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ABSTRACT

This unit considers the application of calculus in determining price discrimination and consumer surplus in a competitive market. Producer surplus and two-tier price discrimination are also developed in problems. It is noted that calculus cannot usually provide numerical answers for practical economic problems. The importance of calculus applications is seen to be in the contributions made to the development of economic theory. It is felt that the examples developed can provide insight into the theoretical relationships in economics and how calculus is used in their development. Exercises and a model exam are included, with answers provided to both at the end of the material. (MP)

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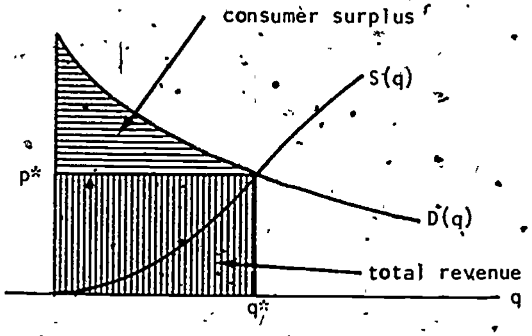
UNIT 294

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

PRICE DISCRIMINATION AND CONSUMER SURPLUS

by Christopher H. Nevison



APPLICATIONS OF CALCULUS TO ECONOMICS

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PRICE DISCRIMINATION AND CONSUMER SURPLUS: AN APPLICATION OF CALCULUS TO ECONOMICS

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Intermodular Description Sheet: *UMAP Unit 294*

Title: PRICE DISCRIMINATION AND CONSUMER SURPLUS

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Review Stage/Date: IV 6/15/79

Classification: APPL CALC/ECONOMICS

Suggested Support Material:

Prerequisite Skills:
1. Differentiation and integration of polynomials.
2. Area between curves by integration.
3. Maximization of a function using the derivative.

Output Skills:
1. Understand the concepts of price discrimination and consumer surplus.
2. Be able to find the roots of polynomial equations.
3. Be able to integrate polynomials and find the area under a curve.
4. Be able to solve maximization problems.

Other Related Units:
Applications to Economics *(Unit 270)*
General Equilibrium-A Leontief Economic Model *(Unit 209)*
The Distribution of Resources *(Units 60-62)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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1. INTRODUCTION

The methods of calculus find many applications in the field of economics. In this module we consider one such application, *price discrimination*, and the *consumer surplus* in a competitive market. In the context of this example we discuss the typical simplifying assumptions which must be made in order to use the calculus. We then develop the particular ideas associated with these concepts. Further variations of these ideas, the *producer surplus* and *two-tier price discrimination*, are developed in problems.

You may wonder whether the methods of calculus can actually provide numerical answers for practical economic problems. The answer is usually no; the importance of applications of calculus to economics such as those discussed here is their contribution to the development of economic theory. Applications can be illustrated with specific numerical examples, as we have, but the information needed to construct the functions for a real situation is in general difficult, if not impossible, to obtain. (The field of econometrics develops techniques for eliciting such information.) Consequently, the examples developed in this module should be studied to gain insight into the theoretical relationships in economics and how the calculus is used in their development.

2. APPROXIMATION OF SUPPLY AND DEMAND

A fundamental assumption used whenever the methods of calculus are applied to economics is that the functions of interest can be closely approximated by *continuous* or, sometimes, *differentiable functions*. Supply and demand theory is no exception.

2.1 The Supply and Demand Functions

Under appropriate assumptions economists tell us that in a competitive market the price at which a good is sold is determined by two functions: *supply* and *demand*. The *supply function* associates with any quantity, q, of the good, a price which will be sufficient to attract exactly that quantity of the good into the market, $p = S(q)$.

It is sometimes easier to think of a supply function as quantity supplied at a given price, $q = \overline{S}(p)$. In the latter form, we can think of a producer deciding how much of a commodity it would be profitable for him to produce at the prevailing market price. We usually assume that the higher the price the larger the quantity supplied, so that the function $q = \overline{S}(p)$ will be increasing. Thus it will be an invertible function, so that we can consider $p = S(q)$, the inverse of the $\overline{S}$ function.

The *demand function* associates with each quantity, q, the price at which the market will be just cleared of that quantity, $p = D(q)$.

Again, it may be easier to think of a demand function in terms of the quantity demanded at a given market price, $q = \overline{D}(p)$. The interpretation in this case is that $\overline{D}(p)$ is the quantity which consumers will buy at the prevailing price p. Demand is normally assumed to be a decreasing function: the higher the price, the less people will buy. Thus $q = \overline{D}(p)$ is also an invertible function and we can just as well consider the inverse function $p = D(q)$.

2.2 Equilibrium Price and Quantity

In terms of the supply and demand functions S(q) and D(q), static analysis suggests that an *equilibrium price*, p*, and *quantity*, q*, occur for the quantity at which the price offered (demand) and the price asked (supply) are equal: $p^* = D(q^*) = S(q^*)$. (Figure 1.)

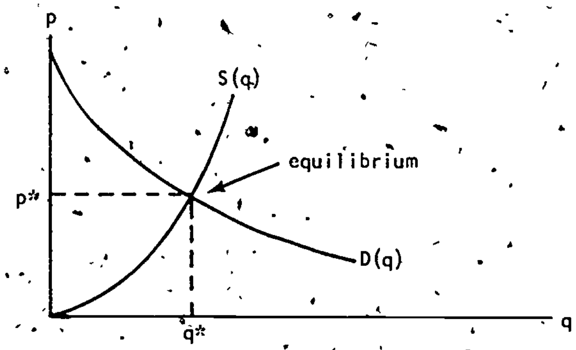

Figure 1. Supply and demand equilibrium.

