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AUTHOR Moser, James M.; Carpenter, Thomas P.
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ABSTRACT

The focus is on the problem-solving behaviors of primary age children on one-step verbal or story problems involving addition and subtraction. When children are given a simple word problem for which they have not learned the necessary algorithms, they are often able to derive a solution on their own. This report focuses on the child-invented processes pupils use on word problems that would normally be solved through use of an addition or subtraction algorithm. Much of the data comes from a three-year longitudinal study of about 100 first-grade pupils from one public and one parochial school in the Madison, Wisconsin area serving middle to upper-middle class neighborhoods. It was felt that invention was a rearrangement of elements into similar structures. The data suggested that instruction has a bearing upon changes in invented behavior. Thus, any characterization of children's formal or invented mathematics concepts and procedures needs to consider the role of instruction. (MP)

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James M. Moser and Thomas P. Carpenter

Wisconsin Center for Education Research
University of Wisconsin-Madison

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Invented Processes in Solution to Arithmetical Problems

James M. Moser and Thomas P. Carpenter

University of Wisconsin-Madison

This paper focuses upon problem-solving behaviors of primary age children on one-step verbal or "story" problems involving the operations of addition and subtraction. In previous writings (Carpenter & Moser, 1982) we have presented a detailed semantic analysis of various verbal problem types. Some might argue that routine verbal problems are not "problems" in the true sense of the word, but are merely exercises that can be answered in a simple fashion by children. There is a degree of truth in this assertion and, for many children, these verbal problems are not problems at all. They know how to analyze and solve them quickly and accurately. However, many primary age children do not have the formal arithmetic skills and procedures to algorithmically solve these problems. For these children, verbal problems do indeed constitute a well defined class of problems for which clearly identifiable problem-solving behaviors have been observed. Greeno (1980) speaks to this point when he writes "...significant processes such as understanding, planning and organizing activity by setting subgoals are very much present in a great many activities that students learn to accomplish routinely. These routine activities, therefore, ought to be counted for what they are, namely, as perfectly legitimate acts of problem solving." (p. 13)

When children are presented a simple word problem for which they have not learned the necessary algorithms, they often are able to derive a solution on their own. This is basically the process of invention described by Resnick (1978). She talks of invention as a process whereby

persons acquire new mathematical knowledge by constructing for themselves new organizations of concepts and new procedures for performing mathematical operations. In essence, it comes to youngsters figuring things out for themselves as opposed to simple application of formally taught facts and skills. That children exhibit inventive behavior when no instruction has taken place has been documented by Groen and Resnick (1977) and by ourselves (Carpenter, 1980; Moser, 1980). In this paper, we examine the procedures that children invent to solve word problems that normally would be solved using an addition or subtraction algorithm. We include in this analysis all behavior that has not been formally taught as part of the curriculum. Some of the "invented behavior" may well have resulted from their learning from others, either in school or at home. Further, inventive behavior will be attributed to children who may use learned behavior, but in a situation different in context from the one in which the behavior was learned. In other words, we do not wish to exclude consideration of learned behavior and the possible effects of instruction.

Although this work is not to be considered as a report of an empirical study, we do need to provide certain background information about the three year longitudinal study that provides the data for our conclusions. This includes a reasonably detailed description of the instruction received by the subjects since we feel this instruction may well have influenced the inventive behavior of the children.

The major section of the paper examines the inventive behavior of primary age children. The discussion is limited to three major categories of behavior observed as the children attempted to solve verbal addition and subtraction problems involving two-digit numbers. The first category of behavior includes solutions that involve knowledge of place value, but are

different from the standard algorithms. The second category involves use of algorithmic behavior, both with and without use of paper-and-pencil. Included here are correct algorithms used by children before they receive any formal instruction in addition or subtraction algorithms and "buggy" algorithms (Brown & Van Lehn, 1982; Resnick, 1982) for subtraction after the addition algorithm has been taught. The third considers processes used on a problem that can easily be solved by using the subtraction algorithm but whose semantic wording is so strongly oriented towards addition and additive strategies such as forward counting that children tended to resist use of the subtraction algorithm.

