought (characterized by the ability to manipulate hypothetical propositions unrelated to reality) since the arithmetic objects and operations are tied in with numbers which can always find concrete representations.

ABSTRACTION AS A THIRD LEVEL OF UNDERSTANDING

The level of understanding resulting from reflective abstraction must be examined within the framework of the child's intellectual development and viewed in terms of the construction of his knowledge. In this respect, Ginsburg and Opper (1979, p. 234) have identified in Piaget's theory of intellectual development three levels of understanding.

> The first of these levels is motoric or practical understanding. This is the level of action. The child can act directly on objects and manipulate them correctly, making the objects do what they are supposed to do. All this indicates that the child has "understood" objects at the level of motor responses. This knowledge is preserved in the form of schemes, which allow the actions to be repeated in identical situations and generalized to new ones.
>
> Another level of understanding is conceptualization. Here the child reconstructs internally the actions that were previously performed directly on objects, and at the same time adds new characteristics to these actions. He organizes the mental activities and provides logical connections. At the same time, much of the child's intellectual work remains unconscious. As we saw in reviewing Piaget's work on consciousness, the child is often capable of mental operations that he is not aware of and cannot express.
>
> A third level of knowledge involves consciousness and verbalizations. Now the child can deal with concepts on an abstract level, and can express his mental operations in words. The child can reflect on his own thought.

This model of understanding does not seem to be completely independent of Piaget's stages since the first level corresponds essentially to the pre-operational one while the other two can be linked to the concrete operational stage. Indeed, at the first level, the child discovers the properties of objects without reconstructing his actions internally. Nevertheless, it is at this level that he develops spontaneously some mathematical intuitions such as classification seriation, partitioning, bringing together or adding to, etc. The second level of understanding describes the internal reconstruction and coordination of actions, and thus corresponds to the concrete operational stage. Finally, "reflecting on his own thought" implies a prior internal reconstruction of the action and thus sets reflective abstraction at the third level.

The Piagetian model described by Ginsburg and Opper is truly a constructivist conception of understanding. There is no question here of "modes of understanding" but of levels of comprehension. In fact, not only is each level constructed by the child, but moreover, there is an embedding of the different levels. Indeed, if at first the child becomes familiar with objects through his actions, he later manages to reconstruct these actions internally and to [illegible]dinate them so that in turn, these actions become the subject of his reflection. If the first phase renders well the child's interaction with his environment, the last two, dealing with his transformations, lead to a logico-mathematical structuring of his thinking (composition of transformations, inverse transformations).

Since Piaget deals with comprehension in general, one should not expect his model to fulfill all the requirements of a particular discipline such as mathematics. In fact, Piaget is quite aware of the technical aspects of the mathematical language and the particular form of its symbolism (Piaget, 1969, pp 44-45). He even discriminates between mathematical "form" and "content" and recommends that the representations used should correspond to the "natural logic of the levels of the pupils" (Piaget, 1973, p.87). Consequently, he suggests that the role of actions should not be neglected by limiting instruction to a verbal form: "Particularly with young pupils, activity with objects is indispensable to the comprehension of arithmetic as well as geometrical relations" (p.80).

This recommendation has important pedagogical implications since quite often the teaching of mathematics is essentially verbal and symbolic while " the child's spontaneous mathematics is informal and unconscious" (Ginsburg and Opper, 1979, p.234). This informal mathematics is based on the child's actions and these can lead to reflective abstraction. These actions can eventually be applied to mathematical symbols but, as long as the pupil is at the concrete operational stage, these symbols must be related to the concrete. (Of course, one cannot ignore the relative nature of "concrete". For instance, following reflective abstraction, numbers tend to be considered as concrete, and addition as a concrete action).

THE ROLE OF SYMBOLISM IN MATHEMATICS

The previous observations bring into question the role symbolization plays in the understanding of a mathematical notion (concept or operation). Symbolization is essential in mathematics since it provides a means of representing a notion detached from its concrete embodiments. However, the introduction of symbols can be premature if an adequate intuitive basis is lacking. This would force the child to function at a strictly formal level which is impossible at the concrete operational stage. Hence, his only choices would be either to learn by rote, or simply to give up. Thus, the question "When to introduce symbolization?" is an important one. Considering that on one hand the Piagetian model suggests going from the concrete to the abstract, and that on the other hand, symbolization is a detachment from the concrete, the introduction of symbols should not, in general, precede the level of conceptualization. For instance, it would be difficult to perceive any pedagogical value in introducing digits before the child can count. However, this principle need not be applied dogmatically. For indeed, there is no harm in a child learning to write numbers greater than 9 in the first grade. But it must be remembered that he then merely learns a writing convention for he is not as yet equipped to understand place value notation.

While symbolization can be described as a detachment of a concept from its concrete representations, we can perceive qualitative differences in the meaning associated with a symbol depending on the levels of understanding. For indeed, what does the digit 7 represent to a child who does not conserve number? He may even count the two rows and write "7" next to each one

• • • • • • • 7
•   •   •   •   •   •   • 7

while claiming that the second row "has more". But then the only thing he can perceive in his digit is the result of his counting and not that of number. In contrast, to a child whose reflective abstraction has led him to conserve, the digit 7 truly represents the number.

Any analysis of the role of symbolization must also take into account the structural differences between the various modes of representation. Each mode (enactive, iconic, symbolic) has intrinsic properties which determine its structure. Although "a picture is worth a thousand words", mathematical symbolism condenses information even more. However, this condensation does not come as a free gift. For instance, even for an elementary notion such as addition (6 + 1 = 7), its symbolic representation is essentially static and has lost the dynamic flavor of action.

Furthermore, for a more advanced concept such as place value notation, its symbolization exceeds conceptually its other representations. Be it the concrete representation (multibase blocks) or its image $\begin{array}{c|c|c} h & t & u \\ \hline 2 & 3 & 4 \end{array}$ neither the material nor the image (defined by the presence of the value indices h,t,u) necessitate any convention regarding the position of the digits. To really value positional notation the student needs to be aware that a permutation of the columns $\begin{array}{c|c|c} u & h & t \\ \hline 4 & 2 & 3 \end{array}$ does not affect the number. It is only when these value indices are discarded that a true need to encode this information arises.

Only then will a pupil appreciate how inspired it was to encode this information in the relative position of the digits and what a turning point this must have been in the history of arithmetic.

It follows from the preceding analysis that symbolic representation has a structure of its own and that for some more advanced concepts it is impossible to completely dissociate content from form. Moreover, since the symbolization of a notion reaches its full significance only after it has been subjected to reflective abstraction, it seems justified to consider such symbolization as part of a fourth level of understanding, that of formalization.

CONCLUSIONS

Because of its constructivist nature, our model of understanding can easily be mistaken for an instructional/learning model which it is not. The latter concerns itself with the pedagogical interventions and the learning processes which bring about understanding. The criteria we have used for our new model evolved from our initial work with our "hybrid" model. The ones we have retained are those which proved to be useful in describing concept formation and we have merely organized them in a constructivist framework. It is indeed quite gratifying and reassuring to see how closely it resembles the Piagetian model described by Ginsburg and Opper.

However the two models have some important differences. Since we wanted a stage-free model we have retained intuitive understanding as our first level without moving into the pre-operational stage. Our second level, initial conceptualization, stresses the "operationalization" aspects in the explicit construction of a concept. In fact, this term was one of our first choices in naming this level. Our third level is clearly identifiable with reflective abstraction Of course, the last one, formalization, answers the specific needs of our discipline.

That the new model can be applied to specific concepts is not very doubtful. After all we are using some of the same criteria as in the old model. However, it is its accessibility to teachers which must be questioned. Can we really convey to them some of the finer points in Piaget's theory which underlie the new model? Can we expect them to discriminate between empirical abstraction and reflective abstraction, and between empirical reversibility and reversibility of thought? Only by trying it out will we be able to tell.

A last point pertains to the change from modes of understanding to levels of understanding. Whereas in the old model levels were implicit in describing the construction of a concept, in the new model they have become explicit. There is thus the danger of perceiving symbolization as corresponding to the level of formalization. However, the symbolic mode of representation can in fact reflect three different things. It could reflect a rote memorization of isolated pieces of knowledge which we do not consider as understanding. Also, as was shown with the conservation of number example, a symbol could reflect a pseudo-physical abstraction rather than a reflective one. To overcome this problem of interpretation teachers must be made aware that their evaluations cannot be limited to the written symbolic form and that a questioning of the child is essential.

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NEEDED RESEARCH IN MATHEMATICS TEACHER EDUCATION

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Recent studies, including the national assessment and other evaluations of mathematics teaching throughout the nation, indicate that present practices are not resulting in the type of learning outcomes most often espoused by mathematics educators. The reports further indicate that both teachers and pupils hold a very narrow view of the nature of mathematics. From such reports come goals and proposals for new directions - most notably the "Agenda for Action." However, no real change will come about merely from the identification of needs or the promulgation of goals. Rather, change depends upon the behavior of individual classroom teachers. Thus it is imperative that deliberate action be directed toward teacher education including both research on teacher education and its outcomes and the development of alternative approaches to designing programs.

This paper attempts to raise some questions that teacher educators might address. These questions arise from several assumptions that underlie the conclusions and suggestions offered here. Those assumptions are as follows: First, teacher education from its earliest preservice phase through the highest level of professional education should be an on-going, developmental process not merely a collection of courses, workshops or classroom experiences. Second, mathematics teacher education must emphasize both knowledge of mathematics and knowledge about mathematics and about how to deliver mathematics to pupils. Third, teacher education programs must relate to the realities and contingencies of teaching and to the instructional and managerial decisions that teachers must make. Fourth, teacher educators should model the teaching strategies and alternatives that are important for teachers to develop.

In reflecting on teacher education it will be helpful to consider a taxonomy of mathematics teaching as a framework both for the development of an on-going teacher education program and for the clarification of research questions. The ordering principle in this taxonomy is teacher behavior since research suggests that teaching behaviors rather than teacher characteristics are most related to learner outcomes. The taxonomy has the advantage of focusing attention on aspects of teaching not readily described in specific behavioral objectives. It also directs attention to the continuity of growth and development in the teacher and it allows one to delineate both entry level goals and goals to be developed during the continuing professional development of the teacher. In particular, it offers a framework for specifying the higher level performances to be expected of experts as opposed to those expected of beginning teachers so that we may formulate more realistic expectations of beginning teachers. Finally, it is hoped that the taxonomy will help teachers to evaluate their own behavior and to plan for their own continuing professional growth. The categories of the taxonomy of mathematics teaching are deliberately named after Bloom's cognitive taxonomy although perhaps better designators could be chosen. Briefly, those categories are as follows:

KNOWLEDGE: Facts, processes, theories, techniques and methodology related to instruction. This level includes knowledge of mathematics and of the curriculum and materials of school mathematics. Knowledge here subsumes all of the levels of Bloom's cognitive taxonomy. This is the component usually associated with the college classroom and usually measured by paper and pencil tests and other conventional classroom methods.

COMPREHENSION: Performance of selected behaviors under controlled conditions, such as peer teaching, microteaching, simulations or role playing. This level involves demonstration that the individual can do something, and the behavior to be demonstrated usually is called for in an explicit manner so that the individual is conscious of the goal of demonstrating the desired behavior.

APPLICATION. Planning and administering learning activities and materials in a classroom setting. This level involves evidence not only that the individual can do but, further, that he/she does do. It involves the use of appropriate teaching skills at the proper time or with the desired frequency as a part of the normal teaching style.

ANALYSIS: Response to pupil, teacher, subject matter and environmental cues to select, organize and administer effective programs and lessons. The teacher recognizes constituent elements of the curriculum and the relationships among them and sees them as an organized whole. The teacher responds spontaneously to students as individuals and the teacher's actions and decisions flow from a consistent and conscious rationale.

<u>SYNTHESIS</u>. Orchestration of one's teaching behavior into a personalized whole. The teacher internalizes and professionalizes the teaching skills and combines the underlying competencies into an effective style unique to the individual.

*<u>EVALUATION</u>. Judgement of the effectiveness of one's teaching according to various internal and external criteria including pupil progress toward the desired goals. The teacher modifies his/her teaching in the direction of greater effectiveness.

One advantage in this taxonomic approach is that it facilitates a kind of reflective research about teacher education and the goals we wish to accomplish. For example, the taxonomy emphasizes not only the cognitive aspects of a teacher education program but also the performance outcomes -- the teaching behaviors -- that we might deem desirable. Second, the taxonomy encourages us to view the development of teaching ability and effectiveness from one level to another. Third, affective teacher characteristics are not overlooked and they can constitute a strand that runs through all levels of the taxonomy. Fourth, because the taxonomy is structured on teaching behavior, a variety of assessment approaches are available. Finally, the taxonomic approach facilitates specification of programmatic goals, instructional alternatives and learning experiences for all stages of a teacher's professional development.

Such a consideration of the goals and alternatives for teacher education also helps in the identification of areas of needed research. This research in teacher education is not that of the engineer or the agronomist, it is the descriptive kind of research that depends on reflection and observation. Following are some of the questions it might address.

The first concern knowledge.- Most mathematics teacher education students have a reasonable to very good knowledge of mathematics as evidenced by ability to calculate, differentiate, integrate, etc. They have learned the techniques and processes of mathematics and they have been successful in the courses where these are taught. This is what we could call knowledge <u>of</u> mathematics. What teachers often lack is knowledge <u>about</u> mathematics. That is, they lack a real understanding of what are the major concepts and principles and relationships in the mathematics they teach. It is as though they have gone from tree to tree to tree in the mathematical forest taking relatively clear still life close-ups of each one, but they have never panned the entire forest with either a wide angle lens or a motion picture camera. They know, for example, numerous techniques for factoring quadratic equations, but they go blank when asked why they spend so much time teaching factoring in algebra class. They can perform accurately and efficiently operations with integers, rationals and reals, but they are hard put to explain in words a seventh grader would comprehend just what they are doing and why.

Therefore, a major contribution would be a clear and through analysis of the concepts and processes of school mathematics and of the relationships both among these elements themselves and between them and the more advanced mathematics we expect our students to study. A second contribution will be the development and testing of effective strategies for facilitating such understandings by teacher education students.

We also must find ways to make mathematics teachers become problem solvers themselves. Music teachers use their own time to engage in musical activities: playing instruments, attending concerts, listening to records. Art teachers similarly engage in painting, photography, pottery making or the like. How many mathematics teachers are there who really do mathematics, who really are problem solvers in the full sense of the word, who elect mathematics as a personal recreation? Unless we can turn teachers on to the real human activity of mathematics, they and their students will continue to see mathematics as something "out there" apart from themselves.

This is related to the next question: How do mathematics teachers perceive mathematics? Recent studies have produced a preponderance of evidence that teachers see mathematics as essentially computational in nature, consisting of precise rules and formulas, and primarily justified by the importance of each topic for subsequent topics and courses. This prompts the additional question: When, how and in what contexts do teachers apply their knowledge to their teaching? How does their knowledge of and about mathematics affect their teaching? For example, do teachers really comprehend the goals of the "Agenda for Action" and similar statements? (A good example frequently occurs when discussing problem solving where mathematics educators mean one thing but many teachers who hear them assume they are talking only about the "story problems" in the texts.) Do teachers recognize mathematical aspects in other domains of human activity and knowledge beyond the very mundane calculations and arithmetic? Do their instructional methods indicate that they differentiate their teaching of concepts or principles from their teaching of skills and algorithms? How do they draw upon their knowledge and experience to interpret pupil behaviors and to diagnose pupil needs? Can they -- more important, do they -- suggest, design and implement alternative teaching strategies or topics including the use of instructional approaches not included in their texts? Or do they, as studies suggest, merely correct the homework, go over a few examples, then give more homework? Do they use materials and technology purposefully in teaching? For that matter, do they use them at all?

Flowing quite readily out of such questions about knowledge come questions about decisions. What curricular, instructional and managerial decisions do teachers make? During the actual teaching activity, which ones and how many of the teacher's actions represent conscious decisions and which ones are automatic responses? What pupil, curricular and environmental variables do teachers respond to and how? Are teachers aware of inconsistencies in their own behavior? For example, how often do teachers realize that they have answered their own questions or that they have accepted a pupil's example when what they asked for was a definition? Do teachers really hear (listen to) what pupils say and do they adjust their behavior and responses in accord with pupil comments?

Both the aspects of knowledge and of decisions recall the research of Perry (1970) who studied the intellectual development of college students. Its essence was to describe stages through which college students pass from dualism where they view the world as black/white, right/wrong, through multiplicity where they can accept uncertainty and diversity of opinion, to relativism and personal commitment. Perry's work raises important questions for mathematics education. For example, one thing frequently observed is typical of undergraduate methods students when they have completed a microteaching lesson. The instructor may ask them to explain why they have used a particular approach or example or activity or instructional sequence. What one is looking for is a rationale that indicates that they have thought about the goals of the lesson and have made conscious instructional decisions. What one finds instead is that as soon as the question is posed, all the notebooks fly open and all the pens come out because the students expect that they are about to be told the "right way" to teach the lesson. This suggests that many preservice teachers may be as yet quite dualistic in their thinking, a position not incompatible with the notion that mathematics is itself right/wrong with precise rules and procedures. We need much more knowledge about the developmental processes of mathematics teachers and the ways in which that development is confounded by the nature of the subject matter in order to design programs that are appropriately aimed for the developmental level of the students. We also shall have to take account of these variables when we set teacher education goals that espouse the open endedness and creativity of problem solving as a central focus for teaching.

We also need to know what teachers think about pupils, what they communicate to pupils about the pupil's ability and goals in mathematics, what they expect from pupils and what they expect from themselves. Further, we need to ask what we expect from teachers before they enter teaching, at the time of entry, during the early years and beyond. For example, many expectations for beginning teachers may correspond to the comprehension and application levels of the taxonomy while many of the higher level competencies of analysis, synthesis and evaluation will be more appropriate expectations for teachers with experience, although our expectations should include some competencies at each level for all teachers.

Which of the goals that we propose are actually observed in the repetoire of mathematics teachers at various stages? How congruent are teaching practices and the goals of education in the 1980's and beyond? How do teachers' attitudes, behaviors and skills change over time with and without educational interventions? What kinds of interventions are most effective in furthering certain changes?

What are the rewards both internal and external that accrue to teaching? What attracts persons to teaching? What keeps some teachers in teaching while others leave? What aspects of education beyond classroom teaching should we be developing and researching? How can we structure our graduate and inservice programs so that they continue the systematic and evolutionary development of teaching competence begun in preservice programs?

How can we model desired teaching styles and strategies in our own teacher educational programs? Do we, for example, teach our college mathematics courses with problem solving is the central focus? Do we use media and materials purposefully? Do we, in our methods courses, employ the same variety of teaching strategies that we talk about? Are the cooperating teachers we select for our student teachers models of these same teaching behaviors? If not, how can we develop cooperating teachers who are?

These are some examples of questions that teacher education research should address if we are to design programs that will equip teachers to meet the needs and challenges of today and tomorrow.

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PROPORTIONAL REASONING OF EARLY ADOLESCENTS: COMPARISON AND MISSING VALUE PROBLEMS IN THREE SCHOOLS

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ABSTRACT

Eight lemonade puzzles requiring proportional reasoning to compare ratios or find missing values were administered to 263 eighth graders in three urban schools with various achievement levels and mathematics programs. Missing value problems with an integral ratio were more difficult than comparison problems with equal integral ratios, but easier than comparison problems with integral and non-integral ratios. There were only minor differences in proportional reasoning between boys and girls in the same school, but among the schools proportional reasoning mirrored the differences in school environment and mathematics achievement.

Recent investigations of adolescents' proportional reasoning have made use of missing value problems (Karplus, Karplus, Formisano, & Paulsen, 1979; Karplus, 1981; Rupley, 1981) and of comparison problems (Noelting, 1978; Noelting, 1980; Karplus, Pulos, & Stage, 1980 -- to be referred to as KPS). The present study was designed to clarify the relationship between these two types and to extend data about both to a more diversified student population than had been employed by KPS and Noelting (1980). This study was designed to answer the following questions:

1. What differences in reasoning are elicited by tasks that require comparison of ratios versus tasks that require computation of a missing value?
2. How can the students' performance lead to a hierarchical scale of proportional reasoning?
3. What differences in proportional reasoning are observed among students from different school environments with differing academic achievement levels?

THE LEMONADE PUZZLES

In the lemonade puzzles, John and Mary prepare lemonade concentrate by dissolving a certain number of spoonfuls of sugar in a certain number of spoonfuls of lemon juice. For instance, John might use two spoonfuls of sugar and ten spoonfuls of lemon juice, while Mary uses three and twelve spoonfuls, respectively

Eight puzzles were presented to each subject, with the amounts of sugar and lemon juice given in Table 1. The numerical values were chosen to include integral ratios in each puzzle in the expectation that this would lead to greater use of proportional reasoning than found by KPS. Two puzzles (WBX and WX-a), which had unequal integral ratios within the recipes, were constructed to determine whether proportional reasoning would be used more extensively than on puzzles with an integral and a non-integral ratio within the recipes (puzzles WX (KPS) and Wx-b). Furthermore, three missing value puzzles were included to provide a more definitive link between them and comparison problems than had resulted from the use of the adjustment question in KPS.

Table 1. Data Used In Lemonade Puzzles

	John's		Mary's			John's		Mary's	
	Sugar	Lemon	Sugar	Lemon		Sugar	Lemon	Sugar	Lemon
Item WB_1	1	2	3	6	Item MWB_1	1	3	2	?
Item W	3	12	5	20	Item MW	4	16	6	?
Item WB_1X	1	3	4	8	Item MB	3	7	6	?
Item WX-a	2	10	3	12					
Item WX-b	4	16	6	20					

W: within recipe ratio integral   X: unequal ratios
B: between recipe ratio integral   M: missing value problem
1: unit amount

THE SUBJECTS

The subjects of this study were 263 eighth graders from three urban schools within two miles of one another in the San Francisco Bay Area. About half the students were boys and half were girls, selected randomly from students who were in attendance in regular mathematics classes while the research was conducted.

The schools were chosen for their academic and ethnic diversity, representative of the high minority population in the city. On the CTBS standardized achievement test (CTB/McGraw Hill, 1975) used by many California school districts, for instance, ninth graders from School A scored at the 70th percentile on the national norms in mathematics and at the 56th percentile in language. School B students scored at the 65th and 43rd percentiles, respectively. School C students scored at the 31st and 29th percentiles, respectively.

Differences for example, among the schools' mathematics programs corresponded to these achievement test results. School A's faculty chose their texts and liked them, and most of the eighth graders studied proportions, equations, and geometry. Schools B's faculty were generally satisfied with their texts and taught proportions in conjunction with fractions, percent, word problems, and other topics. School C's faculty, which included no member with a degree in mathematics, were generally dissatisfied with texts and teaching materials because, among other reasons, they found them unsuitable for the range of their students' skills. They covered proportions in the eighth grade only in a late in-the year ur on percents In turn, school differences consist of differences in faculty, curriculum, teaching styles, achievement, and other factors

PROCEDURE

The Lemonade Puzzles were administered in about 15 minutes during a 40-minute interview The interviewers were male and female members of the ethnic groups represented in the participating schools Each item was read aloud by the interviewer, who asked the following questions

Comparison puzzles --

1 Will the two concentrates taste the same?

2 (after the response to #1) How did you come up with that answer?

3 (If response #1 indicated unequal taste) Which one tastes sweeter?

4 (after the response to #3) Please explain your answer

Missing Value puzzles --

1 How much lemon juice does Mary need to make her concentrate taste the same as John's?

2 (after the response to #1) How did you come up with that answer?

Questions 3 and 4 of the comparison puzzles were sometimes omitted or combined with Questions 1 and 2, because many subjects stated which concentrate would taste sweeter in answer to Question 2.

SCORING OF DATA

We classified the students' explanations in answer to Questions 2 and 4 into the following four categories

Category I (incomplete, illogical)-- don't know, guess, inappropriate quantitative or qualitative operations

Category Q (qualitative)-- qualitative comparison of amounts referring to all four ingredients (more stringent than Category Q in KPS)

Category A (additive)-- using the data to compute and compare differences

Category P (proportions)-- using the data to compute and compare correct ratios, possibly with arithmetic errors (the same as Category R in KPS)

We also identified the data comparisons (sugar/lemon = "within" recipe vs sugar/sugar = "between" recipe) that were used

RESULTS AND DISCUSSION

The distributions of responses among the categories are presented in Table 2 No statistically significant differences were observed for boys and girls, so only the combined data are reported For reference, we include in Table 3 the frequencies of proportional reasoning of eighth graders for the eight comparison puzzles used in KPS (Stage, Karplus, & Pulos, 1980) The frequencies of proportional reasoning observed in the present study were similar to those observed with slightly different lemonade puzzles in a low minority community in our earlier study Puzzles WB and W were the easiest, with more than 50% proportional reasoning. Surprisingly, Puzzle WX a was the hardest, with only 24% proportional reasoning On this puzzle there was an appreciable frequency of additive responses, while the other comparison puzzles elicited additive responses only infrequently.

