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ABSTRACT

The use of counting for subtraction was investigated. Counting for subtraction is related to counting-on for addition and to four skills: the ability to use the subtrahend cardinality to gain entry into the count sequence, the ability to use the minuend cardinality to gain entry into the count sequence, the ability to use the count sequence to represent the difference, and the verbal ability to count backwards. The use of the minuend cardinality to enter the count sequence seems to develop from a coordinated use of the other two nonverbal skills. No relation was found between part/whole class inclusion and the use of counting for subtraction. (Author/MP)

THE USE OF COUNTING FOR SUBTRACTION

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Abstract

The use of counting for subtraction was investigated. Counting for subtraction is related to counting-on for addition and to four skills: the ability to use the subtrahend cardinality to gain entry into the count sequence, the ability to use the minuend cardinality to gain entry into the count sequence, the ability to use the count sequence to represent the difference, and the verbal ability to count backwards. The use of the minuend cardinality to enter the count sequence seems to develop from a coordinated use of the other two nonverbal skills. No relation was found between part/whole class inclusion and the use of counting for subtraction.

Introduction

Research concerning how children solve subtraction problems prior to their use of formal algorithms has found a wide variety of processes: guessing (or estimating) the difference; attempting to count the difference directly; modeling the action of a word problem with concrete objects and then counting the results of the modeling activities; using counting methods which do not rely on the manipulation of concrete objects; heuristic reasoning about number; and memorized number facts (Briars & Larkin, Note 1; Brownell, 1928; Carpenter, Hiebert & Moser, 1981; Carpenter & Moser, 1982, in press; Ilg & Ames, 1951; O'Brien & Casey, Note 2; Secada, Note 3; Steffe, Spikes & Hirstein, Note 4; Steffe, von Glaserfeld & Richards, Note 5). The concern of this paper is the development of the counting procedures which do not rely on the use of concrete materials: counting-up-to, counting-down-from and counting-down-to (terms from Carpenter & Moser, 1982).

Since children invent these various procedures, their use provides evidence that children are creative problem solvers who can invent arithmetic algorithms without instruction (Moser, Note 6). Their use by children has also served as evidence concerning the development of children's number ideas (Steffe, Spikes & Hirstein, Note 4; Steffe, von Glasserfeld & Richards, Note 5; Romberg & Collis, Note 7) and concerning the underlying representations of the semantic structures of word problems (Briars & Larkin, Note 1; Carpenter, Hiebert & Moser, 1981; Carpenter & Moser, 1982, in press).

The persistence of the counting methods, despite classroom instruction on number facts, is another reason for the interest in these procedures. Many second and third grade children will use counting procedures, even though they are capable of employing heuristic reasoning about number and though the numbers involved are within the range of number facts they have memorized (Brownell, 1928; Carpenter & Moser, in press). Among teachers, the reactions to this persistence range from outright antagonism towards the counting procedures (informal observations indicate that some elementary school teachers make their students sit on their hands so that they cannot "use their fingers") to a more laissez faire attitude (which might arise from the recognition that even adults use counting on occasion). Among researchers, the reaction to the counting procedures has ranged from noting their existence without saying much more about them (Brownell, 1928; Ilg & Ames, 1951; Ginsburg, 1977) to attempts at exploiting their development in the classroom enroute to their eventual replacement by memorized number facts (Labinowitz, Note 8; Steffe, Spikes & Hirstein, Note 4). Decisions concerning whether or not to encourage the use of counting in classroom instruction depend on understanding its nature and development more fully than we currently do (Carpenter & Moser, in press). The present paper and the study reported herein are intended to add to that understanding.

The Counting Methods

For a subtraction problem, the three counting methods commonly used by children are counting-up-to, counting-down-from and counting-down-to (terms from Carpenter & Moser, 1982; footnote 1). In counting-up-to, a count sequence starts at the subtrahend and goes up to the minuend. The number of counts taken is the desired difference. For the problem 19 - 6 = ?, counting-up-to would be executed by starting at 6 (the subtrahend) and continuing to 19 (the minuend): "7, 8, 9, ..., 19". The 13 counts which were taken beyond 6 is the desired answer.

