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ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to the basic number work in the elementary school. In addition to an introduction to the unit, the text has sections on numbers, numerals, and numeration (ancient to modern systems), using materials to work in bases other than 10, numeration in the elementary school, and diagnostic and remedial work in numeration. (MP)

NUMERATION

Mathematics-Methods Program
units written under the direction of

John F. LeBlanc or Donald R. Kerr, Jr. or
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MATHEMATICS EDUCATION DEVELOPMENT CENTER

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

Numeration
Addition and Subtraction
Multiplication and Division
Rational Numbers with Integers and Reals
Awareness Geometry
Transformational Geometry
Analysis of Shapes
Measurement
Number Theory
Probability and Statistics
Graphs: the Picturing of Information
Experiences in Problem Solving

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO THE NUMERATION UNIT

Numeration systems have developed from the need to record numbers and to communicate these numbers to others. Throughout the course of time many systems have evolved, and their increasing sophistication and usefulness have accompanied the growth of mathematics itself. These systems have gradually become more efficient as vehicles for quantitative expression and numerical computation.

Since whole-number and decimal computations are based on an understanding of numeration, careful sequencing and development of numeration concepts in the elementary school are essential. Grouping and place value activities follow early number work in the primary grades. At later stages numeration concepts continue to play a significant role in the curriculum as a child learns to compute. In particular, an emphasis is placed on skill with decimals, scientific notation, and estimation. These skills have become more important with the increasing use of the hand-held calculator and with the introduction of the metric system in the U.S.

The <u>Numeration</u> unit is divided into four major sections to correspond to the following points of emphasis:

I. Numbers, Numerals, and Numeration: Ancient to Modern Systems
II. Using Materials to Work in Bases Other Than 10
III. Numeration in the Elementary School
IV. Diagnostic and Remedial Work in Numeration

In the introduction to each section, an attempt has been made to give perspective to the activities included, and questions have been posed to help you focus your thinking on important issues. Teacher Teasers (mathematical puzzles) are interspersed throughout the unit for your interest and enjoyment. Several games have also been included. There are times when a teacher needs to call upon resources to motivate, reinforce, or provide enrichment opportunities for children. Puzzles and games, wisely selected and used, can be quite useful for such purposes. You are encouraged to keep a file of puzzles and games and add to it from your unit experiences and personal study.

Section I

NUMBERS, NUMERALS, AND NUMERATION: ANCIENT TO MODERN SYSTEMS

In this section you will be given a chance to analyze some of the basic characteristics of numeration systems. Our decimal numeration system, which is called the Hindu-Arabic system, possesses some outstanding characteristics. In your everyday living, these characteristics are taken for granted, so you have no need to focus upon them. In teaching children, however, it is important that you are aware of them.

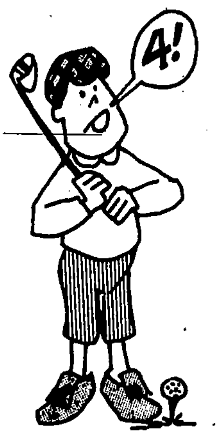
Our numeration system is compared to other, ancient numeration systems for two reasons: one, so that each characteristic of the Hindu-Arabic system may become clearer as you examine a system not having that characteristic; and two, so that you may develop some interest in and appreciation for ancient numeration systems. Some of the characteristics identified will be focused upon and used in Sections II and III, in particular, the concepts of base, grouping, and place value which play an important part in children's learning.

In Activity 1, a distinction is made between the terms "number" and "numeral." Activity 2 places you in the role of a contest judge. You are to choose the best (and poorest) numeration system on the basis of characteristics you deem important. In Activity 3 you will work on a project aimed at broadening your experience with numeration systems.

MAJOR QUESTIONS

What makes a good numeration system? List both mathematical properties and other criteria for judgment.

TEACHER TEASER



Fore_ _ _ _s!

Using four 4's and the arithmetic signs +, -, x, and ÷ name each of the numbers 1 through 20. Here's an example.

$1 = 44 \div 44$ or $1 = (4 + 4) \div (4 + 4)$

Some are tricky--but all can be done!

ACTIVITY 1

NUMBERS AND NUMERALS

FOCUS:

In this activity you will have an opportunity to review the distinction between number and numeral, a distinction you learned and made use of in the elementary school. At the end of this activity you will be asked to summarize the differences between a number and a numeral.

DISUCSSION:

An object and the names given to that object are different. Similarly, a person and the names used to identify that person are different. Some individuals have several names, each used on different occasions, often for different reasons. For example, a father might sometimes be referred to as "Dad" or "Mr. ______" or "the foreman of the first shift" or "the chairman of Elm Township School Board," and so forth. Each name given is different, but the same person is named each time. A similar distinction exists in mathematics between a number and the different numerals which name the number.

DIRECTIONS:

1. Write three or four names by which you are named.
2. A standard numeral for a number is one which is in the simplest form and is most widely recognized as naming the number it represents. The standard numeral for 15 + 7, 2 x 11, 20 + 2, etc. is 22. Write the standard numeral for the number named by each of the following sets of numerals.

	Standard numeral
a) $6 + 3$, 4.5×2, $10^2 - 91$	______
b) $\frac{7}{14}$, $\frac{48}{96}$, $4 \div 8$	______

c) 6 + 7, 16 - 3, 5 + 8 ____________

d) $\frac{21 + 9}{10}$, $3^2 \div 3$, $\frac{15}{7} \times \frac{7}{5}$ ____________

3. Write three other numerals for the number represented by each of the following standard numerals.

   a) 50

   b) 8

   c) $\frac{3}{4}$

   d) 22

4. In the late 50's and early 60's, the elementary school mathematics textbooks made a great issue of the number-numeral distinction. This apparent overemphasis of the distinction seemed to cause some confusion among teachers, who worried, for example, whether the symbol '8' written on the chalkboard should be called "the number 8" or the "the numeral 8." The intention of those who emphasized the distinction was to impress teachers and students that the distinction could be useful. For example, it is useful in adding the fractions $\frac{1}{2}$ and $\frac{1}{3}$ to point out that another name (or numeral) for $\frac{1}{2}$ is $\frac{3}{6}$ and another name for $\frac{1}{3}$ is $\frac{2}{6}$.

   a) Discuss the extent to which a teacher should be concerned about precision in the use of the words "number" and "numeral." (Is there ever a danger in overemphasizing the distinction?)

   b) Discuss and list a few examples for which using the distinction between number and numeral might be useful in teaching children.

5. Briefly describe the difference between a number and a numeral. After you have described the difference, look up the definitions of each in two different sources (one of them a mathematics book or mathematics dictionary). Then revise your description of the difference, if necessary.

ACTIVITY 2

RECORDING NUMERALS: NUMERATION SYSTEMS (THE WORLD NUMERATION CONTEST)

FOCUS:

In Activity 1, you examined the distinction between a number and a numeral. A numeration system is an organized way of recording numbers, no matter how large. In this activity you will become acquainted with various numeration systems and some of the properties of each of these systems. This introduction to different systems will provide illustrations of the importance of the properties that are inherent in our numeration system, the Hindu-Arabic system.

MATERIALS:

Elementary mathematics textbook series.

DISCUSSION:

Human development of number and counting parallels the human development of language. Initially, marks in the sand or tallies on a wall were sufficient. The oral development of numerals was also limited until the need arose for numerals greater than ten fingers and ten toes. Almost every civilization eventually and independently established a base of ten for its numeration system. There are, however, still remnants of base 20 (fingers and toes) in several languages (in French, "quatre-vingt," four twenties; in English, "score" is sometimes used as a substitute base in naming numbers). Studies of primitive tribes find that in their early numeration systems they used the base of two. Their counting was limited to very small numbers and they counted "one, two, two and one, two and two, two-two and one." As civilization developed, the use of the base of ten became widespread. In examining the systems presented in this activity you will note that almost all are based on ten.

This activity is presented as a contest. That is, you will be introduced to several systems; you will be asked to identify the characteristics of each system; you will be asked to judge the systems and pick the "best" system; and finally, you will be asked to defend your choice of the winner. At times, you will be asked to translate numerals in an unfamiliar system into equivalent Hindu-Arabic numerals or vice versa.

To help you keep track of the characteristics of each system a contest worksheet occurs on page 24. You should make notes on it after learning about each system. You will be introduced to the definitions of certain characteristics of numeration systems by analyzing our own system.

DIRECTIONS:

PART A: THE HINDU-ARABIC NUMERATION SYSTEM

1. The Hindu-Arabic numeration system has a base of 10.

A numeration system has a base if it reflects a process of repeated grouping by some number greater than one. This number is called the base of the system, and all numbers are written in terms of powers of the base. If the base is b, numbers are written in terms of 1, b, b^2, b^3, etc.

In the Hindu-Arabic system the base is 10 and numbers are represented by sequences of digits associated with powers of 10. Digits are the individual symbols used alone or in combination to represent a number. In the Hindu-Arabic system, there are ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Since the base in the Hindu-Arabic system is ten, we record a number of objects in groups of ten.

How many groups of ten are there in 87? ________ How many in 43? ________ How many in 60? ________

2. The Hindu-Arabic numeration system is a place value system.

A numeration system is a <u>place value</u> system if the value of each digit is determined by its position in the numeral.

a) For example, each 3 in 333 has a different value. What is the value of the left-hand 3? The middle 3? The right-hand 3?

b) Tell what each digit in 6403 means. How many groups of ten does the 6 in 6403 represent?

3. The Hindu-Arabic numeration system is multiplicative.

A numeration system is <u>multiplicative</u> if each symbol in a numeral represents a different multiple of the face value of that symbol. In a system which has a base and place value the position of a symbol in a numeral determines what power of the base is multiplied by the face value of the symbol to give the place value of the symbol.

In the Hindu-Arabic system, the base is ten, so the digits (symbols) in a Hindu-Arabic numeral represent numbers which are multiples of the digit by one, ten, one hundred, etc. For example, in 367 the 3 represents the product of 3 by 10^2, the 6 represents 6×10^1, and the 7 represents 7×1.

Later, when you study the Oriental numeration system, you will see a multiplicative characteristic that is slightly different from the multiplicative property of the Hindu-Arabic system. In the Oriental system the multipliers appear explicitly; for example, 367 would be written 3(100)6(10)7.

Tell what product is represented by the 5 in each of the following numerals.

653 5210 7513

4. The Hindu-Arabic numeration system is additive.

> A numeration system is additive if the value of the set of symbols is the sum of the values of the individual symbols.

For example, 235 is equal to 200 + 30 + 5. Numerals written in their additive form (such as 235 = 200 + 30 + 5) are called expanded numerals. Write the expanded numeral for each of the following.

758 = ______________________________

2564 = ______________________________

3010 = ______________________________

When we write expanded numerals we see the base, place value, multiplicative and additive features of the Hindu-Arabic system.

5. The Hindu-Arabic system has a zero.

6. The Hindu-Arabic numeration system is a unique-representation system.

> A numeration system is a unique-representation system if each numeral refers to one and only one number.

Discuss: What would be the disadvantages of a numeration system in which each numeral did not represent a unique number?

7. Discuss the Hindu-Arabic numeration system to determine any other characteristics it may have. For example,

   - Is the Hindu-Arabic system efficient; that is, does it use a small number of symbols to write any given number?
   - Is the Hindu-Arabic system convenient, easy to use?

8. List any other characteristics of the Hindu-Arabic numeration system on the Contest Worksheet.

PART B: THE PRIMITIVE NUMERATION SYSTEM

The use of tally marks to record numbers was common among primitive tribes. This system is displayed below.

Primitive Numeral	Hindu-Arabic Equivalent (translation)
/	1
//	2
///	3
////	4
/////	5
//////	6
///////	7
////////	8

1. Is the Primitive system additive? multiplicative?
2. Do we ever use Primitive numerals today? If so, how? when?
3. Suppose you were to write the Primitive numeral for 26. Can you easily tell by looking at the numeral that it stands for 26? What modification of the Primitive system helps to make numerals more easily recognizable?
4. Is there a Primitive numeral for zero?
5. Determine whether the Primitive numeration system has a base and whether it is a place value system. Note the characteristics of the Primitive numeration system on the Contest Worksheet.

PART C: THE EGYPTIAN NUMERATION SYSTEM

The earliest Egyptian numerals were found on papyrus, wood, and pieces of pottery which date to the beginning of the 34th century B.C. This numeration system evolved over a period of three or four centuries, and was used for many centuries thereafter. The characters found on the walls of a temple at Luxor (7th century B.C.) are very clear and precise. Apparently, the numerical hieroglyphics made on stone were made with great care. While the hieroglyphic characters were commonly written from right to left, they were also written from left to right. The symbols for 1 and the first six powers of 10 are shown in the following chart.

Egyptian System		Hindu-Arabic Equivalent
Numeral	Description	
𓏺	(vertical staff)	1
𓎆	(a heel bone)	10
𓍢	(a scroll)	10^2 or 100
𓆼	(a lotus flower)	10^3 or 1000
𓂭	(a pointing finger)	10^4 or 10,000
𓆐	(a burbot fish)	10^5 or 100,000
𓁨	(a man in astonishment)	10^6 or 1,000,000

The examples below indicate the way numbers are written in this system.

EXAMPLES

Egyptian Numeral	Hindu-Arabic Equivalent
𓏺𓏺𓏺𓏺𓏺𓏺	6
𓍢𓎆 or 𓎆𓍢	110
𓆼𓎆𓎆𓎆𓎆𓎆 / 𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺 or 𓎆𓎆𓎆𓎆𓎆𓆼 / 𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺	1059

1. The Egyptian numeration system has a base of ten, as does the Hindu-Arabic system. Discuss how the Egyptian and the Hindu-Arabic systems are alike and how they are different.

2. Using the preceding charts, translate the following numerals from Egyptian to Hindu-Arabic and vice versa as indicated.

Egyptian Numeral	Hindu-Arabic Numeral
a) ∩∩∩∩ |||	
b)	141
c) 𓆼𓆼𓆼𓍢𓍢𓍢𓎆𓎆	
d)	6028

3. Does the Egyptian system have a place value scheme? Does it have have a zero? Is it additive? Is it multiplicative?

4. What advantages do you see that the Egyptian system has over the Primitive numeration system?

5. Record the characteristics of the Egyptian numeration system on the Contest Worksheet. Be sure to include any special features or characteristics of this system that you may have discussed.

PART D: THE ORIENTAL NUMERATION SYSTEM

Perhaps the most unusual feature of the traditional Chinese-Japanese numeration system is the multiplicative scheme it uses. When a smaller-valued symbol precedes a greater-valued symbol, the value of the two symbols is obtained by multiplying (see examples below). The system illustrated here uses the symbols presently used by the Chinese. Numerals may be written from the top down or from left to right.

Oriental Numeral	Hindu-Arabic Equivalent
一	1
二	2
三	3
四	4
五	5
六	6

Oriental Numeral	Hindu-Arabic Equivalent
七	7
八	8
九	9
十	(b) 10
百	(b^2) 10^2 or 100
千	(b^3) 10^3 or 1000

Some examples of numerals written in the Oriental numeration system follow. Note in the first example that we have symbols for 5, 10 and 3. Since the 5 is less than 10 and it precedes the 10, the 5 is multiplied by 10 to get 50. The 3 is then added to get 53.

Oriental Numeral	五	(5)	一	(1)	五	(5)
	十	(10)	百	(100)	千	(1000)
	三	(3)	二	(2)	六	(6)
			十	(10)	百	(100)
			六	(6)	二	(2)
					十	(10)
					五	(5)
Hindu-Arabic Numeral	53	5(10) + 3	126	1(100) + 2(10) + (6)	5,625	5(1000) + 6(100) + 2(10) + 5

1. Earlier, when you studied the Hindu-Arabic system, you noted that it had a multiplicative feature. How does the multiplicative feature of the Hindu-Arabic system differ from the multiplicative feature of the Oriental system?

2. Does the Oriental system have a place value scheme?

3. Translate the following numerals from Hindu-Arabic to Oriental and vice versa as indicated.

Oriental Numeral	七百四十一		三千九十四	
Hindu-Arabic Numeral		2057		415

4. Does the Oriental system have a base? If so, what base does it have? Is the system additive? Does it have a zero?

5. Note the characteristics of the Oriental system on the Contest Worksheet. Be sure to include comments reflecting any special characteristics in your discussion.

6. Looking back at the Primitive, the Egyptian and the Oriental numeration systems, which would you prefer to use to do the following computations? (You do not have to do these computations unless you want to!)

$$\begin{array}{r} 643 \\ + \underline{817} \end{array} \qquad \begin{array}{r} 375 \\ \times \underline{\ 12} \end{array} \qquad 14\overline{)695}$$

PART E: THE ROMAN NUMERATION SYSTEM

The most unusual characteristic of the Roman numeration system is its use of a subtractive principle in the writing of numerals. According to the subtractive principle, a symbol for a smaller number placed before a symbol for a larger number indicates that the smaller number is to be subtracted from the larger number. Thus I has a value of 1 and V has a value of 5, but IV has a value of 5 - 1, or 4. On the other hand, VI has a value of 5 + 1, or 6.

