DOCUMENT RESUME

ED 212 463 SE 035 965

AUTHOR Kieren, Thomas E., Ed.
TITLE Recent Research on Number Learning.
INSTITUTION ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, Ohio.
SPONS AGENCY National Inst. of Education (ED), Washington, D.C.
PUB DATE May 80
CONTRACT 400-78-0004
NOTE 175P.
AVAILABLE FROM Information Reference Center (ERIC/IRC), The Ohio State Univ., 1200 Chambers Rd., 3rd Floor, Columbus, OH 43212 ($5.75).

EDRS PRICE MF01/PC09 Plus Postage.
DESCRIPTORS *Arithmetic; Cognitive Development; *Cognitive Processes; *Developmental Psychology; Educational Psychology; *Educational Research; Elementary Secondary Education; *Learning Theories; *Mathematics Education; *Mathematics Instruction; Models; *Number Concepts; Psychology
IDENTIFIERS *Mathematics Education Research

ABSTRACT

Presented are materials related to the work of the Number and Measure and Rational Numbers working group of the Georgia Center for the Study of the Learning and Teaching of Mathematics. Much of the content reports on attempts to bring constructs from developmental psychology and mathematics to bear in understanding children's ideas of number and measure. The reports included are thought to reflect a stage in a sequence of work, and are presented as a bridge between some of the ideas developed at a 1975 conference and on-going work. Seven individual research reports in mathematics education are included: (1) An Explication of Three Theoretical Constructs from Vygotsky; (2) Quantitative Comparisons as a Readiness Variable for Arithmetical Content Involving Rational Counting; (3) Language and Observation of Movement as Problem Solving Transformation Facilitators Among Kindergarten and First-Grade Children; (4) Aspects of Children's Measurement Thinking; (5) The Rational Number Construct, Its Elements and Mechanisms; (6) Seventh-Grade Students' Ability to Associate Proper Fractions with Points on the Number Line; and (7) The Relationship of Area Measurement and Learning Initial Fraction Concepts by Children in Grades Three and Four. (MP)

RECENT RESEARCH ON NUMBER LEARNING

edited by

Thomas E. Kieren
University of Alberta

May 1980

ERIC® Clearinghouse for Science, Mathematics
and Environmental Education
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Columbus, Ohio 43212

MATHEMATICS EDUCATION REPORTS

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This publication was prepared with funding from the National Institute of Education, U.S. Department of Education under contract no. 400-78-0004. The opinions expressed in this report do not necessarily reflect the positions or policies of NIE or U.S. Department of Education.

FOREWORD

The papers presented in this monograph represent varied efforts which stemmed from and are related to the work of the Number and Measure and Rational Numbers working groups which developed at a meeting of the Georgia Center for the Study of the Learning and Teaching of Mathematics held in the spring of 1975. Many of the discussions at that meeting (see Number and Measure, R. Lesh (ed.), ERIC/SMEAC, 1976) sought to bring constructs from developmental psychology and mathematics to bear in understanding a child's ideas of number and measure. The studies presented here represent explorations in these dimensions as well.

As is naturally the case, when one considers a phenomenon from several perspectives, a new perspective is generated which differs from the original. Thus the deliberations of members of the above working groups have generated concerns which go beyond those of developmental psychology and mathematics as these two relate to mathematical education. Several of the papers here (e.g., Lamb, Owens, Steffe, and Hirstein) attempt to relate developmental and instructional variables. Yet their concern is not for developmental theory but for ways in which one can describe the mathematical thinking of children and the individual child in particular.

This concern for personal mathematical knowledge has led authors of this monograph to bring various philosophical views into play as well. A central assumption, if not underlying the papers, then certainly useful for a reader reading them, is that persons can build up or construct mathematics for themselves. What these constructions look like, as well as their extensibility and their relatedness to other mathematical ideas, is dependent on several things. One component might be characterized as "readiness," the history a person brings to a particular experience. A second aspect is the nature and extent of the experiences of the person. Like any other ideas, mathematical ideas should be "about" something to have validity. Informal mathematical language must be about certain experiences; formal symbolic expressions may be about less formal symbolic expressions; mathematical structural ideas reflect both informal experience and formal symbolic experience. These concerns are treated theoretically and empirically in the papers of this monograph.

These papers reflect a stage in a sequence of work being done by these authors and others. They are presented here as a bridge between some of the ideas developed at the 1975 conference in Georgia and on-going work in this area today.

Beyond the authors there are a number of persons who have worked on this publication. All papers received several reviews and the final form of the papers attempts to reflect reviewer concerns. Without the generous, continuing, and ingenious efforts of Les Steffe this monograph would not exist. He has worked hard in the original stimulation, the organization of critical reviews, and the editing of aspects of the document, as well as sharing some of his own work. Marilyn Suydam has taken the final responsibility to ensure the quality of the publication through ERIC/SMEAC at Ohio State. To all these persons and to the authors who have waited patiently for this publication I give my heartiest thanks.

Thomas E. Kieren
Edmonton
February 1980

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An Explication of Three Theoretical Constructs from Vygotsky

Karen C. Fuson

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Three theoretical constructs discussed by Vygotsky and expanded upon by other Soviet researchers and theoreticians seem to be potentially useful for the field of mathematics, learning, and mathematics teaching. These constructs are the movement from the inter-psychological plane to the intra-psychological plane, the distinction between spontaneous and scientific concepts, and the zone of proximal development. Each of these constructs will be described along with our further analyses and extensions of the constructs. Vygotsky was a seminal thinker and with broad strokes draws a stimulating and thought-provoking picture of the development of human thinking that is recapitulated within each child. However, due to his early death from tuberculosis, most of his ideas were not thoroughly worked out, and he left little published detailed evidence concerning them. Such evidence would have helped to define as well as to support some of the ideas, so its omission is detrimental to comprehension as well as to evaluation of Vygotsky's theoretical points. In addition, the English translations of Vygotsky's work have omitted much of the original, due to translators' attempts to eliminate long digressive passages and to make the work more succinct and pointed. This has resulted in reading which appears at times to be somewhat disjointed, and it has exacerbated the tendency of Vygotsky to concentrate on different aspects of the same concept at different places in his writing. In the original, such shifts might have been 50 pages apart and connected by material which made the transition comprehensible. In the translation, such shifts may be only a few pages apart and contain no such meaningful transitions. For example, the chapter on scientific concepts in the original Russian version is 107 large, fine-printed pages long. In the English translation, this chapter is only 37 small, large-printed pages long. These problems complicate still further the original lack of specificity.

For these reasons, the first step to be taken with respect to increasing the potential utilization of Vygotsky's ideas is to arrive at clarifications and extensions of those

*Special thanks to Richard Lesh, Nancy Stein, Walter Secada, and James Wertsch and to two anonymous reviewers for comments on earlier drafts of this paper. Members of the Number and Measurement Working Group also gave me helpful feedback on an earlier draft.

ideas. Thus, this paper is primarily an explication and an extension rather than a critical evaluation of Vygotsky's work. Examples from the area of early mathematics learning are also provided in some sections. Many of the examples will be quite familiar ones to mathematics educators, but the Vygotskiian frame-work within which they are set may provide a slightly new perspective for them.

Before beginning, it is useful to contrast the different emphases which Vygotsky and Piaget have. For Vygotsky, the paradigmatic learning situation is that of a child interacting with an adult. For Piaget, it is a child alone interacting with objects or a child interacting with peers. Vygotsky was more interested in the cultural learning of culturally important concepts while Piaget focused more upon the natural learning of concepts important regardless of culture (objects important in the natural world). The important learning mechanism for Vygotsky was direct instruction from an older member of the culture. For Piaget, it was the accommodation of one's own views to conflicting ideas of one's peers. These emphases are actually complementary rather than contradictory. First, neither writer would deny the existence of the factors which the other considers. Second, together these two emphases cover most of the important sources of learning in a child's world. This contrast in emphases indicates that one of the important ways in which Vygotsky's ideas can be extended are by consideration of the influences of peers and of the object world.

Movement from the Inter-Psychological Plane to the Intra-Psychological Plane

Vygotsky focuses upon the child as developing within a particular social and cultural context. Development is viewed as a process that is constantly directed by that social and cultural context. The child does not just "develop" spontaneously and unconstrained; the child is also "developed by" her social and cultural context and especially by the older members of that cultural context. This view is specifically related by Vygotsky (1978) to intellectual functioning:

> An interpersonal process is transformed into an intra-personal one. Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals. (p. 57)

This transformation is a gradual one and is the result of the interaction between a long series of developmental events and cultural learning experiences.

Vygotsky does not clearly define inter-psychological processes, intra-psychological processes, or how the movement from one to another occurs. I believe that he is talking about at least three different things when he discusses the inter- to intra-process. He is referring to:

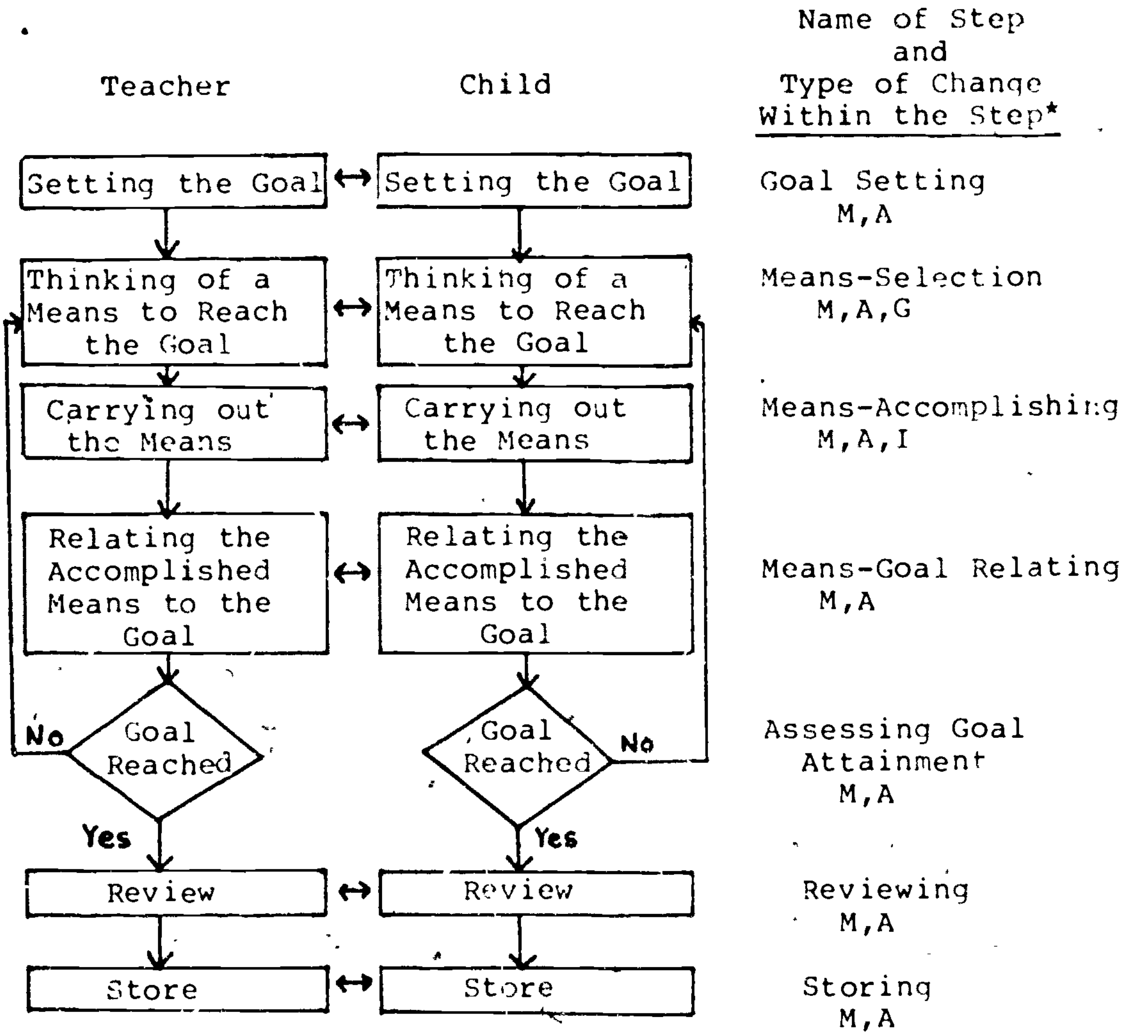
a. the directive function of the adult with respect to the child's attention, actions, and psychological functioning

b. verbal information about concepts or action sequences which the adult may possess and which she or he may pass on to a child

c. cultural and intellectual tools (e.g. language) which the adult helps the child to learn and use

All three of these meanings are evident in an example of a child learning to count. In the many learning trials which a child will experience in learning to count, the adult (or older child) will exercise many directive functions, functions which gradually fade away to come under the control of the child. The adult initially will set the goal of counting, will define the set of countables, will do the whole counting process initially, will have the child practice and imitate, etc. The adult will keep the child's attention on the task, will point out errors, and will help the child to overcome them. Through such efforts over a period of time, the child comes to possess the important cultural tool of counting, i.e., the actual process of counting has passed from occurring for that child only in an interpersonal context and now can occur within that child alone. This type of inter- to intra-movement is the third type above. The second type of inter- to intra-movement is exemplified by possible adult answers to the question, "What is counting?" Here verbal knowledge about counting possessed by the adult is provided to the child. The child, of course, may have limited understanding of the verbal knowledge which is provided, and the semantic representation which the child actually forms and stores may be quite unalike that intended by the adult. Such verbal information given by the adult might be "counting is for finding out how many there are" or "only big girls can count" or "counting is saying your numbers."

The directive function of adults is an extremely important one. This function can be made a bit more specific by considering behavior to consist of sequences of goal-directed activity. Figure 1 presents a very simple sketch of goal-directed activity which is done jointly by a



*M: Mastery
A: Abbreviation
G: Generalization
I: Internalization

Figure 1. Goal-Directed Activity in the Classroom (from Fuson, 1979)

"learner" and a "teacher."* Figure 2 presents a much more complex outline of goal-directed activity, which checks at various points, etc. A given execution of a goal-directed sequence is on the inter-psychological plane if it is a cooperative (co-operative) activity. The directive function of the adult then is to execute with the child (or for the child) any step in the activity sequence which the child cannot do alone, to monitor the progress of the child through the activity sequence, to serve as the external memory for the next step, and to direct the child to it if necessary. The teacher may provide verbal representations of what the child is doing either as directives before the child does them or as descriptions while the child does them (e.g., 'Now you're putting all of the green ones together in a line.") or after the child does them (interpersonal function b). These verbal representations may serve both to direct the child's behavior in some way and also may alter his or her cognitions about the concepts referred to in the verbal representations. Thus such representations may carry both directive and semantic force.

In such a co-operative teaching/learning process, the teacher gradually "fades" from each step and each connection over time, and the activity sequence, as well as skills and concepts within this sequence, formerly possessed only by the teacher, now become possessed by the learner. Other aspects of this type of model of the teaching-learning process are described with respect to mathematics learning in Fuson (1979).

Although many Soviet psychologists discuss or at least allude to this passage from the inter-psychological plane to the intra-psychological plane, the details of this transformation with respect to mathematical ideas have been almost as little studied there as here. Gal'perin (1957), in one of the few such works, postulates a series of five levels in the internalization process. As with Vygotsky, these levels are assumed to be operating in a social learning situation with another person (a teacher) present. Gal'perin here introduces the world of objects, though this world is still set within a social interactive situation. The five Gal'perin levels are:

1. creating a preliminary conception of the task
2. mastering the action using objects
3. mastering the action on the plane of audible speech

*We will use adult (and teacher) to mean any member of the culture who is exercising a directing, teaching function with a child. This frequently will be an older sibling or friend and, more occasionally, a peer.

The cycle may stop at any time either by 1) the intrusion of another event which is attended to or 2) a decision to abandon the goal. Many of the steps become automatized as the activity becomes a familiar one. Some of the evaluation steps may be omitted, especially by younger children.

Goal rected activity occurs as part of the continuing stream of behavior, so this flow-chart contains no start or stop.

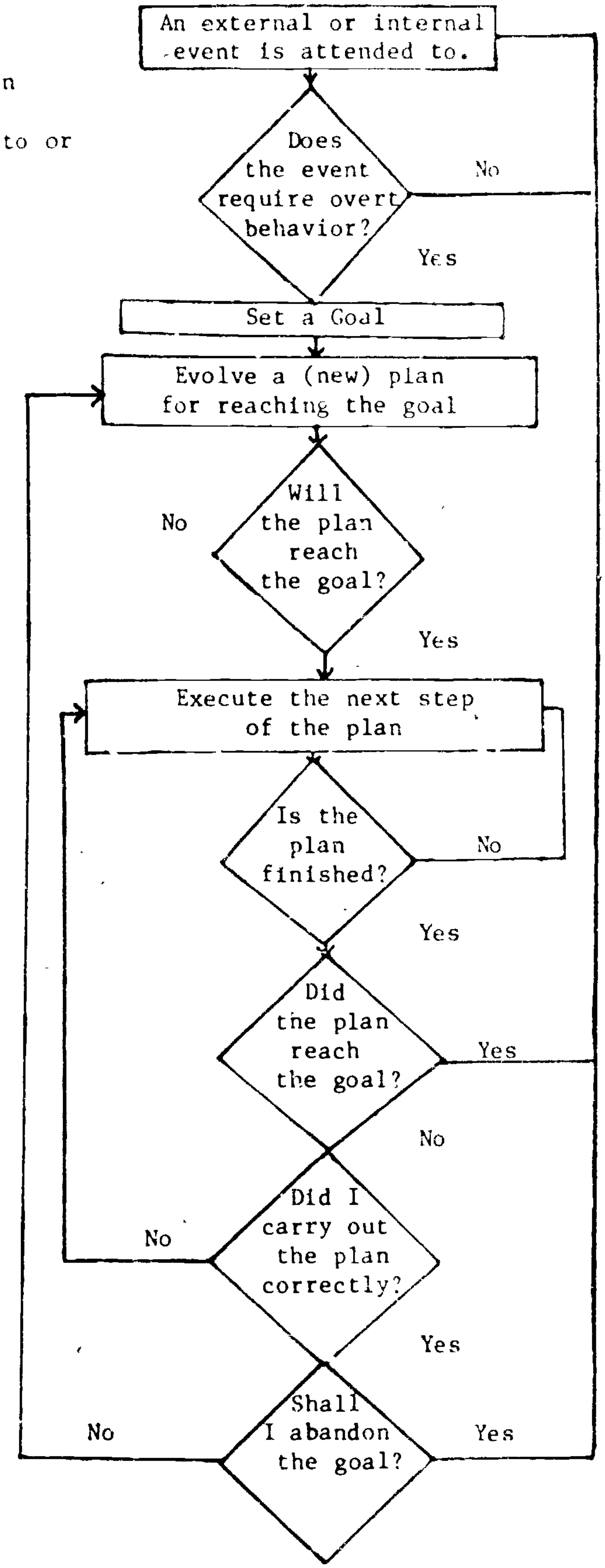


Figure 2. Goal-Directed Activity

4. transferring the action to the mental plane
5. consolidating the mental action

The first level refers to some attempt to ensure that the child understands at least to some minimal extent the nature and the function of the task to be accomplished. The second level is fairly specific with respect to mathematical ideas --it asserts that the first representation which children must build is a concrete one and is built from observations of and actions on objects. With respect to other (nonmathematical) concept domains, this step might be generalized to mean any concrete representation, such as one arising from doing a series of actions for oneself or having certain experiences oneself rather than being told about such actions or experiences. Level 3 may mean several things, and it is not clear which of these Gal'perin meant. This step will be discussed in detail later. Level 4 concerns the internalization of the concrete representation used in Level 2: that is, here an internal representation of any action is substituted for any previous external action and internal images of objects are substituted for real objects. Level 5 refers to effects of practicing which lead to automatization of the mental actions involved in the process.

American research and curriculum efforts in mathematics learning sometimes pay conscious attention to Level 2 and often contain some unconscious efforts at Levels 1, 4, and 5. However, focus on Level 3 is omitted entirely, and sometimes this process is even consciously postponed because of the fear of producing premature and/or empty verbalisms in children.

Failure to achieve adequately Level 1--creating a preliminary conception of the task--probably results in much more difficulty in both research and instruction than is generally realized. Verbal instructions either in a research task situation or in the classroom often fail to convey a message which is meaningful enough for a young child to act upon it accurately. Alternatives which can be helpful here are modeling the type of activity required (this of course cannot be done in some research situations as it would predispose the child to a certain type of response), using objects which by their characteristics will help to create the desired type of response, and using a simpler version of the required task and ensuring that subjects know that the more difficult task should be done in the way that the simpler one was done. A study by Bullock and Gelman (1977) indicates both how little one should assume about a child's preliminary conception of a task and how little one should assume that they spontaneously generalize. Children aged two, three, and four years were reinforced for choosing the smaller (or larger) of one- and two-item arrays. They were then presented arrays of three and four items and asked to "pick the winner." Forty-two percent of the two-year-olds, 75 percent of the three-year-olds, and 87 percent of the four-year-olds responded in a manner consistent with their first condition. This seems to indicate that two-year-olds cannot generalize "more" or "less" relations even for such small numbers as three and four. But in a follow-up study in which the one- and two-item arrays were present (either covered or uncovered) during the testing on the larger arrays, eleven of the twelve children aged 2-1/2 to 3 years responded with the same relationship that had been reinforced for the smaller array. The fact that whether the initial array was uncovered or not was irrelevant to the performance indicates that these young children did not need the information from the small arrays (i.e., their failure was not a result of a lack of memory about which was the "winner" or even what the task had been about); they simply did not realize that the second task was supposed to be related to the first one. For them, the second task was initially not the same task as the first one was.

Gal'perin reported the results of comparing two different methods of becoming familiar with a task. After the teacher's initial explanations, the child either interacted with the materials himself under the direction of the teacher or directed interactions of the teacher with the materials by telling the teacher the next operation to do (but did not interact with the materials himself). The second method proved 'rather more productive" (Gal'perin reported no task or testing details). He hypothesized that not having to perform the action physically freed the child's orienting activity and enabled a fuller and more correct conception of the task. This approach might be examined further. Such effects might also be operating when groups of children work together with objects and watch and direct each other's actions.

More explicit attention needs to be paid by educators towards ways to help a child categorize the learning experiences she is having. Separate mathematics lessons are spread over years of schooling; a child needs help in organizing these lessons so that she can efficiently store new learning and retrieve relevant old learning in order to relate it to new learning. Codes on workbook and text pages might be one such aid. Another one that may be effective is the use of story settings for learning mathematical ideas. This approach was used for some units by Lesh and Nibbelink (1978) in a kindergarten workbook. A given story setting (e.g., stick bugs) is used for a group of five consecutive lessons on measuring. The story situation is motivating to children, the teacher feels natural dealing with a story situation (rather than with just mathematics), and the children related the separate lessons to each other. Re-use of the same setting later in the year would thus seem to function as a strong cue for children. Developing and evaluating the effectiveness of this and other ways to help children sort out and code the various mathematical learning tasks confronting them would seem to be an important research goal for mathematics educators.

Level 2 is regarded by Gal'perin as requiring heavy involvement from an adult (i.e., it has a large interpsychological component), but its primary focus is that the child's interactions with objects is essential for the formation of mental images of the processes:

> This material action is, of course, built up in continuing verbal intercourse with the teacher, under the guiding influence of his instruction, explanations and corrections. But, at this stage, the role of speech, in the case of both teacher and pupil, is limited to indicating objective features of the goal, the objects available and the methods of dealing with them. These instructions, however important, do not take the place of action; the action can be completed only on the level of things, being based on them and determined by them, and remaining essentially an external, material action. ...The kernel of the matter is that this material form of action is not only the inevitable initial form of a child's independent activity, but also the origin of the content and structure of the mental action subsequently elaborated. (Gal'perin, 1959, p.218)

Levels 1, 2, 4, and 5 are not particularly original. Levels 2 and 4 can together be regarded as reflecting the process of the internalization from actions on physical objects in the real world to mental images of actions. Piaget and many others have postulated and researched such a process. The Soviet emphasis here is upon social interactions concerning socially defined objects, but the need for objects in learning is still apparent. Level 3 is Gal'perin's major new contribution to the idea of internalization of mathematical processes. By this I believe he means something rather special to some mathematical processes. This "something special" is neither of the first two interpersonal functions--neither the directive nor the verbal information functions. Rather, Gal'perin here is talking about a problem solution that can be represented verbally--i.e., he is referring to speech which <u>is</u> the actual problem solution. This view, I believe, is that this speech originally accompanies and refers to the actions on objects. Eventually the speech comes to represent the action sequence on symbols, rather than that on objects, and the verbalization actually <u>becomes</u> the solution process. Most computational processes are an example of Gal'perin's Levels 2, 3, and 4. For a child solving 27 + 45 by saying, "Hmmm, seven plus five is ten, eleven, twelve. Put down the two and carry the 1. One and two is three and four is 7. Put down the seven. Seventy-two.", the verbalization <u>is</u> the problem solution process. Such a verbal representation of a cognitive process may be relatively rare. Such representations would seem to be useful for processes involving a serial list of actions that must be performed in a certain order. The verbal representation then serves as a memory aid for each step. According to Gal'perin's scheme, this verbalization eventually becomes internal (Level 4) and finally almost automatic (Level 5).

The directive function of verbalization is also important in the internalization process. Gal'perin sometimes also seems to mean this function when discussing Level 3. However, when the directive function of verbalization is considered, it seems that instead of being a separate level that intrudes into the process of the internalization from objects to mental images, Level 3 might better be conceptualized as a continuum of internalization from "verbal instructions from another" to "verbal instructions to self." This continuum is orthogonal to that representing the internalization from objects to images. If one pictures these two types of internalization as representing perpendicular axes, then various points on the plane suggest research or instructional tasks. A high concrete and high self-verbalization example is an adult mathematics student learning group structures through some physical embodiment of the Klein four-group. A low concrete and high other-verbalization is instruction from a teacher about doing a problem such as 23 + 5 by counting-on using the internalized string of number words (and some acoustic and probably also visual image of this string). Examples in the other two quadrants are obvious.

A recent review of research in the development of self-regulating speech (Fuson, 1980) indicates that training children in the use of self-regulating speech does seem to be effective in some kinds of cognitive tasks. Training studies on self-regulating speech studies in mathematics, particularly in algorithmic processes, would seem to be a valuable contribution to our knowledge about how mathematics is learned.

Verbalization in mathematics learning thus has at least two important functions--a directive one and a representing one. Of course, these functions are not entirely disjoint--any directive involves a representation of what is to be done and any representation carries some implicit directive force. But this distinction is probably a useful one. The directive function of verbalization is used in all subject matter areas while the representing function may not be. Both of these functions deserve the attention of researchers in mathematics education.

Spontaneous Concepts and Scientific Concepts

The second theoretical idea to be discussed is Vygotsky's distinction between spontaneous and scientific concepts.* The main source available in English concerning this distinction is a 37-page chapter on scientific concepts in Thought and Language (1962). This chapter is an edited version of the original Russian 107-page version. As with other Vygotskiian concepts, spontaneous and scientific concepts are fairly broad concepts, and different aspects of them are discussed at different points in Vygotsky's writings. There seem to be at least five separable aspects of these concepts that differentiate them. Describing and discussing these aspects will help to give some specificity to these concepts, but it should be kept in mind that all of these aspects are quite interrelated. Some, in fact, might be taken to be basic and then the others might be argued to be derivable from these basic ones. These aspects also should be viewed as each forming a continuum, with scientific concepts being relatively high on this continuum and spontaneous concepts being relatively low, rather than a more simplistic view of these types of concepts possessing all or none of these characteristics. These five differentiating characteristics are:

1. the level of consciousness at which the concept is understood

2. the presence of a hierarchical (superordinate) system within which the concept is embedded

3. the amount of cultural-historic input into the construction of the concept (as opposed to the ordinary existence of the concept in the natural world)

4. the amount of "mediated" experience (i.e., experience only through verbal definition and discussion) in the learning context as opposed to direct perceptual or other sensory experience

*Vygotsky certainly does not mean to restrict scientific concepts to those concepts used in the natural sciences, for many of his examples come from the social sciences. It is not even clear that he would restrict such concepts to those occurring in the natural and in the social sciences. It is to avoid this ambiguity of common usage that I later suggest the use of the term nonspontaneous concepts instead of the term scientific concepts.

5. the amount of direct and systematic verbal instruction necessary for the learning of the concept

Scientific concepts are relatively high on these characteristics; spontaneous concepts are low. Spontaneous concepts are concepts formed by children through their everyday interactions with their sensory world. These concepts are not organized into a hierarchical system, but are formed and exist in the mind of the child without conscious effort or knowledge and without much (if any) direct instruction by another. Vygotsky specifically mentions the early work of Piaget--on the development of the meaning of concepts like "brother," "because," and "flower/rose"--as providing examples of spontaneous concepts. Spontaneous concepts arise from a rich real-world context and are "saturated with experience."

Scientific concepts are formed consciously by the child from the very beginning. Vygotsky uses consciousness at various times to mean such things as the ability to define a concept in words and to operate with it at will. Also, he asserts that generalization results in consciousness. Vygotsky takes generalization to mean the formation of a superordinate concept that includes the given concept as a particular case. What he really means by this, in terms of the above five aspects, is that if a concept exists within a hierarchical system as <u>one</u> of several exemplars of a subordinate category, then the superordinate category enables a type of meaning to be given to that concept (a certain kind of consciousness) that it cannot get in other ways. We will give examples of this type of consciousness in mathematics a bit later. According to Vygotsky, scientific concepts are not constructed from direct sensory experience but are mediated by other verbal concepts from the very beginning. Children learn about scientific concepts by talking about them or by being talked to about them. In addition, scientific concepts are unlikely to be constructed spontaneously by the child; they result from contributions by past members of the culture and have been maintained as part of the culture's heritage because of their importance or usefulness to that culture. The mediated and abstract nature of scientific concepts would seem clearly to require direct instruction from an older member of the culture. But the manner of this instruction may also, according to Vygotsky, contribute to the consciousness of these kinds of concepts (i.e., 4 and 5 lead to 1). The teacher, working with the pupil, may explain, supply information, question, correct, make the pupil explain, and otherwise have the pupil think about and talk about the concept. In this way scientific concepts may be raised to a level of consciousness not possessed by the unexamined, undiscussed spontaneous concepts. In addition, though not mentioned by Vygotsky, scientific concepts also would seem to possess heightened consciousness in the mind of a child because the child (due to the direction of a teacher) forms a deliberate intent to learn such concepts, i.e., understanding such concepts is set as a goal. Spontaneous concepts seem more likely to be learned by a child within the flow of everyday activity and without the formation of a deliberate, conscious goal to do so.

By all of these means, scientific concepts become what we might call the objects of thought while spontaneous concepts do not; i.e., scientific concepts, because of characteristics 1 through 5, become the objects of conscious reflection. Vygotsky argues that spontaneous concepts originally are not the objects of such conscious reflection, but through the influence of scientific concepts, spontaneous concepts gradually come to rise to the level of consciousness of scientific concepts. That is,

> The formal discipline of scientific concepts gradually transforms the structure of the child's spontaneous concepts and helps organize them into a system; this furthers the child's ascent to higher developmental levels. (1962, p. 116)

Spontaneous concepts also affect scientific concepts. The richness of the context within which spontaneous concepts are embedded is hypothesized by Vygotsky gradually to affect scientific concepts; that is, some of the richness of the meanings associated with spontaneous concepts eventually becomes attached to scientific concepts. Thus, these two types of concepts are said to interact: "... the development of the child's spontaneous concepts proceeds upward, and the development of his scientific concepts downward, ..." (1962, p. 08).

The way in which these two types of concepts interact seems very ill-defined. Are the same hierarchical systems constructed through scientific concepts used to understand spontaneous concepts or does the "hierarchicalness" itself somehow generalize and become applicable for spontaneous concepts? Piaget has an alternative mechanism for the way in which spontaneous concepts come to be objects of conscious reflection. This mechanism is through the conflict and contradiction arising from peer expressions of a differing view on the same concept; the child must accommodate his or her view to that of the contradictory information contained in the peer's viewpoint. Piagetian theory also has an alternative explanation for the increasingly hierarchical nature of spontaneous concepts--the advent of concrete operations which permit class inclusion types of superordinate relations. This does seem to be one of the relatively few places where Piagetian and Vygotskiian theory are directly contradictory.

A further question with respect to the relationship between spontaneous and scientific concepts is whether spontaneous concepts continue to be learned after scientific concepts have created a certain minimal level of mental hierarchicalness. That is, are spontaneous concepts only learned by preschool children? This point is not really clear in Vygotsky's translated writings. If it is true, then the distinction between spontaneous and scientific concepts becomes much less interesting, for it occurs only early in development.