In counting-down-from, the count starts with the minuend and goes backwards by as many steps as the subtrahend. The last word of the sequence is the desired difference. For 19 - 6 = ?, the count starts with 19 and goes down by 6 counts: "18, 17, 16, 15, 14, 13". This procedure is prone to a particular error. If the 6 counts includes the number 19, the person must realize that the next word is the answer, not the last one of the sequence.

Finally, in counting-down-to, the count sequence starts with the minuend and goes backwards to the subtrahend. The number of counts is the desired answer. For the problem 19 - 6 = ?, the count would be "18, 17, 16, ..., 6."

All three methods share two features. The count starts with a number other than one, and there is a need to create some record of the number of counts which are executed in the sequence. The function of this record varies: in counting-up-to and counting-down-to, it produces the desired answer; in counting-down-from, it is used to stop counting. Also, the direction of the count varies: in counting-up-to, it is forward from subtrahend to minuend; in both counting-down methods, it goes backwards from the minuend.

The relationship of Piagetian operativity to early arithmetic has been the subject of many studies resulting in contradictory findings. Mpiangu & Gentile (1975) found no relationship between children's conservation of number and ability to learn addition and subtraction. Using a wider set of measures, Steffe, Spikes & Hirstein (Note 4) found limits to the effects of instruction based on children's performance on numerical equivalence, but none based on class inclusion. Such studies typically use batteries of tests and seek general relationships which might result in the loss of more speci relationships. In subtraction, the counting methods entail the child's use of counting to go from a given part's number to its whole (counting-up-to), to go from the whole to the given part (counting-down-to), or to go from the the whole by as many as the given part (counting-down-by). The use of counting would seem to indicate that the children were using part/whole class inclusion reasoning to plan and execute the count sequence. The relationship of class inclusion knowledge to the use of counting for subtraction was a focus of this study.

Steffe, Spikes & Hirstein (Note 4) have noted the structural similarity between counting-on from the first addend in an addition context and counting-up-to from the subtrahend in a subtraction context. In this study, the relationship between counting-on (for addition) and the use of counting (for subtraction) was investigated.

Secada, Fuson & Hall (Note 10) analysed the subskills underlying the use of counting-on. One of these subskills is the ability to consider simultaneously multiple first addend number word meanings: as the cardinality of the first addend and as a counting word used in the final count sequence (Fuson, 1982). The child who counts-on uses these multiple word meanings to enter the count sequnce at the first addend number word. Similarly, the child who counts-up-to must coordinate the multiple meanings of the subtrahend number word: as the cardinality of the given part in a class inclusion relation and as a count word used within a count sequence. He then uses these coordinated meanings to enter the count sequence at the subtrahend (skill 1). In counting-down, both -to and -from, the child enters the count sequence at the minuend number (skill 2); he must coordinate the multiple meanings associated to this number word: as the cardinality of the whole in a class inclusion relation and as a number word used in a backwards count sequence. Fuson & Hall (in press) review evidence that number words develop initial meanings which are tied to specific contexts and that over time these meanings merge and become flexibly applied. In this study, evidence concerning the existence of these flexible meanings and their relationships to children's use of counting for subtraction were investigated.

In both counting-up-to and counting-down-to, the count sequence is used to create a representation of the difference, i.e., of the unknown part of a class inclusion relation. The relationship of this skill (#3) to the use of counting for subtraction was investigated in this study.

A final skill investigated in this study is the verbal ability to count backwards from the minuend which is necessary for either counting-down procedure. Fuson, Richards & Briars (in press) report that the ability to count backwards from an arbitrary word follows the ability to start counting forwards from that number. Most first grade children, 69 out of a sample of 75 tested in Secada, Fuson & Hall

(Note 10), can count forwards starting with numbers in the range used in this study. Hence, verbal counting forwards which is necessary for counting-up-to was not assessed. Carpenter & Moser (1982) report that 50% of their first grade sample could count backwards. Unfortunately, they failed to relate this verbal skill to the various counting solution procedures used for subtraction. This was done in the present study.

