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IDENTIFIERS Radioactivity

ABSTRACT

The first four units cover aspects of medical applications of calculus: 71-Measuring Cardiac Output; 72-Prescribing Safe and Effective Dosage; 73-Epidemics; and 74-Tracer Methods in Permiability. All units include a set of exercises and answers to at least some of the problems. Unit 72 also contains a model exam and answers to this exam. The fifth unit in this set covers applications to economics: 75-Feldman's Model. This mathematical model describes the behavior over time of a two-sector economy in which sectoral investment allocations are controlled by a central authority according to an overall economic plan. The unit includes exercises and answers. The next three modules focus on Graphical and Numerical Solution of Differential Equations: 81-Problems Leading to Differential Equations; 82-Solving Differential Equations Graphically; and 83-Solving Differential Equations Numerically. The three-unit group contains a total of five quizzes and one exam, and answers are provided for all in appendices. The last unit covers applications of calculus to chemistry: 234-Radioactive Chains-Parents and Daughters. (MP)

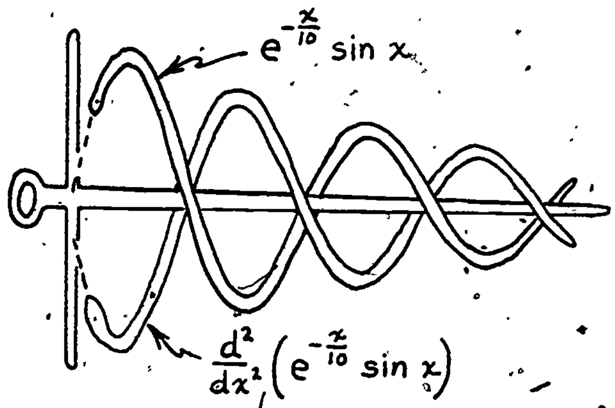
umap

UNIT 71

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

MEASURING CARDIAC OUTPUT

by Brindell Horelick and Sinan Koont



MEDICAL APPLICATIONS OF CALCULUS

Units 71-74

edc/umap/55chapel st./newton, mass. 02160

MEASURING CARDIAC OUTPUT

by

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Intermodular Description Sheet: UMAP *Unit 71*

Title: MEASURING CARDIAC OUTPUT

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Review Stage/Date: IV revision 5/15/79

Classification: MED APPL CALC/CARDIAC OUTPUT (U 71)

Suggested Support Materials:

References:

Guyton, A.C. (1956), Textbook of Medical Physiology, W.B. Saunders, Philadelphia.
Hackett, E. (1973), Blood, Saturday Review Press, New York.
Simon, W. (1972), Mathematical Techniques for Physiology and Medicine, Academic Press, New York.
Schwartz, A. (1974), Calculus and Analytic Geometry, Holt, Rinehart, and Winston, New York.

Prerequisite Skills:

1. Be able to identify $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ as a definite integral.
2. Know how to define the area under a curve. Be able to evaluate definite integrals by antidifferentiation.
3. Be familiar with the trapezoidal and parabolic (Simpson's) rule.

Output Skills:

1. Be able to describe how dye dilution technique is used to determine cardiac output.
2. Be able to explain the setting up of the Riemann sum herein and the derivation of Equations (2) and (3).

Other Related Units:

Prescribing Safe and Effective Dosage *(Unit 72)*
Epidemics *(Unit 73)*
Tracer Methods in Permeability *(Unit 74)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

MEASURING CARDIAC OUTPUT

1. THE TECHNIQUE OF DYE DILUTION

The volume of blood a person's heart pumps per unit time (that is, the *rate* at which it pumps blood) is called the person's *cardiac output*. Normally in a person at rest this rate is about 5 liters per minute. But after strenuous exercise it can rise to more than 30 liters per minute. It can also be raised or lowered significantly by certain diseases of the blood vessels, heart, and nervous system.

In this unit we shall discuss a technique for measuring cardiac output known as *dye dilution*. The technique works as follows. At time $t = 0$ a known amount D of a dye is injected into a main vein near the heart. The dyed blood circulates through the right side of the heart, the lungs, then the left side of the heart, and finally appears in the arterial system. The concentration of the dye is monitored at fixed time intervals Δt at some convenient point in the arterial system. Typically, Δt might equal one second. For purposes of the mathematical development in this unit, we shall assume the monitoring is done in the aorta near the heart. In Exercise 1 it will be assumed that the dye concentration is monitored in a branch artery instead, and you will be asked to make appropriate changes in the analysis that follows.

Normally it will take only a few seconds for the dye to pass through the heart and lungs once and begin to appear in the aorta. A typical set of readings will be as in Table 1, where we see one result of injecting $D = 5$ mg of dye in a main vein near the heart at time $t = 0$ seconds. If we plot these readings on graph paper we get the points shown in Figure 1.

Our question is: How may we use the empirical data given in Table 1 to determine the cardiac output?

TABLE 1

Typical Data for the Dye Dilution Technique

Time (seconds):	0	1	2	3	4	5	6	7	8
Concentration: (mg/liter)	0	0	0	0.1	0.6	0.9	1.4	1.9	2.7
Time (seconds):	9	10	11	12	13	14	15	16	17
Concentration: (mg/liter)	3.0	3.7	4.0	4.1	4.0	3.8	3.7	2.9	2.2
Time (seconds):	18	19	20	21	22	23	24		
Concentration: (mg/liter)	1.5	1.1	0.9	0.8	0.9	0.9	0.9		

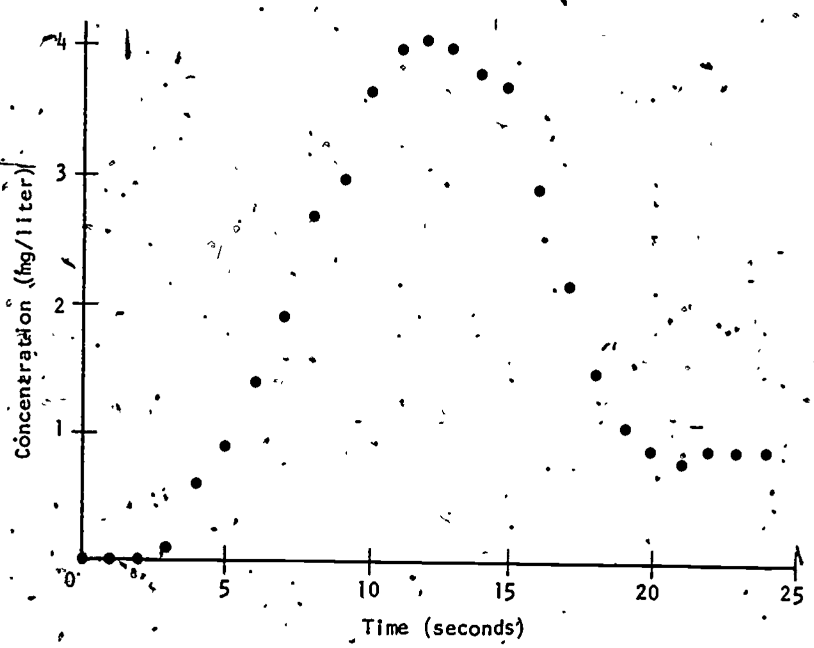


Figure 1. Typical readings in the dye dilution technique when D = 5 mg of dye are injected at time t = 0 seconds.

2. THE FORMULA FOR CARDIAC OUTPUT

2.1 Preliminary Illustration

Let us set the stage by considering a somewhat artificial simplified version of the question. Suppose it were possible to set things up so *all* of the dye flowed through the heart exactly once in a time interval of length T seconds, at a constant concentration of C mg/ℓ. (A record of our observations would look something like Figure 2.) Then we could express the amount of dye by the formula $D = CV$, where V is the volume (in liters) of blood flowing through the heart in this time interval, or $V = D/C$. The *cardiac output* R (the *rate*) would then be given by the formula $V = RT$ (volume = rate × time), which can be written in the form

(1) $$R = V/T = D/CT,$$

where D, C, and T are all known. Notice that CT is the area of the rectangle in Figure 2.

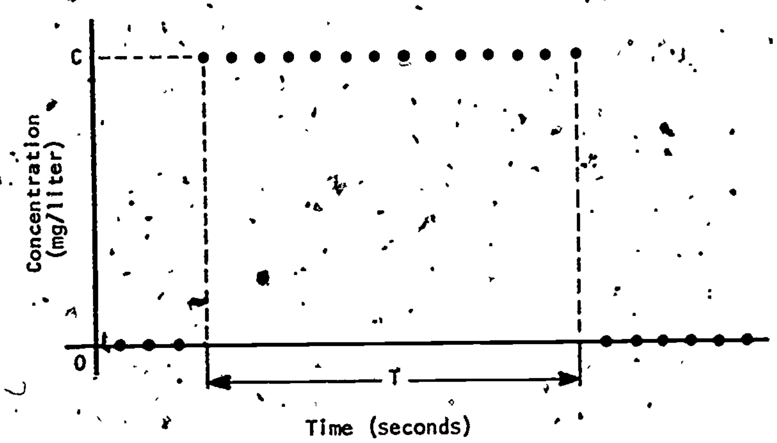


Figure 2. Idealized observations of dye concentration in the aorta.

We cannot achieve this situation. Even if we were to wait a very extended period of time to achieve a constant concentration of dye in the bloodstream, this would be useless, since we would have no way of knowing how long it took all the dye to pass the monitoring point exactly once.

How can we modify this simple algebraic computation to analyze the data of Figure 1, where the dye concentration is not constant?

2.2 Rectangular Approximation

There are two essential differences between the idealized observations of Figure 2 and the realistic observations of Figure 1. One is that in Figure 2 the dye concentration is constant. The other is less striking but equally important—in Figure 2 we can identify a time interval during which we know exactly how much dye has passed by our monitoring point.

Let us consider this second difference first. In Figure 1 we see that the dye concentration rises sharply, then falls sharply, and then, just when we think it is going to fall off to zero, it rises again. This second rise occurs at about $t = 20$ seconds. Now, physiologists know that 20 seconds is just about long enough for some blood passing through the aorta to make a round trip of the body and the lungs, and reappear in the aorta. Apparently what is happening is that most of the dye passes through the aorta in the first 14 or 15 seconds. The dye concentration then falls off rapidly (from $t = 15$ to $t = 21$) as the rest of the dye trickles through. Then, at about $t = 21$, a little dye, having completed its round trip, appears for the second time and mingles with what is left of the "first-time-through" dye to cause the jump in the graph.

We must attempt to pick out what part of the dye concentration after $t = 21$ is due to "first-time-through" dye. Notice that just before $t = 21$ (and especially from $t = 19$ to $t = 21$) the dye concentration is decreasing at a pretty steady rate. *Let us assume that "first-time-through" concentration continues to decrease at this rate.* Then the graph of "first-time-through" concentration, instead of rising at $t = 21$, will pass through the points A, B, and C as shown in Figure 3.

In Figure 3 we simply drew in A, B, and C by eye. They are approximately: A (22,0.5), B (23,0.3), and C (24,0). They represent, at best, a shrewd gue[illegible] there is no point in agonizing over their exact location. By the end of this section we shall see that the portion of the graph after $t = 21$ has only a small effect on our final result.

Now we are ready to confront the first of the two essential differences mentioned at the beginning of this

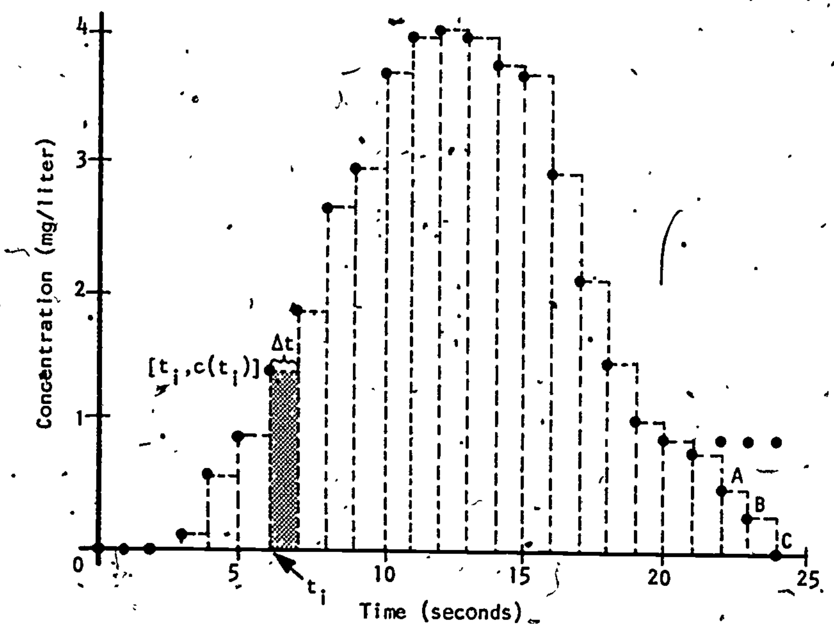


Figure 3. Rectangular approximation of the dye flow.

section. In Figure 3 we have drawn a succession of rectangles. Each rectangle has base Δt (the interval between observations) and height $c(t_i)$ (the observed concentration at time t_i). In our example $\Delta t = 1$. In Figure 3 we have illustrated, for $i = 6$, $t_i = t_6 = 6$ and $c(t_i) = c(6) = 1.4$.

Now let us consider the time interval from t_i to t_{i+1}, of length Δt. At the beginning of that interval the dye concentration is observed to be $c(t_i)$. The volume of blood flowing past our observation point during the time interval is $R\Delta t$. Recall that R is a *rate*. If the dye concentration were constant for this time interval, the total amount of dye flowing through in this interval would be $c(t_i)R\Delta t$, or R times the area of rectangle number i in Figure 3.

The time interval Δt is rather small compared to the total time involved, and the dye concentration never changes abruptly, so the error introduced by making this approximation is not great.

Since we have assumed that the monitoring is done in the aorta near the heart, all the dye must flow by our monitoring point between $t = 0$ and $t = T_0$. If we add all the approximations corresponding to the rectangles from $t = 0$ to $t = T_0$ we must account approximately for the total amount of dye D:

$$D \approx \sum_{i=1}^{n} c(t_i)R\Delta t = R\sum_{i=1}^{n} c(t_i)\Delta t, \tag{2}$$

where n is the number of rectangles. In our example, $n = 23$ if we count the first two "rectangles," from $t = 1$ to $t = 3$, each of which has "height" zero. Thus,

$$R \approx \frac{D}{\sum_{i=1}^{n} c(t_i)\Delta t}, \tag{3}$$

where the denominator is the total area of the rectangles in Figure 3.

2.3 The Definite Integral

So far all our formulas are based strictly upon the observed values $c(t_i)$ and the observer-determined values D and Δt. Now let us note that underlying the empirical values plotted in Figures 1 and 3 there is a function $c(t)$ defined (but not observed) for *all* t between $t = 0$ and $t = T_0$ (see Figure 4). This function may be approximated by fitting a smooth curve to the observed points $[t_i, c(t_i)]$, using the assumed points A, B, and C at the end. The denominator in (3) is then an estimate of the area under this curve. In fact it is one of the approximating sums used in defining the definite integral

$$\int_0^{T_0} c(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} c(t_i)\Delta t,$$

in which we think of n and T_0 as given and set $\Delta t = T_0/n$.

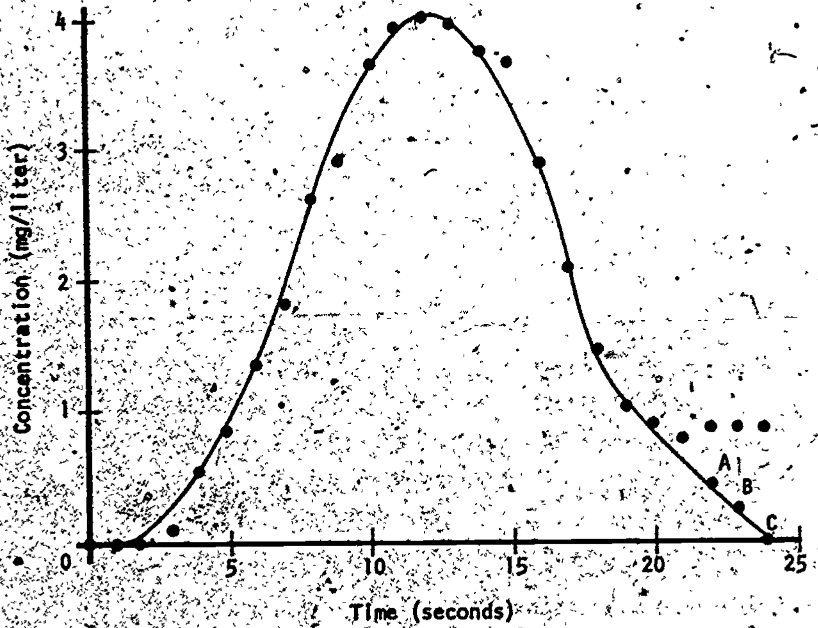


Figure 4. The underlying curve c(t).

We can now write

$$(4) \qquad R \approx \frac{D}{\int_0^{T_0} c(t)dt}.$$

We must use an approximation sign because our curve $c(t)$ is at best a curve which fits the data well. We have no way of knowing if it is exact.

3. COMPUTATION OF CARDIAC OUTPUT

3.1 Antidifferentiation

How we use Equation (4) depends on the nature of the function $c(t)$. It may be that a curve can be fitted to the data points in Figure 1 which is the graph of a function $c(t)$ whose antiderivative $C(t)$ is known. In that case we would use the fundamental theorem of calculus to compute

$$\int_0^{T_0} c(t)dt = C(T_0) - C(0),$$

and then

$$R \approx \frac{D}{C(T_0) - C(0)}.$$

3.2 Numerical Methods

More likely, however, there will be no explicit formula for $c(t)$, let alone for its antiderivative. In this case we use one of a variety of ways to estimate the denominator of Equation (4), and thus obtain an approximation of R. We shall list several, and illustrate some of them with the data of Table 1. Recall that these data were obtained with a dye dosage of D = 5 mg.

(a) We can use the denominator of Equation (3). This uses the areas of the rectangles in Figure 3, rather than the area under the curve in Figure 4.

In our example,

$$\sum_{i=1}^{23} c(t_i)\Delta t = \sum_{i=1}^{23} c(t_i) = 0 + 0 + 0.1 + 0.6 + \ldots$$
$$+ 1.1 + 0.9 + 0.8 + 0.5 + 0.3 = 44.1.$$

Then

$$R \approx \frac{D}{44.1} = \frac{5}{44.1} \approx 0.113 \text{ liters/second}$$
$$\approx 6.8 \text{ liters/minute.}$$

(Although the concentration measurements were taken by the second, output is usually measured in liters per minute.) Notice that in making this computation we replaced the last three experimental points of Table 1 with the points A (22,0.5), B (23,0.3), and C (24,0). The reason for doing this was discussed in Section 2.2.

(b) More laboriously, but also more accurately, we could sketch Figure 4 on a large sheet of graph paper and count the number of squares that fall between c(t) and the horizontal axis. We would then multiply this total by the unit of area represented by a single square. There are also mechanical devices, called planimeters, with which it is possible to trace the boundary of a region and then read an estimate of the area of the region from a meter. We could use one of these instead of counting squares.

(c) If the interval [a,b] is divided into n equal parts $[a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b]$, then the *trapezoidal rule* says:

$$\int_a^b c(t)dt \approx \frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n),$$

where we have written $y_i = c(t_i)$ for $i = 0,1,2,\ldots,n$.

In our example,

$$\int_0^{24} c(t)dt \approx \frac{24-0}{2(24)}(0+0+0+0.2+1.2+1.8+\ldots+2.2$$
$$+ 1.8 + 1.6 + 1.0 + 0.6 + 0)$$
$$= \tfrac{1}{2}(88.2) = 44.1.$$

As in part (a), R ≈ 6.8 liters/minute.

(d) If the interval [a,b] is divided into n equal parts $[a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b]$, where n is an *even* number, then the *parabolic rule*, also known as *Simpson's rule*, says:

$$\int_a^b c(t)dt \approx \frac{b-a}{3n}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots$$
$$+ 2y_{n-2} + 4y_{n-1} + y_n),$$

with the notation of part (c).

In our example,

$$\int_0^{24} c(t)dt \approx \frac{24-0}{3(24)}\Big(0 + 4(0) + 2(0) + 4(0.1) + 2(0.6)$$
$$+ 4(0.9) + \ldots + 2(0.9) + 4(0.8) + 2(0.5)$$
$$+ 4(0.3) + 0\Big)$$
$$= \tfrac{1}{3}(132.2)$$
$$= 44.1 \text{ (to the nearest tenth).}$$

Again, R ≈ 6.8 liters/minute.

4. EXERCISES

1. Assume the dye monitoring takes place at a branch artery which receives only 1/10 of the blood coming from the heart. What changes are necessary in the analysis contained in Sections 2.2 and 2.3? How does this affect Equations (3) and (4)?

2. Suppose c(t) is measured in milligrams/liter, t in seconds, and D in milligrams. In what units should

$$\int_0^{T_0} c(t)dt$$

be expressed.

3. Suppose that at time t the dye concentration is

$$c(t) = -bt(t - T_0) = -bt^2 + btT_0,$$

where b and T_0 are positive constants.

   a. Graph c(t).

   b. Find R in terms of b, T_0, and the total amount D of dye injected.

4. Suppose c(t) is as shown in Figure 5.

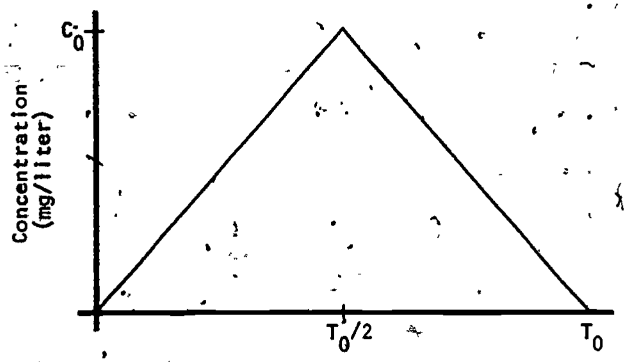


Figure 5. A hypothetical concentration curve.

   a. Find R in terms of T_0, C_0, and the total amount D of dye injected.

   b. How is R affected if (1) T_0 is doubled and C_0 is kept constant? (2) T_0 is halved and C_0 is doubled?

5. Find R in terms of T_0, C_0, and D (the total amount of dye injected) if c(t) is as shown in Figure 6.

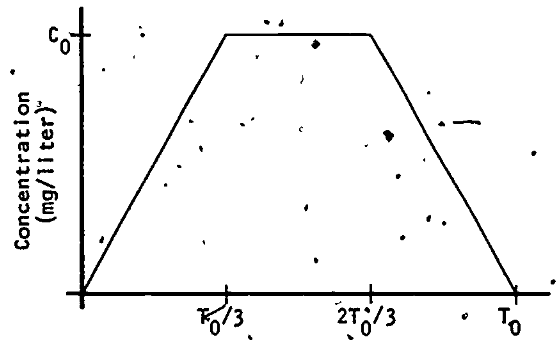


Figure 6. A hypothetical concentration curve.

6. In an attempt to determine cardiac output, 10 milligrams of dye are injected into a main vein near the heart. The dye concentration is monitored at the aorta. The following observations are made:

Time (seconds):	0	1	2	3	4	5	6	7	8
Concentration: (mg/liter)	0	0.1	0.2	0.6	1.2	2.0	3.0	4.2	5.5
Time (seconds):	9	10	11	12	13	14	15	16	17
Concentration: (mg/liter)	6.3	7.0	7.5	7.8	7.9	7.9	7.8	6.9	6.1
Time (seconds):	18	19	20	21	22	23	24	25	26
Concentration: (mg/liter)	5.4	4.7	4.1	3.5	2.8	2.1	2.2	2.1	2.2

a. Plot these observations on graph paper.

b. At what time does recirculation begin?

c. What points would you add to the graph corresponding to A, B, and C in Figure 3?

7. Calculate the cardiac output R from the data in Exercise 6:

a. using Equation (3) directly.

b. using the trapezoidal rule.

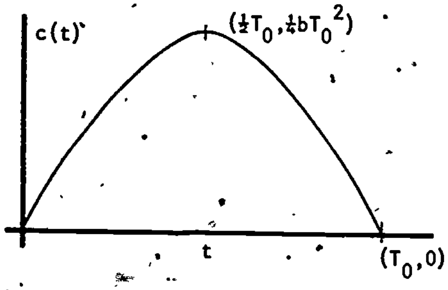
c. using the parabolic rule.

5. ANSWERS TO EXERCISES

1. Throughout Section 2.2, D and R must be replaced by D/10 and R/10, respectively. Equations (3) and (4) are unaffected.

2. Milligram-seconds per liter.

3. a.



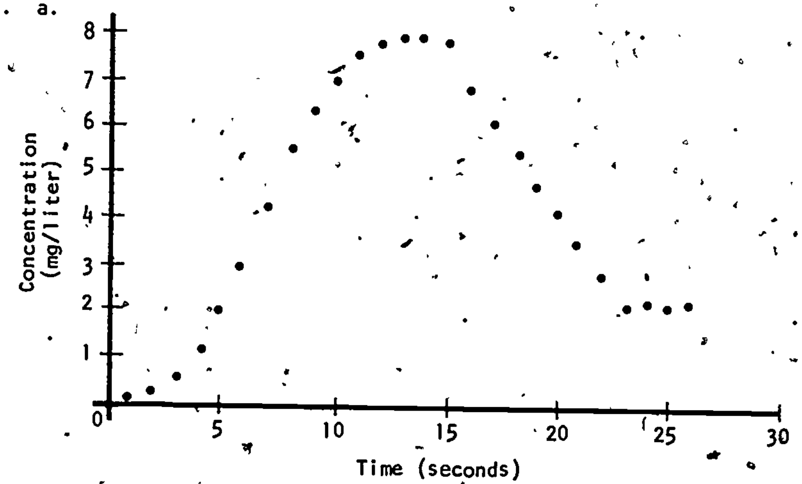
b. $\frac{6D}{bT_0^3}$ $\left(= \frac{D}{\int_0^{T_0} (btT_0 - bt^2)\,dt}\right)$.

4. a. $\frac{2D}{C_0T_0}$ ℓ/sec = $\frac{120D}{C_0T_0}$ ℓ/min $\left(= \frac{D}{A}, \text{ where A is the area of the triangle}\right)$.

b. (1) halved. (2) unchanged.

5. $\frac{3D}{2C_0T_0}$ ℓ/sec = $\frac{90D}{C_0T_0}$ ℓ/min $\left(= \frac{D}{A}, \text{ where A is the area of the trapezoid}\right)$.

6. a.



b. just after 23 seconds.

c. A (24,1.2), B (25,0.7), C (26,0) is one possible answer.

7. Did you remember to replace the last three data points in the table by three points approaching the t-axis? A (24,1.2), B (25,0.7), C (26,0) will do.

a. The denominator of Equation (1) is

$$\sum_{i=1}^{26} c(t_i)\Delta t = \sum_{i=1}^{26} c(t_i)$$

$$= 0 + 0.1 + 0.2 + \ldots + 2.8 + 2.1 + 1.4 + 0.7 + 0$$

$$= 106.7.$$

Therefore,

$$R \approx \frac{D}{106.7} = \frac{10}{106.7} \approx 0.094 \text{ liters/second}$$

$$\approx 5.6 \text{ liters/minute.}$$

b. $$\int_0^{26} c(t)\,dt \approx \frac{26-0}{2(26)}(0 + 0.2 + 0.4 + 1.2 + \ldots + 2.8 + 1.4 + 0)$$

$$= \frac{1}{2}(213.4)$$

$$= 106.7.$$

As in part 7a, $R \approx 5.6$ liters/minute.

c. $$\int_0^{26} c(t)dt \approx \frac{26-0}{3(26)}\left[0 + 4(0.1) + 2(0.2) + 4(0.6) + \dots + 4(2.4) + 2(1.4) + 4(0.7) + 0\right]$$

$$= \frac{1}{3}(320.4)$$

$$= 106.8.$$

Again, $R \approx 5.6$ liters/minute.

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The Project would like to thank Charles Votaw of Fort Hays State University, Hays, Kansas, and Brian J. Winkel of Albion College, Albion, Michigan for their reviews, and all others who assisted in the production of this unit.

This material was field-tested in preliminary form at Russell Sage College, Troy, New York; Northern Illinois University, DeKalb, Illinois; Southern Oregon State College, Ashland, Oregon; California State College at San Bernardino, and; Humboldt State University, Arcata, California.

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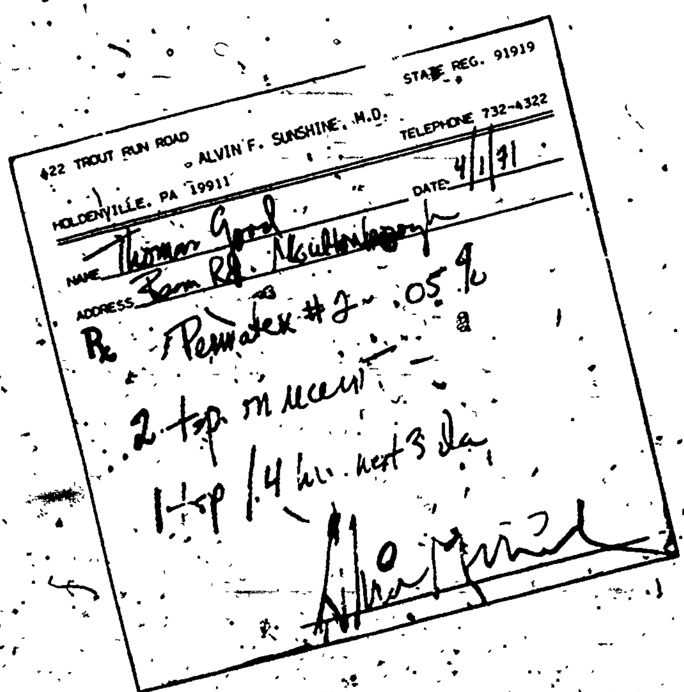
UNIT 72

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

APPLICATIONS OF CALCULUS TO MEDICINE: PRESCRIBING SAFE AND EFFECTIVE DOSAGE



Prepared by: UMAP Staff, based on an earlier unit by
Brindell Horelick and Sinan Koont,
University of Maryland Baltimore County

edc/umap/55 chapel st./newton, mass. 02160

PRESCRIBING SAFE AND EFFECTIVE DOSAGE

9/8/77

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Intermodular Description Sheet: *UMAP Unit 72*

Title: PRESCRIBING SAFE AND EFFECTIVE DOSAGE

Correspondent: Ross L. Finney
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Newton, MA 02160

Review Stage/Date: IV 9/8/77

Classification: MED APPLIC CALC/DRUG DOSE (U72)

Suggested Support Material: Tables of the exponential function or natural logarithms, and/or hand calculator.

Prerequisite Skills:
1. Integrate $C'(t) = kC(t)$.
2. Convert from logarithmic to exponential notation.
3. Compute the sum of the first n terms of a geometric series.
4. Use table of e^x or ln x for calculation.

Output Skills:
1. Describe the accumulated effect of a series of superimposed exponential decay functions beginning at different times.
2. Criticize the fitness of the model above for the description of drug concentration levels in the blood stream.
3. Suggest other phenomena for which the model above might be used.
4. Use the model to determine the desired change in concentration and the interval between doses to keep the concentration between a given upper and lower bound.

Other Related Units:
Introduction to Exponential Functions *(Units 84 - 88, Project UMAP)*. An elementary treatment of the exponential function from definition to integration and differentiation of the function. Many elementary examples and applications.

The following all show additional applications of the exponential function to biology and medicine:

Population Growth and the Logistic Curve *(Unit 68)*
The Digestive Process of Sheep *(Unit 69)*
Epidemics *(Unit 73)*
Tracer Methods in Permeability *(Unit 74)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

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The Project would like to thank Roy E. Collings, Sheldon Gottlieb, Paul Rosenbloom, and Rudy Svoboda for their reviews, and all others who assisted in the production of this unit.

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1. DRUG DOSAGE PROBLEMS

1.1 Gradual Disappearance of a Drug from the Body

The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body. (See Figure 1.)

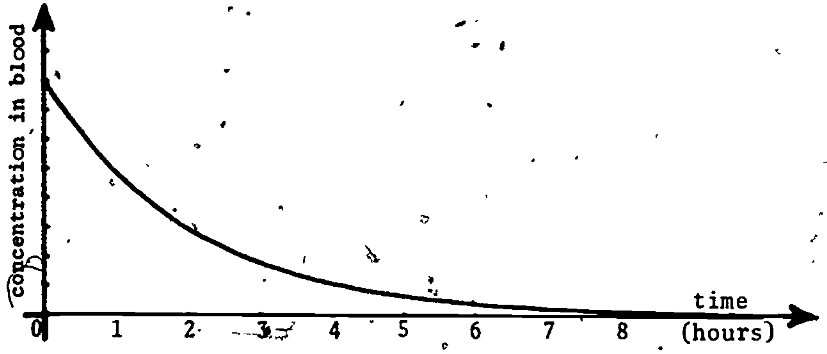

Figure 1. The concentration of a drug in the blood stream decreases with time.

1.2 What is the Effect of Repeated Doses of a Drug?

If doses of a drug were given at regular intervals, what would happen to the concentration of the drug in the blood? Would it behave as shown in Figures 2 or 3, or in some other way?

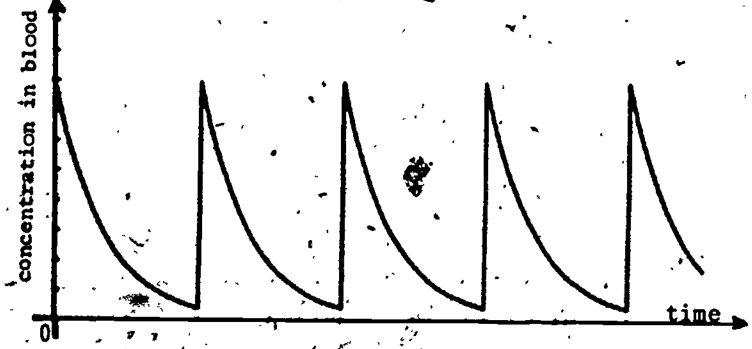

Figure 2. One possible effect of successive doses of a drug.

Figure 3. Another possible effect of successive doses of a drug.

1.3 How to Schedule for a Safe but Effective Drug Concentration

For most drugs there is a concentration below which the drug is ineffective and a concentration above which the drug is dangerous. How can the dose and the time between doses be adjusted to maintain a safe but effective concentration?

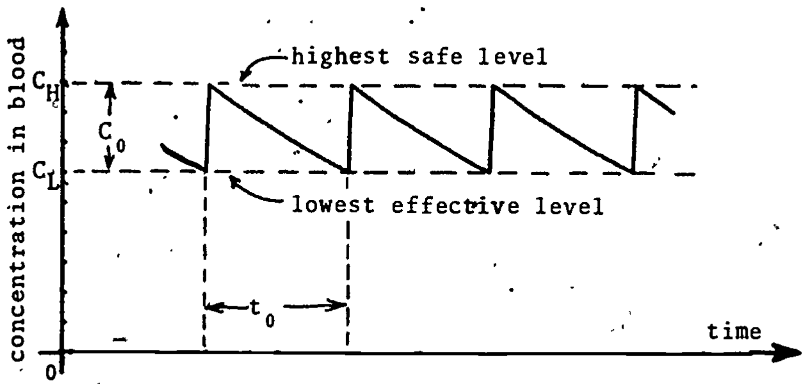

Figure 4. Safe but effective levels.

C_0 = change in concentration produced by one dose

t_0 = time between doses

2. A MATHEMATICAL MODEL OF DRUG CONCENTRATION

To give a reasonable answer to the two questions above, we develop formulas from which we can compute drug concentration as a function of time. The development depends on two assumptions. The first assumption is quite reasonable. The second assumption is reasonable in some circumstances but not reasonable in others, and limits the application of the model we are about to describe.