However, there do seem to be some contributions which this spontaneous/scientific distinction can make. Two changes in terminology will facilitate this usefulness. The first change is to replace the term "scientific" concept with the term Vygotsky used when initially discussing such concepts. He first used the words spontaneous and nonspontaneous concepts. These words seem to include the necessary distinctions without adding possible confusions caused by the use of the word "scientific" with its many other meanings. In addition, it is helpful to differentiate attributes of the concepts themselves from the ways in which those concepts are learned. Thus a nonspontaneous concept is one which is high with respect to characteristics 1, 2, and 3 (from the list of five characteristics of scientific/spontaneous concepts), and a spontaneous concept is one low with respect to the same items. Items 4 and 5 then present important ways in which concept learning can vary. Item 4 can now more clearly be seen to be the same as the second process I discussed with respect to the inter- to intra-movement: b) verbal information about concepts or action sequences which the adult may possess and which she or he may pass on to a child. This indirect verbal means of gaining information about a concept is contrasted with direct perceptual means of operating on objects in the real world. To continue the above terminology, learning from verbal descriptions and definitions is thus non-spontaneous learning and learning through the object world is spontaneous learning. I will return to this distinction as soon as Item 5 has been discussed. Item 5 concerns the amount of direct and systematic verbal instruction that is necessary for the learning of the concept. This item is thus the same as the first type of process identified with respect to the inter- to intra-movement: a) the directive function of the adult with respect to the child's attention, actions, and psychological functioning.

Vygotsky's and Gal'perin's examples of this directive function were all of the direct immediate personal kind: the adult interacted in person with the child. Such immediate direction is very typical of learning in and out of schools. But another way in which the directive function is accomplished is also very typical of school learning--removed direction arising from materials used within the classroom. Textbooks, workbooks, ditto sheets, games, and objects all exercise direction over a child's activities. Thus, there are two types of directive function: immediate and removed. Note that a single adult can reach many more children by a removed directive medium (e.g., a textbook), but that only immediate directive interactions can have a feedback loop that will adapt the directives to the needs of a particular learner.

Now that these distinctions concerning spontaneous and nonspontaneous concepts and spontaneous and nonspontaneous learning have been made, it is possible to discuss certain types of learning not considered by Vygotsky. One can now consider spontaneous and nonspontaneous concepts learned spontaneously or nonspontaneously. In addition, the differentiations made above with respect to Items 4 and 5 permit even more specific types of learning to be discussed. By crossing the verbal information/object information (Item 4) with the directive (immediate or removed)/nondirective (Item 5) (see Figure 3), one obtains Vygotsky's old spontaneously learned category (object information learned with nondirection--the paradigmatic Piagetian learning situation) and the old nonspontaneously learned category (verbal information learned directively--the paradigmatic Vygotskiian learning situation). However, the ability of the directive function to be accomplished in an immediate or a removed fashion actually results in two parallel nonspontaneously learned categories--both of which involve verbal information learned under the direction of an adult, but this direction may be immediate (in person) or removed (or even involve immediate direction about a removed direction, e.g., a clarification of a ditto-sheet instruction).

A new category even more important for mathematics learning results from this crossing of Items 4 and 5. This is the category of object information learned under the direction of the adult. This direction may come in person (i.e., the adult may observe a child interacting with objects and make suggestions, summarize what is happening, etc.), or the directive function of the adult may result from the adult structuring the objects in such a way as to direct what the child will learn from those objects. This latter category thus includes all activities of children with structured objects, i.e., with objects which possess in some aspect of their physical form the properties and relations of the mathematical ideas which it is desired that children learn. For example, multibase blocks, Cuisenaire rods, multiplication pieces, and chip computers all enable children to abstract characteristics of our base ten system of numeration and operations that can be performed within it. This new type of learning thus returns to the two orthogonal axes proposed in the discussion of Gal'perin's levels of internalization--external objects/internal representations of objects and verbal directives from another/verbal directives from self.

The restriction of the meaning of nonspontaneous concepts to the first three characteristics enables us to see more clearly another point--that the "new math" thrust of the late

TYPE OF INFORMATION

TYPE OF DIRECTION		Verbal	Object
	Nondirective	lecture	old "spontaneous"
	Directive		
	Immediate	old "nonspontaneous"	summaries, suggestions, questions
	Removed	books ditto exercise~~s~~	structured materials

Figure 3. Examples of Crossing Item 4 and Item 5

fifties and early sixties was an attempt to make the mathematical concepts learned early in school by children nonspontaneous rather than spontaneous concepts. Sets of objects were not just to be made and then used to aid calculation. Such sets were to be placed within a more general perspective of sets in general, i.e., the sets of objects which have often been used in elementary classrooms were to be viewed by children as specific examples of the more general concept of set. Likewise, nonbase ten systems of numeration were studied in order to give additional meaning to the base ten system. That is, in Vygotsky's terms, via these curriculum modifications children would learn with a greater consciousness the concepts of set and of the base ten system. Whether the fact that these curricular innovations fell short of their expected results was due to the manner in which they were taught or because the goal of such hierarchical generalization was unrealistic is not at present known. Certainly the learning of nonbase ten systems suffered by being approached mainly via translations from the different system into base ten and back again, rather than by working within that system to begin to get a sense of its properties--which could then be generalized to the base ten system. Teaching nonbase ten operations helps adults understand the analogous base ten operations better (e.g., Fuson, 1975). Dienes (1963) also suggests that this is true for children. With structured materials, we now are in a better position to assess Vygotsky's hypotheses about the effects of a superordinate category (base systems in general) upon the consciousness of particular mathematics concepts (base ten operations).

As a final example of a mathematical nonspontaneous concept, let us consider numbers. A cardinal number is the measure of a discrete set of objects. This measure is often arrived at directly by counting, although it may be derived exactly or approximately in other ways. The notion that a cardinal number can be derived by counting (i.e., that the last counting number said when counting a set of objects is the cardinal number for that set), must be constructed by children. Children can be found who count a set and who cannot then answer the question "How many?" about that set (Schaeffer, Eggleston, and Scott, 1974). However, this idea has already been constructed by many three-year-olds and most four-year-olds; thus most children begin school with some notion of a cardinal number. Such notions of a cardinal number derived from counting might thus be considered to be spontaneous concepts.* The usual school curriculum is based

*Because counting is high on Item 3 (it requires heavy artificial cultural learning), it might be considered to be a nonspontaneous concept even though it is low on Item 2 (presence within a hierarchy). This example thus illustrates another difficulty with the notion of non-spontaneous concepts: a given concept may shift categories depending upon which criterion is used.

on the set of natural numbers (or sometimes the set of whole numbers), and it treats all of the operations (+, -, x, ÷) as operations on natural or on whole numbers. Davydov (1979, Note 1) and Gal'perin and Georgiev (1969) propose an alternative approach to number at the beginning of school (age 7 in Soviet schools). This approach assumes that the concept of number derives from the more general process of measurement of a continuous quantity, i.e., a number results when one determines the multiple relationship of some quantity to a part of that quantity used as a measure or a unit of measurement. A very active introductory curriculum is used in which children measure various quantities using various sorts of units. The measure of a quantity is thus always with respect to a particular unit. The counting of discrete sets of objects is a special type of measurement, a type where the unit of measure is the unity (and discreteness) of any single object. Thus from the beginning the natural numbers are viewed as arising from a special sort of measuring process. In addition to this difference, natural numbers are also an example of a relatively rare and "nice" sort of measure--one that comes out even. If one takes any quantity and any measurer, the chances are low that the resulting number (the measure) will be a whole number. Thus, in this approach a natural number is from the beginning a nonspontaneous concept--it is viewed from the perspective of a number as derived from a more general act of measuring.

As with any curricular innovation, we need to obtain information about the immediate and long-term advantages of the inclusion of nonspontaneous concepts in the curriculum. Obtaining such information will be difficult and slow, for the long-term effects may be subtle and far-reaching. But in general, it seems that whenever nonspontaneous concepts rather than the roughly corresponding spontaneous concepts can be learned by children, it would be beneficial to teach the nonspontaneous concepts. The real question then becomes whether and/or when concepts can be taught as nonspontaneous ones.

The Zone of Proximal Development

Vygotsky was interested in the relationships between learning and development. He rejected the view that all development is the result of learning, i.e., that little bits of learning increment are responsible for any intellectual growth of a child. He also rejected the view that development is relatively independent of learning, that developmental processes unfold without regard to and not dependent upon learning that has occurred, and that learning merely uses the achievements of development as these achievements occur. Instead Vygotsky postulates another relationship between learning and development, one that is dependent upon his view of intellectual functioning as moving from the inter-psychological to the intra-psychological plane.

Vygotsky's position results from the observation that with the support of an adult or of more capable peers, children are able to solve problems and accomplish var ous tasks that they are unable to do alone. Vygotsky focuses attention upon this by giving it a name: a zone of proximal development. This zone of a child is

> the distance between his actual developmental level as determined by independent problem-solving and his level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers. (1978, p. 86)

Thus, learning is seen to lead development by creating zones of proximal development in the child:

> that is, learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment and in cooperation with his peers. (1978, p. 90)

The zone of proximal development is where inter-psychological processes first occur. When such processes later become internalized as intra-psychological processes, they become part of the child's developmental achievement and are no longer in the child's zone of proximal development. Thus the relationship between development and learning is a dynamic one, with development providing constraints upon the type of learning that is possible but with learning through the support and guidance of others furthering and contributing to developmental progress.

The notion of the zone of proximal development involves a starting point (actual developmental level, measured by independent problem solving) and an end-point (potential developmental level, measured by cooperative problem solving). This notion thus necessarily involves a learning path, a path from the actual developmental level to the potential level. To me, such a path seems to be constructed in at least three different ways. The first learning path is defined by a task analysis of the subject matter to be learned. Such an analysis specifies a hierarchical set of skills and subskills and marches the child through this hierarchy. Such a learning path in the early number area might move from counting, to symbol recognition, to addition with very small numbers, to addition with larger single-digit numbers, to addition of a double-digit number and a single-digit number without carrying, to such problems with carrying, to the addition of two double-digit numbers without carrying, etc.

A second learning path derives from attempts to trace the spontaneous development of concepts in the mind of the child. This spontaneous developmental path is then used to define the path of instruction for other children. An example of this approach is to attempt to trace a developmental sequence of the counting solution strategies children use in solving addition problems (e.g., Fuson, 1979, Note 2). Here the problem hierarchy is greatly simplified (i.e., the same strategies are used for all of the above types of addition problems), but a solution hierarchy is imposed arising from development within the child. Each of these two approaches determines a different learning path, i.e., the tasks presented for cooperative learning would differ for these two paths.

The Soviet view of the derivation of the learning path is yet another one. This path is not determined just by the nature of the subject matter to be learned nor by the spontaneous sequence of development of a child. The direction of this third path is affected by learning which occurs along it. That is, learning is not assumed merely to help development along in its inevitable path; some learning (in particular some school learning) is hypothesized to change the path of development. Certain concepts are so powerful and general that they become tools of thought, tools that change the course of development. An example of this type of learning path in early mathematics learning might be the concept of ten as a unit. Before this concept is learned, children consider a number such as 37 to be composed only of 37 single units. They would make a set of 37 by taking 37 sticks (rather than three groups of 10 sticks and 7 single sticks), and they would add 37 to another number by counting-on to 37 from the other number either mentally, on a number line, on a hundreds board, on a Chisenbop finger sequence, or on a set of objects, or possibly they might add by a counting-all process using one of these methods. Once the concept of 10 as a unit has been learned, the counting process in all of these circumstances can be changed: counting-on can proceed with jumps of 10 (e.g., 48 + 37 = 48, 58, 68, 78, 79, 80, 81, 82, 83, 84, 85), vertical +10 movements can be made on a hundreds board as well as horizontal +1 movements, tens fingers can be added as well as ones fingers in Chisenbop, and groups of tens can be made and counted by ten. All of these procedures can then lead to the addition algorithm--the adding of like-sized units (ones, tens, hundreds, etc.).

This example indicates that the actual path of learning is actually a composite of all the three single kinds of paths proposed: the subject matter at least partially determines the real learning path (i.e., ten as unit notion is fundamental to the understanding of the addition algorithm); the new idea is not just learned, it is absorbed into the child's own developmentally determined procedure; and the new idea does in fact change the direction of development. In undertaking research on children's learning, it may be necessary to focus upon these three different derivations of learning paths and even to pursue one of them temporarily as though the others did not exist, but ultimately they must be put together. The first two derivations are common in American thinking. The zone of proximal development may help to focus us upon the equally important third type of derivation.

Vygotsky does not further analyze the zone of proximal development. Without additional analysis and specificity, the construct does little more than restate the hypothesized movement from the inter-psychological plane to the intra-psychological plane. This restatement does serve to focus upon the relationship between learning and development and it does enable one to concentrate directly upon that which is of primary importance in education--the effects of instruction upon children's learning and upon their consequent development. However, if this construct can be analyzed further, it may become quite useful. A small first step towards such an analysis is given below.*

Several attributes of the zone seem important. First, the zone is clearly conceptualized by Vygotsky as a distance, in particular as the distance between a starting-point that is the child's present level of achievement and an end-point which is where he or she is able to go with help.

Past achievement Present developmental level State of readiness	D(istance) S(tart) ——— E(nd)	Achievement level with support

Now D might be considered to be dependent upon at least the following:

D(istance) = individual learning Rate
x individual learning Power
x Time spent in cooperative learning
x Level of support necessary for learning to occur

$$D = R \times P \times T \times L$$

Vygotsky dealt explicitly with only one of the factors in this equation--power. His one example concerning the zone of proximal development was of two children who both had tested mental ages of eight years, but one of whom could solve tasks with the aid of the experimenter up to the level of a

*After this manuscript was completed, another such analysis came to the attention of this author. This analysis is in: Brown, A. L. and French, L. A. The zone of potential development: Implications for intelligence testing in the year 2000. Technical Report No. 128. Champaign, Illinois: Center for the Study of Reading, University of Illinois, May 1979. ERIC: ED 170 737.

twelve-year-old and the other who could only solve problems with help up to the level of a nine-year-old.

None of the factors in the distance equation are new ones, but work in this country concerning the factors has tended to come from separate areas of research and remain unconnected. Psychological and psychometric research has examined learning rate (e.g., number of trials to criterion) and power (e.g., total score), though these constructs are often confounded (as with timed tests). Our language contains both of these ideas: we talk about "fast" (Rate) kids and "bright" (Power) learners, but often do not distinguish very carefully between these groups. Educational research, on the other hand, has examined the relationships between time and distance. Traditional schooling has tended to hold time constant (i.e , giving children of varying rates and power the same amount of time on a given subject matter), with the result that distance varies. Approaches like individualized learning and mastery learning have instead held distance constant (i.e., defined certain learning tasks as required for all children), while varying time.

Having all of these constructs in one equation is useful, for this perhaps can facilitate the relating of these too often unrelated concepts. However, in order to utilize the relationships in the equation most effectively, adequate measures of the factors in the equation are needed. IQ is the usual measure of both individual power and of rate. Starting points and ending points are measured by achievement tests and by teacher-made tests, but the former of these especially are usually not at a very detailed level. Time is measured by class periods (or occasionally by time-on-task), and level of support is rarely measured at all. Thus we do not have at present very adequate measures of any of these factors. Adequate measures are necessary if the relations in the equation are to become any more specific.

The distance equation does serve to point out an additional aspect of the learning situation. The starting point, the rate, and the power are all attributes of individual children. These factors are not able to be controlled by the teacher or by the school. However, the end points (and thus the distance), the time, and the level of support are factors which are under the control of the school and the teacher. Explicit realization of the relations among these types of factors may serve to permit sensible adjustments to be made in the factors which are under the control of the teacher.

Little research has examined the level of support conceptualized as such. Even though Vygotsky did not explicitly discuss this factor with respect to the zone of proximal development, it is the most distinctly Soviet factor in the equation. This aspect also may be the largest contribution the idea of the zone of proximal development has to make. The basic notion of the zone of proximal development is that children giving the same response on a given item may have different amounts of remaining untested knowledge. Thus, if provided with a bit of additional knowledge, one such child may be able to continue to respond while another may not. The level of support can be considered either with respect to a given learning process or with respect to a given response measure. Techniques like recognition (rather than recall) tasks, cued recall, and other probed memory tasks are examples of response measures that provide a higher level of support. The level of support may not only provide additional information; it may also involve organizational or meta-level understanding of a task. That is, the adult may possess an understanding of the task as a whole which produces helpful directives that the child is unable to provide for herself or himself. Thus, the level of support might be thought of as the number of steps in a given sequence of goal-directed activity (see Figure 1 or 2) that an adult has to accomplish or as the number of increasingly specific hints a teacher might have to give before a given problem is accessible to solution.

Certain modes of teaching might be related to a continuum of such levels of support. Discovery learning might be characterized as that which is accomplished with the minimal level of support possible, while didactic teaching uses the maximal level of support--the child is explicitly told the whole process, definition, relationship, etc. Sometimes discovery learning and didactic teaching are posed as the only alternatives, while the whole range of decreasingly direct hints, observations, etc., that a teacher may make are ignored. The use of a considerable range of such supports would seem to be particularly important in research which tries to find out what a child can do or can understand. Thus, finding interactions between different levels of support and the distance a child can go in a set of tasks would seem to be quite impc ant as a research goal.

In summary, if we are ble to define aspects of the zone of proximal development more analytically, it may come to serve as a useful theoretical construct. Its main purpose in its presently fairly undefined state is nevertheless important--it suggests that we might profitably turn our research activities toward ascertaining what children can do, especially with adult or with peer help, rather than continue to focus only upon what children in fact do do. Furthermore, Vygotsky's fairly complex notion of the relationship between development and learning ought to help us to steer clear of two naive alternatives: an overemphasis only on 1) the level of a child's development (for example, as in the interpretation of Piaget's theory as dictating that one simply waits for the child to become concrete-operational) or 2) only on what we want children to learn without regard for what the child's developmental level says about such learning.

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QUANTITATIVE COMPARISONS AS A READINESS VARIABLE FOR ARITHMETICAL CONTENT INVOLVING RATIONAL COUNTING*

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Counting has not been considered explicitly in studies of early learning of mathematics to the extent it deserves. One reason counting has not been given a central position is that the mathematics curricula in the United States are, in the main, based on cardinal number for the early years. In Freudenthal's (1973) opinion,

> In the genesis of the number concept, the counting number plays the first and most pregnant role. This should be recognized rather than ignored by developmental psychology and pedagogics." (p. 191)

Freudenthal goes on to claim, "No doubt the stress in psychology on the numerosity aspect is due to Piaget" (p. 192). In the face of Freudenthal's claim, Piaget has claimed that number for the young child is both cardinal and ordinal.

There seems to be a contradiction between the claims of Freudenthal and Piaget. But, in actuality, there is little conflict. Piaget has never studied the development of counting in the same way that he studied the development of the objects one might call number in the child. But Freudenthal's criticism is based, in the main, on the counting number, "mathematically called the ordinal number" (Freudenthal, 1973, p. 171). Essentially, then, Freudenthal's criticism of Piaget is a reflection of the fact that Piaget may not have gone far enough in his studies of the development of the child's conception of number.

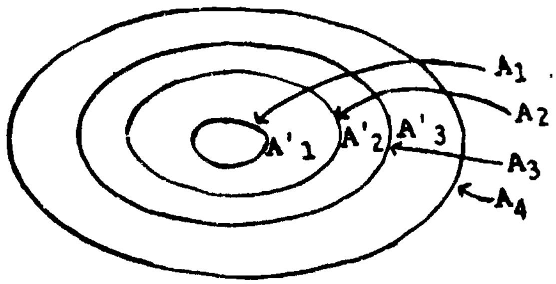
One should not claim that the emphasis on cardinal number in the mathematics curricula of the United States is due to Piaget. The emphasis is based on the theory of cardinal number in mathematics. Freudenthal's criticism of the pedagogs should be interpreted in light of the mathematics involved, even though he does allude to the use of Piagetian theory by pedagogs. To clarify the issues, an overview of number in Piagetian theory and aspects of ordinal number for counting are discussed.

*This paper is based on the following report: Steffe, L. P., Hirstein, J. J., and Spikes, W. C. Quantitative comparisons and class inclusions as readiness variables for learning first-grade arithmetical content. Technical Report No. 9. Tallahassee, Florida: Project for Mathematical Development of Children, 1976. ERIC: ED 144 808.

Number in Piagetian Theory

In his classic work, The Child's Conception of Number (1952), Piaget attempted to show that cardinal and ordinal number are developmental, arising in the child as a synthesis of Grouping I, Primary Addition of Classes, and Grouping V, Addition of Connected, Asymmetrical Relations. While the data presented in this book are "old," the basic theory of the Genevans concerning the development of number in the child has not changed substantially over the last three decades (Piaget, 1970; Beth and Piaget, 1966; Sinclair, 1971). Number, for Piaget (1952), "is at the same time a class and an asymmetrical relation" (p. 184). Two essential conditions for the "transformations" of classes into numbers exist (Piaget, 1952, pp. 183-184). Given a class, all of the elements must somehow be regarded as equivalent, but at the same time distinct. To illustrate these two conditions, imagine some hierarchical system $\emptyset \subset A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_1$ of classes where the following classes contain single elements.

1. A_1
2. $A_1' = A_2 - A_1$
3. $A_2' = A_3 - A_2$
4. $A_3' = A_4 - A_3$



For example, A_1 could be a bead, A_1' a cube, A_2' a bean, etc.

The first condition given is that all elements must be regarded as equivalent (all qualities of the individual elements are eliminated). But, if condition one holds, then, for example, A_2 would not be a class of two elements, but instead only one, for $A_1 \cup A_1' = A_1$--which is to say that the quality of the elements is eliminated. If the differences of A_1 and A_1' are taken into account, then they are no longer equivalent to one another except with respect to A_2. This brings the second essential condition into focus. In effect, the equivalent terms must remain somehow distinct, but that distinction no longer has recourse to qualitative differences. Given an object (the bead), then any other object is distinguished from that object by introducing order--by being placed next to, selected after, etc. "These two conditions are necessary and sufficient to give rise to number. Number is at the same time a class and an asymmetrical relation..." (Piaget, 1952, p. 184). Thus, according to Piaget (1952, p. 184), in qualitative logic, objects cannot be, at one and the same time, classified and seriated, since addition of classes is commutative whereas seriation is not commutative. However, if the qualities of the elements are abstracted, then the two groupings (I and V) no longer function independently, but necessarily merge into a single system.

In Piaget's system, then, number is not to be reduced to one or another of the groupings, but instead is a new construction--a synthesis of Groupings I and V. Elements are either considered in terms of their partial equivalences and are classified, or are considered in terms of their differences and are seriated. It is not possible to do both at once unless the qualities are abstracted (or eliminated); then it is necessary to do both simultaneously.

The only way, then, to distinguish A_1, A_1', A_2', A_3', ...is to seriate them: $A \rightarrow A \rightarrow A \rightarrow$, ..., where $\rightarrow$ denotes the successor relation and A represents A_i' where all the qualities of the element of A_i' have been eliminated. Clearly, Piaget considers each A to be a unit-element, at once equivalent to but distinct from all the others, where the equivalence arises through the elimination of qualities and the distinctiveness arises through the order of succession.

The notion of a unit is central in Piaget's system and is not deducible from the Grouping Structures, but rather is the result of the synthesis already alluded to. Once reversibility is achieved in seriation and classification, "groupings of operations become possible, and define the field of the child's qualitative logic" (Piaget, 1952, p. 155). Here operational seriation has as a necessary condition, reversibility, at the first level of reciprocity.

> A cardinal number is a class whose elements are conceived as 'units' that are equivalent, and yet distinct in that they can be seriated, and therefore ordered. Conversely, each ordinal number is a series whose terms, though following one another according to the relations of order that determine their respective positions, are also units that are equivalent and can therefore be grouped in a class. Finite numbers are therefore necessarily at the same time cardinal and ordinal... (Piaget, 1952, p. 157).

The development of classes and relations does not, as it may seem from the above quotations, precede the development of number in Piaget's theory: those developments are simultaneous. Without knowledge of the quantifiers "a," "none," "some," and "all," which implicitly involve cardinal number, the child is not capable of cognition of hierarchical classifications. A genetic circularity consequently exists in the developmental theory of classes, relations, and numbers.

Given that Piaget so unequivocally states that number for the young child is both cardinal and ordinal, is Freudenthal's criticism justified? Before attempting to answer, aspects of ordinal number theory are discussed.

Ordinal Number

Just as set equivalence is a basic notion for cardinal number, set similarity is a basic concept for ordinal number. For clarity, the order relations discussed below are asymmetric and transitive as well as being connected. Two ordered sets are called similar if there exists a one-to-one correspondence between their elements that preserves the order in the two sets. In symbols, "A is similar to B" is denoted by "$A \cong B$." Hausdorff (1962, p. 51) assigns order types to ordered sets in such a way that similar sets, and only similar sets, have the same order type assigned. In symbols, $r = s$ means $R \cong S$. If a set is well-ordered, then its order type is called an ordinal number. If A is a well ordered set, then A has a first element, say a_0; $A - \{a_0\}$ has a first element, say a_1; $A - \{a_0, a_1\}$ has a first element, say a_2; etc., so that $A = \{a_0, a_1, a_2, a_3, \ldots\}$. The notion used here is that the index of every element is the ordinal number of the set of elements preceeding it. For a_3, "3" is the ordinal number of $\{a_0, a_1, a_2\}$ which is called a segment of A determined by "a_3." In more general terms, each element a of A determines some segment S where $S = \{x \in A: x < a\}$. If $Q = \{x \in A: x \notin S\}$, then $A = S + Q$. Note that $a \notin S$ because $<$ is irreflexive, so a is the first element of Q.

As indicated above, the elements of a set A which is well ordered can be indexed by successive ordinal numbers. If A is a finite set, then $A = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$ and n is the ordinality of A where $\emptyset$ is the ordinality of the empty set. Because any ordering of a finite set is a well-ordering, it is impossible to distinguish the orderings with reference to the ordinal number of the set; i.e., all orderings give the same ordinal number. Thereby, the ordinal and cardinal numbers of finite sets correspond, and it is possible to find the cardinal number of a set by a process of counting, that is, by indexing the elements of the set A by the ordinal numbers $\{0,1,2,\ldots, n-1\}$ by virtue of successive selection of single elements. (Select some a_0, then some a_1, etc., until the last one a_{n-1} is selected.) Then n is called the cardinal number of the set. This process is often referred to as counting.

Concretely, if A is a finite set to be counted, then by successive selection of elements, successive segments of set A are determined. "One" in the selection of the first element has both cardinal and ordinal characteristics in that "one" tells how many elements have been selected and also that the first one has been selected. A subset of the collection A of one element has also been determined. "Two" in the selection of the next element also has both cardinal and ordinal characteristics in that "two" tells how many elements have been selected and also that the second one has been selected. The segment corresponding to "two" is an ordered set, is a subset of the collection A, and contains the set consisting of the first element. It is ordered by the relation "precedes," which is transitive and asymmetrical. If this counting process is continued until A is exhausted, then $A = \{a_1, a_2, \ldots, a_n\}$ has been well-ordered by the relation "precedes." A chain of sets has been established in that if $A_1 = \{a_1\}$, $A_2 = \{a_1, a_2\}$, etc., then $A_1 \subset A_2 \subset \ldots \subset A_n$. In this sense, one can say that one is included in two, two is included in three, etc. If A is counted in a different way, $A = \{a_1^*, a_2^*, a_3^*, \ldots, a_n^*\}$. It must be noted that while a_i^* may not be the same element as a_i, nevertheless a_i^* is the ith element and also i is the cardinal number of $A_i^* = \{a_1^*, a_2^*, \ldots, a_i^*\}$ where $i < n$. While A_i and A_i^* are similar (and therefore equivalent), they are not necessarily equal ordered sets.

Addition and subtraction of ordinal numbers. If A and B are disjoint ordered sets, then the set theoretic sum of A and B/(A + B) is a new ordered set such that the order of the elements of A is retained, the order of the elements of B is retained, and every $a \in A$ precedes every $b \in B$. If a is the order type of A, b the order type of B, then a + b is the order type of A + B. An example of ordinal number addition follows. If a = 5 and b = 3, then 5 + 3 is the ordinal number of the set $\{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}$. To rename 5 + 3, the child could count "one," "two," "three," "four," "five," "six," "seven," "eight," or could count "six," "seven," "eight," which represents a counting-on of B to A. In both cases, 5 + 3 is renamed as 8.

Subtraction of ordinal numbers is possible in special cases. If α and β are ordinal numbers and $\alpha < \beta$, α and β determine a unique ordinal number ξ satisfying the equation $\alpha + \xi = \beta$ (Hausdorff, 1962, p. 74). ξ is of type $W(\beta) - W(\alpha)$ where $W(\beta) = \{\text{ordinal number} < \beta\}$. Clearly, if $\alpha < \beta$, $W(\alpha) \subset W(\beta)$. An example is if α is 7 and β is 9 then $W(\alpha) = \{0,1,2,3,4,5,6\}$; $W(\beta) = \{0,1,2,3,4,5,6,7.8\}$ and $W(\beta) - W(\alpha) = \{7,8\}$. ξ is a remainder in the f owing sense. If a is an element of a well-ordered set P, $S = \{x \in A: x < a\}$ and $Q = \{y \in A: y \geq a\}$, then P = S + Q and S is the segment and Q is the remainder determined by a. Essentially, then, ξ is the ordinal number associated with the remainder of $W(\beta)$ determined by α. The solution ξ of $\alpha + \xi = \beta$ is denoted by $\beta - \alpha$ for finite α and β.

In the case of the equation $n + \alpha = \beta$ where $\alpha < \beta$, the solution is also represented by $\beta - \alpha$ for finite α and β.

However, the solution is arrived at by the following process: $n + (\alpha - 1)$ is the predecessor of β; $n + (\alpha - 2)$ is the predecessor of $n + (\alpha - 1)$; and so forth, until n is reached. Concretely, if $x + 5 = 11$ is the equation, one counts back from 11 to reach 6 (the solution) in the following way: "ten," "nine," "eight," "seven," "six"; so since six is the predecessor of $x + 1$, x must be six.

In the case of the equation $5 + x = 11$, the solution is found by counting the remainder, starting with the first element of the remainder and proceeding to the last. It should be clear that one could also start with the last element of the remainder and count backward to the first. In either case, a double counting process is necessary: ten is one; nine is two; eight is three; seven is four; six is five; so the answer is six. Or, six is one; seven is two; eight is three; nine is four; ten is five; eleven is six; so the answer is six. In the case of counting-back, rather than counting predecessors of elements in the reminder one can count the eleme themselves: eleven is one; ten is two; nine is three, ighc is four; seven is five; so the answer is six.

Comments on Piagetian Theory of Number

From the discussion in the preceding two sections, it can be seen readily that Piagetian theory of number does not include a theory of counting. Counting, however, is an integral part of the theory of ordinal number (and thus cardinal number in the case of finite sets). But neither Grouping I nor Grouping V includes a theory of counting (or of arithmetical operations). However, in Piaget's analysis, a synthesis of Grouping I and V gives rise to number. So, having the Groupings not include counting or operations would not be a shortcoming if Piaget provided a detailed developmental structural analysis of number in a way analogous to that provided for Groupings. But the fact is that no such theory or data exists in Piagetian theory concerning the cognitive development of number beyond the objects called number. Piaget's theory and research concerning number stop with the objects he calls number. He did not go on to investigate, developmentally, counting or operations, although "additive" and "multiplicative" composition of number are discussed. Freudenthal's criticism that Piaget studied only the "numerosity" number is not fully justified in the context of Piaget's studies, as Piaget studied set similarity and relationships between cardinal and ordinal number. But Freudentnal's observation that Piaget did not study coun-.ng is certainly valid. Further, as noted, Piaget did not study addition and subtraction of ordinal numbers per se.

Piaget's Experiments on Cardinal and Ordinal Number

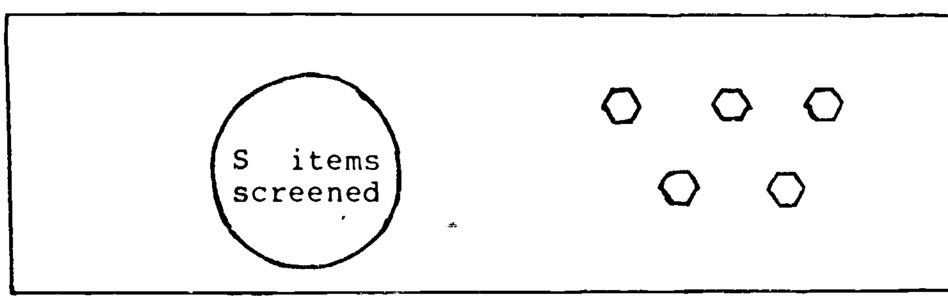
Piaget (1952) did use counting to study development of cardinal and ordinal number. In his study, two problems were of concern. First, a child had to determine a cardinal number given an ordinal number, and second, the child had to determine an ordinal number given a cardinal number. Three experimental situations were employed, one involving seriation of sticks, one seriation of cards, and one seriation of hurdles and mats. In the seriation of sticks experiment, the child was asked to seriate ten sticks from shortest to longest and then was given nine more sticks and was asked to insert these into the series already formed (the material was constructed in such a way that no two sticks were of the same length). He or she was then asked to count the sticks of the series, after which sticks not counted (or sticks the child had trouble counting) were removed, apparently along with one or two he or she did not have trouble counting. The experimenter then pointed to some stick remaining and asked how many steps a doll would climb when it reaches that point, how many steps would be behind the doll, and how many the doll would have to climb in order to reach the top of the stairs formed by the sticks. The series was then disarranged and the same questions as before were put to the child, who would have to reconstruct the series in order to answer the questions.