The subtractive principle was not unique to the Romans. This principle has been found in several primitive cultures. The Zuni Indians used this principle in their use of knot numerals. A medium knot indicated 5 and a small knot before it indicated 5 - 1, while a small knot placed after was 5 + 1.

The following table illustrates the basic symbols used in the Roman numeration system.

Roman Numeral	Hindu-Arabic Equivalent
I	1
V	5
X	10
L	50
C	100
D	500
M	1,000

Note: $\overline{V}$ means 5(1000) or 5,000 in Hindu-Arabic
$\overline{X}$ means 10(1000) or 10,000 in Hindu-Arabic
$\overline{L}$ means 50(1000) or 50,000 in Hindu-Arabic
$\overline{\overline{V}}$ means 5(1,000,000) or 5,000,000 in Hindu-Arabic

Some examples of Roman numerals and their Hindu-Arabic equivalents follow.

EXAMPLES

Roman Numeral	Hindu-Arabic Equivalent
MCMLXXIV	1974
$\overline{X}$MDCX	11610

1. Study the examples above. What value does each symbol represent? Point out where in these examples the subtractive principle is illustrated.

2. In the Roman system, as presented here, a symbol with one bar above it indicates that the face value of the symbol is multiplied by 1000; a double bar indicates multiplication by 1000 x 1000 or 1,000,000. What would a triple bar indicate? What property of the Hindu-Arabic system is similar to this feature?

3. Translate the following Roman numerals to Hindu-Arabic numerals and vice versa as indicated.

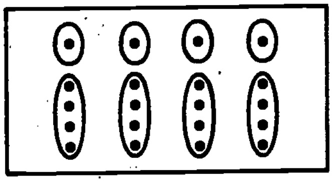
Roman Numeral	Hindu-Arabic Numeral
a) MCLXXIV	
b)	3,402
c) MCCLXXXIII	
d)	17,563

4. Does the Roman numeration system have a place value scheme? Does it have a multiplicative feature? Does it have an additive feature?

5. An important feature of the Roman system is the relationship between the values of the symbols used.

$$
\begin{array}{ll}
\mathrm{I} & \\
 & \times 5 \\
\mathrm{V} & \\
 & \times 2 \\
\mathrm{X} & \\
 & \times 5 \\
\mathrm{L} & \\
 & \times 2 \\
\mathrm{C} & \\
 & \times 5 \\
\mathrm{D} & \\
 & \times 2 \\
\mathrm{M} &
\end{array}
$$

This fact allows computation to be done on a Roman abacus, which is shown in the diagram below.



How is the grouping done in the Roman system? What value is associated with each of the sets of beads in the Roman abacus?

6. Look in some elementary school textbooks to see when and how children are introduced to the Roman numeral system. Why do you think this system is taught in the schools? Why aren't other systems taught? Should any numeration system other than the Hindu-Arabic be taught in the schools? Why?

7. Record the characteristics of the Roman numeration system on the Contest Worksheet. Be sure to include comments on special characteristics of this system.

PART F: THE MAYAN NUMERATION SYSTEM

The American Mayas had a fairly advanced civilization which included an elaborate calendar and a numeration system based on 20. While the civilization was all but destroyed in wars that occurred before the Spanish landings in 1517, the best estimates have asserted that the Mayan numeration system dates back to 3300 B.C. The most astonishing part of their system is that it contains a zero and has a place value scheme.

The Mayan system has a base of 20--well, almost! Working from the bottom up, we have the first place representing 1's, the second place representing 20's, the third place representing not 20 x 20 but 20 x 18. From then on the system is consistent. The fourth place represents 20 x 18 x 20, the fifth 20 x 18 x 20 x 20 or 18 x 20^3, and so on in the

Places

4th	X
3rd	X
2nd	X
1st	X

form $18(20)^n$. One explanation of the discrepancy in the third place may be found in the fact that the official Mayan year consists of 360 days.

The symbols used for the counting numbers less than 20 are very simply written using rods and beads. The symbols are given in the chart which follows.

Mayan	𝋠	𝋡	𝋢	𝋣	𝋤	𝋥	𝋦
Hindu-Arabic	0	1	2	3	4	5	6

Mayan	𝋧	𝋨	𝋩	𝋪	𝋫	𝋬	𝋭
Hindu-Arabic	7	8	9	10	11	12	13

Mayan	𝋮	𝋯	𝋰	𝋱	𝋲	𝋳	𝋡 𝋠
Hindu-Arabic	14	15	16	17	18	19	20

Mayan	𝋡 𝋡	𝋡 𝋢	𝋡 𝋱	𝋢 𝋠	𝋤 𝋠	𝋥 𝋣
Values	20 1	20 2	20 17	2·20 0	4·20 0	5·20 8
Hindu-Arabic	21	22	37	40	80	108

Mayan	𝋱 𝋱	𝋱 𝋲	𝋱 𝋳	𝋡 𝋠 𝋠	𝋳 𝋠 𝋥
Values	17·20 17	17·20 18	17·20 19	20·18 0 0	19(20·18) 0 5
Hindu-Arabic	357	358	359	360	6845

Some examples are provided for additional study.

EXAMPLES

Mayan Numeral		(1) (0) (7)		(6) (8) (4)
Hindu-Arabic Numeral	367	1(20·18) + 0(20) + 7	2,324	6(20·18) + 8(20) + 4

1. The Mayan system is a place value system. It also has a zero. Must a system have a zero to be a place value system? Why?

2. Is the Mayan system additive? Multiplicative? Does each numeral in this system provide a unique representation?

3. Fill in the blanks below with Hindu-Arabic numerals.

   a) 57 = 2(__) + 17

   b) 826 = ____(20·18) + ____(20) + ____(1)

   c) 7240 = ____ (20·18·20) + ____ (20·18) + ____ (20) + ____(1)

4. Translate the numerals below from Mayan to Hindu-Arabic and vice versa as indicated.

Mayan Numeral				
Hindu-Arabic Numeral		5284		2105

5. Use the Contest Worksheet to record the characteristics of the Mayan system. List any additional characteristics that you may have found in your study of the Mayan system.

6. Study the Mayan system symbols. The system has a base and place value of 20; can you also identify a sub-base? Describe the relation between the base and the sub-base? What other system uses this kind of grouping?

7. Complete the cross-number puzzle on pages 22-23, which involves translating numerals from the Egyptian, Oriental, Roman and Mayan numeration systems to their Hindu-Arabic equivalents.

PART G: JUDGING THE CONTEST ENTRANTS

1. Now you should be ready to judge the contestants in the World Numeration Contest. Study the Contest Worksheet to make sure it is completed. If you have some questions, review the work of this activity and/or consult with your instructor.

2. Make a list of the characteristics which you feel are important for a good numeration system. For example, you may want to include "ease with which computation can be performed," etc. (If you add this property, be sure to try some computational examples with systems.) Provide a weighting system, if you wish, and assign a value to each system. Rank the systems in order of your choice, and crown the winner!

CROSS-NUMBER PUZZLE

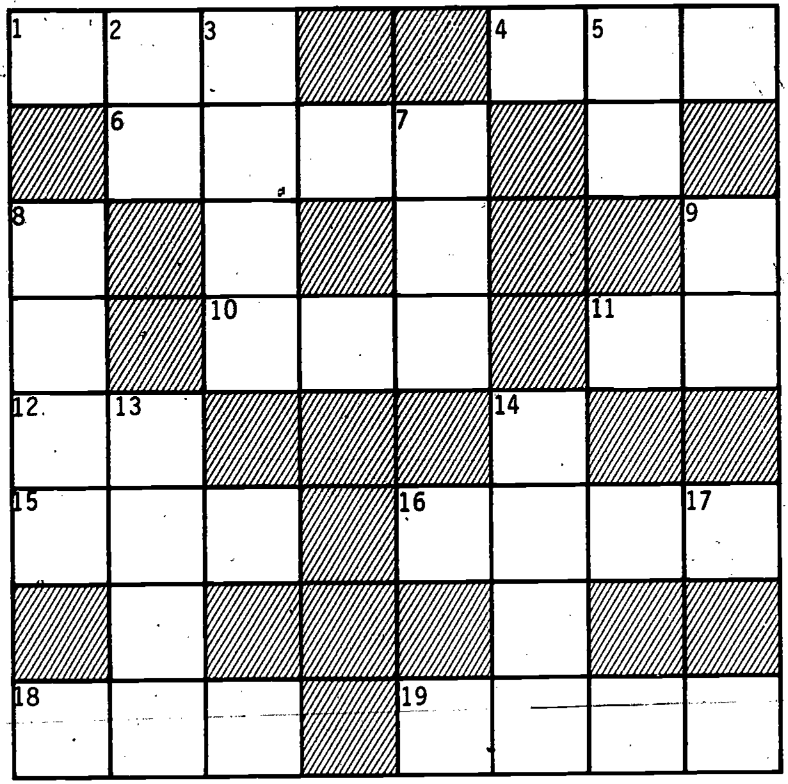
ACROSS

1. 𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺
4. DCLXV
6. 一
千
三
百
一
十
二
10. CCXXIII
11. 𝋢
𝋮
12. 𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺
15. 四
百
六
16. 𝋭
𝋦
𝋯
18. CXX
19. 𓆼𓍢𓍢𓍢𓏺𓏺𓏺𓏺

DOWN

2. 𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓏺
3. MMMMMCCCXII
5. 六
十
八
7. 𝋪
𝋣
8. MCMLXIV
9. 𝋣
𝋮
13. 𝋥
𝋫
𝋧
14. 𓆼𓆼𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓏺𓏺𓏺
17. 五

CROSS-NUMBER PUZZLE



CONTEST WORKSHEET

CHARACTERISTICS OF NUMERATION SYSTEMS

A. Mathematical Properties (✓ check whether the system has the property.)	Hindu-Arabic	Primitive	Egyptian	Oriental	Roman	Mayan
1. Additive						
2. Multiplicative						
3. Generalized grouping scheme or a base						
4. Place value						
5. Has a zero						
Other 6.						
7.						
8.						
B. Other Criteria (Rate the system good, fair, poor.) 1. Convenient, easy to use						
2. Economical in terms of number of symbols						
3. Unique representation						
Other 4.						
5.						
6.						
OVERALL RATING						

ACTIVITY 3

NUMERATION PROJECTS

FOCUS:

In this activity you will be given a choice of several projects which you might undertake to broaden your appreciation of other numeration systems.

DIRECTIONS:

Each group of three or four students should pick one of the following projects on which to prepare a presentation to the rest of the class. If possible, each of the projects should be chosen by some group so that all of the projects can be presented to the class. Your instructor may allow some planning time now and set the dates when the project reports are to be presented.

PROJECT 1. The numeration systems of several ancient cultures have been presented in Activity 2. Write a report presenting a historical development of one of these systems.

PROJECT 2. Trace the historical development of the Hindu-Arabic numeration system. Why is the system called "Hindu-Arabic" rather than "American?" When was zero developed in this system?

PROJECT 3. Create your own numeration system. Include several of the following in your report:

a) Present a table of symbols basic to your system and the Hindu-Arabic equivalents of each symbol.

b) Present examples of large numbers represented by the numerals of your system so that your method of recording is clear.

c) Give a rationale which supports the design of your system.

d) Indicate which of the basic operations (addition, subtraction, multiplication, division) are convenient in your system, and give examples showing how such operations would be performed.

PROJECT 4. Find out which countries in the world use the Hindu-Arabic numeration system. Indicate the type of system adopted by some countries (or regions) that do not utilize the Hindu-Arabic system.

PROJECT 5. Create and display a bulletin board for use in the elementary school depicting some historical aspects of numeration.

PROJECT 6. Trace when and where other numeration systems are used in the schools. Outline a set of activities you might use in a lesson on ancient numeration systems.

Resources for the above projects include elementary school textbooks, encyclopedias, and histories of mathematics. Among the brief histories are the following.

Smith, David E. History of Mathematics, Vol. I and II. New York: Dover Publications, 1958.

Sanford, Vera. A Short History of Mathematics. Cambridge, Mass.: Houghton Mifflin Co., 1958.

Section II

USING MATERIALS TO WORK IN BASES OTHER THAN 10

The work in this section focuses on four basic concepts in numeration: <u>base</u>, <u>grouping</u>, <u>trading</u>, and <u>place value</u>. In Section I you identified certain characteristics of a good numeration system and noticed that having a base was a feature of the better systems. The best systems have a base and use the concept of place value. Grouping and trading are used to translate a numerical situation into a place-value numeral in a given base. Since your understanding of grouping and place value is essential for appropriate development of numeration concepts for children, particular focus is given to these two topics.

Activity 4 introduces grouping and place value through a set of games involving trading. These games use materials found in the elementary school to achieve elementary school objectives. Activity 5 focuses on grouping and place value in bases other than ten. Activity 6 gives you an opportunity to apply your understanding of numeration in solving some puzzle problems.

MAJOR QUESTIONS

1. Two basic concepts of numeration are grouping and place value. What is meant by each and how are they related to each other?

2. What procedure can you suggest for translating a numeral in base n to a numeral in base ten? (Outline such a procedure by means of an example.)

3. Can you outline a procedure for translating a base ten numeral to a numeral in base n?

ACTIVITY 4

GROUPING AND PLACE VALUE THROUGH GAMES

FOCUS:

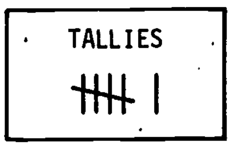
In this activity you will play a game with multibase blocks, a game with colored chips, and a game with an abacus. These games can be used with children to develop three important concepts in numeration: grouping, trading, and place value. These three concepts are essential to understanding number systems which have a base.

MATERIALS:

One set per group of Multibase Arithmetic Blocks* in various bases; sets of poker chips or other colored chips; one abacus per group; two or three dice or a spinner for each group.

DISCUSSION:

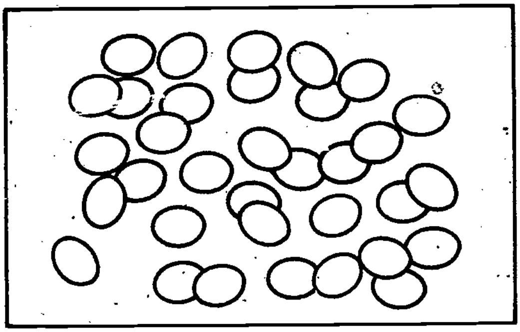
In keeping track of large numbers, a grouping scheme is used. In our numeration system, the grouping is done in sets of ten. (In this activity you will also be asked to group in bases other than ten.) In a grouping experience the representation used must reflect exactly the number in the group. For example, tallies can be used in a grouping experience since if one wishes to represent a group of six objects, six tallies are used. The MAB can also be used in a grouping experience.



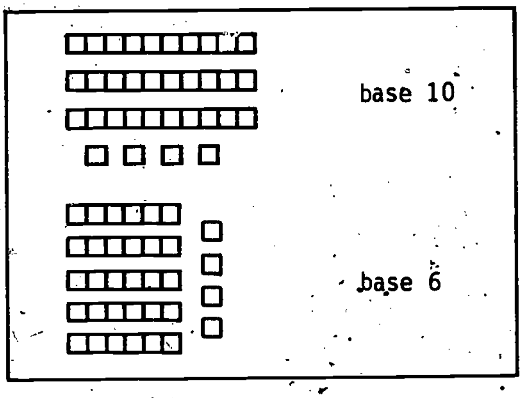
Study the representations on the following page to see if you can see why.

*Multibase Arithmetic Blocks (MAB) were developed by Zoltan Dienes to demonstrate the concepts of base and place value. Appropriate facsimiles may be used in your class or in an elementary classroom. Refer to the illustration of multibase aids on page 35.

OBJECTS



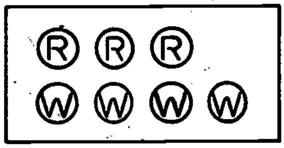
MAB Representation



Trading is an essential concept which relates to both the grouping concept and the place value concept. When we have as many single objects as the base b (10 or 6 or some other number) the single objects are traded for one of the next larger object. With MAB we trade b units for one long, b longs for one flat, and b flats for one block. Game 1 will focus on the grouping principle and the trading involved.

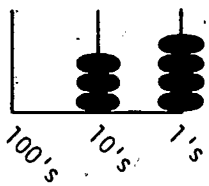
Game 2 represents a transitional step between the grouping principle and place value. Colored chips are used to represent different numbers of objects. After a group of chips has been collected they are traded for a chip of a different color to represent a group. Thus a single chip represents a number of objects rather than a single object. The choice of the particular color of the chips is of course entirely optional. As with the MAB, different bases or groups can be selected. If base 10 is used and if 1 white represents 1 object, and 1 red represents 10 whites, and 1 blue represents 10 reds, then the objects pictured on page 30 would be represented as shown at right. As with Game 1, trading is an essential concept for this game.

Colored Chips

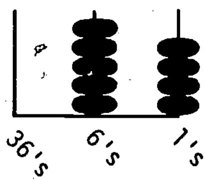


Game 3 highlights place value. In place value, the position that a representation holds determines the value. In our numeration system, we use digits. A 3 may represent 3 or 30 or 300 depending on its position (for example, 333). In this game an abacus is used. A bead on the ones wire represents a single object; a bead on the tens wire represents 10 and a bead on the hundreds wire represents 10 tens. Of course, the number of objects a bead represents depends on both the place value of the wire and the base being used. The objects pictured on page 30 could be represented in any base. The number of objects is shown on abaci below in base 10 and base 6.