Background

The Longitudinal Study

In September 1978, the Mathematics Work Group of the Wisconsin Center for Education Research began a three-year longitudinal study of about 100 first-grade children. The sample was taken from two elementary schools in the Madison, Wisconsin area, one public and one parochial; both serve middle to upper-middle class neighborhoods. Data collected included classroom observations interested mainly in allocated and engaged time and certain teacher behaviors, paper-and-pencil achievement monitoring tasks aimed at assessment of mastery of selected arithmetic objectives, and pupil performance on a set of verbally administered "story" problems in addition and subtraction. The last set of data, which are the basis for this paper, were collected by means of individually administered interviews given in September, January, and May of each of the first three school years, except for May 1981 during third grade. Our interest in this paper is centered upon only the four interviews (January 1980, May 1980, September 1980, and

Table 3

Representative Problem Types

Task 1.	Joining (Addition)	Shelley had 12 coins. Her brother gave her 15 more coins. How many coins did Shelly have altogether?
Task 2.	Separating (Subtraction)	Jim had 35 pears. He gave 21 to Mary. How many pears did Jim have left?
Task 3.	Part-Part-Whole (Subtraction)	There are 28 beads on the neck] e. 16 are red and the rest are white. How many white beads are on the necklace?
Task 4.	Part-Part-Whole (Addition)	Allen has 11 apples. He also has 18 oranges. How many pieces of fruit does Allen have altogether?
Task 5.	Comparison (Subtraction)	Tom has 16 crayons. His friend Sally has 29 crayons. How many more crayons does Sally have than Tom?
Task 6.	Joining Missing Addend (Subtraction)	Jane has 23 markers. How many more markers does she have to put with them to have 37 markers altogether?

January 1981) and only upon those conditions in which the problems contained two-digit numbers.

Six selected verbal problems were administered at each interview. Each of the six was presented in the order as listed in Table 1 under a variety of conditions determined by number size and the availability of problem-solving aids such as manipulative materials or paper-and-pencil. For each, a set of about 60 cubes, divided equally between two colors, and paper and pencil was provided. Under each condition at a particular administration, the wording of the problems was altered to prevent immediate recognition of problem type by the subjects while still attempting to maintain the semantic characteristics of the problems.

Two sets of six problems each were read to the children. The first set involved number pairs for which no regrouping was required for a computational solution while the second used numbers for which regrouping was required. The actual number triples used are shown in Table 2. For the six problems, these six number triples were assigned under a Latin-square design resulting in six sets of six problems each.

Table 2

Two-digit number triples used in Wisconsin longitudinal study

No Regrouping	Regrouping
12, 15, 17	12, 19, 31
12, 16, 28	13, 18, 31
11, 18, 29	14, 18, 32
13, 16, 29	16, 17, 33
14, 21, 35	15, 19, 34
14, 23, 37	17, 19, 36

Children's responses to the individual problems were recorded by the interviewer and were coded into four categories: type of model; correctness; process employed; and, if appropriate, type of error.

Instruction in Addition and Subtraction

Both schools in the study used the Developing Mathematical Processes (DMP) program (Romberg, Harvey, Moser, & Montgomery, 1974) as the basis for mathematics instruction. DMP has a strong emphasis upon problem solving and as a result the children had been exposed to the various problem types used in the interviews. For the longitudinal study, special units were developed for instruction in the computational algorithms for addition and subtraction. These units are briefly characterized here. Of particular interest is the timing of instruction in relation to the administration of the four problem-solving interviews. The first unit was taught in second grade following the January 1980 interview. The unit begins with children counting forward and backward from any number by twos, threes, fives, and tens with care given when the counting bridges from one decade to another. Number patterns and rounding are also taught. Basic problem types including Join, Separate, Combine, and Compare are used. Suggested solution methods include counting by tens and ones as well as use of manipulative materials, especially small counting sticks already grouped together in tens and bound by a rubber band. At first, horizontal number sentences are written to represent problem situations, but the transition is quickly made to the vertical form of writing with the first efforts assisted by the use of a grid labeled with "tens" and "ones". All numbers utilized are two-digit numbers, and no problems require regrouping. Both addition and subtraction receive equal emphasis.