There is no question that aspects of ordinal number and cardinal number were involved in the above experiment. Any conclusion drawn with regard to number, however, by necessity is a function of a capability to construct a series of sticks based on the connected asymmetrical relation "longer than," having little to do with ordinal number. To demonstrate the point more concretely, an eight-year-old child was asked which of a collection of books on a table would be the third one. He answered, "What do you mean, any one could be third." Piaget's experiment with the staircase, then, was more an experiment concerning similarity between a set of n sticks ordered by "shorter than" and the standard counting set {1,2,...,n} than it was an experiment concerning ordination and cardination. A similar analysis holds for the seriation of the cards experiment. While no analysis of the hurdles and mats experiment is given, suffice it to say that it too involves specific relations. In the mathematical development, it is the relation "precedes" which is important, not "shorter than" for sticks, etc. While particular order relations determine order of precedence, precedence is only incidental and not primary in the ordering.

In the three experiments discussed in this section, counting is only incidental. No analysis of counting is provided nor is it at all clear how counting fits into the developmental theory of cardinal and ordinal number. It would seem that counting would be based on ordinal number. But to use it to study developmental relations of cardinal and ordinal number introduces circularity of counting and cardinal and ordinal number in development--a circularity which may not be warranted. Counting typologies exist which may be important for the study of the development of number in the child. These typologies offer a framework for the study of the role of counting in development.

Counting Typologies

Study of the development of children's counting is sorely needed, along with elucidation of its relationship to addition and subtraction. Three types of counting are easily identifiable--rote counting, point counting, and rational counting. The basis in mathematics for rote counting is the set of ordinal numbers $\{1,2,...,n\}$. Behaviorally, rote counting is the recitation of the symbol chain "one," "two," "three," ... The basis in mathematics for point counting is the similarity between a collection of n elements and the set of ordinal numbers $\{1, 2, 3, ..., n\}$ represented by indexing elements: $A = \{a_1, a_2, ..., a_n\}$. Behaviorally, sucessive elements of A are selected until they are exhausted. The basis in mathematics for rational counting is counting-on and counting-back. But it must be understood that, behaviorally, counting-on and counting-back must be associated with mental representations of collections. Behavioral aspects of rational counting are of four identifiable types. The first is rational count-on without a tally. Here, a child must be able to find the number of elements in a given collection P when s elements of P are screened from view and q elements are visible. The task is diagrammed in Figure 1.



P items

Figure 1. Rational Counting-on Without Tally

The child is told the number of elements in S (say seven) and asked to find the number of elements in all (in P). The elements of S are not subject to a point count. The child <u>may</u> start at one and count to seven, but the more efficient behavior is to <u>start</u> at seven and count "eight," "nine," "ten," "eleven," "twelve." There are twelve in all. In this procedure, the child was not required to tally the five visible items by the task demands. Rational counting-on with a tally is demonstrated by Figure 1 if the child is told the number of items in all (12) and asked to count-on to find the number of items under the cover (the number in S).

Rational counting-back without a tally is demonstrated by Figure 1 if the child is told the number of items in all (12) and is asked to count back to find the number of items under the cover. Rational counting-back with a tally is demonstrated by Figure 1 if the visible elements are covered from view and the child is told the number of items in all (12) and the number of items under one of the covers (say five), and is asked to count back to find the number of items under the other cover.

Counting Typologies and Ordinal Number Addition and Subtraction

In a previous example of ordinal number addition (5 + 3), the elements of $\{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$ should be thought of as being similar to the ordinal numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$ where b_1 corresponds to 6, b_2 to 7, and b_3 to 8. Rational counting without a tally is close to the similarity, but the association of visible objects with numbers does not require the mental association of b with 6, etc. In that the task for rational counting-on without tally does not demand the association to be mental, the task for rational counting-on with a tally is the better task to test the child's capability to form a correspondence of the remainder (b_1, b_2, b_3 in the example) of a set with the ordinal numbers corresponding to that remainder.

The following task would seem to be an even more precise test of ordinal number addition. The child is faced with both the segment and remainder in a covered state. He or she is given the number in each and told to find the number in all. This task would represent a distinct improvement over those of the previous paragraph providing that the child did not use fingers to represent one or both collections, but kept a running tally when he or she counted the remainder. In view of this task, a task for rational counting-on with a tally may be too conservative a task for ordinal number addition. But it is better than rational counting without a tally. As there is no way to insure that child will not use fingers as a tally, it is also better than the task just described. Ordinal number subtraction is represented nicely by the task requirements for counting-back with a tally.

Task demands represented by counting-on with a tally are exemplification of conceptual requirements associated with the solution of the equation $5 + x = 11$. Task demands represented by counting-back with a tally are exemplification of conceptual requirements associated with the equation $x + 5 = 11$, each discussed earlier. Counting-on with a tally, then, makes demands different from strictly ordinal number addition. In the former, the child knows the point to which he or she has to count, and constructs the remainder. In ordinal number addition, he or she constructs the containing set.

Research Hypotheses

Piaget's notion of number is quite close to the concept of a well-ordered finite set. But because counting is not fully described by a well-ordered set, the following hypothesis is expected to hold:

> Research Hypothesis 1. Children who are operational with number (in a Piagetian sense) are not necessarily able to rational count-on or rational count-back with or without a tally. But children who are not operational with number (in a Piagetian sense) are not expected to be able to rational count-on or rational count-back with or without tally.

There is little rationale in theory for a relationship to exist between rote counting and number as described by Piaget. Children can learn to rote count through exposure to events on a day-to-day basis--from other children, television, adults, etc.--in a way analogous to language learning. B   learning to recite the number names in proper sequence bears little conceptual relationship wi h being operational regarding number. The second research hypothesis is then:

> Research Hypothesis 2. Numerical skills predicated on rote counting are independent of a child's being operational with number (in a Piagetian sense).

Children's acquisition of rational counting-on and -back would seem to be highly related to whether the children were operational with number (in a Piagetian sense). Children who are operational ut cannot rational count-on or -back would, theoretically, possess the objects called ordinal number, but would not have integrated counting into that conception. Such integration would seem to be accelerated easily by experiences with counting. But children who are not operational (in a Piagetian sense) with number would not yet have developed the objects called ordinal number (modeled by a well-ordered finite set) and would thereby be greatly limited in acquisition of rational counting-on or -back. The third research hypothesis is then:

> Research Hypothesis 3. Acquisition of the ability to rational count-on or -back is highly related to a child's being operational with number (in a Piagetian sense).

A child's ability to obtain cardinal information from ordinal information, and vice versa, has been studied by Piaget (as reviewed earlier). Piaget's studies were criticized on the basis that relations having little to do with number were involved. With tasks designed to reduce the severity of that criticism, it should be the case that the ability to obtain cardinal information from ordinal information (and vice versa) is not a necessary part of a child's being operational with number (in a Piagetian sense). This statement is based on the role of counting in such an ability. The fourth research hypothesis is then:

> Research Hypothesis 4. Children who are operational with number (in a Piagetian sense) are not necessarily able to obtain cardinal information from ordinal information. But children who are not operational with number are not expected to be able to obtain cardinal information from ordinal information.

Because task demands represented by counting-on with a tally are exemplification of conceptual requirements associated with solution of the equation $a + x = b$ where a and b are ordinal numbers and $a < b$, children who are operational with number (in a Piagetian sense) would not necessarily be able to solve problems which are modeled by the equation $a + x = b$, but should be able to acquire such facility. Children who are not operational with number in a Piagetian sense should experience great difficulty in acquiring the ability to solve such problems. The fifth research hypothesis is then:

> Research Hypothesis 5. Acquisition of the ability to solve problems modeled by $a + x = b$ ($a < b$ and orginal numbers) is highly related to a child's being operational with number (in a Piagetian sense). Moreover, children who are operational with number are not necessarily able to solve problems modeled by the equation form $a + x = b$. Children who are not operational with number are not expected to be able to solve problems modeled by the equation form $a + x = b$.

Design of the Study

Sample

The first-grade children in Oglethorpe Avenue Elementary School and Whitehead Road Elementary School, Athens, Georgia, were used as an initial pool of children. All of these children were administered the SMSG Scale 204, Counting Members of a Given Set, and SMSG Scale 205, Equivalent Sets in September 1974. Only children for whom evidence was present that they could point-count to at least seven were considered as a population.

A test of quantitative comparisons (see Appendix 1) was then administered to all of the children in the population. Children were judged to be either extensive quantitative comparers or gross quantitative comparers. If such a judgment could not be made, that child was not considered for the sample. The relationship between extensive quantity and gross quantity and number in Piagetian theory has been explicated elsewhere (Steffe, 1966). An assumption made in this study was that children who were classified as extensive quantitative comparers were operational with number in a Piagetian sense. Children who were classified as gross quantitative comparers were not considered to be operational with number in a Piagetian sense.

Evidence was considered strong for a child to be considered as an extensive quantitative comparer if a child responded correctly in at least five of the eight items on the test of quantity with justification. Evidence was considered strong for a child to be classified as a gross quantitative comparer if a child responded on the basis of perceptual cues and a majority of answers were not correct.

The children were randomly ordered with each group of extensive and gross quantitative comparers within each school. The first 12 children in each quantitative comparison group within each school were considered as the sample--24 extensive quantitative comparers and 24 gross quantitative comparers.

A test of class inclusion was also administered to the population, but 88 of the children scored zero and seven scored one. These 95 children contained the 48 children of the sample. Only nine children showed any evidence (at least two of six items correct) of solving the class inclusion problem and were discarded from the study. This additional characteristic of the sample is mentioned only for informational purposes and is not considered further in the discussion.

Treatment

In order to test Hypotheses 3 and 5, a treatment was included in the study incorporating counting strategies. The treatment was administered by Leslie P. Steffe and W. Curtis Spikes. It began October 1, 1974 and ended January 17, 1975. The children in the experimental group were met four days a week for 50 minutes. The remaining day was spent in their regular classroom.

The instruction in the experimental group was highly individualized for each child, in that very few sessions were held where group interaction or group demonstration was used. Because the instruction was individualized the children were pooled for data analysis.

The first instructional week was spent on classification where the terminology "and," "or," "not," "some," and "all" was introduced. The content of the classifications were dog, squirr , and bird cutouts and balloons, toy soldiers, toy horses, and toy cowboys. The second instructional week was spent on partitioning collections of objects. Three basic activities were designed. The first was designed using two subcollections with counting; the second, three subcollections with counting; and the third, more than three without counting. The third instructional week was spent on loop inclusions and intersections.

The first three instructional weeks were spent on classification activities for two reasons. First, it was felt that such activities may enhance children's acquisition of rational counting-on. Second, an attempt was made in the study to improve classificational activities of children. This attempt is not discussed here.

The remaining instructional time was spent on addition and subtraction and counting activities. The instruction was sequenced according to learning-instructional phases for addition and subtraction. As the instruction was highly individualized, it is difficult to describe any one uniform instructional sequence. However, the learning-instructional phases for addition and subtraction are presented, after which activi ies are elaborated.

In the exploratory phase for the children with rote-counting abilities, addition and subtraction problems were not attempted until they acquired point-counting abilities. This means that children who were rote-counters were given many concrete examples of point-counting to bring their level of counting up to the level of point-counting. This was done in the context of counting all strategies for addition and subtraction exercises at the exploratory phase. The children at this phase were given the problem of determining how many elements there were in two sets, S and Q, when all the elements of both were put together. The elements of S were counted out; the elements of Q were counted out and placed with the elements of S. The children then counted out all of the elements of $S \cup Q = P$. The students continued these types of activities with objects and with their fingers, and worked spontaneously from both verbal and written instructions for basic addition facts. This means that being told: "Solve this problem: How much is six and four?" and being given the symbolized statement--"6 + 4 = ____," elicited the same problem-solving behavior. In the case of using their fingers, the students counted out six fingers, counted out four fingers, and then counted each finger and determined that the answer was "ten." Concrete objects were abandoned by all of the children after about two weeks of instruction on addition and subtraction. Finger dexterity increased if the sums were ten or less.

All of the children in the treatment groups were introduced to the exploratory phase of addition and subtraction. The reason for this was that an attempt was made to let the children differentiate themselves through instruction to the abstraction-representation phase for addition and subtraction. It was expected that the children who were extensive quantitative comparers would enter the abstraction-representation phase more quickly than would the gross quantitative comparers. The abstraction and representation phase is described below.

In the abstraction-representation learning phase for addition, the children used a counting-on strategy to solve the problem $s + q = \square$. Rational counting-on without a tally was most often used, since the children considered either one of the numbers as a starting point and the other number to represent a set of units to be counted. For example, to solve $9 + 3 = \square$, the children selected nine as a starting point and counted-on three units more in the chain: "ten," "eleven," "twelve." There was no need to count to the number nine from one since the children extracted, mentally, the cardinal property of "nineness." The children did not need to count each unit in the problem but did need to point-count a tally of three units. But the tally was constructed before counting.

Missing-addend problems were solved using rational counting-on with a tally. When given the missing addend problem, the missing addend was perceived as part of the total. For example, given the problem $3 + \square = 11$, the children solved it by counting-on from three to eleven and symbolizing the units of the missing addend with a running tally. In finalizing the solution, the children point-counted the tally either simultaneously while counting-on or after.

Subtraction problems were solved using counting-back without a tally. The children solved a problem like $9 - 5 = \square$ by starting at nine to count the units in the backward-ordinal sequence. They counted back five units to the number five, mentally extracted the next number in the backward-ordinal sequence, and named it as the solution to the problem. In this problem situation, the child is asked to solve the problem by counting back.

Instructions on counting-on and counting-back activities were given to each child. The counting-on activities were as follows. A card with three rings on it  was used. Objects were counted out while being placed into one of the rings. These objects were screened from view. Objects were counted out while being placed into the other ring. The children were then asked to find how many were in the big ring. Counting-all strategies could be used to solve the problem as well as counting-on. The goal of such activities was to have the children abstract, through counting activities, that the objects covered did not have to be recounted, but one could start with the number of objects covered and count-on, as described above in the abstraction-representation phase.

The missing-addend problem was first presented using a variation of counting-on without a tally, transforming it to counting-on with a tally. Instead of counting each collection and covering one, the children were <u>told</u> there were a certain number under one cover, a certain number under another, so how many all together? Counting-on with a tally then was modeled by the teachers and by able children for those not able to display it.

Because some of the children had a great deal of difficulty with counting-on, the solution to the missing-addend problem (5 + □ = 7) was modeled using counting-all as a base. In the case of the example, seven objects were counted out, five of the seven were counted-out, and then the remaining two were counted to go into the box. A child with counting-all strategies could execute the solution presented in that way. Efforts were then made to take the able children to solution by counting-on with a tally.

Counting-back activities were also presented, first point counting-back and then rational counting-back without a tally. The counting-back activities were incorporated into subtraction exercises such as 5 - 3 = □ . Structured materials were used due to the great difficulty the child experienced in rational counting-back. The children were given a counting-back board as follows. They were shown that to process 5 - 3 on the board, they would start at five and count off three, to find the answer "two." An attempt was made to emphasize that when "6," for instance, appeared under a particular tile, it told how many tiles were up to and including that tile.

□	□	□	□	□	□	□	□	□	□
1	2	3	4	5	6	7	8	9	10
□	□	□	□	□	□	□	□	□	□
11	12	13	14	15	16	17	18	19	20

All of the children were presented with counting-on and counting-back strategies associated with the three equations $a + b = \square$; $a + \square = b$; and $b - a = \square$. The third learning-instructional phase was also dealt with in <u>instruction</u>. This learning-instructional phase is called the <u>formalization-interpretation</u> phase.

The formalization-interpretation learning phase for addition and subtraction is characterized by the interrelationships of addition and subtraction. The child in this final learning-instructional phase for addition and subtraction can relate problems of the type $9 - 5 = \square$ and $9 = \square + 5$. In relating the two equations, the student must realize that both involve four and five as parts of nine. To move from the former to the latter equation, it was hypothesized that a counting-back with tally would be employed. The student counts-back five units from <u>nine</u> with a mental tally. He or she preserves this tally, <u>five</u>, along with the solution, <u>four</u>, as separate parts of <u>nine</u>.

So the child realizes (by reconstructing the 5 units counted back) that 5 units counted back on to 4 units results in the original 9 units. In this way, addition and subtraction are interrelated. So when a child finds the sum of 4 and 5, he or she also knows the difference of 9 and 5.

The opportunity was given each child in the treatment to enter this learning-instructional phase through written work. Families of equations were presented to the children for solution, such as $4 + 5 = \square$; $4 + \square = 9$; $\square + 5 = 9$; $9 - 4 = \square$; and $9 - 5 = \square$. The children were never told the interrelationships but were left to make the observations. The written work for each child was retained as children differed greatly in the amount of written work they could do.

Addition, subtraction, and missing addend problems were given to the children to solve during instruction on addition and subtraction. The problems were presented in written format. Children who could read the problems were encouraged to work independently. They were encouraged also to write a mathematical sentence for each problem they solved. The problems were read to the children who could not read. These children were also encouraged to write mathematical sentences for the problems they solved.

The children were allowed to use a hand-held calculator during the last four weeks of instruction. The role of the calculator was to check sums or differences.

Interviews

The interviews of interest in this report were part of a larger set of interviews, but only those interviews of interest are discussed. Two missing-addend problems with objects available during solution (see Appendix II) were presented individually to the children in the sample during the first two weeks in October 1976, prior to the administration of the treatment. All of these individual interviews were hand-recorded.

During February 1975, each child was interviewed in three different sittings of not more than 30 minutes per sitting. All missing-addend problems (see Appendix II), the cardinality and ordinality tasks (see Appendix III), the counting-on and counting-back tasks (see Appendix IV), and the just-before and just-after tasks (see Appendix V) were individually administered and audio-taped as well as hand-recorded. The interviews followed the formats given in the appendices. While somewhat structured, the formats were altered whenever necessary to insure that communication was established between the child and interviewer. The just-before and just-after tasks were designed to entail at most rote-counting, and therefore were used in testing Research Hypothesis 3.

Data Sources and Variables

Each videotape was viewed and all data extracted and coded on record sheets. The variables Number in S, Number in P, and Number in S + Number in P were defined using the tasks in Appendix III: Cardinal Information from Ordinal Information. The Number in S variable was scored from response (correct or incorrect) to Question 3 in Task B. The range of scores was $\{0, 1, 2\}$. The Number in P variable was scored from either response (correct or incorrect) to Question 3b in Task A or response (correct or incorrect) to Question 3c in Task A; and response (correct or incorrect) to Question 2 in Task B. The range of scores was $\{0, 1, 2\}$. The Number in S + Number in P variable was not just the sum of the two variables in Number in S and Number in P. The sum variable included responses of children given cues. The sum variables was scored from responses to Questions 3, 3b, 3d, 3f, 3g, 3h of Task A and Questions 2, 2b, 2d, 2e, 3, 3c, 3d of Task B. It should be clear that a given child would not answer all of those questions. The range of the sum variable was $\{0, 1, 2, 3, 4\}$ but, again, was not simply the sum of Number in S and Number in P.

The missing-addend problems were scored on a right-wrong basis. Two scores were obtained, one for each of the two problems in Appendix II. The counting-on and counting-back items (Appendix IV) were also scored on a right-wrong basis. Four scores were obtained: (1) counting-on without a tally, (2) counting-back without a tally, (3) ordinal addition, and (4) ordinal subtraction. On the tasks designed to test just-before and just-after, one point was given if the child could find either the number before (after) 14 or before (after) 11. Zero was awarded otherwise.

The variables were, in summary, Number in S, Number in P, Number in S + Number in P, Missing Addend with Objects, Missing Addend without Objects, Rational Counting-on without a Tally, Rational Counting-back without a Tally, Ordinal Addition, Ordinal Subtraction, Just Before, and Just After.

Research Design and Statistics

The first six children of each of the two quantitative comparison groups (one per school) were assigned to the experimental group and the second six to the control group, as in Figure 1. The children in the Control Group participated in their regular mathematics program, Elementary School Mathematics for Kindergarten through Grade 6 (Eicholz and Martin, 1971). The children in the experimental group participated in mathematics classes conducted by Leslie P. Steffe and W. Curtis Spikes. The 12 experimental children in Oglethorpe School were taught from 10:00 AM to 11:00 AM Monday, Tuesday, Thursday, and Friday; the 12 experimental children at Whitehead Road School were taught from 12:00 PM

School	Oglethorpe		Whitehead	
Quanity \ Treatment	Experimental	Control	Experimental	Control
Extensive	6	6	6	6
Gross	6	6	6	6

Figure 1. Diagram of the Subject Classification

to 1:00 PM on Monday, Tuesday, Wednesday, and Friday. A diagram of the design is given in Figure 2.

September, 1974	September, 1974	October 1, 1974-January 17, 1975	February, 1975
Sample selected	Missing Addend Problems; with Objects administered prior to the treatment	Treatment administered to experimentals Participation in classroom by controls	Post-experimental interviews

Figure 2. Diagram of the Events in the Experiment in Time Sequence

An item analysis was conducted for each test whenever appropriate. Program ANLITH, an item-analysis computer program made available by the Educational Research Laboratory of the University of Georgia, was used to conduct the item analysis. The program was initiated for use at the Educational Research Laboratory by Yi-Ming Hsu and was developed by Thomas Groneck and Thomas A. Tyler.

Item difficulty (p-values) are reported for each item. A p-value is a ratio of the number of correct responses to the total number of responses for an item. Test means, standard deviations, and Cronbach's Alpha reliability coefficient are reported for each test, as well as total score distributions.

Quantity was used as a classification variable (Extensive vs. Gross) and Treatment as an independent variable in all analyses of variance. A univariate analysis of variance is reported for each dependent variable isolated.

Results

Item Analyses

Quantitative comparisons. The test of quantitative comparisons (Appendix I) was administered to 107 children as a pretest. Table 1 contains the difficulty indices for each item, and item characteristics. Items 1, 2, 3, and 6 were of comparable difficulty. These items either had a configuration conducive to solution by visual inspection (triangular or rectangular), had two collections of six objects with a random arrangement (Item 3), or contained a collection which apparently had more than the other (Item 6). These items could be solved by gross quantitative comparisons. The remaining items all demanded an extensive quantitative comparison for correct solution due to difficult geometrical configurations or eight objects in each collection to be compared. They were critical items to separate the extensive quantitative comparers from the gross quantitative comparers.

Table 1

Difficulty Indices and Item Characteristics for Quantitative Comparisons Pretest

Item	Difficulty	Item Characteristic
1	.70	Triangular arrangement; 6 red, 6 green
2	.74	Rectangular arrangement; 6 red, 8 green
3	.73	Random arrangement; 6 red, 6 green
4	.57	Linear arrangement; 6 red, 6 green
5	.49	Linear arrangement; 8 red, 8 green
6	.72	Random arrangement; 8 green, 6 red
7	.59	Circular arrangement; 8 red, 8 green
8	.54	Random arrangement; 8 red, 8 green

The test mean was 5.01, standard deviation 2.58, and internal consistency reliability .84. The reliability of .84 supports the classification into extensive and gross categories. Further justification of the validity of the two quantitative categories is that, if a child scored at least 5 out of 8 correctly with justification for answers, evidence was strong the child would have made an extensive quantitative comparison. (Evidence was strong because at least one of Items 4, 5, 7, or 8 would by necessity have to be answered correctly with justification.)

The distribution of total scores for the eight-item test was as follows: eleven children scored zero, five scored one, five scored two, seven scored three, eight scored four, ten scored five, twenty-one scored six, twenty-one scored seven, and nineteen scored eight. The rather large frequencies for the scores five, six, seven, and eight can be attributed to Items 1, 2, 3, and 6. In retrospect, those items did not necessarily measure extensive quantity.

Number in S and Number in P. Table 2 contains the difficulty indices for the tests of the Number in S and Number in P variables (Appendix III). The first item on Number in S test was more difficult than the second. The first is probably more indicative of the difficulty of the Number in S items due to the fact that the second item was from the second ordinality task and the child had processed a considerable amount of information about the task before asked to find the number in S.

Table 2

Difficulty Indices for Number in S and Number in P Tests

Test Item	Number in S	Number in P
1	.31	.46
2	.54	.44

The frequency distributions, means, standard deviations, and reliabilities for Number in S and Number in P tests are given in Table 3. None of the distributions appear to represent normally distributed variables. The reliabilities are extremely low and are a reflection of the rather large number of children scoring one out of two items correctly.

The items were not homogenous. This heterogeneity may be a result of the items being on different tasks and in different sequences in each task.

Table 3

Frequency Distributions, Means, Standard Deviations and Reliabilities of the Number in S and Number in P Tests

	Frequency Distribution					
	Total Score					
Test	0	1	2	Mean (Percent)	Standard Deviation	Reliability
S	16	23	9	.85 (42)	.71	.15
P	17	19	12	.90 (45)	.77	.33

While the low reliabilities may be attributed to the fact that the tests contained only two items, the tests were administered individually by competent testers. Such individual administration should minimize errors of measurement. This argument strengthens the necessity for better task design for tests of Number in S and P variables.

In the event differences for main effects are detected in the analyses of variance for Number in S or P variables, they can be interpreted. The reason such interpretation is possible is that, given significant differences (say, for quantity), a preponderance of the children scoring zero would have to be in one category and a preponderance of the children scoring 1 or 2 would have to be in another category. For children scoring either zero or two, it is reasonable to conclude that they did not or did have the ability to obtain cardinal information from ordinal information, respectively. For children scoring one, however, difficulties of interpretation are present.

In the event differences are not detected in the analyses of variance for Number in S or P variables, no interpretation should be made.

Missing Addend problems. Table 4 contains the difficulty indices for the missing-addend problem-solving test with and without objects (Appendix II). The missing-addend problems are more difficult on the pretest than on the posttest, as expected.

Table 4

Difficulty Indices for Missing Addend Problem Solving Test

Item	Difficulty	Item Type
1	.19	With Objects: Pretest
2	.15	With Objects: Pretest
3	.58	With Objects: Posttest
4	.50	With Objects: Posttest
5	.54	Without Objects
6	.42	Without Objects

Table 5 contains the frequency distributions, means, standard deviations, and reliability information. None of the distributions appear to represent normally distributed variables. The internal consistency reliabilities are quite substantial, especially for the posttests. Inspection of the frequency distributions for the missing-addend problems show almost an all-or-nothing phenomenon.

Table 5

Frequency Distributions, Means, Standard Deviations, and Reliabilities of the Missing Addend Tests

	Frequency Distributions					
	Total Score					
Test	0	1	2	Mean (Percent)	Standard Deviation	Reliability
With Objects Pretest	36	8	4	.33 (16)	.62	.58
With Objects Posttest	17	10	21	1.08 (54)	.89	.74
Without Objects	22	6	20	.96 (48)	.93	.87

Counting-on and counting-back tests. Table 6 contains the difficulty indices for the counting-on and counting-back tests (Appendix IV). Counting-on items without a tally were each fairly easy items. The counting-on with a tally or ordinal number items were also surprisingly easy. However, the counting-back without a tally items were difficult, as were the counting-back with a tally items (ordinal number subtraction). Item difficulty is somewhat a function of the particular numbers involved.

Table 6

Difficulty Indices for Counting-on and Counting-back Tests

Type	Item Number	Difficulty
Counting-on without a tally	1	.77
	2	.73
Counting-on with a tally	1	.71
	2	.56
Counting-back without a tally	1	.54
	2	.56
Counting-back with a tally	1	.31
	2	.19

Table 7 contains the frequency distributions, means, deviations and reliabilities for the total tests. The reliabilities associated with two tests, counting-on without a tally and counting-back with a tally, are rather low. The former is easy and the latter difficult, each of which contributes to low reliabilities. The analysis of variance for these two tests can be definitely interpreted, but with some caution if no differences are detected in the analyses.

Table 7

Frequency Distributions, Means, Standard Deviations, and Reliabilities for Counting-on and Counting-back Tests

Test	Frequency Distribution Total Score 0	1	2	Mean (percent)	Deviation	Reliability
Counting-on without a tally	6	12	30	1.50 (75)	.71	.50
Counting-on with a tally	14	7	27	1.27 (64)	.88	.84
Counting-back without a tally	20	15	13	.85 (42)	.82	.61
Counting-back with a tally	20	20	8	.75 (38)	.72	.47

Analyses of Variance

The analyses of variance for all variables are summarized in Table 8. Missing-addend problems with objects, administered as a pretest, and the four rational counting tests constitute possible tests to be used in a test of research hypothesis 1:

> Children who are operational with number (in a Piagetian sense) are not necessarily able to rational count-on or rational count-back with or without a tally. But children who are not operational with number (in a Piagetian sense) are not expected to be able to rational count-on or rational count-back with or without a tally.

The missing-addend problems administered as a pretest are included because they were administered close in time to the test of quantitative comparisons. They only constitute a test of the hypothesis in the case of counting-on w[illegible] a tally. Quantity was highly significant for the missing-addend problems on the pretest. The extensive quantitative comparers had a mean score of 41 percent, while the gross quantitative comparers had a mean score of 2.5 percent. The fact that Quantity was significant and the gross quantitative comparers had a mean score of only 2.5 percent supp[illegible] the second statement in hypothesis 1--children who are not operational with number are not expected to be able to rational count-on or -back with or without tally. Table 6 indicates that the counting-back items are at least as difficult as the counting-on items, which makes it feasible to conjecture that the second statement in hypothesis 1 is supported by counting-back scores in September.

The first statement of hypothesis 1 is also supported by the significance of Quantity and the mean score of 41 percent for the extensive quantitative comparers. Most of the 12 children who scored 1 or 2 (see Table 5) had to be extensive quantitative comparers due to the 2.5 percent mean of the gross quantitative comparers. Consequently, at least 12 of the 24 extensive quantitative comparers scored 0 on the missing-addend problems with objects and at most 12 scored 1 or 2. These data clearly support the contention that children who are operational (in a Piagetian sense) with number may or may not possess rational counting-on skills.

Even though the tests of counting-on and counting-back were given in February, they do constitute a test of Hypothesis 1 in those cases where Treatment either is not significant or does not interact with Quantity. In the case of the two counting-back tests, the mean score for counting-back with no tally was approximately 67 percent for the extensive quantitative comparers and approximately 23

Table 8

F - Tests for Quantity (Q); Treatment (T); and Q x T

Source of Variation	Number in S	Number in P	#S + #P	Missing Addend With Objects Pretest	Missing Addend Without Objects Posttest	Missing Addend Without Objects	Counting-on With Tally (Ordinal Addition)	Counting-on Without Tally	Counting-back With Tally (Ordinal Subtraction)	Counting-back Without Tally	Just Before	Just After
Quantity (Q)	1.14	5.01	8.33**	21.19**	43.24**	21.51**	3.36✓	14.47*	19.36**	5.71	<1	2.54
Treatment (T)	1	<1	1	<1	1.13	<1	1.61	<1	2.15	<1	<1	2.54
Q x T	<1	1.85	<1	<1	<1	<1	3.39✓	<1	<1	<1	<1	<1

*(p < .05) **(p < .01) ✓(p < .08)

percent for the gross quantitative comparers. While these data are not as clearly supportive as the data for the missing-addend problems on the pretest, they do not contradict hypothesis 1 due to the fact the children were in a mathematics instructional program for a period of five months and Quantity was significant. The mean score for the counting-back with a tally test was approximately 60 percent for the extensive quantitative comparers and approximately 18 percent for the gross quantitative comparers. These data are stronger in support of hypothesis 1 than the data for the counting-back test without a tally, but still contain a schooling effect.

The patterns of the mean scores for the two counting-on tests were similar. The means are contained in Table 9. These two tests were not used to test hypothesis 1 because of the possibility of a treatment effect.

Table 9

Mean Scores for Counting-on Tests by Quantity (Percents)

Test	Counting-On Without Tally		Counting-On With A Tally	
Q \ T	Exp.	Con.	Exp.	Con.
Extensive	87	88	71	77
Gross	71	50	71	32

The two tests for just before and just after constitute possible tests to be used in a test of hypothesis 2:

> Numerical skills predicated on rote-counting are independent of a child's being operational with number (in a Piagetian sense).

Since Treatment was not significant and did not interact with Quantity for just-before and just-after scores, the evidence is strong that these two variables are not related to Quantity. Consequently, there is no evidence against hypothesis 2 supplied by just-before or just-after tests.

The two tests of missing-addend problems and the four tests of counting represent possible tests to be used in a test of research hypothesis 3:

> Acquisition of the ability to rational count-on or -back is highly related to a child's being operational with number (in a Piagetian sense).

The interaction of Quantity and Treatment was marginally significant for counting-on with a tally. The mean scores are presented in Table 9. The experimental gross quantitative comparers had a mean score of 71 percent, whereas the control gross quantitative comparers had a mean score of only 32 percent--an effect directly attributable to the treatment. The control gross quantitative comparers fared better with counting-on without a tally than they did with a tally, but still were 21 percent below the experimental gross quantitative comparers. These data, taken alone, would not support hypothesis 3. However, Quantity and Treatment did not interact for either missing-addend test, but Quantity was highly significant. The mean scores are presented in Table 10.