Abacus



Base 10



Base 6

The notions of grouping, base, and place value are developed in the elementary school through trading activities. In base 10, 10 ones can be traded for 1 ten and vice versa.

DIRECTIONS:

Play the three games that follow and then answer the questions that follow them. As you are playing the games jot down any notes that might be helpful to you when you work with children. These games can easily be adapted for use with children.

1. A grouping game. "<u>Trade up to a Block</u>"

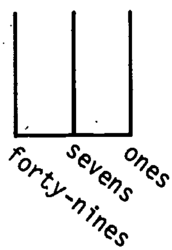
   This game should be played in small groups. Each group should have a set of MAB in a single base and a single die or spinner. (The game can also be played more rapidly with a pair of dice or a pair of spinners.) The object of the game is to win a block. Each player rolls a die and takes as many units as the die indicates. Suppose the MAB are base six. On the first roll, player A rolls a 2 and takes two units. On the second roll, player A rolls a 5. Player A now takes five more units and has a total of seven units. (S)he now trades six units for one long. The game continues, each player trading units for longs, longs for flats, and finally six flats for a block. The first player to get a block wins.

2. A place value game. "<u>Trading to a Blue Chip</u>"

   Each group should have a set of colored chips of three or four different colors. Assume there are three colors, white, red, and blue. Each group should also have a die or a spinner. The object of the game is to trade chips until one player wins a blue chip. Any base can be used, and the group can select its own base. Assume a group has selected base 10. Then 10 white chips can be traded for one red chip, and 10 red can be traded for one blue chip. Each player in turn rolls the die, taking as many chips as the die indicates. Trades should be made whenever possible.

3. Game 3: "<u>Trading on an Abacus</u>"

   This game is played in essentially the same manner as Games 1 and 2, except that the recording of the points should take place on the abacus. You may choose to use any base you wish (try one other than 10). If you should choose base seven, then don't forget



to trade 7 ones for 1 seven and 7 sevens for 1 forty-nine. The winner is then the one reaching 1 forty-nine first.

4. Game 1 is a game illustrating the grouping concept. In a grouping activity, there are exactly as many units (points) in the representation as there are in the number represented. Thus, if there are 42 points (objects) to be represented then 42 units would be used (in appropriate combinations of units, longs, flats) to represent that number.

   Assume that the block game was played in base six.

   a) When you have won a long, how many units have you won? Could you actually count them on the long?

   b) When you have won a flat, how many units have you won? Could you actually count them on the flat?

   c) It would be impossible to count all the units in a block, but you could trade a block for _____ flats. How many units could you count in the set of flats equivalent to a block?

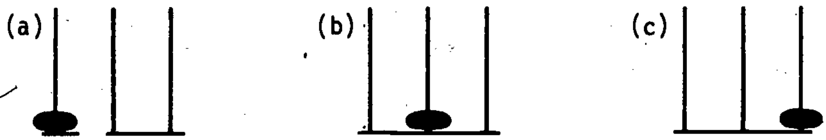
5. Game 2 illustrates a bridge between the grouping concept and the place value concept. Each chip, depending on the color, represents one, ten, or one hundred units. In place value, the position of the digit (rather than the color) determines the value it represents.

   a) Suppose you have two red chips and four white chips. How many units does this represent (using a base of 10)? Can you actually count the units on the chips now?

   b) Suppose you have 1 blue, 4 red, and 18 white chips. How many units does this represent?

6. In Game 3, the abacus represents a place value aid. The position of a bead determines its value. Thus, although each of the abaci on the following page has only one bead on it, each represents a different value.

a) Which abacus represents the greatest value (a), (b), or (c)? Assume that the base is 10, how much larger (smaller) is the number represented in (a) than in (b)? (b) than in (c)? (a) than in (c)?



b) Suppose the base is 8; how much larger (smaller) is the number represented in (a) than in (b)? (b) than in (c)? (a) than in (c)?

c) Discuss why Game 2 is a transition between Games 1 and 3.

7. To show the relationship between the MAB's, the chips, and the abaci, have one member of your group represent a number with MAB's. Another member should use chips to represent the same number, and still another member should display the number on the abacus. Finally, all should write the number in standard notation.

8. As a summary discuss what each of the following terms means and how they are different.

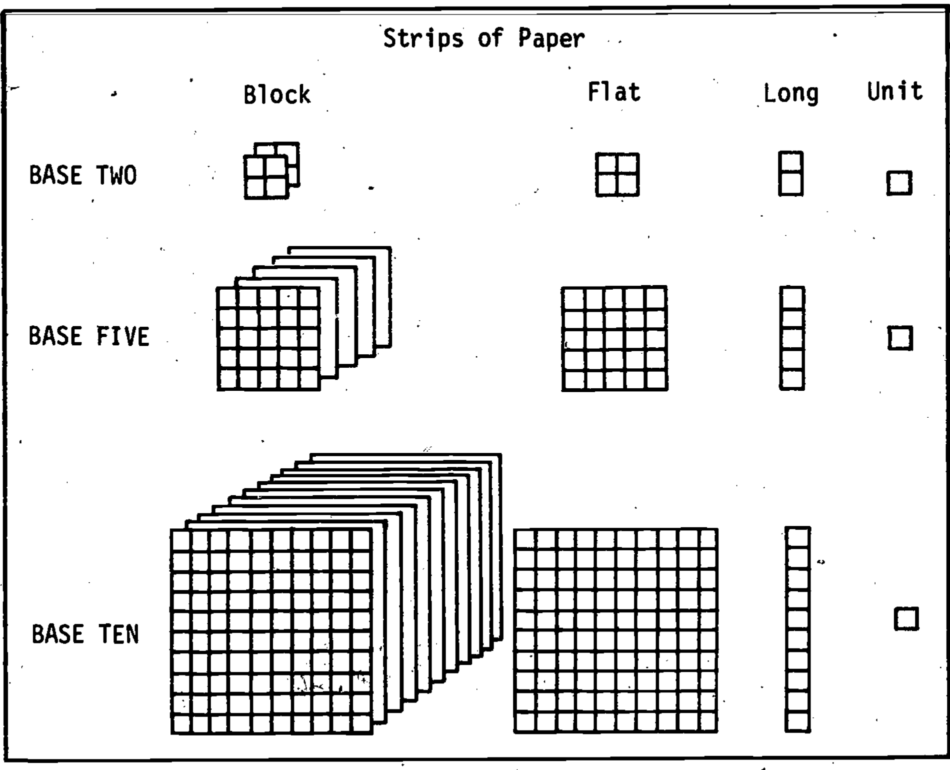
grouping

trading

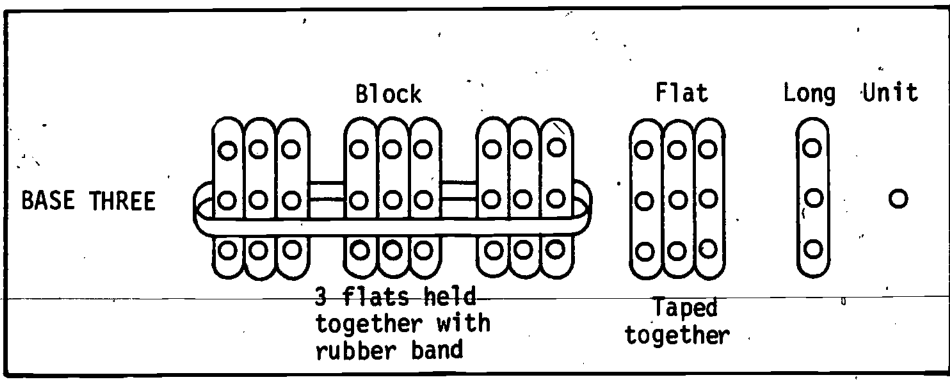
place value

Extend your discussion to include the following:

- What aids might be used with children for each idea?
- How subtle are these ideas for children?
- How might each idea be presented to children?



In place of Multibase blocks, strips of paper, $\frac{1}{2}$ inch or 1 cm. graph paper, or even sugar cubes* could be used. Also, navy beans glued onto tongue depressors or popsicle sticks can be bundled together.



*See Jon Higgins. "Sugar Cube Mathematics," The Arithmetic Teacher, (October, 1969), pp. 427-431.

ACTIVITY 5

GROUPING AND PLACE VALUE IN BASES OTHER THAN 10

FOCUS:

While work with bases other than ten is no longer a central part of the elementary school curriculum, as it was in the early sixties, work with other bases is provided for you for two purposes: one, to help you gain a deeper understanding of grouping and place value; and two, to provide you with some insight and perspective into difficulties children have in understanding our numeration system. This activity is done using aids that you would use in teaching children.

MATERIALS:*

Complete set of Multibase Arithmetic Blocks (all bases); one abacus for each group of students (if possible, pairs of students should work together).

DISCUSSION:

In Activity 4 you played some games that focused on ideas of grouping and place value. Trading was an essential ingredient in both of these ideas. This activity provides additional experience in trading, both in grouping and place value situations.

Remember that underlying both grouping and place value is the notion of base. The base tells how many there are in each group when the grouping is done. In looking at the numeral, the base also tells the value each digit represents. Thus, 5 in 54_{six} tells us there are five groups of six; the 3 in 34_{ten} tells us there are three groups of ten.

*As we have already indicated, less expensive materials may be substituted for the MAB. Refer to the illustration of multibase aids on page 35. Also, if an abacus is not available, a place value chart, a tally chart or other recording device may be used.

The first part of this activity is directed toward grouping, and the second part focuses on place value. As you work through the examples, be alert to possible difficulties that children might have as they are introduced to numeration concepts.

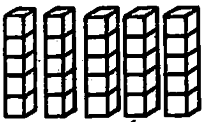
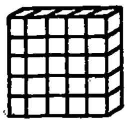
PART 1: THE GROUPING CONCEPT

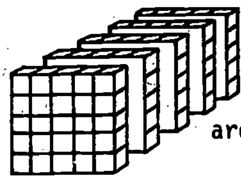
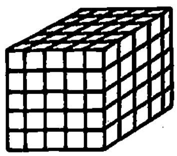
In grouping with MAB the following principle is used:

In base n, whenever there are n like objects, they are grouped and traded to result in the fewest pieces of wood (bundles, etc.).

For example, in base five, when 5 units (or 5 longs, etc.) are collected they are grouped and traded for 1 long (or 1 flat, etc.). Thus in base five:

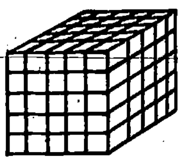
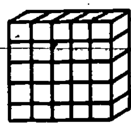
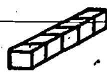
5 units  are equivalent to 1 long 

5 longs  are equivalent to 1 flat 

5 flats  are equivalent to 1 block 

DIRECTIONS:

Use base five blocks for 1 through 5 below.

1. Place 2 , 4 , 3  and 2 

on the table. This is recorded as 2432_{five} and is read "two four three two, base five."

How many units does this numeral represent? (i.e., What is the base ten equivalent of this base five numeral?)

2. Place base five materials on the table to represent the following numerals and determine the total number of units represented (base ten equivalent).

base ten equivalent

a) 343_{five} ______________

b) 2231_{five} ______________

c) 1010_{five} ______________

3. Each person within the group should take turns placing base five materials on the table while others tell the base ten equivalent.

4. Place 3 blocks, 6 flats and 8 units on the table.

a) What number (in base ten) is represented? ______________

b) Make trades until you show the <u>fewest</u> pieces of wood needed to express the number. Write the base five numeral which represents this number.

5. <u>Assume trades have been made whenever possible</u>.

a) With all possible trading completed, what is the largest number that can be represented using base five flats, longs and units only. Write the base 5 numeral _____, the base 10 equivalent _____.

b) With all possible trading completed, what is the largest number that can be represented using base five longs and units only. Write the base 5 numeral _____, the base 10 equivalent _____.

c) Now assume you can use blocks, flats, longs and units and that all possible trading has been completed. Write the base five numeral for the largest number that can be represented. Base five numeral _____ base ten equivalent _____.

d) Write the base five numeral that represents the number of units represented by each piece of wood.

1 unit: _____ base five numeral

1 long: _____ base five numeral

1 flat: _____ base five numeral

1 block: _____ base five numeral

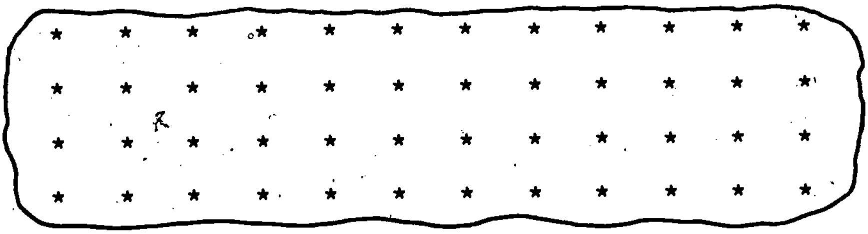
6. Imagine that you are in a World Bank which uses MAB as money! Use the MAB to make the trades. Remember to use the fewest pieces.

a)

	Blocks	Flats	Longs	Units	Numeral
Start in Base Five	4	2	3	3	4233_{five}
First Trade to Base Ten					____ten
Next Trade to Base Six					____six

b)

	Blocks	Flats	Longs	Units	Numeral
Start in Base Two	1	1	0	1	1101_{two}
First Trade to Base Ten					____ten
Next Trade to Base Four					____four

7. 

Represent the number of stars in each base below. Find the number of blocks, flats, longs, and units for the base. Then write the numeral.

	Blocks	Flats	Longs	Units	Numeral
Base Four					____four
Base Ten					____ten
Base Eight					____eight

8. In base ten, only 10 digits are needed. They are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

a) In base five _____ digits are needed. They are __________.

b) In base eight _____ digits are needed. They are __________.

c) In base n _____ digits are needed. They are __________.

PART 2: THE PLACE VALUE CONCEPT

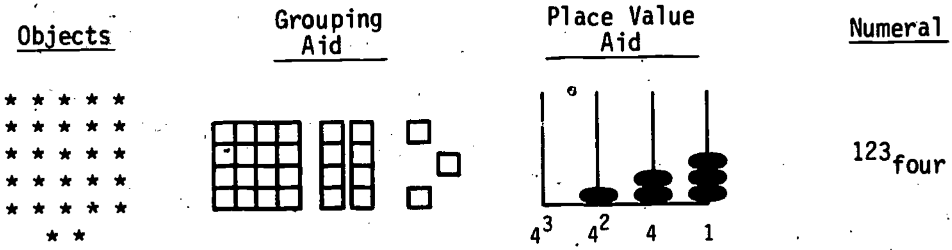
(If possible work in pairs.)

In the previous part, you used aids to focus on the grouping concept. The Multibase Arithmetic Blocks (or similar aids) represented the total number of objects to be grouped. A place value device such as an abacus* can be used to bridge the gap between the total number of concrete objects to be recorded and the recording of the numeral itself.

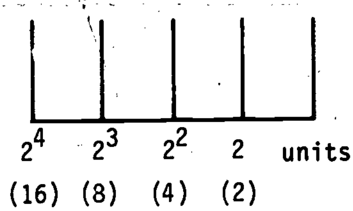
---

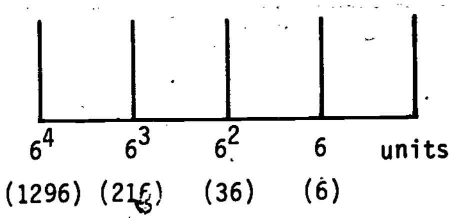
*If an abacus is not available, a place value chart, a tally chart or other recording device may be used. A drawing of an abacus could also suffice.

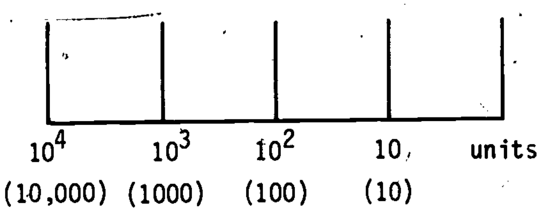
In the illustration below using four as the base, a bead on the units wire represents 1 object, a bead on the b wire represents 4 objects, a bead on the b^2 wire represents 4^2 or 16 objects.



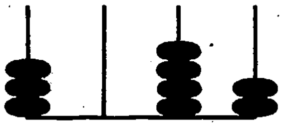
The use of exponents helps one to see the structure of the place value system and the value of each position. You remember that an exponent tells how many times a factor should be multiplied by itself. Thus, 5^4 means 5 x 5 x 5 x 5, that is 5 multiplied by itself 4 times. A place value system has a base and the value of each place is a multiple of the value of the place immediately to its right. For example, the first five positions of bases two, six, and ten are shown on the abaci below.







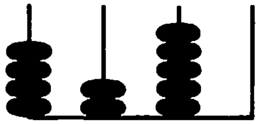
9. Write the numerals represented on the abaci below. Find the base ten equivalent numeral.