2.1 The First Assumption

The first assumption, one that is borne out by clinical evidence, is this: Whatever the mode of elimination, the decrease in the concentration of the drug in the blood stream will be proportional to the concentration itself. If the concentration were doubled, the rate of elimination is doubled also. If the concentration is reduced by a third, the rate of elimination is reduced by a third. The amount being eliminated at any given instant is a fixed fraction of the amount still present.

To model this assumption mathematically, we assume that the concentration of drug in the blood at time t is a function $C(t)$ whose derivative $C'(t)$ is given by the formula

(1) $$C'(t) = -kC(t) .$$

In this formula k is a positive constant, called the *elimination constant* of the drug. Notice that $C'(t)$ is negative, as it should be if it is to describe a decreasing concentration.

2.2 Units of Measurement

We usually measure the quantities in Equation (1) in the following units:

t	hours (hr)
C(t)	milligrams per milliliter of blood (mg/ml)
C'(t)	$\frac{\text{mg/ml}}{\text{hr}}$ or $\text{mg ml}^{-1}\,\text{hr}^{-1}$
k	hr^{-1}

2.3 Drug Concentration Decay as a Function of Time

If we happen to know the concentration of a drug at a particular time, then we can predict the concentration at any later time by integrating both sides of Equation (1). Specifically, if C_0 is the concentration at $t = 0$, then we can calculate $C(t)$ for every $t > 0$ in the following way.

First rewrite Equation (1) to get

$$\frac{C'(t)}{C(t)} = -k$$

Then integrate from 0 to t:

$$\int_0^t \frac{C'(t)}{C(t)}\,dt = \int_0^t -k\,dt$$

$$\ln \frac{C(t)}{C_0} = -kt$$

(2) $$C(t) = C_0 e^{-kt}.$$

Exercise 1. Starting with Equation (1), carry out in detail the steps that lead to Equation (2). [*S-1*]*

To obtain the concentration at time $t > 0$, we multiply the *initial concentration* C_0 by e^{-kt}. The graph of $C(t) = C_0 e^{-kt}$ looks like the one in Figure 5.

* This reference means that there is additional explanation material available in the Special Assistance Supplement at the back of the unit.

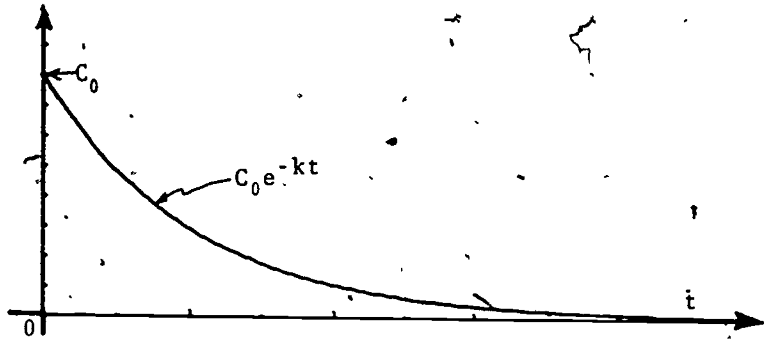

Figure 5. Exponential model for decay of drug concentration with time.

Exercise 2. Suppose that the elimination constant of drug A is $k = 0.2\ hr^{-1}$, and that of drug B is $k = 0.1\ hr^{-1}$. Given the same initial concentration, which drug will have the lower concentration 4 hours later?

2.4 The Second Assumption

Having made an assumption about how drug concentrations decrease with time, we need a companion assumption about how they increase again when drugs are administered. What we shall assume is that when a drug is taken, it is diffused so rapidly throughout the blood that the graph of the concentration for the absorption period is, for all practical purposes, vertical. That is, we assume an instantaneous rise in concentration whenever a drug is administered. This assumption may not be as reasonable for a drug taken by mouth as it is for a drug that is injected directly into the blood stream. [*S-2*]

By combining Assumptions 1 and 2, we arrive at the graphs in Figures 2 through 4.

3. DRUG ACCUMULATION WITH REPEATED DOSES

3.1 Quantities to be Calculated

What happens to the concentration $C(t)$ if a dose capable of raising the concentration by C_0 mg/ml each time it is given is administered at fixed time intervals of length t_0? Does the drug accumulate? If so, to what level? The next graph shows one possibility, and suggests a number of quantities that one should know how to calculate. [*S-3*]

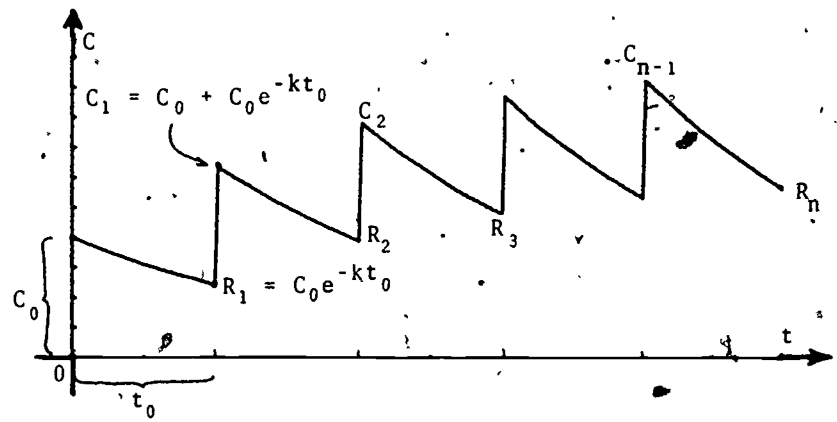

Figure 6. One possible effect of repeating equal doses.

3.2 Calculation of Residual Concentration

If we let C_{i-1} be the concentration at the beginning of the i-th interval and R_i the *residual concentration* at the end of it, we can easily obtain the following table.

TABLE I

CALCULATION OF RESIDUAL CONCENTRATION OF DRUG

i	C_{i-1}		R_i
1	C_0	multiply by e^{-kt_0} $\longrightarrow$	$C_0e^{-kt_0}$
		add C_0	
2	$C_0 + C_0e^{-kt_0}$		$C_0e^{-kt_0} + C_0e^{-2kt_0}$
3	$C_0 + C_0e^{-kt_0} + C_0e^{-2kt_0}$		$C_0e^{-kt_0} + C_0e^{-2kt_0} + C_0e^{-3kt_0}$
.			
.			
.			
n			$C_0e^{-kt_0} + \dots + C_0e^{-nkt_0}$

From the table we see that

$$(3) \qquad R_n = C_0e^{-kt_0} + \dots + C_0e^{-nkt_0}$$

is the sum of the first n terms of a geometric series. The first term is $C_0e^{-kt_0}$ and the common ratio is e^{-kt_0}. Accordingly,

$$(4) \qquad R_n = C_0e^{-kt_0}\left(\frac{1 - e^{-nkt_0}}{1 - e^{-kt_0}}\right).$$

Exercise 3. Calculate R_1 and R_{10} for $C_0 = 1$ mg/ml, $k = 0.1$ hr^{-1} and $t_0 = 10$ hr. (To compare R_{10} with the result of Exercise 4, assume that the data are given to unlimited accuracy.)

To return to Equation (4), notice that the number e^{-nkt_0} is close to 0 when n is large. In fact, the larger n becomes, the closer e^{-nkt_0} gets to 0. [S-4] As a result, the sequence of R_n's has a limiting value, which we call R:

$$R = \lim_{n\to\infty} R_n = \frac{C_0e^{-kt_0}}{1 - e^{-kt_0}}$$

$$= \frac{C_0}{e^{kt_0} - 1} \qquad [S-5]$$

If a dose that is capable of raising the concentration by C_0 mg/ml is repeated at intervals of t_0 hours, then the limiting value R of the residual concentrations is given by the formula

$$(5) \qquad R = \frac{C_0}{e^{kt_0} - 1}.$$

The number k is the elimination constant[7] of the drug.

Exercise 4. Use Equation (5) to find R for the values of C_0, k, and t_0 given in Exercise 3. How good an estimate of R is R_{10}?

3.3 Results for Long Intervals Between Doses

The only meaningful way to examine what happens to the residual concentration, R, for different intervals, t_0, between doses is to look at R in comparison with C_0, the change in concentration due to each dose. [S-6] To make this comparison, we form the dimensionless ratio R/C_0 by dividing both sides of Equation (5) by C_0:

$$(6) \qquad \frac{R}{C_0} = \frac{1}{e^{kt_0} - 1}$$

Equation (6) tells us that R/C_0 will be close to 0 whenever the time t_0 between doses is long enough to make $e^{kt_0} \gg 1$. As for the intermediate values of R_n, we can see from Table I that each R_n is obtained from R_{n-1} by adding a positive quantity ($C_0e^{-nkt_0}$). This means that all the R_n's are positive, because R_0 is positive. It also means that R is larger than each of the R_n's. In symbols,

(7) $\quad 0 < R_n < R$

for all n.

The implication of this for drug dosage is that whenever R is small, the R_n's are even smaller. In particular, whenever t_0 is long enough to make $e^{kt_0} \gg 1$, the residual concentration from each dose is almost nil. The various administrations of the drug are then essentially independent, and the graph of C(t) looks like the one in Figure 7.

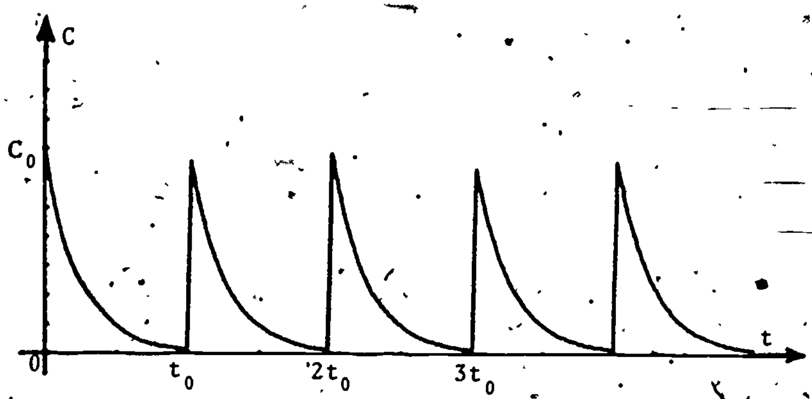

Figure 7. Drug concentration for long intervals between doses.

3.4 Results for Short Intervals Between Doses

If, however, the length of time t_0 between doses is so short that e^{kt_0} is not very much larger than 1, then Equation (6) shows that R/C_0 will be significantly greater than 1. The concentration will build up with repeated doses until it stabilizes into an oscillation between R and $R + C_0$. [S-7] See Figure 8 on page 10.

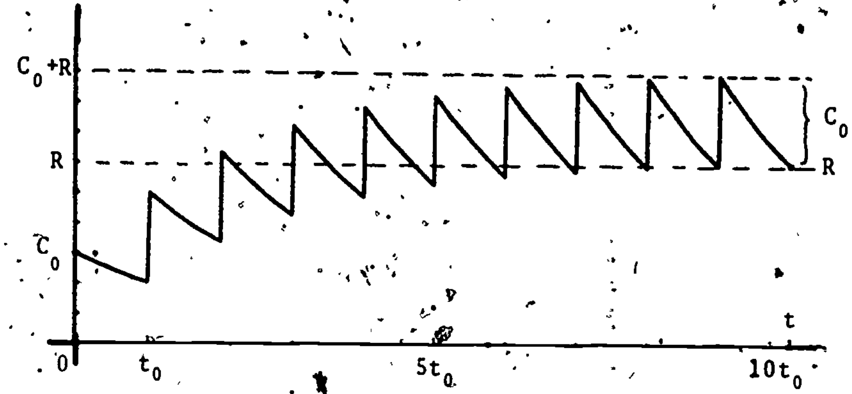

Figure 8. Buildup of drug concentration when interval between doses is short.

4. DETERMINING A DOSE SCHEDULE FOR SAFE BUT EFFECTIVE DRUG CONCENTRATION

4.1 Calculating Dose and Interval

Suppose that a drug is known to be ineffective below a concentration C_L and harmful above some higher concentration C_H. Is it possible to find values of C_0 and t_0 that will produce a concentration C(t) that is safe (not above C_H) but still effective (not below C_L)? To whatever extent the model is valid the answer is YES, and Figure 8 gives us the clue for how to start.

We begin by looking for values of C_0 and t_0 that make

(8) $\quad R = C_L \quad \text{and} \quad C_0 + R = C_H .$

Subtraction then yields

(9) $\quad C_0 = C_H - C_L .$

When these values of R and C_0 are substituted in Equation (5), we find that

(10) $$C_L = \frac{C_H - C_L}{e^{kt_0} - 1}$$

We then solve for e^{kt_0} to obtain

$$e^{kt_0} = \frac{C_H}{C_L}. \tag{11}$$

When we take the logarithm of both sides of (11) and divide both sides of the resulting equation by k, we learn that

$$t_0 = \frac{1}{k} \ln \frac{C_H}{C_L}. \tag{12}$$

Exercise 5. Solve Equation (10) for e^{kt_0} to obtain Equation (11).

Exercise 6. Solve Equation (11) for t_0 to obtain Equation (12).

4.2 Reaching an Effective Level Rapidly

To reach an effective level rapidly, administer a dose, often called a *loading dose*, that will immediately produce a blood concentration of C_H mg/ml. This can be followed every $t_0 = \frac{1}{k} \ln \frac{C_H}{C_L}$ hours by a dose that raises the concentration by $C_0 = C_H - C_L$ mg/ml.

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5. EXERCISES

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7. State two reasons why the model suggested in this unit seems to be a good one.

8. Suggest other phenomena for which the model described in the text might be used.

9. a) If $k = 0.05$ hr^{-1}, and the highest safe concentration is e times the lowest effective concentration, find the length of time between repeated doses that will assure safe but effective concentrations.

   b) Does (a) give enough information to determine the size of each dose?

10. Suppose that $k = 0.01$ hr^{-1} and $t_0 = 10$ hrs. Find the smallest n such that $R_n > \frac{1}{2}R$.

11. Given $C_H = 2$ mg/ml, $C_L = 0.5$ mg/ml, and $k = 0.02$ hr^{-1}, suppose that concentrations below C_L are not only ineffective but also harmful. Determine a scheme for administering this drug (in terms of concentration, and times of dosage.)

12. Suppose that $k = 0.2$ hr^{-1} and that the smallest effective concentration is 0.03 mg/ml. A single dose that produces a concentration of 0.1 mg/ml is administered. Approximately how many hours will the drug remain effective?

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6. ANSWERS TO EXERCISES

For detailed solutions, see the sections of the Special Assistance Supplement referred to in the brackets after each answer.

1. See [*S-1*].
2. $C_A < C_B$, [*S-8*]
3. $R_1 = 0.36788$; $R_{10} = 0.58195$ [*S-9*]
4. $R = 0.58198$; R and R_{10} agree to four decimal places. [*S-10*]
5. See [*S-11*].
6. See [*S-12*].
7. See [*S-13*].
8. See [*S-14*].
9. a) $t_0 = 20$ hours [*S-15*]

   b) No; but the first dose could be as large as 2.72 times the minimum effective dose. [*S-16*]
10. $n = 7$ [*S-17*]

11. Give an initial dose that will bring the concentration to 2 mg/ml. Follow this every 69 hours by a dose that will raise the concentration by 1.5 mg/ml. [*S-18*]

12. About 6 hours. [*S-19*]

WORK SPACE

7. MODEL EXAM

Though you will seldom be asked to take an exam on a single unit, an exam on a cluster of units is usually made up from a pool of questions similar to those below.

1. Assume that the decay in concentration of a drug injected into the blood stream is given by $C = C_0 e^{-kt}$, and that the drug is given in such a way that each dose makes an instantaneous change in the level of concentration of C_0. Write an expression that gives the residual concentration after 3 doses spaced t_0 hours apart; i.e., find the concentration at time $3t_0$.

2. State at least one deficiency of the model described in this unit.

3. Suggest a situation, different from that described in the text, to which this model might be applied.

4. A certain dose of a drug is capable of raising the blood concentration of the drug by 0.5 mg/ml each time it is taken. The decay constant for the drug is 0.1 hr^{-1}; doses are given every four hours.

   a. Find the concentration of the drug just before the third dose.

   b. Find the concentration just after the third dose.

5. Given the drug above and the knowledge that the highest safe level of concentration is 0.9 mg/ml and the lowest effective level is 0.6 mg/ml, devise a reasonable schedule (dose size and time interval) for administering the drug.

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8. ANSWERS TO MODEL EXAM

1. See Table I, page 7 of text.

2. A drug taken orally, such as aspirin, certainly takes a finite time to diffuse into the blood stream. Thus, the assumption of an instantaneous rise in the level of concentration is not realistic for such drugs.

3. The concentration of active developer in a photographic developing solution might vary in a similar way each time replenisher is added to the solution. See [S-14] for other examples.

4a. 0.5598 mg/ml

4b. 1.0598 mg/ml

5. t_0 = 4.05 hr ; C_0 = 0.3 mg/ml .

The first dose could be three times this amount.

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9. SPECIAL ASSISTANCE SUPPLEMENT

[S-1] *Answer to Exercise 1:*

Integration of $\int_0^t \frac{C'(t)}{C(t)}\,dt = \int_0^t -kt$

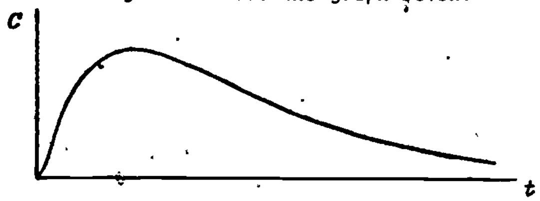
yields $\ln C(t) - \ln C(0) = -kt$

and, letting $C(0) = C_0$, $\ln \frac{C(t)}{C_0} = -kt$

or $\frac{C(t)}{C_0} = e^{-kt}$

and finally, $C(t) = C_0 e^{-kt}$.

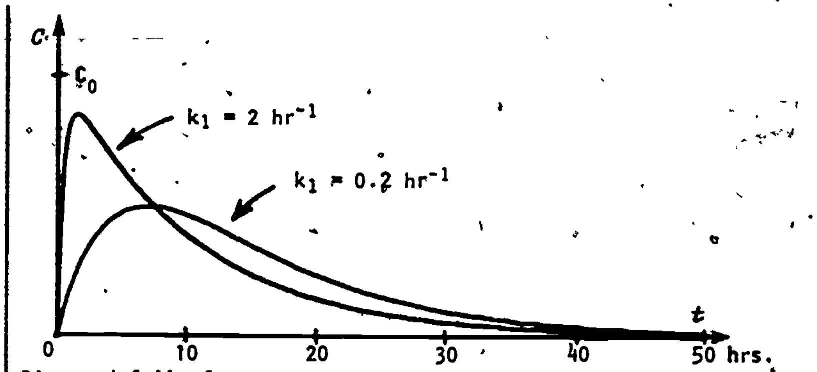
[S-2] If the time for the drug to diffuse through the body sufficiently to affect the desired organ is appreciable compared to the time between doses, then the assumption of a vertical rise in the graph of concentration is a poor approximation. Under these conditions, the graph of concentration versus time for a single dose might resemble the graph below:



After completing this unit, try to sketch how a series of such doses might accumulate. If you would like to pursue this further, the equation of the graph above is

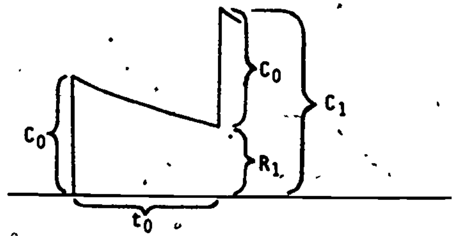
$$C(t) = C_0\left(\frac{k_1}{k_2 - k_1}\right)\left(e^{-k_1 t} - e^{-k_2 t}\right).$$

This equation is plotted at the top of page SA-2 for two different values of the *diffusion constant* k_1. The *elimination constant* k_2 is 0.1 hr^{-1} for both curves.



Rise and fall of concentration when diffusion time is significant.

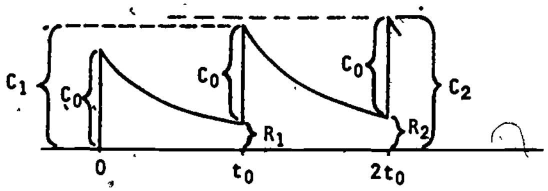
[S-3] Looking at the first two steps of the diagram:



we see that $C_1 = C_0 + R_1$, but $R_1 = C_0 e^{-kt_0}$.

Therefore, $C_1 = C_0 + C_0 e^{-kt_0}$.

Looking at the third step:



we see that $C_2 = C_0 + R_2$, but $R_2 = C_1 e^{-kt_0}$

$$= \left(C_0 + C_0 e^{-kt_0}\right) e^{-kt_0}$$

$$= C_0 e^{-kt_0} + C_0 e^{-2kt_0}.$$

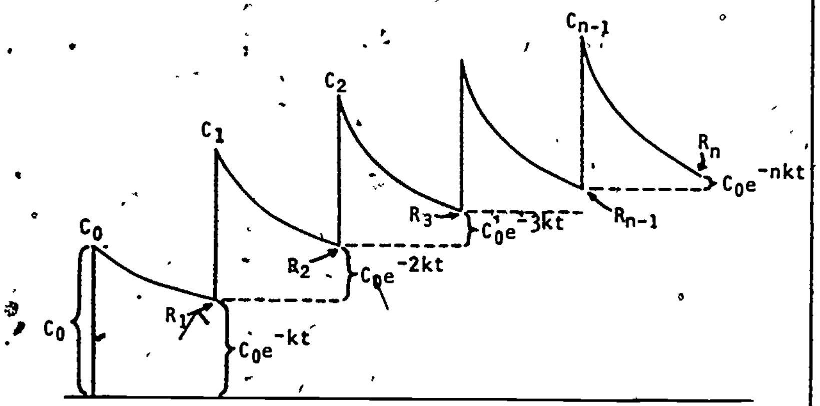
Therefore, $C_2 = C_0 + \left(C_0 e^{-kt_0} + C_0 e^{-2kt_0}\right)$

$$= C_0 + C_0 e^{-kt_0} + C_0 e^{-2kt_0}.$$

We reach the results given in Table I (page 7) by continuing this process.

[*S-4*]

The term C_0e^{-nkt} is the increase in the residual value at the beginning of step n.



Note that at the end of each dose period the residual concentration is greater than the last residual amount by a smaller and smaller increment.

[*S-5*]

Beginning with Equation (4), $R_n = C_0e^{-kt_0}\left(\frac{1 - e^{-nkt_0}}{1 - e^{-kt_0}}\right)$

Use the fact that $\lim\limits_{n\to\infty} e^{-nkt_0} = 0$. $R = \lim\limits_{n\to\infty} R_n = C_0e^{-kt_0}\left(\frac{1}{1 - e^{-kt_0}}\right)$

Eliminate parentheses. $= \frac{C_0e^{-kt_0}}{1 - e^{-kt_0}}$

Multiply numerator and denominator by e^{kt_0}. $= \frac{C_0}{e^{kt_0} - 1}$

[*S-6*]

There are two pitfalls in looking at a value of R of .001 and concluding that it is small. First of all, we do not know what .001 means physically. It might mean .001 kg/ml, which could be a lethal concentration of many drugs, or it could mean .001 mg/ml, which might be an insignificant concentration. The number .001 by itself is devoid of physical meaning or magnitude. The second pitfall is that while .001 mg/ml might be an insignificant concentration of one drug, it might be a very high dose of another drug.

We can avoid both these pitfalls by not looking at the absolute values of R but only at its size in comparison to C_0 by taking the ratio of R to C_0. Thus, if R is .001 g/ml and C_0 is .0002 g/ml, then the ratio

$$\frac{R}{C_0} = \frac{.001 \text{ g/ml}}{.0002 \text{ g/ml}} = 5$$

and we see that R is several times larger than C_0.

[*S-7*]

As R_n becomes larger, the concentration C_n after each dose becomes larger. The loss during the time period after each dose increases with larger C_n (assumption 1, page 3). Finally, the drop in concentration after each dose becomes imperceptibly close to the rise in concentration C_0 due to each dose. When this condition prevails (the loss in concentration equalling the gain) the concentration will oscillate between R at the end of each period and $R + C_0$ at the start of each period.

[*S-8*] *Answer to Exercise 2:*

$$C_A = C_0e^{-k_At_0} = C_0e^{-(0.2\text{ hr}^{-1})(4\text{ hr})} = C_0e^{-0.8}$$

$$C_B = C_0e^{-k_Bt_0} = C_0e^{-(0.1\text{ hr}^{-1})(4\text{ hr})} = C_0e^{-0.4}$$

$e^{-0.8} < e^{-0.4}$; therefore, $C_A < C_B$

[*S-9*] *Answer to Exercise 3:*

$$R_n = C_0e^{-kt_0}\left(\frac{1 - e^{-nkt_0}}{1 - e^{-kt_0}}\right)$$

$C_0 = 1$ mg/ml; $k = 0.1\text{ hr}^{-1}$; $t_0 = 10$ hr.

$$e^{-kt_0} = e^{-(0.1\text{ hr}^{-1})(10\text{ hr})} = e^{-1} \approx 0.36788$$

$R_1 \approx C_0(0.36788)(1) = \underline{0.36788\text{ mg/ml}}$

$$R_{10} \approx C_0(0.36788)\left(\frac{1 - e^{-10}}{1 - e^{-1}}\right) \approx C_0(0.36788)\left(\frac{1 - .00005}{1 - .36788}\right)$$

$$= C_0(0.36788)\left(\frac{.99995}{.63212}\right) = C_0(0.36788)(1.58190) \approx \underline{0.58195\text{ mg/ml}}$$

[S-10] *Answer to Exercise 4:*

$$R = \frac{C_0}{e^{kt_0} - 1}$$

$C_0 = 1$ mg/ml ; $k = 0.1\ hr^{-1}$; $t_0 = 10$ hr .

$$e^{kt_0} = e^{(0.1\ hr^{-1})(10\ hr)} = e^1 = 2.71828$$

$$R = \frac{C_0}{2.71828 - 1} = \frac{C_0}{1.71828} = (0.58198)C_0 = \underline{0.58198\ mg/ml}$$

[S-11] *Answer to Exercise 5:*

Given $C_L = \frac{C_H - C_L}{e^{kt_0} - 1}$; solve for e^{kt_0}.

$$e^{kt_0} - 1 = \frac{C_H - C_L}{C_L}$$

$$e^{kt_0} = \frac{C_H - C_L}{C_L} + 1 = \frac{C_H}{C_L} - \frac{C_L}{C_L} + 1 = \underline{\frac{C_H}{C_L}}$$

[S-12] *Answer to Exercise 6:*

Given $e^{kt_0} = \frac{C_H}{C_L}$, solve for t_0.

Take the logarithm of each side: $\ln(e^{kt_0}) = \ln\left(\frac{C_H}{C_L}\right)$

$$kt_0 = \ln\left(\frac{C_H}{C_L}\right)$$

$$\underline{t_0 = \frac{1}{k}\ln\left(\frac{C_H}{C_L}\right)}$$

[S-13] *Answer to Exercise 7:*

The model appears to be a good one because it is in accord with several common practices of prescribing drugs; i.e., it accounts for the practice of prescribing an initial dose several times larger than the succeeding periodic doses.

The model also provides quantitatively for the prediction of concentration levels under varying conditions of dose rates in terms of a single easily measured parameter, k.

What else would you need to know before you could actually prescribe a particular dose rate?

[S-14] *Answer to Exercise 8:*

Another phenomenon to which the model could be applied is the consumption of alcohol. How often could a can of beer or a cocktail be consumed and still not produce a concentration of alcohol in the blood at which a person is legally drunk?

A very different phenomenon to which this model might also be applied is the burning of an old-fashioned wood stove. Here the rate of burning or heat output is proportional to the charge of wood placed in the stove. There is a maximum safe level of burning to be reached as soon as possible, and a lower level required to keep the cabin up to minimum comfort. As the wood charge is consumed, the rate of burning, heat output, and consumption of wood decrease.

Sketch possible graphs of heat output versus time through several charges of wood. (See Figures 7 - 8, "Heat Output of a Franklin Stove", p. 59 of Jay W. Shelton, Woodburners' Encyclopedia, Vermont Crossroads Press, Waitsfield, VT 05673, 1976.)

[S-15] *Answer to Exercise 9a:*

$$t_0 = \frac{1}{k}\ln\frac{C_H}{C_L}$$

given $\frac{C_H}{C_L} = e$ and $k = 0.050\ hr^{-1}$

$$t_0 = \frac{1}{.05\ hr^{-1}}\ln(e) = (20\ hr)(1) = \underline{20\ hr}$$

[S-16] *Answer to Exercise 9b:*

No, not enough information is given to determine the actual size of each dose. We have only the ratio of the highest safe concentration to the lowest effective concentration. If the value of one of these limits were known, the other could be calculated and the difference in concentration to be produced by one dose determined. However, the actual dose required to produce this change in concentration would depend on the volume of blood in the patient and how quickly the drug would spread through the entire blood system.

[S-17] *Answer to Exercise 10:*

Given: $R_n = C_0 e^{-kt_0}\left(\dfrac{1 - e^{-nkt_0}}{1 - e^{-kt_0}}\right)$

and $R = \dfrac{C_0}{e^{kt_0} - 1}$, find n for $R_n > \frac{1}{2}R$.

The above implies $C_0 e^{-kt_0}\left(\dfrac{1 - e^{-nkt_0}}{1 - e^{-kt_0}}\right) > \left(\frac{1}{2}\right)\dfrac{C_0}{(e^{kt_0} - 1)}$.

Some algebra leads to $-e^{-nkt_0} > \left(\frac{1}{2} - 1\right)$.

Then $e^{-nkt_0} < \left(1 - \frac{1}{2}\right)$,

and $e^{-nkt_0} < \frac{1}{2}$;

but also given were $k = 0.010\ \text{hr}^{-1}$ $t_0 = 10\ \text{hr}$

so $e^{-nkt_0} = e^{-n(.01\ \text{hr}^{-1})(10\ \text{hr})} = e^{-0.1n}$.

Therefore, $e^{-0.1n} < \frac{1}{2}$ and $e^{0.1n} > 2$.

Taking the logarithm of each side: $0.1n \ln e > \ln 2$

or $0.1n > \ln 2$

$n > 10 \ln 2$

$n > 6.9$.

Therefore, the smallest n must be 7.

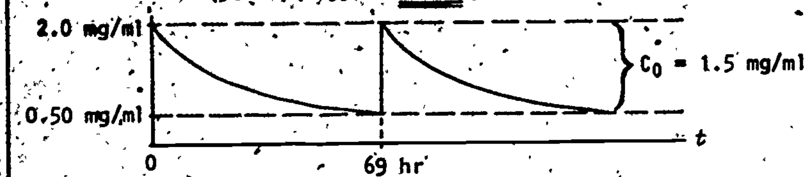
[S-18] *Answer to Exercise 11:*

Given $t_0 = \dfrac{1}{k} \ln \dfrac{C_H}{C_L}$

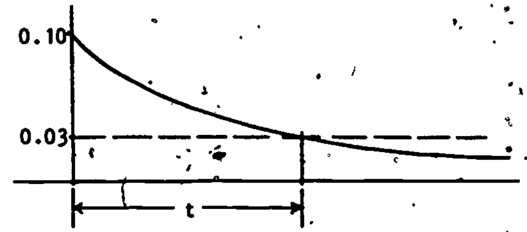
and, $k = 0.020\ \text{hr}^{-1}$ $C_H = 2.0\ \text{mg/ml}$ $C_L = 0.50\ \text{mg/ml}$.

Then, $t_0 = \dfrac{1}{.02\ \text{hr}^{-1}} \ln \dfrac{2\ \text{mg/ml}}{0.5\ \text{mg/ml}} = (50\ \text{hr}) \ln 4$

$= (50\ \text{hr})(1.39) = \underline{69\ \text{hr}}$.



[S-19] *Answer to Exercise 12:*



$t = \dfrac{1}{k} \ln \dfrac{C_H}{C_L} = \dfrac{1}{0.2\ \text{hr}^{-1}} \ln \left(\dfrac{0.10\ \text{mg/ml}}{0.03\ \text{mg/ml}}\right)$

$\approx (5\ \text{hr}) \ln(3.33) \approx (5\ \text{hr})(1.20) = \underline{6.0\ \text{hr}}$.

UMAP MODULE 73

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

Birkhäuser Boston Inc.
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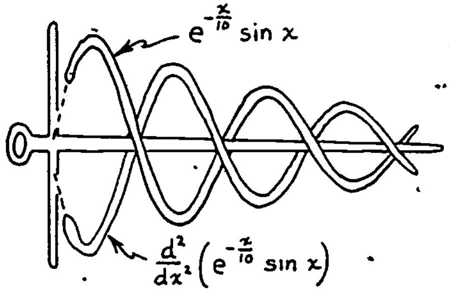
Epidemics

by Bridell Horelick and Sinan Koont



Applications of Calculus to Medicine

EPIDEMICS

by

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 73*

Title: EPIDEMICS

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Review Stage/Date: IV 5/20/80

Classification: APPL CALC/MEDICINE

References:

Bailey, T.J. (1967), The Mathematical Approach to Biology and Medicine, John Wiley and Sons, London.
Batschelet, E. (1971), Introduction to Mathematics for Life Scientists, Springer Verlag, New York.
Olinick, M. (1978), An Introduction to Mathematical Methods in the Social and Life Sciences, Addison-Wesley, Reading, Massachusetts.

Prerequisite Skills:

1. Understand the meaning of $x'(t)$.
2. Know how to antidifferentiate
$$\int \frac{x'(t)}{x(t)}\,dt.$$
3. Be able to solve $\ln y = z$ for y.
4. Know how to determine if $x(t)$ is decreasing.
5. Be able to compute
$$\frac{d}{dx}\,e^u.$$
6. Be able to determine if a graph is concave downward.
7. Know the Maclaurin Series for e^{-x}.
8. Know that a partial sum of a convergent alternating series differs from the series sum by at most the magnitude of the first discarded term.

This unit is intended for calculus students with an active interest in medicine and some background knowledge of biology. Typically this background knowledge may be represented by concurrent registration in a college level introductory biology course.

Output Skills:

1. Be able to describe quantitatively how the course of an epidemic and its control may be modeled mathematically.
2. Be able to criticize the model described herein, naming some strengths and weaknesses.

Other Related Units:

Measuring Cardiac Output *(Unit 71)*
Prescribing Safe and Effective Dosage *(Unit 72)*
Tracer Methods in Permeability *(Unit 74)*

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EPIDEMICS

1. STATEMENT OF THE PROBLEM

An epidemic is the spread of an infectious disease through a community, affecting a significant fraction of the population of the community. Typically, the number of infective persons might rise sharply at first, and then taper off as the epidemic runs its course or is brought under control. Figures 1 and 2 illustrate this.

There are two kinds of steps health authorities can take to control an epidemic. They can attempt to cure those who are sick, and they can attempt to prevent the disease from spreading. Usually, they will attempt both.

Since the disease is infectious, it seems reasonable that reducing contact between those who have or carry it and those who are susceptible to it will help prevent its spread. Another means of controlling some epidemics is to eradicate the source of infection, for example, rodent populations or mosquito breeding grounds. However, this will be of no relevance in the model we shall consider.

Reducing contact may be accomplished by reducing the number of infective persons in any of several ways depending on the nature of the disease and of the community. For example, they may be quarantined, they may be cured, assuming recovery brings immunity and does not leave them as carriers, or, in case of their death, their bodies may be quickly removed.

At what rate will this reduction have to be accomplished to keep the epidemic under control? Can we predict what portion of the community will eventually catch the disease before the epidemic is over?