The structural similarity between counting-on and counting-up-to has been noted. In the development of the counting methods, it is possible that counting-on directly transfers to the subtraction context in the form of counting-up-to. A second possibility is that the skills associated with counting-on are what transfer to the subtraction context, and make possible the various counting procedures. Skills 1 and 2 (above) have already been related to one of the skills underlying counting-on. Skill 3 is related to another counting-on skill identified by Secada, Fuson & Hall (Note 10): using the counting sequence to bridge the first and second addends through the statement of two number words (the first addend and its successor). In both cases, for addition and for subtraction, once the count sequence has been entered, counting is used to refer to elements of the secondary part of a class inclusion relation as if that part were already within the whole. Thus, the relationship of skills 1, 2, and 3 to counting-on was another cern of this study.

In conclusion, the present study had four foci. First was the hypothesis that counting for subtraction is guided by the use of part/whole reasoning. Specifically, we investigated the hypothesis that children who count for subtraction outperform those who fail to do so on a general class inclusion task. Second was the hypothesis that counting-on (for addition) is prerequisite for the use of counting within a subtraction context. Specifically, we hypothesised that all children who count for subtraction also count-on, and that many children who fail to count for subtraction will also count-all (i.e., fail to count-on). Third was the hypothesis that the four counting skills described above underly the use of counting for subtraction. Specifically, we hypothesized that children who count will have these skills; whereas, those who do not will tend to lack them. Finally was the hypothesis that the first three counting skills are derived from skills associated to counting-on. According to the final hypothesis, of the children who fail to count for subtraction, those who count-on should outperform those who count-all on skills 1, 2, and 3, since they have accessible the knowledge bases from which these skills are derived.

Methodology

A subtraction pretest was administered to ascertain which counting procedures children spontaneously use for subtraction. This was followed by an addition test to investigate the use of counting-on. Skills 1, 2, 3, and 4 were assessed in that order, and a part/whole class inclusion task was given last.

Behavioral or probe evidence for the use of solution procedures are commonly used in studies investigating the development of early arithmetic (Carpenter, et. al, 1981; Steffe, et. al, Note 4). For this study, behavioral and converging self report evidence were sought. Even if a child seemed to be following a specified solution procedure in the solution of a task, probes were used in an effort to obtain converging evidence concerning his solution procedures. Probes began by E's asking: "How did you figure that out?" If the child's answer did not specify the procedure used, E sought clarification by incorporating the child's own answer into the next question; e.g., in following up the answer "I counted", E would ask, "What number did you start counting with?" or "How did you count?" Only if both converged, behaviors (when overt) and answers to probes, were procedures for a task classified in assessing the use of counting for subtraction, counting-on for addition, and skills 1, 2, and 3.

This procedure resulted in three children becoming confused. Prior to being asked how they had figured something out, the children gave immediate responses to the trials assessing skills 1 (for two of the children) and 3 (for the third child) which was evidence that they were using the skill in question (see description of skills assessment, below). When asked the probe by E, these children regressed to using counting methods that failed to exhibit the skills. Since converging evidence could not be obtained, these children's performances were excluded from the analyses of skills 1 and 3, respectively.

Prior analyses of the subtraction solution procedures have considered children's work only as they worked on such problems (Steffe, von Glaserfeld & Richards, Note 5). For this study, as in Secada, Fuson & Hall (Note 10), independent evidence for the existence of the skills in question was sought; moreover, the contexts for assessing the skills were similar to the context of the subtraction task.