5^3 5^2 5 units

_____ base five

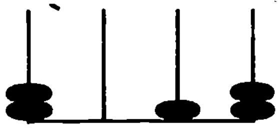
_____ base ten



8^3 8^2 8 units

_____ base eight

_____ base ten



3^3 3^2 3 units

_____ base three

_____ base ten

In base n, the greatest number of beads on any wire is _____ .

10. a) Take turns using the abacus to represent a number. Tell your classmates what base your representation is in. Display the beads to represent your number. Ask your classmates to tell you the numeral in your chosen base. Then ask them to translate it to base ten.

b) Use the reverse procedure; i.e., choose a number and give its base ten numeral. Then ask your classmates to give you the numeral in some other base and to display it on the abacus.

11. Complete the following:

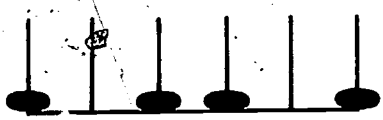
a) Since 4^3 means 4 x 4 x 4, then 8^5 means 8 x _______________.

b) Since 43072_{ten} means $4(10^4) + 3(10^3) + 0(10^2) + 7(10) + 2$, then 23511_{six} means $2(6^4) + 3(\_\_) + 5(\_\_) + 1(\_\_) + 1$, and 61213_{seven} means _______________________________.

12. Sketch the following numerals on an abacus. Be sure to label each wire. Then find the base ten equivalent.

EXAMPLES

101101_{two}



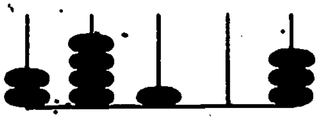
2^5 2^4 2^3 2^2 2 units

base ten equivalent

$32 + 0 + 8 + 4 + 0 + 1$

45

24103_{five}



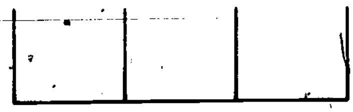
5^4 5^3 5^2 5 units

base ten equivalent

$2(625) + 4(125) + 25 + 0 + 3$

1778

a) 3246_{seven}



base ten equivalent

b) 312011_{four}

base ten equivalent

13. *OPTIONAL:* *While computation in other bases is not a skill that is worth developing, some of you (and some older children, too) may enjoy doing some computation in other bases. As you do these computations, keep in mind that when children are first introduced to standard computation they often experience the same difficulties you may experience in work in other bases.*

EXAMPLE

$$\begin{array}{r} 345_{six} \\ 243_{six} \\ \hline 1032_{six} \end{array}$$

Units: Think 5 + 3 is 1-six and 2. Write 2, carry 1.	Sixes: Think 1 + 4 + 4 is 1-six and 3. Write 3, carry 1.	Thirty-sixes: Think 1 + 3 + 2 is 1-six and 0. Write 10.

a) Add.

$$\begin{array}{r} 675_{eight} \\ 436_{eight} \\ \hline \end{array} \qquad \begin{array}{r} 432_{five} \\ 144_{five} \\ \hline \end{array} \qquad \begin{array}{r} 1011_{two} \\ 111_{two} \\ \hline \end{array}$$

b) Subtract. Use the abacus, chips or MAB to help you do these computations. Don't forget how to trade!

$$\begin{array}{r} 432_{five} \\ 143_{five} \\ \hline \end{array} \qquad \begin{array}{r} 455_{seven} \\ 342_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 675_{nine} \\ 218_{nine} \\ \hline \end{array}$$

c) Multiply.

$$\begin{array}{r} 121_{three} \\ 21_{three} \\ \hline \end{array} \qquad \begin{array}{r} 315_{six} \\ 4_{six} \\ \hline \end{array} \qquad \begin{array}{r} 677_{eight} \\ 5_{eight} \\ \hline \end{array}$$

d) Divide. This is the toughest. Try repeated subtraction!

$32_{four} \div 12_{four}$ $\qquad$ $654_{seven} \div 23_{seven}$

ACTIVITY 6

USING BASES TO SOLVE SOME PUZZLES

FOCUS:

This activity is presented to provide you with a chance to solve some puzzles using bases. It is hoped you will enjoy the challenge!

MATERIALS: Index cards, paper punch, scissors, knitting needle, toothpicks.

DIRECTIONS:

Your instructor will make the assignment for this activity. S(he) may assign a portion or all of the problems to be worked alone or in small groups. Be sure to share not only the results with classmates but also the procedures you used in solving the puzzles.

1. THE RECORDS OF DR. X*

An eccentric mathematician, when he died, left a stack of unpublished papers. When his friends were sorting them, they came across the following statement:

"I graduated from college when I was 44 years old. A year later, I, a 100-year-old man, married a 34-year-old girl. Since the difference in our ages was only 11 years, we had many common interests and hopes. A few years later, we had a family of 10 children. I had a college job, and my salary was $1,300 a month. $\frac{1}{10}$ of my salary went for the support of my parents. However, the balance of $1,120 was more than sufficient for us to live comfortably."

Dr. X was obviously calculating in another base. Can you translate his statements into the equivalent base ten expressions?

*Taken from: Bakst, Aaron. Mathematics--Its Magic and Mastery. New York: Van Nostrand-Reinhold, 1967.

2. Our eccentric mathematician, Dr. X, also played golf and some of his scores were found among his records. He kept records of all his golf matches and his weekly averages too. One weekly record reads this way:

Sunday	11,011	Thursday	11,120
Monday	11,101	Friday	11,021
Tuesday	11,111	Saturday	11,021
Wednesday	11,001		

Question: In what base do you think Dr. X kept his golf scores? Find his score for each day.

3. Age Tables: Age tables or "magic" cards are fascinating. The set of cards below have the numbers arranged in such a way that if a person tells what cards his age appears on, the "magician" can tell him how old he is. The trick is easy. One simply adds the numbers in the upper left-hand corner. For example, if the person says his age is contained on cards E, D, B and A, then the "magician" says with absolute confidence:

"You are 27 years old!" (16 + 8 + 2 + 1).

F

32,	33,	34
35,	36,	37
38,	39,	40
41,	42,	43
44,	45,	46
47,	48,	49
50,	51,	52
53,	54,	55
56,	57,	58
59,	60	

E

16,	17,	18
19,	20,	21
22,	23,	24
25,	26,	27
28,	29,	30
31,	48,	49
50,	51,	52
53,	54,	55
56,	57,	58
59,	60	

D

8,	9,	10
11,	12,	13
14,	15,	24
25,	26,	27
28,	29,	30
31,	40,	41
42,	43,	44
45,	46,	47
56,	57,	58
59,	60	

C

4,	5,	6
7,	12,	13
14,	15,	20
21,	22	23
28,	29,	30
31,	36,	37
38,	39,	44
45,	46,	47
52,	53,	54
55,	60	

B

2,	3,	6
7,	10,	11
14,	15,	18
19,	22,	23
26,	27,	30
31,	34,	35
38,	39,	42
43,	46,	47
50,	51,	54
55,	58,	59

A

1,	3,	5
7,	9,	11
13,	15,	17
19,	21,	23
25,	27,	29
31,	33,	35
37,	39,	41
43,	45,	47
49,	51,	53
55,	57,	59

The problem is to tell why it works. Another way of solving the problem would be to determine how numbers were selected for each card. Clearly it has to do with numeration in another base--base two to be precise. Any further hints will have to be elicited from your instructor.

4. <u>Coding</u>: Can you decode the message in the box below? Each base six numeral stands for a letter of the alphabet:

1 ⟷ A
2 ⟷ B
3 ⟷ C
4 ⟷ D
5 ⟷ E
10 ⟷ F, and so on.

41-23-33 33-22-4-5-30-31-32-1-22-4
22-33-21-2-5-30 2-1-31-5-31
13-10 41-23-33 3-1-22 4-23 32-12-13-31

5. Number Base Square: Change the following base ten numerals to the indicated base and place your answers from left to right in the numbered squares. If your work is correct, the answers should read the same horizontally and vertically.

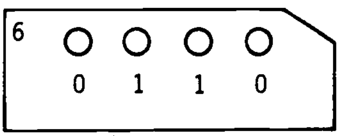
1. 486 = ______ (base five)
2. 1,064 = ______ (base six)
3. 848 = ______ (base seven)
4. 298 = ______ (base six)

1.			
2.			
3.			
4.			

Make up a "number base square" of your own.

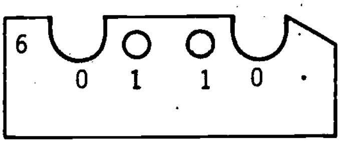
6. Binary Cards: One fascinating way of recording numbers is to use punched cards. The sorting process described below forms the basis of the computer punched-card system. If you do not presently have time to make your own set of binary cards, you might want to recall the idea when you are a teacher, to use as a project with upper grade children.

a) Make a set of sixteen cards and number them from zero to fifteen. Punch four holes in the top of each one and cut a corner off. Write the binary representation of the base ten card numeral, one digit below each hole. Except for the numerals, all cards should be exactly alike.



$(0110_{two} = 6_{ten})$

b) Next cut a slot for each "0" that appears in the binary numeral. (See figure below.)



c) To select any card: Suppose you shuffled the deck of 16 cards and wanted to retrieve the "6" card. Since $6_{ten} = 0110_{two}$, use a knitting needle or nail and put it through the units hole of the deck. Shake so that all the cards with a slot in this position drop out. Since a "6" has a slot in the units place, the six card should be among those that drop out. Use the "dropouts"; lay the others aside. Put the needle through the twos hole, shake and retain those <u>still on the needle</u>. (The six card does not have a punched ~~out twos hole and will stay on the needle~~.) Withdraw the needle and place it through the fours hole. Shake and retain those still on the needle. Finally put the needle through the eighths hole. The only card that will drop out is the six card since it has a slot in the eights place.

d) To place cards in order: First of all shuffle and make sure that the cards are mixed up. Put the needle through the units hole, lift out all those retained and place them <u>behind</u> the others. Repeat for the twos hole, then the fours hole and finally the eights hole. The cards should now be in order from zero to fifteen.

<u>Questions</u>

1. In the card set above, would it make any difference if the slot was meant to represent "1" and the hole to represent "0"?
2. Is it necessary to cut a corner off?
3. Is it possible to modify the process described to make a card set with a base three expression for each number?

7. <u>NIM</u> (A Game for Two): Take as many, e.g., 21, toothpicks (or any other objects) as you please and place them in three different piles. The object of the game is to make your opponent pick up the last stick.

a) Players take alternate turns.

b) Each player picks up as many sticks as he/she pleases (at least one) as long as they come from the same pile.

c) The player who leaves only one stick wins.

Play this game with a classmate and develop a winning strategy.*

TEACHER TEASER

One advantage of base ten numeration is that it provides an easy way to tell whether a numeral represents an odd number or an even number. For instance, you know that 647 represents an odd number just by looking at the numeral.

Write numerals for the numbers from one to twenty in base ten numerals in one column and write numerals for the same set of numbers in base five numerals in a second column. Circle the numerals which represent even numbers in both columns. Do base five numerals for even numbers follow the same pattern as the base ten numerals? See if you can find a pattern for telling whether a number is odd or even by looking at the base five numeral.

Does any other base follow the base ten pattern for odd and even numbers?

*Martin Gardner, in The Scientific American Book of Mathematical Puzzles and Diversions (New York: Simon & Schuster, 1959), pp. 151-157, describes a winning strategy based on the binary (base two) system. This book is available in most libraries.

Section III

NUMERATION IN THE ELEMENTARY SCHOOL

In Section I a number of numeration systems were presented with the intention of helping you focus on the features inherent in our numeration system. Two important characteristics of our numeration system, namely grouping and place value, were singled out for study in Section II. These characteristics were studied using the aids one might use with children in the elementary school. The study of these characteristics was presented using the vehicle of other bases.

Here in Section III the focus of your study is on numeration in the elementary school. This section contains Activities 7 through 14. Activities 7 and 8 provide an opportunity for you to gain a general overview of the scope and sequence of numeration topics in the elementary school.

Activities 9 through 11 contain a sampling of the concepts taught in the primary grades. Activity 9 discusses the importance of classifying, comparing, and ordering in prenumber learning. Activity 10 provides a model for relating a physical referent to number names and numerals. Activity 11 provides an opportunity for you to have some hands-on experience with grouping and place value aids.

Activities 12 through 14 present two concepts developed in the upper elementary school. Activity 12 extends the development of the numeration system to include the representation of rational numbers as decimals. Activity 13 develops the use of the numeration system and exponents to represent large and small numbers in scientific notation. Finally, Activity 14 provides an opportunity for you to summarize and put your learnings on numeration into perspective in a class seminar.

Your experience in Section III will provide a basis for your experience in Section IV, where you will diagnose pupil errors in numeration and suggest appropriate activities to provide remedial help.

As you work through Section III keep in mind the major questions which follow.

MAJOR QUESTIONS

1. What are the most important numeration topics which are dealt with in the elementary mathematics curriculum?
2. Recall your work in Section II relative to the difference between grouping and place value concepts. What materials might be used to develop each concept with children? Describe a sequence which would help children bridge the gap between grouping and place value.

ACTIVITY 7

SCOPE AND SEQUENCE OF NUMERATION TOPICS

FOCUS:

In this activity you are asked to read an essay or view a slide-tape which provides an overview of numeration in the elementary school. Following this you are asked to arrange a set of cards containing specific activities in an appropriate sequence for learning by elementary school children.

MATERIALS:

The Mathematics-Methods Program slide-tape presentation "Numeration in the Elementary School" (optional), sets of current elementary school mathematics textbooks series.

DIRECTIONS:

1. Essay or slide tape

   Read the essay or view the slide-tape entitled "Numeration in the Elementary School" one or more times paying particular attention to both the scope and the sequence of experiences presented.

2. Card Sorting (Groups of four)

   a) After reading the essay or viewing the slide-tape, subdivide your group of four into two pairs. Cut the card pages so you can arrange them in the sequential order you feel is best. These cards represent various numeration activities in grades K-6. Work with a partner to sort the cards and arrange them in what you feel represents an appropriate learning order.

   b) When both pairs have finished the sequencing task, a comparison of learning orders should be made. Resolve any differences within your group, and record the sequence which you decide upon.

c) Check with your instructor who can provide a key which suggests a sequence developed by some mathematics educators.

3. <u>Numeration Activity</u> (Assignment). Do either a) <u>or</u> b) below.

Choose any card from the card set. Outline a numeration activity which:

a) Could be used to <u>motivate</u> the concept or skill task illustrated on the card; or

b) <u>Reinforces</u> the concept or skill portrayed on the card.

You may choose to work alone or with a classmate. Use elementary mathematics texts or other resource books as guides. The activity that you outline should include the:

1. Objective;

2. Materials you would use;

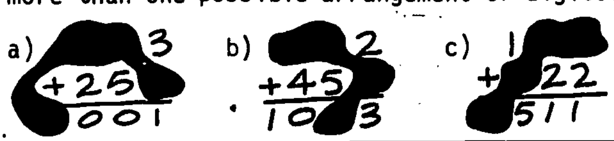
3. Procedure to be followed (outline only).

The outline of the activity should be brief--not much more than one page. If your instructor so directs, you may write your outline on a transparency for presentation to the class.

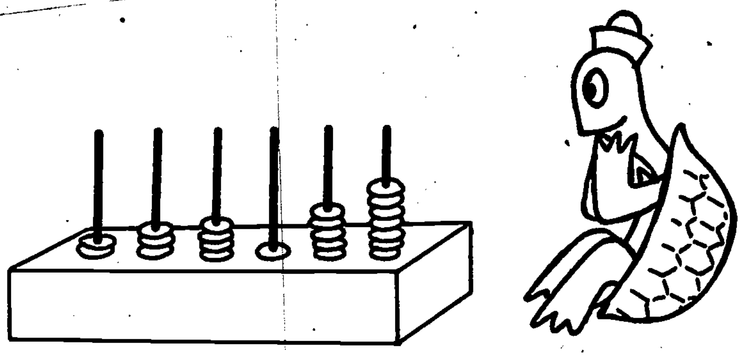
TEACHER TEASER

Ink Blots

If you know that the numerals below are base six representations, and in each case the addends have three digits, can you recreate each problem and its solution? Under the given conditions, is there more than one possible arrangement of digits?

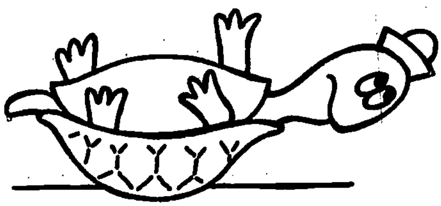


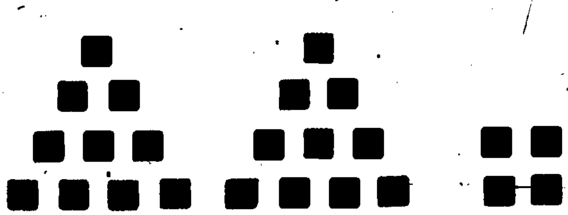
Read the number shown on the abacus.