Table 10

Mean Scores for Missing Addend Problems: Quantity by Treatment

Test	Missing Addend With Objects (Posttest)		Missing Addend Without Objects	
Q \ T	Exp.	Con.	Exp.	Con.
Extensive	79	88	75	69
Gross	17	27	25	14

Because of the significant Quantity by Treatment interaction for count-on with a tally, one would expect at least the same pattern for mean scores for the two tests in Table 10 as in Table 9. In the absence of any such pattern, there is absolutely no basis to the claim that the experimental gross quantitative comparers had obtained a counting scheme in the same way as the extensive quantitative comparers. The extensive quantitative comparers apparently applied their counting schemes to the missing-addend problems whereas the experimental gross quantitative comparers did not. This lack of transfer on the part of the experimental gross quantitative comparers lessens the importance of high mean scores for the experimental gross quantitative comparers on the two counting-on tests. They apparently had learned to execute a solution algorithm in the case of stimuli very close to the experimental counting-on treatment. While problems were presented to the children in the treatment which were missing-addend problems, very few of the gross quantitative comparers could be led to solve them.

The fact that the experimental and control extensive quantitative comparers improved their capability to solve missing-addend problems from the pretest to the posttest* (from a mean of 41 percent to a mean of 78 percent for all problems) and the mean for the gross quantitative comparers was quite low for the missing-addend problems (21 percent for all problems), hypothesis 3 is supported for counting-on. The observation that the experimental gross quantitative comparers evidently did learn to execute counting-on strategies in restricted situations should not be taken lightly and is discussed further in the section on discussion of the results.

The results for the two counting-back tests clearly do not contradict hypothesis 3. Because counting-back activities were given in the experimental group but not the control group and no interaction of Quantity and Treatment existed, one cannot attribute causality to the counting-back activities in the treatment for the relatively high mean scores of the extensive quantitative comparers. As the mean scores in Table 11 indicate, the mathematical experiences of the experimental and control group children together with the fact that they were extensive quantitative comparers led to relatively high mean scores for the extensive quantitative comparers. But it is important to observe that the test of counting-back with a tally was exceptionally difficult for the gross quantitative

Table 11

Mean Scores for Counting-back Tests: Quantity by Treatment (Percents)

Test	Counting-back With a Tally		Counting-back Without a Tally	
Q \ T	Exp.	Con.	Exp.	Con.
Extensive	63	58	59	75
Gross	27	9	23	23

comparers. If a pretest had been administered to the children on counting-back, it would undoubtedly have been very difficult for all the children because only one child in the Treatment group was observed to be able to count-back with a

*While no missing-addend problems without objects to aid solution were administered on the pretest, there is absolutely no reason to believe that they would be easier for the children to solve than those given, especially in view of the data in Table 10.

tally at the start of the treatment. While this claim is only conjectual, the observation cited does lend credibility to the claim that the results for the two counting-back tests do not contradict hypothesis 3. In fact, the observation leads to the stronger claim that the data support the hypothesis.

The three tests for the variables Number in S, Number in P, and Number in S + Number in P represent possible tests to be used in a test of research hypothesis 4:

> Children who are operational with number (in a Piagetian sense) are not necessarily able to obtain cardinal information from ordinal information. But children who are not operational with number are not expected to be able to obtain cardinal information from ordinal information.

In that Quantity and Treatment did not interact for any of the three variables, each can be used to test hypothesis 4. The means are contained in Table 12. The results for the Number in S variable are viewed as inconclusive due to the lack of a significant F-ratio associated with Quantity and the low internal-consistency reliability. A test of hypothesis 4 for Number in S awaits better and more reliable task design.

In case of the Number in P variable, the extensive quantitative comparers outperformed the gross quantitative comparers, especially in the experimental group. An interaction between quantity and treatment is suggested by the means in Table 12, but was not significant statistically. One can say that children who are extensive quantitative comparers can obtain cardinal information from ordinal information better than gross quantitative comparers as long as that information can be obtained from counting forward rather than backward. The effect of Quantity was not as strong for Number in P as it should have been (theoretically). But it must be remembered that the reliability for Number in P variable was low.

Table 12

Means for Tests of Cardinal Information from Ordinal Information: Quantity by Treatment (Percents)

Test	Number in S		Number in P		Number in S + Number in P	
Q \ T	Exp.	Con.	Exp.	Con.	Exp.	Con.
Extensive	53	42	63	50	81	67
Gross	33	41	25	41	48	48

Due to the low reliabilities associated with the tasks, a fair test of hypothesis 4 could not be made. Gross quantitative comparers seemed able to obtain cardinal information to some extent. But an explanation exists for this seemingly good performance. Because the children were told the position of the tenth element in Task A and the fifth element in Task B (see Appendix III) it would be possible for the children to employ point-counting behavior to find the number in P. Moreover, as all of the children could point-count to at least seven (and beyond seven at the time the tasks were administered), the possibility that children used point-counting is very strong. It has also been observed that children who cannot rational count-on or rational count-back can, given a particular number name, orally count-on or count-back from that number. The basis for this observation is the measures for counting-on and counting-back in the preliminary items (see Appendix IV). The child, told that a particular object was tenth or fifth, certainly could have elicited rote counting-back or rote counting-on. That some gross quantitative comparers correctly found the Number in S could be a result of knowing three comes before four and seven comes before eight on a rote-counting basis. Conflict must be introduced into the task design in such a way to separate the false positives (children who scored the item correctly but who could not rational count-back) from the true positives. One way would be to add objects to S and require the children to (1) find the new number of S and (2) find the position of some r of Q. Since the same argument can be applied to the extensive quantitative comparers as was applied above to the gross quantitative comparers, a test of hypothesis 4 awaits better task design.

The missing-addend problems provide tests to be used in a test for research hypothesis 5, stated below:

> Acquisition of the ability to solve problems modeled by $a + x = b$ ($a < b$ and ordinal numbers) is highly related to a child's being operational with number (in a Piagetian sense). Moreover, children who are operational with number are not necessarily able to solve problems modeled by the equation form $a + x = b$. Children who are not operational with number are not expected to be able to solve problems modeled by the equation form $a + x = b$.

Quantity did not interact with treatment nor was treatment significant for any of the missing-addend problems. Quantity was highly significant. These facts, coupled with the data in Table 10 and the mean scores for missing addend problems with objects on the pretest (41 vs. 2.5 percent for extensive and gross quantitative comparers, respectively), supply strong support for each part of hypothesis 5.

Discussion of the Results

Theoretical Observations

Piaget (Beth and Piaget, 196 has distinguished mathematical and genetic structures. In this study, counting was viewed originally as being part of ordinal number theory in mathematics. A fundamental problem investigated was whether counting can be considered as part of genetic structures in the sense of grouping structures. The data strongly suggest the hypothesis that counting is not developmental, but rather that the emergence of the grouping structures in development allows children's culturally induced rote- and point-counting capabilities to be transformed to rational counting-on and -back, a transformation not possible prior to the emergence of the grouping structures. This hypothesis is advanced for several reasons, which follow.

On the pretest of missing-addend problems with objects, the gross quantitative comparers scored essentially zero, but approximately one-half of the extensive quantitative comparers showed evidence of being able to solve the problems. On the posttest of missing-addend problems, the average score of extensive quantitative comparers was approximately 78 percent, whereas the average score of gross quantitative comparers was approximately 21 percent. The latter figure is inflated due to obvious misclassification of two children as gross quantitative comparers. These two children were two of the best students in the experimental group. The mean scores presented for the tests of counting-back with a tally were approximately 61 and 18 percent for the extensive and gross quantitative comparers, respectively. Again, the mean score for the gross quantitative comparers is somewhat inflated. In any case, children who are gross quantitative comparers did not acquire, to any great extent, the capability of applying rational counting-on with a tally or rational count-back with a tally. Through arithmetical instruction, most of the children who were extensive quantitative comparers were able to learn to count-on with a tally but experienced difficulty learning to count-back with tally. Moreover, counting-back with a tally was tested in a restricted situation so that generality of the ability was not in evidence. The rather favorable mean score for the extensive quantitative comparers may be an overestimate of their ability. Instruction did seem to be necessary to solidify counting-on in a deep manner for the extensive quantitative comparers. All of the above facts and observations led to the hypothesis stated.

The observation that Piaget did not go far enough in his study of number seems justified by the results of this study. Based on Piaget's cardinal and ordinal number experiments, one would be led to believe that set similarity, the order on a set, the segment of a set, the remainder of a set, and counting-on and -back would be integrated into an operational system at the level of concrete operations. The data presented here do not support such a belief. Counting-on and counting-back do not emerge together nor do they emerge concurrently with extensive quantity. Counting-back with a tally seems to be a later and more difficult acquisition than counting-on with a tally and both a later acquisition than extensive quantity.

Freudenthal (1973, p. 173) strongly advocates basing addition and subtraction on counting-on and counting-back, respectively. This study shows that basing addition and missing-addend problems on counting-on without and with a tally, respectively, should be advocated for extensive quantitative comparers who are able to count-on prior to instruction. Others should be given instruction on counting-on prior to introduction of addition or missing-addend problems through counting-on. It is now axiomatic that a great deal of instruction on counting-back must precede introduction of subtraction through counting-back. There is no guarantee, however, that, even though children are able to count-on with a tally and count-back with tally, they have the two processes integrated. It is strongly hypothesized that the integration of counting-on and counting-back with a tally is the mechanism through which transfer can take place from knowledge in addition to knowledge in subtraction. The child who "solves" 12 - 4 through counting-on and who also can count-back is well on the way to integrating addition and subtraction. It should be the case that if the latter hypothesis is true, a great deal of instruction will have to take place on counting skills and their integration prior to having children who are capable of extensive quantitative comparisons able to transfer knowledge in addition to knowledge in subtraction. It may be a waste of time to present children who are not capable of integrating counting-on and -back with "families" of number sentences. Knowing how to solve the missing-addend problem by counting-on does not guarantee that children can relate the sentence $5 + \square = 9$ to $9 - 5 = \square$ on an intellectual basis.

This study shows that basing addition and subtraction on counting-on or counting-back for gross quantitative comparers is not possible prior to a great deal of instruction on counting strategies. Even then, the behavior produced is algorithmic and not operational, as evidenced by the failure of the experimental gross quantitative comparers to solve missing-addend problems. Instruction for such children on addition and subtraction should proceed using point-counting until more sophisticated counting techniques are developed. One of the most fundamental problems facing research with young children's acquisition of addition and subtraction is to determine the influence of counting instruction on the ability of gross quantitative comparers to rational count-on.

Observations from the Treatment

As the instruction was individualized for each child in the treatment group, no one instructional sequence may be described. It was the case, however, that each child was presented counting activities which progressed through rote-counting, point-counting, and rational counting. The instruction for addition and subtraction progressed through the learning instructional phases of exploration, abstraction-representation, and formalization-interpretation. The children were programmed through the learning-instructional phases at different rates and did different amounts of work. With few exceptions, the extensive quantitative comparers progressed through the abstraction-representation phase and associated counting activities more rapidly than did the gross quantitative comparers. Even though each child was given the opportunity to progress through the formalization-interpretation phase, only eight of the 48 children in the total sample actually did. It is important to note that tests were given for the formalization-interpretation phase even though they are not reported here.

At the culmination of the learning activities, all children were using rational counting-on to process exercises such as $4 + 5 = \square$. It is interesting to note what seemed to be critical instruction for children who were at most point counters to progress to that level. The instructional procedure used was to direct the children to make marks on their paper to represent the two addends and then gradually lead them into a realization that only marks for one of the two addends would be necessary if one would start counting from the other addend. An analogous procedure was used with finger calculation. Here, children were directed to "put one addend in their head." The children were then encouraged not to mark or use fingers, but to count the smaller addend on the larger mentally (in the case of unequal addends). After the children had mastered the procedure, they seemed very impressed with its powerfulness in calculating sums; they could find sums such as 15 + 4, 25 + 3, etc. Such sums were found even though the children did not know numeration.

Initially, each child was given experience in rote- and point-counting activities. All of the children learned to point-count and write the numerals to at least 50. Point-counting-back activities were also given, first starting h 10 and progressing through 20 or greater, depending on th child. The children, some with great difficulty, learned point-count-back from 20. Addition and subtraction activities were integrated with the counting activities where children used the counting-all procedures with objects to process sums and differences of the basic fact variety ($a + b \leq 10$). The children who were extensive quantitative comparers soon tired of using objects and wanted to use finger calculation. Thereafter, it soon became apparent that all of the children wanted to abandon the physical materials in favor of finger calculation. They were allowed to do so. The extensive quantitative comparers (with the exception of one child) easily learned to process sums such as 4 + 3 by counting-on three to four--"five," "six," "seven"--either through using finger calculation or mental calculation.

The gross quantitative comparers (with the exception of two children, one of whom was one of the best students) used counting-all procedures with finger calculation and did not internalize the counting process until direct instruction was given. It is important to note that trials (on an individual basis) during instruction were provided for these children to give them the opportunity to change counting strategies from counting-all to counting-on while processing sums such as 4 + 3. The trials were used as checks to insure that children were not held to counting-all procedures when in fact they could use more efficient counting strategies.

Several other points are important regarding the gross quantitative comparers. It was not until the last week of instruction that the gross quantitative comparers (with the two exceptions noted) were able to progress to counting-on activities (after approximately six weeks of instruction using counting-all strategies with physical objects and finger calculation). But six weeks should not be considered as a required time. For example, work with the hand-held calculator and problem solving were interspersed during the same six weeks. However, the six-week period does indicate the extreme difficulty children have of acquiring counting-on without tallying if it is not within their cognitive competence. The above procedures of instruction--integrating rational counting with finding sums--may only lead to what one may call algorithms for finding sums for the gross quantitative comparers. The induced counting behavior may not have been counting schemes. In fact, the evidence is strong that gross quantitative comparers did not generalize the counting-on without tallying procedures taught across tasks. But it is important to note that even though instructional procedures on counting-on in addition were effective over a rather narrow range of problems, they gave the gross quantitative comparers a sense of intellectual competence (as observed in instruction) in performing arithmetical exercises.

The effects of instruction on counting-on with tallying and the missing-addend problems were also interesting. The instruction was synthesized so the children were not aware that two different goals were being accomplished with the same activities--the capability to count-on with tallying and the capability to solve the missing-addend problems. The missing-addend problem was initially presented using a counting-all strategy. For example, to solve $4 + \square = 7$, the children were instructed to take seven objects and count out four; the ones remaining would be the answer. Invariably, children who did not possess counting-on with tallying confused the procedure with previously learned counting all procedures for processing sums. That is, to process sums such as represented by the sentence 3 + 8 = □, the children would count out eight objects, count three and the five remaining represented the result of the algorithm. It was necessary to explicitly point out the different appearance of the two types of sentences for these children. Through successive examples, the gross quantitative comparers did discriminate between the two sentence types and apply the correct algorithm. The same learning problem, however, did not occur for the children who were able to count-on with tally. They conceptualized the sentence 4 + □ = 7 as "four and how many is seven--five, six, seven--so it is three." Consequently, no problems in discriminating solution procedures existed for these children for the sentence types represented by the sentences 3 + 5 = □ and 3 + □ = 9.

The counting-all procedure for solving the sentence type 3 + □ = 8 seemed to interfere with the more natural counting-on strategy available to some of the children. After being shown the counting-all procedure, such children seemed to view it as the preferred solution process and were very reluctant to employ counting-on with tallying. It should be recognized that counting-on with tallying requires more mental effort than does the counting-all procedure which may be the cause for some children's great reluctance to use the more sophisticated counting strategy. But it also should be recognized that adults presented the counting-all procedure, which may have given it a status of being the preferred adult solution.

The counting-all procedure for solving missing-addend sentences was used initially, of course, so that the gross quantitative comparers would have a procedure for solving the problems which (it was hoped) could be transformed into a counting-on procedure. In the transformation, an analysis of the counting-all procedure was attempted in the following manner. After a child had solved, say, 3 + □ = 7, by counting out seven, taking three, and then counting the remaining ones to obtain four, he or she was instructed to refocus attention on the three, then count-on the four obtaining seven. This analysis move was not effective for some children who could not count-on without tallying, which was a minimal requirement to conceptualize what was being analyzed. Direct instruction was also given to tie the missing-addend sentence to rational counting-on with tallying. Problems were presented where some of a collection of objects were screened from a child's view. The child was then asked to find how many were screened. He or she had counted all of the objects to find the number in the total collection before some of them were screened. The unsuccessful children were allowed to "peek" behind the screen and count the objects there. These procedures were associated with missing-addend sentences, e.g., 4 + □ = 7, in the obvious ways after the physical problem was solved. Encoding of the physical and mental actions seemed extremely difficult for children who were not able to count-on with tally. These children seemed "lost" in instruction.

The posttest data on the missing-addend problems and the ordinal addition problems showed that the gross quantitative comparers in the experimental group were quite capable of solving ordinal addition problems (mean, 71 percent), but were particularly inept at solving missing-addend problems with objects (mean, 17 percent) and without objects (mean, 25 percent). It was, in fact, surprising that the experimental gross quantitative comparers performed so well on the ordinal addition problems, because during the treatment they seemed particularly inept at doing so. They apparently used trained procedures within a problem context familiar to them. It was particularly pleasing to note that the extensive quantitative comparers in the experimental group performed comparably to these in the control group on the missing-addend problems and ordinal addition problems. The experimental extensive quantitative comparers, when forced to do so, did utilize counting-on with tallying in problem contexts not solvable by counting-all procedures.

Based on experience in instruction with children not capable of counting-on with tally or without tally, it is recommended that teachers not present missing-addend problems to these children until counting-on schemes are acquired either through development or instruction. While such children can learn to solve such missing-addend problems through counting-all procedures, the solution process is algorithmic and conceptualization of the problem is lacking. In the case of children capable of counting-on with tally, the missing-addend problem should be presented with the solution process that of counting-on. These children, in their own time, should produce more efficient solution procedures. It is strongly urged that the child's counting capabilities be the determiner of whether the missing-addend problem is presented or not.

Children who are capable of counting-on, even if it is only without tallying, should be presented with addition through counting-on procedures rather than counting-all procedures. The counting-on procedures should lead to knowledge of basic facts more quickly. Moreover, the children can be exposed to more sophisticated sums (such as 43 + 4 or 56 + 5) and thereby gain a sense of competence not possible through counting-all procedures. Essentially, the exploratory phases of addition and subtraction can be done very minimally with these children. While counting-all procedures should not be forbidden (especially for differences with minuend less than or equal to ten), they should not be emphasized.

Conceptually, counting-back is to differences as counting-on is to sums. While differences may be found by counting-on with tallying, presently no data are available which show that a child is capable of conceptualizing differences in terms of counting-on if counting-back and counting-on are not synthesized (formalization-interpretation phase), one being associated with differences and one with sums. In the instructional activities, counting-back with and without tallying seemed especially difficult for most of the children. Presentation of the activities seemed to cause dissonance, with children refusing to participate mentally. While the extensive quantitative comparers fared much better than the gross quantitative comparers, the instruction on counting-back seemed to be not well-received by the children. But because of its importance to differences, instructional procedures need to be created and tested before definitive recommendations are made concerning the introduction of counting-back with and without tallying.

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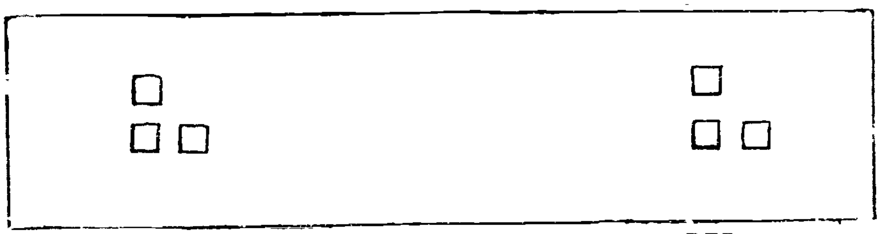
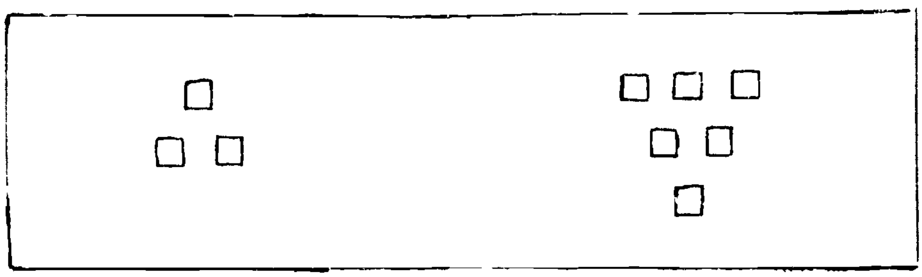
Appendix I. Quantitative Comparisons

Item W-1. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



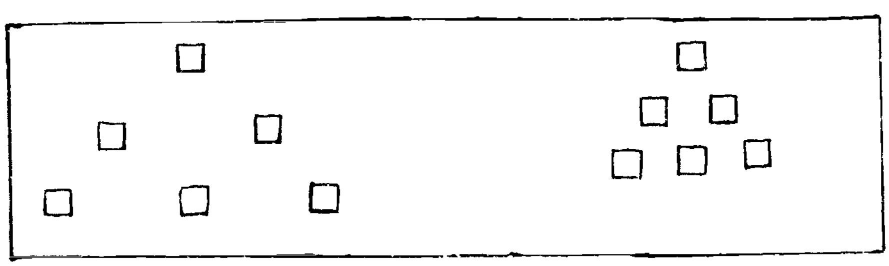
GREEN RED

Item W-2. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



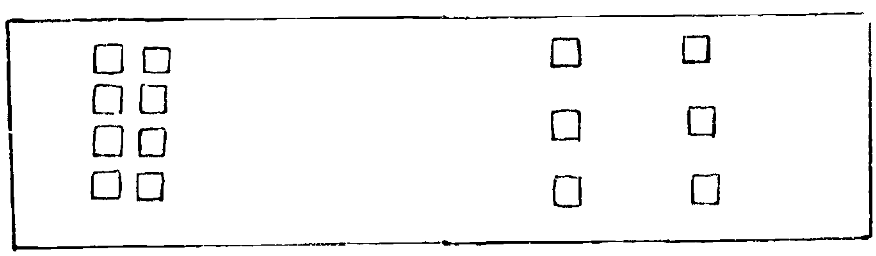
RED GREEN

Item 1. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



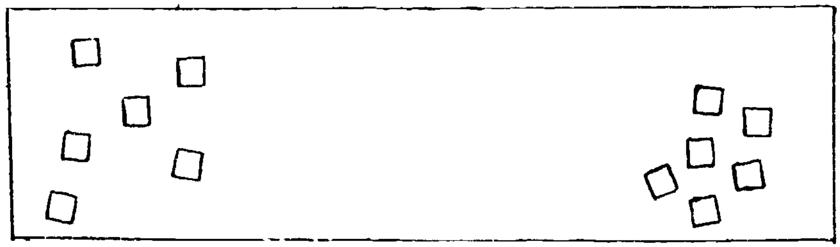
RED GREEN

Item 2. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



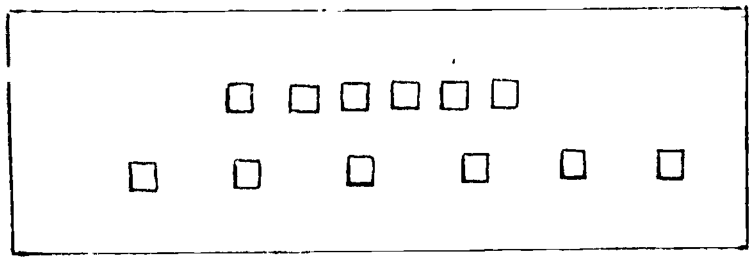
GREEN RED

Item 3. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



RED GREEN

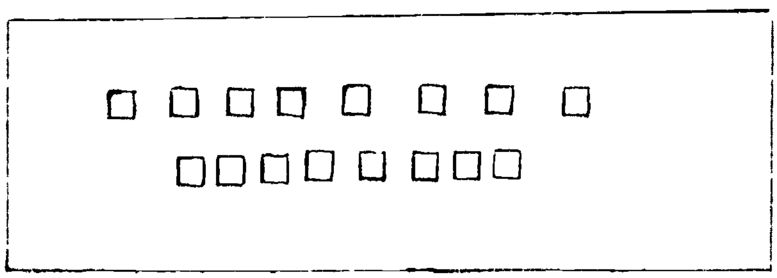
Item 4. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



GREEN

RED

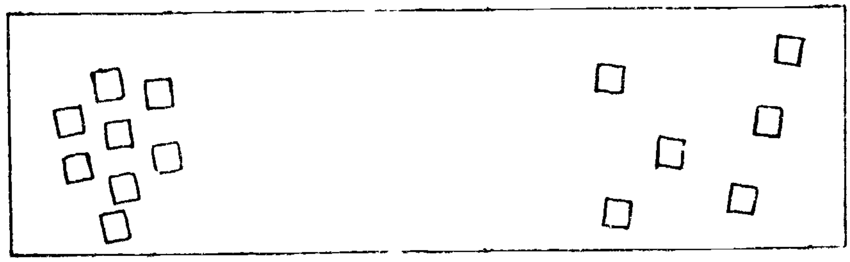
Item 5. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



GREEN

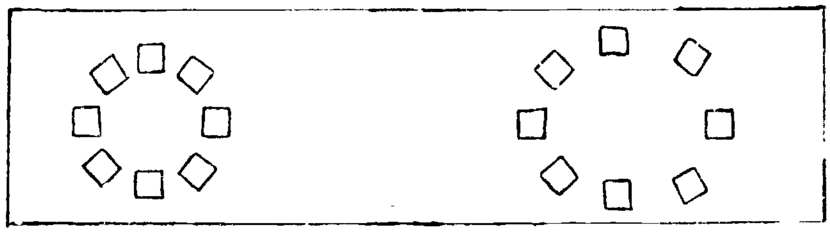
RED

Item 6. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



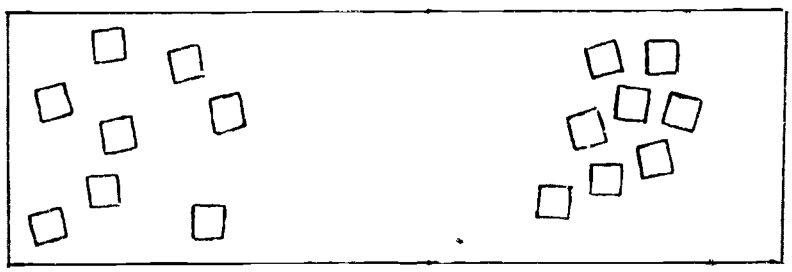
GREEN RED

Item 7. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



RED GREEN

Item 8. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



GREEN RED

Appendix II. Missing Addend Problems

Missing Addend Problems with Objects Present

1. MIKE HAS 5 BLOCKS. HE FOUND SOME MORE. NOW HE HAS 8 BLOCKS. HOW MANY DID HE FIND? (10 BLOCKS PRESENT)

2. LORI HAS 3 JACKS IN HER HAND. SHE PICKED UP SOME MORE AND NOW HAS 7 IN HER HAND. HOW MANY DID SHE PICK UP? (10 JACKS PRESENT)

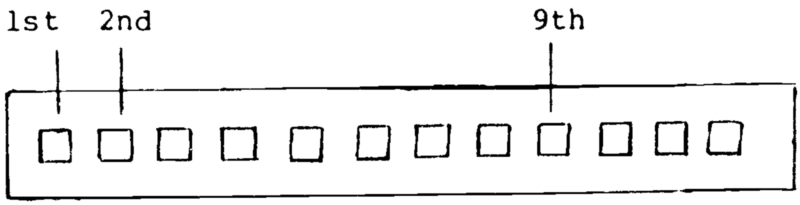
Missing Addend Problems without Objects Present

1. MIKE HAS 3 CATS. HIS MOTHER GAVE HIM SOME MORE. HE NOW HAS 7. HOW MANY DID HIS MOTHER GIVE HIM?

2. TOM HAS 5 COMIC BOOKS. HE GOT SOME MORE FOR HIS BIRTHDAY. NOW HE HAS 8 COMIC BOOKS. HOW MANY MORE DID HE GET FOR HIS BIRTHDAY?

Appendix III: Cardinal Information from Ordinal Information

Task A. (12 counters in a row)



HERE ARE SOME COUNTERS IN A ROW. IF WE START COUNTING FROM THIS END THIS ONE IS FIRST (point), THIS ONE IS SECOND (point), THIS ONE IS THIRD (point).

1. THIS ONE IS NINTH (point). WHICH ONE IS THIS? (point to tenth)

   a. [ ] correct immediate (go to #2)

   b. [ ] correct but counts from the beginning

   c. [ ] incorrect

      THIS ONE IS NINTH (point), THIS ONE IS TENTH (point), WHICH ONE IS THIS? (point to eleventh)

      [ ] correct immediately

      [ ] correct but counts from the beginning

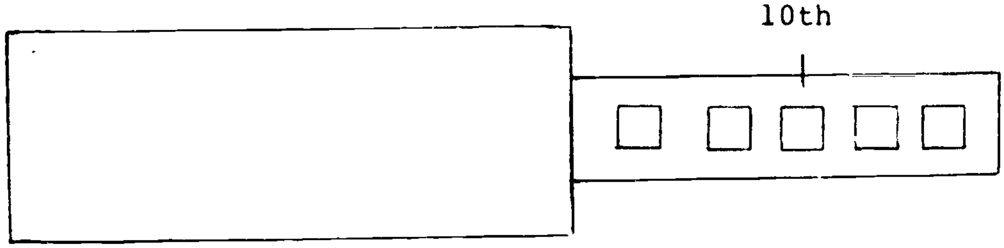
      [ ] incorrect

2. THIS ONE IS NINTH (point). WHICH ONE IS THIS? (point to seventh)

a. [ ] correct immediately

[ ] correct but counts from the beginning

[ ] incorrect



3. (cover seven with cloth) THIS ONE IS TENTH (point). HOW MANY ARE COVERED?

a. [ ] correct - HOW DO YOU KNOW THAT? (go to b)

b. [ ] HOW MANY ARE THERE IN ALL? (stop)

c. [ ] incorrect - THIS ONE IS TENTH (point), HOW MANY ARE THERE IN ALL?

d. [ ] correct - RIGHT, AND HOW MANY ARE COVERED?

e. [ ] incorrect (five) - FEEL THE FIRST ONE. WHICH IS NEXT? (feel second)

f. [ ] correct - HOW MANY ARE COVERED?

[ ] correct

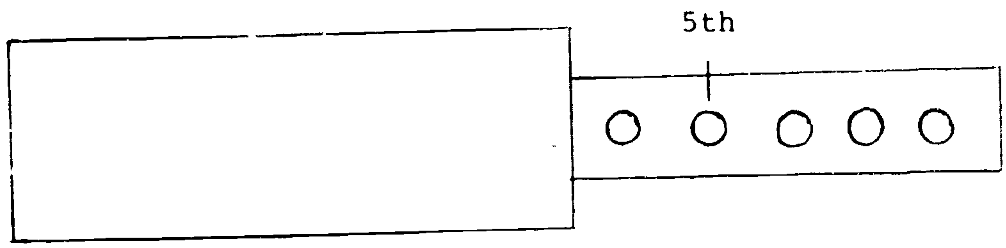
[ ] incorrect (stop)

g. [ ] incorrect (not five) - THIS IS TENTH (point), THIS IS ELEVEN (point), THIS IS TWELFTH (point). HOW MANY ARE THERE IN ALL?

h. [ ] correct - HOW MANY ARE COVERED?

[ ] correct

[ ] incorrect (stop)



HERE ARE SOME COUNTERS IN A ROW. SOME OF THEM ARE COVERED. FEEL THE FIRST ONE HERE.

1. THIS ONE IS FIFTH (point). WHICH ONE IS THIS? (point to sixth)

   a. [ ] correct - got to #2

   b. [ ] incorrect - THIS ONE IS FIFTH (point), THIS ONE IS SIXTH (point). WHICH ONE IS THIS? (point to seventh)

   [ ] correct

   [ ] incorrect

2. THIS ONE IS FIFTH (point). HOW MANY ARE THERE IN ALL?

a. [ ] correct - HOW DO YOU DO THAT?

b. [ ] incorrect (five) - REMEMBER, THERE ARE SOME UNDER THE COVER. FEEL THE FIRST ONE. THIS ONE IS FIFTH (point). HOW MANY ARE THERE IN ALL?

[ ] correct

[ ] incorrect

c. [ ] incorrect (not five) - THIS ONE IS FIFTH (point), THIS ONE IS SIXTH (point), THIS ONE IS SEVENTH (point). WHICH ONE IS THIS? (point to eighth)

d. [ ] correct - HOW MANY ARE THERE IN ALL?

[ ] correct

[ ] incorrect

e. [ ] incorrect - FIFTH (point), SIXTH (point), SEVENTH (point), EIGHTH (point). HOW MANY ARE THERE IN ALL?

[ ] correct

[ ] incorrect

3. THIS IS THE FIFTH ONE (point). HOW MANY ARE COVERED?

a. [ ] correct - done

b. [ ] incorrect - THIS ONE IS FIFTH (point). WHICH ONE IS THIS? (point to fourth)

c. [ ] correct - HOW MANY ARE COVERED?