2.3 Continuous S and D Functions Are Approximations

The supply and demand curves are usually treated as continuous functions. Of course, they are not, since our monetary units are not infinitely divisible. Also some goods can only be sold in discrete quantities. Consequently, the supply and demand functions must be *step functions* or *discrete functions*. (Figures 2 and 3.)

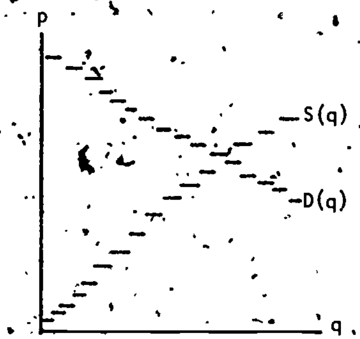

Figure 2. Step functions.

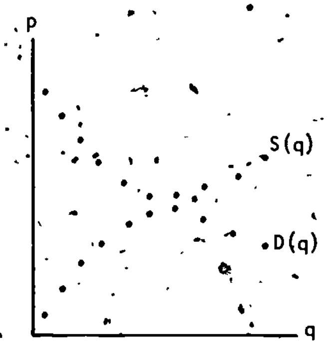

Figure 3. Discrete functions.

When we use a continuous function, we are implicitly assuming that the jumps in these functions are small enough so that *the continuous functions approximate the discrete ones very closely*. (Figure 4.) Only then may we apply the tools of calculus effectively.

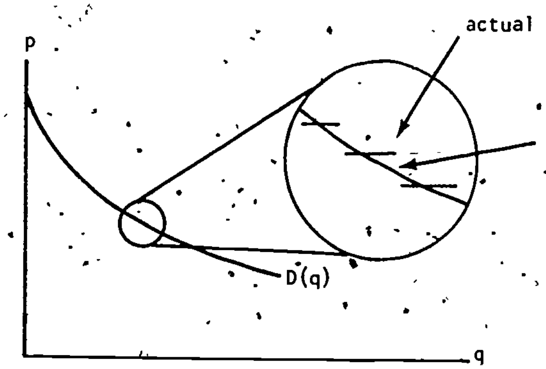

Figure 4. Approximation of demand function.

3. PRICE DISCRIMINATION AND CONSUMER SURPLUS

3.1 Perfect Price Discrimination

A firm selling in the market will naturally try to maximize the revenue which it obtains from selling a given quantity of a good. Ideally, this might be done by charging each consumer as much as that consumer is willing to pay for a good, rather than selling at a uniform price. If it were able to identify the prices which consumers would be willing to pay and discriminate in the prices offered exactly, it would achieve *perfect price discrimination*.

Such discrimination would enable the firm to charge for each quantity, Δq, which it brought to the market the price which consumers would be willing to pay for that quantity, $D(q)$. If we use the discrete form of the demand curve, we see that there is an initial quantity, Δq_0, for which the highest price will be paid, $p_0 = D(q_0)$. Then there is an additional quantity, Δq_1, for which a slightly lower price will be paid, $p_1 = D(q_1)$; a third quantity, Δq_2, for which a yet lower price, $p_2 = D(q_2)$ will be paid; and so on, until the price reaches the point at which the firm is no longer willing to produce (the supply curve). (Figure 5.)

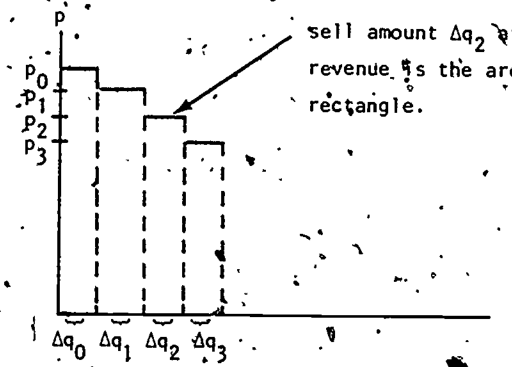

Figure 5. Price discrimination.

3.2 Value To The Consumer

The amount of revenue which the firm takes in from these sales is $p_i \Delta q_i = D(q_i)\Delta q_i$ for each of the incremental amounts Δq_i. Geometrically, these revenues can be represented by the areas of the rectangles -- bounded above by *the actual demand function*. Algebraically, the total revenue is represented by the sum of the areas of these rectangles:

$$\sum_{i=1}^{n} D(q_i)\Delta q_i ,$$

where n is the number of increments necessary to reach the point of intersection of the supply and demand curves.

When we approximate the demand function with a continuous function, this total area is *approximated* by the area beneath the continuous demand function, between the quantities $q = 0$ and $q = q^*$:

$$\int_0^{q^*} D(q)dq = \lim_{n\to\infty} \sum_{i=1}^{n} D(q_i)\Delta q_i .$$

Since in a competitive market firms cannot charge different prices to different consumers, the amount of revenue which could be obtained with perfect price discrimination simply represents the maximum that consumers would be willing to pay. Under the usual economic assumptions of rational behavior, this should represent the *total value* of the good to the consumer.