The second unit was taught several weeks following the first unit and preceding the administration of the second interview in May 1980. In this unit, the children are asked to solve addition problems involving regrouping. The formal algorithmic process is motivated by the use of the bundled counting sticks used in the preceding unit and also in an earlier unit devoted to place value. At first, children solved by counting the sticks one-by-one, then by using sticks grouped in bundles, regrouping as needed. Symbolic recording of the work is done. This learning activity is greatly similar to the mapping technique as described by Resnick (1982). Rather quickly, the children are urged to move to the formal symbolic algorithm, with the transition once again eased by the place value grid described above. The children are reminded to check reasonableness of answers by estimation and by rounding addends to the nearest ten and then mentally adding the rounded numbers. Some subtraction is reviewed, but only with numbers requiring no regrouping.

The third unit was taught in September 1980 when the children were third graders. The third problem-solving interview was given prior to the beginning of this unit. This third unit is similar to the second unit described above, except that the emphasis is upon subtraction and the algorithm for regrouping. One feature of this unit is the checking of addition and subtraction problems by using the inverse operation. In earlier arithmetic units involving "basic fact" addition and subtraction, analysis of verbal problems to decide which operation to use was centered around explicit discussion of the part-part-whole relationship. Appeal to this same part-part-whole analysis is made to make the checking by the inverse operation reasonable.

The final unit was taught during late Fall 1980, preceding the administration of the last problem-solving interview in January 1981. This fourth unit is essentially a review of instruction on the computational algorithms. The expectation was that by completion of this unit, children would be able to correctly apply the two-digit algorithms to a variety of verbal addition and subtraction problems. Not all types of verbal problems were covered, some of the more difficult comparison problems having been delayed.

Solving problems in all areas of mathematics--geometry, measurement, and arithmetic--is the emphasis of the curriculum used by the subjects of the longitudinal study. In particular, verbal problems in addition and subtraction are used early in the program and serve as a motivating rationale for learning standard computational procedures such as memorization of basic facts and algorithms. This technique would appear to be strikingly different from other more traditional instructional programs in which abstract symbolic arithmetic is taught prior to application to problems, generally in a rather context-free setting. Thus, when confronted with a problem situation for which a standard procedure was not available, the children in this study were encouraged to seek a solution using whatever means they had, rather than give up. Consequently, the effect of instruction cannot be ignored as we consider the invented processes that children use in their attempt to solve arithmetic problems.

Inventive Behavior of Primary Age Children

In earlier writings (Carpenter, 1980; Moser, 1980) we presented evidence of inventive behavior by children solving verbal addition and subtraction problems containing one-digit addends. In this section, we examine invented strategies for problems involving two-digit addends. In some cases,

the invented strategies reported here are variations of those observed with the smaller number problems while in other instances, they are unique to two-digit problems.

The extension of counting strategies to two-digit problems provide one example of invention. Children used a variety of counting strategies to solve one-digit problems. Fuson (1982) documents some of the keeping-track mechanisms used by younger children, many of which we also observed during administration of the one-digit problems. With such smaller numbers, keeping track can usually be effected mentally or by using some subset of one's ten fingers. However, with larger two-digit numbers, keeping track can involve using the fingers on at least one hand more than once or some complicated mental operations. These keeping-track mechanisms provide one example of a kind of invention, since they were not formally taught.

One second-grade student used cubes as a tracking device. The problem was a Separate problem (Problem 2 from Table 1) with the numbers 23 and 37 (37 - 23 = □). This child elected to Count Back, beginning with "37, 36, . . ." After several numbers, he seemed to realize that he would have a difficult time keeping track of all 23 words in the sequence. After several moments of silence, he constructed a set of 23 cubes. He then recommenced his downward count, removing one cube from the set for each word spoken. When the set was exhausted, he triumphantly looked up and pronounced the correct answer.