Table 2. Frequency Distributions on the Lemonade Puzzles (percent, N = 263)

	Comparison					Missing Value		
Category	WB_1	W	WB_1X	WX-a	WX-b	MWB_1	MW	MB
I	26	34	43	53	46	36	39	31
Q	6	7	7	5	6	0	0	0
A	3	7	7	18	9	28	22	30
P	65	51	43	24	38	36	39	39

I: Illogical; A: Additive
Q: Qualitative P: Proportional

Table 3. Frequencies of Eighth Graders' Proportional Reasoning by Puzzle Structure, from KPS and Stage, Karplus, and Pulos (1980)

Item	WB	W	B	N	WBX	WX	BX	NX
Frequency	65	72	55	33	47	43	35	17

The three missing value puzzles had very similar distributions in spite of the differences in the occurrence of within or between integer ratios. The degree of difficulty was close to that of the unequal taste Puzzles WB_1X and WX-b. A substantial frequency of additive responses occurred on all three missing value puzzles.

Three of the lemonade puzzles in the present study included one spoonful of sugar for John's lemonade concentrate (Table 1). The presence of this unit amount in either comparison or missing value problems did not make them consistently easier or more difficult than similar puzzles without unit amounts.

Cognitive elements. A comparison of the frequencies of proportional reasoning on the eight puzzles used in KPS led to the recognition that unequal ratios and non integral ratios made puzzles more difficult. Accordingly, we now introduce the following five cognitive elements, which are individual steps in proportional reasoning required on some puzzles and not on others.

IIE - comparing equal integral ratios
IIU - comparing unequal integral ratios
INU - comparing an integral with a non-integral ratio
MVI - finding a missing value for a puzzle with an integral ratio
MVN - finding a missing value for a puzzle with a non-integral ratio

For evidence of competence with respect to any one of these cognitive elements, we examined the students' explanations to find at least one use of the element. For instance, students who solved

Puzzle MB by proportional reasoning using a between comparison were assigned competence in element MVI, while those who solved the same puzzle by means of a within comparison were assigned competence in element MVN. Each cognitive element could be applied on one or more puzzles.

To identify a hierarchical relation among the cognitive elements, we tested them for scalability. We found that elements MVI and MVN formed a perfect scale: all students who were competent in MVN were also competent in MVI (Table 4 -- left side). Elements IIE, IIU, and INU, however, did not scale satisfactorily in any of the schools (Table 4 -- right side): comparable numbers of subjects were competent in IIE and either IIU or INU but not both.

Table 4. Hierarchical Scales of Cognitive Elements, by School (percent)

Element	School A	School B	School C	Total	Element	School A	School B	School C	Total
(none)	(31)	(50)	(73)	(49)	(none)	(16)	(38)	(60)	(32)
MVI	69	50	27	51	IIE	84	71	37	65
MVN & MVI	23	13	6	14	IIE & IIU, not INU	13	8[a]	5[b]	9[a]
					IIE & INU, not IIU	10	[illegible]	3	6
Number of Subjects	102	86	75	263	IIE, IIU, and INU	52	33	13	35

a: 1% omitted IIE
b: 3% omitted IIE

In addition to displaying the cognitive elements' scalability, Table 4 shows the differences in proportional reasoning among students in the three schools. Most of the students in schools A and B used at least one cognitive element, but most of the students in School C did not use proportional reasoning at all. More than half of the students in School A used all the comparison cognitive elements, one third of the students in School B did the same, but only one-eighth of the School C students did so.

Missing value puzzles. Comparing the two sides of Table 4, one finds that cognitive element MVI, used by 51% of the students, was intermediate in frequency between the comparison element IIE (65%) and either IIU (44%) or INU (41%), and comparable to the union of IIU and INU (50%). Cognitive element MVN with a frequency of 14% was less widely used than any of the comparison cognitive elements.

Our efforts to combine the two scales in Table 4 were not successful for the reasons shown in Table 5, which is a contingency table of the two sets of cognitive elements. The hierarchical sequence (IIE) (MVI) (IIU or INU) (MVN) is suggested by these frequencies, but the 14% of subjects who were exceptions precluded accepting it.

Table 5. Contingency Table of Cognitive Elements for Comparison and Missing Value Puzzles (percent, N = 263)

	Comparison Elements					
Missing Value Elements	(none)	IIE only	IIE & IIU	IIE & INU	IIE, IIU, and INU	Total
(none)	29	11	5	2	3	49
MVI only	4	6	3	3	20	37
MVN & MVI	0	0	1	1	11	14
Total	32	17	9	6	35	100

Illogical and additive reasoning Though the frequencies of proportional reasoning of the present study were similar to the findings in KPS, the frequencies of additive and illogical reasoning were not. The use of additive reasoning was much lower than that reported earlier, and illogical reasoning was used more frequently. Furthermore, illogical reasoning varied substantially from a low of 26% on Puzzle WB to a high of 53% on Puzzle WX-a, while illogical reasoning had been approximately uniform for all eight puzzles used in KPS. When one examines Table 2 in more detail, it appears that illogical reasoning increased as proportional reasoning decreased. This result, observed in each of the three schools, differs from the finding of KPS that additive reasoning increased as proportional reasoning decreased.

CONCLUSIONS

By means of the definition of cognitive elements, it was possible to establish separate scales for proportional reasoning on comparison and missing value problems. Equal integer comparison problems were found to be the easiest of all types used. Missing value problems with integral ratios were comparable in difficulty to "unequal" comparison problems with one integral ratio. Missing value problems were therefore also comparable in difficulty to the adjustment problems studied by KPS in conjunction with unequal ratio comparisons.

These results suggest that teaching for proportional reasoning in schools like School C, where achievement levels are low and the usual textbooks are considered inadequate by many teachers, might well begin with equal integer comparison problems that include unit and non unit amounts. The high frequencies of illogical and qualitative responses, however, also lead to the conclusion that the quantitative description of recipe ingredients, amounts of money, times, distances and other variables should receive attention even before specific quantitative relationships like constant ratios, constant sums, or constant differences.

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PARTITIONING AND UNIT RECOGNITION IN PERFORMANCES ON RATIONAL NUMBER TASKS.

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There appears to be a marked contrast between what children and young adults could know about rational numbers and their performance on rational number achievement tests. For example Vinner et al (1981) found numerous errors in simple addition tasks such as 1/2 + 1/4. Two algorithms used by young adults in his sample were $1/2 + 1/4 = \frac{1+1}{2+4} = \frac{2}{6}$ and $1/2 + 1/4 = \frac{4+2}{4} = \frac{6}{4}$. While Vinner et al do not describe the rational number constructs of their sample, one might conclude that the users of algorithms such as those have either do not have a well developed <u>rational</u> number construct or at least do not relate such a construct to the task of adding. For example, if rational numbers (or some subset thereof) had a quantitative base, then a child might know 1/2 + 1/4 = 3/4 as a "basic fact", that is a well known description of experience. A more sophisticated quantity based judgment would be that $1/2 < 1/2 + 1/4 < 1$. Thus the numbers 2/6 (being less than 1/2) and 6/4 (being greater than 1) were not possible candidates for the sum 1/2 + 1/4. In such cases rational numbers would have a meaning beyond a system of two part symbols to be manipulated.

How does a person, in particular a child or young adult in his or her first 9 years of school, build up a meaningful system of knowledge of rational numbers? Kieren (1981) has developed the thesis that such construction results from or at least entails the coordination of at least five dimensions. The first of these is mathematical in nature. Because rational numbers are a complex phenomena, they represent a number of distinct sub-constructs such as measures and operators (Kieren 1976, 1980). Thus a person must have mathematically diverse rational number experiences, representing both the additive and multiplicative nature of these numbers. A second dimension which reflects the first is the diversity of images by which a person can "picture" rational number ideas. The third dimension is symbolic. There are two knowledge building problems associated with this dimension. The first is the use of a pair of integers to depict a single entity. The second is the fact that the formal symbolism (e.g., 3/4) needs to connote and indeed be built upon a variety of informal symbol systems which relate descriptively to diverse physical situations (e.g., 3 divided among 4, three of these fourths of a unit). A fourth dimension is psychological in nature. In particular, one might ask about the capabilities needed to understand a number such as "2/3". One might speculate that processing represented by class inclusion tasks might be basic to being able to use the appropriate unit in developing rational number ideas. Noelting (1978, Noelting and Gogne, 1980) suggest that two staged processing would also be necessary (e.g., $2/3 < 4/7$ because $2/3 = 4/6 \wedge 4/6 < 4/7$). The complex nature of rational numbers mathematically and symbolically suggests complex neurological and psychological processing as well. The fifth dimension concerns the mathematical thinking tools a person uses to cope with and develop rational number ideas. Such tools may result from direct instruction either formally or informally. With respect to whole numbers one such thinking tool or <u>constructive mechanism</u> is counting. Three mechanisms useful in rational number idea construction are unit recognition, equivalence (as a thinking tool rather than a formal mathematical notion) and partitioning.

It is beyond the scope of this essay to further elaborate all five dimensions described above or to detail their interaction. However, much as unit, successor, counting and corresponding are critical to whole number development, the constructive mechanisms mentioned above are critical to rational number development. It is the purpose of this paper to discuss in some detail the mechanisms of unit recognition and partitioning. This discussion will culminate in the consideration of the work of students in grades four through eight on four kinds of rational number thinking.

<u>Constructive Mechanisms</u>

<u>Unit Recognition</u> In a study of performance on number line tasks Novillis (1981 in press) noted that grade 7 students had difficulty in identifying rational number points such as 1/3 on the number line in face of a line of 2 rather than one unit of length. A common error was for the student to indicate the point 2/3 (one third of the way from 0 to 2 on the number line). Babcock (1978) found on basal measurement tasks involving fractional parts that choosing the unit was difficult for students in grades 4 and 6.

There are several ways in which the concept of unit and the mechanism of unit recognition manifest themselves in the context of rational numbers. Hunting (1980) considered the case of unit recognition under a discrete part-whole manifestation of rational numbers. Thus in finding 2/3 of 12 objects one must consider the objects as <u>units</u> then thirds as <u>units of units</u> (sets of 4) and two thirds as a <u>unit of units of units</u>, (a set of 2 sets of 4 out of 3 such sets). In this case choosing or identifying the unit is a "What shall we count" problem.

Although this concept of unit as a set of (one or more) objects has use in a parts of a whole concept, it is based on an atomistic concept of unit. A rational number unit, in general on the other hand is a partible one, at least in theory. Recognizing objects as at once units for counting and as partible units is one level of the unit recognition problem. Pothier (1981) found that young children (5 - 6) had difficulty dividing 7, 5, or even 3 cookies evenly between two children. In each case they found they had an "extra" or "too few" cookies (i.e., they saw cookies as counting units). When asked to divide one cookie between two children, this unit concept suddenly changed and they "shared" the cookie (i.e., they saw the cookie as a partible unit). After this prompt, they could easily solve the other sharing problems (1 1/2, 2 1/2, 3 1/2 cookies per person).

Another rational number unit concept is unit of comparison. Thus when one ties the number 3/4 to some reality one says 3/4 of something (quantitative). If one considers the multiplicative manifestations of rational numbers (ratio numbers and operators) the role of this unit is more abstract and algebraic in nature. If a person is applying rational numbers in a ratio situation, a unit or "whole" must be arbitrarily selected. For example, in a group in which there are 3 girls for each boy one might say that 3/4 of the <u>group</u> are girls but would be less likely to say there are 1/3 <u>boy</u> for each girl. Finally in an operator setting a 3/4 operator maps one quantity to another and the unit of comparison is the unit or identity operator. Here the rational number is describing a relationship and not a quantity, hence unit has a special meaning.

Partitioning

The last thinking tool or thought-action to be considered is partitioning - the <u>act</u> of dividing a quantity into a given number of parts which are themselves quantitatively equal. There are a number of levels of this behaviour (Kieren 1980, Pothier 1981).

The first level of partitioning takes two forms either focussing on the separation into the correct number of peices or an equality of piece or part size without reference to number. A second level is a combination of these two with a division into the given number of parts, followed by an attempt to "even up" the parts. The third level is algorithmic in nature. This is shown, for example, in "dealing out" behaviour in the discrete case or successive halving behaviour in the continuous case. A key feature of this partitioning is that the child is convinced that the action guarantees a partition. It is important to note that a person could have an algorithm for a particular situation (e.g., discrete object, or eighths) but exhibit first level behaviour in other situations (e.g., thirds). A fourth level entails the relating of the result of a partition to number (in particular rational numbers). Thus while an algorithmic partitioner is certain of the equal nature of the partition, the numerical partitioner can answer "how much" or "how big" in rational number terms. Both persons would be able to solve a particular problem. The latter would be able to integrate the experience into a wider mathematical context.

The interaction and the need for coordination of the dimensions of knowledge building exhibits itself in the partition act. Partitioning differs depending on the nature of the material to be partitioned. Some categories of partitioning tasks are discrete (objects which cannot be subdivided), continous (e.g., a rectangular cake), discrete but continuous (e.g., mini-pizzas which are both counting and partible units), continuous but unitized (e.g., connected paper towels), continuous but prepartitioned (e.g., a scored chocolate bar), and discrete but unitized (a dozen eggs).

There is also an obvious connection between partitioning and division. In fact, the standard division algorithm symbolizes standardized partitioning repeatedly by 10 and sharing.

Theoretically the three thinking tools allow a person to generate rational number ideas. Partitioning, equivalence (quantitative), and unit recognition allow a person to realize when 3/4 = 3/4 (e.g., different partitions of the same unit) and that 1/3 can be greater than 1/2 (e.g., 1/3 of a large unit is greater than 1/2 of a smaller unit). Equivalence and partitioning and unit recognition generates order and density properties. Partitioning and first

quantitative equivalence and later formal equivalence allow for the notion of addition, while internal equivalence as well as partitioning can be used to generate multiplication.

The question is - do children and young adults exhibit any of these thinking tools when faced with rational number problem solving tasks? To answer that question the complete (written) protocals of a random sample of 52 grade 6, 7 and 8 students as they worked on four rational number tasks. These protocols were selected from those of all grade 6 pupils in two schools and all grade 7 and 8 pupils in a large junior high school in a small Alberta city with a mining, lumbering and petroleum based economy. These four tasks were part of a larger Rational Number Thinking Test (reliability .92). They required the test taker to figurally share pizzas or chocolate bars among a number of persons and to report the amount of each fair share. From a mathematical point of view, two tasks involved positive rational numbers less than one and two used rationals greater than one. Two tasks involved performing a division of pre-partitioned units (chocolate bars), on one of these the partition could be used to directly solve the problem.

Due to the scope of this paper, the analysis given below is limited in two ways. First, it is limited in the kinds of tasks analyzed. Second the analysis focusses primarily on the partitioning mechanisms used and in a secondary way on unit and equivalence.

Results

The results of the protocol analysis are summarized in Tables 1 - 4. The first categories in the four tables were first derived independently by the two investigators. After classification was attempted using each set consensus was reached on final categories. These categories though grounded in the data, represent a close match with previous theorizing on partitioning (Kieren, 1980). Using derived categories there was a .97 coefficient of agreement between the investigators on the independent classification of individual student task protocals.

Table 1

Percentages of Correct and Incorrect Responses to 2/3 Pizza Problem - Not Partitioned

Categories with prototypical responses	Grade 6 * Male C	Grade 6 * Male I	Grade 6 * Female C	Grade 6 * Female I	Grade 7 ** Male C	Grade 7 ** Male I	Grade 7 ** Female C	Grade 7 ** Female I	Grade 8 ** Male C	Grade 8 ** Male I	Grade 8 ** Female C	Grade 8 ** Female I	TOTAL Male C	TOTAL Male I	TOTAL Female C	TOTAL Female I
No Response	-	12.5	-	-	-	10	-	10	-	10	-	-	-	7	-	4
Halving and Quartering	-	25	-	50	-	20	-	10	-	-	-	-	-	18	-	12.5
Diametric Sixths	12.5	-	-	-	-	-	-	-	20	-	20	-	11	-	C	-
Sequential (Egyptian)	40.4	12.5	-	-	-	10	10	-	-	-	-	-	-	7	4	-
Chords (Evenness)	-	-	-	25	10	-	-	-	-	10	-		3.5	3.5	-	-
2 Radii	12.5	-	25	-	10	-	10	-	-		-	-	7	-	8	-
3 Radii	25	-	-	-	-	-	50	10	20	-	60	-	25	-	45	4
Mathematical	-	-	-	-	10	-	-	-	40	-	20	-	10	-	8	-
TOTALS	50	50	25	75	60	40	70	30	80	20	100	0	65	35	75	25

* Grade 6: 8 Male 4 Female

** Grade 7 and 8: 10 Male 10 Female

Table 2

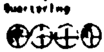
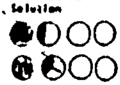
Percentages of Correct and Incorrect Responses to 3/4 Candy Bar Problem - Partitioned

Categories with prototypical responses	Grade 6 *				Grade 7 **				Grade 8 **				TOTAL			
	Male		Female		Male		Female		Male		Female		Male		Female	
	C	I	C	I	C	I	C	I	C	I	C	I	C	I	C	I
No Response	-	12.5	-	-	-	20	-	-	-	-	-	-	-	11	-	-
Halving and Quartering	-	37.5	-	-	-	20	-	10	-	-	-	-	-	18	-	4
Erroneous Interpretation of Problem	-	12.5	-	25	-	10	-	-	-	-	-	-	-	7	-	4
Sequential	-	-	-	-	-	-	16	-	-	-	-	-	-	-	4	-
Repeated Division	25	-	50	-	30	-	-	-	-	-	-	-	[illegible]	-	8	-
Single Unit	12.5	-	-	-	-	-	10	-	-	-	10	-	3.5	-	8	-
Mathematical 24 ÷ 4 / 3 ÷ 3 / 6 ÷ 8	- / - / -	- / - / -	25 / - / -	- / - / -	- / 10 / -	- / - / -	60 / - / 10	- / - / -	70 / 10 / 20	- / - / -	62 / 30 / 10	- / - / -	42.5	-	71	-
TOTALS	37.5	62.5	75	25	50	50	90 ***	10	100	0	100	0	64	36	91 ***	5

* M = 8, F = 4
** M = F = 10
*** Significant difference in percentage at .05 level

Table 3

Percentages of Correct and Incorrect Responses to 4/3 Pizza Problem - Not Partitioned

Categories with prototypical responses	Grade 5 * Male		Female		Grade 7 ** Male		Female		Grade 8 ** Male		Female		TOTAL Male		Female	
	C	I	C	I	C	I	C	I	C	I	C	I	C	I	C	I
No Response	-	25	-	-	-	20	-	10	-	-	-	-	-	14	-	4
Halving and Quartering 	-	25	-	-	-	-	-	10	-	-	-	20	-	7	-	12.5
Ignoring Remainder 	-	12.5	-	-	-	-	-	-	-	-	-	-	-	3.5	-	-
Sequential 	12.5	-	50	-	60	-	60	-	20	-	20	-	32	-	33	-
Repeated Division 	25	-	50	-	10	-	20	-	-	-	-	10	-	11	-	21
Mathematical Solution 	-	-	-	-	10	-	20	-	80	-	50	-	32	-	29	-
TOTAL	87.5	62.5	100	0	80	20	80	20	100	0	80	20	75	25	83	17

* M = 8 F = 4

** M = 10 F = 1[illegible]

Table 4

Percentages of Correct and Incorrect Responses to 4/3 Chocolate bar Problem - Partitioned

Categories with prototypical responses	Grade 6 * Male C	Grade 6 * Male I	Grade 6 * Female C	Grade 6 * Female I	Grade 7 ** Male C	Grade 7 ** Male I	Grade 7 ** Female C	Grade 7 ** Female I	Grade 8 ** Male C	Grade 8 ** Male I	Grade 8 ** Female C	Grade 8 ** Female I	TOTAL Male C	TOTAL Male I	TOTAL Female C	TOTAL Female I
No Response	-	12.5	-	25	-	30	-	20	-	20	-	-	-	21.5	-	12.5
Halving and Quartering	-	12.5	-	-	-	10	-	10	-	10	-	20	-	11	-	12.5
Inverse	-	37.5	-	25	-	-	-	10	-	-	-	-	-	11	-	8
Ignoring Remainder 1 4 3	-	-	-	-	-	-	-	20	-	-	-	-	-	-	-	8
Repeated Division	12.5	-	-	-	-	-	-	-	-	-	-	-	3.5	-	-	-
Sequential 1 + 1/4 + 1/3 (1/4)	-	-	-	-	30	-	30	-	40	-	30	10	18	-	25	4
[1/4 + 1/3(1/4)]4	-	-	25	-	-	-	-	-	-	-	-	-	-	-	4	-
1 + 1/4 + 1/4 (1/4)	-	12.5	-	25	-	10	-	-	-	10	-	-	-	11	-	4
1 + 1/4 + 1/2 (1/4)	-	12.5	-	-	-	10	-	10	-	-	-	10	-	7	-	8
Mathematical 1/3	-	-	-	-	10	-	-	-	20	-	-	10		-	-	4
1 4/12	-	-	-	-	-	-	-	-	10	-	-	-	3.5		-	-
5 1/3	-	-	-	-	-	-	-	-	-	10	10	10	-	3.5	4	4
TOTALS	12.5	87.5	25	75	40	60	30	70	50	50	40	60	36	64	33	67

* M = 8 F = 4
** M = 10 F = 10

Discussion

As one can see the response categories are similar across the four tasks. In fact the active categories could have been collapsed to three.

The first of these would entail a physical exploration with an evening out of a given number of parts used as a strategy. The categories "halving", "chords", "ignoring remainder" would fit here. The second collapsed category would involve behaviours which could be said to entail a physical algorithm and would include the "sequential", "repeated division", "2 radii", "3 radii" categories. The third general category would be mathematical. From a study of protocals this category collected procedures in which the subject appeared to solve the problem symbolically and then draw his or her representation. Such representations always entailed only the fewest physical moves.

The use of a more extensive category system was done to illustrate the wide variety of constructive partitioning behaviour used by these middle school pupils even though the problems should have been relatively simple for them mathematically; with the exception of tha "3/4" problem, over two thirds of this sample attempted to use a physical problem solving approach. These approaches usually showed an attention to the nature of the problem. For example, one grade 7 boy deliberately partitioned each of 4 pizzas in 3 different ways to ensure that a person's fair share of "four thirds" wasn't jeopardized by the systamatic mistake in cutting.

Pothier (1981) in a study of partitioning behaviour in children aged 5 - 8 suggests that non-unit fractional numbers arise from combinations of a unit fraction partition. A study of these categories shows that partitioning in non-unit situations goes beyond the simple repetition of simple partitions (e.g., repeated divisions). Further such acts are influenced by the partitioning situation - contrast the results for the two "4/3" problems.

Two general performance patterns can be observed in the data. With the exception the "4/3 chocolate bar" problem, there is a dramatic improvement in performance with age. The largest change in correct performance occurs from grada 6 to grade 7. Yet in terms of categories the grade 7 subjects differed from the grade 8 subjects. The grade 7 subjects appeared successful through the use of physical algorithms wheras the grade 8 success was based on mathematical solution.

The impact of the physical situation is seen as contrasting performances on the two pre-partitioned problems. In the case of the "3/4" problem the partition fit the solution. In this case many students considered the unit to be the "piece of the bar" and gave the mathematical solution $24 \div 4 = 6$ pieces. This problem and strategy appeared easiest for the subjects, particularly the grade 7 girls who achieved significantly better than their male counterparts. (This was the only significant sex difference in the grade level totals although there are some interesting patterns of category differences.)

In relating division and rational number, the choice of partitionable unit is important. In describing the amount of the fair share students used either a discrete - "6 pieces" or fractional - "3/4 bar" - unit. In this case of the "3/4 chocolate bar", 44 percent used a discrete unit. As noted above many students solved this problem as a whole number division problem. Although the other chocolate bar setting had "pieces" only 22 percent used a discrete unit here and less than 11 percent used such a unit in the other two settings. Thus students seemed to realize the fractional nature of these particular problems as reflected in their unit choice.

The uses of equivalence mechanisms are more difficult to observe in these data. Although this hypothesis would bear testing in careful clinical interviews, it would appear that students using physical algorithms also mainly use quantitative rather than more formal equivalence thinking. This is particularly evident in the sequential category in Tables 3 and 4. There was also evidence of known pair equivalance ($1/3 = 2/6$, $3/4 = 6/8$, $4/12 = 1/3$) in many mathematical category performances and a few physical algorithms (e.g., diameter sixths in Table 1) performances.