Since the focus of this study was the relationship between counting solution processes and underlying skills, different criteria were used in establishing the availability for use of the counting procedures (for subtraction and for addition) on the one hand, and the availability of skills 1, 2, 3, & 4 and of class inclusion knowledge on the other. If a child used a counting procedure on even a single trial, it was considered as evidence that the procedure was available for his use, and hence he was classified as being able to use the procedure in question. The number of times and/or the conditions under which the procedures were used are issues concerning how well they have been learned (i.e., the automaticity of the procedures) or the effects of certain manipulations on their assessment; these latter issues were not a concern of this study. The skills and class inclusion knowledge were considered to differ in kind from the counting procedures which they are hypothesized to underly and make possible. Thus, they were assessed more stringently with the requirement that each skill be exhibited on three trials in a row. The effects of the differential assessment of procedures versus skills is to classify children at the more advanced procedural levels while requiring conservative evidence about the underlying skills and knowledge.

Tasks

The subtraction pretest was a set decompostion task adapted from Steffe, Spikes & Hirstein (Note 4) and used by Secada (Note 3). An unevenly dense array of black dots were glued to a long white card (called an array card). This array card was placed in front of the child while E said, "There are x dots on this card." E then placed an index card with the numeral x written on it (called the numeral card) over the array card. E continued, "But now I'm going to hide some of the dots from you." She turned the array card towards herself, and covered a portion of the card with a covering card. She turned the covered array card back towards the child saying, "There are y dots now showing." A matching numeral card was placed above the visible portion of the array card. E then asked the child, "How many dots did I hide from you? Remember, there <u>were</u> <u>x</u> (E indicated the original numeral card) and now there are only <u>y</u> showing. How many are hidden?" See figure 1 for the task from the child's perspective. This task models subtraction since the minuend is represented by the whole, the subtrahend by the viible part and the difference by the part.

On all eight pretest trials, the minuend was between 16 and 19. On the first four of those trials, the subtrahend (or visible part) was between four (4) and six (6); on the last four trials, it was between eleven (11) and fourteen (14). Numbers this size were used to minimize the possibility of children's using procedures more advanced than counting, while ensuring that their counting would take enough time for E to make observations of their overt behaviors. The manipulation of subtrahend size was undertaken in an effort to induce children to use the counting-down methods, since in prior work, Secada (Note 3) had found that if children use counting-down, they are more likely to use it when the subtrahend is large (i.e., when the difference is small). A single list of random numbers was used subject to the condition that for no two adjacent trials were the minuends, subtrahends or differences respecting equal.

Behavioral evidence that a child was using one of the three counting procedures included his extending fingers one at a time or lip movements from which numbers between the subtrahend and the minuend could be read. A child who provided behavioral and/or probe evidence that he had used one of the counting methods was classified as a count subtractor; if he did not, he was classified as a pre-count subtractor.

A cardinal addition task, adapted from Steffe, et al. (Note 4), and used in Secada, Fuson & Hall (Note 10) was used to investigate children's use of counting-on. An array card and its matching numeral card were placed in front of the child as E said, "There are *m* dots on this card. See, this tells you there are *m*." A second array and numeral card were placed in front of the child while E said that it had *n* dots. E continued by asking the child how many dots were on both cards, and said to remember that the first card had *m* dots so it didn't have to be recounted. Over the six addition trials, the first addend ranged from 12 to 1ˆ, the second from 6 to 9. For no two adjacent trials were the first addends, the second addends or the resultant sums respectively equal; otherwise, a single list of randomly generated numbers was used. A sample task can be seen in figure 2. If a child counted-on before the sixth trial, E continued to the next task. Behavioral evidence for the use of counting-on included a child's doing so out loud, relatively quick response time (the first addend's twelve to eighteen dots provided an easy check on time spent: children counting-on would be quicker than those counting-all), a child's staring intently at the second addend while quickly looking at the first, or a quick indication of the first addend followed by points and/or nods at the second array. Children who gave no evidence, either behavioral or to probes, of using counting-on but who counted all the dots were classified as count-all.