Non-Standard Solutions

The Counting strategies that children use for one-digit problems are tedious for two-digit problems and provide a great deal of opportunity for

counting errors. Since most children had no formal instruction in addition or subtraction algorithms at the time of the first interview, they were forced to either rely on the time consuming counting stategies or invent alternative procedures. Many children used their knowledge of place value together with their understanding of counting strategies or knowledge of one-digit number facts to arrive at unique solutions. Here are some examples of these strategies.

i) For interview task #6 (Join/Subtraction) involving the given numbers 15 and 28: "15, 25, 26, 27, 28. The answer is 13." Counting up from the given addend 15 to the given sum 28 was facilitated by one quick count of 10 (15 to 25).

ii) For interview task #3 (Combination, missing part) involving the numbers 31 and 19: "I think of 31 as 30 and 19 as 20, so the answer is 11." Even though an error was made, this strategy involves knowledge of place value and basic facts (3 - 2).

iii) For interview task #3 involving the numbers 32 and 18: "32 take away 10 is 22. Twenty-two take away 8 is 24!" This example involves breaking 18 into 10 and 8 together with an incorrect application of the subtraction algorithm.

The incidence of observed usage of this type of alternative procedures is listed in Table 3.

Table 3

Percentage of Place-Value Procedures

Interview	Range*
1	6 - 13%
2	1 - 8%
3	5 - 13%
4	3 - 8%

*Ranges of values over the two-digit problems in each interview

When counting was involved as part of these strategies, both upward and downward counting was used on each subtraction problem. The choice of upward or downward counting was not based on problem structure as it was in the smaller number problem interviews. While the frequency of use of these strategies is not overwhelming, it is by no means trivial either. A substantial number of children did use this type of strategy as a construction of a method to solve problems that took into account aspects of place value and other numerical relationships.

Continuing a pattern of behavior that we observed in the earlier phases of the longitudinal study, those children who did use this general type of strategy were selective in that use. That is to say, they did not use this strategy universally over all problems types and all problem conditions. For example, some children were able to use a standard algorithm successfully with subtraction problems for which regrouping was not required but then switched to alternative strategy when confronted with problems containing numbers requiring regrouping. But even then, alternative strategy was not used for all subtraction problems within a single interview. There is an insufficient number of incidences of this behavior to enable us to isolate the factors related to this selective behavior.

Use of Algorithms

As might be expected, the correct use of algorithms increased over time with major advances paralleling the timing of instruction in algorithmic behavior (Moser, 1981). Yet, the incidence of algorithmic use prior to instruction and the emergence of "buggy" algorithms during instruction suggests another occurence of inventive behavior. In this paper, use of algorithms is taken to mean behavior, either written (and easily observable)

or mental (as reported by subjects upon questioning by the interviewer), that takes into account the place value of each digit in a two-digit number and then determines answers for the one's place and the ten's place separately. In identifying algorithmic behavior, we did not differentiate whether operations were done in reverse order (tens before ones) or the method by which the basic fact for each place was determined. If there was a basic fact error made, that was noted but not classified as an incorrect or "buggy" algorithm. Table 4 presents selected data on the use of algorithms on three problem tasks that require regrouping. The actual wording of these tasks is given in Table 1.

Table 4

Use of Algorithms for
Two-Digit Problems Requiring Regrouping

Problem Type	Interview	Problem Correct	Algorithm Used	Algorithm Used Correctly	"Buggy" Algorithm
Join Addition	1	62	25	21	1
	2	69	69	53	7
	3	75	60	45	5
	4	92	92	86	3
Separate	1	45	14	3	8
	2	23	58	2	49
	3	33	40	3	34
	4	75	88	69	18
Join Missing Addend	1	43	10	2	7
	2	39	35	3	21
	3	47	26	3	18
	4	70	54	40	6

Note the percentage of algorithmic use in Interview 1 which took place prior to any formal school instruction in the use of algorithms. Slightly higher percentages were observed for the non-regrouping problems in terms of use while much higher percentages in terms of correctness were seen, particularly for subtraction. It is impossible to tell how much of this behavior was truly invented by the children and how much was due to outside-of-school influence such as parental direction.

There are striking differences between the use of algorithms for addition and subtraction. Before any instruction, about 20 percent of the children were able to use algorithmic behavior to solve two-digit addition problems that required regrouping. However, even after instruction in regrouping for addition, only 3 percent could construct an appropriate subtraction algorithm. In fact, instruction on addition tended to encourage the use of a buggy algorithm that ignored regrouping.

The data from the last three interviews reveals further interesting results. Between Interview 1 and Interview 2, there was direct instruction on the regrouping addition algorithm. Use of the addition algorithm jumps sharply followed by a slight decline over the summer. The large jump from the third to fourth interview for addition appears to represent a consolidation of learning some of which is due to instruction in subtraction and additional review in addition.