3.3 The Consumer Surplus

In a competitive market everyone will pay the same price for the good. That price will be the price p^* associated with the quantity q^* where demand price and supply price are equal, $D(q) = S(q) = p^*$. Thus the total paid by the consumer, p^*q^*, can be represented graphically as the area of a rectangle and the difference between this area and the area under the demand curve represents the surplus value to the consumers beyond what they paid -- the *consumer surplus*. (Figure 6.)

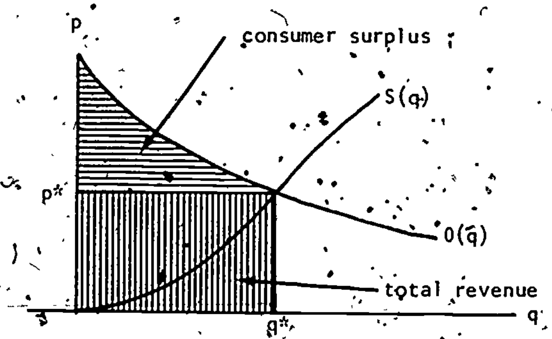

Figure 6. Consumer surplus.

If we are given a supply function, $S(q)$, and a demand function, $D(q)$, then the consumer surplus will be represented by the area between the two functions, $p = D(q)$ and $p = p^*$, on the interval $0 \leq q \leq q^*$. From the integral calculus we know that this area is the integral over the given interval of the difference of these two functions:

$$c.s. = \int_0^{q^*} (D(q) - p^*)\,dq.$$

4. EXAMPLES

4.1 Example 1

Suppose the supply curve is approximated by the function $p = S(q) = \frac{q^2}{500,000}$ and the demand curve by the function $p = D(q) = 144 - \frac{q}{500}$ for $0 \leq q \leq 72,000$, where p is in dollars per unit and q is in units. What are the equilibrium price and quantity and what is the consumer surplus?

We calculate the equilibrium quantity by setting the supply and demand functions equal and solving for q:

$$S(q) = D(q)$$

$$\frac{q^2}{500,000} = 144 - \frac{q}{500}$$

$$\frac{q^2}{500,000} + \frac{q}{500} - 144 = 0.$$

Let $x = \frac{q}{500}$, so that $x^2 = \frac{q^2}{250,000}$ and our equation becomes

$$\frac{x^2}{2} + x - 144 = 0$$

$$x^2 + 2x - 288 = 0$$

$$(x + 18)(x - 16) = 0$$

$$x = 16, -18.$$

Since q and hence x must be positive, 16 is the relevant solution:

$$x = \frac{q}{500} = 16$$

$$q = 8000.$$

Thus the equilibrium quantity is $q^* = 8000$. We can solve for the equilibrium price using either $S(q)$ or $D(q)$:

$$p^* = D(q^*) = 144 - \frac{8000}{500} = 128.$$

The consumer surplus is the area between the curves $p = D(q) = 144 - \frac{q}{500}$ and $p = p^* = 128$ on the interval $0 \leq q \leq q^* = 8000$:

$$\int_0^{8000} (144 - \frac{q}{500} - 128)\,dq$$

$$= \int_0^{8000} (16 - \frac{q}{500})\,dq$$

$$= 16q - \frac{q^2}{1000}\Big]_0^{8000}$$

$$= 128,000 - 64,000 = 64,000.$$

Thus \$64,000 is the surplus value which accrues to the consumer from the competitive market, with equilibrium $p^* = \$128$, $q^* = 8000$.

4.2 Example 2

Suppose that the demand curve is approximated by $p = D(q) = (q - 20)^2$ for $0 \leq q \leq 20$ and the supply curve is approximated by $p = S(q) = q^2$ for $0 \leq q \leq 20$, where p is measured in dollars per unit and q is measured in 100,000's of units. What are the equilibrium price and quantity and what is the consumer surplus?

We calculate the equilibrium quantity by setting supply and demand prices equal:

$$q^2 = (q - 20)^2$$

$$q^2 = q^2 - 40q + 400$$

$$40q = 400$$

$$q = 10.$$

The corresponding price is $S(10) = 10^2 = 100$.

The consumer surplus will be the area between the demand curve and the equilibrium price:

$$\int_0^{10} \left[(q - 20)^2\right] - 100 \, dq$$

$$= \int_0^{10} (q^2 - 40q + 400 - 100) \, dq$$

$$= \int_0^{10} (q^2 - 40q + 300) \, dq$$

$$= \frac{q^3}{3} - 20q^2 + 300q \Big]_0^{10} = \frac{4000}{3} = 1333 \ 1/3.$$

However, the prices used were measured in dollars per unit and the quantity in 100,000's of units. The easiest way to correct this is to multiply the prices by 100,000, so they can be considered as prices in dollars per 100,000's of units. The effect of this is simply to multiply the integrand by 100,000 and consequently, the correct answer for the consumer surplus is given by

$$\int_0^{10} 100{,}000 \, [(q - 20)^2 - 100] \, dq$$

$$= 100{,}000 \int_0^{10} [(q - 20)^2 - 100] \, dq$$

$$= 100{,}000 \ (1333 \ 1/3)$$

$$= \$133{,}333{,}333.33.$$

4.3 Example 3

Suppose that the demand function is given by $p = D(q) = (1 - q)^3 + 35$ for $0 \le q \le 4$ and the supply function by $p = S(q) = 3q^2$, where p is measured in dollars per unit and q is measured in 1000's of units. What are the equilibrium price and quantity and what is the consumer surplus?