To assess skills 1, 2, and 3, an array card composed of blue dots followed by red dots (both sets forming a single array) was placed in front of the child. E placed a numeral card over the array card saying, "There are *x* dots here all together." E then placed a second numeral card over the blue portion of the array while saying, "There are y blue dots. If you counted all the dots, like this (E swept her hand over the entire array in a left to right motion, going from the blue to the red portion of the array), what count number would this dot get?" Depending on which skill was being assessed, E touched either the last blue dot (skill 1), the last dot of the entire array (skill 2) or the next to the last dot of the entire array (skill 3); see figure 2 for the child's perspective.

To exhibit any of the skills, the child had to use the appropriate cardinality to enter the count sequence: the cardinality of the given part for skill 1, the cardinality of the whole for skill 2 and either cardinality for skill 3. Furthermore, for skill 3, the child either had to count up through the unknown part of the array if he had entered the count sequence at the given part (or subtrahend), or to count backwards from the last element of the whole if he had entered the count sequence at the minuend. Behavioral evidence for the skills included the child's spontaneously counting through the unknown part (for skill 3); his responding correctly as soon as E had asked for the count number corresponding to a specific dot, or his doing so after a quick glance at either numeral card. Over four trials for each skill, the whole was between 16 and 19; the given part, between 11 and 14. A single list of random numbers was used, subject to the condition that for no two adjacent trials were the wholes, the given (blue) parts of the unknown (red) parts respectively equal. A child was judged to have a skill if he gave evidence of using it on three trials in a row. If the child exhibited the skill only on the fourth trial, he was given up to two more trials to reach three in a row.

Counting for Subtraction

Skill 4 was assessed by asking a child to count backwards. The first two trials were practice trials using five and seven with E modeling what was meant if necessary. Once the child understood the task, four trials using numbers between sixteen and nineteen were administered. To exhibit this skill, a child had to count backwards from x to x-3. This was to ensure that the child did not give the number which precedes x, an easier task than actually counting backwards (Fuson, Briars & Richards, in press). To have this skill, a child had to exhibit it on three trials in a row.

To assess the child's understanding of part/whole class inclusion, an array card composed of blue and red subarrays was placed in front of him. As in the skills assessment, the whole was between 16 and 19 and the part between 11 and 14. Unlike the skills assessment, numeral cards were not used. E asked the child to determine "which is more, the blue dots or all of the dots." The blue dots were compared to all of the dots in an effort to make sure that the child understood the comparison, i.e., in an effort to maximize performance. To pass this item, a child must have answered correctly on three trials in a row of the four he was given. Only performance data were used due to the finding that requests for justification tend to depress performance on tests of operativity (Brainerd, 1973).

Data Collection

Sixty first grade children aged between 6,9 and 7,5 served as subjects for this study. They had been selected to match the heterogenous ethnic, racial and economic mix of Chicago for attendance in a magnet school. At the time of the April data collection, the children were studying addition facts up to 10, and were learning subtraction meanings. Teachers reported neither encouraging nor discouraging children's use of counting.

Individual children were taken from their classroom for testing which lasted at most 30 minutes. E, a senior education major with extensive training and experience with children in this age range, recorded overt behaviors, answers to tasks, and results of probe questions.

Results

Twenty children counted to solve at least one pretest subtraction trial (count subtractors); thirty-eight failed to use counting on any trial (pre-count subtractors); and two children were dropped from the study for failing to understand the pretest task.

Class Inclusion

Of the twenty count subtractors, eleven solved the part/whole task and seventeen of the thirty-eight pre-count subtractors did so. This chance difference ($p > .50$) is consistent with Steffe, et al.'s (Note 4) conclusion that class inclusion is not a readiness variable for learning early arithmetic.

Counting-on

Of the count subtractors, all twenty counted-on in the addition task; whereas, twenty-two of the thirty-eight (38) pre-count subtractors did so. Chi-square = 11.63, df = 1, $p < .001$. Furthermore, of the twenty count subtractors, nineteen counted-on at the very first addition trial, compared to thirteen of the twenty-two count-on/pre-count solvers who did so. Chi-square = 7.45, df = 1, $p < .01$.