An interesting result is the large increase in the "buggy" SmallerFromLarger (Brown & Van Lehn, 1982) subtraction algorithm at Interview 2. Apparently the instruction on the addition algorithm has caused many children to extrapolate its procedures to subtraction. According to Brown and Van Lehn's Repair Theory, generation of a buggy algorithm can only occur

if there is some sort of algorithm already in existence but that some perturbation in its correct use takes place, the faulty repair of which results in the bug. Most children make a very superficial extension of the addition algorithm to subtraction, ignoring the regrouping involved in the process. The use of buggy algorithms for subtraction was much higher than was the case for addition. In fact, before children had instruction in addition, very few invented buggy addition algorithms.

Why more children don't invent the correct subtraction regrouping algorithm rather than the incorrect buggy one is not clear. It is clear, though, that once the correct algorithm is taught directly (between Interview 3 and Interview 4), most children are successful in its use. A direct corollary to the increased use of the buggy subtraction algorithm at Interview 2 is the pronounced drop in percentage of correct responses for the problems to which the buggy algorithm has replaced counting solution procedures.

Solutions to Join Problems

Children's performance on the Join, Missing Addend problem deserves special mention. Although this problem is solvable by subtraction of the two given numbers in the problem, the semantic structure tended to keep many children from using the subtraction algorithm. On the administration of the smaller number versions of this problem in the other phase of the longitudinal study, almost all children used an additive strategy. This carried over into the two-digit problems. Use of the subtraction algorithm, either the buggy one prior to formal instruction or the correct one subsequent to instruction, was much higher on the other three subtraction

problems (Table 4). The reason for this may be seen in the instruction associated with subtraction. The rationale for the standard algorithm for subtraction is based strictly upon a subtractive or "take-away" notion. For the other three subtraction problems used in the interviews, this take-away interpretation is not inconsistent. However, for a large percentage of the subjects in our study, "take-away" appeared to be contradictory or inappropriate for a problem part of whose wording included "How many more must be added on to...?" Most children realized that the addition algorithm was not appropriate, but they could not relate the additive nature of the action in the problem to the subtraction algorithm. The solution involved a choice strategy, and in this instance, a choice not to use a learned or invented algorithm. Rather, the decision was made to revert to an addition on or counting up strategy, which reflected the problem structure. The data in Table 4 indicate that about one-third of the almost 100 subjects made this choice.

Discussion

A consideration of children's number concepts cannot be divorced from a consideration of the effect of instruction. The alternative counting strategies appear to be on the lower end of the scale along the dimension of being influenced by instruction. They are related to the earlier invented strategies involving use of counting sequences, either forward or backward. Yet, the amplification and extension of the counting sequence strategies to the alternative strategies often involved place value properties which are learned outcomes of instruction. Knowledge of place value also would appear to be a pre-requisite for invention of any algorithms as we have defined them. But this brings us back to the very first discussions

in this paper about the nature of inventive behavior. To think of invention as building out of nothing is too narrow. Invention is a rearrangement of elements into similar structures. For example, for most students, the subtraction "buggy" algorithm is invented only after instruction on the addition algorithm. In most cases, the elements learned in instruction provide some limit to the range of invention.

The data we have presented also suggest that instruction has a bearing upon changes in invented behavior. There was substantial change between interviews, reflecting instruction that took place during that period of time, yet there was almost no change over the summer when no instruction had taken place. What change did take place over the summer was a slight reversion back to earlier behavior.

The effect of instruction can also be observed by comparing our results to those from a study by Collis and Romberg (1981). That study used almost the same procedures as the ones we have used, but found some strikingly different results. Third grade students in Tasmania, Australia were individually interviewed on the same verbal problems with the same number size as reported in this paper. Yet, only one-third used algorithmic procedures, despite the fact that they had received formal instruction on how to compute. Unlike DMP, which was used in the study discussed in this paper, their instruction on computation was taught only in a symbolic context, free of application to "real" problems. This seems to indicate that instruction has a major effect upon the range of application of learned strategies, and any characterization of children's formal or invented mathematics concepts and procedures needs to consider the role of instruction.

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