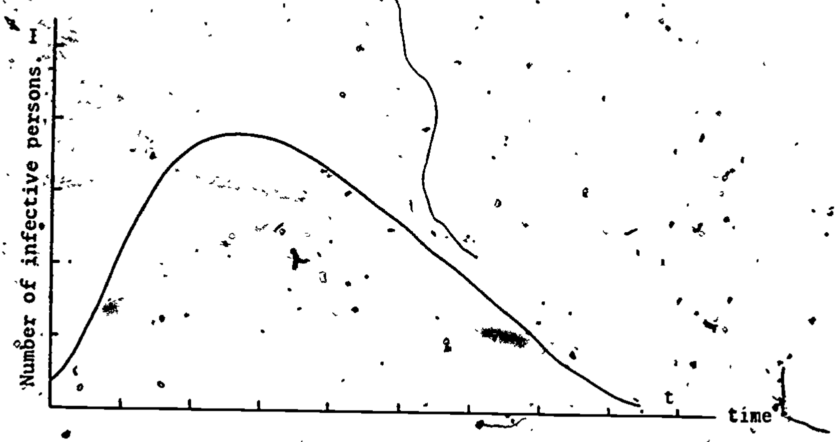


Figure 1. Typical course of an epidemic.

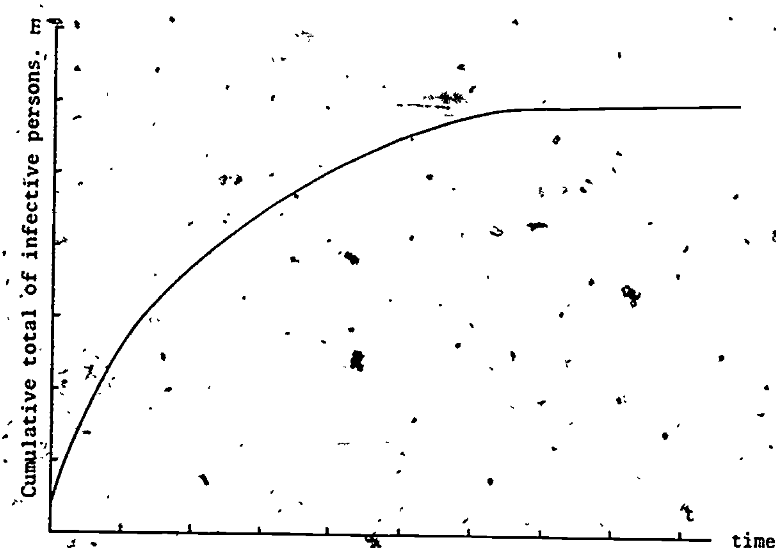


Figure 2. Typical cumulative effect of an epidemic.

2. THE MODEL

2.1 Basic Assumptions

We shall make the following assumptions about the epidemic we are modelling:

(a) The epidemic begins when a small number of infected persons (perhaps returning from a trip abroad) are introduced into a community.

(b) No one in the community has had the disease before, and no one is immune.

(c) The epidemic is spread only by direct contact between a diseased person, or a carrier, and a susceptible person.

(d) All persons who have had the disease and recovered are immune. However, some recovered persons may be carriers.

A simplified description of the progress of the epidemic is shown schematically in Figure 3. In that figure we assume that each person is in exactly one group at a time, and that changes are in the direction of the arrows only. For example, a quarantined person will not be released if he is still a carrier.

We shall also assume, for simplicity, that the total population of groups S, I, and P does not change during or shortly after the epidemic. This means, for example, that there are no births, no deaths from other causes, and no new people moving into the community. This assumption is never realized, of course, but it is a reasonable approximation to the truth if the epidemic is short.

S	I	P
"Susceptibles"	"Infectives"	"Post-infectives"
Persons who have never had disease and are not immune	Diseased persons still at large ——— Recovered persons who are carriers	Recovered persons now immune ——— Quarantined persons ——— Removed bodies
size at time t S(t)	size at time t I(t)	size at time t P(t)

Figure 3. Progress of an epidemic.

2.2 Definition of the Variables

Let us call $t = 0$ the time at which the epidemic begins, and let N = the total population. Let $S(t)$, $I(t)$, and $P(t)$ be the number of persons in groups S, I, and P respectively at any time t. Depending on the nature of the epidemic, t might be measured in hours, days, weeks, or even months. Our basic assumptions tell us among other things, that $S(0) = N$ (the total population), that $P(0) = 0$, and that during and shortly after the epidemic

$$S(t) + I(t) + P(t) = N \quad (1)$$

The number who have caught the disease by time t is $I(t) + P(t)$, or $N - S(t)$.

2.3 The Spread of the Disease

Each time a person catches the disease $S(t)$ decreases by one and $I(t)$ increases by one. How frequently this happens is determined by how frequently a person in group S comes in contact with one in group I.

What is a reasonable formula for the frequency of these contacts? We would expect it to vary directly with $S(t)$ and also with $I(t)$. For example, we would expect that tripling the number of infectives while holding the number of susceptibles fixed would triple the contact frequency. Similarly, we would expect that tripling the number of susceptibles while holding fixed the number of infectives would also triple the contact frequency. The simplest formula which varies directly with S(t) and with I(t) is kS(t)I(t), where k is a positive constant.

We shall assume that a fixed fraction of these contacts results in the disease being transmitted from the infective to the susceptible. Then the frequency with which S(t) decreases by one is $k_1S(t)I(t)$ for some new constant $k_1 (0 < k_1 < k)$. In other words, the rate at which S(t) is changing is $-k_1S(t)I(t)$.

2.4 A Smooth Approximation

The last sentence of Section 2.3 seems to be a statement about the derivative ("rate of change") of S(t). Strictly speaking, S(t) cannot have a derivative, since its graph is not smooth. It must be a step function (Figure 4), with each step being of height one. But it is easy to draw a smooth curve, as shown, which is an excellent approximation to S(t). It will never differ from the true value by more than one, which is *assumed* to be a tiny error compared to the total population. This smooth curve has a derivative, and for it we have

$$(2) \qquad S'(t) = -k_1S(t)I(t)$$

from some constant $k_1 > 0$.

2.5 Removal of Infectives

It seems reasonable that the rate at which victims die from the disease, and thus enter group P, is proportional to the number of infectives at any given time. We shall extend this to an *assumption* that the rate of transfer from group I to group P, for any reason, is proportional to the size of group I. That is, after

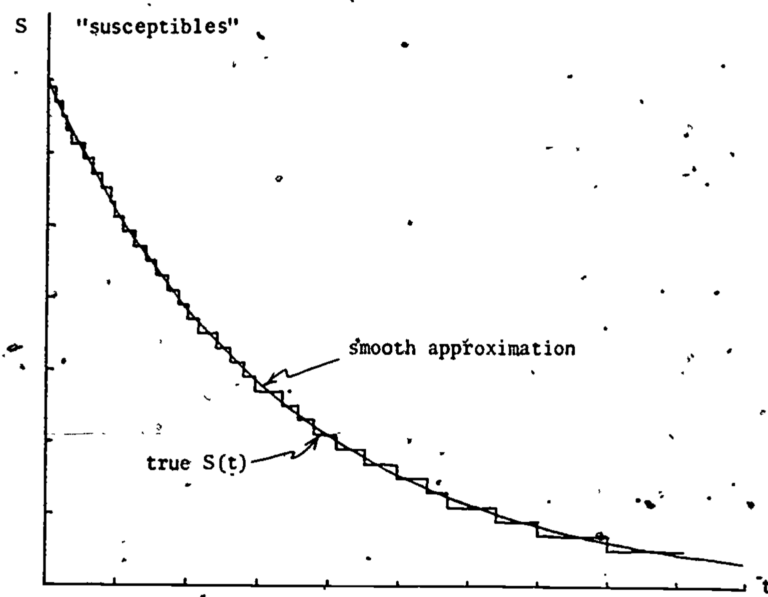


Figure 4. Approximation of S(t) by a smooth curve.

"smoothing" as before,

(3) $$P'(t) = k_2 I(t)$$

for some constant $k_2 > 0$.

Exercise:

1. Criticize this model. For example, are the assumptions realistic? Are they reasonably translated into mathematical terms? What, if any, important aspects of the situation are not represented?

---

3. CONTROLLING THE EPIDEMIC

3.1 Definition of "Control"

Recall that one question we asked was at what rate must persons be transferred from group I to group P to keep the epidemic under control.

6.

So far we have not said precisely what we mean by "under control." Let us recall how the epidemic begins. The disease is introduced into the community by a small number of people. So $I(0)$ is small, $P(0) = 0$, and $S(0) \approx N$.

(A word about the symbol $\approx$: When we say one expression is a good approximation to another, we almost always are thinking of the *percentage* error, rather than the *actual size* of the error. For example, we might well write $1001 \approx 1000$, but would be very unlikely to write $2 \approx 1$, even though $1001 - 1000 = 2 - 1 = 1$.)

The more rapidly $I(t)$ grows, the worse the epidemic becomes. Let us adopt as our definition of "under control" that $I(t)$ stops growing (i.e., $I'(t) \leq 0$) after some time.

3.2 The Threshold Removal Rate

Can control be achieved in our model? We shall present some calculations, and leave it to you to finish them.

Dividing Equation 2 by Equation 3:

$$\frac{S'(t)}{P'(t)} = -\frac{k_1}{k_2} S(t)$$

$$\frac{S'(t)}{S(t)} = -\frac{k_1}{k_2} P'(t)$$

$$\int \frac{S'(t)}{S(t)}\, dt = -\frac{k_1}{k_2} \int P'(t)\, dt$$

$$\ln S(t) = -\frac{k_1}{k_2} P(t) + c.$$

Putting $t = 0$ and recalling that $P(0) = 0$ we get

$$\ln S(0) = c$$

and so

$$\ln S(t) = \ln S(0) - \frac{k_1}{k_2} P(t).$$

Writing $S_0 = S(0)$ and solving for $S(t)$:

(4) $$S(t) = S_0 e^{-k_1 P(t)/k_2}$$

Now it's your turn.

Exercise:

2. (a) Use Equations 1, 2, and 3 to show that $I'(t) = (k_1 S(t) - k_2)I(t)$.

   (b) Show that $S(t)$ is a decreasing function for all t.

   (c) Using (b), show that, if $t > 0$ and $k_2 \geq k_1 S$, then $k_1 S(t) < k_2$.

   (d) Using (a) and (c), show that, if $t > 0$ and $k_2 \geq k_1 S_0$, then $I'(t) < 0$.

Recall that k_2 is the proportionality constant which tells us how fast persons are removed from group I to group P (the one we can influence by quarantine, etc.), and k_1 is the one which tells us how fast the epidemic is spreading. Exercise 2 shows that we can keep the epidemic under control if we can establish $k_2 \geq S_0 k_1$. This critical value $S_0 k_1$ is called the *threshold removal rate*. It varies directly with k_1 and with S_0. But $S_0 \approx N$. So we have the not very surprising result that the threshold removal rate varies directly with the rate at which the epidemic spreads and with the population.

Exercises:

3. A yet simpler (and less realistic) model of an epidemic would be one without any provision for removal. An infective remains an infective. If $N = S(t) + I(t)$, and if we make the same assumptions as before concerning contact between infective and susceptibles, we get:

$$S'(t) = -k_1 S(t)(N - S(t)).$$

Writing $S(0) = S_0$, find an expression for $S(t)$.
(Hint: Antidifferentiate

$$\frac{S'(t)}{S(t)(N - S(t))} = -k_1$$

by using the identity

$$\frac{1}{u} + \frac{1}{v - u} = \frac{v}{u(v - u)}.)$$

4. For the $S(t)$ obtained in Exercise 3, evaluate $\lim_{t\to\infty} S(t)$. What does this imply about the size of the epidemic?

5. For the $S(t)$ obtained in Exercise 3, find the time t when the rate of the spread of the epidemic is at its maximum.

---

4. A MILD EPIDEMIC

4.1 Extent of the Epidemic

Now let us ask the question: suppose k_2 is almost but not quite equal to S_0k_1, so we do not quite "control" the epidemic. For instance, suppose $0.95\ S_0k_1 < k_2 < S_0k_1$. What portion of the community will eventually catch the disease? For $t \geq 0$, we have remarked that the number who have caught the disease by time t is $I(t) + P(t)$. So if the epidemic lasts for time T (i.e., $S'(t) = 0$ for $t > T$), the number we are looking for is $I(t) + P(t)$. Let us call this number the *extent*, and write it E.

4.2 An Equation for the Extent

To find E, we shall begin by observing that $P(t)$ is defined for *all* $t \geq 0$, not just for $0 \leq t \leq T$. Figure 5 shows the graph of a typical step function $P(t)$ starting at time $t = 0$ and extending well beyond $t = T$. It makes it clear that by some time T^*, later than T but not too much later, the slope of the smooth approximation must be close to zero. That is:

(5) $$P'(T^*) \approx 0.$$

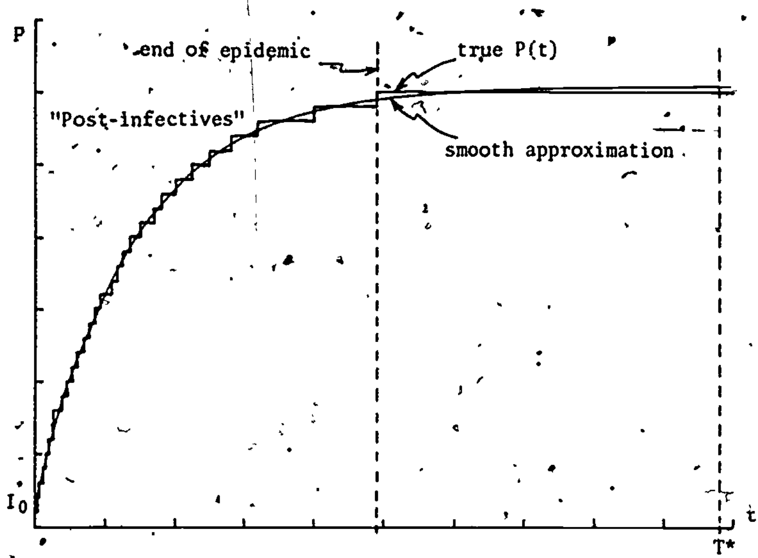


Figure 5. Smooth approximation to P(t).

Equation 3 immediately tells us $I(T^*) = 0$. But the *sum* $I(t) + P(t)$ does not change after $t = T^*$, and so

$$E = I(t) + P(t) = I(T^*) + P(T^*) = P(T^*). \tag{6}$$

We have assumed the total population does not change during or shortly after the epidemic. Specifically, let us take this to mean during the time interval $0 \le t \le T^*$. Then, using Equations 3, 1, and 4 in that order,

$$P'(t) = k_2 I(t) = k_2(N - P(t) - S(t)) =$$
$$k_2(N - P(t) - S_0 e^{-k_1 P(t)/k_2})$$

throughout this interval. Setting $t = T^*$ and using (5) and (6)

$$0 = k_2(N - E - S_0 e^{-k_1 E/k_2}). \tag{7}$$

4.3 An Approximation for e^{-k_1E/k_2}

The appearance of both a linear and an exponential term in (7) makes it very difficult, if not impossible, to solve for E. There is a way to circumvent this difficulty, provided k_1E/k_2 is small. Recall that, for any positive x and any positive integer n

$$e^{-x} \approx 1 - x + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^n}{n!}.$$

with an error of at most $\left|\frac{x^{n+1}}{(n+1)!}\right|$. Setting $x = k_1E/k_2$ and n = 2, we obtain the approximation

$$(8) \qquad e^{-k_1E/k_2} \approx 1 - \frac{k_1E}{k_2} + \frac{1}{2}\left(\frac{k_1E}{k_2}\right)^2,$$

with an error of at most $\frac{1}{6}\left(\frac{k_1E}{k_2}\right)^3$.

4.4 Estimating the Extent

Before we can use (8) we must of course assure ourselves that this error term is small enough for our purposes. Recall that $S_0 \approx N$; that is, initially virtually everyone is susceptible. If we make the very modest assumption that "virtually everyone" means "over 99%" (in other words, the persons who initially introduce the disease constitute less than one percent of the population), then we can show that $E < \frac{1}{4}N$. With this restriction on E the maximum error in using (8) to estimate e^{-k_1E/k_2} works out to be less than one-half of one percent of the true value.

It takes a lot of messy algebra to prove these assertions, and right now that would distract us from the main argument. So we shall leave that algebra for the appendix, and proceed with our estimation.

Replacing e^{-k_1E/k_2} in (7) by the estimate given in (8), and also dividing (7) by k_2, gives us

$$(9) \qquad 0 \approx N - E - S_0 \left[1 - \frac{k_1 E}{k_2} + \frac{1}{2}\ \frac{k_1{}^2 E^2}{k_2{}^2}\right].$$

Since, again, $S_0 \approx N$, we can also replace S_0 by N, obtaining

$$0 \approx N - E - N\left(1 - \frac{k_1 E}{k_2} + \frac{1}{2}\frac{k_1{}^2 E^2}{k_2{}^2}\right)$$

$$E\left[\left(\frac{k_1 N}{k_2} - 1\right) - \frac{N k_1{}^2}{2k_2{}^2} E\right] \approx 0$$

$$E \approx \frac{2k_2{}^2}{N k_1{}^2}\left(\frac{k_1 N}{k_2} - 1\right)$$

$$(10) \qquad E \approx \frac{2k_2}{N k_1}\left(N - \frac{k_2}{k_1}\right)$$

$$\approx 2\left(N - \frac{k_2}{k_1}\right)$$

since $k_2 \approx S_0 k_1 \approx N k_1$.

Exercises:

6. (a) Assume $k_1 = 10^{-6}$, $k_2 = .95$, and $N = 10^6$. Find the approximate value for the extent E of the epidemic.

   (b) Do the same for $k_2 = .99$.

4.5 The Relative Removal Rate

Sometimes k_2/k_1 is called the *relative removal rate.* Its *threshold value* is S_0, which approximately equals N. With this terminology, (10) says that in a mild epidemic, that is, one for which the removal rate is very near its threshold, the total number of persons infected sooner or later is approximately 2δ, where δ is the amount by which the relative removal rate falls short of its threshold ($\delta = N - k_2/k_1$).

5. APPENDIX

In this appendix we shall justify the assertion made in the first paragraph of Section 4.4. Specifically, if

(11) $$0.95S_0k_1 \leq k_2 \leq S_0k_1$$

(the epidemic is nearly but not quite "controlled") and if

(12) $$0.99N \leq S_0 \leq N;$$

(over 99% of the population is initially susceptible), then $E < \frac{1}{4}N$.

We can make a very rough estimate of E graphically. Writing

$$f(x) = N - x - S_0e^{-k_1x/k_2}$$

we see that E is the positive root of $f(x) = 0$; that is, the x-coordinate of the point where the graph of f crosses the positive x-axis.

To get a rough idea what this graph looks like, we first compute

$$f(0) = N - S_0 > 0$$

(note $f(0)$ is small since $S_0 \approx N$) and

$$f(N) = -S_0e^{-k_1N/k_2} < 0$$

(since the exponential function is always positive), thus showing that the graph crosses the x-axis between 0 and N. We leave it to you (see Exercise 7) to show that $f''(x) < 0$ for all x, and that therefore the graph is concave downward and connot cross the positive x-axis more than once.

Exercise:

7. If the function f is defined by

$$f(x) = N - x - S_0 e^{-k_1 x/k_2}$$

for all real x, show that $f''(x) < 0$ for all x. Combine this with the fact $f(0) > 0$ to show the graph of f crosses the positive x-axis at most once.

It follows that if we can find any positive number, M for which $f(M) < 0$, then we can conclude $0 < E < M$ (see Figure 6). We shall now find such a M.

Equation 11 can be rewritten

$$0.95 < \frac{k_2}{S_0 k_1} < 1$$

or, taking reciprocals and reversing the inequalities,

$$(13) \qquad 1 < \frac{S_0 k_1}{k_2} < \frac{1}{0.95} \; .$$

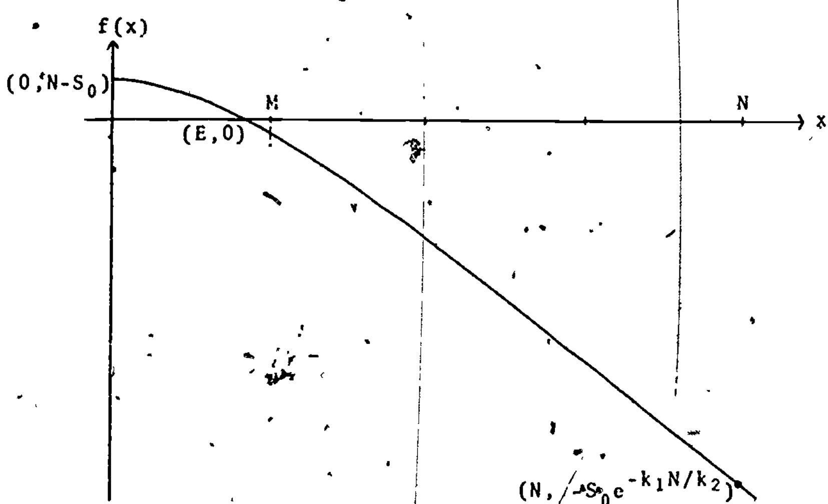


Figure 6. Graph of $f(x) = N - x - S_0 e^{-k_1 x/k_2}$.
(See *S-1* for additional information.) 14

From (12) we get

$$N < \frac{S_0}{0.99}$$

and hence, multiplying by k_1/k_2 and using (13),

$$\frac{k_1 N}{k_2} < \left(\frac{1}{0.99}\right)\left(\frac{k_1 S_0}{k_2}\right) < \frac{1}{(0.99)(0.95)} \tag{14}$$

With this inequality and a calculator, let us calculate $f(\frac{N}{4})$.

$$\begin{aligned} f(\tfrac{N}{4}) &= N - \frac{N}{4} - S_0 e^{-\frac{1}{4}(k_1 N/k_2)} \\ &< \frac{3}{4} N - 0.99Ne^{-\frac{1}{4}(k_1 N/k_2)} \\ &< N(0.75 - 0.99e^{-\frac{1}{4}[1/(0.99)(0.95)]}) \\ &\approx N(0.75 - 0.76) \approx -0.01N \quad 0. \end{aligned}$$

Thus $\frac{N}{4}$ is an example of a point M such that $f(M) \quad 0$. Therefore, $0 < E < \frac{N}{4}$.

Remember (Section 4.3) that the error in our estimate of $e^{-k_1 E/k_2}$ is less than $\frac{1}{6}\left(\frac{k_1 E}{k_2}\right)^3$. As we remarked in Section 4.4, with $E < \frac{1}{4}N$ this works out to less than one-half of one percent of the true value. We'll leave the computational details to you (see Exercise 8).

Exercise:

8. (a) Show that if $0 < E < N/4$, and if k_1 and k_2 are restricted as in the text, then the error in using (8) to estimate $e^{-k_1 E/k_2}$ is less than 0.0032.

   (b) Show that, under the same conditions, the value of $e^{-k_1 E/k_2}$ is greater than 0.76, so that the error of part (a) is less than one-half of one percent.

6. ANSWERS TO EXERCISES

1. The model is (as pointed out in Section 2.1) only reasonable when short epidemics are analyzed. The model does not consider interference from other possible outbreaks or epidemics that might occur during our time interval.

   There is also no accommodation for the spread of disease by infected objects (towels, water, infected air, bathrooms, etc.).

2. a) Given

   (1) $S(t) + I(t) + P(t) = N$,

   (2) $S'(t) = -k_1S(t)I(t)$

   and (3) $P'(t) = k_2I(t)$

   it follows that

$$\begin{aligned} I(t) &= N - S(t) - P(t) \\ \therefore I'(t) &= -S'(t) - P'(t) \\ &= k_1S(t)I(t) - k_2I(t) \\ &= (k_1S(t) - k_2)I(t). \end{aligned}$$

   b) Given $t > 0$, we want to show that $S'(t) < 0$. This is trivial because $t > 0$ implies $I(t) > 0$; the disease is infectious.

   c) Given: $t > 0$ and $k_2 > k_1S_0$ (or we could write $k_2 > k_1S(0)$). At this point the epidemic has not started, but the instant $t > 0$, $S(t)$ decreases implying that $S(t) < S_0$. Substituting $S(t)$ for S_0 we can write $k_2 > k_1S(t)$.

   d) Given: $I'(t) = (k_1S(t) - k_2)I(t)$, $t > 0$ and $k_2 \geq k_1S_0$ (which implies $k_1S(t) < k_2$) it follows that $k_1S(t) - k_2 < 0$

   and $I(t) > 0$

   hence $I'(t) < 0$.

3.
$$S(t) = \frac{NS_0e^{-Nk_1t}}{N - S_0 + S_0e^{-Nk_1t}}$$

4. 0. This implies that eventually everyone becomes ill.

5.

$$-\frac{1}{Nk_1}\ln\frac{N-S_0}{S_0} \quad \text{if } S_0 < \frac{1}{2}N,$$

$$0 \quad \text{if } S_0 \geq \frac{1}{2}N.$$

6. a) 100,000

b) 20,000

7.

$$f''(x) = -\left(\frac{k_1}{k_2}\right)^2 S_0 e^{-k_1x/k_2}$$

8. a) In (8) the error is at most $\frac{r}{6}\left(\frac{k_1E}{k_2}\right)$. We have seen that

$\frac{k_1N}{k_2} < \frac{1}{(0.99)(0.95)}$. (Equation 14).

Let $E = \frac{N}{4}$. Then the error is at most

$$\frac{1}{6}\left(\frac{k_1}{k_2}\cdot\frac{N}{4}\right)^3 = \frac{1}{6}\cdot\frac{1}{64}\left(\frac{k_1N}{k_2}\right)^3 < \frac{1}{6}\cdot\frac{1}{64}\cdot\left(\frac{1}{(0.99)(0.95)}\right)$$

$$\approx .00277$$

b) If $0 < E < \frac{N}{4}$,

then $0 < \frac{k_1E}{k_2} < \frac{k_1\cdot\frac{N}{4}}{k_2} = \frac{1}{4}\frac{k_1N}{k_2}$.

From Equation (14) we then have

$$\frac{k_1E}{k_2} < \frac{1}{4}\cdot\frac{k_1N}{k_2} < \frac{1}{4}\cdot\frac{1}{(0.99)(0.95)}$$

Therefore,

$$-\frac{k_1E}{k_2} > -\frac{1}{4}\frac{1}{(0.99)(0.95)} \approx -0.26581.$$

(rounded *down*, not up, to be sure this inequality is preserved), and

$$e^{-\frac{k_1E}{k_2}} > e^{-0.26581} \approx 0.76658$$

(rounding down again).

The Project would like to thank Vincent J. Grosso of Broward Community College, Fort Lauderdale, Florida; Sidnie Feit of Yale University, New Haven, Connecticut; L.M. Larsen of Kearney State College, Kearney, Nebraska; Thomas J. O'Malley of LeMoyne College, Syracuse, New York; William Glessner of Central Washington University, Ellensburg, Washington; Norman V. Schmidt of North Iowa Area Community College, Mason City, Iowa; Roland F. Smith of Russell Sage College, Troy, New York; Rudy Svoboda and Sheldon F. Gottlieb of Indiana University-Purdue University, Fort Wayne, Indiana for their reviews, and all others who assisted in the production of this unit.

This unit was field-tested and/or student reviewed in preliminary form by Joseph McCormack of the Wheatley School, Old Westbury, New York; Donald R. Snow of Brigham Young University, Provo, Utah; George Akst of California State University at San Bernadino; Jonathan Choate of the Groton School, Groton, Massachusetts; and Peter Nicholls of Northern Illinois University, DeKalb, Illinois, and has been revised on the basis of data received from these sites.

[S-1]

When $\alpha, \beta > 0$, the functions

(14) $\quad f(x) = -\alpha e^{-\beta x}$

have graphs like those shown below in Figure 7.

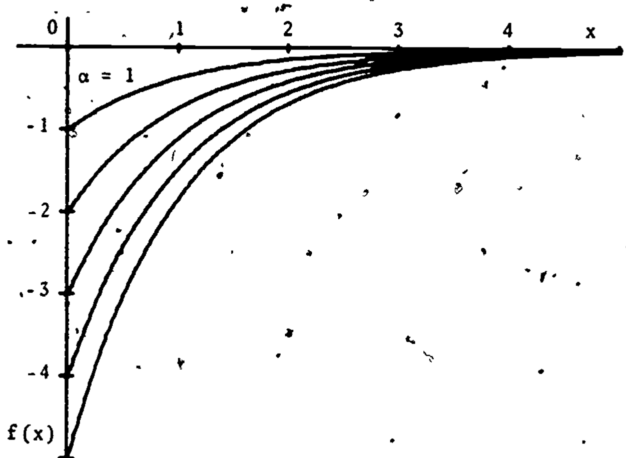


Figure 7.

For example, Figure 8 shows the graph of $f(x) = -e^{-x}$ (obtained by taking $\alpha = \beta = 1$ in Equation (14)) as a smooth curve through the points $(0, -1)$, $(1, -\frac{1}{e})$, $(2, -\frac{1}{e^2})$, ...

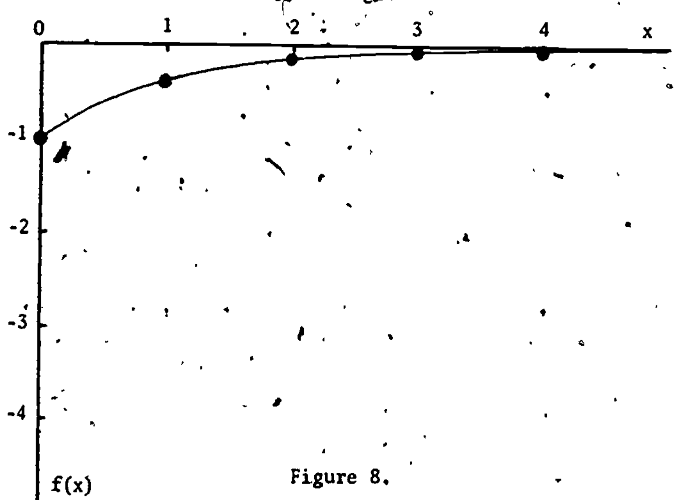


Figure 8.

Subtracting x from $-e^{-x}$ pulls the graph of $-e^{-x}$ away from the positive x-axis: Figure 9 shows the graph of $f(x) = -x - e^{-x}$ as a smooth curve through the points (0,1), $(1,-1-\frac{1}{e})$, $(2,-2-\frac{1}{e})$, ...

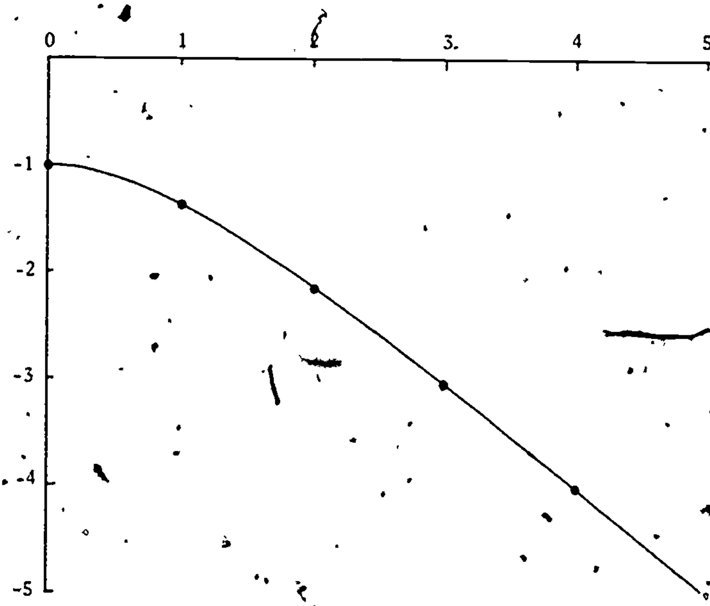


Figure 9.

The graph of

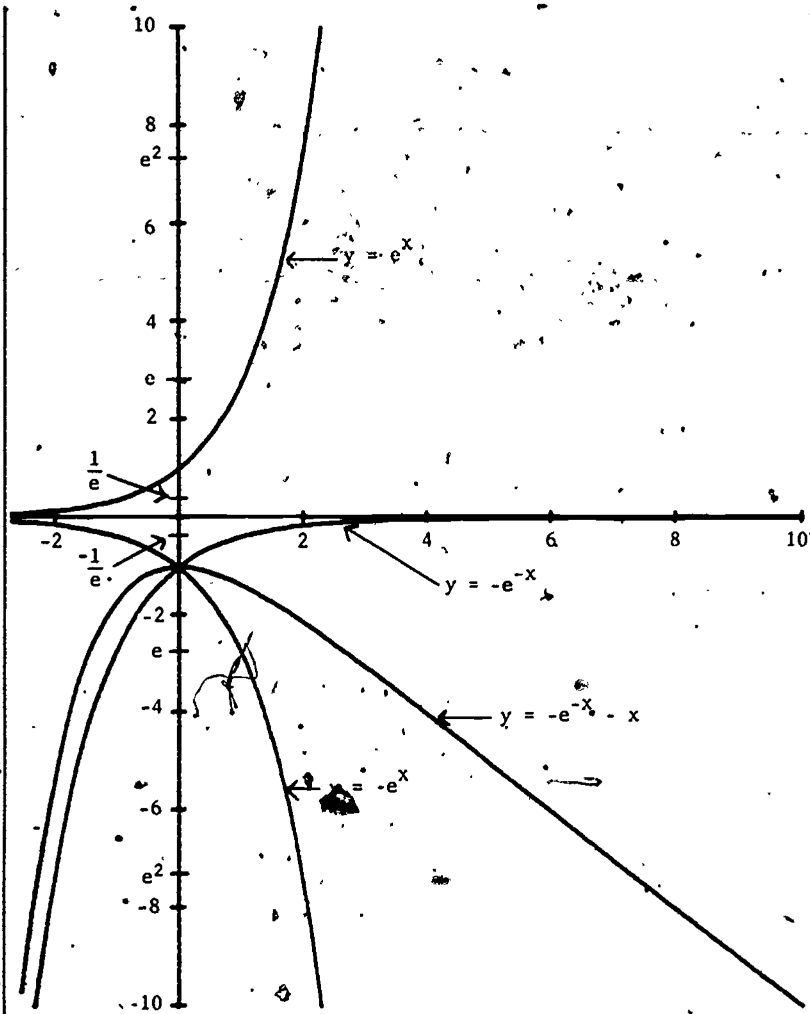
(15) $\quad f(x) = -x - \alpha e^{-\beta x}, \quad \alpha, \beta > 0$

behaves basically the same way.

The addition of a constant N > 0 to the formula for f(x) translates the curve vertically upward to produce a curve like the graph of

$$f(w) = N - w - x_0 e^{-k_1 w/k_2}$$

in Figure 6.



A useful guide for functions involving e.

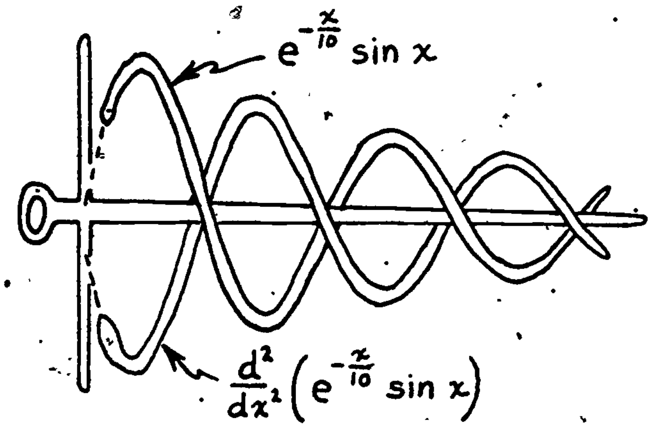
umap

UNIT 74

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

TRACER METHODS IN PERMEABILITY

by Brindell Horelick and Sinan Koont



MEDICAL APPLICATIONS OF CALCULUS

Units 71-74

edc/umap/55chapel st./newton, mass. 02160

TRACER METHODS IN PERMEABILITY

by

Brindell Horelick and Sinan Koont
Department of Mathematics
University of Maryland Baltimore County
Baltimore, Maryland 21228

9/9/77

TABLE OF CONTENTS

Intermodular Description Sheet: UMAP Unit 74

Title: TRACER METHODS IN PERMEABILITY

AUTHORS: Brindell Horelick and Sinan Koont
Department of Mathematics
University of Maryland Baltimore County
Baltimore, Maryland 21228

Review Stage/Date: III 9/9/77

Classification: MED APPLIC CALC/PERMEABILITY (U 74)

Suggested Support Material:

References:

Defares, J.G. and I.N. Sneddon (1961), An Introduction to the Mathematics of Medicine and Biology, North Holland, Amsterdam.
Harris, E.J. (1972), Transport and Accumulation in Biological Systems, Butterworth and Co., London.
Solomon, A.K. (1949), Equations for Tracer Experiments, Journal of Clinical Investigation 28: 1297-1307.
Stein, W.P. (1967), The Movement of Molecules across Cell Membranes, Academic Press, New York.