Counting for Subtraction

The Skills

Of the twenty count subtractors, two children were confused by E's probe questions during the assessment of skill 1, resulting in their performaces being excluded from the analysis. Of the remaining eighteen count subtractors, sixteen exhibited skill 1 (entry into the count sequence at the subtrahend); whereas, twenty-two of the thirty-eight pre-count subtractors did so. Chi-square = 5.38, df = 1, $p < .05$.

On skill 2, entry to the count sequence at the minuend, fourteen of the twenty count subtractors exhibited it compared to fifteen of the thirty-eight pre-count subtractors who did so. Chi-square = 4.88, df = 1, $p < .05$.

On skill 3, using counting within the unknown part, one of the count subtractors was confused by E's probe questions; her data were excluded from this analysis. Of the remaining nineteen count subtractors, all the children exhibited the skill compared to seventeen of the thirty-eight pre-count subtractors who did so. Chi-square = 16.625, df = 1, $p < .001$.

All twenty count subtractors exhibited the ability to count backwards verbally (skill 4) compared to twenty-two of the thrity-eight pre-count subtractors who did so. Chi-square = 9.713, df = 1, $p < .005$.

Error analyses were used in an effort to determine what children did when they failed to exhibit the skills. On skill 1, one of the count subtractors did in fact enter the count sequence at the subtrahend (the given part), but he automatically continued his count and gave the subtrahend's successor for his answer. Secada, Fuson & Hall (Note 10) observed that children who automatically count-on in addition may tend to continue their counting automatically upon entering the count sequence, i.e., the skill which allows the child to enter the count sequence becomes integrated into another skill and withers away as an independently existing skill. That this count subtractor had also abbreviated skill 1 is evident from his counting-up-to on all eight pretest trials and that he counted-on immediately in the addition task (i.e., counting-on was a well developed process for him). Furthermore, he solved the task which assessed entry to the count sequence at the minuend (or the whole, skill 2) by starting to count at the last dot of the given part (i.e., by exhibiting skill 1 as part of another skill). The second count subtractor who failed to exhibit skill 1 alternated between counting all of the given part's blue dots up to the desired dot and in counting backwards starting from the last (red) dot of the whole array until she reached the last dot of the given part (the blue subarray). Interestingly, she did exhibit skill 1 as part of her solution for skill 2. Just as the first child described above, she would start counting from the last blue dot and count the red dots until she reached the last dot. Thus, both these children exhibited the ability to enter the count sequence at the subtrahend (as part of another skill), but simply failed to exhibit it at this task. Of the sixteen pre-count subtractors who failed to exhibit skill 1, one child gave the subtrahend's successor for her answer. She immediately counted-on in the addition task and gave the subtrahend's successor for skill 2 (entry to the count equence at the minuend). The remaining fifteen children counted all of the given part's (blue) dots rather than using the cardinaliy of the given part. In sum, all eighteen count subtractors gave evidence of having the first skill and twenty-three of the thirty-eight pre-count subtractors also did so.

All six of the twenty count subtractors who failed to exhibit skill 2, entry to the count sequence at the minuend, solved the task by counting up to the last dot of the array from the last dot of the known part. They entered the count sequence at the subtrahend (i.e., used skill 1) and counted through the unknown (red) part of the array (i.e., used skill 3). Of the twenty-three pre-count subtractors who did not exhibit skill 2, six also did the same thing. The remaining seventeen simply counted the entire array of the blue dots followed by the red dots.

Of the seventeen pre-count subtractors who failed to exhibit skill 3, one gave the subtrahend's numerosity for his answer; one gave the subtrahend's successor for an answer; and the remaining fifteen children counted all the dots from the first blue dot to the next to last (red) dot to find their answers.