Again, we find the equilibrium by setting supply and demand equal:

$$3q^2 = (1 - q)^3 + 35$$

$$3q^2 = 1 - 3q + 3q^2 - q^3 + 35$$

$$q^3 + 3q - 36 = 0$$

$$(q^2 + 3q + 12)(q - 3) = 0$$

$$q = -\frac{3 \pm \sqrt{9 - 48}}{2}, \ 3; \text{ so } q^* = 3$$

is the only real root. Then $p^* = S(3) = 3(3)^2 = \$27$.

The consumer surplus is the area between the demand function and the line $p = 27$ on the interval $0 \le q \le 3$:

$$\int_0^3 [(1 - q)^3 + 35 - 27] \, dq$$

$$= \int_0^3 ((1 - q)^3 + 8) \, dq$$

$$= \int_0^3 (9 - 3q + 3q^2 - q^3) \, dq$$

$$= 9q - \frac{3q^2}{2} + q^3 - \frac{q^4}{4} \Big]_0^3$$

$$= \frac{81}{4}.$$

Again, our units are incorrect since price is given in dollars per unit while q is in 1000's of units.

This is corrected by changing all price functions by a factor of 1000 to convert the units to dollars per 1000 units. This introduces a factor of 1000 into the integral, so the consumer surplus is

$$c.s. = \frac{81}{4}(1000) = \$20,250.$$

5. A THEORETICAL REMARK

One of the interesting aspects of this particular application of the calculus is that the actual functions we are concerned with are of a discrete nature -- they jump from value to value -- and the approximation is continuous. Thus when we apply the integral to calculate the consumer surplus, the actual answer is the summation of the areas of rectangles,

$$\sum_{k=1}^{n} D(q_k)\Delta q_k,$$

whereas the integral is the approximation. However, the definition of the integral as

$$\int_a^b D(q)\, dq = \lim_{\substack{n\to\infty \\ \Delta q_k \to 0}} \sum_{k=1}^{n} D(q_k^*)\, \Delta q_k$$

tells us that if the jumps Δq_k, are small enough and if the number of rectangles, n, is large, then the integral will be a good approximation to the sum. We are using this tool in a manner reverse to the more usual applications. This idea is characteristic of applications of calculus to economics.

6. EXERCISES

For Exercises 1 - 9 find the equilibrium price and quantity and the consumer surplus for the given supply and demand functions. Assume price is in dollars per unit and quantity is in units, unless otherwise stated.

(1) $S(q) = \frac{q}{5000}$, $D(q) = 4 - \frac{q}{15,000}$

(2) $S(q) = \frac{q}{27,000}$, $D(q) = 10 - \frac{q}{3000}$.

(3) $S(q) = 3q$, $D(q) = 6 - 3q$, where q is in 1000's of units.

(4) $S(q) = 14 + q$, $D(q) = (6 - q)^2$, for $0 \le q \le 6$ and q in 10,000's of units.

(5) $S(q) = \sqrt{q}$, $D(q) = 11 - \frac{q}{100}$, for $0 \le q \le 1100$.

(6) $S(q) = 3q^2$, $D(q) = (1 - q)^3 + 3$, for $0 \le q \le 2$ and q in 1000's of units.

(7) $S(q) = 6q^2$, $D(q) = (2 - q)^3 + 24$, for $0 \le q \le 3$ and q in 100's of units.

(8) $S(q) = \frac{3q^2}{800,000}$, $D(q) = 25 - \frac{q}{200}$.

(9) $S(q) = 3(q - 1)^3 + 3$, $D(q) = 12 - 9q^2$, where q is in 10,000's of units.

7. PRODUCER SURPLUS

7.1 Problem

Make an argument, similar to that of Section 3 above, that the area under the *supply* curve for $0 \le q \le q^*$ represents the value or cost to the producers of the goods sold on the market. Continue by showing that the area below the equilibrium price line, $p = p^*$, and above the supply curve, $p = S(q)$ on $0 \le q \le q^*$ is the surplus value which accrues to the producers in a competitive market, the *producer's surplus*. Sketch a graph illustrating this concept, similar to the graph in Figure 6. Give an integral formula for the producer surplus similar to the formula for consumer surplus at the end of Section 3.3.

7.2 Exercises 10 - 18

Calculate the producer's surplus, as defined above, for each of the Exercises 1 - 9 in Section 6.

8. THE EFFECT OF THE ELASTICITIES OF SUPPLY AND DEMAND

8.1 Linear Supply and Demand Curves

Supply and demand curves are often approximated by *linear* functions as in Exercises 1, 2, and 3 in Section 6. A general form for a supply and demand situation with linear functions would be this:

$$D(q) = A - Bq$$

$$S(q) = Cq$$

where A, B, and C are all positive constants.

8.2 Elasticities of Supply and Demand

The quantity $(1/B)(p/q)$ is the *elasticity of demand* for $D(q)$. The larger this number is, the smaller B is for a particular point (p,q) and consequently the demand curve is relatively flat. This means that small changes in the price are associated with large changes in the quantity demanded. The quantity demanded is sensitive—highly elastic—to changes in price. On the other hand, if the elasticity is small, so B is large, then a large change in price will be associated with a small change in demand. The quantity demanded in insensitive—inelastic—to changes in price.