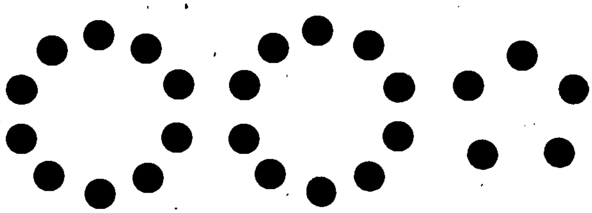
A

How many tens and ones?





____Tens____Ones

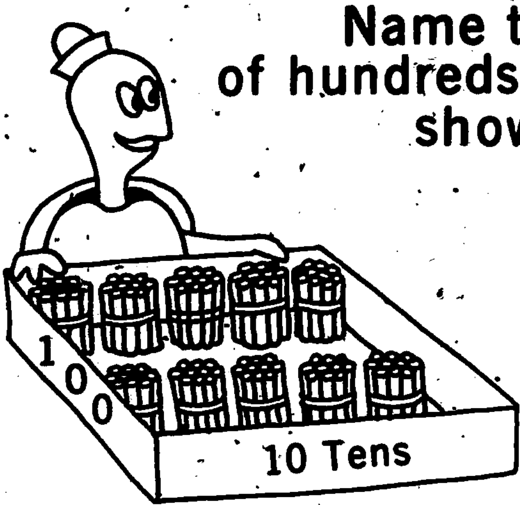


____Tens____Ones

B

NUMERATION
CARD SET

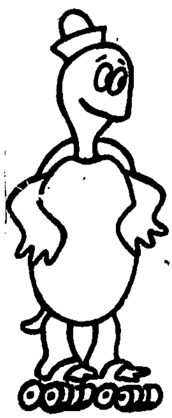
NUMERATION
CARD SET



Name the number of hundreds, tens, and ones shown here.

Hundreds	Tens	Ones

C



DIGIT	4	4	4	4 .	4	4	4
PLACE VALUE	10^3	10^2	10	1	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$
	THOUSANDS	HUNDREDS	TENS	ONES	TENTHS	HUNDREDTHS	THOUSANDTHS

Write in expanded form:

4,327.5 = (__ x 1000) + (__ x 100) + (__ x 10) + (__ x 1) + (__ x $\frac{1}{10}$)

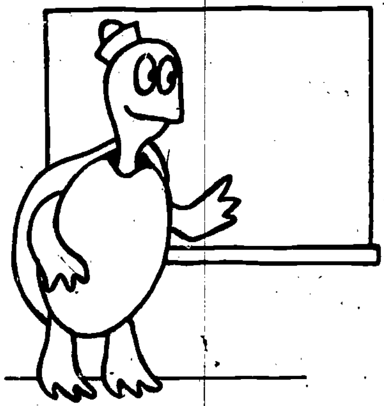
4,327.5 = (__ x 10^3) + (__ x 10^2) + (__ x 10) + (__ x 1) + (__ x $\frac{1}{10}$)

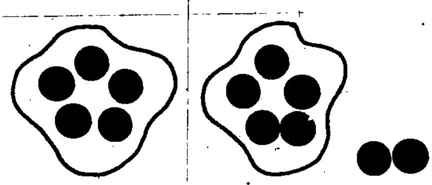
D

NUMERATION
CARD SET

NUMERATION
CARD SET

Write the base five numeral that tells the number of objects.

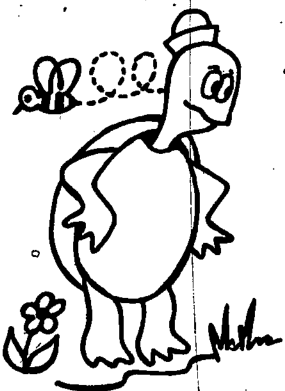


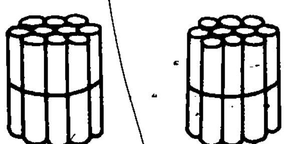


__________ five

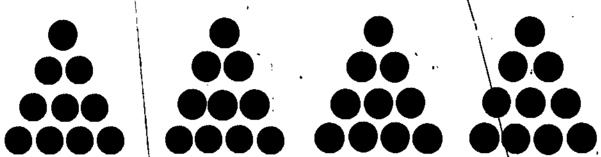
E

How many groups of ten?





______ Tens



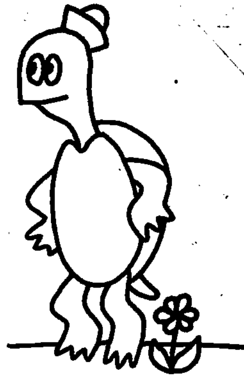
______ Tens

F

NUMERATION
CARD SET

NUMERATION
CARD SET

Find the difference by renaming.



$$\begin{array}{r} 34 \\ -18 \\ \hline \end{array}$$

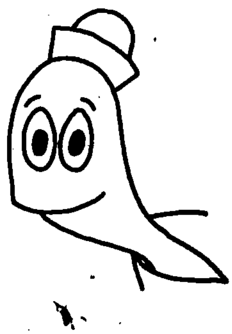
3 tens	4 ones	2 tens	14 ones
-1 ten	8 ones	-1 ten	8 ones

G

$$\begin{array}{r} 24 \\ \times 3 \\ \hline \end{array}$$

24 is renamed as

20 + 4

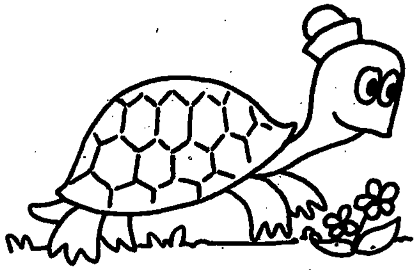


24 x 3 = (x 3) + (x 3)

H

NUMERATION
CARD SET

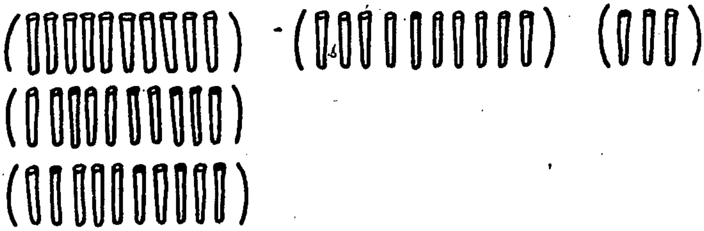
NUMERATION
CARD SET



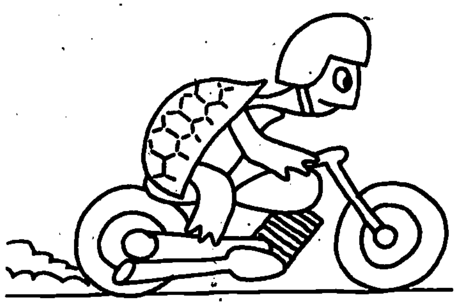
Tell the sum of:

$$\begin{array}{r} 26 \\ +17 \\ \hline \end{array}$$

2 tens	6 ones
+1 ten	7 ones
3 tens	13 ones or ____ Tens ____ Ones



I



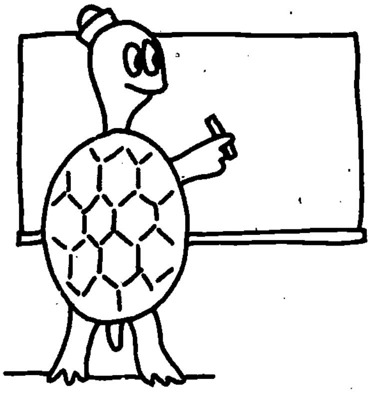
What is the place value of the position shown by each arrow?

8.45 9.141

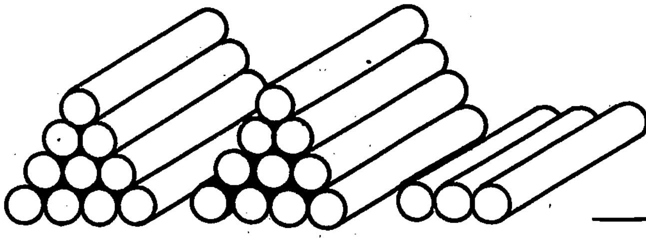
J

NUMERATION
CARD SET

NUMERATION
CARD SET

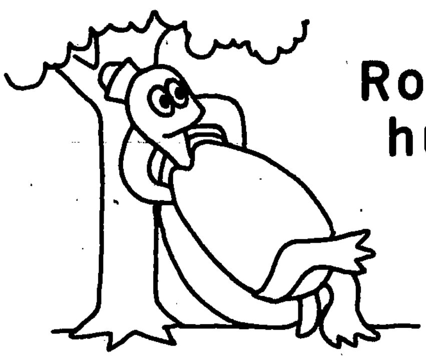


Write the numeral which tells how many in all.



____

K



Round to the nearer hundred thousand.

27,351,421

L

NUMERATION
CARD SET

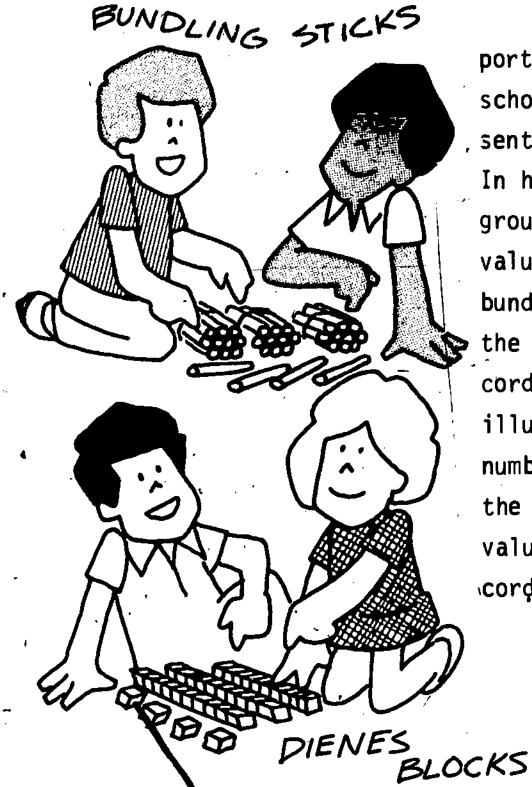
NUMERATION
CARD SET

NUMERATION IN THE ELEMENTARY SCHOOL

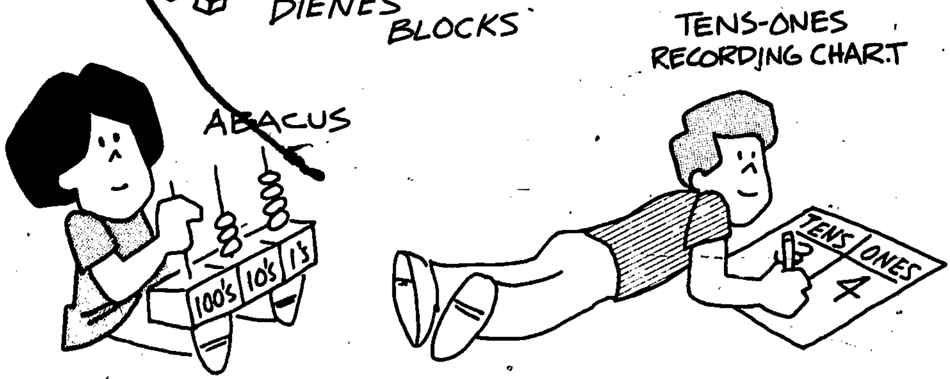
Numeration is without doubt one of the most important topics developed in elementary school arithmetic. In an analysis of errors in the four computational processes Cox* reported that the cause of the greatest number of errors in addition and subtraction was the children's lack of understanding of numeration concepts. The analysis of errors in multiplication and division also revealed that a lack of understanding of numeration played an important part in the children's errors. A careful development of numeration in the elementary school is essential.

Early work in numeration begins with counting. Children often come to school knowing how to count rotely, some to 5, some to 25, and some even beyond 100. Early development work should focus on rational counting, that is, counting objects or selecting a number of objects to match a given number. As the numbers become greater than ten, children should be encouraged to group them by tens and ones. The children should give the standard name, for example, "twenty-three," and the name associated with the grouping, that is, "two tens and three ones."

*Cox, L. S. "Systematic Errors in the Four Vertical Algorithms in Normal and Handicapped Populations," Journal for Research in Mathematics Education (November 1975): 202-220.

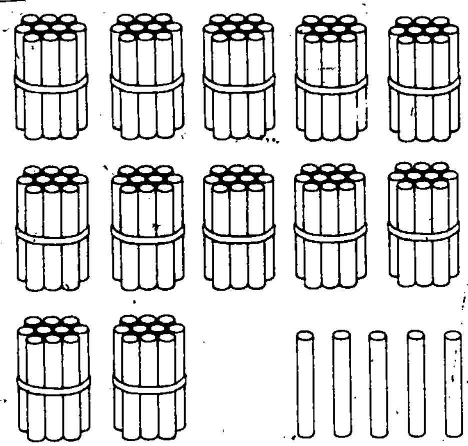


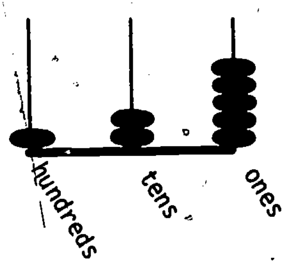
The use of materials is important in all of elementary school mathematics, but it is essential in early numeration work. In helping the child move from the grouping concept to the place value concept, materials such as bundling sticks or Dienes blocks, the abacus, and a "tens-ones" recording chart should be used. The illustrations at the left show a number of aids being used to show the grouping concept, the place value concept, and finally, recording the numeral.



Numeration or recording of numerals should proceed only after the careful development of prenumber activities has taken place. In nursery school or in kindergarten, children should have extensive work in classifying, comparing, and ordering. Although activities which focus on classifying, comparing, and ordering relate to a broad spectrum of knowledge, many are directly related to number. For example, children should compare sets to tell which has more. Counting is one possible extension ~~of the ordering activities~~. Prenumber work must be provided as late as grade 1 for some children.

Beginning in grade 1 and continuing throughout the grades, the relationship between grouping and place value is emphasized in working with larger numbers. For example, in the lower grades the child learns that 10 groups of ten equals 100. Using this information, the child engages in activities which involve the use of bundling sticks or other aids, as well as the abacus, to express such numbers as 125.





hundreds	tens	ones
1	2	5

Aids are used to help children understand the grouping and place value concepts and, in particular, the relationship between them. As the child progresses, more time and energy is spent in analyzing the symbols used in numeration. To emphasize the value of each digit in a numeral, children are asked to rewrite numerals such as 125 in the form 100 + 20 + 5.

Understanding the meaning of place value is a prerequisite to understanding the standard algorithms for computation with whole numbers. In doing the addition on the right, for example, a child must rename 14 ones as 1 ten, 4 ones. The child will eventually learn the "carrying" shortcut, but understanding of the addition algorithm depends upon understanding of place value.

ADDITION

36	3 tens	$\cancel{6}$ ones
+48	4 tens	8 ones
	7 tens	14 ones
	8 tens	4 ones

or

$$\begin{aligned} 36 &= 30 + 6 \\ +48 &= 40 + 8 \\ & 70 + 14 \\ & 70 + 10 + 4 \\ & 80 + 4 \\ & 84 \end{aligned}$$

Similarly, when subtracting, the renaming process is often necessary to complete the problem. In the example at the right, a child must rename 53 as 4 tens and 13 ones.

SUBTRACTION

53	${}^{4}\cancel{5}$ tens	${}^{13}\cancel{3}$ ones
-37	3 tens	7 ones
	1 ten	6 ones
	16	

Besides being used in addition and subtraction, the concept of renaming is used in both multiplication and division. In the multiplication example at the right, the 24 is renamed as 20 + 4.

MULTIPLICATION

$$24 \times 7 = (20 + 4) \times 7$$

or

$$\begin{array}{rl} 24 & \\ \times\ 7 & \\ \hline 28 & (7 \times 4) \\ 140 & (7 \times 20) \\ \hline 168 & \end{array}$$

Several other mathematical skills besides computation are dependent on the concepts of numeration. In working with large numbers, children rely on place value ideas rather than grouping physical objects. For example, to tell which of two numbers is larger, the child must use place value to determine the meaning of the digits. The child might think, "two thousand is less than three thousand, so 2331 is less than 3231."

$2331 < 3231$

Another mathematical skill which depends on place value is rounding. Rounding is an important skill in estimating. For example, to round to the nearer ten or to the nearer thousand, the child must have a clear understanding of the meaning of the digits.

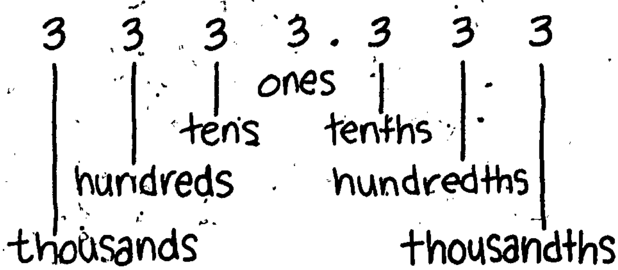
Round to nearer thousand

$53{,}684 \rightarrow 54{,}000$

Round to nearer ten

$53{,}684 \rightarrow 53{,}680$

In the upper elementary grades, the child's study of numeration is extended to the decimal representation of rational numbers. The study of decimals is beginning at an earlier age now as a result of the introduction of the metric system and the hand-held calculator. Children can see the symmetry of the decimal system by studying displays such as that shown below.