Counting-on and the Skills

Of the thrity-eight pre-count subtractors, twenty-two counted-on in the addition task and sixteen counted-all for all six trials. Of the twenty-two count-on children sixteen exhibited skill 1; eleven, skill 2; and fourteen, skill 3. Of the sixteen count-on children six exhibited skill 1; four, skill 2; and three, skill 3. The differences on skills 1 and 3 are significant: skill 1 chi-square = 4.72, df = 1, $p < .05$; skill 3 chi-square = 7.55, df = 1, $p < .01$. On skill 2, the group differences are within chance ($p > .10$); however, six of the eleven count-on children who failed to exhibit skill 2 solved this task by the sequential application of skills 1 and 3 described above.

Discussion

The first hypothesis, that children who use counting as a subtraction solution process are using part/whole class inclusion reasoning to guide their problem solving processes, was not supported. This is consistent not only with Steffe's (Note 4) conclusion that class inclusion is not a readiness variable for the learning of arithmetic, but also with Gelman & Gallistel's (1978) observation that children reason differently contingent on whether specified or unspecified numerosities are implicated in the problem. Future research on subtraction solution procedures might consider the roles of the specified numerosities represented by the subtrahend and the minuend in guiding the solution process.

The data support the hypothesis that counting-on is prerequisite (or necessary) for the use of counting in subtraction. Not only did the count subtractors all count-on, but they counted-on sooner than did the pre-count subtractors, indicating that counting-on is a very well developed addition solution procedure for the more advanced group.

The relationship between counting for subtraction and the associated skills is rather complex. Though each of the four skills differentiate count subtractors from their pre-count cohorts, skills 1, 3 and 4 are more directly associated to the use of counting. <u>All</u> the count subtractors gave evidence of having and using these skills within the skills assessment. The performance of the twelve children (six count subtractors and six pre-count subtractors) who instead of using the cardinality of the minuend to enter the count sequence solved the task for skill 2 by a sequential application of skills 1 and 3 suggests that skill 2's association to the use of counting is mediated by its relationship to these other skills. However, the performance of the one count subtractor who used skills 2 and 3 in solving the task for skill 1 suggests that the relationship among the skills is complex, and worth additional investigation.

In contrast to Carpenter & Moser's (1982) 50%, 71% of our first graders (41 out of 58) could count backwards verbally. However, fewer than 15% (7 of the 41) of those who could, did use their verbal skill to count-down. Additional research will be necessary to determine the relationship of verbal counting skills to their use within subtraction contexts.

Whether counting-on is prerequisite to the use of counting within subtraction because it transfers directly as counting-up-to (Steffe, Spikes & Hirstein, Note 4) or because its subsidiary skills transfer remains open. The performance of the count subtractors relative to the pre-count subtractors is consistent with either possibility. That the count-on children outperformed their count-all cohorts on skills 1 and 3 supports the latter view. Further support for the latter hypothesis comes from the similarity between count-on children and count subtractors on the skills: on skill 1, 89% success for count subtractors versus 73% for count-on; on skill 2, 70% versus 50%; on skill 3, 100% versus 64%; and that six children from each group solved skill 2 through skills 1 and 3. Future research will need to directly relate the skills in counting-on to these three skills.

Though the four skills would seem to underly the use of counting in varying degrees, the high performance by the count-on children on that skills indicates that other skills and understandings are implicated in the transition from the non-use to the use of the counting procedures for subtraction. Just as the skills 1, 2 and 3 are tied to the first addend's solution procedures through counting-on, these other skills and understandings might be tied to the second addend through the use of "keeping track" mechanisms (Fuson, 1982).

In sum, counting-on is necessary for the development of the use of counting for subtraction. The ability to enter the count sequence at the subtrahend and to use counting within the unknown part of part/whole class inclusion relation underly the counting procedures as well. These two skills seem to combine in the development of another skill: the ability to enter the count sequence at the minuend. Whether counting-on directly transfers into the subtraction context, or whether skills associated to its development transfer in the form of the above three skills needs further investigation. The verbal ability to count backwards develops well before its use in the form of counting back procedures. Finally, the use of general forms of class inclusion reasoning do not seem implicated in the development of the counting solutions.

Footnotes

1. There is some evidence that these procedures are not universal. Secada (Note 9) has found that manual deaf children use different counting methods to solve subtraction problems.