[ ] correct immediate

[ ] correct, trial and error

[ ] incorrect

d. [ ] incorrect - FIFTH (point), FOURTH (point).

HOW MANY ARE COVERED?

[ ] correct

[ ] incorrect

Counting-on and Counting-back Tasks

Counting-on

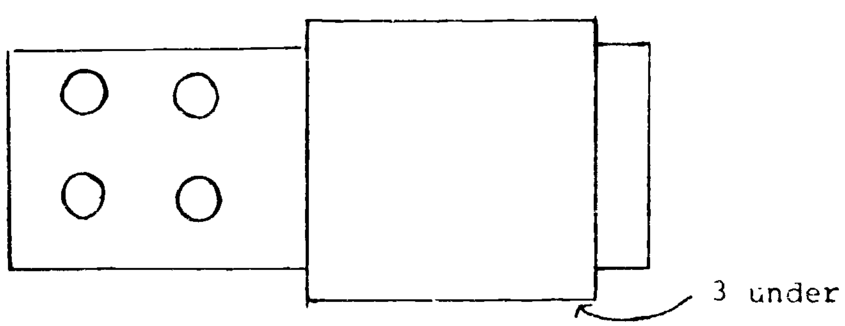
Warm-up Tasks

1. START AT FOUR AND COUNT ON THREE MORE NUMBERS FROM FOUR (If unsuccessful, demonstrate).
2. START AT SEVEN AND COUNT ON FOUR MORE NUMBERS FROM SEVEN (If unsuccessful, demonstrate).
3. START AT TWELVE AND COUNT ON THREE MORE NUMBERS FROM TWELVE (If unsuccessful, demonstrate).

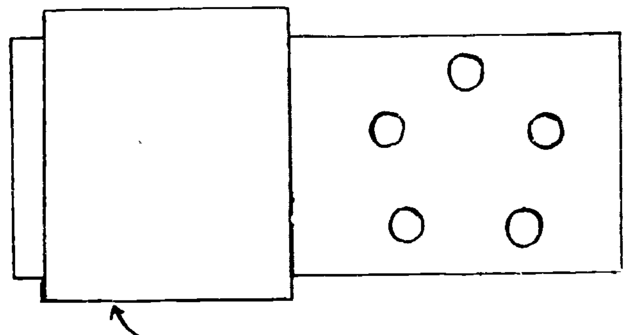
Counting-on without a Tally

1. Three checkers covered with a cloth presented to the child. Four visible checkers arranged randomly are also presented to the child.

E. THERE ARE THREE CHECKERS UNDER THE CLOTH. COUNT ON TO FIND HOW MANY CHECKERS THERE ARE ON THE CARD.



2. The same as (1) except seven checkers were under the cloth and five checkers were visible.

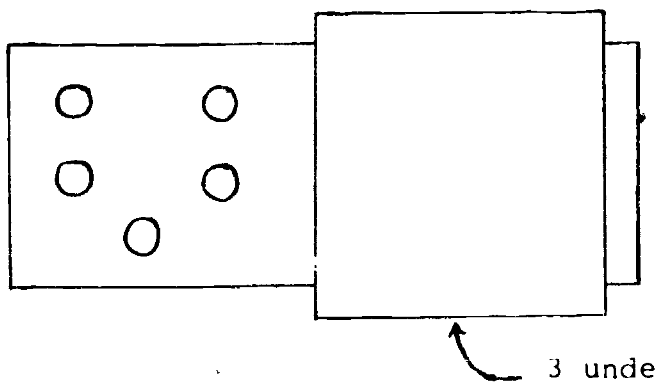


7 under

Counting-on with a Tally

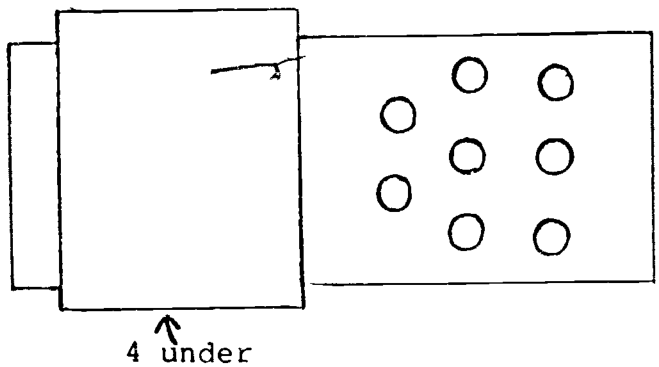
1. Three checkers covered with a cloth are presented to the child. Five visible checkers arranged randomly are also presented to the child.

   E. HERE ARE FIVE CHECKERS. THERE ARE SOME MORE UNDER THE CLOTH. THERE ARE EIGHT CHECKERS IN ALL ON THE CARD. COUNT ON TO FIND HOW MANY CHECKERS ARE UNDER THE CLOTH.



3 under

2. The same as (1), except there are 12 checkers in all, 8 visible.



Counting-back

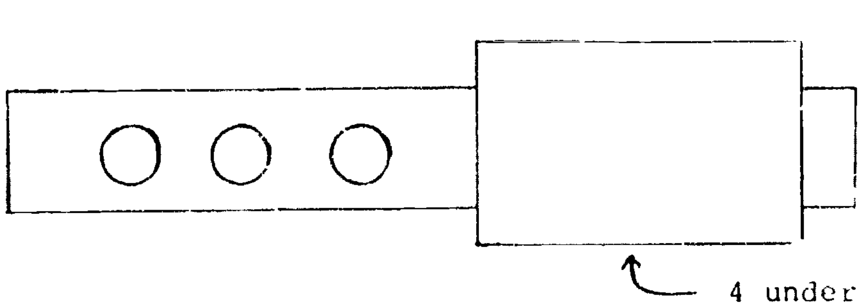
Warm-up Tasks

1. START AT FOUR AND COUNT BACK THREE NUMBERS. (If unsuccessful, demonstrate.)
2. START AT SEVEN AND COUNT BACK THREE NUMBERS. (If unsuccessful, demonstrate.)
3. START AT TWELVE AND COUNT BACK FOUR NUMBERS. (If unsuccessful, demonstrate.)

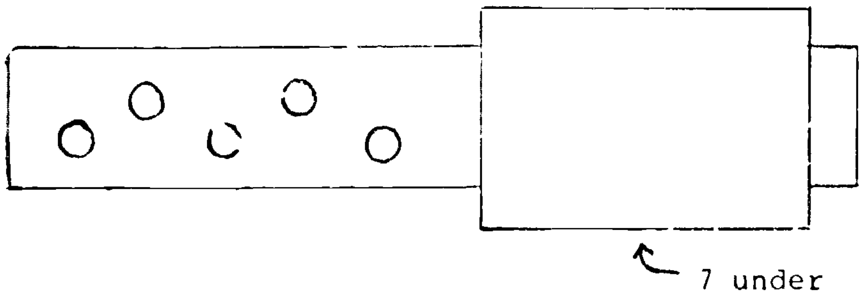
Counting-back without a Tally

1. Four checkers covered with a cloth are presented to the child. Three visible checkers arranged randomly are also presented to the child.

   E. THERE ARE SOME CHECKERS UNDER THE CLOTH. I COUNTED THEM ALL ON THE CARD AND THERE ARE SEVEN. COUNT BACK, STARTING AT SEVEN, TO FIND OUT HOW MANY ARE UNDER THE CLOTH.



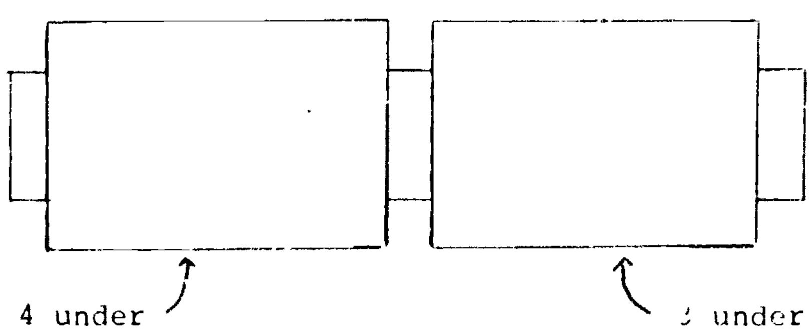
2. The same as (1), except there are seven checkers covered and five visible.



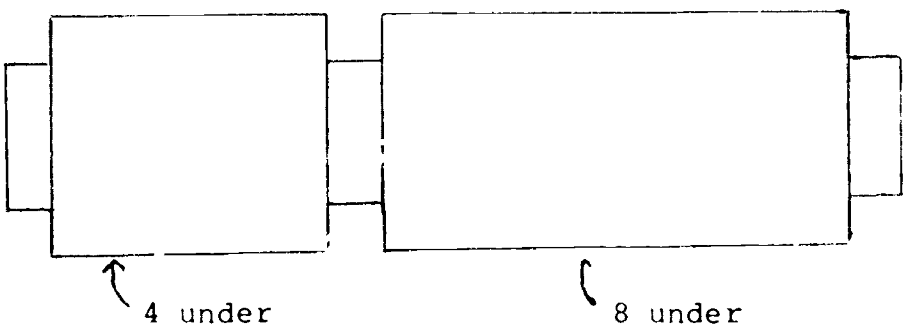
Counting-back with a Tally

1. Seven checkers, four under one cloth and three under another cloth, are presented to the child.

   E. THERE ARE SEVEN CHECKERS ON THE CARD UNDER THESE CLOTHS. THERE ARE FOUR CHECKERS UNDER THIS CLOTH (point). COUNT BACK, STARTING AT SEVEN, TO FIND OUT HOW MANY ARE UNDER THIS OTHER CLOTH (point).



2. The same as (1), except there are four checkers covered under one cloth and eight under the other. The child is asked to count back from 12 to find how many are under the cloth with four covered.



LANGUAGE AND OBSERVATION OF MOVEMENT AS PROBLEM SOLVING TRANSFORMATION FACILITATORS AMONG KINDERGARTEN AND FIRST-GRADE CHILDREN

Jay Shores
University of Houston

Robert Underhill
Kansas State University

The purpose of this study was to ascertain whether the use of overt modeling and/or verbal modeling assists young children to solve four types of mathematical problems.

Theoretical Background

A child's ability accurately to solve basic mathematical problems is known to be affected by both his or her level of cognitive development and the effects of initial school experiences (Underhill and Shores, 1975). In kindergarten and first-grade children aged 5 to 7 years, the ability to conserve numerousness evolves as the children are being exposed to basic mathematical concepts of varying conceptual complexity (Piaget and Inhelder, 1969).

The conservation of numerousness construct was introduced to the mathematics education community from the translated writings of Piaget (1965), the text by Flavell (1963), and the research of Elkind (1961), Dodwell (1960), and Wohlwill (1962). Studies by Van Engen and Steffe (1966), LeBlanc (1968), Steffe and Johnson (1970), and Johnson (1971), among others, have established significant differences between conservers' and nonconservers' problem-solving achievement. In addition, it was found that problems which involved transformation were significantly more difficult than those which did not involve a transformation. Transformational tasks are those which imply movement or action in the context of stated problems (Underhill and Shores, 1976).

The concept of transformation is an important construct in Piagetian research. A transformation is an act or process of alterating, or the changing of one thing into another. A transformation can exist at several different levels. Transformations not only refer to alterations in the physical world, but also to the compensations made by the individual in his or her mental structures. If an object or state is known by an individual, then a transformation in the physical state is accompanied by a transformation in the cognitive structure. In another sense, a transformation occurs when a learner states that 3 + 4 = 7. Piaget and Inhelder (1969) describe operations as reversible transformations, and they use addition of two numbers as a specific example.

The population of interest in the present study was kindergarten and first grade children, so the concept of transformation was defined within the context of pre-operational thought. Piaget (1972) and Sinclair (1971) characterize concrete operational thought as being limited to thinking about object experience through object-invoking mental processing. While concrete operational thought is not limited to thinking-while-manipulating, such thought is characterized by thinking about real present objects and actions. Thus, a meaningful comprehension of $3 + 4 = 7$ suggests that learners may, for example, conjure mental images of sets of real objects with number properties of "threeness" and "fourness." Then if learners comprehend the operation of addition, they conceptualize a transformation in which the two sets with number properties of "threeness" and "fourness" are joined in set union to form a new superordinate set with a number property of "sevenness."

Steffe (1967, 1968) and LeBlanc (1968) defined transformational and non-transformational addition and subtraction story problems as those which do or do not provide movement cues which indicate joining or separating of sets and subsets. Here are addition samples:

> Transformation: Two dogs are in the kennel. Three more dogs are placed in the kennel. Now how many dogs are in the kennel all together?
>
> No Transformation: Bill has three frogs. John has four frogs. How many frogs do Bill and John have all together?

Within the context of earlier statements made by Piaget, one could say that the operation of addition is involved in both types of problems, so both involve transformations. Thus, if the learner is asked to solve the problems, he or she is asked to complete a transformation. The Steffe and LeBlanc tasks might be more appropriately labeled as facilitating and nonfacilitating addition and subtraction types relative to the transformation task to be completed.

It was hypothesized that the modeling of the transformation would assist children who are beginning to conserve numerousness in solving the mathematical problems. The degree of facilitation should fluctuate according to the degree to which the modeling itself varies from fully demonstrated and explained, to fully explained, to simply the implicit movement within the problem statement itself. It was anticipated that a child who observes a transformation will be able to use spatial referents as cues to assist in recalling the untransformed set. Thus, such a child should have less difficulty in solving problems than a comparable child who does not receive a similar modeling experience.

Among the transformational mathematical operations to which young children are introduced were the following. These are of varying conceptual complexity: counting-on, story problems, quantitative comparisons, and ordination. Counting-on requires the formation of one set and the serial addition of elements to it. It is the continuation of a simple counting sequence. Addition story problems constitute a slightly more complex task, namely the establishment of two sets of similar elements and the union of them. Quantitative comparison involves the formation of two sets, the establishment of correspondences between the sets' elements, and a judgment based on equivalence. Ordination, the most complex of the tasks in this study, posits the existence of two sets and relationships of two abstract constructs to them. Both a cardinal and ordinal (spatial position) relationship must be maintained after a spatial transformation.

The Experimental Tasks

Counting-on Tasks. E placed a strip of cardboard containing a row of at least seven chips in front of S. The first n chips were covered with another piece of cardboard. S was told how many chips were covered and was requested to tell how many chips were on the cardboard in all.

Addition Tasks. E placed a cardboard piece with pictures of children and two appropriate sets of chips. E told an addition story problem. S was requested to give the answer.

Quantitative Comparisons Tasks. E placed 2 rows of chips before S. S was asked if there were the same number of chips in each row.

Order Tasks. E placed a strip of cardboard containing four chips of four different colors and a second piece of cardboard containing two chips of two different colors at 135 rotation from the first piece. E gave S a third and a fourth chip to place on the second piece of cardboard.

Procedure

The subjects were presented with two items of each problem type: counting-on, story problems, quantitative comparison, and ordination. Each set of eight problems was presented by three researchers under three modeling conditions in the following order: (1) implicit modeling, in which the subject was presented with a transformed model and simply asked to solve the problem; (2) implied modeling, in which the subject was presented with a transformed model, and the procedure for the restoring transformation was verbally described; (3) overt modeling, in which the experimenter transformed the model as the question was asked and verbally commented on the transformation as it was carried out. A random order of items within each set of eight tasks was used for each subject.

Figure 1 summarizes the three modeling conditions with the four problem types. From a theoretical point of view, the subject must attend to a transformation in each of the twelve tasks. Clearly, the tasks involving auditory and visual cues are much more explicit in their overt manifestations of the necessary transformations. The three cases of each class of concept tasks could be said to depend on 1) attending to, and comprehending, auditorially and visually presented transformation cues, 2) attending to, and comprehending, auditorially presented transformation cues, and 3) spontaneous creation of transformations unaided by experimentally visual or auditory cues.

Knowledge of youngsters' performances on the twelve tasks should clarify researchers' understanding of the roles of language and observed movement in transformational thinking. If these patterns are pervasive, the practitioner is provided with an empirically verified rationale for utilizing modeling procedures during instruction.

Sampling

To obtain a representative sample of kindergarten ($n_K = 20$) and first-grade ($n_1 = 20$) children, a large suburban school system's lists of kindergarten and first-grade pupils were obtained. A random sample of 20 children were drawn from each list to serve as subjects for the study. During testing one child was removed from the sample as deviating from experimental procedures (lifting the cards to count chips). He was replaced by another child drawn at random from the school's roster. Each subject responded to the 24 tasks within a time interval of approximately 30 minutes. All tasks were individually administered.

Analysis

In the 4 x 3 x 2 (Problem Type by Modeling Type by Grade Level) design, the subjects were used as their own controls across problem type and modeling type. An initial factor analysis of the 24 items was conducted in which the items were found to load by problem type. This confirmed the existence of conceptual distinctiveness among the problem types. A subsequent MANOVA was used to determine the effects of problem type, modeling type, and grade level for each of the four problem types.

Modeling Conditions

Problem Types	Overt (Auditory and visual movement cues)	Implied (Auditory movement cues only)	Implicit (No auditory or visual cues)
Counting On Tasks	The cardboard strip was placed before S with all of the chips showing. A second piece of cardboard was used to cover the first n chips, while E explained what he was doing.	The cardboard was placed before S with the first n chips covered. E explained that the first n chips had been covered up.	The cardboard strip was placed before S with the first n chips covered. No explanation of the coverings was given beyond the statement of the problem.
Order Tasks	The cardboard strips were placed initially in parallel positions. As one was rotated through 135°, E described what he was doing.	The cardboard strips were placed initially in 135° positions. E explained how they would match if one were turned.	Same as implied with no explanation.
Story Problem Tasks	A transformation problem (Steffe) was stated as chips were used to demonstrate the action.	A transformation problem was stated. Chips were statically placed in a post-transformational position.	A non-transformation problem stated. Chips were statically placed in two disjoint sets.
Quantitative Comparisons Tasks	Two rows of chips in one-to-one correspondence were presented. One row was linearly dispersed. The action was described.	Two rows of chips were presented statically, one being more linearly dispersed. The dispersion process was explained.	Same as implied with no explanation.

Figure 1. Modeling Conditions and Problem Types

Results

Table 1 presents a summary of the multivariate analysis. For the counting on and story problem items, there was no significant difference across the modeling types. However, for quantitative comparison and ordinal items there was a significant difference ($p < .05$) across the modeling types.

Tables 2 and 3 present a summary of univariate contrasts between grade levels by problem type and modeling condition. In the quantitative comparison type, the significant differences among the kindergarten subjects' responses were found between each of the model types (Overt > Implied > Implicit). The first-grade subjects had a different pattern in their responses, with overt responses being significantly greater than both the implied and implicit responses (Overt > Implied ≈ Implicit).

Table 2

Summary of Univariate Contrasts Between Grade Levels Problem Type and Modeling Condition

			MODELING CONDITION			
			Overt	Implied	Implicit	TOTALS
P	Counting-	$\bar{X}_K$ =	1.01	1.05	0.90	3.05
R	on	$\bar{X}_1$ =	1.75	1.65	1.50	4.90
O	Problems	F =	9.15*	7.02*	5.52*	9.65*
B						
L	Ordination	$\bar{X}_K$ =	0.70	0.60	0.35	1.65
E		$\bar{X}_1$ =	1.10	0.80	0.60	2.50
M	Problems	F =	2.17	0.76	1.55	2.06
T	Quantitative	$\bar{X}_K$ =	1.10	0.70	0.35	2.15
Y	Comparison	$\bar{X}_1$ =	1.30	1.00	0.95	3.25
P	Problems	F =	0.42	1.13	4.97	2.23
E						
	Story	$\bar{X}_K$ =	1.45	1.25	1.50	4.20
	Problems	$\bar{X}_1$ =	1.95	1.90	1.90	5.75
		F =	9.60*	13.90*	5.63*	16.03*
		$\bar{X}_K$ =	4.35	3.60	3.10	
	TOTALS	$\bar{X}_1$ =	6.10	5.35	4.95	
		F =	7.17*	9.92*	10.43*	

* $p \leq .05$

Table 1

Summary of Multivariate Analysis of the Effects of Modeling
Type and Grade Level for Each Problem Type

Problem Type: Counting-on

Source of Variation	SS	df	MS	F
Main Effects	12.85	3	4.28	7.90**
Treatment	.82	2	.41	.75
Grade Level	12.03	1	12.03	22.20**
Interaction	.02	2	.01	.02
Error	61.80	114	.54	

Problem Type: Ordinal

Source of Variation	SS	df	MS	F
Main Effects	6.03	3	2.01	3.61*
Treatment	3.62	2	1.81	3.25*
Grade Level	2.41	1	2.41	4.33*
Interaction	.22	2	.11	.20
Error	63.35	114	.56	

Problem Type: Quantitative Comparison

Source of Variation	SS	df	MS	F
Main Effects	10.39	3	3.46	4.27**
Treatment	6.72	2	3.36	4.14*
Grade Level	3.68	1	3.68	4.53*
Interaction	.65	2	.33	.40
Error	92.55	114	.81	

Problem Type: Story Problem

Source of Variation	SS	df	MS	F
Main Effects	7.53	3	2.57	8.00**
Treatment	.52	2	.26	.82
Grade Level	7.01	1	7.01	22.35**
Interaction	.32	2	.16	.51
Error	35.75	114	.31	

* $p \leq .05$
** $p \leq .01$

Table 3

Summary of Univariate Contrasts Among Modeling Conditions by Problem Type

			MODELING CONDITION		
			Overt	Implied	Implicit
PROBLEM TYPE	Counting-on	$\bar{X}=$	1.38	1.35	1.20
	Implied		.03		
	Implicit		.18	.15	
	Ordination	$\bar{X}=$	.90	.70	.48
	Implied		.20*		
	Implicit		.42*	.22*	
	Quantitative Comparisons	$\bar{X}=$	1.20	.85	.65
	Implied		.35*		
	Implicit		.55	.20*	
	Story Problems	$\bar{X}=$	1.70	1.58	1.70
	Implied		.12		
	Implicit		.00	.12	

*$p \leq$ 05

In the ordinal type, significant differences among the kindergarten subjects' responses were found, with implicit responses being significantly lower than the other two modeling types (Overt $\approx$ Implied > Implicit). With respect to first graders' performances on the ordinal tasks, there were no significant differences.

For the overt items there was a significant difference: children's performance with counting on items and responses to story problem items were significantly higher than their performance on quantitative comparison and ordinal items (CO $\approx$ SP > OR $\approx$ QC). For the implied items, the nature and order of significance was the same as for the overt items. For the implicit items the counting-on, quantitative comparison, and story problem responses were significantly greater than those of the ordinal type (CO $\approx$ QC $\approx$ SP > OR). First graders were significantly better than kindergarteners in counting-on and story problem responses over all types of models.

Significance of the Findings

This study indicates that modeling has an effect upon the subject's ability to solve the two more difficult types o transformation problems, quantitative comparison and ordination. Further, overt and implied modeling significantly affected the subjects' ability to solve counting-on and story problems. The effects were greater for kindergarten than for first-grade children.

These findings support the hypothesis that during the child's transition from nonconserver to conserver, the use of modeling might significantly assist the teacher in facilitation of conservation-related subject matter. Researchers and practitioners need to conduct further investigation to determine the pervasiveness of the differences found in this study. If the significant differences are widespread, then considerable pedagogical change might be warranted.

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ASPECTS OF CHILDREN'S MEASUREMENT THINKING

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Quantity

Human beings normally make decisions (or judgments) during their daily activities. Some of these judgments are quantitative in nature while others are qualitative in nature. For example, one might say that a particular drink is "sweeter" than another. If "sweeter" is determined by a taster, the judgments are of a qualitative nature and it would be difficult for the person to define strictly what was meant by the term "sweeter." But, through taste alone, two or more drinks can be ordered on the basis of "sweeter." This order relation however, does not say how much sweeter one drink is than another. The differences, while they exist, rely on number for their elucidation. If, through chemical analysis, the amount of sugar per unit volume is determined, the differences in the "sweetness" of the two drinks may be determined. Moreover, the drinks can be ordered using the relation "sweeter" through the natural order of the numbers--in which case, the drinks would not have to be tasted.

A quantity is determined by a set of objects and criteria for comparison of those objects--two drinks ordered on the basis of "sweeter" is a quantity. The objects themselves, however, are referred to as quantities. In such references, it will be assumed that a criteria for comparison has been established. Quantities to be "measured" may be categorized into two collections based on whether or not their attributes are additive. Intensive quantities are objects which are compared on the basis of attributes which are nonadditive--for example, temperature. Consider a pail of water with temperature 100°F and a similar size pail of 50°F water. These two nonoverlapping quantities when joined together do not give a quantity of water with temperature of 150°F. Other quantities which are intensive are hardness, softness, density, and intelligence. Extensive quantities are objects which are compared on the basis of attributes which are additive. For example, if the comparison between sticks is length, one could take a stick of length l_1, a stick of length l_2, and join them end-to-end with no overlap. The join would be of length $l_1 + l_2$. Some other quantities which are "extensive" are area, volume, weight, and number.

The primary difference between the two categories of objects is the way in which numbers may be assigned. Intensive quantities are quantities which are "measurable" in the sense that they may be arranged in a series showing differences in degrees of the quantity under consideration. Extensive quantities are "measurable" in the sense of intensive quantities but also in the sense that the attributes for comparison are of an additive nature.

The differences in intensive and extensive quantity may be formalized by properties necessary for the measurement process to have meaning. The first two properties represent minimum conditions in order for numbers to be employed to establish differences.

1. Using a set of objects (say n of them), O_1, O_2, ..., O_n, it must be possible to arrange them in a series with respect to a certain quality. The series requires that the law of trichotomy holds. That is, for any two bodies O_i and O_j, exactly one of the following is true: (a) $O_i > O_j$; (b) $O_i < O_j$; or (c) $O_i = O_j$. "=", ">", and "<" symbolize the relations by which the objects are ordered. Note that ">" and "<" are asymmetrical.

2. If $O_i > O_i$, and $O_j > O_k$, then $O_i > O_k$. This statement is the transitive property of the relation " ." The two properties are sufficient for a collection of objects to be described as intensive quantity and thus be measured. The two properties are not sufficient for quantities to be measured in the extensive case. Four additional properties must hold, all of which concern the physical process of addition of quantities.

3. If $O_i + O_j = O_k$, then $O_j + O_i = O_k$.

4. If $O_i = O_i'$, then $O_i + O_j > O_i'$.

5. If $O_i = O_i'$ and $O_j = O_j'$, then $O_i + O_j = O_i' + O_j'$.

6. $(O_i + O_j) + O_k = O_i + (O_j + O_k)$.

Measurement in the strictest sense is only possible when all six properties hold (Cohen and Nagel, 1934).

Some important examples of quantities are:

1. The positive integers using the natural order relation "greater than."

2. The positive rational numbers using the natural order relation "greater than."

3. Objects (such as sticks or strings) compared operationally by use of the length relation "longer than."

Each of these quantities is extensive.

Measurement

As developed in the preceding section, there is a distinction to be made between measurement and quantity. Quantity (either intensive or extensive) is concerned with a set of objects and a criteria of comparison (note that both the objects and the relations determined by the criteria of comparison are necessary). Quantity is a necessary condition for measurement (in the numerical sense) to take place. If objects are orderable according to some attribute, it becomes possible to assign numbers to the objects of the quantity, or to measure the objects according to that attribute. The assignment of numbers in these situations is called measurement. Quantities, such as the positive integers or positive rationals, are abstract--they are not physical bodies. It is here that quantity and measurement are most easily distinguished. One can select some arbitrary number as a unit and assign numbers to numbers. If unity is selected, the identity mapping is defined. But the domain of the mapping is a set of numbers just as is the range for any selection of a unit number. However, one does not have to measure an object for [illegible]ers to be present, since the objects are numbers. In the c[illegible]f quantities where the objects are physical bodies--phys[illegible]uantities--one can also assign numbers to the objects [illegible]ough selection of some unit body. But the objects to be measured are not numbers. In the case of extensive quantities, the physical bodies can be ordered and combined. Measurement allows one to work in the abstract with physical quantities through working with the numbers assigned to the objects. In measurement of both abstract and physical quantity, the following are present:

1. A set A of objects to be measured (the structural properties of this set are determined by the type of quantity, intensive or extensive);

2. A set B of "measurements" (usually a subset of the positive real numbers); and

3. A process for associating with each element of A an element of B.

This situation can be described through the concept of function. There is a related function for every measurement situation. The domain is the set of objects to be measured, while the range consists of the measurements (usually positive real numbers) to be associated with members of the domain by the particular mapping under consideration.

The mapping must preserve the structure of the domain and range. That is, the quantity under consideration, with its order relation, is mirrored in the range of the function. In fact, the range is a quantity in its own right. The correspondence between domain and range may be used to establish common units of measurement by arbitrarily selecting an object in terms of the arbitrary object's functional value. By selecting different units it is possible to define several functions on the same domain (Blakers, 1967).

Two different types of physical quantities with their associated measurement functions are considered in this study--collections of physical objects ordered by matching relations and linear physical objects ordered by length relations. In the case of collections of physical objects, singular physical objects are taken as the unit. So, given a collection of physical objects, the function assigns the count of the collection to the collection. In the case of linear physical objects, an arbitrary unit is selected so that each physical object in the domain of the function is an integral number of units long. This was done so that, given a collection of units associated with a particular physical object, the count of the collection of units is the length of the physical object.

Reasoning Concerning Measurement

Reasoning in the Domain of the Measurement Function

Two types of comparisons can be made in the domain of the measurement function described immediately above--direct and indirect comparisons. Direct comparisons essentially involve no reasoning because the physical objects are proximal. However, indirect comparisons require transitive or substitutive reasoning because two collections of physical objects are compared by using a third such collection. For example, a child might compare sticks A and B and determine that A is longer than B. Upon comparing B with C, he may determine that B is longer than C. Then, using the transitive property of the relation, it is possible for the child to conclude that stick A is longer than stick C (without overt comparison).

Reasoning in the Domain and Range of the Measurement Function

It is possible for children to compare directly objects in the domain of the measurement function through comparisons of their measurements. For example, suppose some stick A is measured and found to be seven units in length. Then stick B is measured and found to be seven units in length. Using this information, it is possible to conclude that if A and B were to be compared physically, they would be of the same length. It is possible to make indirect comparisons in the following way. Imagine A and B are physically compared and A is found to be longer than B. Then B and C are measured and each is found to be seven units long. Then, because A is longer than B, it is also longer than C. This indirect comparison involved the substitutive property, even though B and C were compared through comparisons of their measurements.

Premise forms. Situations consisting of two instances of a relation or one instance of a relation and another instance of another relation are called premise forms. A transitive premise form is a premise form consisting of two instances of a transitive relation where an implication is possible. A substitutive premise form is a premise form where the substitutive property allows for an implication to be made. An incompatible premise form is a premise form from which no implication is possible on a logical basis. For example, if A is longer than B, and B is shorter than C, no implication can be made about A and C from the information given.

Reasoning concerning the measurement function has been studied in various contexts. Transitivity has been considered for its own sake as well as in comparison to other forms of reasoning such as substitution (Bailey, 1973), conservation (Carey and Steffe, 1968), classification (Johnson, 1971) and seriation (Murray and Youniss, 1968). These studies have included relationships of comparison such as length. Other studies, such as Owens (1972), have included the matching relations as well. The Owens study involved the transitive property across both matching and length relations. Studies such as Bailey (1973), Murray and Youniss (1968), Youniss and Murray (1970), Youniss and Dennison (1971), and Keller and Hunter (1973) provide information of comparative performance on tasks of a transitive and substitutive nature. The present study is concerned with replication inasmuch as further information will be gathered concerning the transitive and substitutive properties across the relations of matching and length.

Previous considerations of measurement topics (Gal'perin and Georgiev, 1975; Wagman, 1975; Carpenter, 1972) have been primarily concerned with the conception of the unit of measurement. Gal'perin and Georgiev considered the unit as it relates to other elementary mathematical notions. Wagman investigated the child's notion of a unit of area. Carpenter considered the unit of measurement and its relationship to conservation of liquid quantity. These studies failed to capitalize on the child's knowledge of the "measurements" of objects (the numbers, in terms of units, in the range of the measurement function). The present study is different from the previous studies in that it involves the child's use of numerical information from the range of the measurement function across the transitive and substitutive properties. Little information exists on this aspect of measurement.

Piaget (Flavell, 1963) has given evidence that development is crucial in the acquisition of measurement concepts. Studies such as those above confirm this claim. One purpose of the present study is to investigate the age characteristics of the measurement concepts of matching and length across the transitive and substitutive premise forms (using comparisons in the domain and range of the measurement function).

In particular, the purposes of the present investigation are: (a) to determine the young child's ability to perform logical reasoning tasks involving the measurement functions associated with collections of objects and linear physical objects; (b) to study the young child's ability to reason with the transitive and substitutive properties of relations; (c) to investigate the child's ability to operate in the range as well as the domain of the measurement function; (d) to study the effect of age on the child's performance of logical reasoning tasks; and (e) to discover interrelationships among the variables of interest.