Conclusion

The data summarized in the tables above illustrated that students can and do use a variety of informal thinking methods in solving problems to which they relate. Students, particularly those in Grade 7, used physical algorithmic partitioning to successfully solve rational number problems. Such partitioning activity related to their perceptions of unit - discrete or partitionable - and seemed to entail a quantitative notion of equivalence. It would appear that these mechanisms can be observed in children's behaviour, might be useful in a description of how a person builds up rational number constructs, and might be more deliberately considered in rational number curriculums even for students in upper middle school grades. Rational number knowledge built under such circumstances might better be about quantitative situations, and not simply about symbolic manipulation.

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PROCESS PATTERNS IN THE SOLUTION OF VISUAL AND NON-VISUAL PROBLEMS

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The protocols and problems from four studies of problem solving in grades 6 through 9 were analyzed. Initially problems were categorized as visual or non-visual using Krutetskii's definitions. Solvers were classified as visual or non-visual according to their reliance on use of drawings and diagrams when solving all problems. Analysis of the process-sequence data indicated that non-visual problems were characterized by less overt representation by non-visual subjects. Subjects persisted and were more likely to succeed in their approaches when the problem type matched their strategy preference.

Krutetskii (1976) assembled four different series of non-routine problems to investigate what he defined as two types of mathematical abilities, namely the visual-pictorial and the verbal-logical components of thinking. One of these series was composed of problems which represented different levels of "visuality," that is problems which, to a greater or lesser degree, suggested a graphic or visual representation and problems which, according to Krutetskii, could be solved with more or less ease by use of a figural or pictorial approach. This investigation followed Krutetskii's theme as it questioned the interaction of specific subject (preference for figural versus analytic strategies) and task variables (visual or non-visual problems).

In particular, the problems used in four different problem solving studies were categorized as being visual or non-visual. These studies all used the "think aloud" methodology to elucidate the processes or strategies employed by students as they solved non-routine mathematical word problems. Visual and non-visual

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solvers were identified. Following coding of the protocols, the process-sequence data were analyzed to determine (1) the ease of representation of visual and non-visual problems by all subjects, (2) the persistence of use of the visual or non-visual solution approach across all types of problems, and (3) the level of success of both the visual and the non-visual solvers when attempting problems representing the different levels of visuality.

It has been established that spatial ability is a good predictor of problem-solving performance (Werdelin, 1958; Wilson & Begle, 1972), especially if the problems are "visual" in Krutetskii's sense (Moses, 1978). Information processing theorists view problem solving as a process of generating successively more elaborate relational networks among the initial problem components and among additional relations and components generated in the process of achieving the problem goal. The network of relations which a problem solver has generated at a given point in the solution process is called the representation of the problem at that point (Newell & Simon, 1972). Research concerning spatial ability has found that subjects with high spatial ability tend to perceive problems holistically, as a gestalt, while subjects with low spatial ability tend to have a more analytic, piecemeal perception of problems, focusing on details (Smith, 1964; Werdelin, 1958). Thus an individual having high spatial ability should be more capable of generating relations and, from the information processing perspective, might be expected to be a more successful problem solver.

Greeno (1977) proposes three criteria for evaluating the "goodness" of understanding in a problem solving situation. Good understanding involves achievement of a <u>coherent</u> representation; representations can vary in the degree to which their components are related in a compact structure. Note that an individual with high spatial ability would be expected to exhibit a high degree of coherence in problem representations. Good understanding requires a close <u>correspondence</u> between the solver's representation and the problem itself; some representations may lead to rather artificial solutions. Good understanding also involves <u>connectedness</u>, the extent to which a solver's representation is related to previous knowledge.

The criterion of special interest here is correspondence. According to Greeno's theory, if a subject is presented a "visual" problem, there will be a better understanding of the problem if the internal representation is spatial rather than numerical or verbal-symbolic (Greeno, 1977, p. 76). In this study it was assumed that such a spatial internal representation would manifest itself through the use of diagrams or figures. Thus the focus of this study was to determine if there was evidence to support the hypothesis of a correspondence between problem type (visual or non-visual) and solution processes (as indicators of internal representation).

METHOD

Data Source. One-hundred and eighty young adolescents (grades 6 through 9) were individually interviewed by one of four interviewers (Days, 1978; Hutcherson, 1976; Kilpatrick, 1968; Konsin, 1981) as they attempted to solve a set of non-routine mathematics problems. The subjects represented a variety of demographic characteristics from four different geographic locations (Indiana, Wisconsin, California and Texas). The interviewers were all educators who were knowledgeable in mathematics and the psychology of mathematical reasoning. These interviews were recorded to produce a set of taped procotols which were forwarded to the Mathematical Problem Solving Processes Project. The protocols were then coded by one of the four investigators using a modification of the process-sequence coding scheme developed by the Smith (1977) group and described in Lucas et al. (1979).

The process data codes were initially analyzed to determine visual (those who consistently relied on drawings or diagrams as their solution approach across all types of problems) and non-visual solvers (those who consistently avoided use of figures and relied upon logical reasoning). Following a ranking, the 27% rule (McCabe, 1980) was followed to identify 48 visual and 48 non-visual solvers from the original population of 180 subjects. The visual solvers used figures or diagrams to solve more than a third of the problems they were presented. The non-visual solvers did not use pictorial representations for any of their problems.

Procedure. The method used to select problems for this analysis was based on Krutetskii's classification according to the degree of visuality in solutions (Krutetskii, 1976). Krutetskii used a series of both geometry and arithmetic problems. The arithmetic problems were divided into three types: visual, average, and mental. Opinions of teachers, as well as consideration of students' solutions, were used by Krutetskii to classify a problem. The visual (V) category contained those problems for which the easiest method of solution was to use a figure or diagram. These problems could also be solved by verbal reasoning, but it was considered to be more difficult to use that method. The problems in the average (A) group were approximately equally easy to solve using either visual or verbal-logical methods. The problems in the mental (M) category were those which did not require visual concepts. These were most easily solved by verbal-logical reasoning.

Some of the problems available on the tapes were identical to the problems in Krutetskii's series. Others had slight changes in syntax or context. The remainder of the problems in this study did not occur in Krutetskii's problem series XXIII, but his definitions were used to classify them initially. Sixteen problems were selected for analysis. For each of these problems, an analysis of the process-sequence codes for all the subjects who attempted to solve the problem was performed to determine the proportion of subjects who used a figure or diagram in their solution. The 16 problems were then ranked according to the proportions. Problems which ranked in the upper 27% were classified as visual problems. Problems ranked in the lower 27% were classified as non-visual problems. Thus four visual and four non visual problems constituted the problem series for the remainder of the analysis.

Examples of visual and non-visual problems are given below:

Visual problem: An airline passenger fell asleep when he had traveled half way. When he awoke the distance remaining to go was half the distance he had traveled while asleep. For what part of the way did he sleep?

Non-visual problem: A can of gasoline weighs 8 pounds. Half the gasoline is poured out of it. The can of gasoline now weighs 4.5 pounds. How much does the can weigh without the gasoline?

The results of this initial analysis suggest that there are some problems for which a relatively high percentage of subjects tend to draw a figure and some problems for which few, if any, subjects draw figures. However, for a number of Krutetskii's problems, evaluation of the solution processes utilized by the subjects in this study did not confirm his classification scheme (see Table 1).

Table 1

Visual	A-1, V1, M1-2, M2-1
Non-Visual	A2-1, A2-2
Not Classified	M2-2, V2, M1-1

Frequency distributions of the process codes for each subject and problem group were performed at each problem solution step. Also, for each process, the mean number of occurences in the coded sequence string was computed.

RESULTS AND CONCLUSIONS

The analyses of process codes support the hypothesis that visual and non-visual problem solvers perform differently. These differences are enhanced on problems classified as either visual or non-visual. Table 2 shows the proportion correct for each group on each problem type. The results clearly indicate that subjects' solution approach matched to problem type produces the best performance.

Table 2

Proportion Correct

		Subjects V	Subjects N-V
Problem	V	.54	.36
	N-V	.23	.46

When strategy and problem type are mis-matched, subjects are more likely to abandon the problem (see Table 3). Subjects who rely on diagrams appear to

Table 3

Proportion Abandoning Problem (Strategy)

		Subjects V	Subjects N-V
Problem	V	.04 (.35)	.13 (.07)
	N-V	.15 (.23)	.00 (.15)

be more likely to change their strategy, especially when solving visual problems. Perhaps the diagrams do not offer sufficient support to finish the problems, making a new approach necessary. At the same time, the external representation makes it possible for the subject to leave the problem without loss of the understanding of problem relationships. The diagram can thus serve as a device which supports flexibility in the solution approach.

The most important function of a diagram is to aid in understanding and representing the problem. Table 4 presents the sequence of solution steps and the use of the process that was coded as "separates", that is, the recognition of problem elements and relationships. It appears that on non-visual problems, the subjects who are less dependent on diagrams exhibit less overt analysis of the problem. They may understand the problem more completely during the

Table 4

Percent of Subject Using "Separates" at Each Step

Subjects	Problem	Solution Step 2	3	4	5	6	7	8	9	10
Visual	V	40	19	12	10	10	12	4	8	12
	N-V	31	31	0	8	23	8	0	8	8
Non-Visual	V	39	13	7	7	7	13	7	7	0
	N-V	21	6	8	0	8	6	10	0	2

initial reading than do the visual solvers. This difference does not appear on visual problems, indicating that the internal representation of these problems may be more difficult. The lack of a difference on problem types for visual solvers provides evidence that any internal visual representation that takes place during the initial reading does not reduce the amount of processing necessary later.

It should be noted that these results do not attempt to categorize or characterize subjects according to visual imagery or spatial ability. Subjects may draw diagrams because they cannot visualize (Moses, 1978) or because they do visualize and wish to represent these images externally. These data are being analyzed further to provide other insights and explanations for the differential success in problem understanding and representation.

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CHILDREN'S CONSTRUCTION OF BI-DIRECTIONAL EQUALITY A TEACHING EXPERIMENT

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This paper describes a teaching experiment with second-grade children designed to provoke a reconstruction of their uni-directional view of equality in the context of equality sentences at higher levels of mathematical relations. Piaget's developmental-constructivist framework served as the basis for designing the intervention and the selection of children. This study sheds light on the process of reconstruction as well as on a method of interacting with children that facilitates this process. Although novel notions of bi-directional equality were demonstrated by the children in different situations during the 12-day intervention, total integration of the concept was not achieved.

INTRODUCTION

Although mathematicians make no distinction between left and right sides of equality sentences, elementary school children do make such distinctions. Whereas mathematicians view equality sentences as equivalence relations having reflexive, symmetric and transitive properties, children read their own meanings into the symbols. Interviews by Behr, Nichols and Erlwanger (1976) and by Van de Walle (1980) reveal that elementary school children read narrow meanings of equality into equality sentences. The researchers also report the children's resistance to those relations that mathematicians find "implicit" in equality sentences. Most children reject equality sentences written in the following forms, 8 = 5 + 3, 5 + 3 = 6 + 2 or 8 = 8, correcting them to conform to a rigid left-to-right format.

Equality Sentence	Child's Reaction	Resistence to Equivalence Relations
8 = 5 + 3	Wrong or backwards/ Corrects as 5 + 3 = 8	Doesn't grasp the symmetric property of equivalence.
5 + 3 = 6 + 2	Wrong/ Corrects as 5 + 3 = 8 and 6 + 2 = 8	Resists the transitivity property of the equivalence relation.
4 = 4	Wrong /Corrects as 4 = 4 + 0 or 4 + 4 = 8	Unable to ascribe meaning to 4 = 4 in terms of the numbers being the same.

Also, instead of talking about the "=" sign in terms of "is the same as" or "is another name for," most children refer to it only as "equals." They explain the sign as a "do something signal" for finding the answer or separating it from the problem. Whereas mathematicians view the equality sentence as a representation of static relations of equivalence, elementary school children tend to view it as a representation of action on objects. An arrow pointed to the right (→) would be more representative of children's understanding of equality than the mathematicians' equality sign (Nichols, 1976, Denmark, 1976). Not only does the concept of bi-directional equality appear to be elusive for elementary school children but for high schoolers as well (Kieran, 1980).

Despite the resistance of elementary school children to novel formats of equality sentences when presented in writing, their responses to oral statements of equality are more flexible (Behr, et al., 1976). Many children will accept statements such as, "Eight is the same as five plus three," as being correct. Thus, in many children's minds there exist opposing views of equality of which they seem unaware.

Children's inability to interpret "=" as a bridge between two quantitatively equivalent expressions may be the direct result of their instruction. A survey of primary school textbooks in mathematics by Denmark (1976) indicated a lack of systematic instruction in equality as a relation. Ginsburg (1977) points out that children's unidirectional view of equality is consistent with the single model presented in the textbooks. Van de Walle (1980) attributes children's narrow view of equality to the format of workbook presentations of exercises as tasks for which answers are to be determined, e.g., 5 + 3 = ____. Therefore, the limited treatment of equality in most elementary textbooks may be the cause of the limited view of equality held by children throughout the grades.

The preceding hypothesis was tested by Denmark (1976) in a teaching experiment with first graders having no prior instruction in equality sentences. For several months, children were exposed to a variety of sentence forms which could be modeled with objects on a balance. At the conclusion of the experiment, the children demonstrated greater flexibility in reading and accepting different sentence forms. Yet, nearly all of them viewed equality as uni-directional, regardless of the written form. The equals sign still indicated the location of the answer, either on the right or left side of the equality sentence. Denmark concluded that the first-grade children's limited view of equality relations was not explained solely by instruction and that their ability and intellectual development were also contributing factors.

Piaget's developmental constructivist position predicts a uni-directional view of equality for preoperational children in terms of irreversibility in their thinking about the part-part-whole relation. On the other hand, it suggests that the emerging logical operations (reversibility, compensation, identity and transitivity) in the concrete operational stage of children's intellectual development are capable of supporting the construction of bi-directional equality. These assumptions[9] were applied in the selection of children for the teaching experiment.

Another assumption of Piaget's developmental-constructivist position is that children construct their own mathematical relations through interaction of existing conceptual structures and their environment. Learning experiments designed within Piaget's equilibration model (Inhelder, Sinclair & Bovet, 1974) have demonstrated how children's reconstructions at higher levels of understanding can be provoked by intensifying this interaction. The teaching experiment, described here, was designed according to Genevan guidelines.

PROCEDURE

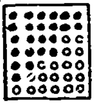
Preliminary Interviews with Children. Since children's new constructions will integrate existing ones, it is essential to first have an intimate knowledge of children's views of equality. A series of three clinical interviews with thirty second-grade children over a four-month period produced a pool of information on children's responses to a variety of tasks pertaining to equality.

Selection of Though-provoking Tasks. Tasks likely to provoke children to rethink and to integrate their existing notions towards bi-directionality were identified. Different combinations of these tasks were tested with the above children in the third interview. Combinations of tasks that resulted in either disequilibrium or partial reconstruction for some children were retained for the teaching experiment. Examples of such tasks are described below.

Task	Rationale for Selection

Representing Partitioning Patterns

The children observe as rows of six Othello pieces are partitioned by flipping increasing numbers of pieces and exposing another color. They are asked to predict "names for six" for subsequent rows. The children write equality sentences showing all the names for six. "If you know that every row has six, can you start your number sentences with six?"



This situation provides a phys- context in which children will be less resistant to writing a single numeral on the left side of the equality sentence.

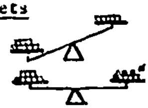
"Read your number sentence without saying the word 'equals.' Make up your own words in its place."

"How come you put a plus sign in all of your number sentences? Did you really put more pieces on the board?"

Children are provoked to rethink of addition and subtraction in a static relationship and to invent synonyms for "equals" in this context.

Representing Equalization of Sets

"How can you make the balance even with the table top? What do you notice about the blocks when the balance is even?"



"Write a number sentence about all that you did. . . I notice that you wrote different sentences. Are they both just as good or does one of them fit better?"

In presenting the problem, the position of the larger set is varied to provoke the child to consider the possibility of matching the order of the symbolic record with the physical context and writing $8 = 5 + 3$. If equalized by taking away blocks, the child could write $5 = 8 - 3$.

Representing Number Story Problems

Children are asked to match different story problems with different physical models for the same equality sentence. The situations modeled are, separating or joining, part-part-whole, equalizing and comparison. (Carpenter et al, 1981)

Children construct physical models for orally-presented story problems and represent them with equality sentences.

In dealing with different classes of problems, the children are provoked to rethink and expand their view of equality to incorporate static relations.

By exposing children to a range of physical tasks, a conflict between opposing interpretations is initiated which may lead to a successful reorganization of ideas.

Pre-assessment. Six end-of-the-year second graders in a textbook-oriented math program were given a pre-assessment of understanding of equality concepts and Piagetian tasks in a series of two clinical interviews. Four of these children were selected for participation in the teaching experiment based on their unidirectional view of equality and their demonstration of at least two logical operations on Piagetian tasks. Although these children lacked sophisticated notions of equality, they were judged to be developmentally close enough to consider the tasks selected for the intervention.

Intervention. During each session an attempt was made to provide optimal possibilities for interaction as the children tested their existing notions of equality in a range of novel physical contexts and were confronted with feedback from a variety of sources. Thus testing, discussing, and evaluating ideas was the setting for reconstruction of equality concepts at higher levels of mathematical relations. The children were encouraged to make their own generalizations following experiences that had the potential for provoking them to do so. The intensive interactions were videotaped and analyzed prior to planning for the next session. Twelve sessions of 45 minutes duration were scheduled over a three-week period.

The role of the investigator included an expansion of the sensitive clinical interviewer's role described by Opper (1977). Further interaction was facilitated by juxtaposing selected children's responses to sharpen the focus on any contradictions and inviting their reaction. These gentle confrontations focused on either conflicting notions expressed by the same child in different contexts, conflicting notions held by other children in the same group, or, conflicting notions held by a hypothetical child of the same age. The latter notion was introduced by the investigator as a counter suggestion. A delicate aspect of this role is to avoid imposing adult authority into the children's discussions, while maintaining a friendly, neutral manner during the intense interactions. The investigator attempted to accept each child's response without any judgment of his own. A further attempt was made to consider each response carefully as a potential indicator of the child's current level of thinking and as a source of hunches for the next question to ask or the next situation to devise.

Delayed Post-assessment A clinical interview, conducted one month following the intervention, employed both familiar and novel tasks to assess the depth and stability of the children's understanding of bi-directional equality.

RESULTS

In the course of the intervention, the children not only demonstrated novel, more sophisticated responses, but also vacillated between sophisticated and unsophisticated responses in different contexts prior to integrating their view of equality. Piaget identified both vacillations and partial constructions as essential, intermediate levels in the construction process. One of the partial constructions demonstrated by the children was the matching of an equality sentence to the order of a specific physical context.

Level	Interpretation
1	Uni directional, context independent. There is only one way to write a number sentence (left to right) regardless of the physical context it represents. (5 + 3 = 8)
2	"Bi-directional," context dependent There is more than one way to write a number sentence, but it is the physical context that determines which way it needs to be written, i.e., the sentence must match the order of the physical context. (8 = 5 + 3)
3	Bi-directional, context independent. (Generalizing beyond the context) There is more than one way to write a number sentence but the context is not important It is the relationship that counts regardless of the order of writing and physical context. The same relationship is expressed by 8 = 5 + 3 and 5 + 3 = 8.

The expression of all three of the above viewpoints within a group of four children led to some lively discussions and eventual reconstructions. Within Piaget's framework, both vacillations and partial constructions are indicators of progress towards higher levels of understanding.

Despite the precautions taken to reduce the influence of adult authority in the discussions and to redirect any differences to the authority of children's own logic, another source of authority interfered with children's progress. With the limited treatment of equality in the children's math textbook, one of the arguments presented against the acceptance of the novel equality sentence formats was, "It's not in the math book!" One child vacillated between rational justifications of the novel formats and concerns for the authority of the textbook. However, the sequence of activities and subsequence discussions provoked another child to confront the authority of the textbook. In a distinct demonstration of intellectual autonomy, she asserted, "It doesn't matter if it's in the math book or not, just as long as what you think if it's right."

The following indicators of bi-directionality were demonstrated within the group of four children, both during the course of the intervention and during the pre-assessment:

- acceptance, justification and continued usage of novel sentence formats,
- invention and continued usage of relational synonyms for "equals",
- generalization of relationships expressed in equality sentences beyond specific physical contexts.

Although demonstrating greater sophistication than the first graders in Denmark's study (1976), these second graders still thought of the equal sign as an indicator of the location of the answer, regardless of its location on the left or right side of the equality sentence.

Bi-directional equality is a multi-faceted concept whose complexity is masked in the economy of the mathematician's representation--the equality sentence. As pointed out by Fuson (1979), the same equality sentence can represent a range of mathematically- and situationally-different types of number-story problems. The interpretation of its elements (+, -, =) will vary with each one. This study shows that second-grade children demonstrating at least two logical operations can construct and grapple with notions of bi-directionality. Although they demonstrated considerable progress, they were unable to coordinate all aspects of the multi-faceted concept within the time constraints of the study.

> Understanding and meaningfulness are rarely if ever "all or none" insights in either the sense of being achieved instantaneously or in the sense of embracing the whole of a concept and its implications at one time (Jones, 1959, p 1).

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APPLIED MATHEMATICAL PROBLEM SOLVING PROCESSES NEEDED BY MIDDLE SCHOOL STUDENTS IN THE SOLUTION OF EVERYDAY PROBLEMS

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Our presentation will discuss several interrelated components of a current NSF-funded research project designed to identify and investigate processes needed by average ability middle school students when they attempt to use elementary mathematics concepts to solve problems in realistic everyday situations. Theoretical perspectives of the project, and past research which contributed to its design and implementation, have been summarized elsewhere (Lesh, Note 1, 1981). This paper briefly describes a portion of the project that will be taking place throughout the fall semester of 1981 (approximately sixteen weeks).

The major goal of the fall segment of the project is to conduct daily observations of approximately eighteen seventh graders as they work, individually or in groups of three, on problems designed by the project staff to elicit important processes, skills, and understandings that are needed in the solution of everyday mathematics problems. A series of "background measures" will be administered during the first two weeks of the fall semester. These measures will concern verbal ability, spatial ability, general intelligence, creativity, field dependence, cognitive restructuring capabilities, memory capabilities, reflectivity-impulsivity, attitudes towards mathematics and problem solving, anxiety about mathematics, and locus of control. They will also measure specific prerequisite understandings and procedural knowledge that underly the problems to be presented in later sessions throughout the fall. The goal is to develop a profile of each student that is as complete and comprehensive as possible within the time constraints of the project.

Some of the background measures will be given as pre- and post tests to help assess the learning effects of the problem solving experiences that the students will be provided Other measures, or the theoretical constructs underlying other measures, will furnish perspectives for interpreting the behavior of individual students in various problem situations At least two days per week the students will work in groups of three on problems designed to require approximately 30-60 minutes of activity One observer will be assigned to each group. Audio tapes will be made of all group problem solving sessions, and videotapes will be made of at least one third of the groups. These tapes will be condensed into written protocols accompanied by observer interpretations. Observational schemes and protocol analysis procedures have been developed during the first year of the project and will be discussed during our presentation.

One day per week will be devoted to worksheets, designed for individual students, focusing on particular problem types, processes, or student capabilities, or on particular stages in the problem solving process. Noncomputational and non-answer giving processes will be given special attention. All of the information used to interpret student performance on these worksheets will be derived from the responses that were given. No attempt will be made to monitor the unrecorded steps that were taken to produce recorded responses. Individual interviews will be required for process oriented testing sessions, which will occupy the remaining two days of each school week. Examples will be given during our presentation to illustrate some of the kinds of interviews we will be using.

Wherever possible, problems and test items were borrowed from existing materials, standardized tests, or past research or curriculum development projects. In many cases, however, new problems had to be created and piloted during the first year of the project.

The kinds of problems we have developed involve no more than elementary arithmetic, measurement, and geometry concepts; they are as much like "candid camera" situations as we were able to devise for use in classroom environment simulations. The problems involve family finances (e.g., balancing a checkbook, planning a vacation, starting a lawn mowing business, estimating the effect of inflation), measurement situations (e.g., estimating distances using different kinds of maps, purchasing enough wallpaper to cover a given room exactly), predicting trends from tables of data, and other problems that might reasonably occur in the everyday lives of youngsters and their families.

The solutions to the problems range in length from sixty seconds to as much as sixty minutes. Often a variety of different solutions are possible, varying in sophistication or complexity, and a variety of solution paths may be available, ranging from using concrete models to using abstract symbols. Special attention has been given to problems in which "non answer giving" stages of problem solving are important, e.g., question asking, problem refinement, modeling (i.e., simplifying the situation to fit available models or creating/modifying models to fit the situation, selecting appropriate representation systems, evaluation trial solutions, etc.).