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FIGURE 1

Subtraction Pretest

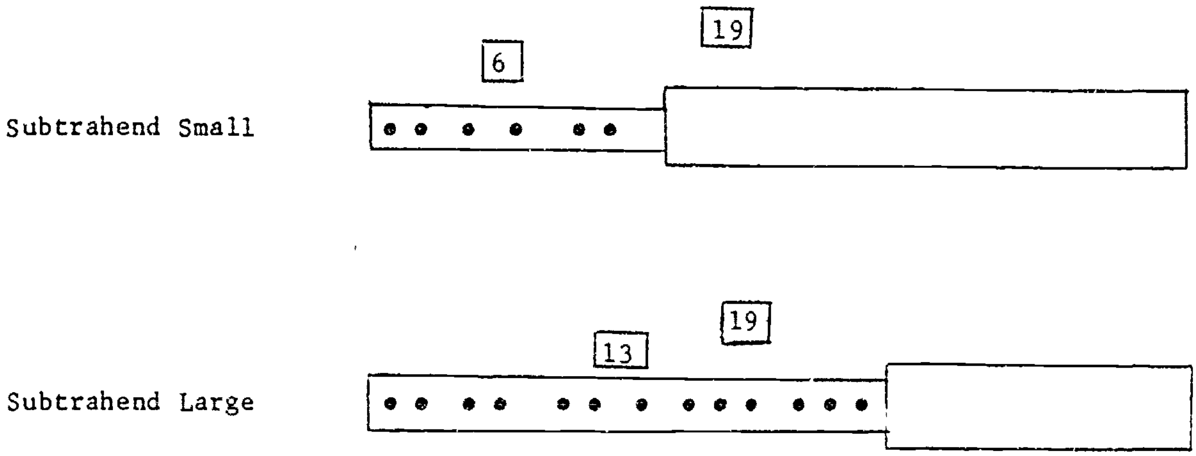


FIGURE 2

Addition Task

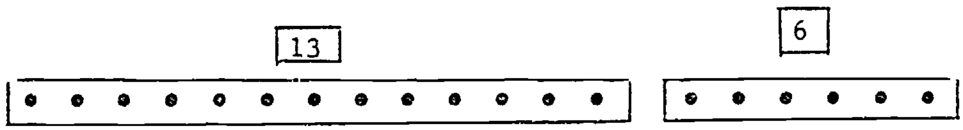
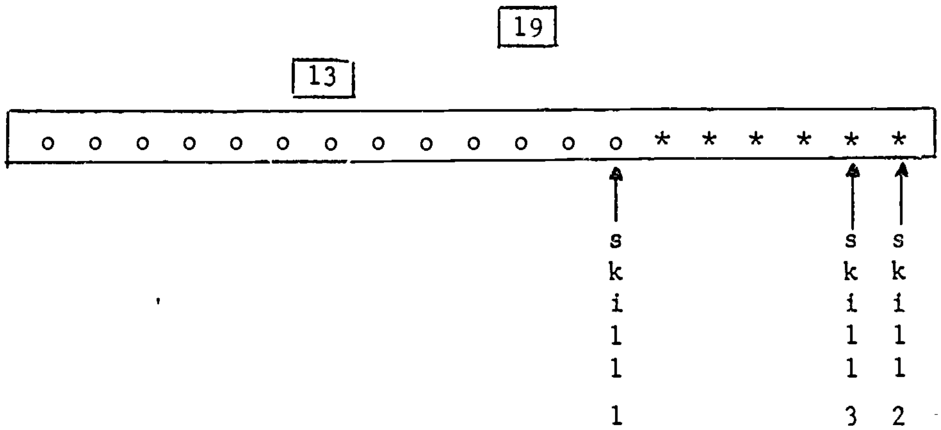


FIGURE 3

Skills Assessment



o = blue subarray

* = red subarray

Note: for the class inclusion assessment, the same array set up was used, without the numeral cards.