Similarly the quantity $(1/C)(p/q)$ is the *elasticity of supply* for $S(q)$. If this elasticity is large, then a small change in price is associated with a large change in the quantity supplied, whereas if the elasticity is small, a large change in price is associated with a small change in supply.

8.3 Problem

Find the equilibrium quantity and price in terms of the constants A, B, C for the general linear supply and demand situation given above. Find the consumer's surplus and the producer's surplus in terms of these constants. Calculate the ratio of the consumer's surplus to the producer's surplus and comment on how this is associated with the elasticities of demand and supply.

9. TWO-TIER PRICE DISCRIMINATION AND MAXIMUM REVENUE

9.1 Two-tier Price Discrimination

Although perfect price discrimination as described in Section 3 is impossible, a monopolist seller or a group of cooperating sellers may be able to do some price discrimination. This can occur, for example, when a producer markets a product under a brand name and also, usually under a house name, in discount stores. This practice discriminates between shoppers who are induced by advertising to buy the "better" brand name and shoppers who look for the bargain goods. Of course, this is an oversimplified description of the situation. Nevertheless, price discrimination does occur in varying degrees in a number of situations due to imperfect information about the market and other factors.

The most simple form of price discrimination would be *two-tier price discrimination* where the seller or cooperating group of sellers charge two prices. Perfect two-tier price discrimination would occur when all those willing to pay the higher price, p_1, do. This occurs for the quantity, q_1, where the line $p = p_1$ intersects the demand curve $p = D(q)$. If we assume that the other price which the sellers charge is the competitive equilibrium price p^* derived earlier, then the total revenue to the seller is $p_1q_1 + p^*(q^* - q_1)$. (See Figure 7.)

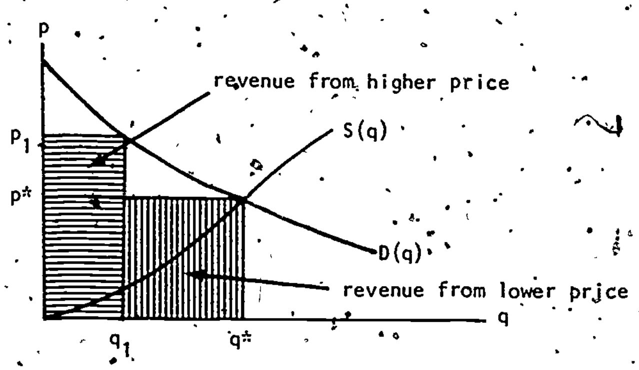

Figure 7. Two-tier price discrimination.

9.2 Problem: Maximizing of Revenue

Under two-tier price discrimination, how should the seller choose p_1 so as to maximize total revenue? What kind of assumption (stronger than the assumptions needed to derive the consumer surplus) about the supply and/or demand functions is needed in order to apply the calculus to this problem?

9.3 Exercises

For each of the following supply and demand functions, find the price p_1 and quantity q_1 which maximize total revenue under two-tier price discrimination.

(19) Supply and demand as in Exercise 1.
(20) S and D as in Exercise 2.
(21) S and D as in Exercise 3.
(22) S and D as in Exercise 5.
(23) S and D as in Exercise 6.
(24) S and D as in Exercise 7.
(25) S and D as in Exercise 8.
(26) S and D as in Exercise 9.

10. MODEL EXAM

1. (a) Sketch the graphs of a typical supply function, S(q), and a typical demand function, D(q), with price as a function of quantity.
   (b) On your graph, indicate the equilibrium quantity, q^*, and price, p^*.
   (c) On your graph, shade the region whose area is the consumer's surplus.

2. Let $S(q) = \frac{q}{500}$ and $D(q) = 6 - \frac{q}{250}$.
   (a) Find the equilibrium quantity and price, q^* and p^*.
   (b) Calculate the consumer's surplus.
   (c) Calculate the producer's surplus.

3. Let $S(q) = \sqrt{q}$ and $D(q) = 6 - q$, where q is in 1000's of units.
   (a) Find the equilibrium quantity and price, q^* and p^*.
   (b) Calculate the consumer's surplus.
   (c) Calculate the producer's surplus.

4. Let $S(q) = q$ and $D(q) = 12 - q^2$.
   (a) Find the equilibrium quantity and price, q^* and p^*.
   (b) Assume that in two-tier price discrimination, one price will be p^*. Find the higher price, p_1, which maximizes total revenues.

11. SPECIAL ASSISTANCE SUPPLEMENT

Discussion of problems:

Problem 7.1: If consumers could collectively discriminate against the producers, then they would purchase each additional parcel of the commodity, Δq_i, at just the price for which the producers would be willing to supply it, $S(q_i)$. The total revenue would be the sum of the amounts received for these parcels,

$$\Sigma\, S(q_i)\Delta q_i \approx \int_0^{q^*} S(q)\, dq.$$

This value of the goods sold to the producers is represented graphically by the area under the supply curve. However, the producers actually receive as revenue the amount corresponding to the rectangular area under the equilibrium price line. Consequently, the difference is the producer's surplus value. (See Figure 8.) This is given by the following integral formula.

$$\int_0^{q^*} (p^* - S(q))\, dq.$$

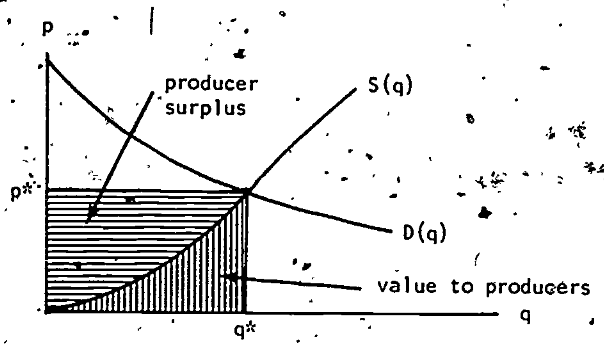

Figure 8. Producer surplus.