Prerequisite Skills:

1. Understand the meaning of $f'(t)$ (rate of change).
2. Be able to antidifferentiate $$\int \frac{f'(t)}{af(t) + b}\, dt.$$
3. Be able to manipulate ln and exp algebraically.
4. Know $\lim_{t\to\infty} e^{-kt} = 0$ if $k > 0$.
5. Know rules of logarithms.
6. Recognize the equation of a straight line.

This unit is intended for calculus students with an active interest in medicine and some background knowledge of biology. Typically this background knowledge may be represented by concurrent registration in a college level biology course.

Output Skills:

1. Be able to describe how radioactive tracer technique is used to monitor substances in the body.
2. Be able to criticize the model described in this unit, naming some strengths and some weaknesses.

Other Related Units:

Measuring Cardiac Output *(Unit 71)*
Prescribing Safe and Effective Dosage *(Unit 72)*
Epidemics *(Unit 73)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is one of many projects of Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research and development in the U.S. and abroad.

The Project would like to thank Rudy Svoboda, Bill Glessner, Leland D. Graber, and Melvin A. Nyman for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED 76-19615. Recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

1. RADIOACTIVE TRACER TECHNIQUE

In the human bloodstream potassium ions (K^+) are constantly moving into and out of the red blood cells (erythrocytes); that is, the surfaces of the erythrocytes are *permeable* to K^+ ions. Ions move from the plasma into the red cells at a certain rate, while other ions within the cells move out into the plasma at a certain rate. The determination of these two rates (that is, of the *permeability* of the cells surfaces to K^+ ions in both directions is of great help to both physiologists and doctors in their efforts to understand the structure and behavior of these cells, and thus ultimately to combat blood diseases.

A technique to determine these permeabilities works as follows. A fixed quantity S of radioactive K^{42+} ions is introduced into the blood. Initially, all these ions are in the plasma. The amount remaining in the plasma at various subsequent times is determined by taking blood plasma samples and measuring the radioactivity present. Our problem is to determine the permeabilities from these raw data.

2. A CLOSED TWO COMPARTMENT MODEL

2.1 Notation

We shall establish a mathematical model depicting those aspects of the situation which interest us. Since we have no need to distinguish one red cell from another, we shall represent the bloodstream schematically by two boxes, one for cells, the other for plasma (Figure 1). If t is the elapsed time since the introduction of the K^{42+} ions, we shall denote the amount of K^{42+} ions in the boxes by $C(t)$ and $P(t)$ respectively. Thus, $C(0) = 0$ and $P(0) = S$.

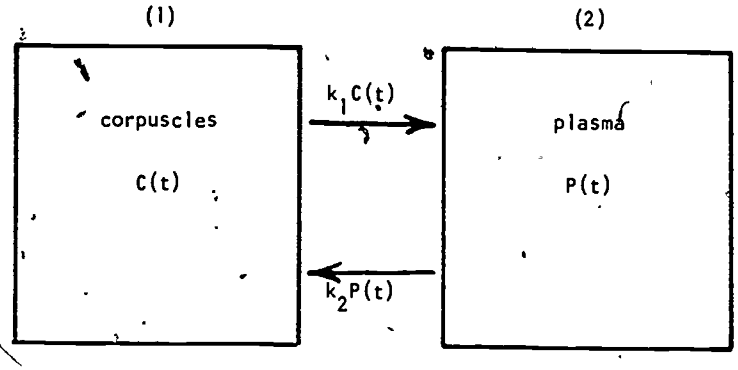


Figure 1. Two Compartment Model of Bloodstream

2.2 Assumptions

We shall assume that this two compartment system is *closed*; that is, there is no loss of K^{42+} from the system. In our notation this says

(1) $$C(t) + P(t) = S.$$

We shall also assume that at time t the number of ions moving from Box 1 to Box 2 (upper arrow) per unit time (the *transfer rate*) is proportional to $C(t)$, while the transfer rate from Box 2 to Box 1 is proportional to $P(t)$. If the respective constants of proportionality (the *coefficients of transfer*) are positive numbers k_1 and k_2, our assumption says

(2) $$P'(t) = -k_2P(t) + k_1C(t).$$

The units of k_1 and k_2 are reciprocals of time (for example, min^{-1} or hr^{-1}). In Equations (1) and (2), S is predetermined by the experimenter and $P(t)$ is observed empirically, so $C(t)$ can be easily computed. We must figure out how to determine k_1 and k_2.

3. THE FORMULA FOR P(t)

We begin by finding P(t) explicitly. It is easy to solve for C(t) in Equation 1 and substitute the result into Equation 2:

$$(3) \qquad P'(t) = -k_2P(t) + k_1(S - P(t))$$
$$= k_1S - (k_1 + k_2)\,P(t).$$

For the moment let us assume that P'(t) is never zero. We can then divide Equation (3) by its right side:

$$\frac{P'(t)}{k_1S - (k_1 + k_2)\,P(t)} = 1.$$

Since P(0) = S, the denominator is negative (= $-k_2S$) when t = 0. We can conclude that it is always negative--it is never zero and therefore cannot change sign. Let us multiply this last equation by -1 and then antidifferentiate both sides:

$$\int \frac{P'(t)}{(k_1 + k_2)\,P(t) - k_1S}\,dt = -\int 1dt = -t + C.$$

To antidifferentiate the left side, write

$$u = (k_1 + k_2)\,P(t) - k_1S$$

$$\frac{du}{dt} = (k_1 + k_2)\,P'(t).$$

Thus, we have

$$\frac{1}{k_1 + k_2}\int \frac{1}{u}\frac{du}{dt}\,dt = -t + C,$$

and since u > 0 we get

$$\frac{1}{k_1 + k_2}\ln u = -t + C,$$

or

$$\frac{1}{k_1 + k_2}\ln\left((k_1 + k_2)\,P(t) - k_1S\right) = -t + C.$$

To evaluate C, we set t = 0 and use P(0) = S:

$$\frac{1}{k_1 + k_2}\ln k_2S = C.$$

Thus,

$$\frac{1}{k_1 + k_2}\left(\ln\left((k_1 + k_2)\,P(t) - k_1S\right) - \ln k_2S\right) = -t$$

$$\frac{1}{k_1 + k_2}\ln\frac{(k_1 + k_2)\,P(t) - k_1S}{k_2S} = -t$$

$$\ln\left(\frac{(k_1 + k_2)}{k_2S}P(t) - \frac{k_1}{k_2}\right) = -(k_1 + k_2)t$$

$$\frac{k_1 + k_2}{k_2S}P(t) = \frac{k_1}{k_2} + e^{-(k_1 + k_2)t}$$

$$P(t) = \frac{k_2S}{k_1 + k_2}\left(\frac{k_1}{k_2} + e^{-(k_1 + k_2)t}\right)$$

$$(4) \qquad P(t) = \frac{k_1S}{k_1 + k_2}\left(1 + \frac{k_2}{k_1}\,e^{-(k_1 + k_2)t}\right).$$

To obtain Equation 4 we had to assume that $P'(t) \neq 0$. But this apparent restriction turns out to be no restriction at all. For it is now a routine computation to show that the function P(t) given by Equation 4 actually satisfies Equation 3 for *all* t (see Exercise 2) and is thus the function we seek. Incidentally, it is also easy to confirm that P'(t) is never zero (see Exercise 2 again).

Exercises

1. a) Obtain an expression for C(t) from Equations 1 and 4.
   b) Obtain the following expression for C(t) by first using Equations 1 and 2:

$$C'(t) = k_2P(t) - k_1C(t) = k_2(S - C(t)) - k_1C(t)$$

   Show that this result agrees with the answer to (a).

2. a) Compute P'(t) if P(t) is given by Equation 4.
   b) Show that, for this P(t), $P'(t) = k_1S - (k_1 + k_2)\,P(t)$.
   c) Show that, for this P(t), P'(t) is never zero.

4. DETERMINING k_1 AND k_2

4.1 Computational Preliminaries

To determine k_1 and k_2 from Equation [illegible] notice that as t approaches infinity, the expression [illegible] parentheses approaches 1, since $e^{-(k_1 + k_2)t}$ approaches ze[illegible]. Therefore,

$$(5) \qquad \lim_{t \to \infty} P(t) = \frac{k_1 S}{k_1 + k_2}.$$

Let us call this value Q. Dividing Equation 4 by Q, we obtain

$$\frac{P(t)}{Q} = 1 + \frac{k_2}{k_1} e^{-(k_1 + k_2)t}$$

$$(6) \quad \ln \left(\frac{P(t)}{Q} - 1\right) = -(k_1 + k_2)t + \ln \frac{k_2}{k_1}.$$

The expression on the left side of Equation 6 is a new function of t. Let us give it a name:

$$(7) \qquad g(t) = \ln \left(\frac{P(t)}{Q} - 1\right).$$

Now Equation 6 tells us that, according to our model,

$$(8) \qquad g(t) = -(k_1 + k_2)t + \ln \frac{k_2}{k_1},$$

so the graph of $g(t)$, theoretically at least, is a straight line with slope $-(k_1 + k_2)$ and y- intercept $\ln \frac{k_2}{k_1}$. So if we could compute $g(t)$ from the experimental data, using Equation 6, and then plot the points $(t, g(t))$, we could accomplish two things:

(a) If the points come close to lying on a straight line, we could use this fact to confirm the accuracy of our model. After all, it is the model which, by Equations 6 and 8 predicts that the points will lie on a line.

(b) If this straight line has slope m and y-intercept b, we could write

$$(9) \qquad \begin{cases} m = -(k_1 + k_2) \\ b = \ln \frac{k_2}{k_1} \end{cases}$$

and then do a little algebra to find k_1 and k_2 (see Exercise 4).

Exercises

3. (a) Evaluate $D = \lim_{t \to \infty} C(t)$.

   (b) Using (a), evaluate $\frac{Q}{D}$.

4. Solve Equations 9 for k_1 and k_2 in terms of m and b.

4.2 Determining Q

There is one catch, though. Equation 7 contains the symbol Q. Now $Q = \lim_{t \to \infty} P(t) = \frac{k_1 S}{k_1 + k_2}$ by definition. But this is a dead end, since we don't know k_1 and k_2 -- in fact, we are trying to determine them.

Luckily, there is a way out. The statement $Q = \lim_{t \to \infty} P(t)$ means, in experimental terms, that the observed amount $P(t)$ of K^{42+} ions in the plasma approaches Q as an equilibrium amount. Experimental evidence confirms that it is feasible to continue monitoring the plasma until $P(t)$ does not change, or changes very little with further passage of time. We can then take this nearly constant value to be Q.

4.3 An Example

To illustrate this method of determining k_1 and k_2, let us consider the data in Table 1 for a hypothetical permeability study.

TABLE 1

Hypothetical Data for Permeability Study

t(min)	0	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
P(t) (mg)	5.00	2.96	2.01	1.49	1.14	1.01	0.97	0.92	0.87	0.85	0.85

It appears that P(t) is settling down to an equilibrium of about 0.85 mg, after t = about 4500 minutes. We shall take Q = 0.85 mg.

We can now use Equation 7 to calculate the values of g(t):

$$g(t) = \ln\left(\frac{P(t)}{Q} - 1\right) = \ln\left(\frac{P(t)}{0.85} - 1\right).$$

We get

t	0	500	1000	1500	2000	2500	3000	3500	4000
g(t)	1.50	0.91	0.31	-0.28	-1.08	-1.67	-1.96	-2.50	-3.75

It is impossible to compute g(4500) or g(5000) on the basis of our data, because $\ln\left(\frac{0.85}{0.85} - 1\right) = \ln 0$ is not defined. This difficulty arises because our data are given to only two decimal places. With more precise measurements we might have found, for example, that P(4500) = 0.854 and P(5000) = 0.851.

The points (t, g(t)) are plotted in Figure 2. They are close to lying on a straight line. In Figure 2 we have drawn in, by eye, what appears to be the "best-fitting" straight line. In doing this it is wise to use a transparent straight edge, so our view of the points is not blocked. We have tried to draw the line so some of the points are slightly above it, some slightly below it, and none too far from it.

The line we have drawn appears to have its y-intercept at about b = 1.55. It also passes pretty nearly through (500, 0.9) and (400, -3.5). Therefore its slope m is about $\frac{-3.5 - 0.9}{4000 - 500} = \frac{-4.4}{3500} \approx 1.26 \times 10^{-3}$.

We could now use Equations 9 to find k_1 and k_2. But if you have done Exercise 3 you have discovered that Equations 9 can be rewritten

$$k_1 = \frac{-m}{1 + e^b}$$

$$k_2 = \frac{-me^b}{1 + e^b}$$

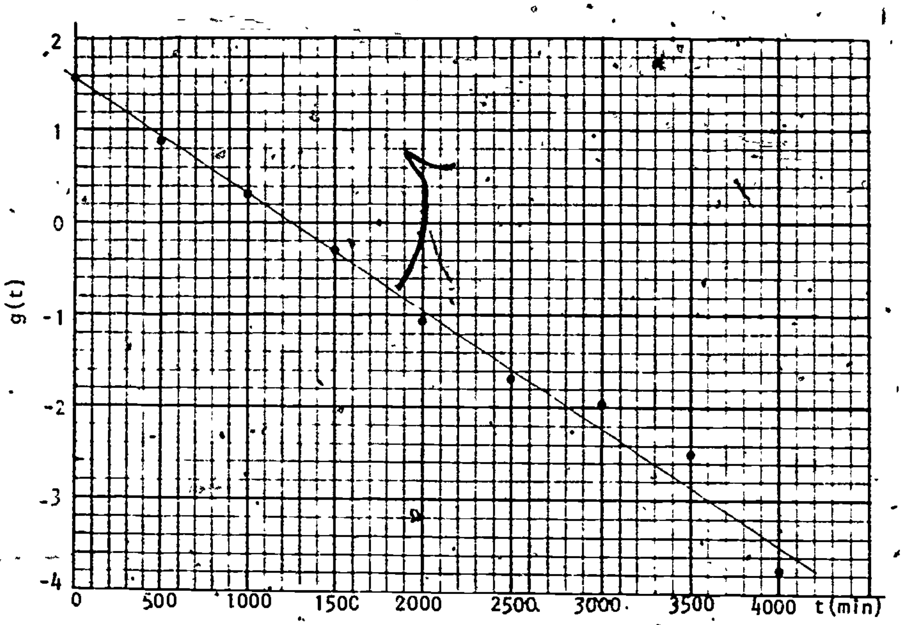


Figure 2.

Plugging in b = 1.55 and m = 1.26 x 10^{-3} we get

$$k_1 = 2.21 \times 10^{-4}\ \text{min}^{-1}$$

$$k_2 = 1.04 \times 10^{-3}\ \text{min}^{-1}$$

4.4 Some Comments on the "Best-Fitting" Line

(a) It is not important whether the line passes through any of the given points. In fact it would be a mistake simply to draw the line determined by two of the points. We might be unfortunate enough to pick two points which are inaccurate because of experimental error or roundoff error.

(b) In finding the slope of the line, use two points on the line itself, rather than two of the given points. As mentioned in 4.4(a), some of the given points are bound to be slightly off. The line in effect "averages out" these errors.

(c) Also in finding the slope, use two points fairly far apart. If they are close together, then the denominator in the slope formula will be small, and a slight error in reading the coordinates of one point may result in a huge error in the slope.

5. EXERCISES

5. In a permeability study the function $g(t) = \ln\left(\frac{P(t)}{Q} - 1\right)$ has been computed and plotted in Figure 3. Determine k_1 and k_2 for this experiment.

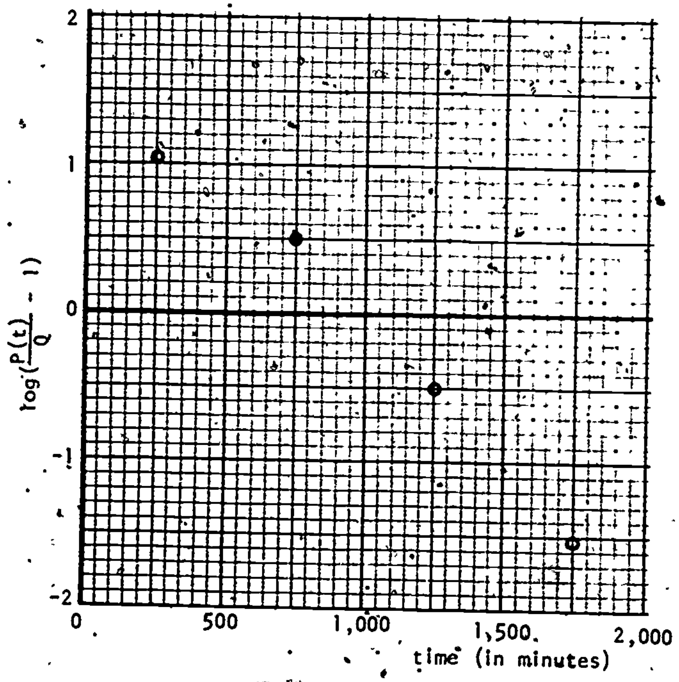


Figure 3.

6. Determine k_1 and k_2 from the following data for a hypothetical permeability study.

t (hours)	0	1	2	3	4	5	6	7	8	9	10
P(t) (mg)	6.00	3.30	2.05	1.55	1.28	1.20	1.15	1.13	1.12	1.10	1.10

7. Consider a two compartment model, not closed, in which $P(t)$ is maintained at a constant level (by continuous adjustment from the outside, for example). If $P(t) = P_0$ the determining equation for $C(t)$ becomes

$$C'(t) = k_1P_0 - k_1C(t).$$

(a) Find $C(t)$ if $C(0) = 0$.

(b) Find $D = \lim_{t\to\infty} C(t)$.

(c) If $Q = \lim_{t\to\infty} P(t)$, find $\frac{Q}{D}$.

8. In the model presented in Exercise 6, find the time at which $C'(t)$ takes on a maximum, and compute that maximum.

6. ANSWERS TO EXERCISES

1. (a) $S - \frac{k_1 S}{k_1 + k_2}\left(1 + \frac{k_2}{k_1} e^{-(k_1 + k_2)t}\right)$

   (b) $\frac{k_1 S}{k_1 + k_2}\left(1 - e^{-(k_1 + k_2)t}\right)$

2. (a) $P'(t) = \frac{k_1 S}{k_1 + k_2}\left[-\frac{k_2}{k_1}(k_1 + k_2)\, e^{-(k_1 + k_2)t}\right]$

   $= -k_2 S\, e^{-(k_1 + k_2)t}$

   (b) $k_1 S - (k_1 + k_2)\, P(t) =$

   $= k_1 S - (k_1 + k_2) \frac{k_1 S}{(k_1 + k_2)}\left[1 + \frac{k_2}{k_1} e^{-(k_1 + k_2)t}\right]$

   $= k_1 S - k_1 S\left[1 + \frac{k_2}{k_1} e^{-(k_1 + k_2)t}\right]$

   $= -k_2 S\, e^{-(k_1 + k_2)t}$

   (c) All three of the numbers k_2, S, and $e^{-(k_1 + k_2)t}$ are positive.

3. (a) $\frac{k_2 S}{k_1 + k_2}$

   (b) $\frac{k_1}{k_2}$

4. $k_1 = \frac{-m}{1 + e^b}$; $k_2 = \frac{-me^b}{1 + e^b}$

5. $k_1 = 3.0 \times 10^{-4}\ \text{min}^{-1}$; $k_2 = 1.5 \times 10^{-3}\ \text{min}^{-1}$

6. $k_1 = 0.14\ \text{hr}^{-1}$; $k_2 = 0.58\ \text{hr}^{-1}$

7. (a) $\frac{k_2}{k_1} P_0 (1 - e^{-k_1 t})$

   (b) $\frac{k_2}{k_1} P_0$

   (c) $\frac{k_1}{k_2}$

8. Maximum occurs at $t = 0$ (endpoint maximum) and equals $k_2 P_0$.

STUDENT FORM 1

Request for Help

Return to:
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55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name________________________________ Unit No.________

Page_____ ○ Upper ○ Middle ○ Lower	OR	Section_______ Paragraph______	OR	Model Exam Problem No._____ Text Problem No._____

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature____________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
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Name________________________ Unit No.______________ Date______________

Institution________________________ Course No.________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much Longer	Somewhat Longer	About the Same	Somewhat Shorter	Much Shorter
____	____	____	____	____

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

UMAP MODULE 75

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

Birkhäuser Boston Inc.
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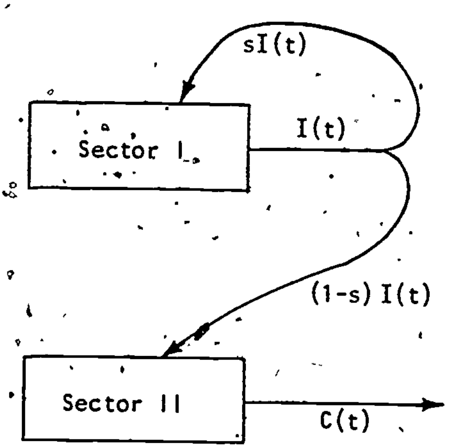
Feldman's Model

by Brindell Horelick
and Sinan Koont



Applications of Calculus to Economics

SE 036 475

Intermodular Description Sheet: UMAP Unit 75

Title: FELDMAN'S MODEL

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Review Stage/Date: IV 8/30/80

Classification: APPL CALC/ECON

Prerequisite Skills:

1. Know the meaning of the derivative and the difference quotient.
2. Be able to evaluate $\int \frac{f'(t)}{f(t)}$ and $\int f'(t)e^{f(t)}dt$.
3. Know $\lim_{t\to\infty} e^{-kt} = 0$ if $k > 0$.
4. Manipulate log and exp algebraically and sketch their graphs.
5. Compute the values of e^t and $\log t$ using tables or calculators.
6. Know how a change in the unit of measurement affects the numerical measure of a quantity.

This unit is intended for calculus students with an active interest in and some background knowledge of economics.

Output Skills:

1. Describe and discuss Feldman's model and derive the expressions for national income, its relative rate of change and the propensity to save.
2. Compute numerical values for rates of output, national income, their rates of change and the propensity to save.
3. Discuss the effects of changes in the parameters of the model or in the units of measurement.

Related Units:

General Equilibrium: Simple Linear Models *(Unit 208)*
General Equilibrium: A Leontief Economic Model *(Unit 209)*
Lagrange Multipliers: Applications to Economics *(Unit 270)*
Price Discrimination and Consumer Surplus *(Unit 294)*
Difference Equations with Applications *(Unit 322)*
Differentiation, Curve Sketching, and Cost Functions *(Unit 376)*
Selected Applications of Mathematics to Finance and Investment *(Unit 381)*

The Project would like to thank John Kenelly of Clemson University, Clemson, South Carolina, and William C. Ramaley of Fort Lewis College, Durango, Colorado for their reviews, and all others who assisted in the production of this unit.

This unit was field-tested and/or student reviewed in preliminary form by David F. Anderson of the University of Tennessee, Knoxville; Murray Eisenberg of the University of Massachusetts, Amherst; and Paul Nugent of Franklin College, Franklin, Indiana, and has been revised on the basis of data received from these sites.

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FELDMAN'S MODEL

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TABLE OF CONTENTS

1. GROWTH MODELS

The behavior of the economy of a given society over time is of interest not only to the economist, but also to all citizens living in that society. After all, the increasing abundance or scarcity of jobs, goods and services depends crucially on how fast the economy is growing relative to the underlying population. Thus it is not surprising that words like growth, stagnation (and more recently stagflation), etc. have become commonplace even in the everyday world of the nightly news broadcast.

For their part mathematical economists have developed many different models, called *growth models*, to describe the expansionary processes in an economy.

In this unit we will be studying a particular model of growth, applicable to planned economies in which all means of production are socially owned. It was developed by the Russian economist G. A. Feldman (1928) in connection with planning for the centrally controlled Soviet economy.

The purpose of this model is to describe the behavior over time of a two-sector economy in which sectoral investment allocations are controlled by a central authority according to an overall economic plan.

The impetus for the construction of Feldman's model came in 1927 when the Soviet Union embarked on a sequence of 5-year plans for the expansion of its economy. The devastation caused by the First World War and the following Civil War had, to a large extent, been overcome and prewar production levels had been restored. The first publication of the model occurred in the November 1928 issue of the Soviet State Planning Commission's journal Planovoc Khoziastvo (The Planned Economy).

Although the model's direct impact on Soviet planning policy formulation is questionable, there is no doubt that its 'flavor' is well in keeping with the initial sequence of Soviet 5-year plans with their strong emphasis on building up the heavy industry sector of the Soviet economy.

As an indication of the model's durability, we note that its main features were duplicated in the 1950's (apparently independently of Feldman's work) by the Indian economist Mahalonobis in his work on a planning model for the Indian economy.

2. DEFINITIONS

2.1 Rate of Output and National Income

As is usually done in Marxian economics we divide the economy into two sectors: the *producer goods sector* (Sector I), producing goods to be invested in production in both sectors, (like drop forges, milling machines and tractors), and the *consumer goods sector* (Sector II), producing goods to be consumed nonproductively by the population (like bagels, television sets, and footballs). We assume that both sectors are producing a continuous stream of goods.

Before we can model the quantitative behavior of the outputs in these sectors over time, we must first define a quantitative measure of how much each sector is producing at any given time. This we proceed to do as follows: let $J(t,u)$ stand for the net output (measured, say, in dollars) of Sector I between time t and time u (measured, say, in years with an appropriate point in time chosen as $t = 0$).

The "net" here means the output remaining after all the "wear and tear" in the investment goods being used in the two sectors has been made good from the "gross" output of Sector I. We use the net output, since only this part of the output of Sector I can be used to expand the economy over time. The rest of the output of Sector I, i.e. the difference between the "gross" and "net" outputs, is required just to keep the economy at the level it had already reached.

If we agree that all of the net output of Sector I goes for investment, it seems appropriate to call the quantity

$$I_{av}(t,u) = \frac{J(t,u)}{u-t}$$

the (average) annual rate of investment or the (average) annual rate of output in Sector I over the time interval from t to u. (Note: the units here are dollars/year.) The *(instantaneous) annual rate of investment* is then defined as

$$I(t) = \lim_{\Delta t \to 0} I_{av}(t,t+\Delta t) = \lim_{\Delta t \to 0} \frac{J(t,t+\Delta t)}{\Delta t}$$

The quantity I(t) measures what the annual rate of output in Sector I is *at a given instant t of time.*

Similarly we can, and do, define C(t), the *(instantaneous) annual rate of output in Sector II.* The sum of the rates of output in the two sectors is then called the *national income:* y(t) = I(t) + C(t).

2.2 An Analogy

You may have noticed some similarity between the procedure we used to define national income and the one used to define instantaneous velocity in first semester Calculus or Physics courses. Your study of velocity probably began by defining a "distance" or "position" function s(t) which described the location of an object at time t -- usually as the direct distance from some arbitrary point called zero. It then defined the *average* velocity over the time interval from t to u as

$$\frac{s(u) - s(t)}{u - t}$$

This average velocity and our average annual rate of output each amount to a ratio of two changes or differences. The numerators look quite different, but they really aren't. In studying velocity it was possible to define a function s of time whose change from time t to time u could be calculated by subtraction and was just what you needed. There does not seem to be a convenient way to do the analagous thing here. Nonetheless, saying "J(t,u) is the net output between time t and time u" is very similar to saying "s(u) - s(t) is the net motion (change in position) between time t and time u."

Thus national income can be thought of as the derivative of a (somewhat fictitious) "cumulative total output" function just as velocity is the derivative of the "cumulative total distance traveled" function. Table 1 further illustrates this analogy.

TABLE 1

Cumulative total over a time interval	Typical units	Average rate over a time interval	Typical units	Instantaneous rate at an instant of time	Typical units
distance traveled	meters	average velocity	$\frac{\text{meters}}{\text{sec}}$	velocity	$\frac{\text{meters}}{\text{sec}}$
total net output in Sector I (=A)	dollars	average rate of net investment	$\frac{\text{dollars}}{\text{year}}$	I(t) = rate of net investment	$\frac{\text{dollars}}{\text{year}}$
total output in Sector II (=B)	dollars	average rate of output in Sector II	$\frac{\text{dollars}}{\text{year}}$	C(t) = rate of output in Sector II	$\frac{\text{dollars}}{\text{year}}$
total output (=A+B)	dollars	average national income	$\frac{\text{dollars}}{\text{year}}$	Y(t)=I(t)+C(t) = national income	$\frac{\text{dollars}}{\text{year}}$

3. THE MODEL

3.1 Assumptions

With the definition of $I(t)$ and $C(t)$ in hand we are ready to start building Feldman's model.

We let $I_P(t)$ and $I_C(t)$ stand for the net rates of investment in Sectors I and II respectively. The entire output $I(t)$ of Sector I is to be invested in Sectors I and II. The central planning authority decides how to split the investment pie between the two sectors and allots a constant (i.e., independent of time), positive fraction s of the output of Sector I to Sector I (see Figure 1).

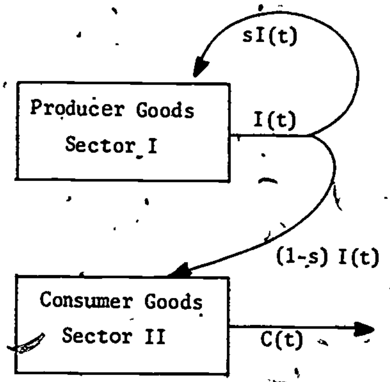

Figure 1. The net output of Sector I is produced at the rate of I(t) dollars per year. It is channeled back into producer goods at the rate of sI(t) dollars per year, and invested in consumer goods at the rate of (1-s)I(t) dollars per year.

Now we need to explore the relationship between $I(t)$, $C(t)$, $I_P(t)$ and $I_C(t)$, all of which are assumed to be differentiable functions of t. We let $K_P(t)$ and $K_C(t)$ stand for the *capital stock*, i.e., the quantity of investment goods invested at time t in the Sectors I and II respectively. By assumption, there is a stable relationship between the capital stock and the corresponding output in each sector, i.e., capital stock is proportional to output:

(i) $\frac{K_P(t)}{I(t)} = v_P =$ constant or $K_P(t) = v_P\, I(t)$

(ii) $\frac{K_C(t)}{C(t)} = v_C =$ constant or $K_C(t) = v_C\, I(t)$

The constants v_P and v_C in equations in (i) and (ii) are called the capital-output ratios (for obvious reasons). For the sake of simplicity, we assume the constants have a common value v:

$$v_P = v_C = v .$$

If we differentiate both sides of the equations in (i) and (ii) and note that the rate of increase in the capital stock k is precisely the corresponding rate of investment, we obtain:

$$I_P(t) = K_P'(t) = v \cdot I'(t)$$

and

$$I_C(t) = K_C'(t) = v \cdot C'(t)$$

We can summarize the discussion above in the following two assumptions of the Feldman model:

Assumption 1: There exists s $(0 < s < 1)$ such that $I_P(t) = sI(t)$ and

$I_C(t) = (1 - s)\, I(t)$.

Assumption 2: $I(t)$ and $C(t)$ are differentiable

and $I'(t) = \frac{1}{v} I_P(t)$ and $C'(t) = \frac{1}{v} I_C(t)$.

The crucial (and distinguishing) assumption of the model is Assumption 1. The claim it makes would almost certainly be false in the absence of controlled allocation according to a plan. It also focuses on the most important parameter of the model, namely s. For, as we shall see in the next section, it is the choice of s that decides how fast the economy grows, both

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overall $(Y(t))$ and in terms of its sectors $(I(t)$ and $C(t))$.

To avoid any confusion let us stress again; in the model presented above, s and v are time-independent constants (parameters), whereas $I_P(t)$, $I_C(t)$, $I(t)$ and $C(t)$ are increasing functions of the variable t(=time).

Finally we note that Assumption 1 can be modified to allow the possibility of $s = 0$ or $s = 1$, in which cases one of the sectors does not grow at all. Both the statement and derivation of the results of the model in the case $s = 0$ or $s = 1$ are considerably simpler than in the case $0<s<1$. We ask you to go through one such derivation in Exercise 12.

3.2 Derivation of Results

We suppose, at some starting point in time (conveniently taken to be $t = 0$) the values of I, C, and Y are known: $I(0) = I_0$, $C(0) = C_0$ and $Y(0) = Y_0$.

Now the equations of Assumptions 1 and 2 can be combined and rewritten as:

(1) $$I'(t) = \frac{s}{v} I(t)$$

and

(2) $$C'(t) = \frac{1-s}{v} I(t).$$

Dividing both sides of Equation 1 by $I(t)$ and integrating we obtain

$$\int_0^t \frac{I'(t)}{I(t)}\,dt = \int_0^t \frac{s}{v}\,dt$$

$$\log I(t)\Big|_0^t = \frac{s}{v} t$$

$$\log I(t) - \log I_0 = \log \frac{I(t)}{I_0} = \frac{s}{v} t$$

$$\frac{I(t)}{I_0} = e^{\frac{s}{v}t}.$$

For notational convenience we define $k = \frac{s}{v}$ and obtain

(3) $$I(t) = I_0 e^{kt}.$$

Substituting Equation (3) into Equation (2) and keeping in mind that $k = \frac{s}{v}$, we obtain after some algebraic manipulation

$$C'(t) = \frac{1-s}{v} I_0 e^{kt}$$

$$= \frac{1-s}{s} \frac{s}{v} I_0 e^{kt}$$

$$= \frac{1-s}{s} I_0 ke^{kt}.$$

Now we can integrate easily to get an expression for $C(t)$.

$$\int_0^t C'(t)dt = \frac{1-s}{s} I_0 \int_0^t ke^{kt} dt$$

$$C(t) - C_0 = \frac{1-s}{s} I_0 e^{kt} \Big|_0^t$$

(4) $$C(t) = C_0 + \frac{1-s}{s} I_0 (e^{kt} - 1).$$

Adding Equations (3) and (4) yields

$$Y(t) = I(t) + C(t)$$

$$= I_0 e^{kt} + C_0 + \frac{1-s}{s} I_0 (e^{kt} - 1)$$

$$= I_0 (e^{kt} - 1) + I_0 + C_0 + \frac{1-s}{s} I_0 (e^{kt} - 1)$$

$$= Y_0 + (I_0 + \frac{1-s}{s} I_0)(e^{kt} - 1),$$

since $I_0 + C_0 = Y_0$, and finally

(5) $$Y(t) = Y_0 + \frac{I_0}{s} (e^{kt} - 1).$$

The relative rate of growth of a differentiable quantity $f(t)$ is $f'(t)/f(t)$. It measures *not* how fast the quantity is changing in absolute terms, but *rather* how fast it is changing relative to its own size. From

Equations (1), (2), (4) and (5) we can obtain expressions for the relative rates of growth for I(t), C(t) and Y(t):

(6) $$\frac{I'(t)}{I(t)} = \frac{s}{v} = k \qquad \text{(directly from (1))}$$

(7) $$\frac{C'(t)}{C(t)} = \frac{k}{\left(\frac{Y_0}{I_0}\frac{s}{1-s} - \frac{1}{1-s}\right) e^{-kt} + 1} \qquad \text{(Exercise 1)}$$

(8) $$\frac{Y'(t)}{Y(t)} = \frac{k}{\left(\frac{Y_0}{I_0} s - 1\right) e^{-kt} + 1} \qquad \text{(Exercise 2)}$$

Exercise 1

(a) Prove Equation (7).