The decimal feature of our numeration system can best be shown by writing expanded numerals using exponents. Exponents are commonly used in business and science to show very large and very small numbers.

DECIMAL: POWERS OF 10

$$34{,}125 = (3 \times 10^4) + (4 \times 10^3) + (1 \times 10^2) + (2 \times 10) + (5 \times 1)$$

Again the use of the hand-held calculator has emphasized the need to teach scientific notation, because very large and very small numbers are displayed on many calculators in scientific notation.

SCIENTIFIC NOTATION

$$93000000 = 9.3 \times 10^{7}$$
$$0.000047 = 4.7 \times 10^{-5}$$

Finally, children's learning of numeration includes some work in other bases (Section II) and some historical aspects of numeration (Section I). Numeration is often called a "strand" in elementary school mathematics since various aspects of this topic are presented throughout the elementary school.

ACTIVITY 8

GRADE-LEVEL PLACEMENT OF NUMERATION TOPICS

FOCUS:

This activity will provide a means of helping you refine your knowledge of the scope and sequence of numeration topics in the elementary school.

MATERIALS:

Several sets of elementary mathematics textbook series.

DIRECTIONS:

1. Looking through an elementary mathematics text series, identify three or four topics related to numeration which are developed at each grade level.

2. Make a chart as shown below. Fill in the topics and give an example of each topic. Indicate the approximate number of textbook pages devoted to the development of each topic. To help you get started, a partial list of topics (not necessarily in proper sequence) follows.
   - Grouping by tens
   - Counting by fives
   - Writing four-digit numerals in expanded notation
   - Renaming tens as ones and vice versa
   - Writing fractions as decimals
   - Reading and writing numerals to one million

	Topic	Example	Number of Pages
Kindergarten			
Grade 1			
etc.			

3. Compare your chart with the charts of classmates who used a different textbook series. Summarize the major differences in the grade placement of topics.

85

ACTIVITY 9

CLASSIFICATION, COMPARING AND ORDERING

FOCUS:

In Activities 7 and 8 you had an opportunity to gain a perspective on the numeration concepts developed throughout the grades. Now in Activities 9 through 11 you will have a chance to look more carefully at numeration experiences that are appropriate for the primary grades. In this activity you will become familiar with some nursery school and early primary experiences that are prerequisites for numeration, and with number readiness activities in general.

DISCUSSION:

There are some general concepts and vocabulary that must be understood and used by children before any number work can begin. These concepts are generally developed through asking children to classify, to compare, and to order. Before a child can group objects (as required in early numeration), he must know what a group is. Asking children to (classify) sort objects into groups using a property such as color, size, etc., not only brings home the concept of a group but also helps the child focus on the particular property and the term associated with it. Comparing objects or groups of objects to determine which is taller, which is bigger, or which has more members likewise helps the child focus on numerical vocabulary. Ordering objects, say from smallest to largest or from shortest to tallest, or ordering groups from the group having the least number of objects to that having the most, provides an opportunity for the child to gain and show his understanding of numerical concepts and terminology.

This activity will provide you with an opportunity to gain some insight into experiences which might help young children acquire the concepts and terms which are prerequisite to prenumber learning. The development here is meant to be an introduction. You may want to delve much more deeply. An excellent and more thorough development is presented in the 37th Yearbook of the National Council of Teachers of Mathematics in an article by E. G. Gibb and A. M. Castaneda entitled "Experiences for Young Children."

DIRECTIONS:

1. Three activities for children are outlined on pages 65-68. For each activity identify the purpose: classifying, comparing, or ordering. One of the parts of each activity is missing: the "Focus," "Introducing the Activity," or the "Suggestions for Follow-Up Discussion." Supply the missing parts in each case.

2. Write one activity which focuses on classifying, comparing or ordering. Your instructor may want you to share this activity with your classmates by preparing it on an overhead transparency. Some helpful references follow.

   a) NCTM 37th Yearbook: Mathematics Learning in Early Childhood.

   b) Lorton, Mary Baratta. Workjobs. Addison Wesley Publishing Company, Menlo Park, California, 1972.

   c) Platts, Mary E. Launch. Stevensville, Michigan: Educational Service, Inc., 1972.

   d) Teachers editions of selected K and Grade 1 mathematics texts.

Comment: While many commercial materials and resources are available for early childhood activities, the most available (and perhaps best) are simple objects from the child's environment: rocks, bottle caps, popsicle sticks, straws, shells, beads, buttons, milk cartons, animal figures, balls, ... Every teacher of young children needs a junk box of such items. (S)he also needs to develop an adequate picture file which children can use for sorting, comparing, and ordering, and later for simple number work.

PRENUMBER ACTIVITY A

Purpose of Activity: ____________________

Focus: Child will separate objects according to their color.

Materials: Box with assorted small objects--reds, blues, greens, and yellows (beads, blocks, buttons, combs, balls, balloons, pencils, colored macaroni, bits of cloth, ...); four pie tins; a tagboard label for each pie tin: red, blue, green, yellow. Each label should have a bit of cloth glued to it which illustrates the color word.

Description of Activity: The child ~~takes objects~~ ~~objects~~ takes objects from the box, one at a time, and places them in the pie tin labeled with the matching color.

Introducing the Activity: Spread out the four color labels on the floor, and have a child place a pie tin near each. Examine the labels with the child. Have him feel the cloth and tell you the color if he can. If he isn't sure of the colors, simply name each plate with him: "This is the plate for blue things, this one is for red things..." Then bring out the box of objects. "Can you find something blue to go in this plate?" "Look at the color of the other things and put them in the matching plate."

Suggestions for Follow-Up Discussion:

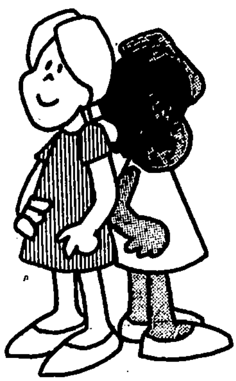
PRENUMBER ACTIVITY B

Purpose of Activity: ______________________

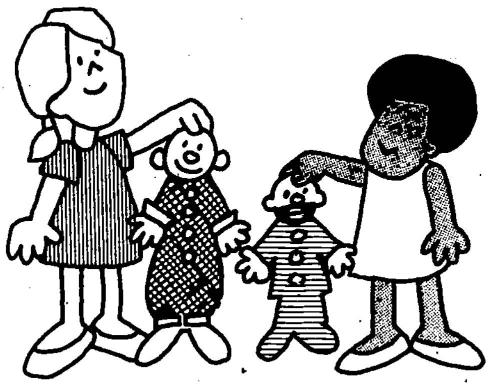
Focus:

Materials: 10-12 envelopes, each of which contains two construction-paper clowns (without hats). In each envelope one clown is taller than the other. Box to hold envelopes.

Description of Activity: Two children work on this activity together. They first measure each other to determine who is taller (or have a friend/teacher tell them who is the taller). They then take turns drawing envelopes, and deciding which of the clowns inside each is taller. At each turn the taller child keeps the taller clown; the shorter child keeps the other. When the activity is over the children fill the envelopes again, placing a short and a taller clown in each.



Introducing the Activity: Choosing two children of distinctly different heights, ask the others in the group: "Who is taller? How do you know?" Repeat with two other children, getting them to stand back-to-back or finding how each measures up to a strip of tape fixed to a wall. Working with one pair of children, bring out the box of envelopes. Opening one, ask: "Which clown is taller? How can we tell?" (One way: place one clown on top of the other, making sure feet (or heads) line up.) The taller child should take the taller clown; the shorter child keeps the other.



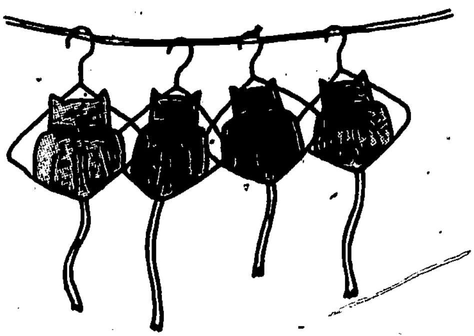
Suggestions for Follow-Up Discussion: If the children have not yet replaced the clowns in the envelopes, talk to them about what they are doing. Ask one child how his clowns compare with the ones the other child holds. If the clowns have already been returned to the envelopes, open one of them and lay both clowns beside each other on the floor/table. Then, moving one clown away, ask: "When you played the game, who got this one? Why?" If the children measured themselves independently of your help ask how they decided who was taller. Have each child show you something (or someone) taller and something shorter than he is.

PRENUMBER ACTIVITY C

Purpose of Activity: ____________________

Focus: Child will show his understanding of "short-long" by arranging objects using the criterion of shortest to longest.

Materials: Six to eight clothes hangers; box for storing hangers. Staple a cardboard cat figure on each hanger, using twine for the tail. The cats' tails should all be different in length.



Description of Activity: The child compares the lengths of the cats' tails and places the hangers on a line so that the lengths of the tails range (left to right) from shortest to longest.

Introducing the Activity: Have a child open the box and take out two "cats." Talk about the cats, and note how the lengths of their tails compare. "Put both hangers on the line, but put the cat with the shorter tail on the left." (Provide a clothespin marker to guide the child who still confuses right and left.) "Can you put the other cats on the line so that each tail gets longer as you walk to the right (walk away from the clothespin)?"

Suggestions for Follow-Up Discussion:

ACTIVITY 10

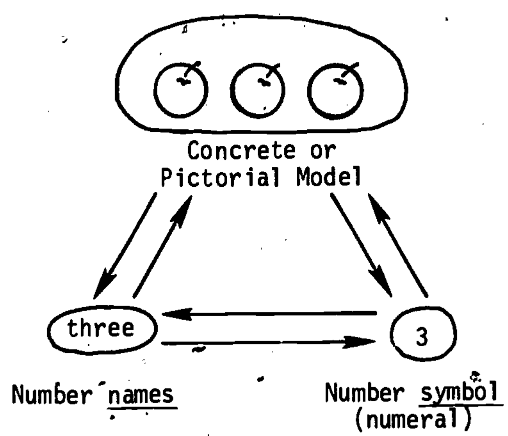
RELATING NUMBERS TO NUMBER NAMES AND SYMBOLS

FOCUS:

In this activity a model for relating numbers, number names, and symbols will be illustrated. This same model can be applied to several other numeration concepts.

DISCUSSION:

Many children come to school with the ability to count by rote (say the number names in correct sequence) to ten or more. Many of these children do not, however, know the meaning of these numbers or the symbols for them. Helping the child understand the meanings of numbers, the names for numbers, and the written symbols for them is the objective of the following model.



In this model, the child is given one form and is asked to show his understanding by providing the other two forms. For example, a child might be shown a picture of three apples and asked

a) "How many apples are there?" ("Three")

b) "Can you show me the numeral that tells us how many apples there are?" (Child points to a card with a '3' written on it. Or later, the child writes the numeral '3.')

DIRECTIONS:

1. Each teaching situation below presents either the pictorial/concrete model, or the number name, or the numeral to represent a specific quantity. Questions or directions then follow which you (as a teacher) might give the child to test his/her understanding of number. Fill in the boxes in the right-hand column to indicate the relationship you are testing. The first one is done for you.

Questions or Directions	Relationship Being Developed
The child is given a set of three toy cars.	
a) How many cars do you have?	a) model → verbal name
b) Point to the numeral that shows how many cars you have.	b) model → symbol
The teacher points to the numeral '4.'	
a) Make a set to show how many.	a) ☐ → ☐
b) Tell how many cars.	b) ☐ → ☐
The teacher says, "two."	
a) Make a set to show "two" cars.	a) ☐ → ☐
b) Write the numeral which shows how many cars you have.	b) ☐ → ☐

The child is given a set of five.

a) How many do you have?

b) Write the numeral that shows how many cars you have.

a) [ ] → [ ]

b) [ ] → [ ]

2. In the examples below, the relationship being developed is provided. You are to supply the appropriate questions or directions.

Questions or Directions	Relationship Being Developed
	a) model → verbal name b) model → symbol
	a) verbal name → model b) verbal name → symbol
	a) symbol → model b) symbol → verbal name

ACTIVITY 11

GROUPING TO PLACE VALUE

FOCUS:

In this activity you will have an opportunity to use the aids appropriate for grouping and place value activities. Each aid is designed for a specific purpose, that is, to develop grouping concepts or to develop place value concepts. It is important to know when and how to use these aids to develop early numeration concepts.

MATERIALS:

Bead frames, "trading chips," Dienes blocks, bundling sticks, Unifix cubes, bean sticks, place value charts, abaci, or other grouping-place value aids for each group of students.

DISCUSSION:

One goal of teaching numeration in the elementary school is to enable the child, given a set of objects, to identify the number of that set with a base ten place value representation. To achieve this goal in some meaningful fashion, a variety of carefully sequenced activities (over a considerable time span) are used.

Numeration work begins with concrete manipulatives which allow the child to:

a) Choose a number as a base, group the objects of a set into groups with the base number in each group.

b) Count how many groups (with the base number of objects in each) have been formed.

c) Count how many single objects are left over.

The most fundamental of those manipulatives would include a set of similar objects, probably sticks or pipe cleaners, with some means (such as rubber bands) of bundling them into groups.

After this initial or grouping stage, other materials are introduced which are designed to bring the child to the final place value stage. In this latter stage the child is able to represent the number with symbols meaningfully. Materials used at this stage would indicate the positional or place value significance of the digits.

It should be noted that both grouping and place value activities take place in several elementary grades. In grade 1, for example, children study the relationship between ones and tens. They are given, say, 37 bundling sticks and asked to put them into bundles of tens and ones. The abacus or some other place value aid is used by the children to express the number. Finally, the symbols are used. In succeeding grades, children study larger numbers and (in grades 5 and 6) decimals. Throughout the grades, both the grouping and the place value concepts are emphasized. Care should be taken to use appropriate aids.

Let us reflect for a moment upon the developmental approach suggested.

Suppose that "thirty-four" is to be represented.

At the "grouping" stage a child might illustrate "thirty-four" by forming three bundles of 10 sticks, noting that 4 single sticks are left over. Figure 1 below represents the completed task.

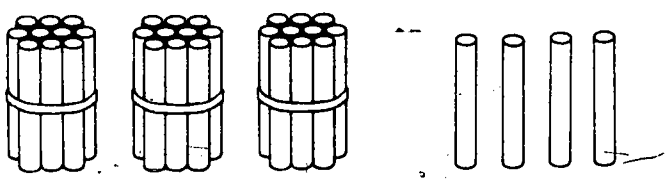
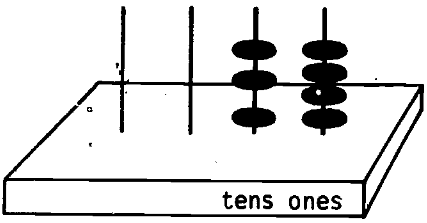


Figure 1

When grouping notions are well grasped, a child might choose to represent "thirty-four" on the abacus, a device which emphasizes place value concepts. A child as this stage would represent "thirty-four" in the manner illustrated in Figure 2, where three discs represent the number of "tens" and four discs indicate the number of "ones."

Figure 2



The abacus thus arranged represents the standard numeral, 3 groups of ten and 4 ones or 34.

DIRECTIONS:

Before you proceed, be sure you have read the Discussion above carefully.

1. Identify the following materials as either grouping aids or place value aids. Do some materials convey both ideas?

   Bead frames
   Trading chips (varied colors)
   Dienes Blocks
   Bundling sticks
   Unifix cubes
   Bean sticks
   Place value charts
   Abaci

2. Choose two or three grouping aids and two or three place value aids. Show each of the following numerals on each. Demonstrate to a classmate how you might explain the bridge between a grouping display and a place value display.