Problem 8.3: $S(q) = Cq$ and $D(q) = A - Bq$, so we solve $Cq = A - Bq$ to get $q^* = \frac{A}{B + C}$. From $S(q^*)$ we get $p^* = \frac{AC}{B + C}$.

The consumer's surplus is then given by

$$C.S. = \int_0^{\frac{A}{B+C}} \left(A - Bq - \frac{AC}{B + C}\right) dq = B\left(\frac{A^2}{2(B + C)^2}\right),$$

and the producer's surplus by

$$P.S. = \int_0^{\frac{A}{B+C}} \left(\frac{AC}{B + C} - Cq\right) dq = C\left(\frac{A^2}{2(B + C)^2}\right).$$

Comparing these we have the following formulation:

$$C.S./P.S. = B/C = \frac{\text{Elasticity of Supply}}{\text{Elasticity of Demand}}$$

Consequently, the larger the elasticity of supply is relative to the elasticity of demand, the greater the portion of the surplus value which accrues to the consumers is. This will happen if producers are more sensitive to price changes than the consumers are.

Problem 9.2: The producer should choose q_1 and p_1 to maximize the quantity $p_1q_1 + p^*(q^* - q_1) = q_1(p_1 - p^*) + p^*q^*$. But the discriminatory price, p_1, and quantity, q_1, are related by $p_1 = D(q_1)$, so the quantity to be maximized can be written as

$$f(q_1) = q_1(D(q_1) - p^*) + p^*q^*, \quad 0 \leq q_1 \leq q^*$$

This can be maximized by the usual methods of differential calculus -- find the critical points where $f'(q_1) = 0$ and check these and the endpoints, 0 and q^*, for the maximum value. Geometrically, it is clear that the maximum will always be interior, so we may omit checking the endpoints. In order to apply this technique, the demand function, $D(q)$, must be differentiable. It is usually reasonable to approximate the actual discrete demand function with a differentiable function, $D(q)$.

12. SOLUTIONS TO EXERCISES

1. $S(q) = \frac{q}{5000}$, $D(q) = 4 - \frac{q}{15000}$.

$\frac{q}{5000} = 4 - \frac{q}{15000} \Rightarrow q^* = 15000$. $p^* = S(q^*) = 3$.

$$C.S. = \int_0^{15000} (D(q) - p^*)dq = \int_0^{15000} \left(4 - \frac{q}{15000} - 3\right)dq = \$7500.$$

2. $S(q) = \frac{q}{27000}$, $D(q) = 10 - \frac{q}{3000}$.

$\frac{q}{27000} = 10 - \frac{q}{3000} \Rightarrow q^* = 27000$. $p^* = S(q^*) = 1$.

$$C.S. = \int_0^{27000} \left(10 - \frac{q}{3000} - 1\right)dq = \$121,500.$$

3. $S(q) = 3q$, $D(q) = 6 - 3q$, q in 1000's.

$3q = 6 - 3q \Rightarrow q^* = 1$. $p^* = S(q^*) = 3$.

$$C.S. = 1000 \int_0^1 (6 - 3q - 3)dq = \$1500.$$

4. $S(q) = 14 + q$, $D(q) = (6 - q)^2$, $0 \le q \le 6$, q ib 10,000's

$14 + q = (6 - q)^2 \Rightarrow q^2 - 13q + 22 = 0 \Rightarrow (q - 11)(q - 2) = 0$.

$\Rightarrow q^* = 2$. $p^* = S(q^*) = 16$.

$$C.S. = 10000 \int_0^2 ((6 - q)^2 - 16)dq = \$186,667.$$

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5. $S(q) = \sqrt{q}$, $D(q) = 11 - \frac{q}{100}$, $0 \le q \le 1100$.

$\sqrt{q} = 11 - \frac{q}{100} \Rightarrow q = 121 - \frac{22q}{100} + \frac{q^2}{100^2}$.

Let $x = \frac{q}{100}$ so $x^2 = \frac{q^2}{100^2}$ and this becomes

$x^2 - 122x + 121 = 0 \Rightarrow (x - 121)(x - 1) = 0 \Rightarrow x = 1, 121$

$q^* = 100, 12100$, so $q^* = 100$, $p^* = S(q^*) = 10$.

$$C.S. = \int_0^{100} \left(11 - \frac{q}{100} - 10\right)dq = \$50.$$

6. $S(q) = 3q^2$, $D(q) = (1 - q)^3 + 3$, $0 \le q \le 2$, q in 1000's

$3q^2 = (1 - q)^3 \Rightarrow q^3 + 3q - 4 = 0 \Rightarrow$

$(q - 1)(q^2 + q + 4) = 0 \Rightarrow q = 1, \frac{-1 \pm \sqrt{1 - 16}}{2}$.