(b) Show $\frac{C'(t)}{C(t)} \to \frac{s}{v}$ as $\to \infty$.

Exercise 2

(a) Prove Equation (8).

(b) Show $\frac{Y'(t)}{Y(t)} \to \frac{s}{v}$ as $t \to \infty$.

3.3 The Average Propensity to Save

Another important quantity is $\alpha(t)$, defined as the ratio of investment (rate of output of Sector I) to the national income:

$$\alpha(t) = \frac{I(t)}{Y(t)} = \frac{I(t)}{I(t) + C(t)}$$

Thus $\alpha(t)$ measures the fraction of the total output of the economy, at a given point in time, which is saved (invested) rather than consumed. So its name, although cumbersome, should not come as a surprise: $\alpha(t)$ is called the average propensity to save.

If we let $\alpha_0 = \alpha(0)$ = the ratio of I_0 to Y_0, then

$$\alpha(t) = \frac{I_0 e^{kt}}{Y_0 + \frac{I_0}{s}(e^{kt} - 1)}$$

$$= \frac{s}{(\frac{s}{\alpha_0} - 1)\, e^{-kt} + 1}$$

If $s > \alpha_0$ one can show (Exercise 3) that $\alpha(t)$ is an increasing function of time. In any case, since $e^{-kt} \to 0$ as $t \to \infty$, we see that

$$\lim_{t\to\infty} \alpha(t) = s.$$

That is, the average propensity to save approaches the fraction of investment devoted to Sector I as time goes on.

Exercise 3

Calculate $\alpha'(t)$. Use your answer to show that

(a) $\alpha(t)$ is an increasing function if $s > \alpha_0$.

(b) $\alpha(t)$ is a decreasing function if $s < \alpha_0$.

Exercise 4

Compute $\alpha(t)$ for $t = 5$ and $t = 10$ if $\alpha_0 = 0.05$, $s = 0.5$, and $v = 5$.

---

4. NUMERICAL EXAMPLES

In this section we provide the results of calculations of the relative rates of growth of $I(t)$, $C(t)$, and $Y(t)$ and of the average propensity to save $\alpha(t)$ for some reasonable values of the parameters of the model; namely, $v = 1, 5$; $\alpha_0 = 0.1, 0.3$; and $s = 0.3, 0.7$.

It is certainly not unusual for an economy to be reinvesting 10% ($\alpha_0 = 0.1$) or 30% ($\alpha_0 = 0.3$) of its total output. As for the reasonableness of $v = .1$ or 5 (when time is measured in years [see Exercise 5]), we can refer to empirical determinations of the marginal capital coefficient. In fact, (with time measured in years), Feldman estimated the capital-output ratio v to be 2.4 in the years 1928-33. Leontief (1939) found comparable ratios to range from 0.076 to 7.1 in various branches of the American economy.

We have used one high (Table 2) and one low (Table 3) value of s in the computations to illustrate the effect of favoring one or the other of the two sectors for investment.

TABLE 2

Sector I Favored for Investment (s = 0.7)

$v = 5$, $\alpha_0 = 0.1$					$v = 5$, $\alpha_0 = 0.3$			
t (years)	I'/I (%)	Y'/Y (%)	C'/C (%)	α	I'/I (%)	Y'/Y (%)	C'/C (%)	α
1	14	2.3	0.8	0.11	14	6.5	2.9	0.32
5	14	3.5	1.3	0.18	14	8.4	4.4	0.42
20	14	10.3	6.3	0.51	14	13.0	11.0	0.65
50	14	13.9	13.8	0.70	14	14.0	13.9	0.70

$v = 1$, $\alpha_0 = 0.1$					$v = 1$, $\alpha_0 = 0.3$			
1	70	17.5	6.4	0.18	70	42.1	21.8	0.42
5	70	59.3	43.6	0.59	70	67.7	61.7	0.67
20	70	70	70	0.70	70	70	70	0.70
50	70	70	70	0.70	70	70	70	0.70

TABLE 3

Sector II Favored for Investment (s = 0.3)

$v = 5$, $\alpha_0 = 0.1$					$v = 5$, $\alpha_0 = 0.3$			
t (years)	I'/I (%)	Y'/Y (%)	C'/C (%)	α	I'/I (%)	Y'/Y (%)	C'/C (%)	α
1	6	2.1	1.6	0.10	6	6	6	0.30
5	6	2.4	1.9	0.12	6	6	6	0.30
20	6	3.7	3.2	0.19	6	6	6	0.30
50	6	5.5	5.3	0.27	6	6	6	0.30

$v = 1$, $\alpha_0 = 0.1$					$v = 1$, $\alpha_0 = 0.3$			
1	30	12.1	9.6	0.12	30	30	30	0.30
5	30	20.7	18.3	0.21	30	30	30	0.30
20	30	29.9	29.8	0.30	30	30	30	0.30
50	30	30	30	0.30	30	30	30	0.30

Exercise 5

(a) What are the units of v?

(b) How does v get affected if we switch from dollars to dimes as the measure of output?

(c) How does v get affected if we switch from years to months as the measure of time?

Exercise 6

Assume $\alpha_0 = 0.2$, $v = 3$. Evaluate $\frac{I'}{I}$, $\frac{C'}{C}$, $\frac{Y'}{Y}$ at $t = 1$ for

(a) $s = 0.1$

(b) $s = 0.6$

(c) $s = 0.9$

5. CONCLUDING REMARKS

In concluding, we make three remarks on this model:

1) I'/I is given by $\frac{s}{v}$, and does *not* depend on α_0; i.e., it is quite possible to have a very fast growing sector I, while the initial average propensity to save is low. This seems to have happened in the Soviet Union.

2) The relative rates of growth of the national income and the consumer goods sector eventually approach the growth of the producer goods sector (Exercises 1 and 2).

3) The average propensity to save rises to s (provided $\alpha_0 < s$). The empirical verification of this prediction of the model seems to have been a source of controversy among economists. Different analyses of the Soviet economy have led to widely differing estimates: from α remaining essentially constant at .23 in the years 1928 to 1937, to α increasing from .17 to .37 in the same time period. (See Exercise 11.)

6. EXERCISES

Exercise 7

Given: $C_0 = 2I_0$, $s = 0.75$ and $v = 3$

(a) How many years must pass before the value of $I(t)$ catches up with the value of $C(t)$.

(b) Sketch the graphs of $I(t)$ and $C(t)$. (Assume $I_0 = 1$.)

Exercise 8

Under what circumstances will

$$\frac{C'(t)}{C(t)} = \frac{Y'(t)}{Y(t)} = \frac{I'(t)}{I(t)}$$

for all values of t?

(Hint: First see when $\frac{C'(t)}{C(t)} = \frac{I'(t)}{I(t)}$, and then when $\frac{Y'(t)}{Y(t)} = \frac{I'(t)}{I(t)}$.)

Exercise 9

What is the relationship between $\frac{Y'(t)}{Y(t)}$ and $\alpha(t)$?

Exercise 10

In Feldman's model is it possible to have

$$\frac{C'(t)}{C(t)} < \frac{I'(t)}{I(t)} < \frac{Y'(t)}{Y(t)} \ ?$$

(Hint: First see when $\frac{C'(t)}{C(t)} < \frac{I'(t)}{I(t)}$, and then when $\frac{I'(t)}{I(t)} < \frac{Y'(t)}{Y(t)}$.)

Exercise 11

Discuss possible reasons for the mistaken estimation of $\alpha(t)$ if $I(t)$ and $Y(t)$ are measured in current prices and the inflation rates in the two sectors are different.

Exercise 12

Suppose $s = 0$ and derive a formula for $C(t)$.

Exercise 13

It takes not only capital but also labor to produce output. What are the implicit assumptions of the Feldman Model about the supply of labor?

Exercise 14

What are the implicit assumptions of the Feldman model about international trade?

Exercise 15

Suppose we drop the assumption $v_P = v_C = V$, i.e., v_P and v_C are now two different constants. How does this modify Equation (4)?

7. REFERENCE

Domar, E. D. (1957), *Essays in the Theory of Economic Growth*, Oxford Press, Oxford.

8. ANSWERS TO EXERCISES

3. $\alpha'(t) = \dfrac{ks(s/\alpha_0 - 1)e^{-kt}}{[(s/\alpha_0 - 1)e^{-kt} + 1]^2}$ is positive if $s > \alpha_0$, and negative if $s < \alpha_0$.

4. $\alpha(5) = 0.077$
   $\alpha(10) = 0.116$

5. (a) Unit is time; e.g., years.
   (b) Stays same.
   (c) $v(\text{months}) = 12\ v(\text{years})$

6. (a) $I'/I = 3.33\%$ $C'/C = 7.2\%$ $Y'/Y = 6.5\%$
   (b) $I'/I = 20\%$ $C'/C = 3.9\%$ $Y'/Y = 7.6\%$
   (c) $I'/I = 30\%$ $C'/C = 1.1\%$ $Y'/Y = 8.4\%$

7. (a) 3.66 years

8. $\alpha_0 = s$

9. $\alpha(t) = v \cdot \dfrac{Y'(t)}{Y(t)}$

10. No.

12. $I'(t) = \frac{s}{v} I(t) = 0$

$\therefore I(t) = I_0 = \text{constant}$

$C'(t) = \dfrac{1 - s}{v} \quad I(t) = \dfrac{I_0}{v}$

$C(t)\big|_0^t = \dfrac{I_0}{v} t\big|_0^t$

$C(t) = C_0 + \dfrac{I_0}{v} t$

15. $C(t) = C_0 + \dfrac{1 - s}{s} \cdot \dfrac{v_p}{v_C} I_0 \left(e^{\frac{s}{v_p}} - 1\right).$

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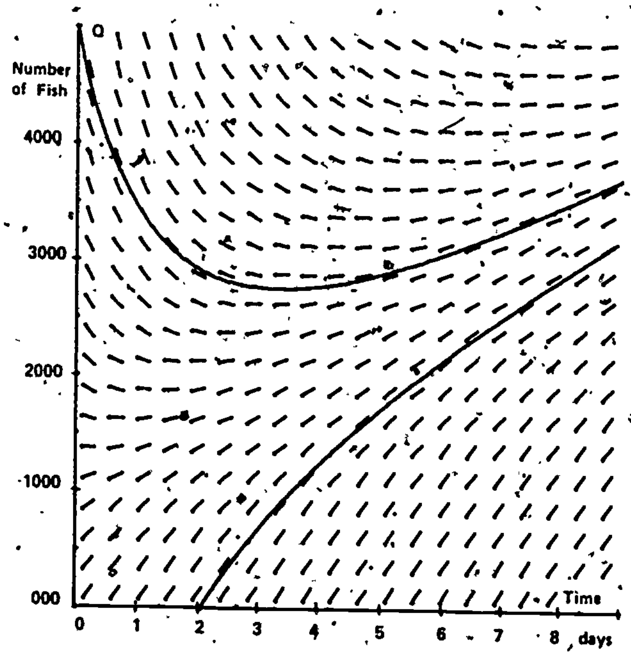
umap

UNITS 81, 82, 83

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

GRAPHICAL AND NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS



August 1977

edc/umap/55chapel st./newton. mass. 02160

GRAPHICAL AND NUMERICAL SOLUTION
OF DIFFERENTIAL EQUATIONS

by

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Intermodular Description Sheet: UMAP Units 81-83

Title: GRAPHICAL AND NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Author: Paul Calter
Vermont Technical College
Randolph Center, Vermont 05061

Review Stage/Date: IV 6/30/77

Classification: DIFFERENTIAL EQUATIONS, GRAPHICAL & NUMERICAL SOLUTIONS

Suggested Support Material:

1. A calculator will greatly reduce the drudgery of computation.
2. A computer, if one wishes to program Euler's Method.
3. Optical Filter Problem: Desk lamp, exposure meter, tracing paper, sheet of glass.
4. Beam Deflection: Hacksaw blade, notebook-sized piece of wood, two nails, graph paper.

References:

Stark, Peter, Introduction to Numerical Methods, MacMillan, 1970.
Reddick and Miller, Advanced Mathematics for Engineers, Wiley, 1955.
Scarborough, Numerical Mathematical Analysis, Johns Hopkins, 1966.

Prerequisite Skills:

1. Differentiate and integrate simple rational, logarithmic and exponential functions.
2. Identify the derivative as a rate of change of a function, and as the slope of a line tangent to graph of the function.
3. Determine the derivative of a curve at a point by measuring the slope of a line tangent at that point.
4. Use the Δx notation for an increment of, or a finite change in, x.
5. Given the forces on a beam (all the forces perpendicular to the beam), find the total moment of force about a given point.

Output Skills:

1. Given a verbal description of a simple situation that can be described by means of firdt order differential equation, write such an equation.
2. Given an equation, a graph, or a table of data points, determine whether they represent solutions to a given differential equation.
3. Draw a tangent field for a given first order differential equation.
4. Sketch several possible solutions through a given tangent field.
5. Solve a first order differential equation by a graphical application of Euler's Method.
6. Carry out a numerical solution of a differential equation by Euler's Method, either by hand, by using a calculator or by computer.

Preliminary versions published as Project CALC Units 81, 82, 83 with the title:
Solution of Differential Equations by Graphical and Numerical Means; or A CALCulated Plot

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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PREFACE

In your mathematical studies up to this point, you have often been called upon to make a mathematical "description" of a situation. This description usually consisted of an equation of some sort. The familiar "word problems" in an algebra course gave rise to one or more linear equations, or perhaps a quadratic or exponential equation.

When the situation to be described contains a non-uniform *rate of change*, the equation will contain a *derivative*. It is then called a *differential equation*.

It is the purpose of this module to show how to describe certain physical situations by means of differential equations and how to solve these equations by simple graphical or numerical techniques.

The differential equations (DE's) treated in this module will be *ordinary* (containing no partial derivatives), *first order* (will contain only *first derivatives*), and of *first degree* (the derivative will not be raised to any power higher than *one*).

I would like to express my appreciation to Mary Jane Neuendorffer and William U. Walton of Project CALC for their extensive help in preparing this module, and to the many people who reviewed the draft and offered valuable suggestions.

Unit 81: PROBLEMS LEADING TO DIFFERENTIAL EQUATIONS

Chapter 1

THE OPTICAL FILTER PROBLEM

I threw down my pencil in frustration. Taking my action as a signal, Polly and Herb did the same.

Professor Arclet didn't notice. He had dozed off, his habit when he gave a test, and he was snoring lightly now, forming silent equations with his lips.

Herb held the book poised above the floor. His chubby face showed both discouragement and apprehension. He looked at me with a questioning smile. That book. Another reason for our discontent. Heavy. Complicated. Eighteen dollars. I knew it was unkind, but I gave Herb the nod.

Five pounds of calculus came crashing to the floor. Arclet sprang to his feet, banging his knees painfully on the desk. He staggered to the blackboard and began to lecture, a continuation of the derivation begun in his sleep.

"Professor Arclet," I interrupted. "We can't do this test."

The exam is what he calls a *pre-test*. It was the one on differential equations. Arclet always gave us a pre-test when we got to a new topic, and those of us that passed didn't have to come to class until the next topic came up. I never passed any of these tests, but they did give me some ideas of what was coming, and what I was supposed to be able to do later on. Sometimes he used the same test after we finished the topic.

"Well," he said, rubbing his sore knees and absently staring out at the Vermont landscape, "that means we need to go over this material."

There was a groan from Polly, who had freckles and wavy auburn hair. "But professor," she said, "differential equations seem so, so--nowhere! We spend all that time and effort learning this stuff, and I'll bet we never use it in a million years." Polly's face was flushed and her voice shook.

"Yeah, yeah," from Herb and me.

Encouraged by this support, Polly went on. "What is the stuff good for? Can you give us a single example of where someone used a differential equation to solve a real important problem?"

Polly sat down to our applause.

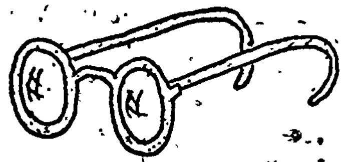
Arclet was silent for a long time, his thick, bushy brows drawn together in concentration. This was our first open rebellion, and I wondered how he would handle it.

He shuffled around to the front of the desk and sat on it. We knew this was symbolic--removing the barriers between student and teacher. This usually meant that he was going to tell us not to think of ourselves as students versus teacher, but as a group of people seeking the truth together. Instead, he just sat there, kicking his heels against the side of the desk.

"Let me tell you a little story," he began after a minute or two of thumping. "It's about a young friend of mine, named Denis Dropmore."

I settled comfortably into my seat, ready to doze.

"Young Denis," Arclet said, "was a proposal writer at the Deadly Nightshade Sunglass Company, back in the 'sixties.' One day, the company received a request from the Department of Defense to prepare a quotation for a badly needed item. Some unmentioned foreign power had developed a new weapon, the Ballistic Laser for Infantry Neutralization and Destruction, code name BLIND. Denis's company was asked to produce a protective eyecover to be worn by soldiers: Sunglasses, Head Attached, Defensive, Expendable, Shatterproof, code name SHADES.



"The DOD had sent along a small sample of the lens material to be used. Only the thickness could be changed. Another requirement was that the glasses reduce the intensity of the laser beam to 10 percent of its value.

"Testing the sample in the laboratory, the company discovered that the 1mm thick material would reduce the intensity of a laser beam by 15 percent. 'Well, good,' said Al, the company's softspoken chief engineer. 'If one millimeter of that material will remove 15 percent, then six millimeters will remove the required 90 percent. Denis, write into the proposal that we will make the lenses six millimeters thick.'

"Denis chuckled. 'Al,' he said, loud enough for everyone in the office to hear, 'you don't know anything about optical filters, do you?'

"Al smiled up at Denis. 'You're fired,' he said softly."

The class bell rang. Arclet's story grabbed us, but so did lunch; we all ran out.

Chapter 2

THE SAGGING BEAM PROBLEM

Today, Polly, Herb and I were not late for class for the first time in the semester. Professor Arclet was seated at the edge of the desk, exactly as we had left him yesterday.

"Denis Dropmore sat drinking beer at home," Arclet began, "dreading the return of his wife. How could he break the news? At 5:30, Angelica Dropmore came home. She looked distressed. Denis took his young wife by the hand. 'Honey,' he stammered. 'Today I ...'

"'Denis, I got fired,' wailed Angelica.

"Denis was what you'd call flabbergasted. Angelica poured out her story.

"'I was at the bank, as usual, today, tidying up my desk after the last customer left, and couldn't help hearing Nichols and Dymes, those two creepy vice presidents, having a battle. You know the ones--always trying to outdo each other.

"'Nichols shouted, "Under the safe."

"'Dymes screamed, "No, you twit, it goes under the midspan."

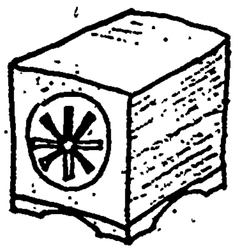
"Angelica raced on, 'It all had to do with that heavy safe coming in next week. They found a location directly over a thirty-foot-long steel beam, but couldn't get it closer than six feet from the end support. Old Mr. Usury, the bank president, was afraid that bending of the beam under that massive load would cause cracking of the expensive marble floor tiles. He told the two VP's to install a third vertical column under the beam at the point where the sag would be greatest if the third column wasn't there. Nichols wanted to put it directly under the safe, while Dymes said no way, that the greatest sag would be at the midpoint of the beam.

"'Remember,' Denis, I studied structures in architecture school, so *I knew better*.

"'The column should go *between* the safe and the midpoint,' I blurted.

"'Two pairs of eyes turned on me. Disbelief and contempt, I felt it in their stare. Nichols and Dymes spoke in unison, agreeing with each other for the first time since they joined the bank.

"'"You're fired," they said.'"



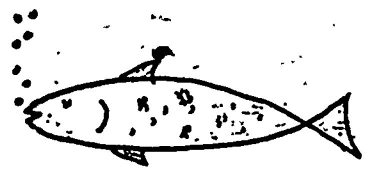
Chapter 3

THE FISH POND PROBLEM

"Denis hated to burden his wife with further bad news," Arclet said, "but hated more to keep anything from her, so he told Angelica the events leading up to his own firing. However, after publicly apologizing to Al for his brash words, Denis had been told that he could have his job back if he brought in, by Monday morning, a computation of the correct lens thickness. That was fine, but even though he knew that Al's answer could *not* be right, he, himself, was not sure how to go about finding the correct answer.

"'Tears formed in Angelica's eyes but she brushed them away. 'Well, at least we have plenty to eat. For a while, anyway.'

"She was referring to the fish.



"Ten days previous, old Cy Seepage had put a dam across a stream running through his property in order to trap some trout. Incensed by the blocking of their favorite trout stream, Denis and Mike Mossy had, the following day, opened the drain. Water flowed from the pond, carrying with it trout. Denis and Mike collected them in a wire basket hidden by the dense, hillside brush. They had left the basket in place day and night, occasionally emptying it and storing the trout in Mike's freezer. Every day, old Seepage checked the level of the pond, planning to net all the trout as soon as the pond was full. The level continued to rise, for water flowed in faster than it could be drained out. Seepage had no reason to suspect foul play.

"The Dropmore's telephone rang.

"It's for you.' Angelica handed the receiver to Denis. 'It's Mike Mossy.' She wrinkled her nose.

"'Hi, Mike, what's up?'

"'The pond is. Nearly full. Seepage says he's going up tomorrow to catch and count the fish, and half the town is going up to watch.'

"'That's great, Mike. All we have to do is divvy up the fish we swiped.'

"'Er--that's why I called. I've got a proposition. Let's each of us give a guess at the fish left in the pond. The one coming the closest gets *all* the fish. How about it?'

"Denis was hesitant. Mike wasn't too bright, but he *was* lucky; always winning contests. Denis was finally shamed into agreeing.

"'How could you be so stupid,' Angelica cried, when she heard what Denis had done. 'Don't you remember that both our families are coming to dinner tomorrow night, and I was going to serve that trout? Ten guests coming and no main dish, and no money left either. What are we going to do?'"

Chapter 4

MODELLING THE OPTICAL FILTER PROBLEM

By now, Herb and Polly and I were all involved with Denis and Angelica Dropmore. We begged Arclet to continue the story even after the bell rang. He just shook his shaggy head, so off we went, discussing their problems continuously until class the next day.

Arclet came into the classroom with a cardboard carton, and dumped the contents onto the desk. It was a funny collection of junk. There was a desk lamp, measuring spoons, wood blocks, a bag of peas, a hacksaw blade, and lots of other things. Arclet stood beside the heap, looking arch.

"Aren't you going to continue the story today?" we wanted to know.

"This *is* the continuation of the story," he said mysteriously, indicating the pile of junk with a grand sweep of his arm. He was a pain when he put on the theatrics.

He cleared his throat. "To continue, we left Denis on a Friday night, with a highly troubled mind. At eight o'clock, he received another 'phone call. It was Mr. Usury, the president of the bank. He was very kind, and said he suspected that what Angelica said about the beam was correct. If she could come in by 8 AM on Monday with something that would convince him that she was right, Mr. Usury would not only rehire her, but would give her a raise as well.

"The challenge was now clear. Angelica and Denis were faced with three problems, one of which had to be solved by three o'clock the following day, and the other two by Monday morning.

"When Angelica came down Saturday morning, she found Denis asleep at the kitchen table. Heaped on the table and the counters were the very items you see before you now, and sheets of paper covered with scrawled figures and graphs lay strewn about the floor. He had devised ways to *model* the three problems."

"Is that what you're going to show us now?" I asked Professor Arclet.

"Not exactly," Arclet said. "That is what *you're* going to show *me*."

Not only did Arclet make me perform an experiment right on the spot, but he make me write it up as well, complete with objective, steps, conclusion--the whole bit. Much of the following description is lifted right out of my notebook.

<u>Title</u>: THE OPTICAL FILTER EXPERIMENT

<u>Objective</u>: *To learn about filters.*

At this point Arclet objected to my objective as being too "narrow." I crossed it out and wrote:

<u>Objective</u>: *To see how different equations can arise from a physical problem.*

Now Arclet objected to my objective as being too objective.

"Make your objective more subjective," he urged.

"What I mean," he said in response to my vacant look, "is that you should phrase your objective in terms of what you will be able to do *after* the lesson, that you couldn't do before."

So I wrote:

<u>Objective</u>: *After this lesson, I should be able to write a differential equation to describe a physical problem.* (If it's real easy.)

Arclet sniffed at this, but let me go on to the next step.

<u>Materials</u>:
1. Desk lamp with 60 watt bulb.
2. Photographic exposure meter with a scale graduated in EV (exposure value).
3. About 40 sheets of translucent tracing paper, or typist's onion skin. These will be our filters.
4. A sheet of glass large enough to cover the front on the lamp housing.

<u>Procedure</u>:

Step 1: Swivel the lamp upward, so that it points at the ceiling. Place the glass over it, and tape it in place. Place one sheet of paper (filter) on the glass.

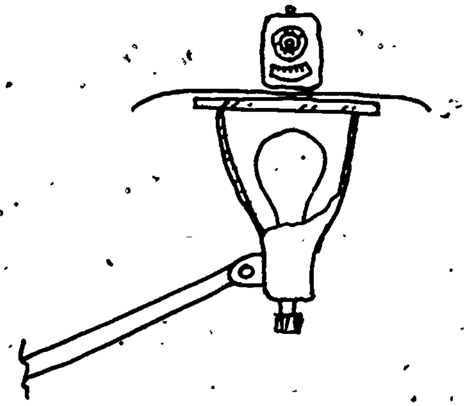


Figure 1: Apparatus for the Optical Experiment

Step 2: Holding the meter right against the filter take a first reading. This will be the starting value, so it should produce a needle deflection close to the top of the scale. If it's too high, cut down the light by using more filters or a smaller bulb. If too low, use a larger bulb.

Step 3: Once you get this large deflection, tape down the filters and mark the outline of the meter on the top one so the meter can always be returned to the same location. Rotate the meter dial until the number *ten* of the EV scale is opposite the needle.

Turn off the lamp when not taking readings so the paper won't overheat.

Step 4: Add filters, singly or in groups, each time noting the meter reading and the total number of filters added. Do *not* count the filters taped to the lamp, as these were only used to adjust the starting intensity, and to control the color of the light reaching the meter.

Step 5: Keep adding filters until the light reaching the meter is so weak that no further readings can be obtained.

I wound up with a two-column table of data looking something like this:

Experimental Data:

t Number of Filters	L Light Level (EV)
0	10.0
1	7.5
2	5.5
.	.
.	.
.	.

Plot of the Data:

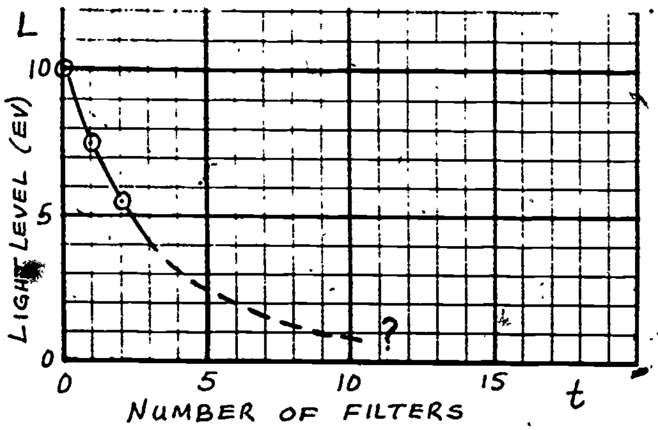


Figure 2: Plot of Light Transmission Vs. Number of Filters

After plotting the graph, I attempted to explain to the others who were looking over my shoulder.

"Now, it's pretty clear that not every filter is absorbing the same amount of light, for if it was, the curve would be a straight line sloping downward. It's plain that each filter is absorbing *less* light than the one before it, because the reduction in light per filter gets smaller, for each additional layer."

"That makes sense," responded Arclet. "When we say that a filter absorbs, say, 25 percent, it means 25 percent of the light *reaching the filter*. Since less light reaches the farther filters, they absorb less.

"In other words: *The amount absorbed depends upon how much is present.*

"In other words: *The rate of decrease of the light level is proportional to the light level.*

"In other words: *The rate of change of L is proportional to L, if we let L represent the light level.*"

"How can you talk about a *rate*?" asked Herb. "The filter isn't moving."

"Quite so," said Arclet, "but do we need movement to have a rate of change? What about *interest rate*, eh?"

"Oh, yeah," said Herb.

Arclet went to the board and wrote,

[The rate of change of *L*] [Is] [Proportional to] [*L*]

"Now for an equation," he said. "There are four expressions, each enclosed in brackets, and four of us: Each of us will write the proper mathematical symbol below one of the expressions. I'll go first." Under [Is] he chalked in a huge equals sign, and, with sickening coyness, handed the chalk to Herb. "Next," he said.

Herb dashed to the board and, under [*L*], wrote "*L*" and tossed the chalk to Polly.

Under [Proportional to] Polly wrote the proportionality symbol $\propto$, but just stood there with a dissatisfied wrinkle in her nose. "How can we have both an equals sign *and* a proportionality sign in the same equation?" Her question was directed at the blackboard, but Arclet answered instead.

"You cannot have both. Instead of the proportionality symbol, how about a ...."

"Of course," said Polly, and she wrote a constant of proportionality, *k*. The equation now read,

[The rate of change of *L*] $= kL$

and the chalk was passed to me.

Remembering that *rate of change* always meant a derivative, I wrote,

$$\frac{dL}{d?}$$

under the remaining expression.

"Oh, I see," said Arclet mockingly. "The derivative of *L* with respect to I-don't-know-what. Very resourceful."

Smarting, I wrote,

$$\frac{dL}{dt} = kL$$

where t = total thickness of filters (number of filter sheets).

"Just one thing missing now," said Arclet and, after waiting a moment for one of us to supply the missing thing, went to the board and wrote a minus sign with a big flourish. "The slopes are all negative, right?"

The equation was then,

$$\boxed{\frac{dL}{dt} = -kL}$$

To see if this equation really held for my data, I sketched in the tangent to the curve at several points. I then measured the slope of those tangents (rise over run). At each point where I measured the slope, I read the ordinate *L*, and divided the slope, (dL/dt), by the ordinate to obtain *k*.

$$k = -\frac{dL/dt}{L}$$

I got approximately the same value for k at all of the points taken.

"Not bad, not bad at all," Arclet said, as I returned to my seat. "Now take another look at that equation. Is there anything different about it, or is it like all the other equations we've seen so far?"

"It's got a derivative in it," Herb said.

"Right!" Arclet was encouraged by the alert response. "And any equation containing a derivative is called......." He waved his arms like a conductor, trying to get us to sing out the remainder of his sentence. We sat there silent.

"A DIFFERENTIAL EQUATION," he hissed through his teeth. "Class dismissed."

Chapter 5

MODELLING THE SAGGING BEAM

"Today, we will do another experiment," Arclet said after we'd settled into our seats. "To begin,"

"But the story," we howled.

"Are you here for learning, or for entertainment?" the professor growled. (We knew we had gone too far.) Then, more softly, "We'll get back to the story. But first it's important for you to understand the experiments that Denis Dropmore stayed up all night to perform in his kitchen.

"Now, who will do this one? Polly, my dear, come up here."

Polly went up to the desk and performed the experiment with some assistance from Arclet. The following outline is almost word-for-word from the notes I took that day:

Title: THE BEAM EXPERIMENT

Objective: *To be able to make a model of a beam deflection problem.*

Materials:
1. Hacksaw blade
2. Sheet of wood, at least 8½ by 11 inches, and at least ½ inch thick
3. Two nails
4. Sheet of rectangular graph paper
5. Ruler

Procedure:

Step 1: Tape the graph paper to the board, so that the edge of the paper is even with the edge of the board.

Step 2: Draw a line down the middle of the paper, parallel to the long edge.

Step 3: Drive the two nails into this mid-line. Let them extend about 3/4 inch. The distance between the nails should be 10 inches.

Step 4: Prop the board into a vertical position. Lay the hacksaw blade across the nails.

Step 5: One and a half inches from one of the nails, push against the blade with the eraser end of a pencil. Get an assistant to trace the shape of the blade on the graph paper.

Step 6: Changing the pressure against the blade, draw several such deflection curves.

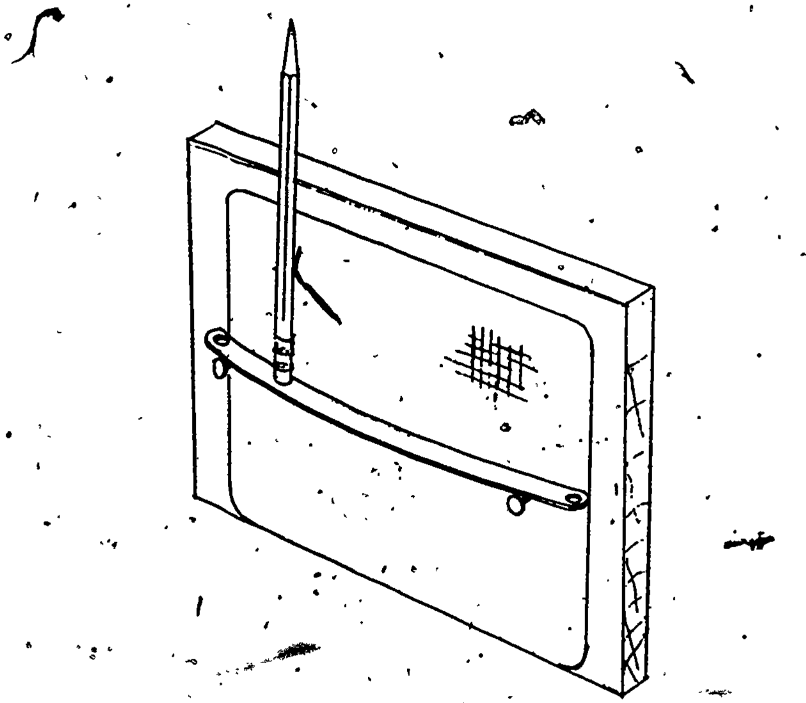


Figure 3: Apparatus for the Beam Deflection Experiment

Arclet pointed out that many physical problems required special knowledge of the field in order to even write the equation, and that this was one such case.

"Fortunately, I happen to have a broad engineering background in addition to my mathematical training," said Arclet. "Now write this down." He tilted his head back and defocussed his eyes, as if he were reading something off the inside of his skull. "The second derivative of the vertical displacement y with respect to the position x along the beam is directly proportional to the bending moment M and inversely proportional to the product of the modulus of elasticity E of the material and the moment of inertia I of the beam cross-section."

This entire sentence was delivered without taking a breath and in a flat monotone, like a third-grader reciting a poem on Parent's Day.

As with the optical filter problem, we wrote the expression in brackets and, with less difficulty this time, wrote the symbols beneath.

$$\left(\begin{array}{c}\text{The second derivative}\\ \text{of } y \text{ with respect}\\ \text{to } x\end{array}\right) \left(\begin{array}{c}\text{is directly}\\ \text{proportional}\\ \text{to}\end{array}\right) \left(M\right) \left(\begin{array}{c}\text{and inversely}\\ \text{proportional}\\ \text{to } EI\end{array}\right)$$

$$\frac{d^2y}{dx^2} \quad = k \quad M \quad \frac{1}{EI}$$

or,

$$\frac{d^2y}{dx^2} = k\,\frac{M}{EI}$$

Arclet then pointed out that E was, in fact, the constant of proportionality in this equation, so that a separate k was not needed. Our final equation was then,

$$\boxed{\frac{d^2y}{dx^2} = \frac{M}{EI}}$$

"Don't worry about this equation now," Arclet said. "We'll go into it in detail when I explain later how we solved this problem numerically. For now, it's enough that you find the point of maximum deflection on your hacksaw curve, and verify that it lies between the load and the midpoint, as Angelica had claimed."