Unlike the case with many typical word problems, <u>reading</u> the problem is not expected to be a source of difficulty. The questions emerge from concrete situations that are quite familiar to the students. The problem is to find a solution, not to interpret the meaning of the problem situation. Unlike many mathematical puzzles, no "clever mathematical tricks" are needed in the solution procedures. All of the problems involve straightforward uses of easy to identify ideas from arithmetic, measurement, or intuitive geometry. Further, a variety of "outside resources" will be available, including calculators, resource books, other students, and teacher/consultants who will supply facts and information upon request. So solution attempts will not be blocked by deficient technical skill (e.g., computation) or memory capabilities (e.g., measurement facts).

Whether we are observing individuals or groups, the goals are: (a) to describe the processes, skills, and understandings that are used to solve various problems, (b) to create problems that are realistic, mathematically rich, and psychologically interesting--and to describe the most important characteristics of each problem; (c) to obtain a profile of the individual and group characteristics, understandings, and capabilities that influence solution; and, (d) to observe the problem solving activities from a variety of perspectives, characterizing differences in performance across problems and across time The expected products include a classification scheme of problem types, behaviors, and individual and group characteristics, where we assume that one of the most important dimensions distinguishing organisms (either individuals or groups) concerns the "conceptual models" (see Lesh, 1981 for a definition and brief discussion of conceptual models) available to the organism.

Our work suggests that the kinds of "problems" that are included in most textbooks are quite unlike the major problem types students commonly encounter in everyday situations (Bell, Note 2; Lesh, 1981). And, the processes that average ability youngsters need in order to use mathematical ideas to solve realistic everyday problems are seldom represented among the priorities listed by spokespersons for either "basic skills" or "problem solving" (Lesh, 1981).

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THE ROLE OF METACOGNITIVE DECISIONS IN MATHEMATICAL PROBLEM-SOLVING

(OUTLINE OF THE SESSION)

Presenters: Richard Lesh, Northwestern University
Frank Lester, Indiana University
Edward Silver, San Diego State University

The recent research literature on memory and cognitive development has contained numerous references to terms such as metamemory, metamnemonics, metalinguistics, and metacognition. Even more recently, considerable interest in metamemory and metacognition has been generated within the mathematics education community. For example, at a conference on problem-solving held at Indiana University in the late Spring of 1981 one of five discussion groups devoted exclusive attention to metacognition.

Before proceeding to describe the nature of this session, a definition of metacognition is in order. This definition has been given by the noted developmental psychologist, John Flavell, who has been most prominent in making metacognition a legitimate area for research. While the definition lacks precision, it serves as a clear description of the types of processes involved in metacognitive behavior:

> "Metacognition" refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data. For example, I am engaging in metacognition. . . if I notice that I am having more trouble learning A than B; if it strikes me that I should double-check C before accepting it as fact; if it occurs to me that I had better scrutinize each and every alternative in any multiple-choice type task situation before deciding which is the best one; . . . if I sense that I had better make a note of D because I may forget it. . . . Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognition objects on which they bear, usually in the service of some concrete goal or objective (Flavell, 1976, p. 232).

Contemporary research in mathematical problem-solving is rife with studies the "tactical" behavior of problem-solvers and of the ability or inability of problem-solvers to use various heuristics. While many of these investigations have been carefully conceptualized and painstakingly conducted, they have focused on cognitive behavior only. Furthermore, the methodology most often employed, protocol analysis, deals with problem-solving behavior at a microscopic level only (cf. Schoenfeld, Note 1).

It is our bias that metacognitive processes constitute the essence of real problem-solving and their role can no longer be ignored. The primary purpose of this session is to initiate serious dialogue about the role of metacognition in problem-solving and about how a metacognitive dimension can be incorporated into future problem-solving research.

The session will be organized as follows:

I. Theoretical considerations about metacognition.
   A. What is its role in problem solving?
   B. How is it related to cognition?
   C. What is going on in other fields (e.g., developmental psychology, reading).
   (Lesh & Silver)

II. A tentative scheme for categorizing metacognitive behavior. (Lester)

III. Description of two studies currently underway.
   A. The Applied Problem-Solving Project (Lesh)
   B. Developmental changes in metacognitive behavior (Lester)

IV. Difficulties in studying metacognition (Silver)

V. Group discussion (Note: A large portion of the session will be devoted to discussion).

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THE EMERGENCE OF ALGORITHMIC PROBLEM SOLVING BEHAVIOR

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The primary focus of the research program of the Mathematics Work Group of the Wisconsin Research and Development Center is the study of the development of addition and subtraction concepts and skills in young children. The major vehicle for this investigation is a three-year longitudinal study begun in September 1978 with first grade children with an average age of 6 1/2 years. The final data collection point for the study took place in January 1981. A number of variables are under investigation including problem solving behaviors on a specific set of verbal problems, selected cognitive skills, performance on written arithmetic tasks, and the nature of classroom interactions observed in the classrooms of the subjects in question. Details of the study and some earlier results are contained in previous papers presented to the PME (Carpenter, 1980; Carpenter & Moser, 1979; Moser, 1980). In this paper only children's performance on problem solving tasks will be considered. A further restriction is the limitation to problems involving addition and subtraction of two-digit numbers.

BACKGROUND INFORMATION

Subjects. Subjects for the study consist of about 100 children from six classrooms in two elementary schools that all draw from predominantly white middle to upper-middle class neighborhoods. All received instruction from the Developing Mathematical Processes (DMP) program, an activity oriented instructional program developed at the University of Wisconsin. DMP has a strong emphasis on problem solving and during the time period reported here, subjects were instructed in the analysis and solution of verbal problems of the type used in this study.

Data to be reported were taken from four individually administered problem solving interviews that were given in January 1980, May 1980, September 1980, and January 1981. At the time of the first interview, all subjects were in the middle of second grade; thus, by the time of the final interview, all were in the middle of third grade. At the time of the first interview, no formal instruction in the use of computational algorithms had been given. Between the first and second interview, introduction to addition and subtraction without regrouping and addition with regrouping was taught. Summer holidays occurred between the second and third interview. Between the third and fourth interview, the regrouping algorithm for subtraction had been taught.

Problem solving interviews. Each interview includes six problem types, two with an additive structure and four with a subtractive structure. Representative problems and the order in which they are given to a child are presented in Table 1.

Each interview consisted of two parts, the first with the six problems containing two-digit numbers for which no regrouping (borrowing or carrying) is required to compute the answer [hereafter described as the "d" problems] and the second part with six problems containing two-digit numbers for which regrouping is required [hereafter described as the "e" problems]. Six different number triples were used for each part. They are listed in Table 2. The assignment of number triples to problem types involved a six-by-six Latin square design resulting in six sets of six problem tasks which were uniformly and randomly distributed across subjects. Problem wording was systematically changed, while retaining the essential semantic structure. The interviews were conducted in a quiet room separated from the child's actual classroom. The child was presented with paper and pencil, and a large set of plastic cubes. Problems were read to the children by the interviewer and repeated as necessary.

Table 1

Representative Addition and Subtraction Verbal Problems

1.	Joining (Addition)	Jacques had 12 pennies. His father gave him 15 more pennies. How many pennies did Jacques have altogether?
2.	Separating (Subtraction)	Marie had 29 candies. She gave 18 of them to Collette. How many candies did Marie have left?
3.	Part-Part-Whole (Subtraction)	There are 31 children in the classroom. Nineteen of them are girls and the rest are boys. How many boys are in the classroom?
4.	Part-Part-Whole (Addition)	Jean-Paul has 17 red marbles. He also has 19 blue marbles How many marbles does Jean-Paul have altogether?
5.	Comparison (Subtraction)	Chantal has 16 tickets. Her friend Michel has 29 tickets. How many more tickets does Michel have than Chantal?
6.	Joining, missing addend (Subtraction)	Diane has 23 strawberries. How many more strawberries does she have to put with them so she has 37 strawberries altogether?

Table 2

Number Triples Used in Verbal Problems

"d" Problems		"e" Problems	
12,15,17	12,16,28	12,19,31	13,18,31
11,18,29	13,16,29	14,18,32	16,17,33
14,21,35	14,23,37	15,19,34	17,19,36

RESULTS

One of the major questions of interest in this particular set of problem solving tasks was whether subjects would exhibit similar types of solution strategies as they had used with smaller number problems (sums between 5 and 16 and all addends being one-digit numbers). For those problems, a great deal of direct modeling and use of a variety of forward and backward counting techniques had been observed. Or would children resort to algorithmic behavior? A child was coded as using an algorithm if he/she gave direct written or verbal evidence [illegible] place-value consideration had been made and that computations were made [illegible]ately for the ones' and tens' places. We did not record how the actual [illegible]tation within a particular place was carried out. If, for example, the problem involved the sum $\frac{15}{+19}$, we did not attempt to determine how the child would get the sum of 5 + 9, either by a known or derived fact, or by some counting method. Table 3 presents the results for the four interviews for all six problem types and for both number sizes. Both the percentage of children who used an algorithmic behavior and the percentage of correct answers from among the algorithm users are given.

Table 3

Percentage of Children Using Algorithmic Behavior

Problem type	Interview 1 (Jan. 80) d	e	2 (May 80) d	e	3 (Sept. 80) d	e	4 (Jan. 81) d	e
1 Joining (Addition)	24 *(20)*	25 *(21)*	61 *(58)*	69 *(53)*	67 *(56)*	60 *(45)*	90 *(88)*	92 *(86)*
2 Separating (Subtraction)	19 *(17)*	14 *(3)*	65 *(51)*	58 *(2)*	58 *(54)*	40 *(3)*	87 *(85)*	88 *(89)*
3 Part-Part-Whole (Subtraction)	18 *(15)*	14 *(2)*	64 *(51)*	52 *(2)*	48 *(39)*	32 *(2)*	89 *(86)*	80 *(59)*
4 Part-Part-Whole (Addition)	32 *(31)*	24 *(19)*	70 *(65)*	72 *(60)*	66 *(62)*	61 *(44)*	92 *(87)*	95 *(85)*
5 Comparison (Subtraction)	16 *(14)*	14 *(2)*	50 *(38)*	45 *(3)*	41 *(35)*	27 *(3)*	78 *(73)*	81 *(65)*
6 Joining, missing addend (Subtraction)	18 *(17)*	10 *(2)*	39 *(27)*	35 *(3)*	28 *(23)*	26 *(3)*	59 *(54)*	54 *(40)*
Actual number of subjects	96		96		93		93	

(Numbers in parentheses represent percentage of total subjects who used algorithmic behavior who also solved the problem correctly.)

The immediate impression is that the increase in frequency and correctness of use of algorithmic behavior mirrors instruction in computational algorithms. Paper-and-pencil arithmetic skills tests administered independently of the problem solving interviews give exactly the same results in terms of ability to use a computational algorithm correctly. The great majority of errors made with the regrouping subtraction algorithm in the early stages prior to formal instruction with that algorithm were of the type well know to teachers, which is exemplified by $\begin{array}{r} 31 \\ -18 \\ \hline 27 \end{array}$, where in the ones' place the child follows the rule of "subtract the smaller from the larger" without any regard to the meaning of the entire number.

Of more interest than simple correctness is the different pattern of use of algorithmic thinking for problem 6, the Joining, missing addend task. A reasonable explanation for the much lower incidence of algorithmic solution is the semantic structure of the problem. Using the specific example given earlier, the wording strongly suggests that the best literal translation of that problem is the number sentence $23 + \square = 37$. However, of those children who elected to use symbolic representations almost all chose to use the vertical computational form rather than horizontal sentences. The vertical counterpart to the sentence written above is an awkward one, totally unfamiliar to children who had only seen the traditional form. It would appear that the children who realized this fact decided to not proceed in an algorithmic fashion, even though their behavior on other subtraction problems indicated that they could correctly use the subtraction algorithm. The most frequently used alternative strategy for this sixth task was Counting Up.

Another facet of the study was to investigate the relationship between the use of written symbolic representations and the use of algorithmic solution processes. If a sentence, either horizontal or vertical, was written, it was almost always the case that the sentence was written before the solution process was initiated. This is contrary to the results of an earlier pilot study (Carpenter, Moser, & Hiebert, 1981) where, when smaller numbers were involved, the children wrote the sentence after solving. In this latter study, however, the experimenter directed the child to write a sentence. In the present study, using written symbolism was at the discretion of the child. There was a very large number of children who did not write sentences, but still solved algorithmically. This was especially true with the "d" problems. In fact, at the time of the fourth interview, almost half of the subjects did not write a sentence for the addition problems without regrouping. The success rate for algorithmic students who did not write sentences was very high, due to the fact [illegible] these were probably the brighter students who are likely to solve correctly [illegible].

While much of the discussion has dealt with the use of algorithmic solving, it is appropriate to briefly characterize the behavior of those children who did not use algorithms. The results are essentially similar to those we have gathered using the same subjects, but with the smaller sized number problems. Problem structure appears to be the most powerful factor in determining the choice of strategy. Subtractive strategies seem to predominate for the Separating problem while additive strategies are most evident for the Joining, missing addend problem. Again, the only place where Matching appears is with the Comparison problem. As noted earlier, problem 6, the Joining, missing addend was the task that had the least number of algorithmic solutions. As a result, it was also the problem with the greatest frequency of so-called "Heuristic" (Carpenter, 1980) strategies. I take this as further evidence that children are capable of inventive behavior (Moser, 1980).

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LEVEL AND METALEVEL IN DEVELOPMENT AND THE PASSAGE FROM A STAGE TO ANOTHER

by Gerald Noelting

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The two-level theory of information-processing, (Sternberg, 1977, 1979) finds its counterpart, in problem-solving research, with the distinction betweed tactical and strategic decisions (Schoenfeld, 1981). A totally new perspective is thus brought to research on the working of intelligence, with emphasis put on "managerial" decisions as opposed to "episodes".

When examined in the perspective of cognitive development, which surely must interest math educators, various questions can be posed. How are "managerial" skills developed during growth? Do they appear at a certain age? Can they be found in young children? How do problem-solving strategies differ at any 4 years as opposed to 9 years, to 11, to 17?

Working out the steps in problem-solving at different ages for an overview of our research on the development of proportional reasoning (Noelting, 1980; in press), we were surprised to find that, at all the stages which had been identified, problem-solving consisted in an interaction between two levels of relations. The first level corresponds to the problem as seen by the subject, consisting in data that are encoded with their 1st-order relationship to one another. The second level corresponds to 2nd-order strategies put into use by the subject to sort out these relations and combine them, leading up to a solution. We have called the 1st-order, noted as R_1, the *factual* level. Here the data are encoded and directly related as objective <u>facts</u>. We have termed the 2nd-order level, R_2, the *inferential* level. It is here that the information brought to the subject's attention is processed through combination of relations, leading to a solution of the problem. Here one finds <u>composition</u> and <u>control</u>.

The interesting fact is that, when passing to a new stage, the *inferences* of the former stage become simple *facts*, which can be worked out inside a *new* system of inferences, consisting in novel combinations of relations, leading to a higher-order solution. Rather, one should say, it is when inferences are properly worked out, that they become part of the subject's mental set of schemes, procedures which are immediately put into action when similar data are found. Thus a new level of strategic processing can be set into motion, which can be called a new stage of information processing.

However this explanation is not sufficient to explain the passage from a stage to another; further mechanisms of differentiation and combination of two types of strategies intervene.

At each stage in the construction of *proportional reasoning*, in fact, we had found two types of strategies. The "external" strategies bear on elements of the same kind making up a <u>class</u>, the "internal" strategies bear on elements of different kinds making up a <u>relation</u>. A proportion such as $(a,b) = (c,d)$, where the ordered pairs are ratios, combine these two types of relations. In the experiment alluded to, ratios consisted in glasses of orange juice and water, which are mixed and compared, 1st term being orange juice and 2nd term water. Here *a:c* are "external" relations between numbers of glasses of orange juice, and *a:b*, "internal" relations between numbers of glasses of orange juice and water. Dual strategies are found at each stage as can be seen in the examples given.

When the subject masters the processing strategies of a stage, these are internalized as mere factual data. The external and internal relations can be differentiated and recombined, giving a new base level to work from. This process, given as a tentative explanation for the passage from a stage to another, finds some factual evidence.

Passage from stage 0 (Figures 1 and 2) to stage IA consists in a combination of *prototypical class* and *order of succession*, to yield *number* at stage IA, also found at stage IB (Figures 3 and 4).

Passage from stage IB to IIA (Figures 5 and 6) consists in the differentiation and ulterior combination of internal and external relations, one fixed as an <u>invariant</u>, the other mobilised as a <u>variant</u>, to yield the equivalence class of ratios or the transposition of relations.

Passage from stage IIA to stage IIIA (Figures 7 and 8) consists in differentiation and combination of an equivalence class and an "agreement and difference" scheme, to yield the common denominator algorithm or the unit-factor method.

Diagrammatic representation of information-processing at two levels: a "factual", 1st-order level and an "inferential", 2nd-order level, is a useful tool to show how intelligence proceeds at each stage of development. A complement would be to show that problem-solving consists also in working out relations *within* sub-wholes of a problem and *between* sub-wholes. This is offered as a new aspect of the model, describing processes involved in problem-solving in various fields.

STAGE 0: PREOPERATIONAL SYMBOLIC STAGE

a) Identification: construction of the prototype (class-type strategy)

Nicolas, 2;9	H9: (1,2) vs (2,1) Failure	"-Where will it taste stronger? -*This* (points at glass of orange juice in A and 2 glasses in B). *And this is water* (2 glasses in A) *and this is water* (glass in B)."

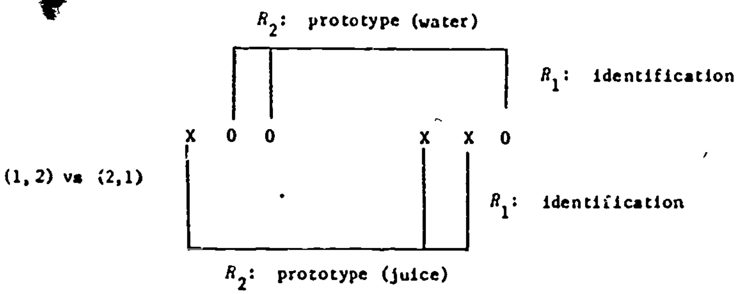


Figure 1. Problem-solving hierarchy at stage 0 (prototype)

R_1: identification

R_2: prototype with common name

X : juice, 0 : water

b) Localisation: construction of succession (relation-type strategy)

Jacques, 4 years	V: (3,1) vs (1,3) Failure	(Points successively at the glasses in B) "*There cherry. Water. This tastes water. This tastes water.*"

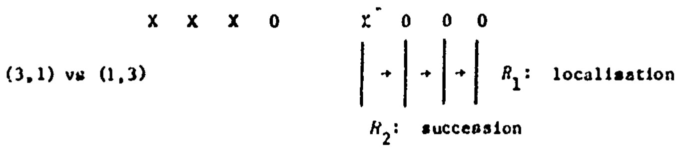


Figure 2. Problem-solving hierarchy at stage 0 (succession)

R_1: localisation of a locus

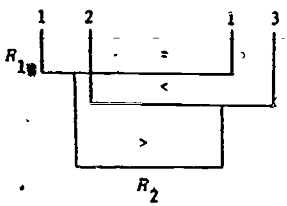
R_2: succession of loci

STAGE IB: PREOPERATIONAL INTUITIVE STAGE

a) External strategy

Stéphane, 5;0	B7: (1,2) vs (1,3) Chooses A (success)	*"There are 2 glasses of water, there* (A); *3 glasses of water there* (B). *There's 1 glass of juice there and there* (A and B)."

(1,2) vs (1,3)



Thus (1,2) > (1,3)

Figure 3. Problem-solving hierarchy at stage IB (external). Construction of the agreement and difference scheme.

R_{1e}: external agreement

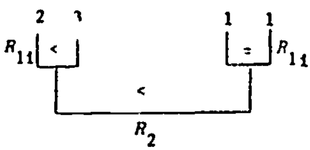
R_{1e}: external difference

R_2 : agreement-difference scheme

b) Internal strategy

François, 6;0	B8: (2,3) vs (1,1) Chooses B (success)	*"In* B, *there is 1 glass of juice, 1 glass of water. In* A, *there are 2 glasses of juice and 3 glasses of water. There is too much water taste in* A."

(2,3) vs (1,1)



Thus (1,1) > (2,3)

Figure 4. Problem-solving hierarchy at stage IB (internal). Internal compensation vs non-compensation.

R_{1i}: internal compensation

R_{1i}: internal non-compensation

R_2 : internal compensation scheme

STAGE IIA: CONCRETE OPERATIONS

a) External strategy: construction of the equivalence class.

Johanne, 11;0 | A7: (1,1) vs (2,2) Chooses E (success) | "Each glass dilutes 1 glass, so A has a glass of juice and B 2 glasses. A has a glass of water and B 2. They remain equal, only there is more liquid mixed together in B."

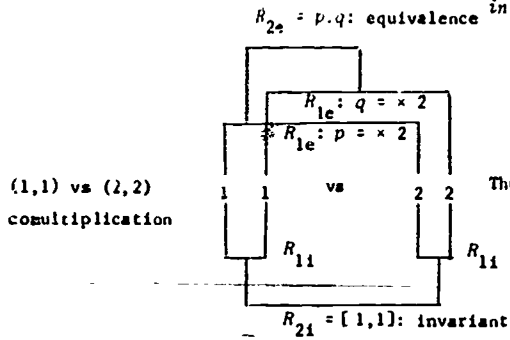


Figure 5. Problem-solving hierarchy at stage IIA (equivalence class) Relations are fixed, terms are comultiplied.

b) Internal strategy: transposition of relations

Anne-Marie, 9;0 | B11: (1,1) vs (2,2) Chooses E (success) | "One for one in A equals 2 for 2 in B."

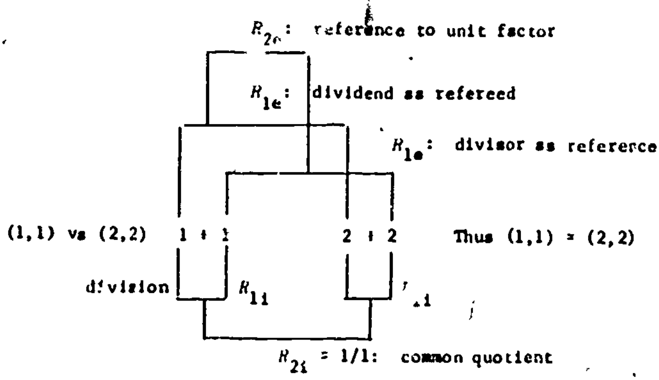


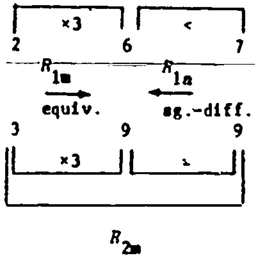
Figure 6. Problem-solving hierarchy at stage IIA (unit factor). Terms are fixed, relations become operations.

STAGE IIIA: EARLY FORMAL OPERATIONS

a) External strategy: equivalence with difference

Yves, 15;8	L8: (2,3) vs (7,9) Chooses B (success)	*"2 glasses of juice for 3 glasses of water (A), while its the same except there is 1 glass of juice more (B)."*

(2,3) vs (7,9)



Thus (7,9) > (2,3)

Multiplicity

Figure 7. Problem-solving hierarchy at stage IIA (common denominator). Combination of equivalence and agreement-difference.

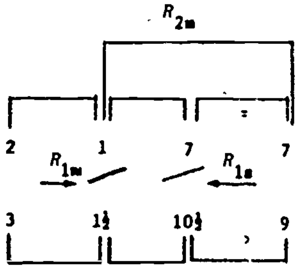
1m: multiplicative; 1a: additive (1st-order)

b) Internal strategy: unit-factor method

Anne, 16;3	L8: (2,3) vs (7,9) Chooses B (success)	*"Because there is 1½ glasses of water less."*

Transposition of relation

(2,3) vs (7,9)



Thus (7,9) > (2,3)

Figure 8. Problem-solving hierarchy at stage IIIA (unit-factor) Unit-factor reduction then transposition.

COGNITIVE FUNCTIONING AND PERFORMANCE ON ADDITION AND SUBTRACTION TASKS

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and

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In a project carried out in Tasmania in 1980, we were able to classify a population of children aged 4 to 8 years into groups according to their Cognitive Processing Capabilities (CPC)—this was done by giving two batteries of tests, one to measure M-space (Case, 1978) and one to measure Cognitive Developmental Level. The tests in both batteries were intended to bear implicitly on the early learning of mathematical material. The CPC measures and their descriptions were to be obtained by combining the information obtained from the two batteries of tests: Following this classificatory procedure, the relationship between each distinct group with particular CPC characteristics and various experiences incorporating mathematical content in the beginning school years was to be examined. A preliminary examination of some of this data was reported at Berkeley last year (Romberg & Collis, 1980a). Let us summarize the results of grouping according to CPC measures.