In the past, logical reasoning in middle childhood has been extensively studied by psychologists. These studies are important for mathematics education in that they are at least relevant to children's measurement behavior. It is true, however, that close investigation of the constructs studied by the psychologists must be made in order to ascertain their applicability in mathematics education. Smedslund's (63a) study is no exception. He has argued that in order to assess concrete reasoning, one must make a clear distinction between percept, goal object, and inference pattern. Percept deals with the set of properties inherent in the stimulus situation as presented to the child. Goal object is that which the child is told to obtain, for example, number or length. An inference pattern is formed by a set of premises and a conclusion.

Transitivity, although considered to be an inference pattern by Smedslund, is not thought of as such in formal logic (except for hypothetical syllogism). The inference scheme in formal logic closest to transitivity, in the sense that Smedslund talks about it, is Modus Ponens. This scheme, when it involves transitivity, is as follows:

1. If aRb and bRc, then aRc.

2. $\dfrac{\text{aRb and bRc.}}{\therefore \text{ aRc}}$

One of the premises is a statement of transitivity of the relation "R", and the other is a conjunctive statement of two instances of the relation. What is usually assessed in tasks of transitive reasoning is the capability of a child to make the conclusion based on a knowledge of the second premise. Knowledge of the first premise is not directly assessed, but inferred upon evidence of a correct conclusion. In the work of Piaget (1952), it is not assumed that the child is consciously aware of statement forms. Rather, the statement forms are models for the child's thought. Consequently, it is too strong to say that in tasks of transitive reasoning an inference pattern is being assessed. More correctly, the behavior being observed is that of the child being able to make an implication or a conclusion. Therefore, in this study, implications involving the transitive property and substitutive property are dealt with rather than a transitive or substitutive inference.

The two implications mentioned above are logically fundamental to operational knowledge of matching and length relations by children. Only by being able to work successfully with these implications will children possess operational knowledge of the relations. Moreover, knowledge of these relations and their properties is essential due to their close relationship with measurement. The importance of including measurements of objects in the logical reasoning tasks has been pointed out by Osborne (1975). He suggested the need to examine how children tie relations and operations in the range space of the measurement function to operational definitions of relations and operations in the domain of the function.

Framework and Hypotheses

Murray and Youniss (1968) conducted a study of the child's achievement of transitive reasoning and its relation to seriation behavior. The relational category used was that of length. The sample consisted of kindergarten, first-grade, and second-grade children. As part of the study, variations on the classical transitivity paradigm were included. The purpose of inclusion of premise forms A=B and B > C, and A > B and B=C, along with the standard form A > B and B > C, was to help control for non-transitive solutions. As expected from a logical point of view, seriation behavior was found to be a prerequisite for transitive reasoning. When the three premise forms were compared, it was found that they were ordered in difficulty from least to most difficult, A > B and B > C, A > B and B=C, and A=B and B > C. These differences in difficulty suggest a hierarchical development of relational reasoning with transitivity appearing prior to substitution. Apparently, tasks using two different relations are more difficult for young children than tasks using only one relation. Of course, difficulty levels cannot be used to determine hierarchical development, but the results are suggestive.

Youniss and Murray (1970) conducted another study to investigate the effects of efforts to control non-transitive solutions for transitive reasoning tasks. An attempt was made to force use of a middle term for measurement purposes. The premise forms A > B and B > C, and A > B and B=C, were used again. Children questioned were kindergarteners, first graders, and third graders. Performance was age-related and, again, a difference in difficulties for premise forms was found. Premise forms which required the use of two relations were more difficult than premise forms which require the use of one relation.

Youniss and Dennison (1971), in a later study, tested kindergarten, first-, and third-grade children using the same three premise forms. As in the earlier studies, results again showed different difficulties for premise forms. However, the order from least to most difficult was A > B and B=C, A > B and B > C, and A=B and B > C. A study by Keller and Hunter (1973) was designed to test task variations on conservation and transitivity items. The tasks with premise forms A > B and B > C, and A=B and B > C, were of particular interest. First-grade children were used as subjects. No significant difference was found between the two types of tasks.

Studies reported in the preceding paragraph were concerned with "task variations" of transitivity problems. From a mathematical point of view, these tasks involved use of the transitive property and the substitutive property. Logically speaking, it should be the case that transitive reasoning precedes substitutive reasoning in development. Several of the studies reported indicate empirical confirmation of this hypothesis. However, the studies by Keller and Hunter (1973) and Youniss and Dennison (1971) do not show different performance for these two premise forms. Based upon the mixed available evidence, there is no reason to advance one particular hypothesis over another, using similar type tasks. However, none of the studies included tasks where the children were asked to reason on the basis of transitivity or substitution after they had physically compared physical bodies A and B, measured physical bodies B and C, compared them on the basis of the measurements, and then were required to compare A and C through reasoning. Because of the added dimension of measurement in the tasks, and the fact that some studies have shown transitive reasoning to appear before substitutive reasoning, it is hypothesized that there will be a sequential development for the premise forms in the case of measurements of B and C, where transitivity precedes substitution.

Piaget (1952) has studied the development of number and measurement. In Piaget's theory, number is derived by a synthesis of operations dealing with classes and those dealing with relations. For example, if one considers a finite collection of objects in light of their number, it is necessary (according to Piaget) to eliminate all qualities of objects so that they become identical and interchangeable. However, it is still possible to arrange objects into classes so that the classes are included in one another (serially inclusive). Although all qualities have been eliminated, the elements must somehow be kept separate or some objects might be counted twice. Using both class inclusion and serial order, (*) is contained in (**), (**) is contained in (***), and so on.

Piaget's (1952) research shows that very young children experience difficulty with both the class and serial aspects of number. When considering the class notion of number, one might present a child with two collections of cubes (12 red cubes and 5 blue cubes). A one-to-one correspondence is then set up between the blue cubes and a subset of red cubes. The child may observe this action, but still refuse to believe that the one-to-one correspondence has produced equal amounts of reds and blues. In particular, the child might comment "the reds are more, they came from a bigger pile." This "faulty" reasoning is due to the child's misconceptions concerning classes and subclasses. Misconceptions concerning serial order might take the following form: when constructing a set of objects with nine members (by adding one object at a time), the child may fail to recognize that at some point it is necessary to form a set of eight objects.

Piaget (Sinclair, 1971) suggests that the development of spatial concepts parallels that of classes, relations, and numbers. The difference is that the spatial concepts (length, etc.) involve continuous objects. In length measurement, there are several steps to be considered. First, a unit must be partitioned off and then displaced without gaps or overlaps. This corresponds to a seriation. Second, the continuous units form inclusions--one piece included in two, and so on. Therefore, measurement is constructed from a synthesis of displacement and partitioning of an additive nature. This parallels the seriation and inclusion which constitutes the number concept. Research results indicate that measurement lags behind number in development. Although the construction of ideas is parallel, the introduction of continuous objects makes the topic of linear measurement more difficult.

Even though the results of a study by Lamb (1975) do not show transitive or substitutive reasoning developing for matching relations prior to length relations, the introduction of numerical information into the tasks may affect transitive and substitutive reasoning. It should be the case that transitive and substitutive reasoning develop for matching relations prior to that of length relations when numerical information is present.

Carpenter (1972) conducted a study in which he investigated the effects of numerical cues on liquid quantity conservation. The study involved first and second graders. The results showed that children did attend to numerical cues. However, numerical distractors (incorrect numerical cues) produced approximately the same number of errors as did perceptual distractors. The results do indicate that correct numerical information may aid the young child's reasoning using transitive and substitutive premise forms, as it was the case that well over 90 percent of the subjects recognized that, in measurement, the greater number of units measured the greater amount of that quantity.

The latter results also suggest that children six years of age may be capable of performing logical reasoning tasks which involve use of numerical information. However, Piaget's (1952) work suggests misconceptions could hinder the acquisition of the capability to perform tasks using the numerical information from the range of the measurement function. Therefore, it is hypothesized that numerical cues (numbers in the range of the measurement function) will aid reasoning for children who are at a level where meaning is established for number (around seven or eight years of age). For other children, the cues will either hinder or offer no aid in reasoning.

The following are the hypotheses used in this study:

1. There is a hierarchy for the development of premise forms where transitivity precedes substitution. The evidence presented does not support this hypothesis for the comparisons involving only the domain of the measurement function. However, introduction of tasks involving use of the "measurements" for objects makes this hypothesis reasonable.

2. Evidence does not support the hypothesis of hierarchical development of matching and length relations (using only the domain of the measurement function) across the transitive and substitutive properties. However, it is hypothesized that introduction of numerical information (from the range of the measurement function) will affect performance. It will be the case that reasoning will appear for matching relations prior to that for length relations.

3. It is hypothesized that the introduction of numerical information (from the range of the measurement function) will aid the reasoning of children with a well-developed conception of number (around seven or eight years of age). For other children, the cues will either hinder reasoning or offer no aid.

Procedures

Different explanations have been offered for incorrect answers given by children in tasks of transitive reasoning. In particular, Smedslund (1963b) analyzed classical transitivity tasks and gave three reasons why children who are able to reason transitively might fail to give correct responses. They are: (a) the child misunderstands the question; (b) the child fails to make the initial comparisons correctly; or (c) the child forgets the initial comparisons. Smedslund also considered three possibilities for incorrectly inferring the presence of transitive reasoning in subjects: (a) guessing; (b) perceptual cues; and (c) the child constructs nontransitive hypotheses on his or her own (i.e., A > B so A > C without regard for comparison of B to C).

The role of memory in transitive reasoning has been studied extensively. Owens and Steffe (1972) were of the opinion that memory is not a crucial factor if children are allowed to make the initial comparisons themselves. However, a study by Roodin and Gruen (1970) was designed to measure the effect of presence of a memory aid on children's ability to make judgments of a transitive nature. This procedure involved the use of an additional comparison stick (as an aid) in tasks concerned with the transitive property of length relations. Half the children tested were allowed to use the memory aid while the other half were not. At each level (five, six, or seven years), the children using the memory aid made significantly more correct responses. These children were also able to make more correct verbal explanations of the transitive process.

Another important procedural question is that of the type of stimulus situations presented to the child. Divers (1970) presented children with three perceptual stimulus arrays: (a) neutral, where the arrangement of objects produces no apparent bias; (b) screened, where the objects are removed from direct sight at the time of response; and (c) conflictive, where the objects are arranged to give bias to the responses which are incorrect. The results of Diver's study show that children were more successful with the neutral stimulus display. Owens and Steffe (1972) used similar stimulus conditions and once again found the neutral conditions the most productive, but not significantly so over the other stimulus situations.

The final methodological variable to be discussed is that of requiring a rational, verbal explanation in accompaniment with correct response to determine presence of a cognitive structure. Brainerd (1973) presented a summary concerning this methodological dispute. He concluded that: (a) the use of explanations as sole determiner of presence or absence of cognitive structure is appropriate; and (b) the use of judgment as sole criterion seems to be most appropriate. Brainerd does suggest that the explanation could be used to advantage as an amplifier of the structures present in a child's thought. This combination of judgment and explanation criteria received support from Roodin and Gruen's (1970) study. They found that virtually all children who could give verbal justification for the transitive process also made correct responses. The converse was not true.

In the tasks constructed for the present study, children were required to make the initial comparisons, and were allowed to recompare objects when forgetting apparently took place. In regard to stimulus condition, there were no intended conflicting stimuli. However, a form of screening was used to help eliminate the possibility of judgment based solely on perceptual cues. Tasks in the present study required a verbal justification as well as judgmental response from the child.

Each child received 28 tasks, 14 matching tasks and 14 length tasks. Seven of the matching tasks contained no numerical cue and seven involved a numerical cue. A similar split was present for the length items. The items in each of the categories were designed using the following premise forms.

1. $A = B$ and $B = C$;
2. $A < B$ and $B < C$;
3. $A > B$ and $B > C$;
4. $A = B$ and $B < C$;
5. $A = B$ and $B > C$;
6. $A < B$ and $B = C$; and
7. $A > B$ and $B = C$.

The child was allowed to compare the five red and five blue discs by means of one-to-one correspondence. Upon completion, the child was asked to describe what had happened. The red discs were then screened from the child's view using a large orange sheet of cardboard. The child then compared the five blue discs with five green ones. Again the child was asked to judge the outcome. At any time, if the child established a wrong relationship, the interviewer helped to correct it. The five green discs were also screened from the child's view. The child was then asked to predict the outcome of a comparison between the red and green collections of objects. Following the subject's response, a justification was requested. If the child obviously had forgotten information, or acted in a confused manner, the task was repeated by re-establishing relationships and then continuing as before. The remaining six items in this section were constructed along similar lines.

Matching with cue tasks (using both the domain and range of the measurement function) were similar in makeup to those of matching without cue in that the child compared sets A and B by way of one-to-one correspondence. However, instead of physically comparing sets B and C, the child counted them. After counting C, the objects of C were screened from the child's view. Other aspects of the tasks were identical to those of matching without cue.

Conduction of tasks for the length categories was analogous to that of the matching tasks. However, in these tasks the comparisons were made in terms of the length of sticks. For physical comparisons, sticks were laid side-by-side. For the analogue of counting activities, the child used a ruler made up of distinguishable units; the child had to count the appropriate number units. Before proceeding to the matching and length tasks, each child was asked a series of preliminary questions. These questions were asked to determine suitability for including the child in the study. In order to eliminate color discrimination problems from the study, each child was given a test of color recognition. Six square pieces of construction paper (red, blue, yellow, green, black, and white) were placed before the child. The child was then asked to identify the colors of the squares. The order of questioning was random for each child. If there was confusion concerning the colors, there was dialogue between the subject and interviewer. For example, if the child responded "purple" for a dark blue square, the experimenter and child discussed color to see it the child would agree that the square could also be called blue. If the child still refused to call the square blue, another object of that color was tried. A child unable to respond to the color questions, even after coaxing, was excluded from the study.

The second portion of the preliminary interview consisted of determining whether the child could count a collection of objects (up to at least ten toy animals and was asked to count the objects. The animals were arranged in an approximately straight line. No attempt was made to confuse the child. A child having trouble was allowed to try the task again. A child still unsuccessful was excluded from the study.

Training tasks were then presented for the relational categories of matching and length. For the matching relations, a collection of six red blocks was presented to the child. The interviewer had a collection of six blue blocks. The interviewer and child then made pairs of blocks (one-to-one correspondence). Upon completion of the pairing, the child was questioned concerning the relation that existed between the collections of blocks. The child was asked if one person had more or if they both had the same. Incorrect responses were corrected. The child was then given six blocks and the experimenter took four. Again, after pairing, the child was asked for a judgment of the outcome. Appropriate corrective procedures were used if necessary. These experiences were used to insure familiarity with the relations of "as many as," "more than," and "fewer than." If the child was unable to make correct judgments after the training experience, the child was excluded from the study.

In length relations, the child was first presented with two sticks that were the same length. The child was then asked to compare the sticks physically and what relation existed. Corrective dialogue was used with the child if necessary. The child was then asked to compare two sticks of different lengths. These experiences were used to insure familiarity with the length relations. Failure in these tasks was grounds for exclusion from the study.

As well as receiving the aforementioned preliminary tasks, the child had experience with a unit ruler. The child placed a stick next to the ruler and counted the number of units necessary to determine the length of the stick.

The main 28 tasks were given in two sessions. During the first session, the child receiving all 14 tasks for length or matching. The decision as to which relational category was to be used first was made randomly for each child. In the second interview period, the child received the remaining 14 items in a similar manner. Within each of the four separate categories, the child was questioned randomly on the seven items. The entire interview, including preliminary tasks, was audio-taped to allow for checks on the scoring procedure. Checksheets were used to keep score.

Scores of 0, 1, or 2 were assigned for each task in the study. If the child could not respond correctly to the tasks, a score of zero was assigned. A correct response but failure to give a rational justification earned a score of one. Two was the score for both a correct judgment and a rational justification.

The study was conceived with the idea of spanning the years when children are at some stage of concrete operations. Children were selected from kindergarten, second, and fourth grades. Age restrictions were also placed on the selection of students. Kindergarten pupils were chosen so that, at the time of testing, their age was between 5.5 and 6.0 years. Second graders were selected so that the age at test time was from 7.5 to 8.0 years, while fourth graders were between 9.5 and 10.0 years. Twenty children were randomly selected from those available at each grade level, constituting a sample of 60 subjects for study.

Kindergartens were chosen from three private day schools. From the three schools used, there were 63 children engaged in the program. Of these, 30 met the age requirement; 20 students were randomly selected. The sample consisted of an all-white selection of students. Sex distribution was 13 girls and 7 boys.

Second-grade students were selected from the primary school of a small county school system. Of 266 available second graders, 97 met the age criterion; 20 students were randomly selected for the sample. Of the 20, 13 were girls and 7 were boys; 9 were black and 11 were white.

Fourth graders were selected from the elementary school in the same county. Of 286 fourth graders, 76 met the age criterion; 20 students were randomly selected. Of the 20, 10 were boys and 10 were girls; 16 were white and 4 were black.

Analysis

Initially a score of 0, 1, or 2 was assigned to each of the 28 tasks used in the study for each child. The scores 0, 1, and 2 represent a relatively arbitrary classification scheme. A categorical scaling technique (Kundert and Bargmann, 1972) was used to replace 0, 1, and 2 with scaled scores which approximate an interval scale. The assigned scaled scores were determined in such a way that differences between age bands were maximized. This was essentially a problem in discriminant analysis--what scaled scores should be assigned to columns (raw scores) so that a linear combination of these scaled scores would best differentiate between rows (age bands)?

These scaled scores were determined so as to have mean zero and variance one. Since the raw scores 0, 1, and 2 were ordinal in character, the scaled scores (i.e., the scores used in place of the 0's, 1's and 2's) should exhibit the same order. In cases where the data did not bear out this assumption (for example, if scaled scores for 0 and 1 were reversed), the reversed numbers were given the same scaled score. Kundert and Bargmann (1972) suggest the equating of inverted scaled scores, because an inconsistent result should be replaced by the nearest consistent one. Similarily, if particular cells were essentially empty, adjacent categories were combined. Wherever irregularities occurred, the scaling procedure was conducted again. The newly determined scaled scores were used in the remainder of the analysis.

The three major hypotheses proposed are repeated below:

1. There is a hierarchy for the development of premise forms where transitivity precedes substitution. The evidence presented does not support this hypothesis for the comparisons involving only the domain of the measurement function. However, introduction of tasks involving use of the "measurements" for objects makes this hypothesis reasonable.

2. Evidence does not support the hypothesis of hierarchical development of matching and length relations (using only the domain of the measurement function) across the transitive and substitutive properties. However, it is hypothesized that introduction of numerical information (from the range of the measurement function) will affect performance and it will be the case that reasoning will appear for matching relations prior to that for length relations.

3. It is hypothesized that the introduction of numerical information (from the range of the measurement function) will aid the reasoning of children with a well-developed conception of number (around seven or eight years of age). For other children, the cues will either hinder reasoning or offer no aid in reasoning.

In order to test hypothesis one, scores for the following variables were formed (by combining individual scaled scores):

1. Matching no cue - Transitivity (MNCT)--scores from the three matching transitivity items with no measurements involved.

2. Matching no cue - Substitution (MNCS)--scores from the four matching substitution items with no measurements involved.

3. Matching with cue - Transitivity (MCT)--scores from the three matching transitivity items with measurements involved.

4. Matching with cue - Substitution (MCS)--scores from the four matching substitution items with no measurements involved.

5. Length no cue - Transitivity (LNCT)--the length transitivity items with no measurements involved.

6. Length no cue - Substitution (LNCS)--the length substitution items with no measurements involved.

7. Length with cue - Transitivity (LCT)--the length transitivity items with measurements involved.

8. Length with cue - Substitution (LCS)--the length substitution items with measurements involved.

These composite scores were used in an analysis of variance across grade levels (age bands). Inspection of the means from the analysis of variance (for differences and sequence) was conducted. The transitivity means (MNCT, MCT, LNCT, and LCT) were inspected for amount of increase (or decrease) from grade to grade. The substitution means (MNCS, MCS, LNCS, and LCS) were inspected in a similar manner. If the data were to support the hypothesis of earlier development for transitivity over substitution, the means for the transitivity items should show a lesser increase, from grade to grade, than the substitution items, especially for items involving numerical cues.

Hypothesis number two, comparison of matching and length relations, was tested in a similar manner. The means for matching (MNCT, MNCS, MCT, and MCS) were compared with the means for length (LNCT, LNCS, LCT, and LCS). If the hypothesis of earlier development for matching relations over length relations were borne out by the data, the increase in means, from grade to grade, should be less for the matching items than for length items, especially for the items involving numerical cues.

In order to test hypothesis number three, the means of items involving no numerical cues were compared with the means of items involving numerical cues. If the data were to support the hypothesis, the means for the items involving numerical cues (MCT, MCS, LCT, and LCS) should show a sharper increase, from grade to grade, than the means for the items involving no numerical cues (MNCT, MNCS, LNCT, and LNCS).

Critical F-values for the ANOVA's were computed in the traditional manner at the .05 level of significance. As part of the scaling procedure, an analysis of variance (across grades) was run for each individual task variable. This information helped to determine which variables best discriminate between grades, The critical F-values for the ANOVA's were computed in relation to the maximum characteristic root distribution as suggested by Kundert and Bargmann (1972). Heck charts were used as an aid in this computation (Morrison, 19(7), Significance was determined at the .05 level.

Results

Table 1 contains the scaled scores for all of the tasks except LC7. LC7 was dropped because inspection of Table 2 revealed nodifferences between grades. The scaled scores are presented in order to give a listing of the scores to be used in further analysis of the data. The contingency tables (Table 2) give the distribution of raw scores for each variable for each of the 28 tasks across grades:

1. Transitivity
2. Substitution
3. Matching
4. Length
5. No cue
6. Cue

The raw scores were used in determination of scaled scores.

Inspection of the scaled scores reveals that in nine of 27 cases, it was necessary to collapse the categories for the raw scores of 0 and 1. The necessity of collapsing categories for 0 and 1 suggests the importance of justification in assessing performance in logical reasoning tasks. This fact is further supported by inspection of the contingency tables.

TABLE 1

Scaled Scores

item	Raw Score	0	1	2
MNC 1	(A = B and B = C)	-1.53	-1.53	.65
MNC 2	(A < B and B < C)	-1.11	-1.11	.90
MNC 3	(A > B and B > C)	-1.22	-1.22	.82
NNC 4	(A = B and B < C)	-1.74	-1.12	.78
MNC 5	(A = B and B > C)	-1.77	-1.05	.80
MNC 6	(A < B and B = C)	-1.03	-1.03	.97
MNC 7	(A > B and B = C)	-1.53	-1.00	.78
MC 1	(A = B and B = C)	-1.31	-1.31	.76
MC 2	(A < B and B < C)	-3.70	.07	.40
MC 3	(A > B and B > C)	-2.73	- .20	.68
MC 4	(A = B and B < C)	-1.70	-1.00	.72
MC 5	(A = B and B > C)	-2.46	-1.26	.69
MC 6	(A < B and B = C)	-1.37	-1.23	.79
MC 7	(A > B and B = C)	-1.41	-1.41	.71
LNC 1	(A = B and B = C)	-2.28	- .80	.70
LNC 2	(A < B and B < C)	-1.72	- .81	.93
LNC 3	(A > B and B > C)	-1.35	- .90	.96
LNC 4	(A = B and B < C)	-1.14	-1.14	.87
LNC 5	(A = B and B > C)	- .94	- .94	1.07
LNC 6	(A < B and B = C)	-1.93	.07	.71
LNC 7	(A > B and B = C)	-1.50	- .80	.85
LC 1	(A = B and B = C)	-1.14	-1.14	.87
LC 2	(A < B and B < C)	-1.30	- .53	1.27
LC 3	(A > B and B > C)	- .82	- .82	1.22
LC 4	(A = B and B < C)	-1.60	- .82	1.01
LC 5	(A = B and B > C)	-1.34	-1.09	.90
LC 6	(A < B and B = C)	-1.12	- .66	1.17

Table 2

Contingency Tables

MNC 1

(A = B and B = C)

Grade	0	1	2	Totals
K	2	5	13	20
2	2	6	12	20
4	0	3	17	20
Totals	4	14	42	60

MNC 2

(A < B and B < C)

Grade	0	1	2	Totals
K	0	13	7	20
2	0	12	8	20
4	0	2	18	20
Totals	0	27	33	60

MNC 3

(A > B and B > C)

Grade	0	1	2	Totals
K	2	8	10	20
2	0	12	8	20
4	0	2	18	20
Totals	2	22	36	60

MNC 4

(A = B and B < C)

Grade	0	1	2	Totals
K	3	8	9	20
2	2	8	10	20
4	0	2	18	20
Totals	5	18	37	60

MNC 5

(A = B and B > C)

Grade	0	1	2	Totals
K	4	7	9	20
2	1	10	9	20
4	0	2	18	20
Totals	5	19	36	60

MNC 6

(A < B and B = C)

Grade	0	1	2	Totals
K	6	6	8	20
2	3	10	7	20
4	2	2	16	20
Totals	11	18	31	60

MNC 7

(A > B and B = C)

Grade	0	1	2	Totals
K	4	6	10	20
2	7	5	8	20
4	0	1	19	20
Totals	11	12	37	60

MC 1

(A = B and B = C)

Grade	0	1	2	Totals
K	3	7	10	20
2	3	7	10	20
4	1	1	18	20
Totals	7	15	38	60

MC 2

(A < B and B < C)

Grade	0	1	2	Totals
K	4	7	9	20
2	0	11	9	20
4	0	5	15	20
Totals	4	23	33	60

MC 3

(A > B and B > C)

Grade	0	1	2	Totals
K	5	8	7	20
2	1	9	10	20
4	0	6	14	20
Totals	6	23	31	60

MC 4

(A = B and B < C)

Grade	0	1	2	Totals
K	8	4	8	20
2	2	7	11	20
4	0	0	20	20
Totals	10	11	39	60

MC 5

(A = B and B > C)

Grade	0	1	2	Totals
K	2	9	9	20
2	0	8	12	20
4	0	1	19	20
Totals	2	18	40	60

Table 2 continued

MC 6

(A < B and B = C)

Grade	0	1	2	Totals
K	4	9	7	20
2	2	8	10	20
4	0	0	20	20
Totals	6	17	37	60

MC 7

(A > B and B = C)

Grade	0	1	2	Totals
K	3	7	10	20
2	3	5	12	20
4	1	1	18	20
Totals	7	13	40	60

LNC 1

(A = B and B = C)

Grade	0	1	2	Totals
K	2	7	11	20
2	4	6	10	20
4	0	3	17	20
Totals	6	16	38	60

LNC 2

(A < B and B < C)

Grade	0	1	2	Totals
K	3	11	6	20
2	3	7	10	20
4	0	5	15	20
Totals	6	23	31	60

LNC 3

(A > B and B > C)

Grade	0	1	2	Totals
K	5	9	6	20
2	2	9	9	20
4	1	3	16	20
Totals	8	21	31	60

LNC 4

(A = B and B < C)

Grade	0	1	2	Totals
K	4	7	9	20
2	4	8	8	20
4	1	2	17	20
Totals	9	17	34	60

LNC 5

(A = B and B > C)

Grade	0	1	2	Totals
K	4	9	7	20
2	6	8	6	20
4	2	3	15	20
Totals	12	20	28	60

LNC 6

(A < B and B = C)

Grade	0	1	2	Totals
K	3	7	10	20
2	8	5	7	20
4	1	5	14	20
Totals	12	17	31	60

LNC 7

(A > B and B = C)

Grade	0	1	2	Totals
K	6	5	9	20
2	3	8	9	20
4	1	3	16	20
Totals	10	16	34	60

LC 1

(A = B and B = C)

Grade	0	1	2	Totals
K	6	8	6	20
2	3	6	11	20
4	2	1	17	20
Totals	11	15	34	60

LC 2

(A < B and B < C)

Grade	0	1	2	Totals
K	6	13	1	20
2	4	9	7	20
4	0	6	14	20
Totals	10	28	22	60

LC 3

(A > B and B > C)

Grade	0	1	2	Totals
K	2	13	5	20
2	0	12	8	20
4	0	9	11	20
Totals	2	34	24	60

Table 2 continued

	LC 4 (A = B and B < C)				LC 5 (A = B and B > C)				LC 6 (A < B and B = C)			
Grade	0	1	2	Totals	0	1	2	Totals	0	1	2	Totals
K	4	11	5	20	1	13	6	20	5	9	6	20
2	1	12	7	20	1	10	9	20	6	8	6	20
4	0	3	17	20	0	2	18	20	2	5	13	20
Totals	5	26	29	60	2	25	33	60	13	22	25	60

	LC 7 (A > B and B = C)			
Grade	0	1	2	Totals
K	[illegible]	8	4	20
2	6	7	7	20
4	7	4	9	20
Totals	21	19	20	60

Only a small percentage of the responses were given a raw score of 0 in the original scoring. An overwhelming number of students were able to respond correctly. Without justification, it would have been difficult to discriminate between groups in the study.

Brainerd (1973) made a strong case for using both response and justification as opposed to requiring only a response when assessing the child's performance on logical reasoning tasks. This position is supported by the present data. The requirement of a rational justification made it possible to gain more insight into children's reasoning processes. If it had not been used, much valuable information might have been lost.

Consideration of Tables 1 and 2 suggests certain trends in the data. A comparison of transitivity raw scores (Table 1) with substitution raw scores (Table 2) indicates that the percentage of children receiving 0's was greater on the substitution items than on the transitivity items, and the percentage of children receiving 1's was less for the substitution category than for the transitivity category. This result gives some indication that, on the whole, transitivity items are slightly less difficult for children than are the substitution items. A similar comparison of matching and length items indicates that children have less difficulty with the matching items. Comparison of the tables for cue items vs. no cue items indicates that, on the whole, there is approximately the same level of performance.

The fact that, on the whole, transitivity items appear to be less difficult for children is attributable to the fact that in a transitive item the child is required to reason with only one relation at a time. However, in substitutive tasks, two relations are being considered simultaneously. This indicated trend in the data is consistent with the results of Murray and Youniss (1968) and Youniss and Murray (1970), but inconsistent with the data of Youniss and Dennison (1971) and Keller and Hunter (1973).

Matching relations items appear easier for children (on the whole) than length relations items. This result is consistent with the discrete objects vs. continuous objects (length items) discussion presented earlier. Use of continuous objects makes the logical reasoning tasks more difficult for children than is the case for discrete objects (matching items).

Similar performance for no cue vs. cue items is attributable to the fact that the advantage obtained by older children (who could use the cue effectively) may have been negated by the children who were confused by the numerical cue. For the children who had an incomplete conception of number, the introduction of numerical cues may have hindered their performance. The selected age bands may have affected this result as well.

The combined scores MNCT (Matching no cue transitivity), MNCS (Matching no cue substitution), LNCT (Length no cue transitivity), LNCS (Length no cue substitution), LCT (Length cue transitivity), and LCS (Length cue substitution) were used in an analysis of variance design across grade levels (age bands). F-values are presented in Table 3. The critical value for F at .05 is 3.16.

Table 3

ANOVA (Combined Variables)

Variables	F
MNCT	7.62*
MNCS	11.02*
MCT	9.36*
MCS	14.22*
LNCT	6.61*
LNCS	7.50*
LCT	9.89*
LCS	12.18*

*(P < .05)

The results of the analysis of variance clearly show that performance on logical reasoning tasks is age related. The means (by grade level) are presented in Table 4. It should be noted that, in some cases, the means do not present an ordered sequence from least to greatest for grades K to 2 to 4. The three cases where this occurs are involved with no cue variables. The cue items do present an ordered sequence for grades K to 2 to 4. Apparently, the introduction of numerical cues aided the second-graders. This point will be discussed further in consideration of the three major hypotheses of the study.

Table 4

Means (ANOVA--Combined Variables)

	K	2	4
MNCT	- .72	- .93	1.64
MNCS	-1.17	-1.29	2.47
MCT	-1.34	.06	1.29
MCS	-2.00	- .59	2.60
LNCT	-1.00	- .53	1.54
LNCS	- .68	-1.36	2.03
LCT	-1.52	- .09	1.60
LCS	-1.25	- .67	1.92

Hypothesis 1

It was hypothesized that transitive reasoning would develop earlier than substitutive reasoning. If this hypothesis were to be borne out by the data, the increase in means from grade to grade (especially from K to 2) would be smaller for transitivity than for substitution. The transitive and substitutive means are presented in Tables 5 and 6.

Table 5

Transitivity Means (Combined Variables)

Variables	K	Grades 2	4
MNCT	- .72	- .93	1.64
MCT	-1.34	.06	1.29
LNCT	-1.00	- .53	1.54
LCT	-1.52	- .09	1.60

Table 6

Substitution Means (Combined Variables)

Variables	K	Grades 2	4
MNCS	-1.17	-1.29	2.47
MCS	-2.00	- .59	2.60
LNCS	- .68	-1.36	2.03
LCS	-1.25	- .67	1.92

For the matching no cue categories, both transitivity and substitution show reversals in means from grades K to 2. In the matching cue categories, the increases are approximately the same. On the length of no cue items, transitivity means increased while substitution means showed reversal of order. On length cue items, the transitivity increase was larger than the substitution increase. The results do not clearly support the hypothesis as stated. There is insufficient evidence to conclude that transitivity develops earlier than substitution. The data are at odds with that of Bailey (1973). This apparent discrepancy in results may be due to the differences in task design. (Bailey used polygonal paths, constructed of several sticks for comparison.)