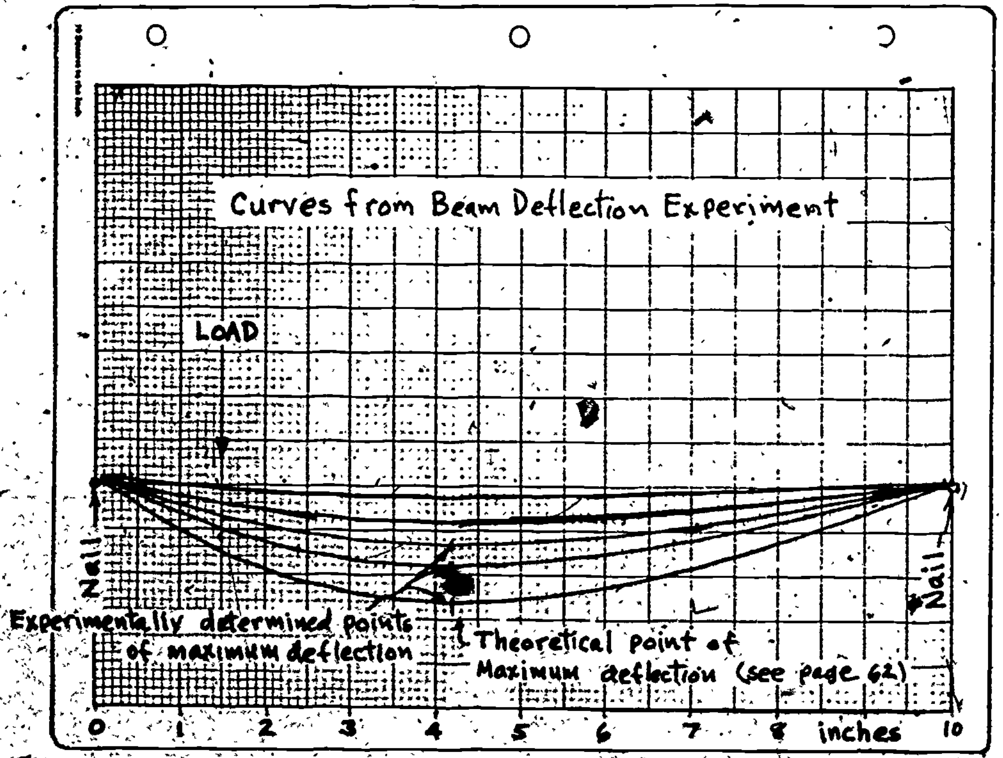


Figure 4. The curves Polly drew of the deflected hacksaw blade.

At the end of our second lesson in writing differential equations for different physical problems, Arclet gave us the following quiz.

Quiz #1

1. If the rate at which heat is lost by a body is proportional to the temperature difference between the body and its surroundings, write a differential equation to describe this situation.
2. An object moves in a medium offering resistance proportional to the square root of its velocity at any instant. Describe this situation with a differential equation.
3. State in words what a differential equation is.

For answers see page 67.

Unit 82: SOLVING DIFFERENTIAL EQUATIONS GRAPHICALLY

Chapter 6

PROFESSOR ARCLET TO THE RESCUE

Today, Arclet was seated on the edge of his desk, a signal that the story was to resume. We took our seats in a hurry.

"You will recall," he began, "that we left Denis Dropmore asleep in a kitchen filled with the debris of the two experiments we have just performed. Sweet Angelica, fearing that her husband's mind might have snapped under the stress of their troubles, woke him gently.

"'Careful of the fish!' he shouted, pushing her arm away from the little pile of green, split peas with which he had unsuccessfully tried to simulate the fish pond problem.

"'There, there, Denis dear. Of course I won't hurt your little fishies. Now, you come upstairs and lie down while I make some 'phone calls.'

"'No time, no time,' he cried, and began rushing about the kitchen, rumpled and unshaven, counting piles of split peas, and holding sheets of tracing paper up to his swollen, red eyes.

"'So much to do, and so little time. I feel that I'm so close to an answer but I don't know what to do next.'

"'Now, Denis, please lie down, and let me call the doctor.'

"Denis stopped dead in his tracks.

"'The doctor? The doctor! His eyes were wild, and he shook the hacksaw blade menacingly. 'Of course! I need *the doctor*. Angelica, you're a genius.' He ran to the phone and called the only doctor of mathematics he knew. Me, Arclet, of course. Realizing the urgency of his problems, I went over at once.

"At Denis' house, we exchanged the briefest of greetings, and went right to work. I scanned the results of his experiments, and was astounded at their cleverness. Realize, of course, that he had no equations written, as we have done, and I set about writing these. I sent Denis off to take some measurements at the pond.

"He returned in an hour with the measurements and estimates, and with the news that people had already begun to gather at the pond. Mike Mossy was there, annoyed that Denis didn't have his guess ready yet.

"Since the fish problem required a solution sooner than the other two, we tackled that first. We solved it simply by drawing a tangent field.

"Class dismissed."

Chapter 7

TANGENT FIELDS, AND SOLUTIONS TO DE'S

Arclet was so obvious. He had left us hanging yesterday, with that reference to tangent fields: today we were suposed to rush into class and yell, "What is a tangent field? What is a solution? Teach us, teach us. I hate to admit it, but I *was* a little curious. Arclet came into the room.

"Lesson today, right?" we asked. He winked and began writing on the blackboard.

Title: THE MEANING OF A SOLUTION TO A DIFFERENTIAL EQUATION. INTRODUCTION TO TANGENT FIELDS.

Objective: *By the end of this lesson, you should be able to:*

a) Verify whether a particular equation, graph, or table of point pairs is a solution to a given differential equation.

b) Draw a tangent field.

c) Use a tangent field to sketch a solution.

These preliminaries out of the way, Arclet began to lecture.

"When you *solve* algebraic equations, what do you get, aside from a headache?"

It looked as if we were in for one of Arclet's lighter lectures.

"You get some number, the *root*, which is the value of x at which the plot of the equation crosses the x-axis.

If the curve crosses in more than one place, as a quadratic may, you get a root at each crossing. Fine.

"But what does it mean to solve a *differential* equation? What sort of answer can we expect? A number? Several numbers? An equation? One objective of this unit is to help you to understand what the solution to a differential equation means, and how to recognize one. After completing this unit, you should be able to examine what is claimed to be the solution of a particular DE, and say for certain whether it is an impostor or not. We will not *find* any solutions. Not yet.

"Look at this differential equation:

$$\frac{dy}{dx} + 2y - x = 0 .$$

If we think in graphical terms, dy/dx is the slope of some curve $y = f(x)$. We can represent this slope by m. For simplicity, we rewrite the DE using m.

$$m + 2y - x = 0$$

"Any three numbers, m, x and y that will satisfy this equation will be a solution. Take, for example, $m = 1$, $x = 5$, and $y = 2$. Substituting, we get

$$1 + 2(2) - 5 = 0 .$$

"It should be apparent that there are infinitely many such combinations that will satisfy our equation. Just pick any values for x and y out of the air, and solve the equation for m.

"Suppose that we pick the values 1 and 3 for x and y.

Then

$$\begin{aligned} m &= x - 2y \\ &= 1 - 2(3) \\ &= -5 \end{aligned}$$

Thus the set of numbers -5, 1 and 3 will satisfy our equation. Now compute three or four more sets of numbers that will also work. To keep from getting too spread out, take values of x and y between 0 and 10.

"Got the numbers? All right, now plot them on a sheet of graph paper, using x for the abscissa and y for the ordinate in the usual way. Alongside each point, write in the value of m. You should now have something looking more or less like this.

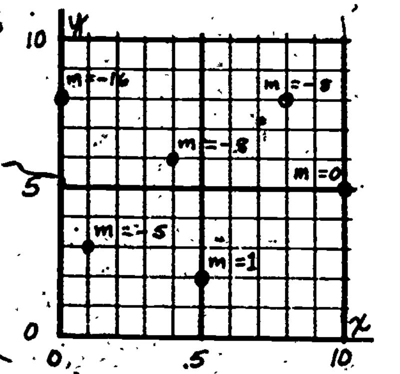


Figure 5: Slopes at Various Points

"Draw a smooth curve through the points. In fact, try to draw in your curve so that it will have the *slope* required at each point."

"It doesn't work, right? Don't worry, I didn't expect it to. But now I ask you the following question. Instead of choosing x and y at random, can you think of some clever way of picking them so that when all the points are connected, the slopes at each point are precisely those required by the equation? Why should you go to all this trouble? Because

> WHEN YOU DO THAT, YOUR COLLECTION OF POINTS WILL BE A SOLUTION OF THE DIFFERENTIAL EQUATION.

I wouldn't ask you to do something that wasn't important.

"At this point, take five minutes to struggle with that problem. You can work by trial and error if you like. A good way to start would be to draw a short line through each of your points, with a slope of the proper value.

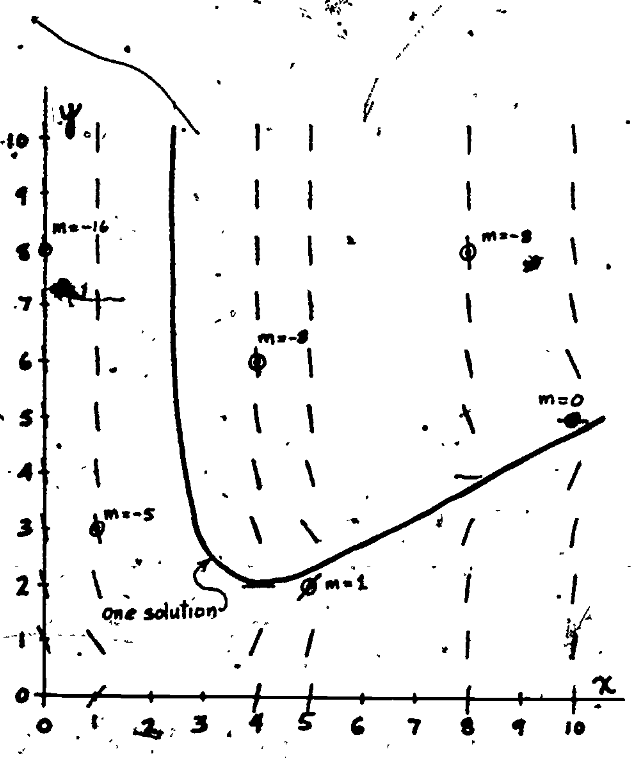


Figure 6: Part of the tangent field for $\frac{dy}{dx} + 2y - x = 0$

After five minutes, Arclet wrote on the board, Tangent Fields. He then continued to lecture.

"If you had graphically indicated the slopes at your points, and had added more points to your plot, you would arrive at a diagram known as a *tangent field*, or a *direction field*. If we sketch in a curve whose *slope* is always the same as that of the surrounding tangent field, we know that any point on the curve will satisfy the differential equation. The collection of points on the curve is therefore one solution of the DE.

"Now go ahead and complete your own tangent field, and sketch in several possible solutions."

He gave us a few minutes to do that before continuing.

Different Forms of a Solution

"The curves that you sketched previously are solutions to the differential equation. For the moment, don't worry that there are an infinite number of such curves. We will get to that problem later. The information in these curves can be presented in three different ways.

a) Graphically, as they are now.

b) Numerically, as a table of x,y pairs.

c) Analytically, as the equation of the curve connecting the points.

"In our next lesson we will solve a DE *graphically*, and obtain a curve. Later, we'll do a *numerical* solution and arrive at a table of point pairs for an answer. An *analytic* solution is obtained by manipulating the differential equation. We're not going to do that."

I raised my hands to clap, but Arclet's icy stare stopped me cold.

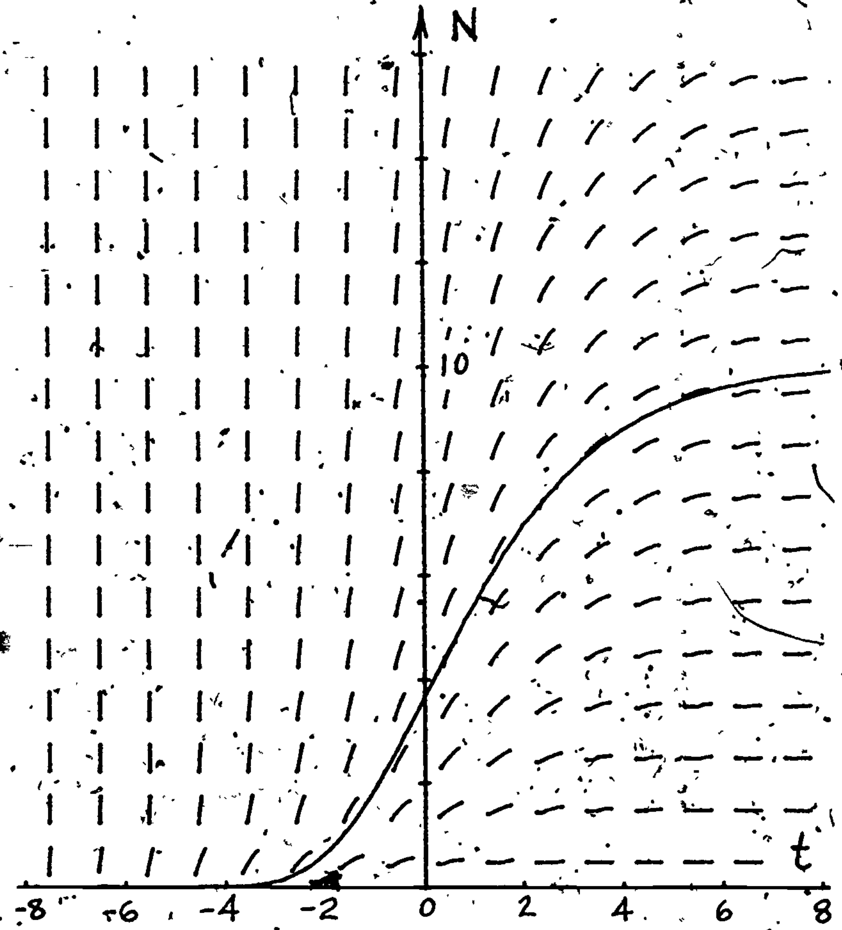
Boundary Conditions

"Why so many solutions? At this stage in your math career you should not be too surprised to get an infinite number of possible answers when you reverse a mathematical operation. Look what happened when you took the inverse of a trig function; or an anti-derivative. In solving a differential equation, integration must be performed, and it is the unknown constant of integration which makes the result indefinite. As with the anti-derivative, some additional information is needed before the unknown constants can be evaluated. These additional facts are called *boundary conditions*.

"For example, suppose I told you that, in our previous problem, the slope had to be -7 when x was zero. Wouldn't you merely draw in the one curve having the required slope where it crossed the y-axis, and ignore the rest? Or, if you knew that the curve had to pass through the point (4, 5), this information would exclude all possible curves but one.

"If you have access to a computer, you might enjoy writing a program to plot tangent fields. Have the program print numbers giving the slope, at the proper coordinates on the paper. If you have a plotter available, you even can have it draw in the tangent lines. Here is a tangent field produced by such a program."

Figure 7. A Computer-Generated Tangent Field



$\frac{dN}{dt} = bNe^{-kt}$, where $k = 0.5$, and $b = 0.5$.

Arclet concluded the lesson by giving us the following quiz.

Quiz #2

1. Verify that the equation

$$y = \frac{x}{2}(\ln x + 1)$$

is a solution to the differential equation

$$\frac{dy}{dx} = \frac{x + 2y}{2x}$$

by plotting it for values of x from 1 to 5, and measuring slopes of at least three points.

2. Given the differential equation

$$\frac{dy}{dx} = \frac{2x - y}{x}$$

plot the tangent field, letting x range from 1 to 5 and y range from 0 to 5. Sketch in a possible solution.

For answers see page 68.

Chapter 8

THE FISH POND PROBLEM SOLVED WITH A TANGENT FIELD

"Time was of the essence," Arclet said, continuing the tale. "Mike Mossy had just called, reminding Denis of their agreement; if one of them failed to make a guess by the time the fish were counted, he forfeits his share automatically. I worked furiously over the graph while Denis fed me the measurements he had taken and Angelica did the computations on a pocket calculator.

"Here is what Denis found at the pond.

a) Water flowed into the pond at an estimated rate of three gallons per minute.

b) Each ten gallons flowing in brought with it two fish.

c) Greedy old Cy Seepage had gotten the Fish and Game Department to stock the pond free of charge by telling them he was going to allow public fishing. They had put in one thousand trout the same day that the dam was finished.

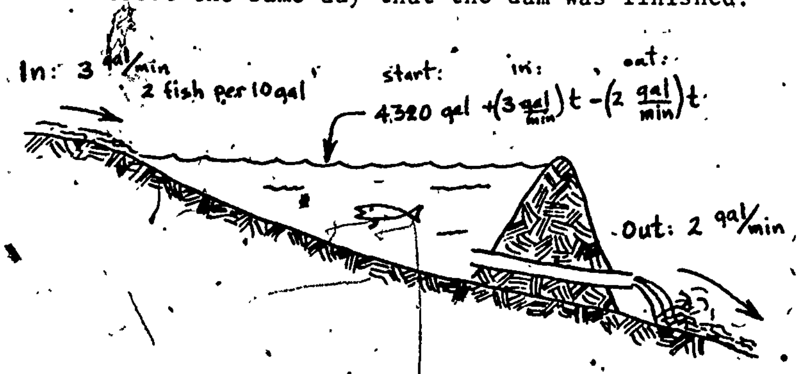


Figure 8. Fish Pond Flow Rates

d) Since Denis and Mike had opened the drain after one day of filling, the pond would then have contained 4320 gallons.

e) Water ran out the drain at the rate of 2 gal/min.

f) The pond was completely filled nine days after the opening of the drain, at which time it contained 17,280 gallons.

"Denis and Angelica watched with awe and gratitude as I developed my clever solution.

"Let Q represent the number of fish in the pond at any instant. Then,

$$\frac{dQ}{dt} = \text{rate of change of the number of fish}$$

Also, let

$$C = \text{the concentration of fish;}$$

that is, the number of fish per gallon of water, at any instant. We assume that the fish are evenly distributed throughout the pond. Since water containing 0.2 fish per gallon was entering at the rate of three gallons per minute, fish were coming in at the rate of $3 \times 0.2 = 0.6$ fish per minute.

"Since water containing C fish per gallon was leaving at the rate of two gallons per minute, fish were being drained off at the rate of $2C$ fish per minute. Here we are making the assumption that the drained water contains the same concentration of fish as in any other place in the pond.

"The rate of change of the number of fish in the pond is obviously [fish in] minus [fish out] or,

$$\frac{dQ}{dt} = 0.6 - 2C. \qquad \text{(equation 1)}$$

The total amount of water, W, in the pond, if we start reckoning time at the opening of the drain, is,

$$W = 4320 + 3t - 2t = 4320 + t,$$

where t is in minutes.

"The concentration of fish is then the total number of fish divided by the total amount of water, or

$$C = \frac{Q}{W} = \frac{Q}{4320 + t}. \qquad \text{(equation 2)}$$

By substituting equation 2 into equation 1, the rate of change of the number of fish becomes,

$$\frac{dQ}{dt} = 0.6 - 2C = 0.6 - \frac{2Q}{4320 + t},$$

which was the differential equation describing the fish population in the pond.

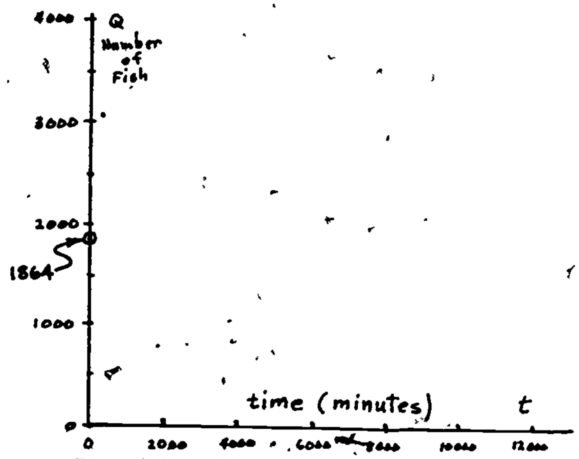


Figure 9a: The Start of Arclet's Graph

"The next step was to plot a tangent field. I let the abiscissa be time, in minutes, extending from zero to

$$9 \text{ days} \times 24 \text{ hr/day} \times 60 \text{ min/hr} = 12{,}960 \text{ min}.$$

I could also compute the number of fish in the pond at time zero. It was,

$$1000 + 4320 \text{ gal} \times 0.2 \text{ fish/gal} = 1864 \text{ fish}.$$

(See figure 9a.)

"I next computed slopes at various points on the graph, naturally taking them only where I expected the curve to be. I computed them from the differential equation

$$\text{slope} = 0.6 - \frac{2Q}{4320 + t}$$

Now this was the graph:

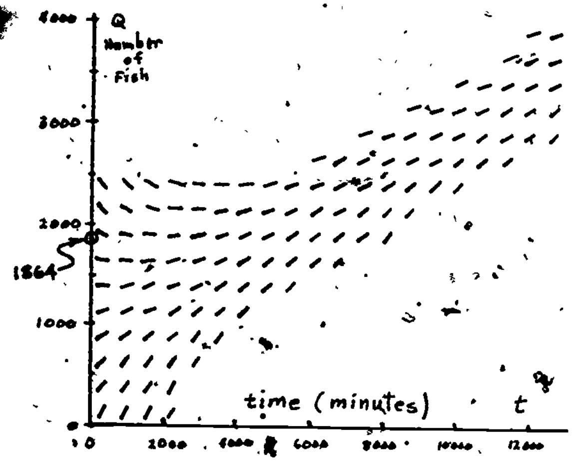


Figure 9b: Arclet's Tangent field for the Fish Problem

and I could easily sketch in the curve showing the growth of the fish population. Where the curve intersected the 12,960 minute line was the number of fish at the time the pond was full.

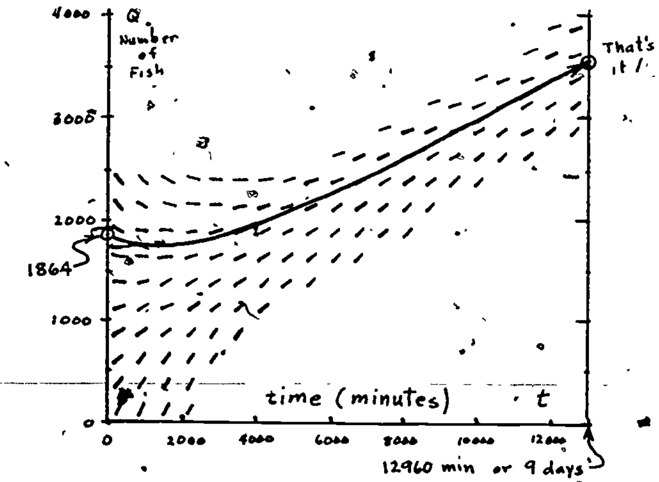


Figure 9c: Arclet's Solution to the Fish Problem

"Denis startled me by tearing the sheet from my hand before the ink was dry, and flew out the door. Angelica and I waited for several hours. Their dinner guests started to arrive, and we made feeble excuses for his absence! He returned home at six. We had a very fine dinner of boiled potatoes, broccoli with hollandaise sauce, pink chablis, and, oh yes, broiled trout."

Arclet stopped speaking. His eyes were glazed as he smacked his lips and recalled that fabulous dinner. Herb wanted to drop his calculus book again, but I said no. It frightened me when Arclet got like this.

We slipped out after the bell rang, leaving the professor sitting on the edge of the desk, muttering something about lime sherbet.

Chapter 9

SOLVING DIFFERENTIAL EQUATIONS GRAPHICALLY

Today Arclet was all business. He strode briskly into the room, always a bad sign, and began a rapid-fire barrage of questions. What was wrong with his solution of the fish problem? Why draw in all those tangents when only a few are needed? How could you limit your work to only those needed? And so on.

I slumped in my chair. It was a good day to lie low.

Confronted with a counter-barrage of silence, Arclet sullenly began the lecture he had been softening us up for. He wrote on the board,

Title: GRAPHICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Objective: *Upon completion of this lesson, you should be able to solve a differential equation by a graphical application of Euler's Method.*

"In the last unit we solved a differential equation graphically by plotting a tangent field," Arclet began. "This is fine, but it takes more work than is really needed. By the end of this unit you should be able to produce a graphical solution by the much easier Euler's method.

"I will illustrate the method by doing an example. We wish to solve the equation:

$$\frac{dy}{dx} = \frac{x^2}{y},$$

with the boundary condition that $y = 1$ when $x = 1$. Suppose, also, that we're interested only in the region between $x = 1$ and $x = 10$.

"We start by plotting our starting value, (1,1) on rectangular coordinate paper. Now our differential equation tells us what the slope at any point should be:

$$\text{slope} = m = \frac{x^2}{y} .$$

At (1,1) the slope should be

$$m = \frac{1^2}{1} = 1 ,$$

so, through (1,1) we draw a short line with a slope of 1. Our construction should be:

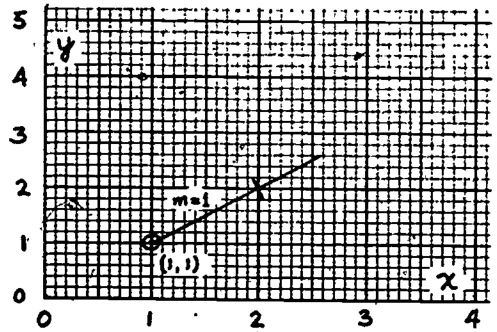


Figure 10a: Construction for a Graphical Solution

"Now we step along the x-axis to a new value of x. The size of the step is important; large steps give inaccurate results and small steps are a lot of work. I'm going to take steps of 1 for this demonstration.

"At $x = 2$, the value of y as read from our previously drawn line, is 2. Mark this point on the graph. Through this point, we draw another short line having the required slope at this point, which is:

$$m = \frac{2^2}{2} = 2 .$$

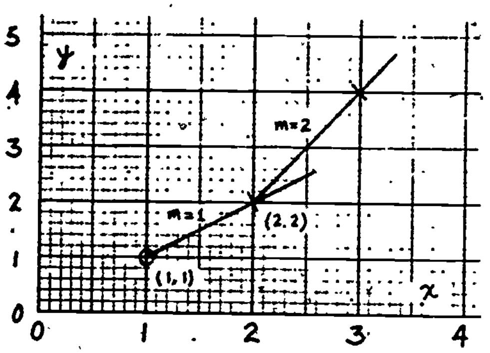


Figure 10b. Continuation of the Construction

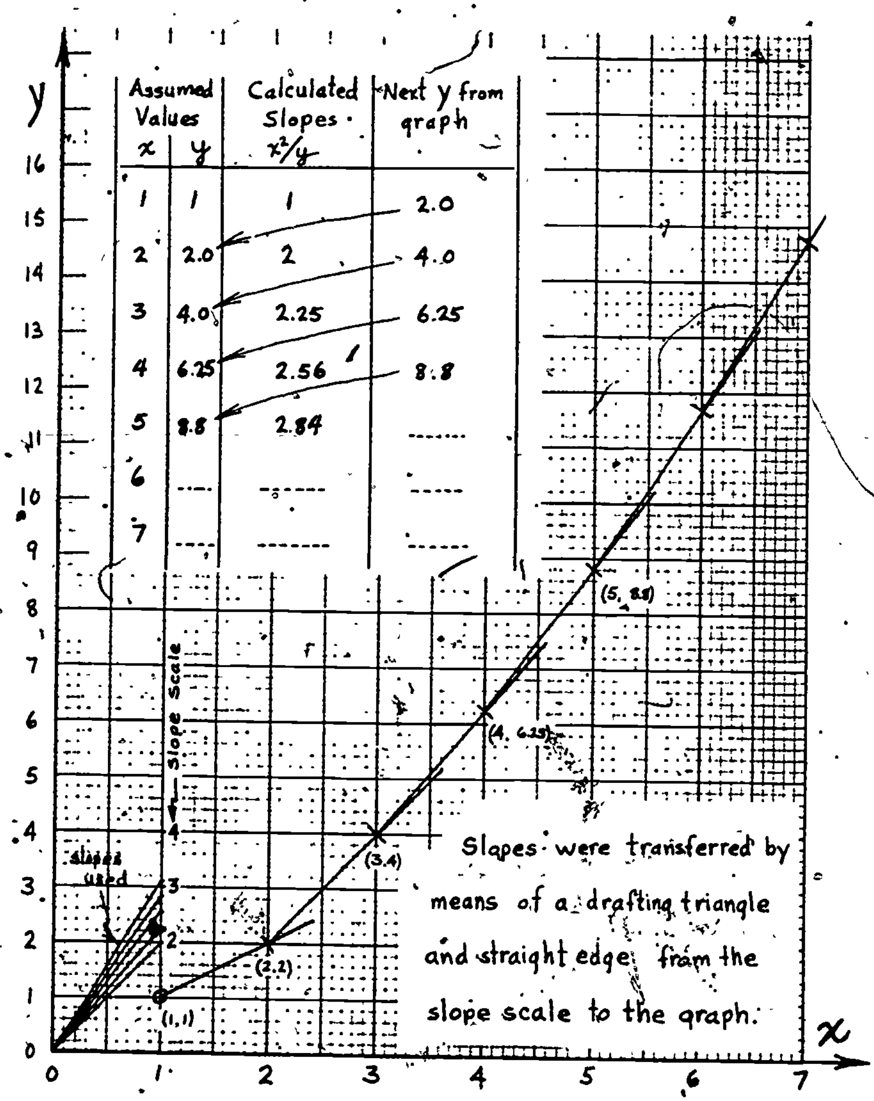
At $x = 3$, y has the value of 4, so the slope of the next segment is

$$m = \frac{3^2}{4} = 2.25 .$$

"The process is repeated as far out as is needed, in this case until $x = 10$."

At this point Arclet paused dramatically. Then he flipped on his ever-ready overhead projector, pulled out a transparency, and said triumphantly, "Viola, the final solution!" (see figure 11)

We all groaned.



Assumed Values x	y	Calculated Slopes x^2/y	Next y from graph
1	1	1	2.0
2	2.0	2	4.0
3	4.0	2.25	6.25
4	6.25	2.56	8.8
5	8.8	2.84	------
6	---	------	-----
7	---	------	-----

Figure 11. Graphical Solution of the Differential Equation $\frac{dy}{dx} = \frac{x^2}{y}$ through the point (1,1).

For our quiz, Arclet gave us the following problem.

Quiz #3

1. Given the differential equation

$$\frac{dy}{dx} = \frac{x + 2y}{2x}$$

with the boundary conditions, $y = 1$ when $x = 1$, solve graphically by Euler's method, taking intervals, delta x of 1, and find the value of y when x equals 5.

For answer see page 70.

Chapter 10

A GRAPHICAL SOLUTION OF THE FILTER PROBLEM

Wasn't it strange how Arclet's story was requiring so much work from us to be able to understand it? Before he could begin his narrative the following day, I had my hand up.

"Professor, is this a true story?"

Arclet looked over my head, focussing his eyes on infinity, and sighed. "My boy, what is truth?"

"Heavy," breathed Polly.

"Bull," muttered Herb.

"To continue," said the professor, "Denis, Angelica, and I were groggy from the huge meal, the chablis, and the cognac I had thoughtfully remembered to bring, so we didn't tackle the remaining two problems after the guests left but waited until the next morning.

"We decided the next most urgent problem was to determine the thickness of the sunglass lens.

"Angelica had persuaded Mr. Usury to let her into the bank, so while Denis and I worked on the lens problem, she went off to take measurements on the steel beam.

"We had decided to do a graphical solution. On a sheet of rectangular paper, we took lens thickness x in millimeters along the abscissa, and light transmission L along the ordinate, graduated from zero to 100%. We marked out two known points: a transmission of 100 percent at a thickness of zero, and a transmission of 85 percent at a thickness of 1mm. We connected the two points, and measured the slope of the line. Our graph was then:

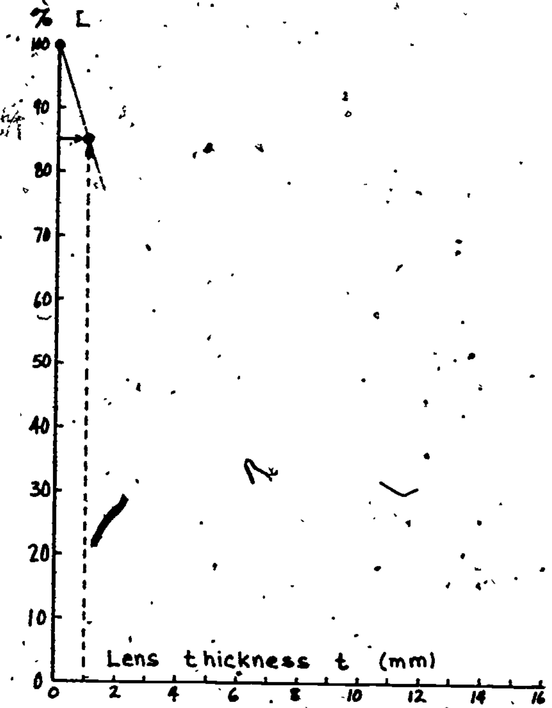


Figure 12: Start of the Graphical Solution

"If you recall from our filter experiment, the differential equation we arrived at was:

$$\frac{dL}{dt} = -k = \text{slope},$$

where L is the light transmitted, t is the filter thickness, and k is a constant. We had to find k before we could proceed with the construction. We solved DE for k,

$$k = -\frac{\text{slope}}{L},$$

at the first point on our graph.

Here, $L = 100$ and the slope is -15, so

$$k = -\frac{-15}{100} = 0.15,$$

and we used that number throughout the construction.

"Through our second point (1,85) we drew a line with a slope of:

$$\text{slope} = -0.15(85) = -12.75,$$

and saw that it intersected the line $t = 2$ at a transmission of 72.25. We kept repeating the computations and the construction until our curve dropped below the ten percent transmission level. Here is the complete graph.

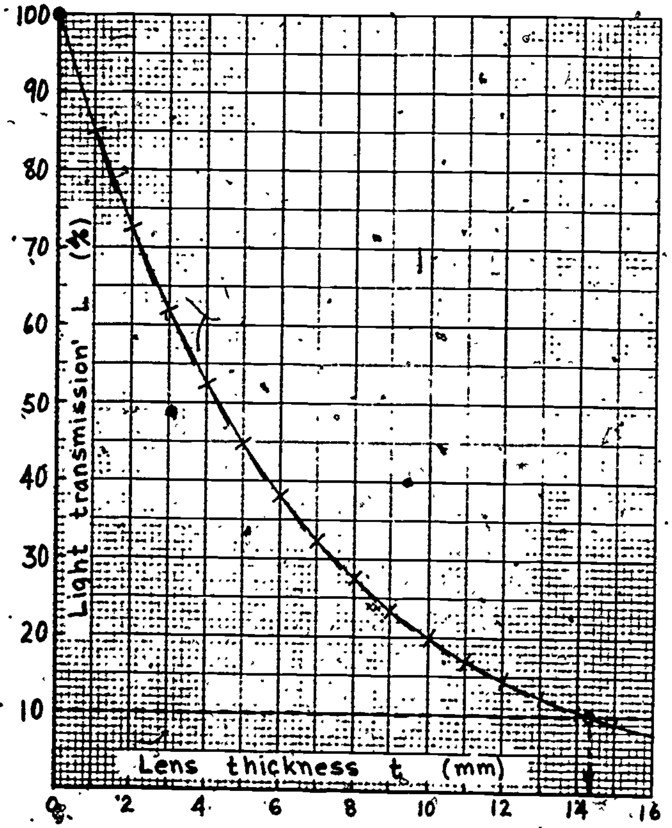


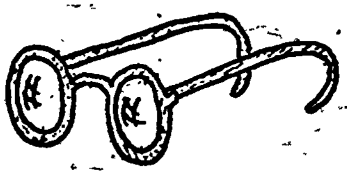
Figure 13. Graphical Solution of $\frac{dL}{dt} = -kL$ for $k = 0.15$

"As you can see, the lens thickness required was 14.4mm.