It was found that, using Factor Analytic techniques and a cluster analysis procedure (for details, see Romberg & Collis, 1980b, c), the population of 4 to 8 year olds could be assigned to six groups with the associated characteristics summarized in Table 1.

The tables following Table 1 indicate the direction in which the analyses are leading us with respect to the relationship between CPC level and the following three variables:

(i) achievement on elementary addition and subtraction problems,
(ii) strategies used by pupils on elementary addition and subtraction problems, and
(iii) pupil use of addition and subtraction algorithms.

(i) <u>Achievement</u>: Table 2 shows the percentage correct by CPC level on addition and subtraction problems using numbers up to 20 (see Carpenter &

Table 1

M-space Groupings with Associated Characteristics

Group	M-space Measure	Characteristic
1	1	elementary qualitative comparisons only, lack quantitative and logical ability
2	2	qualitative correspondence, lack specific quantitative and logical skills
3	2 *S+	high qualitative correspondence, have certain specific quantitative skills (i.e., counting for specific purposes), do not reach criterion on logical skills
4	3 *S-	high qualitative correspondence, high on quantitative skills, do not reach criterion on logical skills
5	3 *S+	ceiling on qualitative correspondence, high on quantitative skills, high on logical skills
6	4 *S-	

*S+ and S- represent the presence/absence respectively of a spatial ability as measured by one of the tests.

Table 2

Achievement on Tests C+ Tasmanian Data
(% Correct, Total Population)

CPC Level	% Correct Responses	No. of Trials Involved
1	22	180
2	65	450
3	81	39
4	83	264
5,6	96	252
Total	72	1542

Moser, 1979). The results are the combined scores for two tests, one in which physical material was available and one in which physical material was not available. The results show a significant increase in achievement by CPC level—the biggest gains being made between levels 1 and 2 and again between levels 2 and 3.

(ii) Strategies: Table 3 shows the percentage of the various kinds of strategies used by the same children on the same problems as were involved in the results in Table 2.

Table 3

Pupil Strategies on Tests C+ Tasmanian Data
(% of Times Strategy Used; Total Population)

CPC Level	Direct Modeling[1]	Counting	Routine Mental Operation[2]	Non-routine Mental Op.[3]	Inappropriate
1	28	0	1	0	70
2	36	18	13	6	27
3	18	33	26	10	13
4	11	30	35	14	9
5,6	13	40	42	6	0

[1]Using physical material, e.g., counters, fingers, etc.

[2]Using known number facts or relationships

[3]Innovative use of number facts or relationships

There are several interesting features of Table 3 which need careful consideration, three of which will be mentioned here. First, there is a very significant drop in the use of inappropriate strategies employed from levels 1 through to level 3; second, the use of direct modeling goes down as CPC level rises and the use of counting and routine mental operations rises; third, the reduction in use of inappropriate strategies is spread over the other categories, with counting and routine mental operations taking the largest and almost equal shares.

(iii) Pupil Use of Algorithms: Table 4 shows the percentage of third-grade children, by CPC level, who, having been taught the algorithms for addition and subtraction, actually used them to obtain the answer to a problem.

Table 4

Use of Known Algorithm to Solve Problem Tasks D and E
Tasmanian Data
(% Using Algorithm, Total Population of Subjects Who Had Learned Algorithm)

CPC Level	% Using Algorithm	% Using Inappropriate Strategy	% Using Counting
2	24	37	8
3	19	19	22
4	20	19	18
5,6	25	3	32

Children at all CPC levels use the taught algorithm infrequently, between one-fifth and one-fourth of the number of times when it is appropriate. They appear to prefer to fall back on more "primitive" strategies such as counting which they have used successfully previously. It can be seen that the data on these tests parallel those in Table 3 in that, with rise in CPC level, the use of inappropriate strategies decreases significantly at the same time as use of counting strategies increases. It is of interest to note that when the children cease to use inappropriate strategies they do not, in the main, turn to the algorithm which has been taught as the appropriate strategy. In fact, for this population, the use of the algorithm does not increase significantly with increasing CPC level. It is interesting to speculate on the reasons for this. Perhaps the emphasis on understanding the relationship between the algorithm and its application is misplaced at least at this early stage; perhaps we should treat problem solving strategies and algorithmic procedures as discrete entities, teach them separately and worry about bringing them together at a later stage in the child's mathematical development.

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HEIKR; HUMAN EXPERIENTIAL- & IMAGERY-KNOWLEDGE REPRESENTATION

Abraham Shammas

Abstract: Mathematics can be viewed as a language. But the existing theories of semantic and lexical memory, though each well established in terms of their supporting data, fall short of explaining the overall data at hand. HEIKR is a model of cognitive mind that successfully competes with many theories and rectifies their shortcomings. It turns out that HEIKR can be also used to clarify a series of other concepts in theories of mental ability and learning.

Keywords: Mathematics education, cognitive psychology, models of mind, Turing machines, artificial intelligence, semantic memory, concept formation.

#1. INTRODUCTION

Semantics can be defined as the study of the regularities of "meanings." But what is the nature of these meanings and how are they organized? Theories of semantic memory are mainly aimed to these questions. A brief review of these theories will be introduced in #2. To rectify some of the shortcomings of these theories, a new approach seems justified (Shammas, 1981). In #3 a short description of the new model, HEIKR, is presented. It turns out that, in addition, HEIKR can be a useful tool to clarify some notions of learning and abilities. #4 will conclude this report with short reflections on some of these concepts.

#2. REVIEW

Recently much has been said about the complexity of our mental abilities. The amount of knowledge (Anderson & Bower, 1973; Fahlman, 1979; Guilford, 1979), its diversity and effectiveness in facing the reality (Chadwick-Jones, 1975 a,b; Guilford, 1979; Newell & Simon, 1976), and its organizational and inferential significance (Anderson & Bower, 1973; Collins & Loftus, 1975; Conrad, 1975; Fahlman, 1979; Schank, 1977; Woods, 1975) have absorbed much attention. Factors like imagination (Paivio, 1974; Pressley, 1977; Pylyshyn, 1980; Warnock, 1978), associations and mechanism (Anderson & Bower, 1973), intentions and actions (Boden, 1979; McGinn, 1979; Pettit, 1979), perception (Hochberg, 1974; Gibson, 1974; Pick, 1974), and conception (Fahlman, 1979; Quillian, 1969) have found strong supporters and opponents (see reviews in Anderson and Bower, 1973; Gibson, 1969; and Lachman et al., 1979).

In cognitive science there seems to be little doubt about the existence of representations and memory. Several theories - in form of computer programs - try to specifically explicate the structure of semantic memory (Collins & Loftus, 1975; Fahlman, 1979; Quillian, 1969; Schank, 1977; Smith et al., 1974). According to TLC[1] (Quillian, 1969), the semantic memory is structured upon sub-/super-ordinate relation with strict economy. Later, when it was clear that some instances may be more typical than others (of a category), or some attributes closer (in terms of semantic distance) than others (to each object), two essentially different theories emerged as alternatives. According to TSA[2] (Collins & Loftus, 1975), semantic memory has internal structure similar to TLC, with less strict economy. FCM[3] (Smith et al., 1974) did not "impose" any structure on the store, but assumed specific mechanism for comparison of two concepts in terms of their defining features. CD[4] (Schank, 1977) rejected the idea that concepts constitute the semantic units. Rather, primitive ACTs, and consequently, conceptualizations are the essence of semantic networks. Several other specific theories of semantic memory have highlighted specific aspects of this limited store. But, altogether, the structure and the nature of semantic memory remains as mysterious as two decades ago (Lachman et al., 1979). The lexical store has had almost a similar story (Carrol & White, 1973; Lachman et al., 1974; Oldfield & Wingfield, 1965). To the author (Shammas, 1981), it seems that the overall data at hand do not justify presumptions about the structure of either store, or their total separation. A new approach is needed. HEIKR[5] (Shammas, 1981) is a candidate. According to HEIKR, some "creative" hardware-

1 - Teachable Language Comprehendar
2 - The Theory of Spreading Activation
3 - The Feature Comparison Model
4 - Concept Dependancy
5 - Human Experiential- & Imagery-Knowledge Representation

process (intentionalization) is needed to "abstract" concepts from the subjective experiences, and construct a phenomenal world (of attributes) with growing congruency with the external world (of properties). HEIKR has very few primitives with only one general memory. But still it seems to be powerful enough to compete with the specific theories of semantic and lexical memory. In addition, it seems that HEIKR may be used to clarify other concepts of mental ability, teaching, and development.

#3. HEIKR; THE MODEL

a - Structure

The only store presupposed in HEIKR is a 3-D array MxNxT, where

M is the order in which attributes are constructed at each level N

N is the complexity rank of the attribute, and

T is the chronological measure of time in steps of dt.

b - Processes

At each entry mxn there is an arc (attribute-arc) parallel to T starting at t=0 at an indicator which can reflect whether the arc is being activated or not. The set of all these indicators is called the attribute analyzer. Attribute arcs of the lowest degree are directly connected to the receptor cells (sensory arcs). At each moment t_0 the next available image-base (array MxN) is devoted to representation of the scene at t_0. This is done by linking each entry-register mxn (knowicle) to its corresponding attribute arc according to the momentary activation present in its arc (v=o, no activation; v=-1, negative activation; and v=+1 for positive). LTM is defined as the total set of the momentary representations.

Whenever a pattern of attributes is common in exactly K representations (e.g. K=80), the process of intentionalization finds N, the highest rank of complexity, in the attributes composing the pattern. Then the next available arc at level N+1 is devoted to the pattern (as pattern's intentional arc). At the same time, the next available image-base is devoted to the pattern as the intentional image of the K images sharing the pattern, and is positively linked to the new intentional arc.

In sum, HEIKR has 3 main parts (attribute-arcs, image-bases, and analyzers), and 4 primitive processes (activation, map, intentionalization, and selective attention).

#4. SOME IMPLICATIONS OF HEIKR

Ordinarily, HEIKR may act as a video camera. When your cat, Mutzi, is in the scene for j units of time (dt), the j consecutive snapshots of the scene will include his images. Their "quality" depends on HEIKR's previous knowledge: whenever a property, P, is incurred which has already been intentionalized, P's intentional-image may be activated (P is perceived), and its attributes will be copied on the snapshot along the incoming information. Thus, later images of Mutzi are likely to be "richer" in terms of higher-order links. At any rate, the process of intentionalization searches for "regularities" of patterns of attributes. If Mutzi's tail has been mapped K times in a particular angle, then it is intentionalized. Thus, not only the object is being abstracted, but so are its parts, functions, and features. Hence, HEIKR uses more or less abstract images (of Mutzi), but when needed, more concrete images of Mutzi, his parts, functions, and features are available to support inferences (e.g. 'Mutzi has ears.' can be supported by the link of his image to the image of Mutzi's ear)[1]. If Mutzi was shown to HEIKR always in a cage, then in HEIKR's mind, the cage will be an "attribute" of Mutzi. Although the cage may independantly be intentionalized, its "association" to Mutzi is very strong: the presence of one cause HEIKR to imagine the other. Here, then, the cage acts as a symbol for Mutzi. Note that the actual presence of both Mutzi and the cage is not necessary. To create temporal correspondence, it is enough that the experimenter (instructor) manages to have HEIKR imagine (indirectly activate) the image of the cage (by implimentation of selective attention and rehearsal). It is obvious that if the word 'Mutzi' is substituted in place of the cage, it would act as a name. We could continue to elaborate on this example and to see the relations that can develop between signs and meanings. Rather, it seems that examples could be used by the reader to verify the following:

1 - HEIKR may even visualize the shape, size, and functions of Mutzi's ears.

- Upon experience, recurrent enough properties of the scene are abstracted into attributes that "chunk" together several features.

- Upon activation of an aggregation of attribute arcs, all the images which share the pattern may be activated (memory; access by content self-addressing). If the aggregation is sensory due to presence of an object O, the direct activation of its images is perception.

- Activation of an image may spread to similar images (imagination). A family resemblance is enough to ensure spreading activation through a whole file of images (adjacent snapshots are locally similar).

- The recurrent intentionalized pattern might be a conventional regularity, rather than reflect domain organization. Thus HEIKR can develop symbols, language, and mathematical concepts. There is no need for structural transformations (surface-deep).

- In HEIKR relations between concepts may be analyzed either by comparing their attributes (intellectual analysis) on the attribute analyzer, or by comparing the images themselves (experiential analysis). This mental orientation should not be confused with the notions concrete-abstract.

- The closer the attributes to the sensory ones the more concrete the image. The attributes with higher-degree of complexity are more abstract in two senses: (1) the higher attributes "chunk" several details into fewer number of arcs and (2) each such attribute is more "remote" from sensory ones, detected by receptors.

- The intentionalization index K may indicate the "cast of mind" (Krutetskii, 1976). Smaller K in HEIKR would mean that the subject tends to generalizaitons and analytical oriantation. Larger K would mean that the subject tends to imagery and concrete analysis.

- An unexpected implication of HEIKR is the extention of the propositional theory "down" to the atomic propositions (Tarski, 1972). Now the truth value attached to primitive propositions is no longer arbitrary (Woods, 1975): "snow is white" for HEIKR, if the attribute "white" is among the features of "snow."

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YOUNG ADULTS' THINKING ABOUT RATIONAL NUMBERS

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ABSTRACT

Three written tests of fraction understanding and competence were administered to 220 young adults. In addition, 20 subjects participated in clinical interviews to probe their understandings and misunderstandings about rational numbers. Most of the subjects had a limited understanding of basic concepts associated with rational numbers. Their thinking tended to be characterized by representational rigidity - a reliance on a single model to interpret fractions. Furthermore, their algorithms for comparison and addition of fractions tended to be dissociated from the representations they used in interpreting fraction situations.

It is well-known that many persons reach adulthood with neither computational skill nor with conceptual understanding of rational numbers (cf., Carpenter, Coburn, Reys, & Wilson, 1978). The point of view that led to the development of this study was that a great deal could be learned by studying the knowledge possessed by young adults who had been exposed to typical school instruction in rational numbers. Two particular benefits to the Rational Numbers Project were foreseen. By identifying the understandings and misunderstandings of these individuals, useful insights could be gained that would guide the instructional development components of the Rational Number Project. Furthermore, the instructional routines developed in other components of the project could be used with subjects in this study to try and correct the misunderstandings.

In this paper, some of the data is reported on young adults' thinking about basic fraction representations. In order to give a reasonably complete picture of the data on which this report is based, the methodology will be discussed first in some detail.

METHOD

This study consisted of three parts: written testing, clinical interviews, and instructional intervention. Only the first two parts will be discussed in this paper.

<u>Subjects.</u> A total of 161 community college students, 76 men and 85 women, and 59 college students, 6 men and 53 women, volunteered to take the three written tests. The community college students had failed an Arithmetic Screening Test, consisting largely of whole number and rational number computation, and they were enrolled in five sections of a Basic Arithmetic class. The college students were elementary education majors enrolled in two sections of a required course on Number Systems. About 85% of the total sample was between the ages of 18 and 24 years old.

Twenty-nine of the community college subjects and 27 of the college subjects volunteered to participate in the rest of the study. Of those volunteers, 14 community college students and six college students were chosen for clinical interviews. The interview sample consisted of 4 students who had scored quite high in the written tests, 6 students whose scores were average, and 10 students who performed poorly on the tests. The "interesting" errors made by subjects on the written tests were well represented in the interview sample.

<u>Written Tests.</u> Three multiple-choice written tests were administered[2]:

1. Assessment of Rational Number Concepts (RNC) -- a 60-item test that focused on basic conceptual understanding of individual fractions and ratios,
2. Assessment of Rational Number Relationships (RNR) -- a 56-item test that focused on comparisons of fractions and ratios, formation of equivalent fractions and ratios, ordering of fractions, and proportions, and
3. Assessment of Rational Number Operations (RNO) -- a 35-item test that focused on addition, subtraction, and multiplication of fractions.

A more complete description of the written tests, including a discussion of the emphasis on translations between modes of representing rational numbers, is found in Lesh and Hamilton (Note 1).

Each test was administered to subjects in their classroom groups by their regular instructor. Due to scheduling differences between the college and community college, timing of the test administration was different for the two groups of subjects. Community college subjects took each test on a different day, and test days were separated by approximately one week. College subjects took the RNC and RNO tests on the first day of testing, then took the RNR test approsimately one week later.

Clinical Interviews. Individual tape-recorded interviews were conducted with each subject. A typical interview session lasted from 45 to 50 minutes; for some subjects a second interview session of about 20-25 minutes was necessary. The clinical interview consisted of a review of selected problems from the written tests and the completion of additional task. designed to probe a subject's rational number understanding. The total time for the interview varied from subject to subject depending on the time taken to complete the given tasks or to explain the basis for a response.

For each subject, problems from the written tests were selected that might illuminate the processes the subject used to arrive at an erroneous answer (or a correct answer, in some cases) or which might be used as probes of the depths of a subject's understanding. In the typical interview, a subject was presented with 6-8 problems from the written tests, asked to solve each problem and to explain the basis for the solution, and confronted with discrepancies between written test answers and interview solutions. Whenever a discrepancy occurred a subject was asked to resolve it. During the interview session, the interviewer remained non-directive until the subject arrived at a solution with which she or he felt comfortable, then the interviewer actively probed to determine the underlying basis for the subject's answers.

In addition to the problems from the written tests, each subject completed several additional tasks designed to probe the subject's understanding.

Some interview tasks were developed specifically for this study, and other tasks were chosen from an interview protocol developed in another component of the Rational Number Project (Landau, Hamilton, & Hoy, Note 2). In some of the tasks, subjects were instructed to close their eyes and to describe what they "see in their mind's eye" when the interviewer said the name of a fraction or a statement about fractions, such as "one-third plus one-sixth" or "which is larger, three-fifths or five-eighths?". Subjects were encouraged to describe the evoked image in as much detail as possible, using pictorial, physical, and/or verbal descriptions. Each subject was also asked, "Is a fraction a number?," and was asked to explain fully his or her response. Other tasks were chosen on the basis of being appropriate to probe the underlying conceptual basis of a subject's behavior.

RESULTS/DISCUSSION

Kuder-Richardson reliability estimates for the three tests were acceptably high. For the community college sample the reliabilities for RNC, RNR, and RNO were 0.96, 0.95, and 0.93; for the college sample, the reliabilities were 0.84, 0.98, and 0.87, respectively.

The general performance of the subjects on the written tests is summarized in Table 1. The college sample did considerably better than the community

TABLE 1

Mean Performance on Written Tests

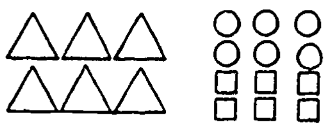
	RNC (Max = 60)	RNR (Max = 56)	RNO (Max = 35)
College (N = 59)	52.8 (5.4)	42.2 (15.5)	27.3 (5.6)
Community College (N = 161)	41.5 (14.1)	33.7 (13.2)	18.1 (9.0)

college sample, t-tests of differences in the mean performance were all significant ($p < .01$). Of course, since the community college students were enrolled in a Basic Arithmetic class, it is not surprising that they performed poorly on the written tests; nevertheless, it was heartening to find the relatively strong performance (e.g., 88% mean success rate on RNC) of the elementary education majors. Since most of the errors were made by community college subjects, error rates will be reported only for that population.

On the RNC Test, performance was better for fraction items than for ratio items. Ratio errors sometimes involved reversals (i.e., 12 to 1 instead of 1 to 12), but they often were associated with perceptual distractors that led to a fraction interpretation. For example, in a problem that presented a regular pentagonal region, half of which was shaded, and asked for the ratio of shaded to unshaded parts, about 54% of the community college subjects chose the answer "1 to 2." Interviews revealed that subjects looked at the picture and immediately "saw ½," thereby triggering the choice, despite their ability to answer some other ratio questions correctly.

The importance of perceptual distractors was also evident in subjects' responses to the following question:

39. What fraction of the set of objects are triangles?



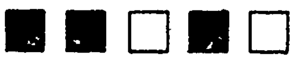
a. $\frac{6}{6}$ b. $\frac{1}{2}$ c. $\frac{1}{3}$ d. $\frac{2}{1}$ e. not given

Less than 40% of the community college sample correctly answered the question, and it apparently generated much confusion. About 20% of the subjects chose "½" as the correct answer. There were two apparent reasons, determined in the interviews, for this response. Some subjects apparently viewed the triangles as occupying half of a rectangular region that contained the entire pictorial display. Their thinking was obviously influenced by the region interpretation of fraction. Another, though less common, reason for the response "$\frac{1}{2}$" was based on a "part-part" ratio interpretation. Subjects apparently compared 6 triangles to 12 non-triangles, obtained $\frac{6}{12}$ and reduced to $\frac{1}{2}$. It is interesting to note that several of the 18% of subjects who chose the answer "not given," also applied the ratio interpretation to obtain $\frac{6}{12}$, but did not recognize or accept the equivalence of $\frac{1}{2}$ as an answer.

A "part-part" interpretation was also apparently associated with some of the 23% of subjects who chose the answer "$\frac{6}{6}$." Explanations for this answer are still being sought through analysis of the interview and written test responses. One subject reported that the circles and squares each took up as much space as 3 triangles; thus, the comparison of triangles to non-triangles was 6 to 6. In that analysis, one sees elements of both the region misinterpretation and the "part-part" ratio interpretation discussed above. For some subjects at least, this error represented the confluence of the misunderstanding of one interpretation and the misapplication of another.

The prevalence of "part-part" ratio interpretations in this question was striking, since there had been a very high success rate on earlier problems requiring a fraction interpretation of a set of discrete objects. For example, the following question was correctly answered by 93% of the community college sample:

3. What fraction of this picture is shaded?



a. $\frac{3}{2}$ b. $\frac{5}{2}$ c. $\frac{2}{5}$ d. $\frac{3}{5}$ e. not given

The only other problem that elicited a large number of "part-part" interpretations was one that presented a picture of eight balls - 3 footballs, 2 tennis balls, and 3 basketballs - and asked, "What fraction of the balls are tennis balls?". That problem generated far less confusion than problem 39, but 32% of the community college subjects chose the answer "$\frac{2}{6}$". Interviews strongly suggested that the basis for that response was a "part-part" interpretation.

Dees (1980) had reported a large percentage of her disadvantaged high school sample has inappropriately applied "part-part" ratio interpretations to fraction problems involving descrete objects, and it was not uncommon for such errors to be made on NAEP items (Carpenter, et al, 1978). Nevertheless, it was puzzling that so many students could be successful on items like problem 3, correctly applying a "part-whole" interpretation, but err on item 39 and the tennis ball problem, often applying a "part-part" interpretation. The interviews provided a plausible, though somewhat surprising, perceptual explanation for these findings. In problem 3, and other problems requiring a "part-whole" interpretation, some of the discrete objects were shaded! For many students, the presence of shaded parts apparently triggered an appropriate "part-whole" response - a direct analogue of the shaded portions of a geometric region. On the other hand, when no shading was present, as in question 39 and the tennis ball problem, the "part-part" ratio response was triggered.

To test this hypothesis, a variant of problem 3, in which the shaded squares were replaced by unshaded circles, was administered to interviewees. Each subject was asked, "What fraction of the set is squares?". Each interviewee who chose an answer of "$\frac{2}{3}$" had also missed the tennis balls problem but had correctly answered the original version of question 3. Thus, it would appear that many of the subjects answered questions about basic fraction and ratio concepts not on the basis of the question asked but simply on the basis of perceptual cues.

The tendency to respond on the basis of perceptual cues was not confined to interviewees who performed poorly on the written tests. Half of the interviewees with middle-range performance on the tests also exhibited the response pattern discussed above. Since there were only a few opportunities in the test for these errors to be made, overall performance was not unduly affected by the perceptually-cued errors.

Further information concerning subjects' thinking about basic fraction representations was obtained from the "imaging" tasks. When asked to close their eyes and think about the fraction "three-fourths," 15 of the 20 interviewees reported a "pie" or circle subdivided into four congruent parts, with three shaded parts. So dominant was this image of a fraction that 10 of these subjects were unable to report any secondary image when asked to "think about another way different from the way you first 'saw' it." As one of the subjects aptly put it: "I just keep seeing that pie in four pieces. I can't shake that picture."

When asked to report their image for statements of fraction equivalence and fraction addition, the circular region image was also the most frequently reported. Two aspects of the addition data are worth noting: (1) the nature of the circular region image for fraction addition and (2) the abandonment of the circular region image by some of the most capable interviewees.

The eight subjects who reported circular region images for fraction addition were almost uniform in their reported images. For example, for "$\frac{1}{3} + \frac{1}{6}$," these subjects reported two circular regions - one cut into three pieces, with one piece shaded, and the other cut into six pieces, with one piece shaded - and did <u>not</u> report any addition action until asked to do so. When asked to "tell what the answer is," seven of the subjects reported "two-ninths" on the basis of counting shaded and unshaded regions. It should be noted that four of these seven subjects were able to add "$\frac{1}{3} + \frac{1}{6}$" correctly when it was presented in that form. Only one of the subjects realized that "two-ninths" was not correct but did not know how to resolve the difficulty. Her comments are instructive: "I guess the pies just don't work for addition. They work o.k. in the beginning but not at the end." A more complete discussion of subject's responses to questions about fraction addition is given in Silver (Note 3).