The latter roots are imaginary, so

$q^* = 1$. $p^* = S(q^*) = 3$.

$$C.S. = 1000 \int_0^1 ((1 - q)^3 + 3 - 3)dq = \$250.$$

7. $S(q) = 6q^2$, $D(q) = (2 - q)^3 + 24$, $0 \le q \le 3$, q in 100's.

$6q^2 = (2 - q)^3 + 24 \Rightarrow q^3 + 12q - 32 = 0 \Rightarrow$

$(q - 2)(q^2 + 2q + 16) = 0 \Rightarrow q = 2, \frac{-2 \pm \sqrt{4 - 64}}{4}$.

$q^* = 2$ and $p^* = S(q^*) = 24$.

$$C.S. = 100 \int_0^2 ((2 - q)^3 + 24 - 24)dq = \$400.$$

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8. $S(q) = \frac{3q^2}{800,000}$, $D(q) = 25 - \frac{q}{200}$.

$\frac{3q^2}{800,000} = 25 - \frac{q}{200}$, letting $x = \frac{q}{200}$, $\Rightarrow$

$3x^2 + 20x - 500 = 0 \Rightarrow (3x + 50)(x - 10) = 0 \Rightarrow$

$x = 10, -50/3 \Rightarrow q = 2000, -\frac{10000}{3}$, so

$q^* = 2000$; $p^* = D(q^*) = 15$.

$$\text{C.S.} = \int_0^{2000} \left(25 - \frac{q}{200} - 15\right) dq = \$10,000.$$

9. $S(q) = 3(q - 1)^3 + 3$, $D(q) = 12 - 9q^2$, q in 10,000's

$3(q - 1)^3 + 3 = 12 - 9q^2 \Rightarrow 3q^3 + 9q - 12 = 0 \Rightarrow$

$3(q - 1)(q^2 + q + 4) = 0 \Rightarrow q = 1, \frac{-1 \pm \sqrt{1 - 16}}{2}$, so

$q^* = 1$; $p^* = D(q^*) = 3$.

$$\text{C.S.} = 10000 \int_0^1 (12 - 9q^2 - 3)dq = \$60,000.$$

10. $S(q) = \frac{q}{5000}$, $D(q) = 4 - \frac{q}{15,000}$

and from #1, $q^* = 15000$, $p^* = 3$.

$$\text{P.S.} = \int_0^{q^*} (p^* - S(q))dq = \int_0^{15000} \left(3 - \frac{q}{5000}\right)dq = \$22,500.$$

11. $S(q) = \frac{q}{27000}$, $D(q) = 10 - \frac{q}{3000}$

and from #2, $q^* = 27000$, $p^* = 1$.

$$\text{P.S.} = \int_0^{27000} \left(1 - \frac{q}{27000}\right)dq = \$13500.$$

12. $S(q) = 3q$, $D(q) = 6 - 3q$, q in 1000's

and from #3, $q^* = 1$, $p^* = 3$.

$$\text{P.S.} = \int_0^1 (3 - 3q)dq = \$1500.$$

13. $S(q) = 14 + q$, $D(q) = (6 - q)^2$, q in 10000's

and from #4, $q^* = 2$, $p^* = 16$.

$$\text{P.S.} = 10000 \int_0^2 (16 - (14 + q))dq = \$20,000$$

14. $S(q) = \sqrt{q}$, $D(q) = 11 - \frac{q}{100}$

and from #5, $q^* = 100$, $p^* = 10$.

$$\text{P.S.} = \int_0^{100} (10 - \sqrt{q})dq = \$333.33$$

15. $S(q) = 3q^2$, $D(q) = (1 - q)^3 + 3$, q in 1000's

and from #6, $q^* = 1$, $p^* = 3$.

$$\text{P.S.} = 1000 \int_0^1 (3 - 3q^2)dq = \$2000$$

16. $S(q) = 6q^2$, $D(q) = (2 - q)^3 + 24$, q in 100's

and from #7, $q^* = 2$, $p^* = 24$.

$$\text{P.S.} = 100 \int_0^2 (24 - 6q^2)dq = \$3200.$$

17. $S(q) = \frac{3q^2}{800,000}$, $D(q) = 25 - \frac{q}{200}$,

and from #8, $q^* = 2000$, $p^* = 15$.

$$\text{P.S.} = \int_0^{2000} \left(15 - \frac{3q^2}{800,000}\right)dq = \$20,000$$

18. $S(q) = 3(q-1)^3 + 3$, $D(q) = 12 - 9q^2$, q in 10,000's
and from #9, $q^* = 1$, $p^* = 3$.

$$P.S. = 10000 \int_0^1 \left[3 - \left(3(q-1)^3 + 3\right)\right] dq = \$7500.$$

19. $S(q) = \frac{q}{5000}$, $D(q) = 4 - \frac{q}{15000}$,
and from #1, $q^* = 15000$, $p^* = 3$. Maximize
$f(q_1) = q_1\left(\left(4 - \frac{q_1}{15000}\right) - 3\right) + 45000$ on $0 \le q_1 \le 15000$
$f'(q_1) = 1 - \frac{2q_1}{15000} = 0$ if $q_1 = 7500$.
Thus, $q_1 = 7500$ and $p_1 = D(q_1) = \$3.50$.