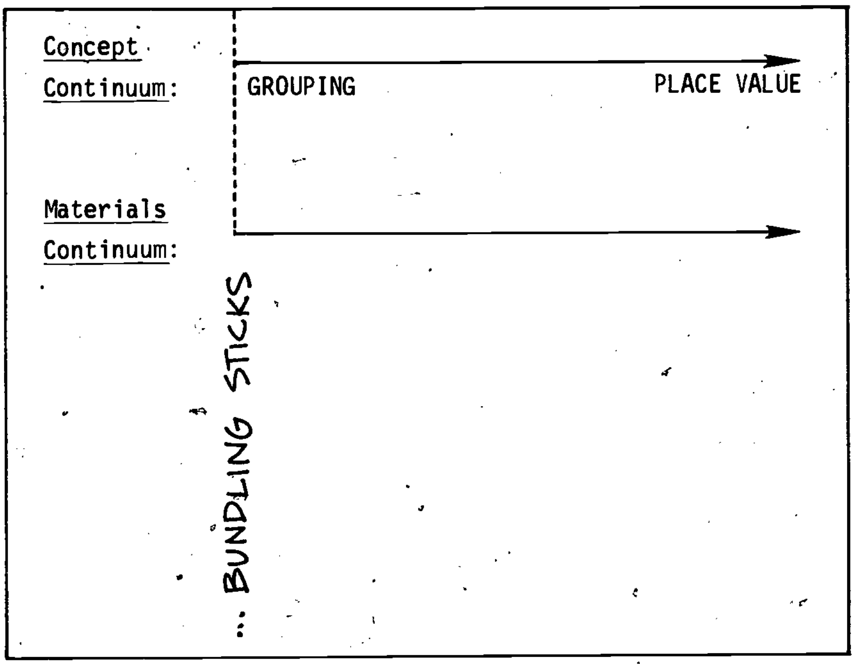
   a) 17 b) 40 c) 34 d) 29

3. Use a pair of aids (one grouping and one place value) to show how you would rename after addition or how you would rename in preparation for subtraction. Demonstrate the examples on the following page to another student.

a) 8 + 16 b) 19 + 23 c) 18 + 34

d) 23 - 9 e) 31 - 18 f) 43 - 24

4. The top line of the continuum pictured below, a CONCEPT continuum, ranges from grouping to place value notions. We might construct a corresponding MATERIALS continuum on which we would locate the various manipulative aids (listed in 1 above) often used in the development of numeration concepts and skills.



Bundling sticks might logically appear first on the "materials continuum." Decide where you would locate the other materials, arranging them in a manner which would naturally extend grouping to place value ideas. Check your ideas with others in your group. After your group has decided on an arrangement, list the aids on the "materials continuum."

5. (Class Discussion) Activities 9 through 11 fave focused on the primary-grade numeration concepts. While these activities have presented only a few of the numeration concepts taught in the

primary grades, they have focused on the essential ones--grouping and place value. A class discussion with your instructor should provide a chance for you to ask any questions you may have.

TEACHER TEASER

A Weighty Problem

A man had 40 sacks of gold dust which ranged in weight from 1 to 40 ounces. (No two sacks weighed the same, and there were no fractional weights.) One day, after returning from a long trip, he suspects that a thief has pilfered some of the gold, so he hires you to weigh each sack. He provides you with a balance scale and a box of 40 weights, ranging from 1 ounce to 40 ounces.

1. If you are allowed to place weights on only one side of the balance, what is the minimum number of weights you'll take from the box in order to weigh the sacks of gold? How many ounces does each of these weights weigh?

2. If you are given the same task but are allowed to use both sides of the balance, what is the minimum number of weights you'd need to use? How heavy is each weight?

3. With the weights in (1) above, every counting number 1-40 can be represented. To what base are these weights related? What about the weights in (2) above?

ACTIVITY 12

EXTENDING NUMERATION TO RECORD NUMBERS LESS THAN ONE

FOCUS:

Activities 9 through 11 focused on some numeration concepts appropriate for the primary grades. Activities 12 and 13 will focus on two numeration concepts taught in the upper grades. This activity takes you through some developmental experiences which are appropriate to use in applying numeration concepts to decimals.

MATERIALS:

(The letters preceding each group of materials correspond to the four parts of the activity.)

A. Three pieces of (construction) paper for each group of students (10 x 10 graph paper would be ideal), scissors, and ruler.

B. An abacus for each group.

C. Centimeter grid paper, heavy paper, felt-tipped pen, glue.

D. Dienes Blocks

DISCUSSION:

In the upper elementary grades children learn that numbers between whole numbers can be represented using decimals. The place value concept emphasizing ones, tens, hundreds and thousands is extended to include ones, tenths, hundredths and thousandths. Children should see and use the symmetry of our numeration system in learning decimal notation.

6527.684

6	5	2	7.	6	8	4
thousands	hundreds	tens	ones	tenths	hundredths	thousandths

$$6 \times 1000 + 5 \times 100 + 2 \times 10 + 7 + 6 \times \frac{1}{10} + 8 \times \frac{1}{100} + 4 \times \frac{1}{1000}$$

Note: The point of symmetry is the "ones" place, not the decimal point.

As with most other mathematical developments in the elementary school, the use of materials can help in making initial notions about decimals clear.

DIRECTIONS:

Work through each part, imagining you are working with children or preparing materials for children. Try to anticipate difficulties or questions children might have; also record any other activities that suggest themselves to you as you work.

PART A: CUTTING TENTHS AND HUNDREDTHS

(Your group will need three pieces of construction paper.)

1. Take one of the three pieces of construction paper and cut it into 10 congruent (same size and shape) pieces. Make use of your ruler to determine points of division. What part of the original piece of construction paper does each piece represent?

2. Throw away six of the congruent pieces.

3. Select one of the four remaining pieces and cut it into 10 congruent parts. (Use a ruler if necessary to determine points of division for cutting.)

4. What part of the original piece of construction paper does each piece represent?

5. You should have three different-sized pieces of construction paper left. Sort them into three piles, corresponding to the three sizes.

6. Record on the following page the number of pieces you have in each pile:

Number of Full-sized Sheets	Number of Middle-sized Pieces	Number of Smallest-sized Pieces

7. You no longer have three whole sheets of construction paper since you have thrown some pieces away. Can you think of an efficient way to represent the amount of construction paper you haven't thrown away in terms of the number of sheets? ((6) above should serve as a hint.)

8. How much paper have you thrown away?

PART B:

USING AN ABACUS TO SHOW THE SYMMETRY OF OUR NUMERATION SYSTEM

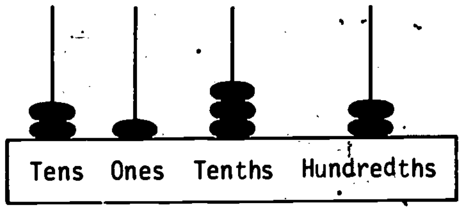
(You will need an abacus to complete this activity.) The display below illustrates the symmetry of the base ten system.

3	3	3	3.	3	3	3
thousands	hundreds	tens	ones	tenths	hundredths	thousandths

$$(3 \times 10^3) + (3 \times 10^2) + (3 \times 10) + (3 \times 1)$$
$$+ (3 \times \frac{1}{10}) + (3 \times \frac{1}{10^2}) + (3 \times \frac{1}{10^3})$$

Note that the "ones" place occupies a central position in this symmetry. By relabeling the wires on an abacus, the aid can be used to represent numbers such as that shown above. If you use a commercial abacus, you are limited to four-digit numbers, since most commercial products have only four wires. An alternative is to make your own abacus by sticking knitting needles in clay, straws in styrofoam, or wires in wood. Beads can then be used to represent the value of the digits.

1. What number is represented on the abacus to the right?

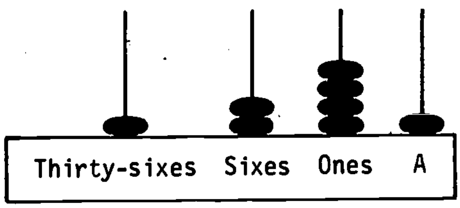


2. What is the largest number that can be represented on this abacus, using the labeling shown?

3. Discuss with others in your group the modifications that would be necessary to express 0.6128 on the abacus.

4. Suppose you were to use a "base six" abacus.

   What would be the appropriate label for Wire A?



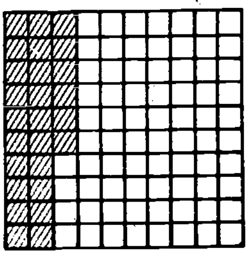
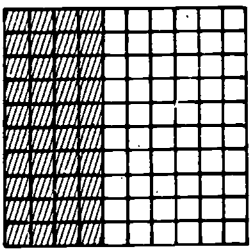
   Use a mixed fractional number to represent in base ten the number shown.

PART C: MATCHING PICTURES, NUMERALS AND NAMES

MATCH is a card game. The cards are teacher-made. Sample cards are shown below. Centimeter grid paper can be glued to the cards and shaded with a felt-tipped pen. The deck consists of 36 cards: 12 cards, each of which contains a number expressed in tenths and hundredths (e.g., .14, .86, .07, .5); a second set of 12 cards, each of which contains a grid that corresponds to a number of the first set;

a third set of 12 cards, each of which contains the verbal description of a card in the first set.

SAMPLE CARDS

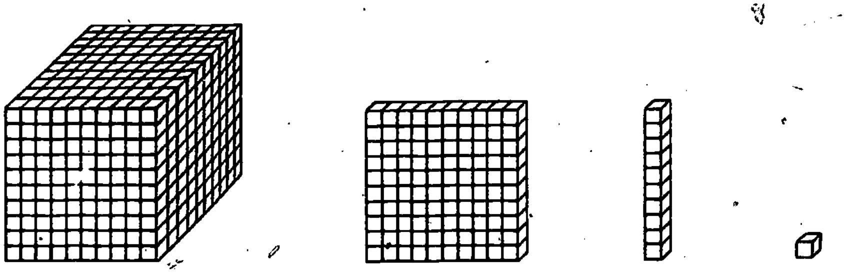
	.26	twenty-six hundredths
	.40 or .4	forty hundredths or four tenths

MATCH can be played by three or four people. Shuffle all 36 cards and deal them to the players. The player to the left of the dealer starts by drawing one card from the dealer. This player can then lay down any set of three cards which match. (Set (a) above is a match. So is set (b).) Play continues in this way, each player drawing a card from the person on his right and laying down matches. The first person to run out of cards wins.

Construct the cards and try it. Can you suggest any modifications to the game?

PART D: USING DIENES BLOCKS

Using the Dienes blocks, think of the block as one, the flat as one-tenth, the long as one-hundredth and the unit as one-thousandth.



1.0 0.1 0.01 0.001

1. Display the following numbers using Dienes blocks.

    a) 0.25

    b) 0.043

    c) 1.141

    d) 0.305

2. Use the Dienes blocks as a grouping aid and the abacus as a place value aid to display the following numbers. Explain to a classmate the relationship between the grouping display and the place value display.

    a) 0.015

    b) 1.032

    c) 0.055

    d) 0.307

PART E: DECIMALS IN THE PRIMARY GRADES?

1. The use of the hand-held calculator has become widespread. Recently a child was posed the problem: "A father gave 58 cents to be shared equally among seven children. How much should each child receive?" One child had a hand calculator and correctly

entered the division example 58 ÷ 7. The answer 8.2857142 which appeared on the display startled the child. "Wow! Look how much each child gets," was his reaction. Discuss how would you help the child understand what the "answer" 8.2857142 really represents.

2. Some educators believe decimal notation soould be introduced in the primary grades. Discuss this issue, focusing on the following points:

   a) What is your opinion of this belief?

   b) Outline briefly a sequence which might be followed in such a development.

ACTIVITY 13

EXPONENTS AND SCIENTIFIC NOTATION

FOCUS:

For a number of reasons decimals now play a greater role in people's lives than ever before. Scientific notation utilizes decimals and is very convenient for representing very large numbers or numbers that are very near zero. Exponents, especially powers of 10, are essential to scientific notation. In this activity you will have an opportunity to learn or review exponents and then apply them in the use of scientific notation.

DISCUSSION:

One can write 32000000000000 or one can write 3.2×10^{13}. The latter is easier. In the upper grades children learn that very large and very small numbers can be more easily expressed using scientific notation. This notation emphasizes the base ten nature of our numeration system. For children the series of ideas which culminate in expressing numbers in scientific notation is developed over a period of time. Three stages in the development are presented for you: exponents, expanded notation using exponents, and writing numbers in scientific notation.

DIRECTIONS:

PART A: EXPONENTS

1. The use of exponents

4^3 means $4 \times 4 \times 4 = 64$ $\qquad 64 = 4^3$ ← exponent (3), base (4)

64 is the 3rd power of 4

Complete the following

a) 4^5 = ____ x ____ x ____ x ____ x ____ = ____

b) 10^6 = ____ x ____ x ____ x ____ x ____ x ____ = ____

c) $7^3 =$ ________

d) $8^2 =$ ________

e) $10^5 =$ ________

2. Some properties of exponents

When multiplying two numbers whose base is the same, the exponents are added.

$$3^3 \times 3^2 = (3 \times 3 \times 3) \times (3 \times 3) = 3^{(3 + 2)}$$
$$= 27 \times 9 = 243 = 3^5$$

a) $6^2 \times 6^4 = 6^{—}$ b) $4^3 \times 4^3 = 4^{—}$

c) $3^2 \times 3^4 = 3^{—}$ d) $10^3 \times 10^4 = 10^{—}$

e) $64 = 4^{—}$ f) $81 = 3^{—}$

Check one or two of the above examples to convince yourself that exponents are added in multiplication.

When dividing two numbers whose base is the same the exponents are subtracted.

$$5^3 \div 5^2 = (5 \times 5 \times 5) \div (5 \times 5) = 5^{(3 - 2)}$$
$$= 125 \div 25 = 5 = 5^1$$

a) $6^4 \div 6^1 = 6^{—}$ b) $8^4 \div 8^3 = 8^{—}$

c) $5^5 \div 5^4 = 5^{—}$ d) $10^5 \div 10^2 = 10^{—}$

By definition, any whole number (except 0) to the 0 power equals 1. Thus $10^0 = 1$, $7^0 = 1$, etc. Show why it makes sense to define it this way. For example, compute $8^6 \div 8^6$ by dividing and then by using the subtraction rule for exponents.

3. Exponents to express rational numbers

Study the display below. Notice that we can write these rational numbers as powers of ten by means of exponents.

$$\begin{array}{ccccccc}
1000 & 100 & 10 & 1 & \frac{1}{10} & \frac{1}{100} & \frac{1}{1000} \\
| & | & | & | & | & | & | \\
10^3 & 10^2 & 10^1 & 10^0 & \frac{1}{10^1} & \frac{1}{10^2} & \frac{1}{10^3}
\end{array}$$

The same rational numbers less than 1 can be written using negative exponents. Note the symmetry in the display below.

$$\begin{array}{ccccccc}
1000 & 100 & 10 & 1 & \frac{1}{10} & \frac{1}{100} & \frac{1}{1000} \\
| & | & | & | & | & | & | \\
10^3 & 10^2 & 10^1 & 10^0 & 10^{-1} & 10^{-2} & 10^{-3}
\end{array}$$

The number 100 is expressed as 10^2 while the number $\frac{1}{100}$ is expressed as 10^{-2}.

Complete the following.

a) $64 = 4^3$ so $\frac{1}{64} =$ ____

b) $625 = 5^4$ so $\frac{1}{625} =$ ____

c) $100000 = 10^5$ so $\frac{1}{100000} =$ ____

d) $6^{-3} =$ ____

e) $10^{-4} =$ ____

f) $8^{-2} =$ ____

g) $3^{-4} =$ ____

PART B: EXPANDED NOTATION USING EXPONENTS

Exponents can be used in expressing a numeral in expanded notation. Study the example for 234.1067 shown below.

$234.1067 =$

(1) $200 + 30 + 4 + .1 + .00 + .006 + .0007$

(2) $(2\cdot 100) + (3\cdot 10) + (4\cdot 1) + (1\cdot\frac{1}{10}) + (0\cdot\frac{1}{100}) + (6\cdot\frac{1}{1000}) + (7\cdot\frac{1}{10000})$

(3) $(2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0) + (1 \times \frac{1}{10^1}) + (0 \times \frac{1}{10^2}) +$
$(6 \times \frac{1}{10^3}) + (7 \times \frac{1}{10^4})$

(4) $(2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0) + (1 \times 10^{-1}) +$
$(0 \times 10^{-2}) + (6 \times 10^{-3}) + (7 \times 10^{-4})$

Write the following numerals in expanded notation in two ways using forms (3) and (4) above.

a) 42.671

b) 5178.071

PART C: SCIENTIFIC NOTATION

In scientific notation every number is written as the product of a number between one and ten times ten to some power.

$$\boxed{\_\_\_.\_\_\_ \times 10^{\_\_}}$$

EXAMPLES

(1) $810 = 8.1 \times 100$
$= 8.1 \times 10^2$

(2) $78{,}000 = 7.8 \times 10000$
$= 7.8 \times 10^4$

1. Study and find a pattern for placing the decimal point and determining the power of ten.

$$3{,}120 = 3.12 \times 10^{3}$$

$$218{,}400 = 2.184 \times 10^{5}$$

$$45{,}800 = 4.58 \times 10^{4}$$

$$309 = 3.09 \times 10^{2}$$

$$2{,}004{,}000 = 2.004 \times 10^{6}$$

Describe the pattern you have found.

2. Each of the following expressions represents a very small number. Study the examples and find a pattern to help in locating the decimal point and determining the power of ten.

$$.0273 = 2.73 \times 10^{-2}$$

$$.00120 = 1.2 \times 10^{-3}$$

$$.000143 = 1.43 \times 10^{-4}$$

$$.0000297 = 2.97 \times 10^{-5}$$

$$.0000039 = 3.9 \times 10^{-6}$$

Describe the pattern you see.

3. Convert any numbers in scientific notation to standard form. For c) and d) also answer the specific questions asked.

a) Suppose it were possible to stack a billion pennies one on top of the other. They would form a stack 9.56×10^2 miles high. If they were placed end to end they would extend a distance of 1.184×10^4 miles.