It was interesting to observe that two of the most capable interviewees shifted from a circular region image, which they had reported for a fraction and fraction equivalence, to a "measuring cup" image. These subjects represented "$\frac{1}{3} + \frac{1}{6}$" dynamically - the second fractional quantity was added to the first by pouring the second amount into the cup containing the first. Although the subjects were unable to give a specific reason for their shift to the measuring cup image, they reported feeling more comfortable thinking about addition of fractions in that way. Another of the most capable subjects also reported the "measuring cup" image, but had done so consistently from the beginning.

Other information concerning subjects' thinking about rational numbers was obtained from tasks in which subjects were asked to explain (use of pictorial or physical models was encouraged) the basis for the symbolic algorithms they used to compare and to add fractions. The general finding was that subjects, except for the most capable, were unable to give any physical of pictorial description that corresponded to the algorithms they used. Even subjects who could reliably and consistently name the larger of two fractions or find their sum, and who used sensible and correct procedures to do so, were unable to relate those procedures to pictures or models of circular regions or any other model of fraction. A few subjects were able to give a weak explanation based on fraction representations, but most were not. On the other hand, the most capable subjects were often able to explain their procedures using several different fraction models or interpretations. The "explanations" that most subjects were able to give were simply verbalizations of the steps in the algorithm.

It would be hard to argue that the subjects in this study are generally representative of young adults. Nevertheless, their thinking about rational numbers is probably representative of a sizeable segment of the adult population. Despite my natural reluctance to overgeneralize, I can't resist the temptation to interpret the findings in a fairly general way.

All of the above data (and other data that are not reported here) seem to point to the conclusion that most young adults have a limited understanding of the concepts and procedures associated with rational numbers. Most of their thinking seems to be based on a single model of fraction - the circular region - or on no model at all. Very few of the young adults in this study were able to give plausible justifications for the procedures they used to compute with fractions. We have seen that the reliance on a single model led many subjects to make perceptually-cued errors in interpreting fraction and ratio situations. Furthermore, we have seen that most of the subjects were unable to connect this dominant model to the algorithmic procedures they use.

The instructional implications of these findings seem to be obvious. Unless instruction provides for both "internal and external connectedness" among fraction representations and procedures, it is unlikely that much general understanding can be attained. Students need an intensive exposure to a variety of models for interpreting rational numbers, so that students can choose flexibly among alternate models when interpreting a fraction situation. In addition to this "internal connectedness" among fraction representations, instruction needs to emphasize the basis for algorithmic procedures for comparing and combining fractions with respect to the models a student has studied.

NOTES

1 This paper was prepared with the support of NSF Grant No. SED79-20591. Any opinions, findings, or conclusions expressed herein are those of the author and do not necessarily reflect the views of the National Science Foundation.

2 I would like to thank Verna Adams, Jan Ford, and Sybil Rogert for their assistance in collecting the written test data.

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SYMBOLIC UNDERSTANDING

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Abstract

The nature of symbolic understanding is discussed, and the many powers which it confers are listed. An explanation is offered why many learners fail to achieve symbolic understanding as here defined, and four ways are suggested by which they may be helped.

In this paper I offer a further contribution to a series of discussions about the nature and varieties of mathematical understanding which has taken place over the past five years. (See bibliography.) By 1978 seven categories had been proposed, which I subsequently suggested (Skemp, 1979b) could be re-arranged into a table showing three kinds of understanding and two modes of mental activity. However, as I was aware at the time, my analysis of formal understanding was incomplete, since the words 'form' and 'formal' are used with two distinct meanings, of which I only dealt with the first. (i) There is 'form' as in 'formal proof'. This is the meaning used by Buxton (1978), and I have already suggested (op. cit.) that we distinguish this one by calling it 'logical understanding'. (ii) There is 'form' as used in statements such as 'This equation can be written in the form $y = mx + c$'. This is the meaning used by Backhouse (1978), and is also that in the first part of the definition given by Byers and Herscovics (1977): "Formal understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas ...". I now suggest that we distinguish this meaning by calling it

SYMBOLIC UNDERSTANDING.

'Symbolic' here refers to a symbol-system, not to a collection of isolated symbols. A symbol system consists of a set of symbols, corresponding to a set of concepts; together with relations between the symbols corresponding to relations between the symbols. (Example: 2, 3 are separate symbols. When we write them like this 2^3, we use two relations between these symbols, one of size and one of position, which correspond to two relationships between their corresponding numbers. So here we have two distinct schemas: the symbol system, and the structure of mathematical concepts. This suggests the provisional formulation: symbolic understanding is a mutual assimilation between a symbol system and an appropriate conceptual structure.

[Rece]ntly I have been emphasising that the achievement of new understanding gives [new a]bilities (Skemp, 1980). So what can we do when we have symbolic understand[ing] that we could not do before? The power of mathematical symbolism is a special case of the power of language, so we would expect this power to be great. Here are ten functions of symbols (there may be others) which I listed some years ago (Skemp, 1971), though without then seeing the use of symbols as conferring a different kind of understanding because at that time none of the present series of discussions had taken place.

1. Communication.
2. Recording knowledge.
3. The formation of new concepts.
4. Making multiple classification straightforward.
5. Explanation.
6. Making possible reflective activity.
7. Helping to show structure.
8. Making routine manipulations automatic.
9. Recovering information and understanding.
10. Creative mental activity.

I'd like briefly to up-date this earlier thinking by connecting it with the new model. Here I suggest that symbols act as an interface, in two ways: between the delta-ones of different people, and between delta-one and delta-two in the same person. The first interface makes possible the functions numbered 1, 2, 5, on the list above, and the second interface, between delta-one and delta-two, relates to all the others. (I think there is some overlap.)

The powers conferred by symbolic understanding are immense, though we are so used to them that we tend to take them for granted. The task of acquiring it is also a considerable one, and we easily overlook the achievement of (almost) every child in learning to speak his mother tongue with considerable mastery by the age of five. But we cannot overlook the difficulties which many children have in learning to understand mathematical symbolism. Part of the difficulty lies in the fact that understanding the symbols depends on being in possession of the conceptual structure. But the conceptual structure has to be acquired largely by means of the symbol structure (though not exclusively in this way). So each has to help the other to develop, and we need to know how we as teachers can provide conditions which facilitate this.

There is a new factor to be taken into account here. In the earlier discussions of understanding, we were concerned with the assimilation of concepts to schemas, of small entities to large ones. But now we are concerned with the mutual assimilation of two schemas: of two entities which are comparable in size. A comparable event in the history of mathematics can be found in the great achievement of Descartes, who assimilated two major structures, geometry and algebra, to each other. When something like this happens, as well as an increase of power, there is also the possibility of a partnership in which one organisation (in this case mental) dominates the other. Whether or not this is desirable may vary in different instances. Since Descartes there has been, it appears to me, a progressive takeover in this partnership by the algebra. We can now find points defined as ordered pairs, triplets, or n-tuples of numbers; and books, whose titles indicate that they are about geometry, in which one finds nothing but algebra, and not a single drawing, diagram, or geometrical figure.

We may or may not think that this is good; and although I think myself that it is not, I accept the opposite as a tenable position. But I trust that none of us is happy with a partnership in which the conceptual structure is dominated by the symbol system, and mathematics is little or nothing more than the manipulation of symbols. The power and also the beauty of mathematics is in the ideas. Symbols help us to use this power by helping us to make fuller use of these ideas. Yet the situation I have just described is the way it is for all too many children.

Where there is isomorphism between the two structures, it may matter little which one dominates, either in an individual or collectively. Part of the success and beauty of algebraic geometry lies in the closeness of this isomorphism, so that each structure helps to increase our understanding of the other. But between the symbol systems and the conceptual structures of mathematics, we find local isomorphisms only. Overall there are many inconsistencies. For example, the spatial relationship <u>is next on the left to</u> means three different things in these three cases:

2^3 $\qquad\qquad$ $2\frac{1}{2}$ $\qquad\qquad$ $2a$

Another example. The ordered pair of numerals (2, 3) can signify a rational number, a point in a plane, or a free vector. With the first meaning, we add like this.

$(2, 3) + (4, 5) = (2 \times 5 + 3 \times 4, 3 \times 5)$

With the second meaning, we cannot add it all.
With the third meaning, we add like this: $(2, 3) + (4, 5) = (2 + 4, 3 + 5)$ (which is the way some children find it more natural to add rational numbers). And this is not just carelessness in our choice of symbol systems. It is inescapable, because the available relations between symbols are quite few. Left/right, up/down, big and small (as in indices and suffixes), bold face and light (which we can't speak, only print) — we soon run out of these. But the relations between mathematical concepts which we are trying to represent are many, and continue to increase as our knowledge advances.

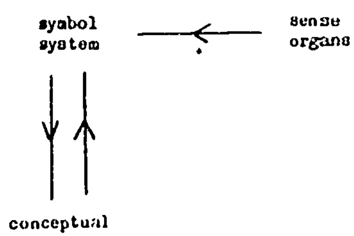
how can we help children to build up an increasing variety of meanings for the same symbols? How can we prevent them from becoming progressively more insecure in their ability to cope with the increasing number, complexity, and abstractness of the mathematical relations they are expected to learn?

As a help towards answering this question, I would like to introduce another part of my model. This is based on the well-known phenomenon of resonance.

> "The starting point is to suppose that conceptualised memories are stored within tuned structures which, when caused to vibrate, give rise to complex wave patterns. ... Sensory input which matches one of these wave patterns resonates with the corresponding tuned structure, or possibly several structures together, and thereby sets up the particular wave pattern of a certain concepts." (Skemp, 1979a)

A schema, being a conceptual structure stored in memory, thus corresponds (in this model) to a particular, complex, tuned structure. We all have many of these, and sensory input will be interpreted in terms of whichever one of these resonate with what is coming in. What is more, for different people, different structures may be activated - caused to vibrate - in this way by the same input. Even for the same person and the same input, different structures may be activated at different times. Thus 'field' will cause different vibrations according as the schemas with which it resonates are mathematical, electromagnetic, farming, academic, or cricket. Whichever schema resonates most easily will attract the input.

In the case we are discussing, there are two contenders: the symbol system and the conceptual structure.



Since communication is by the utterance of symbols, all communication whether verbal or written first goes into a symbol system. To be understood relationally, it must be attracted to an appropriate conceptual structure. What is more, the input must be interpreted in terms of the relationships within the conceptual structure, rather than those of the symbol system. (Example: 572 must be interpreted, not as a succession of three single-digit numbers, but as a single number formed by the sum $5 \times 10^2 + 7 \times 10 + 2$.)

This requires (i) that the conceptual structure is a stronger attractor than the symbol system; (ii) that the connection between the symbol system and the conceptual structure are strong enough for the input to go easily from the first to the second.

How can we help this to happen? I have four suggestions to offer: briefly, for each could usefully be expanded into a chapter of a book.

(i) We have noted that the symbol system has a built-in advantage, that all communications necessarily go there first. And for the conceptual structure, there is a point of no return. In the years long process of learning mathematics, if those conceptual structures are not formed early on, they will never get the chance to develop as attractors. The effort to find some kind of regularity is strong. If the conceptual structure is absent or weak, the input will be assimilated to the symbol system. But this guarantees problems, for we have seen that the symbol system is inconsistent. Learning at this level may be easy short-term, but it becomes impossibly difficult long-term. In contrast, the conceptual structure is (or should be) internally consistent. Of all subjects, relational mathematics is one of the most internally consistent and coherent, so long-term it is much easier to learn and retain. So part of the answer is that by careful analysis of the mathematical concepts, we must sequence material in such a way that new material is presented which can always be assimilated conceptually. (See Skemp, 1971, Chapter 2 & 3.)

(ii) Especially in the early years, we can work first with physical embodiments of mathematical concepts and activities, so that the sensory input goes first to the conceptual structure and is then connected with its symbolic representation.

(iii) Again especially in these all-important early years, I think that we should stay with spoken language much longer. Recently I came upon a nice quotation from Sartre. "On parle dans sa propre langue, on écrit en langue étrangère." (Sartre, 1964). The connections between thoughts and spoken words are initially much stronger than those between thoughts and written words or mathematical symbols. Spoken words are also much quicker and easier to produce. So in the early years, we need to resist pressures to have 'something to show' in the form of pages of written work.

(iv) Some notations, such as the use of parentheses to denote the order of operations, can be seen to arise out of the needs of a situation. (E.g. Kieran, 1979.)

(v) We should use transitional, informal, notations as bridges to the formal, highly condensed notations of established mathematics. By allowing children to express thoughts in their own ways to begin with, we are using symbols already well-attached to their conceptual structure. These ways will probably be lengthy, ambiguous, and different between individuals. By experience of these disadvantages, and by discussion, children may be led gradually to the use of conventional notation in such a way that they experience its convenience and power.

CONCLUSION

In the light of the foregoing discussion, I offer the following revised formulation.

> Symbolic understanding is a mutual assimilation between a symbol system and a conceptual structure, dominated by the conceptual structure.

Symbols are magnificent servants, but bad masters, because by themselves they don't understand what they are doing.

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EARLY ADOLESCENTS' ATTITUDES TOWARD MATHEMATICS: FINDINGS FROM URBAN SCHOOLS

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ABSTRACT

Attitudes towards mathematics were assessed using a questionnaire and an interview. Eighth graders in three urban schools were found to have generally positive attitudes towards mathematics. Their perceptions of what it takes to do well in mathematics included more behavioral responses (e.g. "listen to the teacher") than intellectual responses (e.g. "understand the material"), however, and this was particularly evident at the school where fewest students demonstrated proportional reasoning. There were few sex differences in attitudes towards mathematics and none in performance.

Investigation of attitudes towards mathematics has been a source of concern to researchers for many years, despite the difficulties pointed out recently by Kulm (1980) in a review of the research. He notes that there are many possible explanations for the lack of decisive statements about attitudes, including the problems of validly and reliably measuring attitudes, the many influences on mathematics attitudes, the many influences on mathematics achievement, and so on. On balance, the research in this area is sustained by the strong sentiment among mathematics educators that attitudes are probably important points of intervention if only we understood them better. Further interest in attitudes is generated by of studies of sex differences in mathematics that generally find sex differences in mathematics achievement only in conjunction with sex differences in mathematics attitudes (e.g. Fennema and Sherman, 1978).

Promising results in sex difference studies have been obtained when investigators have used separate scales that test such components of attitude as enjoyment, valuing, and confidence in mathematics separately, (Fennema & Sherman, 1978; Sherman, 1980; Lantz & Smith, 1981). Given the measurement problems of relying exclusively on paper and pencil surveys, it is also advisable to examine attitudes with more than one instrument. Interviews, observations, and projective techniques have been suggested as alternatives (Kulm, 1980) and have been used with some success (e.g. Wolleat et al., 1980). Thus, it was the purpose of this study to examine the relationship between achievement in a particular area of mathematics, proportional reasoning, and attitudes toward mathematics, using a differentiated construct of attitude and two attitude measures.

THE ATTITUDE MEASURES

Two attitude measures were used in this study--a questionnaire and an interview. Both were based on previous work (Stage, Karplus, & Pulos, 1980), revised to strengthen the subscales.

The Mathematics Attitude Survey 2 (MAS2) is presented in Table 1. It is a 25 item questionnaire with a four point Likert scale from "disagree a lot" to "agree a lot." The items constitute four categories

Table 1. Mathematics Attitude Survey (n=422)

Statement	Mean	S.D.
	(on a 4-point scale where 1 = disagree a lot 4 = agree a lot)	
1. I like to do math puzzles. E	3.15	.82
2. I think I could do well in more advanced classes. C	2.76	.90
3. Most jobs require some math. N	3.60	.70
(4.) I will stop taking math as soon as I can. O	1.49	.90
5. Playing math games is fun. E	3.20	.89
(6.) You have to be super smart to learn math. C	1.50	.84
7. I will need math to do well in school. N	3.52	.76
(8.) I wish math weren't so important. O	2.08	1.10
9. Once I start working on a math puzzle, I find it hard to stop. E	2.51	1.02
(10.) No matter how hard I study, I will get low grades in math. C	1.62	.89
11. Knowing math helps me in many ways. C	3.70	.64
(12.) Math makes me nervous. O	1.85	.94
13. Math is my favorite subject. E	2.61	1.06
(14.) You need to have a good memory to be good in math. C	2.67	1.00
15. I use math outside of school. N	3.39	.86
(16.) Math is a lot of rules for numbers. O	3.22	.93
(17.) I think math is boring. E	1.89	.95
18. Math is easy for me. C	2.74	.84
19. I will need math for my future work. N	3.58	.76
20. I think math is fun. E	2.99	.88
21. I think everyone can learn math. C	3.64	.70
(22.) After I finish school I can forget about math. N	1.37	.78
(23.) I wish I were smarter in math. O	3.41	.86
24. I like math classes. E	3.00	.87
25. I can do well in math if I want to. C	3.50	.80

E = enjoy, C = confident, N = need, O = other

( ) = negatively worded items

N, need mathematics, E, enjoy mathematics, C, feel confident in mathematics, and O, other items of interest to the investigators The MAS2 was administered in classroom groups as part of a 45 minute battery of paper and pencil tasks for this research program

The student interview contained general questions about school and was administered individually, at the beginning of a 40 minute interview that included the lemonade puzzle, a proportional reasoning task (Karplus, Pulos, & Stage, 1981) The interviewer's interest in mathematics was not revealed to the subjects, as they were asked what their favorite and least favorite parts of school are, which of their academic subjects they like best and least, do best and worst in, work hardest and least hard in, what they think it takes to do well in English and in mathematics, whether it matters if they get good grades in school and why, and what they do after school

The subjects for this study were the 230 students in three urban schools described in Karplus, Pulos and Stage (1981) who completed the lemonade puzzles and the two attitude measures For the MAS2, data are reported for these students and their classmates (n=422) who took the paper and pencil portion of the battery

RESULTS AND DISCUSSION

Mathematics Attitudes Survey

The responses to the MAS2 are reported in Table 1, which shows that students' attitudes towards mathematics are generally positive Students are in strong agreement with statements that "Knowing math helps me in many ways," and "I think everyone can learn math" Similarly, they are in strong disagreement with the statements, "After I finish school I can forget about math," and "You have to be super smart to learn math" These positive findings for eighth graders are in agreement with the National Assessment for Educational Progress findings for 13 year olds (Carpenter, et al , 1980)

Analysis of variance found only two items on which there were sex differences that were significantly different at the 01 level or less, items #9 and #25, on which the girls' responses were more positive than the boys There were also two items, #6 and #14, on which there were differences among the three schools that were significant at the 01 level, however, the order on those items was different

Of particular interest on the MAS2 was the relationship among subscales--Need, Enjoy, Confident to see if they replicated the finding of a Guttman scale (Stage, Pulos, & Karplus, 1981) Using 80% as a criterion, subjects were categorized as "needing math" if they agreed with 4 of the 5 needing statements, "enjoying math" if they agreed with 6 of the 7 enjoyment statements, and "feeling confident" is they agreed with 6 of the 8 confidence statements We replicated our earlier result from a substantially different school population that needing precedes enjoying, enjoying precedes confidence The scale had a coefficient of reproducibility of 92, indicating a valid scale, and a coefficient of scalability of 73, indicating that the scale is unidimensional and cumulative Further discussion will report students' responses to the MAS2 in terms of the subscale scores

School Attitudes Interview

The response to the school attitudes interview is reported in Table 2 It shows that students have fairly positive attitudes towards mathematics, particularly in comparison with other school subjects Forty eight percent of the students volunteered mathematics as the subject that they like best, 36% of them reported that they do best in math, and 55% reported that they work the hardest in math The comparable figures for English are 5%, 28%, and 18%, respectively

Using a Chi square test, there were only three questions on which the the distributions of the responses were significantly different at p< 01 by sex boys enjoy physical education more than girls do (35% to 21%), boys play sports after school more than girls do (54% to 26%), and girls are more likely to mention doing their homework spontaneously (81% to 73%) There were several items on which the responses differed by school however Most prominent were those that occurred on the questions "What do you think it takes to do well in math?" and "What do you think it takes to do well in English?" The results are reported in Table 3 where the students' first and second open ended replies were categorized as abilities (e g brains, being smart), mental skills (e g know how to add well) and behavior (e g do your homework, listen to the teacher)

Table 2. School Attitude Interview (N=230) (percentages)

Favorite part of the school day	Least Favorite	Not Mentioned	Favorite
Social	0	82	18
Physical Education	11	60	29
Nonacademic Subject	5	82	13

Comparison of academic subjects	Least/Worst Part of Day	Not Mentioned	Most/Best Part of Day
Like math	20	32	48
Perform in math	26	38	36
Work in math	9	36	55
Like English	88	7	5
Perform in English	22	50	28
Work in English	18	64	18
Like other subject	50	40	10
Perform in other	39	40	21
Work in other	53	32	15

What does it take to do well in	Ability	Mental Skills	Behavior
English, first answer	26	19	55
English, second answer	25	18	57
Math, first answer	40	9	52
Math, second answer	32	4	64

Does it matter if you get good grades? Yes 95% No 5%

Why? Other people, 16%; My own reasons, 38%; To achieve a goal, 46%

What do you do after school?	Mentioned Spontaneously	Mentioned After a Probe	Do not Do
Homework	77	21	2
Housework	32	51	17
Work at a job	11	16	73
Play sports	41	36	23
Play games	42	40	18
Play musical instrument	67	25	8
Watch TV	10	38	52
Read	21	62	17
Listen to music	11	71	18
Be with friends	24	47	29
Hobby	3	42	55
Other	31	8	61

Table 3. Responses to the Question
"What does it take to do well in...?" (percentages)

Question	School A n=101	School B n=58	School C n=79
English, first answer			
ability	33	16	26
mental skill	12	34	17
behavior	55	50	57
English, second answer			
ability	30	20	24
mental skill	13	34	10
behavior	57	46	66
Mathematics, first answer			
ability	42	43	34
mental skill	6	19	4
behavior	52	38	62
Mathematics, second answer			
ability	39	40	15
mental skill	3	3	5
behavior	53	57	80

There are some differences in the responses by school to the English question, but the prominence of the "behavior" response by students from School C is particularly evident in mathematics, an average of 71% of their answers.

Relationships among the attitude measures

From the MAS2, there are three subscales (Need, Enjoy, Confident) and a composite scale score (from 0 to 3, one point for each of the subscales). From the interview there are composite variables that represent liking mathematics (Like), performing well in mathematics (Perform), and working hard in mathematics (Work). The relationships among the questionnaire and interview variables are shown in Table 4, together with the scores for proportional reasoning from the lemonade puzzles (Karplus, Pulos, & Stage, 1981). The competence score (0,1) indicates whether or not the student used proportional reasoning on any puzzle in the set of eight. The performance score (0,8) indicates the extent to which the student used proportional reasoning on the puzzles.

Of the MAS2 subscales, Confidence is the most consistent correlate, both with other affective variables and with performance. Of the interview variables, Like and Perform are stronger correlates with other affective variables than Work, [illegible] the interview variables show low correlations with performance. It is interesting to note that affective variables were more strongly related to performance than to competence, replicating results from a different population (Stage, Karplus, & Pulos, 1980).

There is only one significant correlation with the school variable, indicating that the students from School C are less aware of the need for mathematics than students from School B and A. There are no significant correlations with sex.

Table 4. Correlations Among Affective and Achievement Variables

	From MAS2			From Interview			From Lemonade Puzzles		School	Sex
	Enjoy	Confidence	MAS2	Like	Perform	Work	Competence	Performance		
From MAS2										
Need	.16	.27	.55	.02	.08	.05	.06	.13	-.17†	-.05‡
Enjoy	-	.35	.78	.34	.24	-.09	.03	.20	.04	.09
Confidence		-	.78	.30	.24	.02	.08	.27	-.12	-.03
MAS2			-	.34	.28	-.02	.07	.28	-.10	.02
From Interview										
Like				-	.51	.06	-.03	.13	.13	.04
Perform					-	.00	-.16	.00	.13	.04
Work						-	.04	-.04	.09	.07

n=230; r=.15, $p \leq .05$ † ordered A,B,C
r=.18, $p \leq .01$ ‡ ordered M,F

School and sex differences in attitude and achievement

As reported in the companion paper (Karplus, Pulos, & Stage, 1981) there were no sex differences in the use of proportional reasoning. There were also no substantial sex differences in attitudes towards mathematics.

There were substantial school differences in proportional reasoning: students in School A used proportional reasoning more frequently than students in School B, students in School B used proportional reasoning more frequently than students in School C. These performance differences are, to a limited extent, reflected in the students' responses to the attitude measures. Students in School C report that good behavior, rather than skill or ability, is required for success in mathematics. Although they like mathematics, they are also less likely to see the need for mathematics than students in Schools B and A.