"Denis could now prove to his boss Al that the required lens thickness was not 6mm, save his job, save the eyesight of our fighting men, and keep his family together.

"At that moment Angelica came home with measurements of the steel beam. Her display of gratitude and affection more than repaid me for my efforts."

"We'll get him for this," I whispered to Herb.



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Unit 83: SOLVING DIFFERENTIAL EQUATIONS NUMERICALLY

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Chapter 11

SOLVING DIFFERENTIAL EQUATIONS NUMERICALLY

"Yesterday you expressed some doubts about the veracity of my little story," Arclet began with some sternness. "Well, this should settle the matter." He tossed the 'phone book into my lap. I opened it where the page had been turned down, and gazed at the name circled in red.

Denis Dropmore, it said. Arclet would live to regret showing me that 'phone book.

"I would really like to continue the story," Arclet said, "but before you can understand what happened next, it will be necessary first to have a lesson on the numerical solution of differential equations."

I smiled. It was all so clear now.

Arclet walked to the board and began to write:

Title: NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Objective: *Upon completion of this lesson, you should be able to carry out a numerical solution of a differential equation by Euler's Method, by hand, on a calculator, or by computer.*

Then Arclet began the lecture.

"If you've understood the graphical solution of a differential equation by Euler's method, the way to perform a numerical computation should be obvious. We will

use the same equation we had for the graphical solution as an example:

$$\frac{dy}{dx} = \frac{x^2}{y} = \text{slope } m$$

with the starting value, $y = 1$, when $x = 1$. We now increment x by some chosen amount Δx. Let's take 1 for our step size now, and later we'll see the effect of making it smaller.

"The slope of the curve, at our first point is,

$$\text{slope} = \frac{x^2}{y} = \frac{1^2}{1} = 1 ,$$

and the change in ordinate going from the first to the second point will be the slope times the change in x, or

$$\Delta y = m\Delta x = 1(1) = 1$$

so the ordinate at the second point will be

$$y + \Delta y = 1 + 1 = 2 .$$

The second point on our curve is thus (2,2).

"Repeating the computation, the slope at the second point is,

$$\text{slope} = \frac{x^2}{y} = \frac{2^2}{2} = 2 .$$

The change in ordinate is,

$$y = m\Delta x = 2(1) = 2 .$$

So the ordinate at the third point is

$$y + \Delta y = 2 + 2 = 4 .$$

The third point is thus (3,4).

"We continue the computation in a similar way as far as we need to go, obtaining each ordinate y_n from the ordinate y_{n-1} of the preceding point, and the slope

m_{n-1} at the preceding point by the equation:

$$y_n = y_{n-1} + m_{n-1}\,\Delta x\,.$$

"A computer really comes in handy here. The following short program, written in BASIC, is designed to perform this computation, taking a step size Δx of 1.

```
10 PRINT "X", "Y", "SLOPE"
20 S=1
30 Y=1
35 FOR X=1 TO 10 STEP S
40 M=X+2/Y
50 PRINT X, Y, M
60 Y=M*S+Y
70 NEXT X
80 END
```

"This program will also print the slopes, as an extra dividend. Here is a RUN:

X	Y	SLOPE
1	1	1
2	2	2
3	4	2.5
4	6.25	2.56
5	8.81	2.837
6	11.647	3.090
7	14.738	3.324
8	18.063	3.543
9	21.606	3.748
10	25.355	3.943

Step size=1.0

"Now that we have a program, it is an easy matter to determine the effect of changing the step size. Notice that the program was written so that only line 20 need be changed. Taking a step size of 0.5 results in the following table:

X	Y	SLOPE
1	1	1
1.5	1.5	1.5
2	2.25	1.777
2.5	3.138	1.991
3	4.134	2.176
3.5	5.222	2.345
4	6.395	2.501
4.5	7.646	2.648

Step size= 0.5

5	8.970	2.786
5.5	10.364	2.918
6	11.823	3.044
6.5	13.345	3.165
7	14.928	3.282
7.5	16.569	3.394
8	18.267	3.503
8.5	20.018	3.609
9	21.823	3.711
9.5	23.679	3.811
10	25.584	3.908

"Now we reduce the step size to 0.1, at the same time rigging the program to print only at integral values of x, to save time and space.

X	Y	SLOPE
1	1	1
2	2.357	1.696
3	4.253	2.116
4	6.525	2.452
5	9.111	2.743
6	11.975	3.006
7	15.091	3.246
8	18.440	3.470
9	22.006	3.680
10	25.777	3.879

Step size= 0.1

"In one final RUN, let's reduce the step size by another order of magnitude.

X	Y	SLOPE
1	1	1
2	2.378	1.681
3	4.278	2.103
4	6.554	2.441
5	9.143	2.734
6	12.010	2.997
7	15.128	3.238
8	18.479	3.463
9	22.048	3.673
10	25.821	3.872

Step size= 0.01

"At this point, it might be interesting to compare the final value of y obtained using the various step sizes. Let's make another table.

step size	ordinate at $x = 10$
1.0	25.355
0.5	25.585
0.1	25.777
0.01	25.821
Theoretical	25.826

"Not too bad. Even with our coarsest step, our final value is less than two percent different from the theoretically correct answer, and with our smallest step, our answer is correct to four significant figures."

Polly interrupted. "Where did that theoretical value come from?"

"I was hoping you would be curious. Remember when I spoke about analytic solutions? Now would be a good time for you to scan that chapter in your text, and we'll get to it in a week or so."

"I'll do that tonight, professor," said Polly.

Herb muttered something inaudible.

At the end of the lesson, Arclet gave the following take-home quiz.

Quiz #4

1. Given the differential equation

$$\frac{dy}{dx} = \frac{y}{x - 2x^2y}$$

and the boundary condition

$$y = 3 \text{ when } x = 1$$

do a numerical solution by Euler's method, taking intervals no larger than 1, and find the value of y when x equals 5. You may program this problem on the computer if you wish.

For answer see page 71.

Chapter 12

A NUMERICAL SOLUTION TO THE SAGGING BEAM PROBLEM

Today Arclet began, "It was now Sunday night, and we had just finished a supper of pan fried trout served with a savory made of soy sauce, chablis and a touch of dill. We attacked the last and most difficult of the three problems.

"Angelica had done an admirable job of collecting the data for which I had sent her. We began by making a diagram and carefully listing all we knew about the problem.

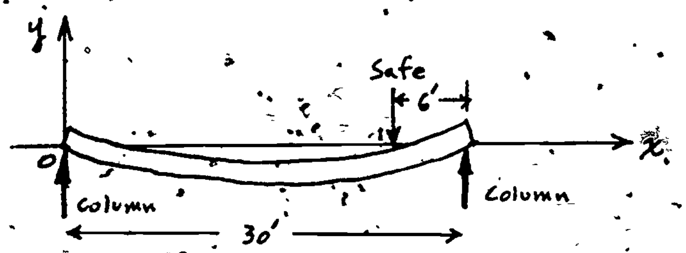

Figure 14. Free Body Diagram of the Beam

"I drew in coordinate axes as shown, and indicated the forces acting on the beam. We felt it reasonable to assume that all other loads, including the weight of the beam itself, would be negligible in comparison to the weight of the huge safe.

"We found the dimensions of the beam cross-section in one of Angelica's architecture books, and it gave the moment of inertia.

$$I = 12{,}000 \text{ inch}^4$$

We also found the modulus of elasticity of steel,

$$E = 30{,}000{,}000 \text{ psi}$$

"I took as my starting point the differential equation for the deflection curve of a beam, which we wrote in class the other day:

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

where y is the vertical deflection at any point x along the beam, M is the bending moment acting on the beam, and E and I are as defined above.

"Before solving the DE, it was necessary to know how the bending moment M varied with position x. I considered a section of beam lying to the left of the safe,

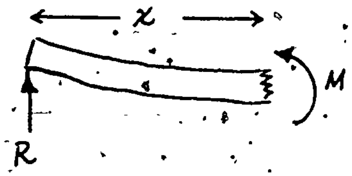

Figure 15. Moments at the Left End

and asked, 'What moment M would be required to keep that section of beam in equilibrium (keep it from rotating)?'

"It would have to be

$$M = Rx$$

where R is the vertical reaction at the left end of the beam, and x is the distance to the section being considered. We can find R without too much trouble, by asking what force R is needed to keep the entire beam from rotating about the right hand support point.

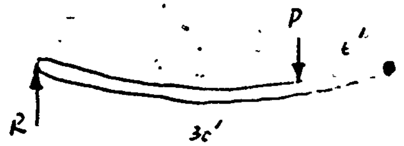

Figure 16. Taking Moments About the Right End

"For equilibrium, we must have

$$30R = 6P ,$$

where P is the weight of the safe. The bending moment is then,

$$M = Rx = \frac{P}{5} x ,$$

where x is in feet, so our differential equation is

$$\frac{d^2y}{dx^2} = \frac{Px}{5EI} .$$

"Angelica had found the weight of the safe from the manufacturer's catalog, and we added some weight for contents, and took $P = 10{,}000$ pounds. Putting in the known values, I computed

$$\frac{P}{5EI} = \frac{(10{,}000 \text{ lb})}{5(30{,}000{,}000 \text{ lb/in}^2)(12{,}000 \text{ in}^4)}$$

$$= \frac{(1 \times 10^4)}{5(3 \times 10^7)(1.2 \times 10^4) \text{ in}^4} = \left(\frac{1}{18} \times 10^{-7}\right) \text{in}^{-2}$$

and converting to dimensions in feet

$$\frac{P}{5EI} = \left(\frac{1}{18} \times 10^{-7}\right) \text{in}^{-2} \left(\frac{144 \text{ in}^2}{1 \text{ ft}^2}\right) = 8 \times 10^{-7} \text{ ft}^{-2}$$

So our DE becomes,

$$\frac{d^2y}{dx^2} = (8 \times 10^{-7})\, x$$

with all distances expressed in feet.

"If we now take the integral of both sides, we get

$$\frac{dy}{dx} = (8 \times 10^{-7}) \frac{x^2}{2} + C = \text{slope},$$

an equation giving us the slope at any distance x along the beam. This is the differential equation we must solve.

"I decided to do a numerical solution by Euler's Method. For a starting value, I knew that y had to be zero when x was zero.

"At this point, Denis, who had followed my previous solutions with much care, interrupted me.

"'How can you find the slopes from that equation when you don't know the value of C?'

"'We have another boundary condition we haven't used yet,' I replied. 'We know that the deflection y also has to be zero at the other support, where $x = 30$. All we have to do is keep guessing at C until we find one that gives a displacement of zero at the right end. We ought to get it in about fifteen tries.'

"'But don't you have to do the entire computation, with each guess at C, before you know whether it is good or not?' Denis asked in apprehension.'

"That's right."

"'And how long will each computation take?'

"About an hour.

"There was a long silence. It was now a little past midnight.

"It was time to play my trump card. I went to my car and returned with a portable teletype and an acoustic coupler. A quick phone call to the computation center brought the full power of the college's massive computer into the Dropmore's kitchen, and the little teletype was soon spitting out columns of figures.

"Here, in BASIC, is the program I wrote:

```
 5 PRINT "X", "Y", "SLOPE"
10 C = 11.7E-5
20 S = .5
30 FOR X = 0 to 30 STEP S
40 M = 4E-7 * X↑2 - C
50 PRINT X, 1E5*Y, 1E5*M
60 Y = Y + M*S
70 NEXT X
```

"To get a first guess for C that would not be too wild, I assumed that the slope would be zero when x was about 20 feet. Solving the DE for C,

$$0 = 4 \times 10^{-7}(20)^2 + C$$

$$C = -16 \times 10^{-5}$$

"I entered this guess for C on line 10, ran the program, and observed the final value of y. I then changed C in a way that kept reducing the final deflection.

"The value of C that I finally used (11.7×10^{-5}) gave a deflection at the end of 0.05×10^{-5} ft, as compared to the maximum deflection of 136×10^{-5} elsewhere on the beam.

"Here is a copy of the final RUN.

X	Y x 10^5	SLOPE
0	0	-11.7
.5	- 5.85	-11.69
1	-11.695	-11.66
1.5	-17.525	-11.61
2	-23.33	-11.54
2.5	-29.1	-11.45
3	-34.825	-11.34
3.5	-40.495	-11.21
4	-46.1	-11.06
4.5	-51.63	-10.89
5	-57.075	-10.7
5.5	-62.425	-10.49
6	-67.67	-10.26
6.5	-72.8	-10.01
7	-77.805	- 9.74
7.5	-82.675	- 9.45
8	-87.4	- 9.14

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	X	Y x 10^5	SLOPE
	8.5	-91.97	- 8.81
	9	-96.375	- 8.46
	9.5	-100.605	- 8.09
	10	-104.65	- 7.7
	10.5	-108.5	- 7.29
	11	-112.145	- 6.86
	11.5	-115.575	- 6.41
	12	-118.78	- 5.94
	12.5	-121.75	- 5.45
	13	-124.475	- 4.94
	13.5	-126.945	- 4.41
	14	-129.15	- 3.86
	14.5	-131.08	- 3.29
	15	-132.725	- 2.7
	15.5	-134.075	- 2.09
	16	-135.12	- 1.46
Maximum	16.5	-135.85	- .81
Deflection	17	-136.255	- .14
Here →	17.5	(-136.325)	.55
	18	-136.05	1.26
	18.5	-135.42	1.99
	19	-134.425	2.74
	19.5	-133.055	3.51
	20	-131.3	4.3
	20.5	-129.15	5.11
	21	-126.595	5.94
	21.5	-123.625	6.79
	22	-120.23	7.66
	22.5	-116.4	8.55
	23	-112.125	9.46
	23.5	-107.395	10.39
	24	-102.2	11.34
	24.5	- 96.53	12.31
	25	- 90.375	13.3
	25.5	- 83.725	14.31
	26	- 76.57	15.34
	26.5	- 68.9	16.39
	27	- 60.705	17.46
	27.5	- 51.975	18.55
	28	- 42.7	19.66
	28.5	- 32.87	20.79
	29	- 22.475	21.94
	29.5	- 11.505	23.11
	30	5.00000000E-02	24.3

"Notice that the maximum deflection occurs just over 17 feet from the end of the beam, between the safe and the midspan, as Angelica had predicted."

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Arclet rose slowly to his feet. "And that, dear class, is the end of the story. Denis and Angelica both regained their jobs, and were more highly regarded than ever before. Convinced of the power of mathematics--in the hands of an expert--they are both studying calculus in their spare time. I visit them often, giving them small tips to facilitate their studies, and the affection and respect they shower upon me is almost embarrassing.

"Have a good weekend, and remember that we have a test on differential equations on Monday." He strutted out.

We sat there silent for a long time; heads hung down and shaking slowly from side to side.

Finally Herb spoke, "Mat, didja call?"

"Yeah. They never heard of Arclet!"

"Cheap trick," Polly said.

"We've been had. Taken, conned, duped, used."

"Should we go through with the plan?"

"Let's vote."

Three thumbs pointed downward. That night, I phoned my brother in Seattle.

Chapter 13

THAT EXAM AGAIN

Today, Arclet seemed a different person. His hair was brushed, he wore a well-pressed suit with a clean shirt, and his shoes were shined. He strode into the classroom, carrying a smart little attache case with metal trim instead of his battered, old briefcase.

"Who cares about differential equations?" he asked. "I'll tell you who cares. Boeing Aircraft Company on the West Coast cares, that's who." He waved a telegram at us. "Listen."

HAVE URGENT PROBLEM INVOLVING DIFF EQN. YOUR NAME REF TO US. HOPE YOU CAN COME ASAP. YOUR USUAL FEE PLUS TRAVEL EXPENSES PAID.

"I've arranged for a substitute teacher, who will be here starting tomorrow. See you in a week."

Halfway out the door, he stopped short. "I nearly forgot. Finish this test today and leave it with my substitute."

It was exactly the same exam that Arclet had used as a pre-test for this topic. In fact, the same test that had prompted his whole ridiculous story.

We had come full circle.

THAT EXAM

1. Given the differential equation

$$\frac{dy}{dx} = \frac{3y^3 - x^3}{3xy^2}$$

determine whether the equation

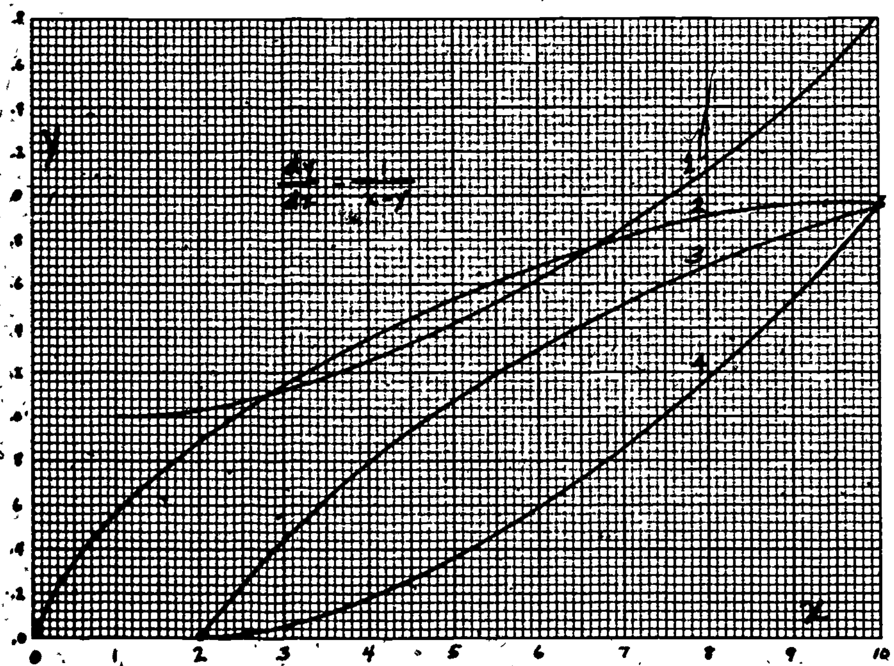
$$y = x\sqrt[3]{1 - \ln x}$$

is a solution.

2. Only one of the curves graphed below is a possible solution to the differential equation,

$$\frac{dy}{dx} = \frac{1}{x - y}$$

Check enough points to determine which curve represents a solution.



3. Given the differential equation

$$\frac{dy}{dx} = \frac{3 - xy}{2x^2}$$

verify whether the following table of points represents a solution by checking at least 3 points.

X	Y
0.5	-4.586
1.0	-2.000
1.5	-1.184
2.0	-0.793
2.5	-0.568
3.0	-0.423
3.5	-0.323
4.0	-0.250
4.5	-0.195
5.0	-0.153

4. Plot a tangent field for the differential equation

$$\frac{dy}{dx} = \frac{x^2 - 2y}{x}$$

where $1 \leqq x \leqq 5$ and $0 \leqq y \leqq 5$.

Sketch in 2 possible solutions.

5. Given the differential equation

$$\frac{dy}{dx} = x^2 - y,$$

with the boundary conditions

$$y = 1 \text{ when } x = 1,$$

find a solution graphically, taking intervals, delta x of 1 and find the value of y when $x = 7$.

6. Given the differential equation

$$\frac{dy}{dx} = \frac{x}{x+y},$$

and the boundary conditions

$y = 1$ when $x = 0$,

do a numerical solution, taking intervals delta x of 1, and find the value of y when $x = 10$.

7. Write a differential equation to describe the following situations. Be sure to define your symbols.

a) A body falls in a medium offering resistance proportional to the speed at any instant.

b) A particle moves in a horizontal line acted upon by an attractive force which varies inversely as the cube of the distance from a fixed point.

c) The rate of flow from a tank of uniform cross-section is proportional to the square root of the liquid depth.

d) Radium decomposes at a rate proportional to the present amount.

8. Poke a hole in Arclet's fish story by calculating how many fish Mike's freezer would have to hold.

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APPENDICES

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Answers to Quiz #1

1. Let: u = temperature at any time t ;
   u_0 = temperature of surroundings

   then,

   $$\frac{du}{dt} = -k(u - u_0)$$

   which is known as Newton's law of cooling.

2. Let R be the resisting force of the medium on the object. The statement of the problem gives

   (1) $$R = -k\sqrt{v}.$$

   Assuming R to be the only force acting on the object, Newton's second law gives

   (2) $$ma = R,$$

   where m is the object's mass and a is the object's acceleration. Substituting (2) into (1) gives

   (3) $$ma = -kv.$$

   Acceleration is defined as the rate of change of velocity, or

   (4) $$a = \frac{dv}{dt}$$

   which when substituted into (3) gives

   (5) $$\boxed{m\,\frac{dv}{dt} = -kv,}$$

   a differentiated equation describing the system.

   If additional forces acted on the body, equation (2) would have to be modified to account for them. One additional force, F, driving the object through the medium would lead to the following differential equation:

   $$m\,\frac{dv}{dt} = F - k\sqrt{v}.$$

3. See text page 17.

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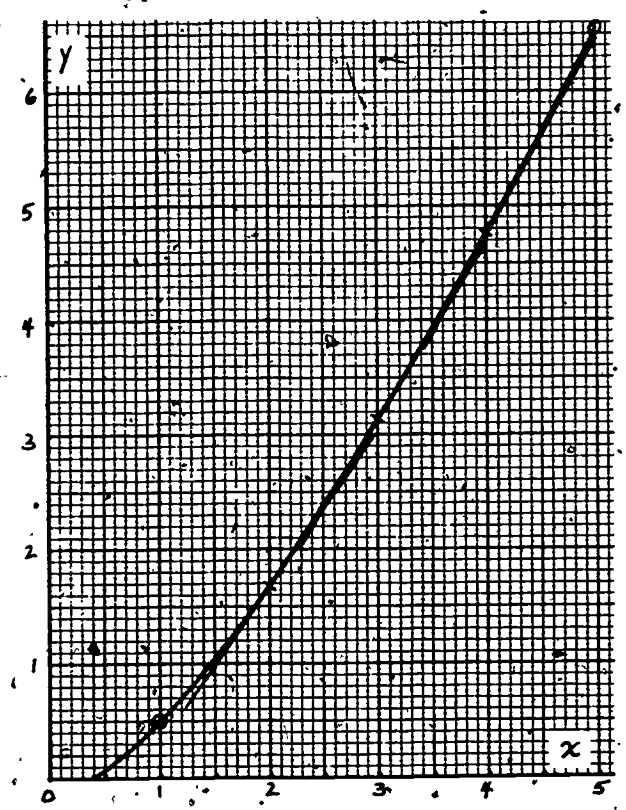
Answers to Quiz #2

1. The equation $y = \frac{x}{2}(\ln x + 1)$ is plotted below, and the slopes are measured at $x = 2$, 3, and 4.



A comparison with the slopes calculated from the DE is given in the table.

x	y	$\frac{x + 2y}{2x}$	Measured Slope
2	1.69	1.35	1.32
3	3.15	1.55	1.55
4	4.77	1.69	1.75

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Since the agreement is excellent, we may assume that the equation

$$y = \frac{x}{2}(\ln x + 1)$$

is a solution of the differential equation

$$\frac{dy}{dx} = \frac{x + 2y}{2x}$$

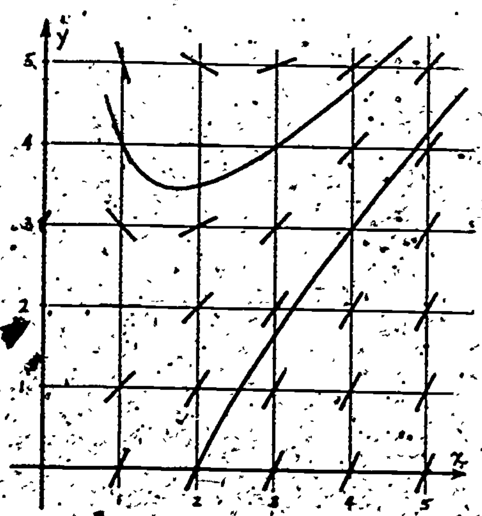
in the domain shown by the graph.

2. Calculation of $\frac{dy}{dx}$ for $\frac{dy}{dx} = \frac{2x - y}{x} = 2 - \frac{y}{x}$

x	y	$\frac{y}{x}$	$\frac{dy}{dx}$
1	0	0	2
	1	1	1
	2	2	0
	3	3	-1
2	0	0	2.0
	1	0.5	1.5
	2	1.0	1.0
	3	1.5	0.5
4	0	0	2.0
	1	.25	1.75
	2	.50	1.50
	3	.75	1.25

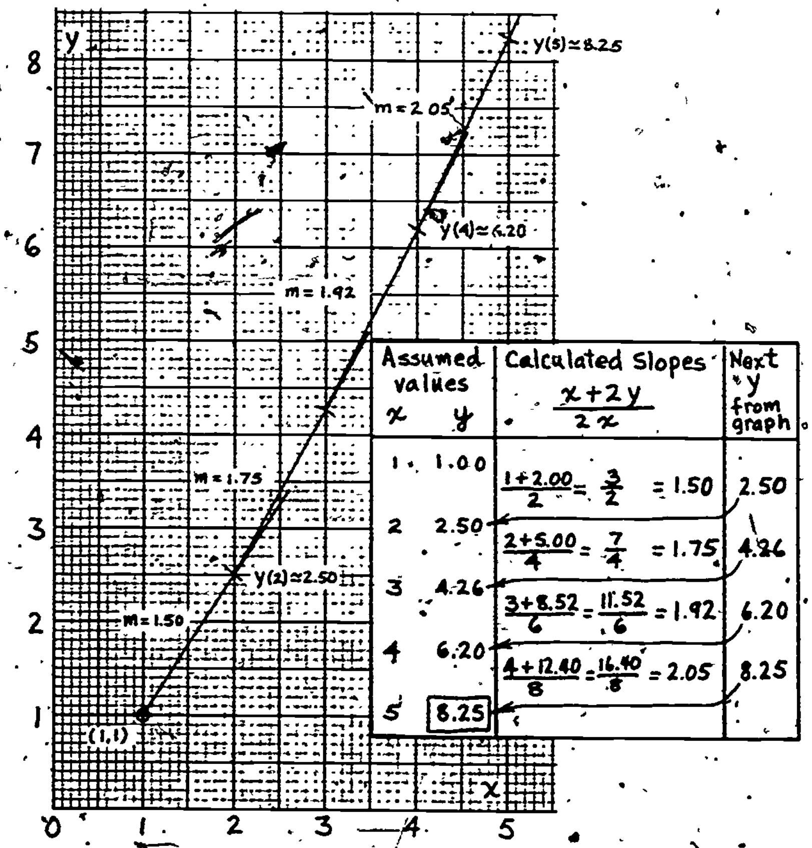
etc.

Tangent Field for $\frac{dy}{dx} = \frac{2x - y}{x}$ with two possible solutions.



Answer to Quiz #3

1. Graphical solution of $\frac{dy}{dx} = \frac{x + 2y}{2x}$ through point (1,1)



Assumed values x	y	Calculated Slopes $\frac{x+2y}{2x}$	Next y from graph
1	1.00	$\frac{1+2.00}{2} = \frac{3}{2} = 1.50$	2.50
2	2.50	$\frac{2+5.00}{4} = \frac{7}{4} = 1.75$	4.26
3	4.26	$\frac{3+8.52}{6} = \frac{11.52}{6} = 1.92$	6.20
4	6.20	$\frac{4+12.40}{8} = \frac{16.40}{8} = 2.05$	8.25
5	$\boxed{8.25}$		

Answer to Quiz #4

1. Given: $\frac{dy}{dx} = \frac{y}{x - 2x^2y} = m,$

and Euler's Method: $y_n = y_{n-1} + m_{n-1}\,\Delta x,$

we let $y_0 = 3$ when $x_0 = 1$ and $\Delta x = 1$ for this calculation.

It is easiest to make a table for calculation as follows:

n	x_n	y_n	$\frac{y}{x - 2x^2y}$	m	$m\Delta x$	$y + m\Delta x$
0	1	3	$\frac{3}{1 - 6}$	$-\frac{3}{5}$	-0.600	3.000 - 0.600
1	2	2.400	$\frac{2.40}{2 - 19.20}$	$-\frac{2.400}{17.20}$	-0.1395	2.400 - 0.140
2	3	2.260	$\frac{2.260}{3 - 40.68}$	$-\frac{2.260}{37.68}$	-0.0599	2.260 - 0.060
3	4	2.200	$\frac{2.200}{4 - 70.40}$	$-\frac{2.200}{66.40}$	-0.0331	2.200 - 0.033
4	5	2.167				

The value of y obtained when $x = 5$ will depend upon the step size used, as shown in the table.

step size	$y(5)$
1	2.167
0.1	2.521
0.01	2.549
0.001	2.551

Answers to That Exam

1. Differentiating the proposed solution,

$$\frac{dy}{dx} = \frac{2 - 3\ln x}{3(1 + \ln x)^{2/3}}$$

Substituting y and its derivative into the DE, we get,

$$\frac{2 - 3\ln x}{3(1 + \ln x)^{2/3}} = \frac{3\{x(1 - \ln x)^{1/3}\}^3 - x^3}{3x\{x(1 - \ln x)^{1/3}\}^2}$$

$$= \frac{3x^3 - 3x^3\ln x - x^3}{3x^3(1 - \ln x)^{2/3}}$$

$$= \frac{2 - 3\ln x}{3(1 - \ln x)^{2/3}}$$

which verifies the proposed solution.

2. Curve 3 is the only curve shown that can be a solution. To determine if a curve is a possible solution, pick a convenient point on the curve, measure the slope of the tangent line at that point and compare the measured slope with the derivative calculated by substituting the coordinates of the point into the equation for the derivative,

$$\frac{dy}{dx} = \frac{1}{x - y}.$$

Curve 1: At the point $(-1,1)$ the measured slope ≈ 0, but

$$\frac{dy}{dx} = \frac{1}{1 - 1} = \frac{1}{0} = \text{undefined}.$$

The measured slope of 0 is not undefined, therefore the solution cannot be curve 1.

Curve 2: At $(9.50, 1.96)$ the measured slope ≈ 0 (at least $< |0.02|$), but $\frac{dy}{dx} = \frac{1}{9.50 - 1.96} = \frac{1}{7.54} = 0.13,$

and 0.13 is not $< |0.02|$, therefore the solution is not likely to be curve 2.

Checking another point on curve 2 we have at $(0,0)$ the measured slope ≈ 1.1, but

$$\frac{dy}{dx} = \frac{1}{0 - 0} = \frac{1}{0} = \text{undefined},$$ and 1.1 is not undefined, therefore the solution cannot be curve 2.

Curve 3: At (2,0) the measured slope ≈ 0.5 and

$$\frac{dy}{dx} = \frac{1}{2 - 0} = \frac{1}{2} = 0.5.$$

Therefore the solution could be curve 3.

Curve 4: At (2,0) the measured slope ≈ 0 and

$$\frac{dy}{dx} = \frac{1}{2 - 0} = 0.5.$$

Therefore the solution cannot be curve 4.

Since the solution is supposed to be one of the curves, and it cannot be curve 1, 2, or 4, but could be curve 3, we might assume that it must be curve 3. It would be wise, however, to check another point or two on curve 3.

Curve 3: At (4, 0.8) the measured slope ≈ 0.32 and

$$\frac{dy}{dx} = \frac{1}{4 - 0.8} = \frac{1}{3.2} = 0.3125,$$

which is in close agreement with the measured slope.

At (7.5, 1.6) the measured slope ≈ 0.175, and

$$\frac{dy}{dx} = \frac{1}{7.5 - 1.6} = \frac{1}{5.9} = 0.1695,$$

again in close agreement with the measured slope.

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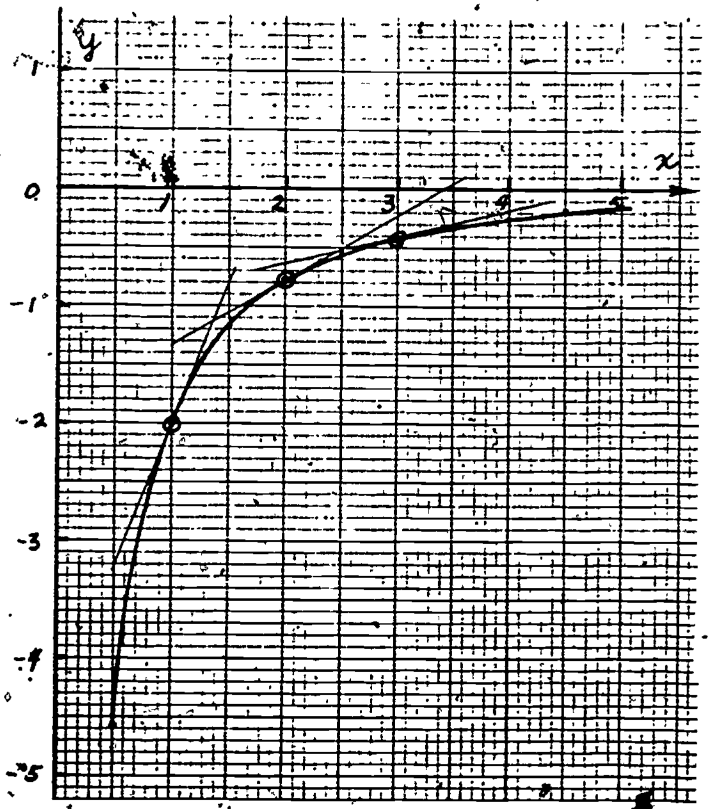
3. The given points are plotted below, and the slopes at $x = 1, 2,$ and 3 are measured. These compare well with the slopes computed from the DE, as shown in the table.



x	y	Measured Slope	$\frac{3 - xy}{2x^2}$
1	-2.000	2.40	2.50
2	-0.793	0.55	0.57
3	-0.423	0.22	0.24

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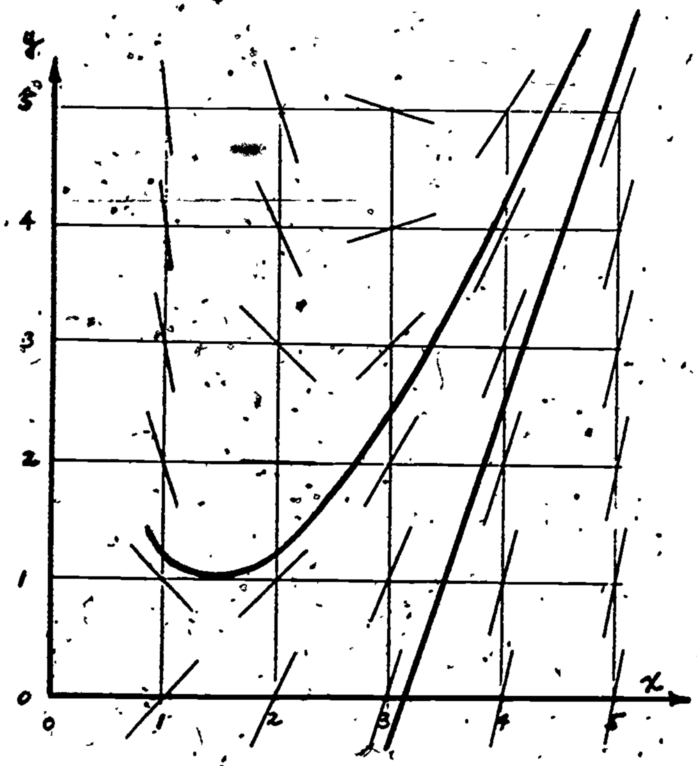
4. **Calculation** of slopes at different points for

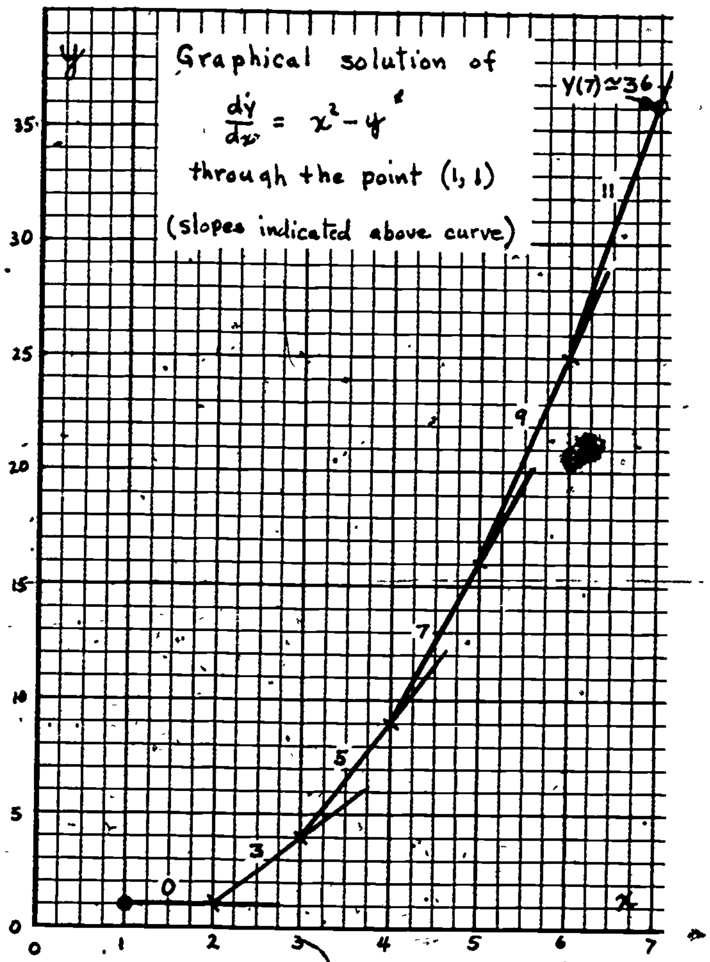
$$\frac{dy}{dx} = \frac{x^2 - 2y}{x}$$

x	y	$x^2 - 2y$	$\frac{dy}{dx}$
1	0	1 - 0	1
	2	1 - 4	-3
	4	1 - 8	-7
3	0	9 - 0	3
	2	9 - 4	5/3
	4	9 - 8	1/3

This table is continued and the results plotted below.