20. $S(q) = \frac{q}{27000}$, $D(q) = 10 - \frac{q}{3000}$
and from #2, $q^* = 27000$, $p^* = 1$. Maximize
$f(q_1) = q_1\left(10 - \frac{q_1}{3000} - 1\right) + 27000$.
$f'(q_1) = 9 - \frac{q_1}{1500} = 0$ if $q_1 = 13500$.
Thus, $q_1 = 13500$ and $p_1 = D(q_1) = \$5.50$.

21. $S(q) = 3q$, $D(q) = 6 - 3q$, q in 1000's
and from #3, $q^* = 1$, $p^* = 3$. Maximize
$f(q_1) = q_1(6 - 3q_1 - 3) + 3$.
$f'(q_1) = 3 - 6q_1 = 0$ if $q_1 = \frac{1}{2}$.
Thus, $q_1 = \frac{1}{2}$ and $p_1 = D(q_1) = \$4.50$.

22. $S(q) = q$, $D(q) = 11 - \frac{q}{100}$
and from #5, $q^* = 100$, $p^* = 10$. Maximize
$f(q_1) = q_1\left(11 - \frac{q_1}{100} - 10\right) + 1000$.
$f'(q_1) = 1 - \frac{q_1}{50} = 0$ if $q_1 = 50$.
Thus, $q_1 = 50$ and $p_1 = D(q_1) = \$10.50$.

23. $S(q) = 3q^2$, $D(q) = (1-q)^3 + 3$.
and from #6, $q^* = 1$, $p^* = 3$. Maximize
$f(q_1) = q_1\left((1 - q_1)^3 + 3 - 3\right) + 3$.
$f'(q_1) = (1-q)^3 - 3q_1(1-q_1)^2$
$= (1-q_1)^2(1-4q_1) = 0$ if $q_1 = \frac{1}{4}, 1$.
Thus, $q_1 = \frac{1}{4}$ and $p_1 = D(q_1) = \$3.42$.

24. $S(q) = 6q^2$, $D(q) = (2-q)^3 + 24$
and from #7, $q^* = 2$, $p^* = 24$. Maximize
$f(q_1) = q_1\left((2-q_1)^3 + 24 - 24\right) + 48$.
$f'(q_1) = (2-q_1)^3 - 3q_1(2-q_1)^2$
$= (2-q_1)^2(2-4q_1) = 0$ if $q_1 = \frac{1}{2}, 2$.
Thus, $q_1 = \frac{1}{2}$ and $p_1 = D(q_1) = \$27.38$.

25. $S(q) = \frac{3q^2}{800,000}$, $D(q) = 25 - \frac{q}{200}$
and from #8, $q^* = 2000$, $p^* = 15$. Maximize
$f(q_1) = q_1\left(25 - \frac{q_1}{200} - 15\right) + 30000$.
$f'(q_1) = 10 - \frac{q_1}{100} = 0$ if $q_1 = 1000$.
Thus, $q_1 = 1000$ and $p_1 = D(q_1) = \$20$.

26. $S(q) = 3(q-1)^3 + 3$, $D(q) = 12 - 9q^2$
and from #9, $q^* = 1$, $p^* = 3$. Maximize
$f(q_1) = q_1(12 - 9q_1^2 - 3) + 3$.
$f'(q_1) = 9 - 27q_1^2 = 0$ if $q_1 = \sqrt{1/3}$.
Thus, $q_1 = \sqrt{1/3}$ and $p_1 = D(q_1) = \$9$.

13. SOLUTIONS FOR THE MODEL EXAM

1. See Figure 6.

2. $S(q) = \frac{q}{500}$, $D(q) = 6 - \frac{q}{250}$.

(a) $\frac{q}{500} = 6 - \frac{q}{250} \Rightarrow q^* = 1000,\ p^* = S(q^*) = 2.$

(b) $$\text{C.S.} = \int_0^{1000} \left(6 - \frac{q}{250} - 2\right)dq = \$2000$$

(c) $$\text{P.S.} = \int_0^{1000} \left(2 - \frac{q}{500}\right)dq = \$1000.$$

3. $S(q) = \sqrt{q}$, $D(q) = 6 - q$, q in 1000's.

(a) $\sqrt{q} = 6 - q \Rightarrow q = 36 - 12q + q^2 \Rightarrow q^2 - 13q + 36 = 0$

$\Rightarrow (q - 4)(q - 9) = 0 \Rightarrow q = 4$ or 9,

so $q^* = 4$, $p^* = S(q^*) = 2$.

(b) $$\text{C.S.} = 1000 \int_0^{4} (6 - q - 2)dq = \$8000.$$

(c) $$\text{P.S.} = 1000 \int_0^{4} (2 - \sqrt{q})dq = \$2666.67.$$

4. $S(q) = q$, $D(q) = 12 - q^2$.

(a) $q = 12 - q^2 \Rightarrow q^2 + q - 12 = 0 \Rightarrow (q + 4)(q - 3) = 0$

$\Rightarrow q = 3, -4$, so $q^* = 3$ and $p^* = S(q^*) = 3$.

(b) Maximize $f(q_1) = q_1(12 - q_1^2 - 3) + 9$

$f'(q_1) = 9 - 3q_1^2 = 0$ if $q_1 = \pm\sqrt{3}$ so

$q_1 = \sqrt{3}$ and $p_1 = D(q_1) = \$9$.

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