CONCLUSIONS

An earlier study of sixth and eighth graders from a middle-class, suburban school with an ethnically mixed population (Stage, Pulos, & Karplus, 1981) had found generally favorable attitudes towards mathematics and a Guttman scale of need, enjoyment, and confidence. These results were replicated with eighth graders from three lower-class, urban schools with high minority enrollments. School differences in attitudes towards mathematics were not strong but they were consistent with performance differences in proportional reasoning.

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INVESTIGATING LEARNING DIFFICULTIES IN ALGEBRA

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Algebra has traditionally been the mathematics course in which success or failure determines whether a student can go on to higher level mathematics courses or is forever limited to a knowledge of basic arithmetic. Data from the second National Assessment of Educational Progress in mathematics show that only one-half of the 17-year-olds in the sample took courses in mathematics beyond the level of elementary algebra. Many factors contribute to this statistic, but one significant factor must be the difficulty students have in learning algebra.

MATHEMATICAL CONTENT OF ALGEBRA

The content of elementary algebra appears, for the most part, in one of two forms: in expressions (combining or simplifying terms, operations on polynomials, operations on rational expressions, etc.) or in equations (solving equations and inequalities, graphing of functions, solving systems of equations, etc.). Both of these forms rely upon the use of variables (literal symbols, *x*, *y*, *z*, ...) for their written expression.

Variables. Literal variable symbols play a multitude of roles in algebra and are referred to in as many different ways — as unknowns, generalized numbers, indeterminates, independent variables, dependent variables, constants(!), parameters, and so forth. Kuchemann (1978) has developed a taxonomy of six different uses for variables, and Tonnessen (1980) has investigated college students' understanding of variables. Wagner (1977, 1981b) studied middle school and high school students' ability to conserve equation and function under transformations of variable.

An analytical framework that has recently been developed (Wagner, 1981a) is useful in guiding the formulation of questions to use in investigating the interaction between students' interpretations of the roles of variables and their ability to work with them in the contexts of expressions and equations. A deeper insight into students' understanding of variables may eventually suggest alternative teaching strategies, such as that recently developed by Herscovics and Kieran (1980).

Expressions and equations. All research related to algebra deals directly, or indirectly, with expressions or equations. Most recent research has focused primarily on equations or functions. However, Rachlin (1980, 1981) has conducted a study that focused almost entirely on polynomial, rational, and radical expressions. He found that above-average algebra students make the same types of errors when working on tasks that are difficult for them as other algebra students make on more routine problems.

A few studies, notably Davis, Jockusch, and McKnight (1978) and Matz (1979), have identified particular points of confusion between expressions and equations and have proposed ways of accounting for them. For example, Matz distinguishes two uses of the equals sign (=) in algebra, one as a relation sign in an equation and one as a sign of equivalence between expressions in a chain of reductions. The kinds of operations that are appropriate in each case are very different, but students are rarely conscious of the distinction between the two uses of the equals sign. Davis, meanwhile, speaks of visually moderated sequences and the effect that the appearance of an algebraic form has on the student's determination of the appropriate next step in a problem. Putting the ideas of Davis and Matz together provides a plausible explanation for students' continuing confusion between operations on rational expressions and solving rational equations. That is, when the student completes the first step in adding two rational expressions and writes "=" in the sense of equivalent expressions, the result looks the same as a rational equation, so the student may then follow a visually moderated sequence and begin "clearing the equation of fractions," completely forgetting that the original problem was simply to combine terms and simplify.

This error, and others like it, are all too familiar to any algebra teacher, but until the causes of such errors can be better identified, it is difficult to improve the teaching of algebra so as to enhance students' understanding. Having students solve problems involving expressions and equations that are purposely similar in form may help in identifying particular learning difficulties associated with these forms.

PSYCHOLOGICAL PROCESSES IN LEARNING ALGEBRA

The learning of algebra, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. Even the application of formulas to "routine" textbook exercises involves some degree of problem-solving activity on the part of most students, at least for a while. Thus, one approach to the investigation of learning difficulties in algebra is to use certain well established problem-solving processes to guide the initial selection of interview tasks. Two basic problem-solving processes identified by Krutetskii (1976) in his model of mathematical abilities are those of generalization and reversibility.

Generalization. Krutetskii considered the ability to generalize mathematical material to operate on two levels:

a) the ability to see something general and known in what is particular and concrete (subsuming a particular case under a known general concept), and

b) the ability to see something general and still unknown in what is isolated and particular (deducing the general from particular cases, to form a concept). (p. 237)

The first of these levels has been characterized by Dienes (1965) as an extension of an already-formed class. This notion of generalization is commonly reflected in the ordered series of exercises found in most mathematics texts, in which increasingly more complicated extensions of a form are made. Graded sequences of problems within a topic and similar forms of problems across topics can be used to measure this aspect of generalizing.

Krutetskii's second level of the ability to generalize mathematical material is closely related to Dienes's definition of abstraction as a process of class formation. This aspect of the ability to generalize algebraic ideas can be measured by various kinds of concept attainment tasks, such as conservation tasks, sorting tasks, and triplet comparison tasks.

Reversibility. Krutetskii (1969) considers reversibility an essential aptitude for the formation of algebraic concepts. He defines the basic concept of reversibility as follows:

> By reversible (two-way) associations (and series of associations) we mean those associations in which the thought or realization of the second element (or of the last element) evokes the thought or realization of the first element (p. 51)

Inhelder and Piaget (1958) distinguish between two forms of reversibility — negation (or inversion) and compensation (or reciprocity). Negation reverses an operation by "undoing" it. Compensation reverses an operation by leaving it alone and cancelling its effect.

Collis (1975a, 1975b) has investigated students' ability to apply reversibility in the context of linear equations with a single missing term. However, complete reversibility of addition in an expression of the form $a + b = c$ incorporates three possible variations: one in which c and a are known, one in which c and b are known, and one in which c is known but neither a nor b is given. These three variations can be used to investigate students' ability to apply reversibility in the context of polynomial and rational expressions, as well as simple linear equations.

METHODOLOGY

With all of its inherent limitations, the clinical approach is generally recognized by mathematics educators as the best available methodology for investigating internalized operations of thought. The Soviet ascertaining experiment may be especially useful for investigating learning difficulties in algebra because:

a) The use of thinking aloud procedures mixed with retrospection through directed questioning provides traces of thought not available through paper-and-pencil means. Written tests provide snapshots of the student's thinking process. Verbal protocols also provide snapshots, but the intervals between pictures are so abbreviated that the viewer obtains more of a motion picture of the thought process.

b) The flexible nature of the Soviet ascertaining experiment permits the interviewer to use impromptu questioning to follow the flow of a student's thought as it is in the process of forming.

c) A student's understanding of algebra changes with time and experience. The longitudinal nature of the Soviet ascertaining study is designed to capture this dynamic quality of learning.

Studies of problem-solving processes often involve mostly above-average and gifted students because these students are generally better able to articulate reasoning processes. On the other hand, error analysis studies often involve mostly average and below-average students ~~because these~~ students make more mistakes. In order to identify learning difficulties from a psychological perspective, it is important to analyze the reasoning processes used by capable students and compare them to the processes used by less capable students. Tasks that range in difficulty all the way from standard textbook exercises up to moderately difficult nonstandard problems should enable investigators to obtain a reasonable amount of information about the reasoning processes used by both above-average and below-average students.

A STUDY IN PROGRESS

A study of learning difficulties in algebra is currently being conducted with 12 high school freshmen in Athens, Georgia. Half of the students are above average and half are below average. Each student is being interviewed for one hour every three weeks over the course of an academic year, using the Soviet ascertaining methodology. Several pretest instruments are being administered to assess the students' initial understanding of variables, expressions, and equations, as well as their ability to apply the processes of generalization and reversibility.

Interview tasks. Both standard textbook problems and nonstandard, related problems are being included in the interviews. The standard problems provide the below average students with some problem-solving challenge and insure that these students experience some success in obtaining answers. The nonstandard problems provide the above-average students with a problem-solving challenge. It is through the tracing of thought processes during problem-solving activity that the investigators hope to obtain insight into learning difficulties.

Interview tasks represent the cells of a 3-dimensional content x process x form matrix in which the content topics are operations with polynomials, algebraic fractions, and radicals, the processes are generalization and reversibility, and the forms are expressions and equations. Standard probitems are used as foundation tasks upon which generalization and reversibility tasks are constructed. Wherever possible, expression tasks and equation tasks resemble each other.

Analysis of data. Initially, the oral and written responses of all students will be analyzed for regularities in observable sequences of behavior across individuals. Next, the responses for each student will be carefully examined with reference to several questions of interest, such as the following.

a) To what degree of variation do the students generalize their solutions? On tasks with multiple correct responses, do the students make general statements?

b) Do the students generalize processes or operations from one task to the next? Do the students transfer the processes or operations from one content domain to another?

c) How does increased difficulty affect a student's application of generalizations? If the student focuses on a particular difficulty, are incorrect connections more likely to occur?

d) Is there a pattern to the students' behavior in terms of the types of reversibility they apply to the algebra tasks?

e) How persistent are students in attempting to use a particular approach when experiencing difficulty in solving a problem?

f) How do students react to hints that suggest a different approach?

g) If a student solves a problem using one approach, can he or she solve the same problem another way? For instance, if a student uses negation to solve a problem, can he or she use compensation as well, or vice versa?

The analysis will range far beyond these particular questions. In general, the investigators will be especially interested in identifying learning patterns that seem contrary to the standard curriculum. It is these patterns that may be especially helpful in suggesting alternative, more effective methods of teaching algebra.

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CLASSROOM STUDIES USING FEATURE IDENTIFICATION TASKS

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Feature identification tasks have a potential to clarify the value of examples and non-examples in the classroom. The use of examples and non-examples has been identified as a critical problem area in the learning of mathematical concepts (Sowder, 1980). Psychological research on conjunctive feature identification tasks favor sequences of all positive instances over mixed positive and negative instances (Bourne and Dominowski, 1972, Erickson and Jones, 1978). However, research involving mathematical concepts found the opposite. Mixed positive and negative instances were favored over the all positive sequences (Shumway, 1971, 1974, 1977).

Shumway and White tried to identify a critical variable that accounts for the difference in the mathematical and psychological research. They found that students had difficulty using negative instances in developing a concept (1977). In addition, frequency levels of irrelevant dimensions seemed to be one of the factors influencing the usefulness of non-examples. If the frequency level was maintained at a 50-50 chance of either level occurring, the traditional psychological results were obtained showing that sequences of all positive instances are more helpful in learning a concept However, if the frequency level is manipulated so that one level occurs more often (90% of the time), mixed positive and negative instances were better than all positive instances (Shumway et al, 1981).

Intuitively, the idea of different frequencies for different levels of a dimension seem to imitate the classroom. Typically in the classroom, some irrelevant features are not varied (i.e. orientation of a figure in a textbook, names of variables). Two attempts were made to replicate the 1981 study (Shumway et al, 1981) using topics that were closer to classroom activities than the letter strings used previously. The first was the geometric concept of the altitude of a triangle and the second was a computer simulated qualitative analysis chemistry experiment.

GEOMETRIC CONCEPT

A pilot study was run which failed to replicate the results of the 1981 study. Five dimensions of the triangle varied in two levels each. Ninth grade algebra students were instructed on how to use negative instances and also on the five dimensions which were manipulated. The subjects were given geometric sequences which replicated the treatments of the previous study.

The results showed no significant differences in the treatments. There was a slight favoring of mixed positive and negative sequences for both treatments. The subjects had a difficult time focusing on the five bi-level dimensions that were manipulated. Some Ss focused on irrelevant attributes other than the five intended by the researcher and some attended to only a subset of the five dimensions.

QUALITATIVE ANALYSIS CONCEPT

An attribute identification concept learning task was designed for administration by use of the Apple II microcomputer. The task was related to the processes of qualitative analysis in chemistry. A simulation of a chemical system was programmed so as to present the results of a chemical reaction to Ss. The program included graphics animated to represent chemical and physical changes which are often observed in qualitative chemical analysis.

The dimensions of focus for the Ss were:

dimensions	features
- formation of a precipitate	yes or no
- formation of a gas	yes or no
change in temperature	increase or decrease
- color change of solution	yes or no
- rate of reaction	fast or slow

Five dimensions were chosen in order to replicate the study of Shumway et al (1981). The microcomputer simulated chemistry experiments were presented in sequences which were replications of the sequences used by Shumway.

The Ss received instruction and practice on tasks similar to the experimental task as a group. The instruction included the group solution a task involving three factors with two levels of each factor. The stimulus was a string of 3 letters and the Ss were asked to identify what two features defined the concept. This was an attribute identification task with the rule conjunction given. The strategy used was discussed as the group worked thru the task.

The second practice task was done individually by each subject. The instances for this task included levels of attributes from four factors, with two levels for each factor possible. After the Ss completed the task the results were discussed along with questions and answers related to strategies for solution.

Following the two practice tasks the subjects responded to a pretest designed to determine if the Ss were able to use information from the negative instances. The results showed that 17 of the 23 Ss were able to use information from negative instances while 6 were not. There were no significant ($p < .05$) difference among the four treatment conditions on pretest scores. See table 1 for means and standard deviations by cell.

Table 1

Means and Standard Deviations for Pretest Scores by Cell

Sequence Condition		Frequency 50/50		90/10	
	+	M	5.4	M	4.2
		SD	3.65	SD	4.55
	±	M	5.8	M	5.9
		SD	4.32	SD	3.94

The specific task for the Ss using these sequences of experiments was to identify the two features that identify the presence of a certain substance. If the substance is present the experiment is an example of the concept, if the substance is not present in the system the experiment is a non-example of the concept.

The simulation begins with a few drops of a reagent added to a beaker filled with a blue solution. The subject observed the results of the experiment and was asked to circle the changes observed on a response sheet. After the chemical experiment was completed the Ss were asked to guess if the experiment was an example or non-example of the concept. The Ss then enter their guess into the microcomputer which immediately indicates whether the instance was an example or non-example. At this point the Ss record the microcomputer feedback, make a guess as to what two features are needed for the instance to be an example and request the next experiment by entering a 1 on the keyboard.

There were 20 different sequences of 40 instances (experiments) each. The average time required by the Ss to complete this attribute identification task was about 20 minutes.

The Ss for this research were 23 masters and Ph.D. candidates in a course on learning theory. The Ss were teachers of science and/or mathematics at the secondary level. The Ss were randomly assigned to the 20 sequences. All sequences were responded to by at least one subject and where more than one S responded to a sequence, the average value was determined for further analysis.

The 20 sequences were arranged as shown in table 2.

Table 2

Feature Proportion

Instance Type		50/50	90/10
	+	Sequence #01 02 03 04 05	Sequence #06 07 08 09 10
	±	Sequence #11 12 13 14 15	Sequence #16 17 18 19 20

The criterion for the concept learning task was the number of experiments needed for the Ss to determine the features which identify the presence of the specific substance.

The means and standard deviations for the four cells of this design are given in table 3.

Table 3

Means and Standard Deviations
for Criterion Scores by Cell

Frequency	
50/50	90/10
X = 7.60	X = 15.80
SD = 9.24	SD = 16.10
X = 10.00	X = 11.80
SD = 2.00	SD = 10.73

A 2x2 two way analysis of variance of the data resulted in non significant sequence condition (+,±) by frequency (50/50,90/10) interaction and main effects. A plot of the mean is given in figure 1.

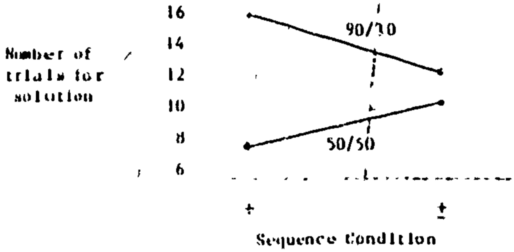

Figure 1

Although non significant, the lower line of the graph appears to be consistent with the usual psychological results for conjunctive/attribute learning tasks, that is to say, sequences of all positive instances are favored over sequences of positive and negative instances. The upper line appears to be consistent with the influence of high frequency (90/10) of irrelevant features being more difficult and consistent with the mathematical concept learning tasks in which sequences of positive and negative instances are favored over sequences of all positive instances. It is not clear from this study that the effect of negative instances is increased when the frequency of levels of irrelevant attributes is other than 50/50. The replication of this study to include 5 or more Ss per cell should provide the statistical power needed to make a more valid interpretation.

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LATE ARRIVALS

THE EFFECTS OF DIFFERENT STUDENT INTERACTION PATTERNS ON LEARNING OUTCOMES IN MATHEMATICS

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How students perceive each other and interact with each other during instruction has considerable impact on learning outcomes. How well a student understands the material and remembers it, how the student feels about the subject and the teacher, how the students feel about each other and themselves as learners are all influenced by the pattern of interactions between students. There are three basic types of interaction patterns that can be implemented during instruction (Deutsch, 1962, Johnson & Johnson, 1975): Competitive, Individualistic and Cooperative. In a Competitive situation an individual's goal achievement is negatively correlated; when one person achieves his goal all others with whom he is competitively linked fail to achieve their goal. In a math class if one student has the highest score on an assignment, no other student can have the highest score. In an Individualistic situation an individual's goal achievement is independent from others; the goal achievement of one person is unrelated to the goal achievement of others. In a math class structured individualistically each student is working toward a set criterion and are not linked together in any way. In a Cooperative situation an individual's goal achievement is positively correlated with group member when one person achieves his goal all others with whom he is cooperatively linked achieve their goals. In a math class structured cooperatively each student is working to make sure he masters the material and that the others students in his group also master the material.

It is important to realize that there may be a significant difference between having students "work in a group" and stucturing students to "work cooperatively". An operational definition of cooperation includes positive interdependence (that group members see themselves in a sink or swim together situation) and individual accountability (each student should master the material). The operational definition of competition has each student working individually to try and do better than the other students and in the individualistic situation each student is working individually toward a stated criteria which he will reach, or not reach, depending on his own efforts.

There is a great deal of research available which examines t' effects of the different interaction patterns on instruction (Johnson & Johnson, 1975; Johnson, et.al. 1981). In this paper a set of field studies will be featured which focus on the learning of mathematics in cooperative, competitive and individualistic settings. The studies cover a range of ages in students and content in math. Three studies were done in a First Grade setting and examined both drill and problem solving tasks (Johnson, Johnson & Skon, 1978; Johnson, Skon & Johnson, 1980, Skon, Johnson & Johnson, in press). One study was done in Third Grade with the regular math curriculum, primarily drill activity (Johnson & Johnson, in press). Three studies were done at the Fifth-Sixth Grade level examining both drill and problem solving activities (Johnson, Johnson and Scott, 1978; Johnson, Johnson and Tauer, 1979; Johnson and Johnson, 1979). One study was done at the Eleventh Grade level using the regular general math curriculum (Johnson & Johnson, submitted). A variety of dependent measures were used in these studies, but they primarily fall into three broad categories: Achievement, Attitudes and Acceptance of Differences.

ACHIEVEMENT

In a Meta-Analysis of studies comparing the effects of cooperative, competitive and individualistic interaction patterns on achievement, it was determined that having students work cooperatively will result in higher achievement than having students work individualistically or competitively (Johnson, et. al., 1980). These results were consistent across age groups and subject matter, including the area of mathematics. The Meta-Analysis also indicated that there seems to be a stronger relationships between cooperation and problem solving tasks than cooperation and very simple drill-review.

This finding was mirrored in the field studies. Each of these studies followed a strict Post-Test Only Design with students being randomly assigned to treatment and teachers rotated across conditions. In the First Grade studies students in the cooperative condition performed ~~significantly~~ better on math drill tasks, story problem tasks, spacial reasoning and visual sorting tasks, and math equations. There was also evidence that on problem solving tasks, not only do low ability and middle ability students do better in cooperative groups, but the high ability students in the cooperative condition achieved significantly higher than high ability students working alone, and less errors were made by students working in cooperative groups. The Third Grade study did not have achievement data.

Three studies at the Fifth-Sixth Grade level also indicated that cooperation tends to promote higher achievement on daily math achievement and on end of unit tests in another study. One of the studies dealt with several different kinds of tasks including math drill-review (two-place multiplication problems) and problem solving (finding a number of triangles in a figure). On the drill-review students in the cooperative and individualistic conditions did better than students in the competitive condition. On the problem solving task students in the cooperative condition did significantly better than students in the other two conditions with the lowest performance by the students in the individualistic condition. In analyzing the responses it was found that few errors were made by students in cooperation and they tended to pursue the problem longer while the students in the individualistic, and somewhat in the competitive, conditions would find the obvious answers and stop.

In the Eleventh Grade study achievemnt was again higher for students in the cooperative condition. Since this study dealt with the issue of mainstreaming learning disabled students into regular math classes, it is interesting to see that both handicapped and nonhandicapped students did better in the heterogeneous, cooperative setting than handicapped and nonhandicapped students working alone. There was not a competitve condition in this study.

Overall, there is growing evidence that cooperation promotes higher achievement in math than having students work competitively or individualistically and this is especially true when the task is something more than simple drill-review.

ATTITUDES

A consistent finding in student interaction studies is the positive affect associated with working cooperatively. These studies on math classes continue to support these prior findings. The First Grade students in the cooperative conditions viewed math as less difficult than did students in the competitive and individualistic conditions, and perceived themselves as having more peer support. The students in the Third Grade Study cooperative condition also perceived more peer support and encouragement for learning. In the Fifth-Sixth Grade studies, students in the cooperative conditions felt more teacher support and encouragement, tended to perceive more peer support and caring, tended to feel more relaxed and comfortable in math class, and tended to view the tasks as shorter, easier and more enjoyable than students in the competitive and individualistic conditions. In the Eleventh Grade study students in the cooperative condition perceived the math assignments to be less difficult, perceived more peer support for learning, and tended to be more motivated to be on task than students in the other two conditions.

Students in the cooperative conditions tend to feel more positive about each other, the teacher and math class than students who are competing or working individualistically.

ACCEPTANCE OF DIFFERENCES

All of these studies addressed the question of ho[illegible] dents perceive each other and three of them examined students attitudes about other students who are different than they are. Students who were learning disabled and mainstreamed into the math classes was a major theme of the Third Grade and Eleventh Grade studies and different ethnic background was a focus in one of the Fifth Sixth Grade studies. In the Third Grade study where a few students who were identified as having severe learning and behavior problems were mainstreamed into cooperative and individualistic conditions, there was far more interaction between the handicapped and nonhandicapped students and that these interactions were almost entirely positive in nature. In addition it was found that these relationships generalized to free time situations where there were more cross handicap positive interactions than in the individualistic group, and students in the cooperative condition made significantly more cross handicap choices for friends on a sociometric nomination.

In the Eleventh Grade Study there were over four statements from nonhandicapped students to handicapped peers to every one made in the individualistic condition. Students in the cooperative condition indicated more cross handicap helping and made more cross handicap choices on a sociometric rating scale than did students in the individualistic condition.

In the Fifth-Sixth Grade Study which looked at acceptance of differences, cooperative learning experiences in math tended to promote more motivation to be part of a learning group with persons who were different sexually, ethnically, and culturally, with the expectation that the heterogenuity would increase the learning and enjoyment of the class.

A number of other studies also are in agreement with these studies that cooperation promotes an acceptance of differences among peers.

CONCLUSIONS

The series of math studies reported here all look at the effects of having students work cooperatively, competitively or individualistically in math classrooms. Three major learning outcomes are discussed: achievement, attitudes, and acceptance of differences.

These studies in math support the data of other studies which indicates that achievement is enhances by having students work together cooperatively rather than having students work alone competitively or individualistically. The significance of this finding is in the fact that most classrooms tend to emphasize individual achievement and teachers are still being taught in pre and inservice to separate students from one another and make sure they do their own work. In addition to math achievement, students in cooperative groups tend to make fewer errors and are more persistent and creative in problem solving situations.

There is considerable concern at this time about the attitudes of students toward math class and studying mathematics. These studies indicate that students working in cooperative learning groups feel more positive about studying math, feel more support from the teacher, and more support from peers to learn math than do students learning competitively or individualistically.

There is also evidence from these studies that learning math in a classroom that uses cooperative learning groups often can build an acceptance of difference among the students so that they can not only work effectively in a mixed ability group, a mixed sexually group, a group that includes different ethnic backgrounds or mainstreamed handicapped students, but that they gain an appreciation for the differences that exist and select to interact heterogeneously beyond math class. There will be increasing importance attached to building this kind of climate for acceptance of differences as schools continue to deal with sexism, racism and rejection of handicapped in classrooms. Structuring heterogeneous cooperative groups will not only tend to build the acceptance of differences but will also provide higher achievement and more positive feelings in all students.