Hypothesis 2

It was hypothesized that performance for matching relations would develop earlier than performance for length relations. If the data were to bear out this hypothesis, the increases in means from grade to grade (especially from K to 2) would be smaller for matching than for length. The matching and length means are presented in Tables 7 and 8.

Table 7

Matching Means (Combined Variables)

Variables	K	2	4
MNCT	- .72	- .93	1.64
MNCS	-1.17	-1.29	2.47
MCT	-1.34	.06	1.29
MCS	-2.00	.59	2.60

Table 8

Length Means (Combined Variables)

Variables	K	2	4
LNCT	-1.00	- .53	1.54
LNCS	- .68	-1.36	2.03
LCT	-1.52	- .09	1.60
LCS	-1.25	- .67	1.92

As in hypothesis 1, there is no clear trend for earlier development of one category over another. For the transitive variables, matching no cue shows a reversal in means, whereas the length no cue variable shows an increase of .47. For no cue substitution, both matching and length variables show a reversal in means. With the cue transitivity variables the gains are similar, while for cue substitution the matching gain is larger than the length gain. When reversals occur, it is difficult to determine differences. As Kundert and Bargmann (1972) suggest, the reasonable approach is to replace an inconsistent result with the nearest consistent one (equate the means in this case; producing no gain). There is insufficient evidence to confirm the hypothesis of earlier development for matching relations over length relations.

Hypothesis 3

It was hypothesized that the introduction of numerical cues would aid the reasoning of children who have a well-organized conception of number (around second grade) and would possibly hinder or not aid the younger children. If the hypothesis were to be borne out by the data, the means for the cue items (MCT, MCS, LCT, and LCS) would show a sharper increase from grade to grade than the no cue means (MNCT, MNCS, LNCT, and LNCS). The no cue and cue means are presented in Tables 9 and 10.

Table 9

No Cue Means* (Combined Variables)

Variables	K	2	4
MNCT	- .72	- .93	1.64
MNCS	-1.17	-1.29	2.47
LNCT	-1.00	- .53	1.54
LNCS	- .68	-1.36	2.03

Table 10

Cue Means (Combined Variables)

Variables	K	2	4
MCT	-1.34	.06	1.29
MCS	-2.00	- .59	2.60
LCT	-1.52	- .09	1.60
LCS	-1.25	- .67	1.92

In this case, the trend is clearly established. The cue variables show sharper increase, in all cases, than the no cue variables. This is consistent with the hypothesis as stated. The introduction of numerical cues (numbers from the range of the measurement function) apparently aids the child of approximately second-grade level (between 7.5 and 8.0 years of age). The sharpness of the increases in means from grades K to 2 indicates the possibility that cue information hinders young children whose number concepts are not clearly established.

Discussion

Previous studies such as Bailey (1973), Murray and Youniss (1968), and Youniss and Murray (1970) had indicated that a hierarchical development for premise forms should exist. These studies were at odds with the results of Youniss and Dennison (1971) and Keller and Hunter (1973). Due to the introduction of numerical cues into logical reasoning tasks, it was hypothesized that reasoning in case of the transitive premise form would develop earlier than reasoning for the substitutive premise form. The hypothesis was not supported by the data of this study which indicate no precedence in the development of premise forms. As discussed earlier, possible explanation for conflicting results is difference in task design.

It was hypothesized that the introduction of numerical cues would enable the child to reason logically with matching relations before length relations. The hypothesis was not confirmed by the data from the study. The result of no precedence for one relational category over another is now generalizable to tasks involving measurements as well as those involving no measurements.

Carpenter's results (1972) suggested that children could use numerical information in measurement situations. On that basis, it was hypothesized here that introduction of numerical cues would aid the logical reasoning of children around seven or eight years of age. This hypothesis was firmly supported by the data. Children of seven or eight years of age or older are capable of thinking in terms of both the domain and range of the measurement function in that they are able to use the function and its properties in order to perform logical reasoning tasks of a transitive and substitutive nature. The presence of numerical information significantly aids children in logical reasoning tasks if the children are at an age where they most likely have a true understanding of number. Results indicated that very young children might be hindered by such numerical information.

Suggestions and Recommendations

As with most other research studies, a portion of this report gives direction for further investigation into the topics of logical reasoning and measurement:

1. Studies should be designed to investigate the child's ability to operate with logical premise forms in the domain and range of the measurement function. This should be done using the functions for number (counting) and length. A study of this type would serve to replicate the present study. Following this, it would be advisable to conduct studies using different measurement functions such as area, volume, and weight. Investigation of measurement functions such as area, volume, and weight would help to provide important insights into the understanding of the child's acquisition of measurement ideas.

2. Lamb (1975) used an incompatible premise form in his study of the functions for number (counting) and length. Studies should be done which use this logical reasoning form across measurement functions such as area, volume, and weight. The Lamb study was done using only relations from the domain of the given measurement functions. Introduction of numerical cues would provide new and interesting information.

3. Replication studies could be conducted with the data being subjected to a factor analysis. These results would show important interrelationships among the variables of interest.

4. The effects of item design (task construction) as a variable in this and related studies should be studied more specifically.

Based upon the evidence from the present study, the following classroom recommendations are in order:

1. Teachers should expect similar development of the matching and length relation as age increases. Likewise, similar developmental characteristics for the transitive and substitutive properties exist. As teachers spend time with children who are acquiring relational properties, it is appropriate to give experiences of a varied nature across the relations of matching and length as well as across the premise forms of transitivity and substitution.

2. As children gain experience and competence with the various aspects of number, the introduction of numerical cues into the measurement process will enhance the child's ability to reason logically. This is true for at least the transitive and substitutive properties of matching and length relations. The younger children did not benefit from introduction of numerical information. In fact, numerical cues may have hindered the younger children on the tasks. This latter point indicates teachers should be on the alert if they use measurement as the basis for number acquisition, as they may impede progress in number development. In particular, aids such as the number line and other length models for number (sticks, rods, etc.) should not be used too early in the elementary school mathematics curriculum.

3. Based on 1 and 2 above, one might suggest that children should gain experience in making comparisons, ordering, subdividing, and interating. That is, development of a unit may be essential before a child can make good use of numerical information in measurement situations. Note that all of these behaviors relate either directly or indirectly to the tasks used in this study.

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THE RATIONAL NUMBER CONSTRUCT--ITS ELEMENTS AND MECHANISMS

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I. Constructs and Mechanisms

The term mathematical concept is used in many ways. It can refer to an object or a class of mathematical objects. Most frequently in mathematics a concept is associated with a formal defining statement. Thus, a rational number is "any number x which satisfies $ax = b$ where a and b are integers $(b \neq 0)$."

Yet such a definition does not tell as much about the notion of rational numbers particularly as it exists as personal knowledge. "Knowing" rational numbers can mean a large number of things. In fact, Wagner (1976) suggests that for the person rational numbers should be a mega-concept involving many interwoven strands.

Margenau (1961), the eminent philosopher of science, has analyzed the cognitive component of such complex knowing. He sees knowledge as a continuum between two extreme types--facts, which apparently exist independently of our control, and abstract concepts, which owe their existence purely to human invention. To avoid certain logical and psychological pitfalls, Margenau sees all knowledge as attributable to human construction, but sees these constructs as bounded by and rooted in the realm of facts which, he suggests, function as protocols against which our ideas or constructs are functionally tested. Hence Margenau would picture knowledge (in our case rational number knowledge) in the following way:

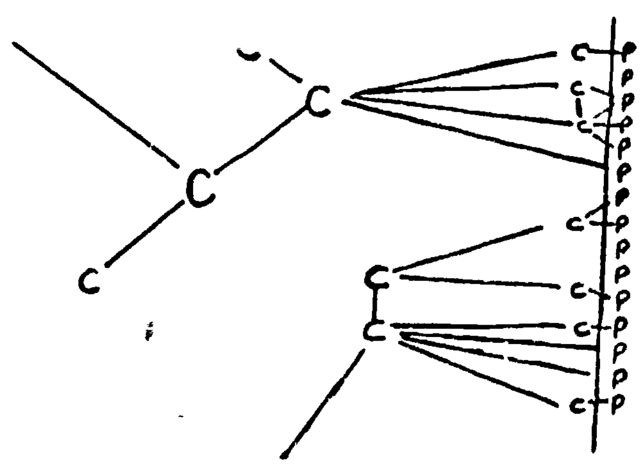


Figure 1

Several things are obvious from Figure 1. Some constructs are very close to the P-plane; these act directly on the plane but have little or no explanatory power. For example, knowledge of the algorithm for adding fractions acts on a subset of the P-plane, but may be a disconnected piece of knowledge with little explanatory power.

Other constructs are more distant from the P-plane. These constructs generally have more explanatory power and are connected in a wide sense. These connections are of great importance, for they allow for empirical verification in scientific terminology or application and problem solving in the nomenclature of mathematics instruction. A problem, the purchase of 3.5 metres of cloth at $4.75/m, arises in the P-plane. The solution is arrived at through traversing a path among constructs (ratios, multiplying decimals, relating decimals and money) and arriving at a dollar figure back in the P-plane.

Van Engen (1953) has described this phenomenon from a mathematical-psychological perspective using the notions of "meaning" and "understanding." In terms of Figure 1 above, "meaning" applies to the process of building up or developing the elements in the C field. "Understanding" applies to the development and maintenance of the interconnections and more particularly the use of the paths which allow the application of ideas back into the P-plane. Two criteria for constructs which optimize applications are inter-connectedness and extensibility. That is, the constructs are both connected to many other constructs and apply to a large segment of the P-plane.

In light of this rather complicated picture of knowledge developed above, what does it mean to "know" rational numbers? Put plainly, what is it that a person must functionally know about rational numbers to be numerate? In the plane of protocols, rational numbers are involved in representing and controlling part-whole situations and relationships. Rational numbers are fundamental to measuring continuous quantities. If quantities, particularly those continuous, are divided, rational numbers are involved. Finally, rational notions are involved in any quantitative comparisons of two qualities (ratios). Thus one's general rational number construct should allow a person to control such P-plane events.

At a construct level, knowing rational numbers entails control over two-dimensional symbols in various forms (fractions, decimals). Operations on rationals, while at a low level involve knowledge of conventional algorithms, more generally entail control of primitive forms of vector addition and function composition. Knowledge of rationals also requires functional capability with equivalence classes and quotient fields. Such constructs also entail connections with those of earlier natural number notions and a more general construct of number which also includes the real numbers.

Given the above description of rational number knowledge, the question of its acquisition arises. With natural or counting number knowledge, this is at least partly a natural induction process. A child, before and outside of school, has a large number of contacts with situations to which natural number applies. Thus, school mathematics can build upon the natural knowledge, both in terms of constructs and protocols (quantifying discrete situations: more than, less than, sorting, counting). This requires elaborating and generalizing the counting mechanisms and thence moving to primitive algebraic (e.g., ordering) and numeration constructs to help a child develop a more extensive control over discrete quantitative situations.

The experience base of school children with respect to rational number ideas is much more limited. This is true both in terms of contact with the quantification of continuous phenomenon and the language of rational numbers. (There is very little contact with fraction words beyond "half," "third," "quarter," and "percent" even for children of age 11 or 12). Thus the process of developing mechanisms for building rational number concepts presents the school with a more complete task to accomplish. In addition, the general rational number construct is a more inclusive one than that of natural numbers. Hence school must provide children with experience with mechanisms (such as partitioning to be explored more fully below) in a variety of construct contexts (e.g., rational numbers as measures, to be elaborated below), as well as providing elementary language experience with words relating to fractional phenomenon.

The remainder of this essay discusses the attempts of instruction over the past century and a half to provide rational number experiences. After analyzing these attempts (both of "old" and "new" mathematics), a picture of a more complete rational number construct is developed and elaborated. The essay concludes with a discussion of the implications of this construct of rational numbers, that is, that which a person knows when he or she can function maturely with rational numbers.

1. The "Old" Mathematics Constructs

De Morgan (1943), writing in 1831 for the Society for the Dissemination of Useful Knowledge, stated that even then fractions were a topic immensely difficult to learn. To alleviate this difficulty, De Morgan showed a method by which fraction knowledge could be developed as an extension of whole number knowledge. His vision of the fraction construct was a set of computational algorithms and his development focused on these, particularly addition. (Actually De Morgan's development of addition was quite "modern").

For the 120 to 130 years following De Morgan's writing, mathematics instruction addressed itself to building up the same construct as De Morgan saw for fractions--the task of computation with fractions and decimals. As schooling became universal and occurred for a longer period of a person's life, fractions came to occupy a substantial position in the mathematics curriculum of what we now know as the middle school. Research (Kieren, 1976) focused on a detailed analysis of computation tasks. As pictured below, rational number work (fractions) was seen to derive from whole number computation and base 10 numeration.

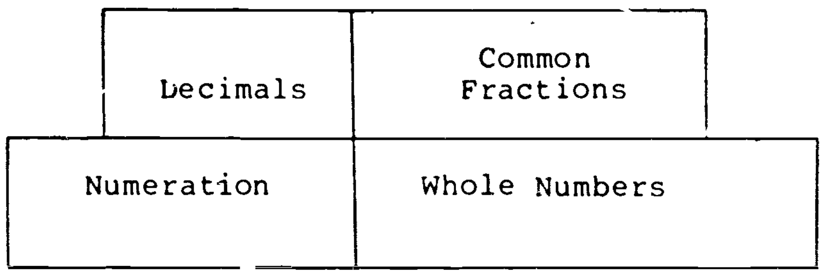


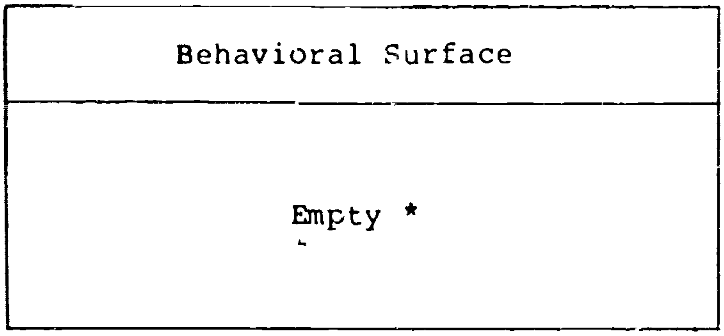
Figure 2

The detailed analysis of computational tasks into atomistic sub-tasks as well as the everyday observation of the "fraction" curriculum in action indicated that this instructional approach focused on what we might term a "behavioral surface" of the rational number construct. For example, the "unlike denominators" division task was seen as very difficult. One of the bases for this difficulty was a need for the learner to use equivalent fractions. Yet there was no stress in the "old math" on the general construct of equivalence, nor was there an attempt to consider rational numbers in their algebraic framework.

Margenau (1961) discusses a similar problem in the area of science when he suggests:

> The errors we are endeavoring to expose originate in a disregard of theory, in a belief that facts have feet on which they can stand. Actually, they are supported in a fluid medium called theory, or theoretical interpretation, a medium which prevents them (facts) from collapsing into insignificance. (p. 29)

Applying this thinking to the problem of a developing rational number construct, the picture below emerges.



*Little or no knowledge of the objects or structure of rational numbers; also limited experience with language development.

Figure 3

One effect of this "empty" construct--that is, one devoid or lacking in higher level support constructs--is a collapse in a person's functional ability with fractions and rational numbers. The behavioral surface breaks down or at least exhibits severe cracks in the form of poor performance or rational number tasks. As suggested by De Morgan's comments cited earlier, this has been an age-old problem.

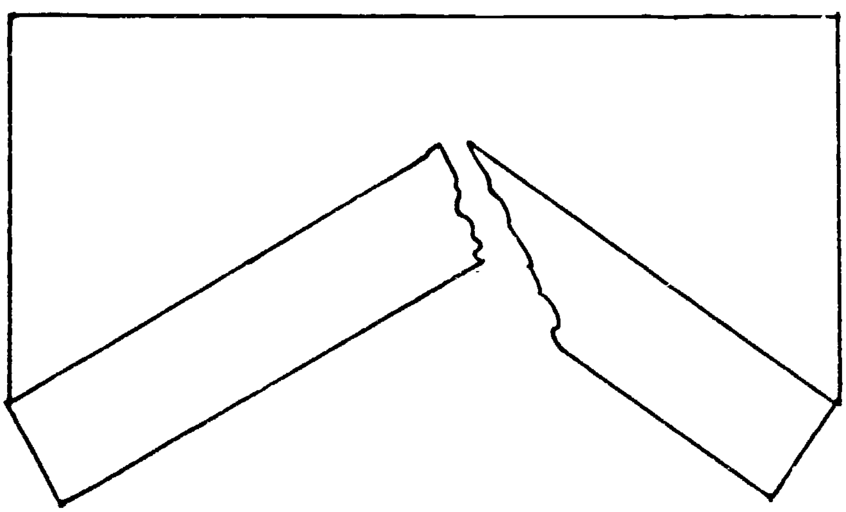


Figure 4

2. <u>The New Mathematics Constructs</u>

The "modern" mathematics movement sought to alleviate this problem by giving some depth to the rational number constructs of children and adolescents. One of the curricular mechanisms was to have children interact and build up their own ideas of the mathematical structure (i.e., fields) underlying the rational numbers. The movement also addressed itself (in a way) to the language aspect by clarifying (or attempting to clarify) the distinction between numbers and their names. Thus, a rational number, 2/3, could have many names--4/6, .6, 18/27, 66 2/3 percent. It was hoped that this careful development of structure and the related language would provide a sound basis for the rational number concept.

The structure manifest in the "new" mathematics curriculum was that of the quotient field. Explicit instruction was given to the construct "equivalence" and attention given to the solution of equations of the form $ax + b = c$. The rational number construct of "new mathematics" might be pictured as in Figure 5:

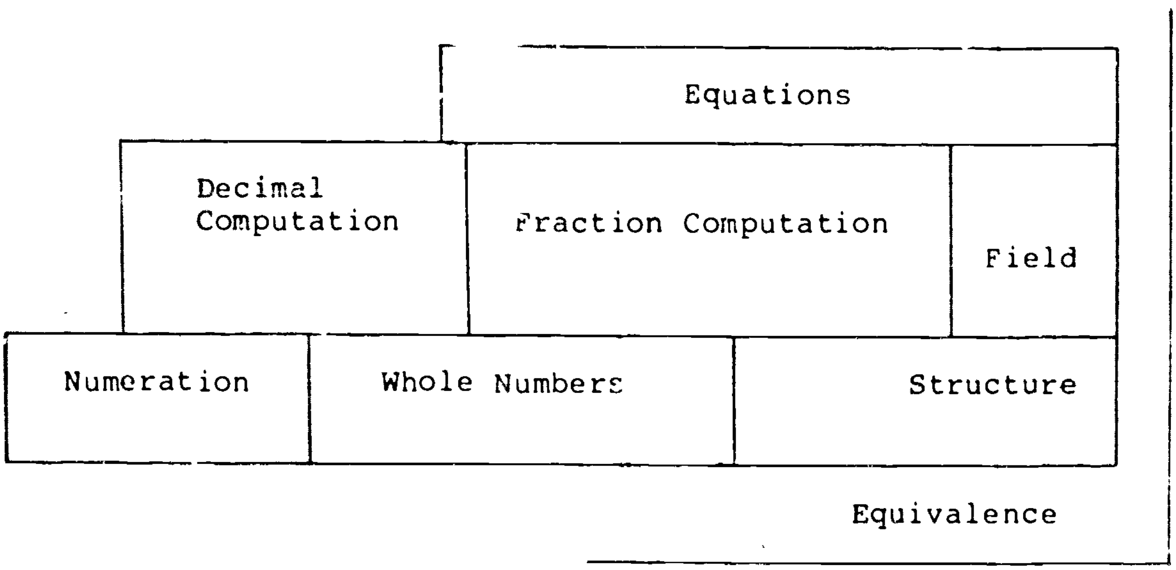


Figure 5

In comparing textbooks of 1975 with those of a century earlier, Kieren (1976) saw no essential differences. In light of the above discussion, this observation is shown to be a half-truth. It is certainly not true that the construct of rational numbers addressed by 1975 textbooks is identical to that of 1875 textbooks. What is true is that emphasis in many current curriculums is a reduction of Figure 5 to one of the following configurations (see Figures 6 and 7).

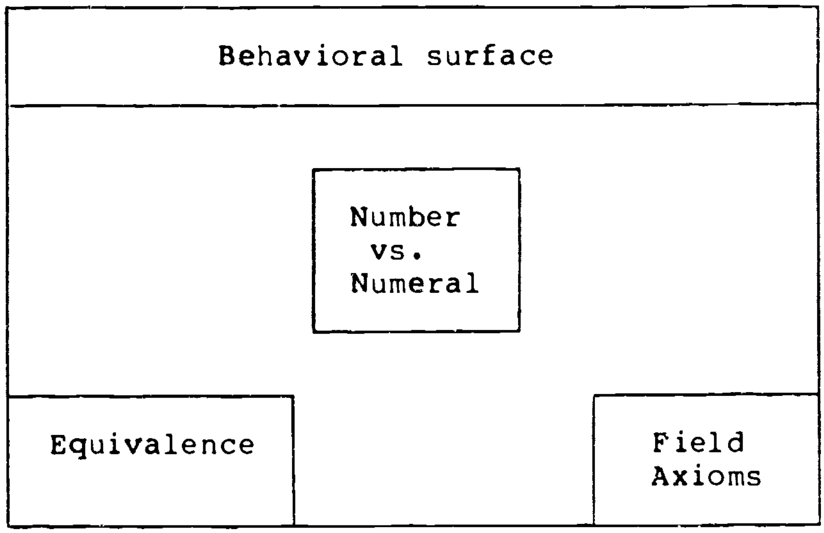


Figure 6

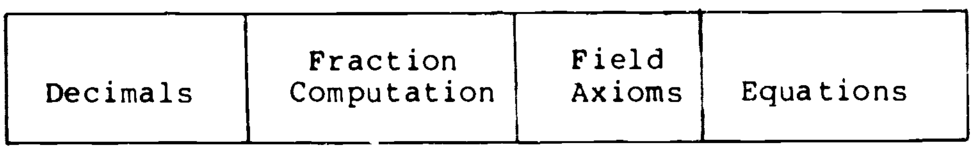


Figure 7

Figure 6 represents a situation in which basic constructs are developed (and taught) in isolation from one another, from the behavioral surface of computation and from the functional reality of mathematics as it is applied. Thus, the theoretical constructs developed are destined to become unused relics in the mind of the learner. A situation such as depicted in Figure 6 has often led to a curriculum having as its view a rational number construct as pictured in Figure 7. This is an extended behavioral surface, with more or less factual knowledge of axioms and equations appended to the "old math" surface. (Indeed some critics would say that elements of the fractional and decimal components have been replaced.) Thus, particularly in some recent curriculum objectives lists, the intent of a complete rational number construct in reality is a "surface" of new facts with little more support than the "old" mathematics construct.

The weakness of such a behavioral surface construct has been predicted above and should manifest itself in relatively poor performance by adolescents and adults on rational number tasks and settings. The reality of such poor performance has been documented for a long time. The recent NAEP data suggest that, while adolescents are functional with whole numbers, their performance on fractions tasks is at a much lower level (Carpenter et al., 1975). Even adolescent students in good programs considered to be "modern," at least by design, do not perform well on rational number tasks. This fact is documented by Ginther, Ng, and Begle (1976) in the findings of their survey of 95 eighth-grade mathematics classes. Although the above research was not done in such a way as to directly provide proof, the data are indicative that even instruction toward an "extended behavioral surface" construct of rational numbers does not indicate mature functioning on the part of the older adolescent.*

3. Alternatives

One reaction to the prolonged history of poor results in rational number instruction is that the rational number construct as developed above is accessible only to more mature students. Thus, one plausible alternative is the postponement of rational number instruction until the secondary school. Put more generally this hypothesis might read as follows:

> Instruction in rational numbers should be postponed until the student has reached the stage of formal operations.

This hypothesis and its curriculum implications are not new. Washburne (1930) suggested delaying the teaching of the meaning of fractions where groups had to be considered as units until the age of 11 years 7 months. The intent of this suggestion was to sequence instruction so as to allow for the mastery of the tasks involved. From a very different point of view, Freudenthal (1973) argues for the postponement of the teaching of the addition algorithm for rationals until it can be developed as a consequence of algebraic ideas from which it arises (see Kieren, 1976, pp. 118-120).

The first of the suggestions above appears to suggest that older students will be able to induce the broader construct of rational numbers even from a curriculum based on a "behavioral surface" view of the construct. The second

*It should be noted that most curriculums in mathematics and fractions and rational numbers were not developed using an analysis of how children or adolescents "thought" about the subjects at hand or how they could go about building up systematic mechanisms for developing desired skills, concepts, or abilities.

suggests that rational numbers, in operation at least, derive their meaning from algebraic structure and hence instruction should be based on providing students the opportunity to deduce rational number constructs from more general ones using the mechanisms of logic.

II. A More "Complete" Rational Number Picture

An important assumption in the above calls for "postponement" is that older adolescents will be able to develop functional rational number constructs even from a limited instructional basis. Yet from Margenau's point of view, it is questionable whether a construct, developed from a narrow instructional base, will be logically potent or extensive enough to be practical and viable.

Given the history of rational number instruction over the past 150 years, or over the past 20 years for that matter, are there alternatives to the "postponement" hypothesis stated above? Generally, one might say that better instruction for younger children might be an alternative. But what is the basis of such instruction? How can it be directed toward the development of a functional construct, potent and extensive? To do so, the basis for improved fractional and rational number instruction needs to take into account Wagner's (1976) view of the rational numbers as a mega-concept. That is, instruction needs to address itself implicitly to the many components or strands which comprise the rational number construct. In addition, such instruction needs to consider the interrelationships among the major components or strands.

In an analysis of rational numbers, Kieren (1976) suggested seven interpretations for fractional and rational numbers:

- fractions
- decimals
- ordered pairs (equivalence classes)
- measures
- quotients
- operators
- ratios

This analysis further suggested that these interpretations were or should be isomorphic. From the point of view of mathematical structure, this trivial representation theorem is true (with the exception of certain ratio interpretations). It is this representation theorem which has provided the basis for the postponement argument. At least implicitly, this theorem is responsible for the most current developments (and also forms a basis for the "go decimal now that metrics are here" rationale). It follows the dictum of economy of thought to select one or two interpretations at most and provide explicit rational number instruction under these. From this instructional base, it is hoped that a cognitive counterpart to the above representation theorem will provide for transfer needed for a fully finctional rational number construct.

Yet this paper and other sources (e.g., Wagner, 1976) have suggested that this bold leap from the mathematical mechanism of a representation to a parallel cognitive mechanism is not yet proven. Thus, the basis for the rationals needs to involve more than one or two of the above interpretations.

1. The Major Components

A mathematical analysis of rational numbers ("What kind of mathematical objects are these?") leads to numerous interpretations logically simplified by a representative theorem. These interpretations form a conceptual pool for the building of related cognitive and instructional structures. From this pool, five ideas of fractional numbers emerge as a basis for a rational number construct, as pictured in Figure 8.

Part-Whole	Quotients	Measures	Ratios	Operators

Figure 8

These five--part-whole relationships, ratios, quotients, measures, and operators--are not mathematically independent and, indicated by the dotted lines, are not psychologically independent either. Yet they represent five separate fractional or rational number thinking patterns.

a. Part/Whole, Ratio

The first two of these patterns, part-whole and ratio relationships, are closely related. These have formed the traditional and modern bases for developing fraction meaning. In the first, some whole is broken up into "equal" parts. Fractional ideas are used to quantify the relationship between the whole and a designated number of parts. It is important to note that this representation is bi-partite both in words and symbols (seven-eighths, 7/8). While three hundred and three hundredths have parallel designatory and literal structures (three of something), the numerical interpretations (3/100 or .03) of three hundredths show the part-whole relationship to be related to the ordered pair notion, while 300 does not. More importantly, part-whole and set-set relationships generalize and hence psychologically highlight the notion of equivalence ($2/3 \approx 4/6$). Some of the current curricular models of this phenomenon have been set-subset, dissected and shaded regions, and number line relationships. Yet most of these models have been but brief stepping stones to the formal symbolic computation which formed the implicit construct of rationals.

The ordered pair notation takes on new significance with respect to ratio relationships--the quantitative comparisons of two qualities. Three-tenths (3/10) of a floor surface has a very different meaning than 3/10 which compares the number of girls and boys on a soccer team. This distinction has been blurred (7 deaths per 1000, 450 automobiles per 1000) by the concept of equivalence. While we represent 3 hits in 4 bats (3/4) and 30 hits in 40 bats (30/40) with the decimal .750, they are clearly very different phenomena. However, 75/100 [75 centimetres and 750/1000 of a metre (750 millimetres)] are the same measure.

Another reason for this blurred distinction is the problem of class inclusion. Piaget (1952) has discussed this ability at length with respect to whole number development. Yet the ability to handle class inclusion may be more important for fractional and rational number development.

It might be said that the part-whole number relationships are a special case of ratio relationships. While formal notion of equivalence is the same for both, the psychological one is different. Further, the notion of additivity in the two settings is different. Thus, while the two relationships share many characteristics and fall under the rubric of the rational number construct, for the learner they represent different if related subconstructs and lead to different concepts and functioning.

b. Quotient

The sub-construct "rational number as a quotient" is closely related to part-whole relationships. Yet for the learner it arises from and is applied in a different context. It allows for quantification of the result of dividing a quantity into a given number of parts and is related ultimately to the algebra of linear equations. While dividing a unit into fourths and designating 3 (3/4) leads to the same quantity as dividing 3 units into 4 parts (3/4) it is clear that these are different problems for the learner. In fact, it is a genuine instructional task in the mathematical education of 10- to 14-year-olds to develop a rational number construct and accompanying language which can relate these two sub-constructs.

c. Measure

The sub-construct "rational numbers as measures" is again closely related to the part-whole relationship. However, the measurement tasks means the assignment of a number to a region (taken here in the general sense of this word; may be 1-, 2-, or 3-dimensional or have some other characteristic). This is usually done through an iteration of the process of counting the number of whole units usable in "covering" the region, then equally subdividing a unit to provide the appropriate fit. The focus here is on the arbitrary unit and its subdivision rather than on part-whole relationships. It has been seen in the research of Washburne (1930), Novillis (1976), and Babcock (1978) that the identification of the whole (unit) in part-whole situations is difficult because the "whole" is implicit as opposed to the explicit unit of the measurement sub-construct.

Rational number as measures is a natural setting for two important aspects of the rational number construct. The joining of two measures to find a "sum" measure exhibits the vector additions* aspect of rational numbers. Using the metre as a unit provides a natural entre to decimal notation, with decimetres, centimetres, and millimetres serving as physical models for tenths, hundredths, and thousandths (.1, .01, .001).

d. Operators

The operator sub-construct portrays rational numbers as mechanisms which map a set (or region) multiplicatively onto another set. Thus a "2 for 3" operator maps a domain element 12 to a range element 8 and a "2/3" operator maps a region onto a similar region of reduced size. That this sub-construct provides a viable approach to rational numbers is well illustrated in numerous German school texts and particularly in the work of Griesel (1974) and Dienes (1971). This sub-construct focuses attention on the rationals as elements in the algebra of functions. Composition of operators provides a very simple foundation for multiplication of rational numbers.

2. Reality and the Five Sub-Constructs

In the first section of this paper, several tasks were presented as representing mature functioning with rational numbers. These tasks, the control over part-whole relationships, measuring, and quantitative comparisons, represent a reasonable core of what a person should expect to master as a result of instruction in the rational number

*This should not be confused with the normal ordered pair addition algorithm for vectors.

construct. In Margenau's (1961) terms these tasks and their doing represent the use of an interconnected net of potent sub-constructs which form a viable rational number construct. It is reasonable to ask if the five sub-constructs identified above provide a sufficient basis for mature functioning. In the final analysis, this is an empirical question in which the construct is validated in terms of the performance of persons who have been identified as possessing these sub-constructs. However, a face validation of these constructs is provided in Figure 9.

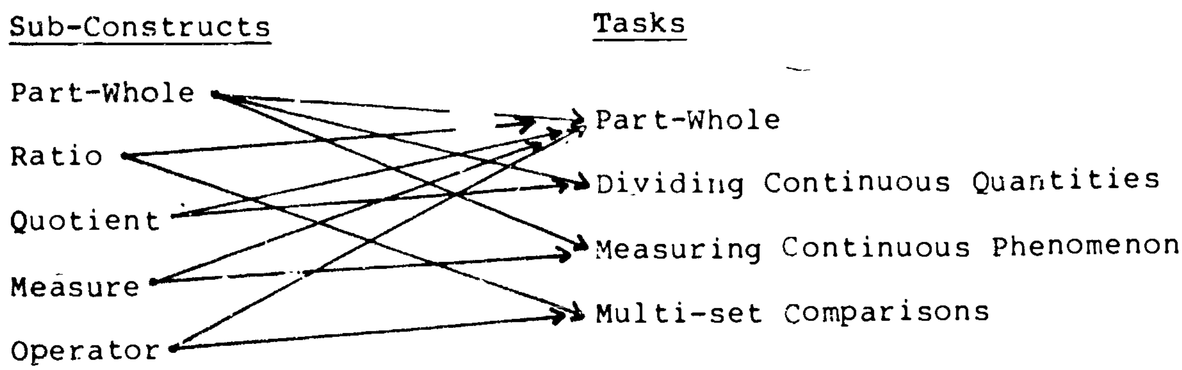


Figure 9

The defense of the interrelationships pictured in Figure 9 is taken directly or by direct implication from the discussion of each of the five sub-constructs above. From this picture it would appear that the part-whole sub-construct is of central importance as a basis for mature functioning. Again this hierarchical theorem is empirically testable. However, because of this potency of the sub-constructs and their connection to other ideas discussed below, the hierarchy issue probably is not of critical importance. The important conclusion to be drawn from the relationships pictured in Figure 9 is that the sub-constructs form a sufficient basis for mature functioning while each individually does not.