9.56×10^2 = ____________________

1.184×10^4 = ____________________

b) During a meteor shower the earth's atmosphere collects as much as 2×10^{10} pounds of interplanetary dust in one day.

2×10^{10} = ______________________________

c) The mass of a neutron is 1.675×10^{-24} grams and the mass of an electron is 9.107×10^{-28}. What is the difference in their masses?

1.675×10^{-24} = ______________________________

9.107×10^{-28} = ______________________________

Difference = ______________________________

d) The average distance of the planet Pluto from the sun is 3.67×10^{9} miles. If one light-year equals 5.88×10^{12} miles, how many light years is Pluto from the sun?

3.67×10^{9} = ______________________________

5.88×10^{12} = ______________________________

Pluto is ____________________ light years from the sun.

4. What is the largest number that could be recorded using scientific notation on a calculator? (Assume that the calculator has an eight-digit display plus a two-digit exponential display.)

ACTIVITY 14

SEMINAR

FOCUS:

This seminar provides an opportunity for you to summarize your study of numeration in the elementary school.

DIRECTIONS:

Take another look at the chart of topics you prepared in Activity 8.

1. Ask yourself whether there are any topics in your chart which you have not studied and which you would like to discuss in class. Mention these to your instructor.
2. As a class, outline the major ideas in the numeration strand of elementary school mathematics. Identify those with which you think children will have the most difficulty.
3. Review the various aids used in numeration. Discuss purposes, advantages, and shortcomings of each.

Section IV

DIAGNOSTIC AND REMEDIAL WORK IN NUMERATION

Much of your work as a teacher will come in the form of diagnosing learning problems and providing appropriate remedial work for children who are having difficulty. In fact, one of the major differences between an excellent teacher and an average teacher lies in their sensitivity to children's difficulties or potential difficulties in learning. The excellent teacher will become aware of areas of potential difficulty and gear the instruction to prevent learning problems from arising. Problems always arise, however, and the sensitive teacher will attack them with a problem-solving attitude. Nothing is more satisfying to a teacher than to see growth resume after a child's progress has been stopped because of some learning difficulty.

The previous sections have provided mathematical and pedagogical background for you to make instructional decisions. This section will provide an opportunity to apply your knowledge by analyzing pupil pages which have been selected from children's work at both the primary and upper grade levels. The section concludes with an assignment in which you are asked to write a lesson on a numeration topic.

MAJOR QUESTIONS

1. What are two or three common numeration errors made in the primary grades? What kinds of things can be done about these errors?
2. What are two or three common numeration errors made in the upper grades? What kinds of things can be done about these errors?
3. Outline a general procedure for helping children who are having learning problems related to numeration.

ACTIVITY 15

CHILD ERRORS: DIAGNOSIS AND REMEDIATION

FOCUS:

Brownell pointed out several years ago that child errors provide rich opportunity for constructive teaching.* It is in this spirit that the present activity is written. The following pages present the work of children at various grade levels. After you study the child pages and locate specific errors, you will be challenged to design activities to help a child overcome such errors.

DISCUSSION:

The work of four children is depicted on the following pages. The table below gives an indication of the learning level of each child and of the nature of the numeration concept or skill on each child page.

WORKSHEET	CHILD	PAGE
A	Jana - early primary	Writing and interpreting two-digit numerals
B	Jack - upper primary	Review Sheet: Reading and writing numerals; renaming; understanding hundreds
C	Tony - lower intermediate	Review Sheet: Telling the greater of two numbers; rounding; reading and writing large numbers
D	Jerry - upper intermediate	Understanding the meaning of decimals

*William A. Brownell and Gordon Hendrickson. NSSE 49th Yearbook, Part I: Learning and Instruction. Chicago: National Society for the Study of Education, 1950, p. 115.

Each of these children is having a problem with some phase of numeration, and this difficulty is portrayed on the child's worksheet.

DIRECTIONS:

1. For each of the four worksheets on the following pages, carry out the steps below. (Steps (a) and (b) should be completed individually.)

    a) Circle any child errors you find.

    b) Identify any error pattern(s) the child seems to display. An example of how to do this is given in the Analysis of Errors section of Jana's worksheet, page 96.

    c) If a possible cause of error(s) is supplied, discuss whether you agree with the cause given--and if you disagree, supply your analysis. If a possible cause of error(s) is not given discuss this issue with your group, and write your analysis.

    d) If some suggested remedial activities are supplied, follow the procedure outlined in (c) above. That is, evaluate the remedial activities suggested, and write your suggestions where you disagree.

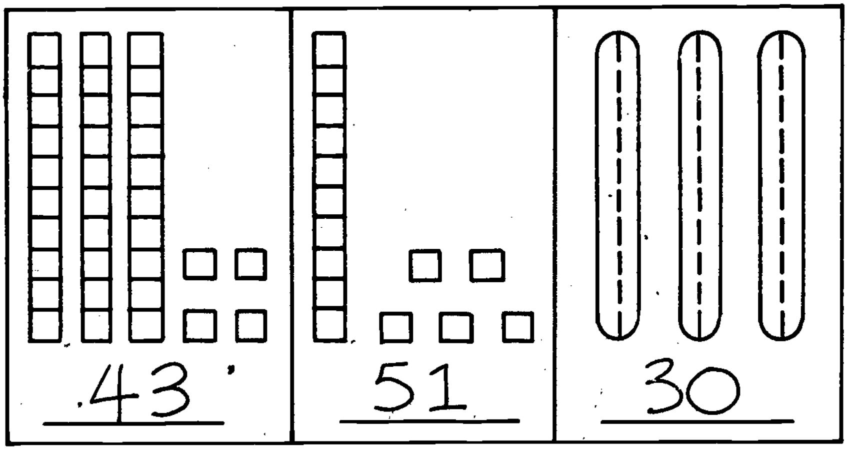
2. Discuss as a total class the analysis of the errors; give possible causes, and make suggestions for remediation.

3. List other errors you think children might make in numeration. In this discussion you can use your own or your instructor's experiences with children. You might wish to interview some children or their teacher in a visit to an elementary school.

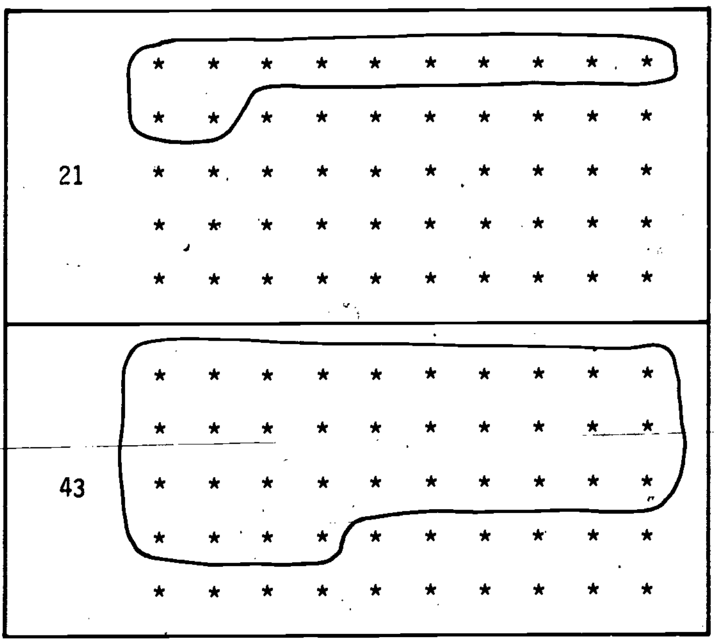
WORKSHEET

NAME Jana

1. Write the numeral for each.



2. For each numeral circle the correct number of stars.



Analysis of Errors--Jana

Error Pattern(s)

In (1), Jana seems to have the place value of ones and tens reversed. She may write 30 correctly because 03 doesn't make much sense (or she may even be counting only the sticks rather than the beans. In that event, there are 3 ones and 0 tens!) The same error pattern is found in (2). She consistently sees the numeral in the ones place representing tens and vice versa.

Possible Cause of Errors

Jana may be reading numerals from left to right (as she does in reading a book). Thus the ones come first, and then the tens.

Suggested Remedial Instruction

1. Ask Jana to count and write numerals up to 25. If she does this correctly, go to (2) below. If not, open a book with which she is familiar (one that has at least 25 pages). Ask her to count the pages and point to the page number as she counts. Point out that the ones are on the right while the tens are on the left: 10, 11, 12, etc.

2. Since Jana did not make any grouping errors she needs reinforcement of the place value concept. Ask her to count through 25 on the abacus, flipping a bead with each number. At 10 and 20 make sure she trades 10 ones for 1 ten.

3. As a test of her understanding ask her to display 18 (26, 42, etc.) on the abacus. Conversely, display 23 (41, 36, etc.) on the abacus and ask her to tell you how many are displayed.

WORKSHEET

NAME Jack

1. Write the numerals for:

Seventeen 71

One hundred one 1001

Two hundred forty-five 20045

2. Complete:

174 means	16 tens	$\boxed{4}$ ones
255 means	$\boxed{5}$ tens	15 ones
392 means	38 tens	$\boxed{2}$ ones

3. Write the number that is a hundred greater.

3450 4450

700 800

2321 3321

Analysis of Errors--Jack

Error Pattern(s)

Possible Cause of Errors

Jack may have a reversal problem. The 71 for 17 may be accidental, but it should be checked out through an interview. He certainly does not understand place value. In (3), for example, he thinks adding a 1 to the leftmost digit increases it by one hundred. More likely, he has a faint recollection of seeing the teacher add 1 to a number and he is making a stab at completing the assignment. His work in (1) suggests he knows something about writing numerals but has no concept of place value.

Suggested Remedial Instruction

1. After checking Jack to see if he has a reversal problem, you could use Dienes Blocks and an abacus and go through the following sequence using several different examples.

   a) Display a number of blocks.

   b) Work with Jack to display numbers on the abacus.

   c) Write the numeral showing the relationship between each digit and the abacus.

2. Ask Jack to add 100 to the Blocks, to the abacus, and to the numeral, comparing each aid with each step.

3. In Jack's present stage of learning, renaming (as exhibited in (2) of his Worksheet) is not an appropriate activity. After Jack has a better understanding of place value, begin to show "trading" a ten for 10 ones for a given number displayed on the abacus (say 342). Ask Jack to count the number of ones and the number of tens, and then record them. Much practice with this sort of activity must precede pure pencil-and-paper activity.

WORKSHEET

NAME Tony

1. Circle the numeral which is greater.

   3449 or 3481

   2632 or 2658

2. Round to the nearer hundred.

   A. 42,371 42,300

   B. 37,418 37,400

   C. 81,693 81,600

3. The teacher dictated the following:

   A. Sixty-seven hundred five

   B. One hundred sixty thousand

   C. Sixteen hundred fifty

   The child wrote:

   A. 67,005

   B. 160,000

   C. 16,050

Analysis of Errors--Tony

Error Pattern(s)

Possible Cause of Errors

Suggested Remedial Instruction

WORKSHEET

NAME Jerry

1. Circle the greater number.

A. 0.8 0.42

B. 0.086 0.091

C. 0.177 0.2

2. Which is closer to 0.83?

0.8 or 0.9

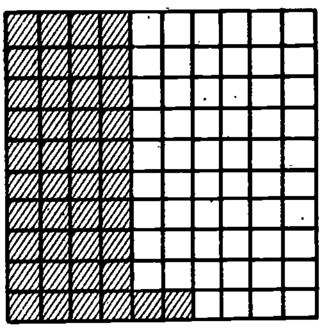
3. What is the place value of the position shown by each arrow?

0.76 — tens

0.45 — ones

4. Write a decimal for each picture.

A.  .5

B.  .42

Analysis of Errors--Jerry

Error Pattern(s)

Possible Cause of Errors

Jerry seems to have a conceptual understanding of decimals (see (4) on the worksheet), but he seems to have no understanding of the way decimals are symbolized. His work in (1) and (3) suggests that he is viewing these decimals as whole numbers rather than as a fractional part of a whole number. He completely ignores the decimal point and treats each number as a whole number. Perhaps he did (4) by counting the shaded areas as whole numbers and putting a decimal point in the answer because he was told to "write a decimal"! It would be useful to listen to Jerry explain how he did these problems.

Suggested Remedial Instruction

ACTIVITY 16

DEVELOPING A NUMERATION LESSON

FOCUS:

The sequencing exercises of Activities 7 and 8 provided you with a broad frame of reference for numeration activities in the elementary school. Activity 15 provided an opportunity for you to become acquainted with some common errors in numeration. In this activity you are asked to write a numeration lesson, taking note of the sequential development of numeration <u>and</u> incorporating special emphasis on ideas that might present difficulty to children.

MATERIALS:

Several elementary mathematics text series (teacher's editions).

DIRECTIONS:

1. A number of situations related to numeration which have occurred with elementary school children are presented on page 106. Read this page carefully and then choose <u>one</u> situation which is at a grade level of your choice or which concerns a problem you would like to become more familiar with.

2. Develop a lesson which you could use to help children learn the numeration concept or skill described. Focus upon the particular aspects of the concept (skill) embodied in the situation you selected. Be sure to anticipate possible barriers to child learning, and shape your lesson accordingly. You may wish to refer to the teacher's editions of several elementary mathematics texts to get ideas for developing your lesson, but be sure to mingle your own bit of creativity with the suggestions in these texts. Your lesson format should include:

   a) <u>Objectives</u>;

b) Materials you would use;

c) Outline of motivational activity (if used);

d) Outline containing the sequence of key questions and/or major ideas you would address yourself to in developing such a lesson with children.

e) Follow-up exercises or activities you would assign. (Provide samples of the type of work you would assign. It is not necessary to develop full worksheets unless you arrange to do (3) below.)

f) Suggest how you might evaluate whether the children attained the objectives you set for the lesson.

If you are concise and use an outline format, your lesson will probably not exceed two pages.

3. If possible, arrange through your instructor to teach your lesson to an elementary school child.

NUMERATION SITUATIONS

1. Counting Past 100

   Children often have difficulty counting past one hundred. For example, it is not uncommon for children to count: 99, 100, 200, 300, ... (Devise a lesson which focuses on enabling a child to count: 99, 100, 101, 102, ...)

2. Numeral Reading of Larger Numbers

   When asked to write six thousand forty-one, Tom writes "641." He similarly records "220" for two thousand twenty. (Write a sequence of activities to help Tom.)

3. Ordering Numbers

   Some children, in responding to the number pair (310<u>9</u>, 316<u>1</u>), identify "3109" as representing the greater number. They compare only the digits in the ones place. Other children become confused in ordering numbers when "0" appears among the digits. (Keeping in mind the types of problems children experience, develop a lesson on ordering numbers in the thousands.)

4. Rounding Numbers

   When children are asked to round to the nearer hundred they sometimes give responses such as:

   a) 3531 ⟶ 3500

   b) 3591 ⟶ 3500

   In such an instance they have concentrated only on the digit in the hundreds place, regardless of what followed it. (Develop a lesson which reviews the basic concept of rounding and applies this to finding the nearer hundred.)

5. Extension of the Numeration System to Decimals

   Some students have difficulty understanding place value to the right of the ones place. Given the number pair .43 and .8, they may mistakenly identify .43 as the greater number. Or they might say .52 is greater than .6. (Develop a lesson which extends the decimal system to tenths and hundredths, keeping this problem in mind.)

6. Exponential Notation

When children see 4^3 (four to the third power, or four cubed), they often think this means "4 x 3." (Develop a lesson which will clarify the use of an exponent.)

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Fitzgerald, et al. Laboratory Manual for Elementary Mathematics. Boston: Prindle, Weber & Schmidt, Inc., 1969.

Greenes, Carole E. Problem Solving in the Mathematics Laboratory. Boston: Prindle, Weber & Schmidt, 1972. (Contains several numeration games.)

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Rinker, Ethel. "Eight-ring Circus: A Variation in the Teaching of Counting and Place Value," The Arithmetic Teacher. (March, 1972), pp. 209-216.

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REQUIRED MATERIALS

ACTIVITY	AUDIO-VISUAL AND OTHER RESOURCES	MANIPULATIVE AIDS
2	Elementary mathematics textbook series.	
4		Multibase Arithmetic Blocks in assorted bases, poker or colored chips, abaci, dice, spinners.
5		Multibase Arithmetic Blocks (all bases), abaci, or substitutes.
6		Index cards, paper punch, scissors, knitting needle, toothpicks.
7	Slide-tape: "Numeration in the Elementary School," cassette recorder, slide projector, (optional); set of current elementary school mathematics textbooks.	
8	Several sets of elementary mathematics textbook series.	

ACTIVITY	AUDIO-VISUAL AND OTHER RESOURCES	MANIPULATIVE AIDS
11		Bead frames, "trading chips," Dienes blocks, bundling sticks, Unifix cubes, bean sticks, place value charts, abaci, or other grouping-place value aids.
12		Construction paper, scissors, ruler, abacus, centimeter grid paper, heavy paper, felt-tipped pen, glue, Dienes blocks.
16	Several elementary mathematics textbook series (teacher's editions).	

Continued from inside front cover

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory, Probability and Statistics,* and *Experiences in Problem Solving.*

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