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Teachers' Critical Moments: The Other Side to Student Understanding

Janet C. Shroyer

In his address to the Fifth International Group for the Psychology of Mathematics Education, Kilpatrick (1981) calls for more research to be conducted in the classroom. His recommendation comes at a time when there is an increasing use of the clinical interview technique to obtain information about students' understanding of mathematics. As a result of investigations of students' thinking, some interesting error patterns have been isolated and described. For example, in their study of students' knowledge of rational numbers, Behr and Post (1981) report the effect of visual-perceptual distractors on students' understanding of fractions. One way of lending credibility to such findings obtained outside the classroom is to find examples of the same or similar error patterns occurring during classroom instruction. Studies on teacher thought and behavior provide one source of classroom data from which to seek such examples. The purpose of this paper is to illustrate how this might be done and how the findings from the otherwise divergent lines of research might be used to increase our understanding about the teaching and learning of mathematics.

In her study of critical moments in the teaching of mathematics, Shroyer (1981) isolated and examined incidents of student difficulty and insight for which teachers experienced momentary crises. Evidence of the teachers' cognitive difficulties and emotional discomfort were sought from their thoughts and feelings reported through a process known as stimulated recall (Kagan, 1975). From her process tracing study of three teachers teaching units on rational numbers, Shroyer observed that the distinctive types of student difficulties and insights which caused teachers' problems were also indicative of students understanding of mathematics. For example, one critical moment illustrates the impact of visual-distractors, similar to those reported by Behr and Post (1981), and student misconceptions in identifying fractions from models. One type of visual distractor was noted by Behr and Post when students were distracted from modeling a fraction by more divisions that were necessary. For example, students might have trouble representing 1/3 in a rectangle which was already divided into six squares (see Figure 1).

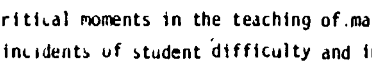

Figure 1.

Shroyer's study also offers some clear indicators of what teachers need in the way of assistance if they are to effectively cope with these distinctive student difficulties and insights during interactive instruction. Before describing the critical moment and the student difficulty it illustrates, some remarks need to be made about the setting in which Martha, the teacher who experienced the critical moment, was teaching.

Setting

Martha was teaching an introductory unit on fractions in which she relied al most entirely on the use of concrete and pictorial tasks. Six days of lessons were included in the study. On the first five days she focused on developing the concept of fractions beginning with unit fractions, then moving to other proper fractions and, finally, examining improper fractions. Cuisenaire rods were used for many of the different activities, both as rods to manipulate and as models of rectangles to draw on paper. The first day, students were asked to show unit fractions for designated units of Cuisenaire rods (e.g., 1/3 of dark green), about the rods (e.g., red is 1/8 of brown), and to model 1/3 in different ways.

Students were asked to compare or order fractions on the second, third, and fourth days and to identify equivalent fractions on the fifth day of the unit. Addition of fractions was introduced on the fifth day and was the only topic on the sixth day. Martha relied on total class instruction for most of the unit. She provided one separate work period for adding fractions and gave three short tests on the first three days. Students were very much involved with the tasks, and they eagerly participated in the classroom interaction. There were no disruptive incidents of student misbehavior and very few minor infractions of classroom decorum. The following description of Martha's critical moment which occurred on the second day will include relevant information about the anticedent behaviors and conditions, the specific student difficulties, the teacher's elective actions taken in response to these student difficulties, and the consequences to the instructional flow of the same lesson and the next day's lesson.

Critical Moment

Martha's critical moment involves more than one incident of student difficulty. Several students had trouble naming parts of rectangles or circles which had been subdivided after one portion had been shaded or had the value written on it. For a rectangle cut into five equal pieces in which the first piece was identified as 1/5, for example, the next or remaining portion was identified as 1/4 (see Figure 2).

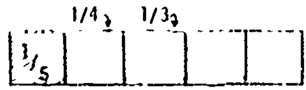

Figure 2.

The student's response was based on the fact that four pieces still remained. The next piece was labeled 1/3.

The first error in Martha's critical occurred during the first planned activity on the second day of the unit. The task was essentially the same as was given for a test on the first day Students were answering questions about the fractional values of parts of a rectangle formed by drawing around the dark green rod on grid paper with squares of 1 sq. cm. to match the face of a white rod. On the grid paper, each rectangle was automatically divided into six squares. Students had been asked to divide the rectangles and shade 1/2, 1/3, 4/6, 3/6, and 2/6 which they did with few difficulties. Referring to the rectangle in which the first square had been shaded as 1/6, Martha asked Rodney what was not shaded, and he responded with 1/5 instead of 5/6.

The second instance of this error pattern occurred when Martha was reviewing the same problems by asking questions about the figures she had already drawn on the overhead projector. Pointing to the same picture shown in Figure 1, Martha asked Rodney what was not shaded, and he replied 1/4. He had forgotte, or was unable to see from her drawing that the rectangle had been subdivided into thirds, he was responding 1/4 because he could count four squares that were not shaded. Before the reader forms any firm opinion about Rodney's ability or attention to the task from these two errors, it should be pointed out that during the same activity, he volunteered the first equivalent value for a fraction. He suggested 1/2 as another name for the model of 3/6.

For the first error, Martha acknowledged that Rodney was seeing five unshaded parts, reminded him they were working with sixths, and repeated her question. He answered correctly, but it appeared as though his helper may have whispered the answer to him. During this activity, helpers had been designated to assist those who had some difficulty with the test the first day. For the second error, Martha continued to ask questions when Rodney reported that the unshaded portion was 1/4 instead of 2/3 (see Figure 1). In this case, the helper was no longer present.

T. How many parts did we divide this [rectangle] into, Rodney?
S. Three.
T. Three. How many of them are shaded?
S. Two [referring to the two squares rather than one of the three divisions].

At this point, Martha began to realize how the small squares were interfering with Rodney's understanding of thirds.

> T. I see why you said two. One of the parts is shaded, isn't it? How many of the parts are not shaded?
> S. Two

Rodney's response was correct, but Martha seemed unaware of it at the time. Instead, the cause of his error was still on her mind as she went on to say,

> T. You are counting each little box, aren't you?
> S. No, but ..
> T. [going right on] I can see why you are confused; it's confusing. Rodney, tell me again how much is not shaded. If we divide that into thirds, how much is not shaded? [Pause and no response.] OK, I'll tell you--2/3. How many thirds are in the whole thing?
> S. Three.
> T. One is shaded; how many are not shaded?
> S. 2/3.

Two-thirds was not really the answer to the last question, but Martha did not seem to mind since Rodney had indicated the unshaded part was 2/3.

If the above two incidents had not qualified as a critical moment for Martha, two similar errors which occurred during the next activity probably would have.

and fourth instances of the same error pattern came during the next activity. Martha had switched to drawing circles and rectangles on the board and asking students to label each of the parts. Milton expressed some disbelief that each of the four equal pieces of a circle were 1/4 (Figure 3). After obtaining his

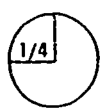

Figure 4.

agreement that each piece was 1/4, Martha drew another picture (see Figure 3), established the small portion was 1/4, and asked what all the rest would be. Milton replied 1/3, as though the other pieces were still showing, his error pattern was the same as Rodney's. In response, Martha offered him a word of encouragement and told him the correct response: "You are on the right track; it would be 3/4." For the next problem, Martha drew a candy bar in the shape of a rectangle, divided it into fifths, and asked Todd to label the first piece. She noticed him counting the pieces and drew this to the attention of the class before he correctly identified the first piece as 1/5, telling her to "scratch out one [of the pieces]." He proceeded to label the next two parts as 1/4 and 1/3. Martha acknowledged this was confusing and called on Milton. He responded 1/4 for the second piece as well. Another redirect elicited the correct response of 1/5, and the dialogue continued:

> T. You would label it 1/5, too. How can that be? Kevin?
> S. Because all those things [squares] are all the same.
> T. All right, every one of them is the same size, isn't it. And every one of them is called...?
> S. 1/5 [called out by a number of students in the class].
> T. What if I am talking about two of them? Steve? [No response.] Milton?
> S. 2/5.
> T. That's right, that is 2/5. So what am I going to label this piece right here [pointing to the next square], Rodney?
> S. 1/5.
> T. And what about this one, Rodney?
> S. 1/5.

Martha responded affirmatively and moved on to another picture. When reviewing how many thirds, fifths, fourths, and halves were in each of the pictures, she called on both Todd and Milton again. This time her answers were correct.

Much might be said about the particular actions that Martha took in response to these errors, but such discussion must necessarily be delayed. For right now, it is important to note that while Martha may not have done a lot to correct the students' misconceptions at the time they occurred, she was persistent in trying to find out if they understood. She continued to call on the same students until they were able to give correct responses.

Several aspects of Martha's mental processing of this critical moment were apparent from the stimulated recall data. First, she was not as cognizant of the error pattern as it may have appeared with the first incident. It took the second and possibly even the third and fourth instances of the same error pattern before she was fully aware of the misconception and its significance to the student. This was apparent when she did not talk about the error from the first instance but talked instead of the one from the second. In emphasizing her fascination with the recurring incidents of the same error pattern, however, she appeared to the reporting reflective rather than a recalled reaction.

The only indication that Martha may have been alerted to the difficulty students had in identifying fractions from the picture models came from one remark she made during the recall session on the first day and the opportunity she had for seeing some of tne erros they made on the first test. Neither seemed adequate, however, for her to anticipate and prepare for such difficulties.

Second, there were several reasons why Martha responded to the students in the manner that she did, including (a) her perception of the students' abilities in mathematics, (b) her own lack of knowledge about what to do, and (c) her conscious awareness of the error pattern produced by the critical moment itself. Martha evidenced more reluctance to push for correct responses from students she perceived as being less capable in mathematics which is similar to other findings of the effects of teacher expectation (Brophy & Good, 1974). Although she did not admit to any difference in her treatment of these students, differences in her actions during critical moments involving students perceived as having more or less ability were apparent. Martha also spoke of not wanting to confuse a confused student even more as the result of her own learning experiences. She told of her mind's feeling like an "eggbeater" when teachers tried to explain mathematics to her.

Martha acknowledged she did not know how to respond to the difficulty students were having in labeling fractions from the models. She really had little alternative other than to continue much as she had been doing which was to emphasize the equivalence of the portions into which the figures had been subdivided. Her conscious awareness of the errors these students were making was evident in the persistence of Martha's efforts to elicit correct answers to subsequent questions from the same students. This was not at all typical of student difficulties which were routinely processed with less conscious awareness. At the same time, Martha found satisfaction in getting correct answers from these same students.

As a consequence of the teaching and recall experience and the test results from the second day, Martha planned another activity for the next day to clear up students' misunderstanding of fractions. She eliminated one visual-distracter by drawing around the rods and leaving the portioning to the students. Students did so well with this activity that Martha became distressed because she was not doing more to help the students. It was a differenty type of critical moment brought about by the discrepancy between her expectation for and the reality of the students' performance. The other problem of failing to recognize the unit was either corrected or failed to resurface.

Discussion and Implications

The critical moment just described compliments the work by Behr and Post (1981) in at least two important ways. First, the error pattern of Martha's students was somewhat different than what Behr and Post describe in the early version of their final report. This is at least partially due to the fact that students were not asked the same questions. As a consequence, somewhat different difficulties may have emerged from the two studies. Behr and Post describe how the presence of more subdivisions than are necessary to model a fraction can confuse the students and cause errors, but only one of the four errors in the critical moment occurred in the presence of such distractors. It was the second and also the most salient to Martha. Students in Martha's class may have been demonstrating the impact of a different visual-distractor as there was a perceptual cue present in each of the four cases. Students misinterpreted the fractional values of pieces of rectangles and circles from which one or more piece had already been shaded or had a number written on it.

The second way in which the description of Martha's critical moment compliments the work by Behr and Post is that it offers a teacher's perspective. As already described, the teacher's perspective includes her interpretation of the students' errors, her actions taken in response to the errors, reasons for the actions she took, and some possible consequences of the critical moments. Both teacher and student perspectives are needed to understand the teaching and learning of mathematics.

As is apparent in the critical moment just described and in the other critical moments from the Shroyer study, teachers need help in interpreting and evaluating the more distinctive student responses or contributions. They also need suggestions on how to respond to them. This requires task- and topic-specific information about the difficulties and insights students might encounter and prescriptions for teacher responses.

Investigations of student understanding of mathematics and teachers' critical moments have proved to be a valuable source of information about possible student difficulties and insights. However, such studies do not reveal which teacher actions are effective or even appropriate in the interactive teaching situation. Teachers' critical moments occur because their actions are not routinely determined by the task and questioning patterns of an activity. Teachers are concerned not only with their mathematical response to unpredictable and distinctive student performance, but also with having to respond in the presence of and in a way which involves the class.

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CHILDREN'S THINKING ABOUT ADDITION, SUBTRACTION, AND ORDER OF DECIMAL FRACTIONS

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The purpose of this paper is to report the rationale which children use in thinking about selected decimal concepts and addition and subtraction algorithms. The larger study of which this was part included pretests, instruction, and posttests on decimal fraction concepts and notation, addition and subtraction algorithms for decimals, and multiplication concepts and algorithms for decimals. ~~Children were selected for interviews on the basis of the posttest~~ scores. An effort was made to choose a representative cross section of interviewees with respect to performance on decimal concepts and decimal algorithms.

One class of grade five children was selected at each of two schools in Clarke County, Georgia. Eight of the 23 subjects of school 1 and nine of the 25 subjects in school 2 were selected to be interviewed. The mean posttest score of the group chosen was 49% on decimal concepts and 78% on addition and subtraction algorithms while the mean score for the entire sample was 48% on decimal concepts and 74% on addition and subtraction algorithms.

The instructional sequence began with a review of whole number place value and the role of digits in numeration. Then decimal fractions were introduced based on partitioning a unit into 10 or 100 equal parts. Base ten models were employed as the major concrete embodiment. Place values and word and numeral names were introduced. Common fraction notation was avoided throughout instruction. Base materials were traded to show equivalence, for example, of 32 hundredths and 3 tenths, 2 hundredths. Order was shown by placing numbers on a number path. This unit of nine instructional periods was followed by a test on decimal concepts and notation.

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The second unit of instruction on addition and subtraction algorithms lasted about eight periods and was followed by a test on these algorithms. Instructional strategy for addition and subtraction relied heavily upon the base ten materials and place mats in which lines were drawn to indicate places, addends and sum. No labels were indicated for the places, but the children were asked to think of the names and to keep the columns straight -- tenths under the tenths, etc., analogous to tens under the tens, etc., for whole numbers.

The addition-subtraction section was followed by a section on multiplication and in one school a section on division as well. Two to four weeks after the final posttests the interviews were held.

INTERVIEW PROCEDURE

Each child was interviewed individually by the investigator who was also the teacher for the lessons. Present at the interview was another adult whose task it was to record the interview in shorthand. Transcripts were later typed by the recorder from the shorthand version. As each child entered the interview space, he/she was introduced to the recorder and told of her role to make notes. The tasks were introduced as some questions like those we had been doing in class. The child was made aware that the investigator would be asking for an explanation of his/her thoughts about doing the decimal questions. Each child was asked for cooperation and each one agreed to do his or her best. The questions were presented one at a time and the child was given an opportunity to decide a plan of attack. Often the child would obtain an answer and the interviewer would then ask, "How do you know how to do that?" or "Explain what you did." Following the child's response, more specific questions and probing ensued.

RESULTS

Task A: 6.3 + 824.43 + 32.
Nine of the 17 children interviewed answered correctly. Among them, four mentioned aligning the places of the numerals. As Suzanne said, "You line the tenths up with the tenths and the whole numbers with the whole numbers."

Four children mentioned aligning the decimal points. Donna put it succinctly, "You have to line up the decimals because you are adding ... so I can keep track."

Among the children who got the question correct, five annexed a decimal and zeros to make 32.00 from 32. These children were asked to justify this move or made a spontaneous comment. Mark said, "It makes it easier to keep the tenths and hundredths places. It's the same value." Tracey said, "The zeros just hold the place, they don't change the amount. You still have 32 and no more."

Of the 9 children whose responses were classified as incorrect, six of the children aligned the right-hand digit of each numeral as if adding whole numbers without regard for the decimals at all. Four of these then counted decimal places in the question and arrived at the answer of 82.538. Tom verbalized the procedure as "You line up the back numbers ... you count up the decimals."

The remaining three children who missed the question aligned the decimals in 6.3 and 824.43. Then 32 was placed at the right-hand side and one actually wrote .32. Another, Beth, in response to the investigator's questions said, "Three is in the tenths [place] and two is in the hundredths." However, in further probing Beth said that the number 123 has no places because there is no decimal. She correctly named the placed in 123.0 but thought it was bigger than 123.

Task B: 6 UNITS AND 4 TENTHS TAKE AWAY 1 UNIT AND 9 HUNDREDTHS.

Nine of the 17 children got this task correct. Each one translated to usual notation, annexed a zero (6.40–1.09), and wrote it in vertical format. Only one child mentioned "... I think you are supposed to line up the decimals" but of course all did. Most of the questions and responses had to do with zeros holding places. Jerome read them correctly "Six and four tenths ... six and 40 hundredths", then added "But it's the same. You could keep on adding zeros and you would still have the same because zeros count as nothing." Mark said, "It makes it easier and you couldn't do it without the zero because you have nothing to take away the nine and I have to borrow." Regarding 1.09 Donna said, "The 9 would be in the wrong place, tenths place, without a zero" and Mark said, "You couldn't put 9 hundredths here and leave the tenths place blank."

Among the eight children whose responses were classified as incorrect, there were three classes of errors.

$$\begin{array}{lrlrlrlr} (1) & \begin{array}{r} 6.4 \\ -\ \underline{1.09} \\ 5.49 \end{array} & (2) & \begin{array}{r} 6.4 \\ -\ \underline{1.9} \\ 4.5 \end{array} & \text{or} & \begin{array}{r} 6.4 \\ -\ \underline{1.9} \\ .45 \end{array} & (3) & \begin{array}{r} 1.09 \\ -\ \underline{6.4} \\ 4.5 \end{array} \end{array}$$

Four children made an error of type 1. Joey said, "I could have added a zero [to 6.4] ... That [6.40] would be different." The first one is correct because "it tells you six and four tenths and that is six and 40 hundredths." Two children who got the result 4.5 said, "I don't know" or "I can't remember" when asked to justify their answer. Tammy decided upon .45 and confidently said, "Because there's two numbers at the end of the decimal point." The only student who made the type 3 error when finished, spontaneously said it would be easier to put this (6.4) on top. After responding correctly to questions about the relative size of 6.4 and 1.09, she did the original question again, correctly.

Task C: TAKE AWAY 4 - 2.47.

Eight of the 17 children interviewed got this question correct. As in the previous question each one did so by rewriting as 4.00 - 2.47 stacked vertically. In response to questions each one rationalized the annexing of zeros. For example Suzanne said, "You can't take 47 from nothing, you need the place values." Only [illegible] was questioned about making 4.00 the minuend. She replied, "Because you couldn't put 2.47 on top because it is less than 4." In response to a question about where to place the decimal, Mark replied, "In adding you have the decimal lined up and you put the decimal in the answer under the decimal in the problem."

Among those whose responses were classified as incorrect, there were three clusters of errors.

$$\begin{array}{lrlrlr} (1) & \begin{array}{r} 2.47 \\ -\ \underline{4} \\ 2.43 \end{array} & (2) & \begin{array}{r} 4. \\ -\ \underline{2.47} \\ 2.47 \end{array} & (3) & \begin{array}{r} 4.00 \\ -\ \underline{2.47} \\ \end{array} \end{array}$$

Four children made the error of type 1. In rationalizing her work, Tracey said, "You don't have a decimal in four and four is bigger than two. So you don't say two take away four because two can't take away four. I can't put my four under my two." After further questioning which provided some hints, Tracey and two others did the question correctly. Beth correctly named the places in 2.47 but maintained that "The four doesn't have a place because there's no decimal".

Two youngsters made errors of the second type. They had both made similar mistakes on Task B. Neither could complete the task correctly, due to subtraction mistakes, even after being told by the interviewer to annex a decimal point and zeros and after aligning the numerals correctly. Three subjects set the question correctly as indicated in error type 3, but did not arrive at a correct solution. Two of these made subtraction mistakes and the third placed the decimal point incorrectly. After obtaining .153 he annexed a zero (.1530) "Because you need four places to ... Because there are four in the question, I need to point off four in the answer."

After observing the kind of mistakes made and the follow-up questions asked, it appears that it would be beneficial to be more consistent in the type of questions asked of those who got a correct answer. Perhaps everyone should be asked about which number should be the minuend and subtrahend and perhaps about the relative size of these. Everyone should be asked about annexing zeros and aligning the numerals properly in columns. Finally everyone might be asked about the decimal point in the answer.

Task D: ADD 12 TENTHS TO 13 HUNDREDTHS.

Most of the nine (of 17) students who successfully completed this task wrote

1.20
+0.13 , although some omitted one or the other of the zeros. Of course the most interesting aspect of this question is the equivalence of 12 tenths and 1.2 or 1.20. Seven of these children used a rationale similar to Tracey's. "You can't write 12 tenths so you have to regroup 10 tenths making one unit and 2 tenths left. ... Put a zero up there to hold the hundredths place." One student eliminated .12 as 12 hundredths and the other observed that 12 tenths would have only one decimal place.

Among the eight subjects who did the question incorrectly, three added .12 and .13, two added .12 and .013, and the rest had variations on the same theme. When questioned about their work, no pattern emerged and most children were unsure. Only Donna had conviction that .12 could be either 12 tenths or 12 hundredths. Seth thought .12 would be 12 hundredths, 12. would be 12 tenths and 1.2 would be one and two tenths. Dawn said 1.2 "Would be one and two tenths and 12. would be just twelve." The investigator asked four of these children to use the base ten models to help with the question. Two, who had been successful on the previous three tasks, were successful in using the blocks and in writing the representation on the paper. Two children, who had been unsuccessful on the previous three tasks, did not use the base ten materials consistently, nor arrive at a correct result. It might have been revealing to have routinely asked the children to use the materials to justify their work. As it was, it seemed inappropriate to use the materials as a further probing device when a child was satisfied with the result. In summary, about half of the children appreciated the equivalence of 12 tenths and one and 2 tenths, and were able to solve the problem. Children who did not understand this equivalence were unsuccessful.

Task E: WHICH IS LARGER 2.45 or 2.5?

This task was surprisingly easy as 13 of the 16 children who were asked, responded correctly. Eight of these children thought of annexing a zero to 2.5. For example, Tammy said, "If I add a zero then that would be two and 50 hundredths and two and 50 hundredths is larger than 2 and 45 hundredths." Four of the children compared the numbers of the tenths places. Roderick said, "Because four is smaller than five .... I look at the tenths place." Tracey said it strangely, "The more numbers you have to the right of the decimal, the less they are." Further probing revealed that she understood. Suzanne had a third unique strategy. "You say this 2.45 is like four and one-half tenths and you compare ... to two and five tenths and two and five tenths is more." The first and more common strategy was to be expected because equivalence was emphasized in instruction. The second strategy was not emphasized in instruction and must have been a generalization by the children from their knowledge of whole number place value.

Only three children believed that 2.45 is larger than 2.5. One said, "Because hundredths make it larger than tenths" and a second said, "forty-five is bigger than five." In both of these cases the children seemed to be focusing on a comparison of five and 45 without particular regard for the place values. The third child could not verbalize a rationale.

It was surprising, based on past general impressions, that 0.81 of the subjects were successful on Task E. Perhaps this was due to the instruction or perhaps to the nature of the task. These children were aware of the equivalence of 2.5 and 2½, and ½ is the easiest of all fraction concepts for children to grasp. Further, 2.45 is halfway between 2.4 and 2.5 as Suzanne pointed out.

DISCUSSION

The tasks that were chosen forced the children to alter or transform the problem before proceeding in algorithmic fashion, and as a result were relatively difficult. However, most of those who answered correctly were able to give a rational explanation for their work. Sometimes the explanation was different from that given in instruction. For example, in addition and subtraction the students had been shown to align like places in columns analogous to the algorithms for whole numbers. Some subjects verbalized this as "line up the decimals", a perfectly acceptable rule. A common mistake was to invoke some part of a rule for a multiplication algorithm like "count the decimal places."

The tasks chosen for this study gave emphasis to the concept and rules for equivalent decimal expressions. One type, expressing a whole number such as 32 as a decimal, say 32.00, was especially troublesome. Most who were successful on Task A used this move. Not surprisingly, all who were successful on Task C expressed 4 as 4.00. This algorithm is easily stated. However, the more substantive concept that whole numbers are a part of the same rational number system and have many artibrary decimal expressions is more difficult.

Another error which is interesting is reversing the order in subtraction. Some children ignored the order specified in the task and devised ways to subtract what they thought was the smaller number from the larger. The need for changing the order was no doubt necessitated by their lack of skill in determining order and equivalent forms.