5. Graphical solution of $\frac{dy}{dx} = x^2 - y$ through the point (1,1).



Use the same method as shown in the answer to the problem in Quiz #3, page 74.

6. A numerical solution with intervals of 1 unit yields the following table of point pairs.

x	y
0	1.00
1	1.00
2	1.50
3	2.07
4	2.66
5	3.26
6	3.86
7	4.48
8	5.08
9	5.69
10	6.31

(For method see answers to Quiz #4, page 75.)

7a. Let W be the weight of the body, and R be the resisting force. Then by Newton's second law of motion

$$\Sigma F = ma$$

$$W + R = m\frac{dv}{dt}$$

but $R = -kv$ where v = instataneous velocity, so

$$W - kv = m\frac{dv}{dt}$$

since the mass $m = W/g$

$$\frac{dv}{dt} + \frac{kg}{W} - g = 0$$

7b. Let x be the distance from the fixed point, and F the attractive force. Then,

$$F = -\frac{k}{x^3} = ma$$

where m is the particle mass and a is the acceleration, $\frac{d^2x}{dt^2}$, then,

$$-\frac{k}{x^3} = m\frac{d^2x}{dt^2}$$

7c. Let R = the rate of flow from the tank, and y = the liquid depth. Then, from the statement of the problem,

$$P = C_1\sqrt{y}.$$

But, the rate of flow must also be proportional to the rate at which the water level is changing. Thus,

$$R = -C_2\frac{dy}{dt}.$$

Setting the two expressions for rate equal to each other we have

$$-C_2\frac{dy}{dt} = C_1\sqrt{y}.$$

Dividing by $-C_2$, and letting $\frac{C_1}{C_2} = k$ we have finally

$$\frac{dy}{dt} = -k\sqrt{y}.$$

8. The number of fish collected in Mike's freezer would equal the difference between the number of fish that came into the pond and the number left in the pond when Cy Seepage counted them,

or $$\begin{pmatrix}\text{fish in}\\ \text{freezer}\end{pmatrix} = \begin{pmatrix}\text{fish into}\\ \text{pond}\end{pmatrix} - \begin{pmatrix}\text{fish left}\\ \text{in pond}\end{pmatrix}.$$

Arclet used the tangent field method and calculated the fish left in the pond to be about 3500 fish (see figure 9c).

The fish into the pond would be the number that were carried in by the water plus the number put in by the Fish and Game Department, that is

(12,960 min)(0.6 fish/min) + 1000 fish = 7776 fish + 1000 fish
= 8776 fish
≐ 8800 fish

Then the total number of fish caught in the basket would be

8800 fish - 3500 fish = 5300 fish

Some freezer!

Some dinner!

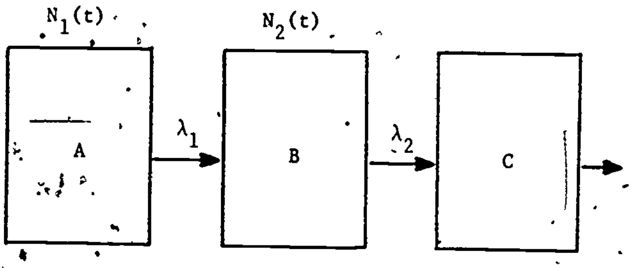
umap

UNIT 234

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

RADIOACTIVE CHAINS: PARENTS AND DAUGHTERS

by Brindell Horelick and Sinan Koont



APPLICATIONS OF CALCULUS TO CHEMISTRY

edc/umap /55chapel st./newton.mass. 02160

RADIOACTIVE CHAINS: PARENTS AND DAUGHTERS

by

Brindell Horelick
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University of Massachusetts
Amherst, Massachusetts 01003

TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 234*

Title: RADIOACTIVE CHAINS: PARENTS AND DAUGHTERS

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Review Stage/Date: III 10/30/79

Classification: APPL CALC/CHEM

Suggested Support Material: A hand calculator with the exponential function, and with capacity from 10^{-99} to 10^{99}.

References:

Friedlander, G., J.W. Kennedy, and J.M. Miller (1964). Nuclear and Radiochemistry. John Wiley & Sons, New York.

Harvey, B.G. (1962). Introduction to Nuclear Physics and Chemistry. Prentice-Hall, Englewood Cliffs, New Jersey.

Kaplan, I. (1962). Nuclear Physics. Addison-Wesley, Reading, Massachusetts.

Prerequisite Skills:

1. Ability to integrate $\int_0^t \frac{f'(t)}{f(t)}\,dt$.
2. Ability to use first and second derivatives as aids in graphing functions.
3. Knowledge of $\lim_{t\to\infty} e^{-kt}$ (k positive).

Output Skills:

1. Know equations governing radioactive chains ("parents and daughters").
2. Know the meaning of *transient equilibrium* and *secular equilibrium*.
3. Know the approximations relevant to transient and secular equilibrium, and know the circumstances under which they are applicable.

Other Related Units:

Kinetics of Single Reactant Reactions (U232)

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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RADIOACTIVE CHAINS: PARENTS AND DAUGHTERS

1. INTRODUCTION

1.1 Radioactive Decay

Radioactive decay is a *first order* reaction. This means that if a radioactive substance is not being replenished in any way, then its amount (number of atoms) $N(t)$ decreases at a rate proportional to that amount:

(1) $$N'(t) = -\lambda N(t),$$

where λ is a positive constant known as the *disintegration constant* or *decay constant*.

The elementary consequences of Equation (1) are discussed in many elementary calculus textbooks. In our Unit 232 (*Kinetics of Single Reactant Reactions*) we discuss first order reactions in greater detail. In that unit you can discover how experimenters determined empirically that radioactive decay is a first order process, and what this suggests about the mechanism of radioactivity.

1.2 Chains

When a radioactive substance A decays into a substance B, A and B are referred to as the *parent* and the *daughter*. It may happen that B itself is radioactive and is the parent of a new daughter C, and so on. In fact, this is a very common situation. There are three chains like this, beginning respectively with U^{238}, U^{235}, and Th^{232}, whose lengths are 19, 17, and 13. They do not overlap, and together account for all naturally occurring radioactive substances beyond Thallium (atomic number 81) on the periodic table. Each of these chains ends with a stable (non-radioactive) form of lead.

2. SETTING UP THE EQUATIONS

2.1 Notation and Assumptions

We shall consider the relationship between one parent A and her radioactive daughter B. We shall write $N_1(t)$ and $N_2(t)$ for their amounts at time t, and λ_1 and λ_2 for their decay constants. Figure 1 may help you

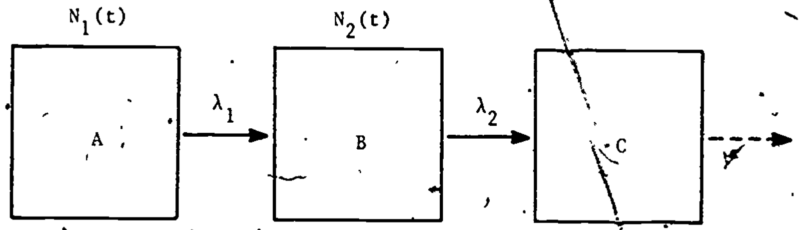

Figure 1. Schematic representation of a radioactive chain.

remember this notation. Since the rate of decay of B into C depends only on the amount of B present, and not on the amount of C, we do not care whether C is stable or radioactive.

Now imagine that at the instant t = 0 we have a freshly prepared amount N_0 of A, and none of B. That is, $N_1(0) = N_0$ and $N_2(0) = 0$. Imagine that the chain of reactions in Figure 1 then proceeds without external interference.

2.2 The Equations

Since A is not being replenished, Equation (1) applies directly, and we have

$$N_1'(t) = -\lambda_1 N_1(t).$$

If B were not being replenished, Equation (1) would apply again, and $N_2(t)$ would be changing at the rate $-\lambda_2 N_2(t)$. But B *is* being replenished. Each atom of A which decays becomes an atom of B, and this is happening at the rate $\lambda_1 N_1(t)$. So altogether we have

$$N_2'(t) = \lambda_1 N_1(t) - \lambda_2 N_2(t).$$

We are confronted with the following system of equations:

(2) $N_1'(t) = -\lambda_1 N_1(t)$

(3) $N_2'(t) = \lambda_1 N_1(t) - \lambda_2 N_2(t)$

$N_1(0) = N_0$

$N_2(0) = 0.$

3. SOLVING THE EQUATIONS

3.1 Solving for $N_1(t)$

It is fairly straightforward to solve Equation (2) for $N_1(t)$. This was done in Unit 232 and is probably done in your calculus textbook. We just divide through by $N_1(t)$ and then integrate from 0 to t:

$$\int_0^t \frac{N_1'(t)}{N_1(t)}\,dt = -\int_0^t \lambda_1\,dt.$$

This leads to the equation:

$$\ln(N_1(t)) - \ln(N_1(0)) = -\lambda_1 t,$$

or

$$\ln\left(\frac{N_1(t)}{N_0}\right) = -\lambda_1 t,$$

since $N_1(0) = N_0$. The usual absolute value signs are not needed, because the quantities involved are positive. Finally,

(4) $N_1(t) = N_0 e^{-\lambda_1 t}.$

Exercise 1.

Find a relationship between λ_1 and the half life of A (the half life is the time t^* at which $N_1(t^*) = \frac{1}{2}N_0$).

3.2 Solving for $N_2(t)$

Finding $N_2(t)$ is a bit more tricky. Applying Equation (4) to Equation (3) we get

(5) $N_2'(t) = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2(t).$

Equation (5) probably looks quite different from any you have seen before. Let's try to make a shrewd guess what kind of solution it has. It says that the derivative of $N_2(t)$ is the sum of two terms, $\lambda_1 N_0 e^{-\lambda_1 t}$ and $-\lambda_2 N_2(t)$. With luck, this might remind us of the product rule:

(6) if $N_2(t) = u(t) \cdot v(t)$

then $N_2'(t) = u(t) \cdot v'(t) + v(t) \cdot u'(t).$

Can we pick u(t) and v(t) so the terms in Equation (6) match up with the terms in Equation (5)? In other words, can we pick u(t) and v(t) so that

(7) $u(t) \cdot v'(t) = \lambda_1 N_0 e^{-\lambda_1 t}$

and

(8) $v(t) \cdot u'(t) = -\lambda_2 N_2(t)$?

Since $N_2(t) = u(t) \cdot v(t)$, Equation (8) can be rewritten

$$v(t) \cdot u'(t) = -\lambda_2 u(t) v(t)$$

and we are in business! The v(t) factors cancel out, leaving us with

$$u'(t) = -\lambda_2 u(t)$$

which looks very much like Equation (2) and can be solved in the same way. First,

$$\int_0^t \frac{u'(t)}{u(t)}\,dt = -\int_0^t \lambda_2\,dt.$$

Then, writing $R = u(0)$,

$$\ln\left(\frac{u(t)}{R}\right) = -\lambda_2 t$$

$$u(t) = Re^{-\lambda_2 t}.$$

Putting this into Equation (7) gives

$$Re^{-\lambda_2 t}v'(t) = \lambda_1 N_0 e^{-\lambda_1 t}$$

$$v'(t) = \frac{\lambda_1 N_0}{R} e^{(\lambda_2-\lambda_1)t}.$$

If $\lambda_1 = \lambda_2$ we feel confident you can complete this solution yourself (see Exercise 2).

If $\lambda_1 \neq \lambda_2$, then $\lambda_2 - \lambda_1 \neq 0$ and we can write

$$v(t) = \frac{\lambda_1 N_0}{R(\lambda_2-\lambda_1)} e^{(\lambda_2-\lambda_1)t} + K,$$

where K is the constant of integration. Then

$$N_2(t) = u(t) \cdot v(t) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1} e^{-\lambda_1 t} + KRe^{-\lambda_2 t}.$$

Using the fact that $N_2(0) = 0$, we get

$$0 = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1} + KR$$

$$KR = -\frac{\lambda_1 N_0}{\lambda_2-\lambda_1}$$

$$(9) \quad N_2(t) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}).$$

Exercise 2.

Find $N_2(t)$ if $\lambda_1 = \lambda_2$.

Exercise 3.

Assuming C is stable, find the time at which the total radioactivity (i.e., the total number of disintegrations of A-atoms and B-atoms per unit time) is greatest.

3.3 Graphs of $N_1(t)$ and $N_2(t)$

It is easy to confirm from Equation (4) that $N_1'(t) < 0$ for all t, that $N_1''(t) > 0$ for all t, and that $\lim_{t\to\infty} N_1(t) = 0$. With a little more work (see Exercise 4) it can be confirmed that $\lim_{t\to\infty} N_2(t) = 0$ and that

$$(10) \quad N_2'(t) \begin{cases} > 0 \text{ if } t < t_0 \\ = 0 \text{ if } t = t_0 \\ < 0 \text{ if } t > t_0 \end{cases}$$

and

$$(11) \quad N_2''(t) \begin{cases} < 0 \text{ if } t < 2t_0 \\ = 0 \text{ if } t = 2t_0 \\ > 0 \text{ if } t > 2t_0, \end{cases}$$

where

$$t_0 = \frac{\ln\lambda_1 - \ln\lambda_2}{\lambda_1-\lambda_2}.$$

Therefore, the graphs of $N_1(t)$ and $N_2(t)$ have the shapes shown in Figures 2 and 3.

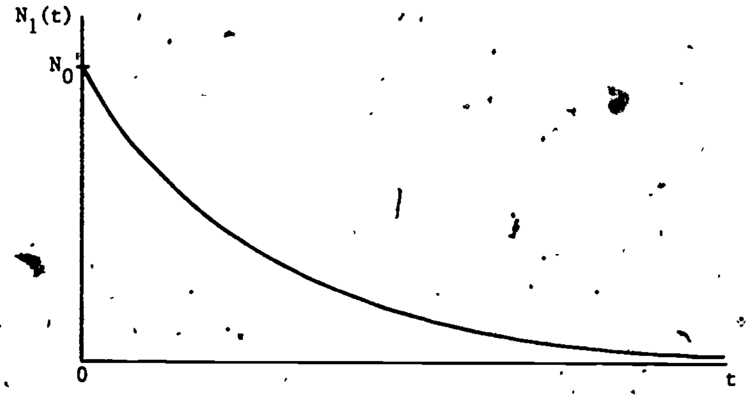

Figure 2. Typical graph of $N_1(t)$ (amount of A as a function of time).

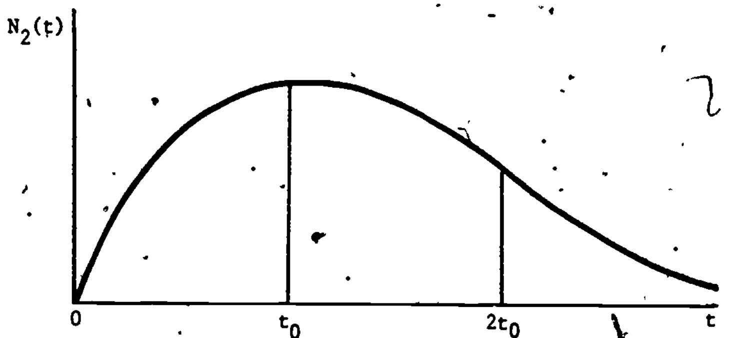

Figure 3. Typical graph of $N_2(t)$ (amount of B as a function of time).

Exercise 4

a. Show that $\lim_{t\to\infty} N_2(t) = 0$.

b. Confirm Equation (10).

c. Confirm Equation (11).

Exercise 5

Find the time at which the greatest amount of B will be present.

Exercise 6

For the chain $Bi^{210} \to Po^{210} \to Pb^{206}$, $\lambda_1 = 1.37 \times 10^{-1}\ \text{day}^{-1}$ and $\lambda_2 = 5.1 \times 10^{-3}\ \text{day}^{-1}$.

a. Use Exercise 5 to determine when the amount of Po^{210} will be greatest.

b. If initially there are 10^{-8} grams of Bi^{210}, how many grams of Po^{210} will there be when it is at its maximum amount?

4. EQUILIBRIUM

4.1 What is Equilibrium?

In a continuing process such as the one we are discussing, it is natural to ask about "equilibrium" of the process. Webster's Seventh New Collegiate Dictionary (1965) defines equilibrium as "A static or dynamic state of balance between opposing forces or actions." To a scientist, "state of balance" means that certain measurable quantities remain constant. But in practice scientists frequently use the word "equilibrium" when the measurements under consideration are nearly constant rather than actually constant. There is good reason for this. The process we are discussing illustrates that reason nicely. We have been devoting our attention to $N_1(t)$ and $N_2(t)$. For either of these actually to be constant over any time interval, its derivative would have to be zero throughout that interval. But $N_1'(t) < 0$ for all t, and $N_2'(t) = 0$ for only one t. So, strictly speaking, it is impossible for either $N_1(t)$ or $N_2(t)$ to be "in equilibrium."

But $N_1(t)$ and $N_2(t)$ involve negative exponential functions. In fact, functions involving negative exponentials

occur fairly commonly in the description of physical and chemical processes. The most basic negative exponential function is e^{-t}, and if you know anything at all about it you know that it approaches zero very fast, so that, although it is never constant, it is before long practically equal to zero and therefore practically constant. This characteristic ("never constant but practically constant") carries through to many of the more complicated functions involving negative exponentials. The quantities they describe never actually reach their limiting values, but usually come (and remain) extremely close to them within a reasonable length of time--perhaps even so close that the difference is not measurable. Scientists often apply the words "equilibrium" or "steady state" to this situation.

There is one more thing we should say about the word "equilibrium" before we move on to discussing specific cases of it. The functions $N_1(t)$ and $N_2(t)$ are examples of functions which come and remain extremely close to a constant value (zero). But it would be wrong to say that they are therefore in equilibrium, even allowing for the stretching of the definition which we just discussed. The reason lies in the other part of the definition: "opposing forces or actions." There are no "opposing forces or actions." Rather than having two things happen which cancel each other out, we have nothing happening at all (in the limit). In plain English, virtually all of A will have decayed into B and then into C, so that there will be virtually none of A or B left. It takes no fancy mathematics to see this. If C is radioactive it will eventually decay, and so on, so that the limiting situation is that only the stable substance at the end of the chain will remain.

There are, however, two situations involving radioactive chains to which the word "equilibrium" is usually applied. The first of these is known as *transient equilibrium*. Another, known as *secular equilibrium*, can be regarded as a special case of the first.

4.2 Some Comments on Approximation

Before we get into the mathematics of transient and secular equilibrium, it will be wise to take a moment to discuss just what we mean by a "good approximation." When scientists say two numbers r and s are approximately equal, they almost always mean that the difference between r and s is small *compared to* either of the numbers. For example they might say $1002 \approx 1000$ (depending on the context), but would almost never say $2 \approx 1$.

Saying $r - s$ is small compared to (for example) s means $\frac{r-s}{s} = \frac{r}{s} - 1$ is small, or $\frac{r}{s}$ is near 1. In the numerical examples we just gave, $\frac{1002}{1000} = 1.002$, which is very near 1, but $\frac{2}{1} = 2$, which is much further from 1.

This interpretation of approximation can be applied to functions too. Let's look specifically at negative exponential functions. If P and Q are any non-zero constants, and if a and b are constants such that $0 < a < b$, then

$$\frac{Pe^{-at} + Qe^{-bt}}{Pe^{-at}} = 1 + \frac{Q}{P}e^{(a-b)t} \to 1 \tag{12}$$

as $t \to \infty$, since $a - b$ is negative. Therefore, for t large enough, $Pe^{-at} + Qe^{-bt}$ can be approximated by Pe^{-at}. We shall use this fact in the next section.

4.3 Transient Equilibrium

It has often been observed that in many chains involving a parent A and a radioactive daughter B, after a while both parent and daughter appear to be decaying at the same rate, in the sense that in any given time interval (say from t_1 to t_2, with t_1 large enough) parent and daughter each lose the same *fraction* of their initial amount — $N_1(t_2)/N_1(t_1) = N_2(t_2)/N_2(t_1)$. This phenomenon is called *transient equilibrium.* Let us try to explain it mathematically.

We can rewrite the equation of the preceding paragraph $N_2(t_2)/N_1(t_2) = N_2(t_1)/N_1(t_1)$. In other words, the observed result is that $N_2(t)/N_1(t)$ is a constant. (This is why the word "equilibrium" is used in describing this phenomenon.) Why should this be so? Let's investigate this quotient, starting with the formulas for $N_1(t)$ and $N_2(t)$ given in Equations (4) and (9) respectively.

We know from Equation (9) that

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}).$$

If $\lambda_1 < \lambda_2$, Section 4.2 tells us that for t large enough

$$N_2(t) \approx \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}. \tag{13}$$

We also know, from Equation (4), that

$$N_1(t) = N_0 e^{-\lambda_1 t}$$

(which is exact). Dividing Equation (13) by Equation (4), we get

$$\frac{N_2(t)}{N_1(t)} \approx \frac{\lambda_1}{\lambda_2 - \lambda_1}. \tag{14}$$

On the other hand, if $\lambda_2 < \lambda_1$ then this does not go through as neatly. Equation (13) has to be replaced by

$$N_2(t) \approx \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2} e^{-\lambda_2 t}$$

and then Equation (14) becomes

$$\frac{N_2(t)}{N_1(t)} \approx \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{(\lambda_1 - \lambda_2)t}.$$

This is a *positive* exponential. It does not have a finite limit.

So the mathematics tells us that transient equilibrium should be observed when $\lambda_1 < \lambda_2$, but not otherwise. Sure enough, this is exactly what happens.

Another way of looking at transient equilibrium is to compare the approximation given in Equation (13) directly with Equation (4). The exponent $-\lambda_1 t$ is the same in both cases. So for t large enough B behaves as if it had the same decay constant (and therefore the same half life*) as A. Since $\lambda_1 < \lambda_2$ this apparent half life is longer than B's natural half life, an observation which should appeal to your common sense even with no mathematics at all. After all, two things are happening to B. It is decaying at its natural rate, and it is being replenished at a certain rate. Therefore you would expect its actual rate of disappearance to be somewhat slower than if it were not being replenished. This common sense observation may help you remember which way the inequality $\lambda_1 < \lambda_2$ goes for transient equilibrium.

4.4 Secular Equilibrium

We have said that transient equilibrium occurs when λ_1 is *smaller* than λ_2. Now let us suppose that λ_1 is very small, and very much smaller than λ_2. (Scientists write $\lambda_1 << \lambda_2$ to mean λ_1 is very much smaller than λ_2.) This is actually a very common occurrence. For example, in the chain

$$Ra^{226} \rightarrow Rn^{222} \rightarrow Po^{218},$$

Ra^{226} has a half life of about 1620 years. The decay constant for the first step is $\lambda_1 = 4.28 \times 10^{-4} yr^{-1} = 1.17 \times 10^{-6} day^{-1}$. In contrast, Rn^{222} has a half life of 3.83 days, so that $\lambda_2 = 0.181 \ day^{-1} = 1.81 \times 10^{-1} \ day^{-1}$.

We know that whenever $0 < \lambda_1 < \lambda_2$, $e^{-\lambda_1 t}$ decreases to zero more slowly than $e^{-\lambda_2 t}$. If $\lambda_1 << \lambda_2$, the difference in these rates is so great that long after $e^{-\lambda_2 t}$ has become tiny enough to neglect in Equation (9), we can still say $e^{-\lambda_1 t} \approx 1$. (When t = 200 days in the example given above, $e^{-\lambda_1 t} \approx 0.99977$ and $e^{-\lambda_2 t} \approx 2 \times 10^{-16}$.) Then we would have

(15) $$N_2(t) \approx \frac{\lambda_1}{\lambda_2 - \lambda_1} N_0$$

and

(16) $$N_1(t) \approx N_0,$$

both approximately constant.

*For the definition of half life and its relation to the decay constant, see Exercise 1. For more about half life, see UMAP Unit 232 (*Kinetics of Single Reactant Reactions*).

Not only are $N_1(t)$ and $N_2(t)$ decreasing at the same rate, but this rate is so slow that they are in fact virtually constant. This situation is known as *secular equilibrium*. Again, as remarked in Section 4.1, we are stretching the term a bit, since the quantities involved are not really constant. Here there is yet another abuse of terminology, in that the "virtually constant" values of $N_1(t)$ and $N_2(t)$ are *not* their limiting values. Eventually (although perhaps none of us will live long enough to see it) these amounts will begin to decay noticeably, and ultimately they will approach zero.

Remember we also remarked in Section 4.1 that for all their abuse of the term, scientists do agree that it is wrong to apply the word "equilibrium" to a situation in which "nothing is happening" (there are no opposing forces or reactions). This is not a problem here. Plenty is happening. New B nuclei are being formed, and old ones are decaying. The total number of B nuclei remains the same, but they are not at all the same nuclei. (The total number of people in New York City is about the same as forty years ago, but they are certainly not the very same people.)

4.5 More Comments on Approximation

One thing about Section 4.4 may puzzle you. Adding the approximations given in Equations (16) and (15) we get

$$N_1(t) + N_2(t) \approx N_0 + \frac{\lambda_1}{\lambda_2 - \lambda_1} N_0 > N_0.$$

But the total number of atoms, including those of C and possibly later substances, must always equal N_0. How can this be?

What has happened is that $N_1(t)$ has decreased by a certain amount while $N_2(t)$ has increased by a lesser amount. But there was a lot of A to begin with, so the decrease is small compared to the original amount, and $N_1(t)/N_0 \approx 1$. On the other hand, there was none of B to begin with, and even at secular equilibrium there is very little. Compared to this amount, the increase is significant. It's as if Exxon Oil (A) were to pay you (B) \$10,000 and you were to use \$10 of that money to bribe your math teacher (C). Exxon will still have essentially the same amount of money as before, and you will be much better off financially, even though the total of Exxon's money and your money, when calculated precisely, will be less than before.* The numerical calculations in Exercise 7 may help you see what is going on.

Exercise 7

a. Use Exercise 5 to show that the amount of Rn^{222} in the chain of Section 4.4 is greatest at about t = 66 days.

b. (Requires a calculator.) For the values of t given below, compute the precise amounts of Ra^{226} and Rn^{222} as given by Equations (4) and (9); and also the sum of these amounts. Then compute the approximations given by Equations (15) and (14), and also their sum. Tabulate and compare these results. Take $N_0 = 10^9$ atoms, and make all computations to the nearest integer.

Use t (in days) = 1,2,3,4,5,6,7,8,9,10,20,30,40,50,60,66,70, 80,90,100,200,300,1000.

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*You will note that for this analogy it does not matter whether your math teacher is stable or not.

5. ANSWERS TO EXERCISES

1. $N_1(t^*) = N_0 e^{-\lambda_1 t^*} = \frac{1}{2} N_0$.

$$e^{-\lambda_1 t^*} = \frac{1}{2},$$

$$-\lambda_1 t^* = \ln\frac{1}{2} = -\ln 2$$

$$t^* = \frac{\ln 2}{\lambda_1}.$$

2. Writing $\lambda = \lambda_1 = \lambda_2$:

$$v'(t) = \frac{\lambda N_0}{R} e^{(\lambda-\lambda)t} = \frac{\lambda N_0}{R}$$

$$v(t) = \frac{\lambda N_0}{R} t + K$$

$$N_2(t) = u(t)\cdot v(t) = Re^{-\lambda t}\left(\frac{\lambda N_0}{R} t + K\right) = \lambda N_0 t e^{-\lambda t} + RKe^{-\lambda t}.$$

Since $N_2(0) = 0$, we have $RK = 0$ and

$$N_2(t) = \lambda N_0 t e^{-\lambda t}.$$

3. Set $D'(t) = 0$, where $D(t) = \lambda_1 N_1(t) + \lambda_2 N_2(t)$.

$$D'(t) = \lambda_1 N_1'(t) + \lambda_2 N_2'(t)$$

$$= -\lambda_1^2 N_0 e^{-\lambda_1 t} + \frac{\lambda_1\lambda_2 N_0}{\lambda_2-\lambda_1}\left(-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right)$$

$$= \left(-\lambda_1^2 N_0 - \frac{\lambda_1^2\lambda_2 N_0}{\lambda_2-\lambda_1}\right)e^{-\lambda_1 t} + \frac{\lambda_1\lambda_2^2 N_0}{\lambda_2-\lambda_1} e^{-\lambda_2 t}.$$

$D'(t) = 0$ when

$$\left(\lambda_1^2 + \frac{\lambda_1^2\lambda_2}{\lambda_2-\lambda_1}\right)e^{-\lambda_1 t} = \frac{\lambda_1\lambda_2^2}{\lambda_2-\lambda_1} e^{-\lambda_2 t}$$

$$\frac{\lambda_1^2(\lambda_2-\lambda_1) + \lambda_1^2\lambda_2}{\lambda_1\lambda_2^2} = e^{(\lambda_1-\lambda_2)t}$$

$$e^{(\lambda_1-\lambda_2)t} = \frac{\lambda_1(\lambda_2-\lambda_1) + \lambda_1\lambda_2}{\lambda_2^2} = \frac{\lambda_1}{\lambda_2^2}(2\lambda_2-\lambda_1)$$

$$t = \frac{1}{\lambda_1-\lambda_2}\ln\left[\frac{\lambda_1}{\lambda_2^2}(2\lambda_2-\lambda_1)\right] = \frac{\ln\lambda_1 + \ln(2\lambda_2-\lambda_1) - 2\ln\lambda_2}{\lambda_1-\lambda_2}.$$

4. a. Use $e^{-\lambda_1 t} \to 0$ and $e^{-\lambda_2 t} \to 0$.

b. $N_2'(t) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\left(-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right) = 0$

when $\lambda_1 e^{-\lambda_1 t} = \lambda_2 e^{-\lambda_2 t}$

$$\frac{\lambda_1}{\lambda_2} = e^{(\lambda_1-\lambda_2)t}$$

$$(\lambda_1-\lambda_2)t = \ln\frac{\lambda_1}{\lambda_2} = \ln\lambda_1 - \ln\lambda_2$$

$$t = \frac{\ln\lambda_1 - \ln\lambda_2}{\lambda_1-\lambda_2}.$$

Since $N_2(0) = 0$, $N_2(t_0) \geq 0$ (from physical considerations), and $\lim_{t\to\infty} N_2(t) = 0$, the desired inequalities for $N_2'(t)$ follow.

c. $N_2''(t) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\left(\lambda_1^2 e^{-\lambda_1 t} - \lambda_2^2 e^{-\lambda_2 t}\right) = 0$

when $\lambda_1^2 e^{-\lambda_1 t} = \lambda_2^2 e^{-\lambda_2 t}$.

$$\left(\frac{\lambda_1}{\lambda_2}\right)^2 = e^{(\lambda_1-\lambda_2)t}, \text{ etc.}$$

(Use the fact that $\ln\left(\frac{\lambda_1}{\lambda_2}\right)^2 = 2\ln\frac{\lambda_1}{\lambda_2}$.)

5. This occurs when $N_2'(t) = 0$. That is, at $t_0 = \frac{\ln\lambda_1 - \ln\lambda_2}{\lambda_1-\lambda_2}$, by Equation (10).

6. a. By Problem 5,

$$t_0 = \frac{\ln(1.37\times 10^{-1}) - \ln(5.1\times 10^{-3})}{1.37\times 10^{-1} - 5.1\times 10^{-3}} = 24.95 \text{ days.}$$

b. $N_2(t_0) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$

$$= \frac{(1.37\times 10^{-1})\times 10^{-8}}{5.1\times 10^{-3} - 1.37\times 10^{-1}}\left(e^{-1.37\times10^{-1}\times 24.95} - e^{-5.1\times10^{-3}\times 24.95}\right)$$

$$= 8.81 \times 10^{-9} \text{ grams.}$$

7. Column A gives the exact amount of Ra^{226}.
Column B gives the exact amount of Rn^{222}.
Column C gives the sum of columns A and B.
Column D gives the approximation of the amount of Ra^{226}.
Column E gives the approximation of the amount of Rn^{222}.
Column F gives the sum of columns D and E.

t	A	B	C	D	E	F
1	999,998,830	1070	999,999,990	10^9	6464	1,000,006,464
2	999,997,660	1963	999,999,623			
3	999,996,490	2708	999,999,198			
4	999,995,320	3330	999,998,650			
5	999,994,150	3849	999,997,999			
6	999,99[illegible]980	4282	999,997,262			
7	999,991,810	4643	999,996,453			
8	999,990,640	4945	999,995,585			
9	999,989,470	5196	999,994,666			
10	999,988,300	5406	999,993,706			
20	999,976,600	6291	999,982,891			
30	999,964,901	6436	999,971,336			
40	999,953,201	6459	999,959,660			
50	999,941,502	6463	999,947,965			
60	999,929,802	6464	999,936,266			
66	999,922,783	6464	999,929,247			
70	999,918,103	6464	999,924,567			
80	999,906,404	6464	999,912,868			
90	999,894,706	6463	999,901,169			
100	999,883,007	6463	999,889,470			
200	999,766,027	6463	999,772,490			
300	999,649,062	6462	999,655,523			
1000	998,830,684	6457	998,837,141.			

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ____________________ Unit No. ________

Page ____		Section ____		Model Exam Problem No. ____
○ Upper	OR	Paragraph ____	OR	Text Problem No. ____
○ Middle				
○ Lower				

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ____________________

Please use reverse if necessary.

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

STUDENT FORM 2

Unit Questionnaire

Name________________ Unit No.________ Date________

Institution________________ Course No.________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much Longer	Somewhat Longer	About the Same	Somewhat Shorter	Much Shorter
____	____	____	____	✓

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)