3. Mechanisms for Construct Development

The previous sections have been devoted to the development and defense of five basic sub-constructs of the rational number construct. Whatever the outcome of empirical studies, it will be true that at least some elements of the construct will prove valid. Given these, a significant problem is: How do these sub-constructs develop in a person or how does a person build them up? How does a child or adolescent move from the experiences of Margenau's P-plane to constructs which support mature functioning?

The mechanisms for this movement probably fall into two categories, developmental and constructive. The former, although influenced by social interaction, are also quite dependent upon maturation. Further, they take the form of schema usually identifiable only by performance on classic development mechanisms, including a well-developed schema of reversibility, the ability to handle class inclusion, the ability simultaneously to manipulate and cross-compare two sets of data, and proportionality. The former two of these are generally developed in children late in Piaget's stage of concrete operations (ages 9-11) while the latter are indicators of the stage of formal operations (ages 11-14). All four of the above can be thought of as internalized schemes which sponsor actions as opposed to conscious mechanisms.

The connection between reversibility and the rational number construct with its two types of inverses is obvious. The notion of "reciprocal" is important both in general development and in the development of all of the sub-constructs noted above. The discussion of part-whole and ratio constructs indicated the importance of the class inclusion notion. Ability to apply this mechanism is likely central to the ability to identify the unit, a key to the part-whole, quotient, and measurement sub-constructs. The cross-comparison of two sets comes into play in recognizing rational numbers in particular settings, in developing equivalence classes, and particularly in the composition of operators. The proportionality scheme is central to a generalized notion of ratio and equivalence.

The constructive mechanisms are to a larger extent products of experience. They are deliberate procedures used by the learner in coping with rational number settings and hence in building up the rational number sub-constructs. They have parallels with respect to the development of whole number constructs, the most prominent of which are the various forms of counting. Two such deliberate constructive mechanisms are ordered pair language and partitioning.

The use of ordered pair language is central to development of rational number subconstructs at many levels of sophistication. The whole issue of attaching bi-partite number names to fractional settings is one of the keys to the development of meaning of the various sub-constructs. This has been discussed by the Gundersons in 1957 and more recently carefully considered by various researchers at the University of Michigan in their development of the Initial Fraction Sequence (Payne, 1976).

Although this is mainly speculative at this stage, it may be that the second mechanism, partitioning, may play the same role in the development of rational number constructs that counting does vis-a-vis the natural numbers. Partitioning is seen here as any general strategy for dividing a given quantity into a given number of "equal" parts. Thus, it can be seen as important in developing all of the five sub-constructs. In fact, it may act as a primitive substitute for the proportionality schema in such tasks as finding fractions equivalent to a given fraction. It is also useful in the iterative division of a unit in the measure sub-construct.

Two other mechanisms which play roles in the early and more formal rational construct development, respectively, are the identification of the unit and the application of mathematical structural properties and the accompanying formal logic.

Aside from the work of the Piagetian school on proportionality and region subdivision, little is known about the mechanisms used in rational number construct development. The eight mentioned above are only some of the possible candidates and the relationships between these and rational number task performance remains to be empirically verified.

A recent exploratory study by the author lends some support to the existence of and hypothesized relationship between the mechanisms and rational construct development. In this study, random samples of five students in Grade 4 and ten students in Grades 5, 6, 7, and 8 were drawn from the population of a small county school system. In a clinical setting, using the mechanism of a simulated packing machine, each subject was asked to react to instances of the operator sub-construct. These included the operators 1 for 2, 1 for 3, 2 for 3, 3 for 4, the composition of 1 for 2 and 1 for 3, and the composition of 1 for 2 and 3 for 4 as well as their inverses. For each operator, the subject received up to six trials with feedback and then was asked to write down his or her predictions of machine performance in 10 cases (5 direct and 5 inverse). The experimenter then asked for an explanation of how the machine worked and probed for the most elaborate answer. To summarize the results briefly, there appeared to be categories of subject performance and thinking:

1. Reacted correctly to less than 10 percent of the items and hence considered nonfunctional on the tasks.

2. Could handle 1 for 2 and its inverse but no other settings. Appeared as though their fractional recognition in these settings was "1/2".

3. Could handle unit fractions and inverses.

4. Could handle non-unit and unit fractions.

5. Could handle simple and composed unit fractions.

6. Used a fractioning approach to handle simple and composed non-unit fractions.

7. Functioned using operators as proportions.

The categorized results across grade levels are shown in Figure 10.

Grade Level

Categories	4	5	6	7	8
7					X
6				XXX	XX
5				XXXX	XXXX
4			X		
3		XXX	XXXX	X	XX
2	XXX	XXXXX	XXXXX	XX	X
1	XX	XX			

Figure 10

Several things in this analysis bear on the issue of mechanisms. With respect to the developmental mechanisms, it can be noted that the <u>inverse</u> notion did not prove to be a major difficulty in this task set. In nearly all cases, including 70 percent of the Grade 4 responses, whenever a subject could produce the direct operator, he or she could also produce the inverse. Levels 4 and above required the <u>simultaneous or at least related comparison of two sets of data</u>. While 70 percent of the Grade 7 and 8 subjects fell in categories 4 and above, only 1 student in 25 at the lower grades did so. Only one student, in Grade 8, used the operator as direct <u>proportions</u>: "Oh, yes, they're all like 3/4."

The constructive mechanisms were also discernible. Subjects categorized in Category 2 tended to see only <u>1/2 as a fractional mechanism</u>. It was almost as if a sub-construct "1/2" formed a primitive fractional number construct. Many other subjects would also resort to saying "half" or "doubling" when they were confused by a situation, even when they would verbalize that the situation was not like the "1 for 2" situation. This phenomenon was prominent prior to Grade 7 and occurred even with a few Grade 7 and 8 students.

Subjects in Categories 4 and 5 made extensive use of partitioning to find results in the packaging process. For example, one Grade 8 subject in looking at a given 3-for-4 example (12/9) reacted, "2 sixes in 12, but 2 in 9 doesn't work, 3 fours in 12, 3 threes in 9--oh, I see." She then proceeded to test her hypothesis using partitioning in another situation. In general, students in Category 5 used partitioning as a substitute mechanism for proportionality to complete most of the tasks required. Subjects in Categories 3 and 4 did not seem explicitly to exhibit partitioning behavior and seemed to "see" the tasks in subtractive terms.

In summary, the mechanisms discussed in this section of the essay are seen to serve two purposes. They are used or are a cognitive basis for the building up of rational number constructs. Because they are general in application, they also may serve to unify the basic sub-constructs into the general rational number construct.

4. The "Complete" Rational Number Construct

The previous sections of this essay have presented some broad goals, some constructs, and some mechanisms related to rational number learning. It will be the purpose of this section to present a representation of the object of rational number knowing. As suggested earlier, Wagner (1976) has pictured the rational number "mega-concept" as a bundle of strands each representing a sub-concept. The representation in Figure 11 takes a different approach to illustrate the supportive role of the sub-constructs and the interactive role of the mechanisms discussed in detail above.

The complexity of the diagram in Figure 11 is only in part due to the author's inability neatly to represent mental constructs in two dimensions. The rational number construct of a maturely functioning person is complex. It subsumes the control functions outlined at the beginning of this paper and again in Figure 9. It also forms a basis for more abstract functioning in the areas of algebra and analysis as indicated to the right of the broken line in the top level of the diagram. There are numerous skills such as computation, setting up proportions, determining equivalence, solving equations, and measuring which are implied by the structure. Many of these rest on the notion of equivalence and the implicit operations of vector addition and function composition. The mature competencies, as shown in Figure 9, and a generalized idea of equivalence are dependent upon the underlying development of the five constructs which have a genesis in primitive fraction notions such as 1/2, division of regions, and subdivision. The boxes labeled "m" as well as the large word "PARTITIONING" show the pervasiveness of these mechanisms, particularly those of inverse and partitioning. Thus, the construct of rational numbers represented here is an integrated complex whole and not a behavioral surface without support.

5. On Connectability with Other Mathematical Ideas

The potency of the rational number construct in generating mature functioning has been discussed at length in the previous sections of this paper and it is clear that the construct does not stand independently of its important applications. But as Margenau (1961) suggests, the rational number construct must also be extensive and connectable, that is, related to other constructs and generative of some. Clearly, whatever its form, the construct or scheme of rational numbers does not stand alone or isolated in a person's mind. The general construct pictured above might be thought of as a part of a net of interconnected mathematical constructs. In particular, the operator sub-construct can be connected to those of transformation and the more general function construct. It can also be connected to the group construct. The measure sub-construct is connectable to general measurement construct as well as to the formal mathematical construct of measure through the real number construct; the useful related instructional structure here is various number-line forms. The quotient sub-construct is by definition related to linear equation solving and hence represents a point of connection to the algebra of equations as well as to the field structure. The ratio sub-construct is connected to the many forms of proportional constructs and in particular to probability and descriptive and inferential statistics. The part-whole sub-construct is internally connected in that it serves as a source of language and symbolism in the other constructs.

This brief analysis shows the rational number construct developed above to be potentially robust in terms of its relationship to other mathematical constructs. This very feature leads to questions of instructional sequencing. For example, should instruction in the operator sub-construct precede or come after the study of other transformations--for example, geometric transformations. In the latter case, the rational operator becomes an abstraction from the similarity or size transformation. Another question is, "how does the measure sub-construct fit into general measurement instruction; to what extent does one make use of the other?" That these are important questions is seen in the fact that curriculums based on inducing a rational number construct through its connection to natural numbers or integers, while axiomatically valid, have not been successful when measured against the goal of mature functioning.

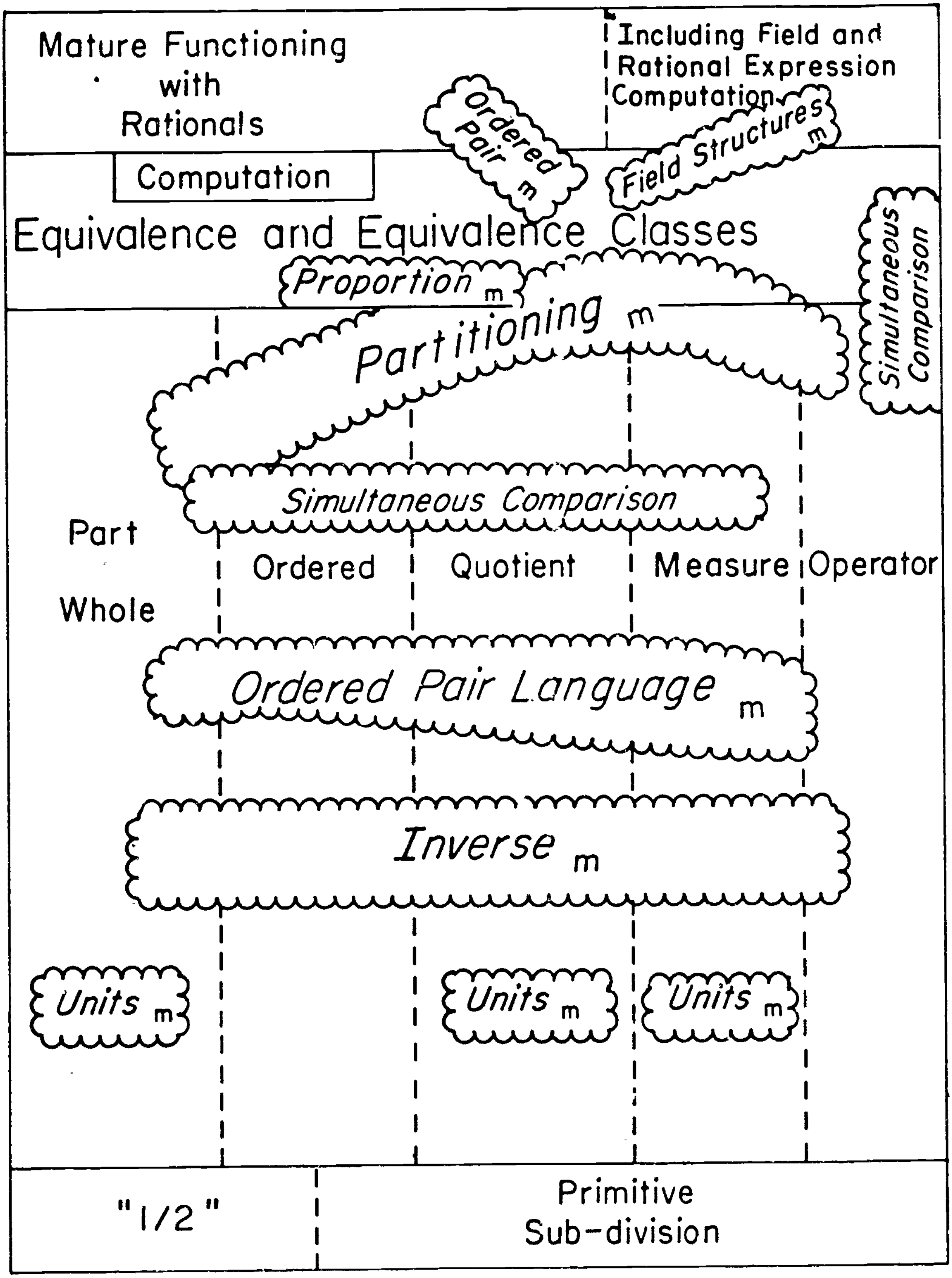

Mature Functioning
with
Rationals
Including Field and
Rational Expression
Computation
Ordered Pair m
Field Structures m
Computation
Equivalence and Equivalence Classes
Simultaneous Comparison
Proportion m
Partitioning m
Simultaneous Comparison
Part
Whole
Ordered
Quotient
Measure
Operator
Ordered Pair Language m
Inverse m
Units m
Units m
Units m
"1/2"
Primitive
Sub-division

Figure 11

6. A Digression on Decimals

The purpose of this section is to test the above model of the rational numbers construct against the supposition that rational numbers be developed using the decimal notation, without using or at least substantially deferring use of ordered pair notation. What is the effect of this approach on the development of the five basic sub-constructs and the mechanisms discussed above? As has been suggested above and in an earlier paper (Kieren, 1976), the use of the metre as a unit provides a natural mechanism by which the measure sub-construct could be developed using decimal notation. This would require attention to a standardized partitioning mechanism (into 10 parts). The actual algorithm paralleling the addition of measures would of course be the trivial extension of the whole number algorithm. Likewise the computational mechanism in the quotient sub-construct could be considered the obvious extension of a whole number algorithm. However, the interpretation of dividing 3 objects into 4 equal parts benefits from a fractional understanding of rationals which a 10's partitioning would at best cloud. Decimal notation does not highlight the reciprocal notion of multiplicative inverse so useful in and indeed a highlighting contribution of the quotient sub-construct.

Similarly, the development of the part-whole and ratio sub-constructs are clumsy under decimal notation. Of course, one might argue that common fractional notions such as halves, fourths, and thirds could be taught as special mathematics. But this hardly helps develop a construct which allows a person to control part-whole and particularly set-set multiplicative comparisons. Thus, the ratio sub-construct would particularly be under-developed. One might also argue that later notions of rational expressions would suffer from a decimal notation development. However, necessary ordered-pair rational notions could be introduced as a prerequisite to this study as they could prior to or with any quotient field study. However, a general decimal approach would hinder an important application of the ratio sub-construct--probability.

To the extent that a rational operator can be conceptualized in the parametric sense--e.g., 2y = .25x for the "1 for 4" operator--a decimal approach is not inimicable to the operator construct. One can easily see that this representation, and even functional conception of the operator approach, is at a sophisticated level. It does not well allow for the use of the partitioning mechanism. Hence the contribution of this sub-construct, multiplication as function composition, would have to be delayed until late in the curriculum or lost entirely.

Figure 12 summarizes the hypothesized effects on the development of the five sub-constructs and selected mechanisms of a solely decimal approach to rational numbers.

Code:

Part-Whole --- E	E+ : somewhat stronger development
Ratio -------- W	E : equal development
Quotient ----- E	E- : somewhat weaker development
Measure ------ E+	W : substantially weaker development
Operator ----- E-	
Partitioning - W	
Inverse ------ E-	
Unit --------- E	
Proportions -- E- (because more abstract)	

Figure 12

As can be seen, it is hypothesized that a solely decimal approach will profit the additive construct of measure, slightly weaken the constructs involving the notion of inverse, and substantially weaken the multiplicative and proportional aspects of sub-constructs. Further, the mechanism of partitioning would have a much more limited effect on the rational number construct development of the individual.

It has been suggested that, with the onset of the metric system, fraction instruction could be eliminated and that indeed student learning in the area of decimal fractions would be greatly enhanced. This hypothesis is rather obviously true if one's construct of the rational numbers consists of the behavioral surface of computation with rational numbers. However, if one's instructional aim is to allow students to build up a construct that allows for control over a wide variety of rational number problem settings, as well as gaining some basis for further algebraic and analytic work, then the analysis above suggests that there are hypothesized costs as well as benefits to a decimal approach. It should be emphasized that the effect of a "decimal only" or "decimal mainly" instructional experience has not really been empirically tested. Such a treatment has not gained currency in Europe, which has been metric a long time (see European texts or Friesel, 1971). Thus, the conclusion of this digression awaits empirical data.

III. Implications of the Above Theory for Research

Even a casual reading of the above discussion shows a large number of hypotheses demanding further exploration and testing. Central to the above development are the five basic sub-constructs. Each of these constructs is in need of several kinds of explication. While each has been described in some detail here and in other places, relatively little is known about these constructs as they exist for or are developed by children and adolescents. This is particularly true for the quotient, measure, and operator sub-constructs. Thus, research is needed, delimited on sub-construct lines, which gives a clear picture of the shape and developmental pattern of these constructs in young persons.

The five basic sub-constructs also require other forms of validation beyond the content and, to a certain extent, construct validation suggested above. One such task is to relate construct capability and rational number achievement. As suggested in Figure 9, the sub-constructs appear to be logically related to various rational number task settings. It would be important to define achievement in terms of these tasks as well as in more conventional ways.

As suggested early in the above discussion, the rational number constructs are the products of deliberately arranged experiences for the individual. This suggests that the above constructs are in need of curricular validation. That is, it must be shown that they each form the basis for deliberate instructional activities. Further, it must be shown that <u>these activities allow the large majority of the population to develop the particular sub-construct</u>. To a certain extent, this has been implicitly carried out in various curriculums to date for the part-whole and operator sub-constructs. However, even these efforts obviously have not been directed towards the sub-constructs as developed in this paper.

Figure 9 is also suggestive of yet another form of construct validation. This would involve a study of the interrelationship of the sub-constructs and would have as its goal a parsimonious description of the rational number construct: Are all of the sub-constructs necessary? Do some subsume others?

This interrelationship study could also lead to an expanded list of sub-constructs. This might come about in two possible ways. First it might be shown that the net of sub-constructs does not account for some important rational number task. Thus, some elaboration of the given constructs or extension to new constructs would be necessary. Further, new mathematical ideas might suggest the development of a new mechanism or sub-construct useful to persons in handling rational number tasks. The development in this paper and others (Dienes, 1971; Griesel, 1971; Wagner, 1976) show that the mathematical topics of fields, analysis, and operators provided bases for the personal constructs of quotients, measure, and operator. Thus, such extension of the rational number construct through use of newer mathematical thinking is obviously possible.

The mechanisms persons use in building up a rational number construct present many other avenues of research. While there has been considerable work in mathematics and science education as well as developmental psychology on the proportionality scheme, there has been almost no deliberate study of partitioning and its related notion, use of units. The same is true for the study of inversing, particularly as it relates to rational or fractional settings. In line with the construct theory above, research is needed showing the role of various mechanisms in each of the constructs above. Again a focus of the search will be for a more precise explication of the rational number construct. Thus it may be that the mechanisms, or some of them, will prove to be more important than the basic sub-constructs. Under any circumstance, it will be important to define curricular conditions for the development of these mechanisms.

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SEVENTH-GRADE STUDENTS' ABILITY TO ASSOCIATE PROPER FRACTIONS WITH POINTS ON THE NUMBER LINE

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Geometric regions, sets, and the number line are the most commonly used semi-concrete models for fractions in elementary school textbooks. Novillis (1976) analyzed the fraction concept into 16 less complex related subconcepts where each one was associated with one of these three semi-concrete models. She reported that associating a proper fraction with a point on a number line was more difficult for intermediate grade students than associating a proper fraction with a part-whole model where the unit was a geometric region and with a part-group model where the unit was a set. Payne (1976) also reported that elementary students had more difficulty with the number line model than with the area model (part-whole) model.

An obvious question raised by these studies is: Why was the number line model so much more difficult for students than the other two types of models? In the Novillis 1976 study, the number line test items utilized number lines of length one, two, and three. It could be that the length of the number line might be a relevant variable. It was observed that when the number line is of length greater than one, some students disregard the scaling and treat the number line that is depicted as a unit regardless of its length. Muangnapoe (1975) reported third and fourth graders exhibiting this behavior. An important difference between a part-whole model and a number line model is that in the number line model the students need also to attend to the scaling. Hence a number line model implies a length greater than one. Whenever a number line of length one is used, then the number line model is not being completely tested. In this case, the number line is really just another part-whole model where the unit is not in question, being the "whole."

Novillis (1976) also tested intermediate grade students' ability to associate proper fractions with part-whole, part-group, and number line models where the number of equivalent parts in the unit was a multiple of the denominator. The mean for 279 intermediate grade students on the 20-item subtest that measured students' recognition of equivalence with these three models was 1.85. The mean score on the four-item Number Line, Equivalence Subtest was 0.29.

Portions of this article have been previously published: Larson, C.N., Locating fractions on number lines: Effect of length and equivalence, School Science and Mathematics, 1980, 80, 423-428.

Payne (1976), summarizing a series of studies done at the University of Michigan from 1968 to 1975, reported that in all of the studies equivalent fractions was troublesome for most students, especially reducing to lowest terms. These results seem to suggest that further investig. ion is needed in this area.

Purposes

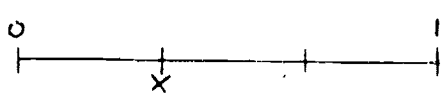
The purposes of this study were to investigate: 1) seventh-grade students' ability to associate a proper fraction with a point on a number line when the number line is of length one and of length two; 2) seventh-grade students' ability to associate a proper fraction whose denominator is b with a point on a number line, when the number of line segments into which each unit segment has been separated equals b and 2b; and 3) the hierarchical dependencies among the four types of number line subconcepts that occur when both length of number line (one or two) and number of line segments in each unit line segment (b or 2b) are considered.

Definitions

This study deals with four subconcepts of the fraction concept. All four subconcepts involve the number line model. The behavior that is related to each subconcept is described below:

Subconcept L1: Number line, Length 1.

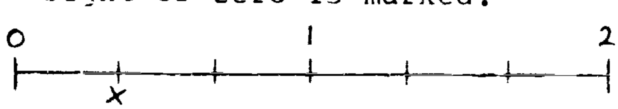
The student associates together the proper fraction a/b and a point on a number line of length one, where the unit segment has been separated into b equivalent line segments and the a^{th} point to the right of zero is marked.

Example: 

The point on the number line marked by X can be named by the fraction 1/3.

Subconcept L2: Number Line, Length 2.

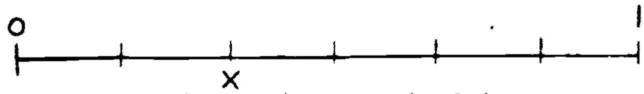
The student associates together the proper fraction a/b and a point on a number line of length two, where each unit segment has been separated into b equivalent line segments and the a^{th} point to the right of zero is marked.

Example: 

The point on the number line marked by X can be named by the fraction 1/3.

Subconcept EL1: Number line, Equivalence, Length 1.

The student associates together the proper fraction a/b and a point on a number line of length one, where the unit segment has been separated into 2b equivalent line segments and the 2ath point to the right of zero is marked.

Example: 

The point on the number line marked by X can be named by the fraction 1/3.

Subconcept EL2: Number line, Equivalence, Length 2.

The student associates together the proper fraction a/b and a point on the number line of length two, where each unit segment has been separated into 2b equivalent line segments and the 2ath point to the right of zero is marked.

Example: 

The point on the number line marked by X can be named by the fraction 1/3.

Method

Instrument

A sixteen-item multiple choice test--Locating Fractions on the Number Line--was constructed by the investigator to measure the behavior related to Subconcepts L1, L2, EL1, and EL2. The test contains four subtests of four items each; each subtest corresponds to one subconcept. The 16 items were randomly ordered in the test. The fractions 1/3, 1/5, 2/5, and 3/8 were used in each subtest. These same four fractions were previously used by Steffe and Parr (1968) and Novillis (1976). Each subtest contained two test items of each of the following types: a) given a fraction, the student chooses the correctly marked number line; b) given a number line with a point marked, the student chooses the correct fraction.

The reliability of the subtests, as determined by the Hoyt procedure, was r = .86 for subtest L1, r = .76 for subtest L2, r = .85 for subtest EL1, and r = .80 for subtest EL2.

Sample

Seventh-grade students in the fall of the year were selected as the population in order to evaluate students at the end of the elementary school years. The sample consisted of 382 seventh-grade students, approximately half of the seventh-grade students in a predominantly middle class junior high school in Miami, Florida. The students were assigned to three different tracks for mathematics instruction. The only class at the highest level, Pure Math, and half of the class sections at the other two levels, Structures and Whole Numbers, were tested. A total of 13 class sections was tested, the one highest level class, Pure Math (n = 31), five sections of the second level, Structures (n = 156), and seven sections at the third level, Whole Numbers (n = 195). The test was administered by the investigator from October 22-28, 1975.

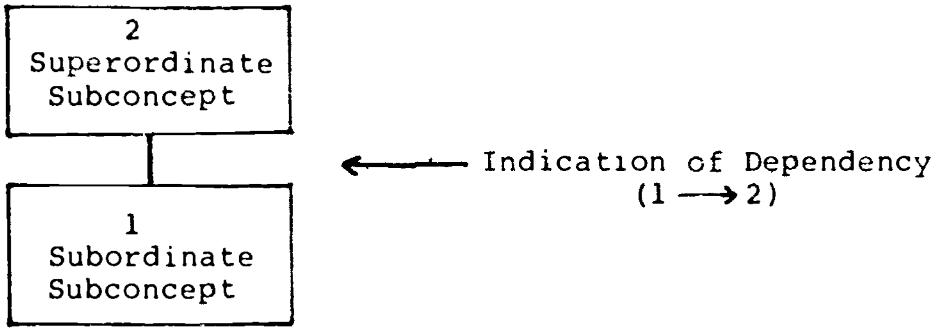
Procedure for Establishing Hierarchical Dependencies

Given the nature of the four subconcepts that were being investigated, it seemed reasonable that a hierarchical relationship might exist. In the past, learning hierarchies have been hypothesized, instruction given based on the hypothesized hierarchy, and then a test administered to assess each cell in the hierarchy (Gagné et al., 1962; Eisenberg and Walbesser, 1971). Gagné et al. (1962) and Walbesser (1968) have developed numerical procedures for testing the validity of hypothesized dependencies in learning hierarchies. Novillis (1976) adapted the ratios used by Gagné et al., (1962) and Walbesser (1968). Her study differed from traditional hierarchical studies in that no instruction was given and the research hypothesis predicted a relationship in only one direction. In using ratios to validate hierarchies, a transitivity inference is usually made that if A is established to be subordinate to B and B is established to be subordinate to C, then A is accepted as being subordinate to C.

In the present study a hierarchy was not hypothesized; instead, two methods of analysis were selected to test for all possible dependencies among the four subconcepts in order to construct a feasible hierarchy. In Method 1, the two ratios adapted by Novillis (1976) were used to test all pairs of the four subconcepts for possible dependencies.

A criterion level of 75 percent was established for each subtest and the scores changed to a binary scale where 1 denotes reaching criterion on the subtest and by inference acquisition of the related subconcept; similarly, 0 denotes nonacquisition of the subconcept. Figure 1 illustrates the four possible categories of students for a pair of subconcepts.

The (0, 0) category is not considered in the analysis of the data as there is no way of knowing which subconcept each



Subordinate Subconcept	Superordinate Subconcept: Nonacquisition	Superordinate Subconcept: Acquisition
Nonacquisition	(0,0)	(0,1)
Acquisition	(1,0)	(1,1)

Figure 1. Four Categories of Students Associated with Each Hierarchical Dependency.

student will acquire first. In this study the hierarchical analysis of the data is essentially concerned with deciding when the number of students in category (0, 1) as compared to the number of students in categories (1, 0) and (1, 1) is of a magnitude that does not contradict the hypothesis of a hierarchical dependency.

Following are the two ratios used by Novillis (1976) with minimum levels of supporting a hierarchical dependency:

$$\text{Ratio 1} = \frac{n(1, 0)}{n(1, 0) + n(0, 1)} \geq .75$$

$$\text{Ratio 2} = \frac{n(1,1) + n(1,0)}{n(1,1) + n(1,0) + n(0,1)} \geq .90$$

Ratio 2 is a test of the dependency only when Ratio 1 is at the .75 level. The rationale for the ratios and the levels of acceptance is described in Novillis (1976).

A hierarchy was then constructed based on the results of this analysis and the transitivity inference. Method 2 was to test the transitivity inference of the hierarchy that results from use of Method 1. Method 2 consisted of generalizing the two ratios used in Method 1 in order to deal with all four subconcepts at once rather than with pairs of subconcepts.

When considering the four subconcepts simultaneously there are 16 categories that result for the quadruple of subconcepts. These 16 categories have been partitioned into three classes: the Null Class--(0,0,0,0); the Mastery Class--(1,1,1,1); and the Intermediate Class, which contains all categories where students reach criterion on from one to three of the subtests. Ratio 1', an extension of Ratio 1, deals solely with the 14 categories in the Intermediate Class. The numerator is the number of students in the categories of the Intermediate Class supporting the hypothesized hierarchy--Positive Categories (P). The denominator is the number of all of the students in the Intermediate Class (I). The two new ratios are:

$$\text{Ratio 1'} = \frac{n(P)}{n(I)}$$

$$\text{Ratio 2'} = \frac{n(P) + n(1,1,1,1)}{n(I) + n(1,1,1,1)}$$

Results

One of the main purposes of this study was to compare seventh graders' performance on various fraction/number line tasks where length and equivalence were varied. To accomplish this a 2 x 2 repeated measures ANOVA was performed on the students' scores on the four subtests. The mean and standard deviation for each subtest are summarized in Table 1; the 2 x 2 repeated measures ANOVA indicates: 1) that associating proper fractions with points on number lines of length one was significantly easier for the students tested than on number lines of length two; and 2) that associating proper fractions with points on number lines where the number of equivalent line segments in each unit segment is the same as the denominator was significantly easier for the students tested than on number lines where the number of equivalent line segments is twice the number in the denominator. Even though the interaction was significant, the means for each subtest listed in Table 1 show: 1) that the means for both levels of length one (L1 and EL1) are higher than for both levels of length two (L2 and EL2); and 2) that the means for both levels of the number of equivalent segments in a unit equaling the denominator (L1 and L2) are higher than for both levels of the number of equivalent line segments in a unit being twice the denominator (EL1 and EL2).

Table 1

Means and Standard Deviations on Four Fraction/Number Line Subtests

(n = 382)

Subtest	Mean	Standard Deviation
L1	2.67	1.59
L2	2.32	1.50
EL1	1.62	1.62
EL2	1.51	1.53

Table 2

ANOVA for Length by Equivalence

Source	SS	df	MS	F
Length (L)	20.50	1	20.50	22.46*
Error	347.75	381	.91	
Equivalence (E)	327.13	1	327.13	185.71*
Error	671.12	381	1.76	
L X E	5.42	1	5.42	12.30*
Error	167.83	381	.44	

$P < .001$

The second type of analysis used in this study was the testing for possible hierarchical dependencies using Methods 1 and 2, previously described. In order to test for dependencies using Method 1, all possible pairs of subconcepts were identified. The order of the subconcepts in each pair was determined by the magnitude of the means for each related subtest in the pair. The data listed in Table 3 were used to compute Ratios 1 and 2; Table 4 displays the results of this computation. Hierarchy 1 is illustrated in Figure 2. It consists of the dependencies that were supported and the level of each subconcept relative to the others based on the subtest that measured the associated behavior.

Table 3

The Number of Students in Each of Four Categories for Each Possible Pair of Subconcepts
(n = 382)

	Categories			
Dependency	n(0,0)	n(0,1)	n(1,0)	n(1,1)
L1 → L2	129	15	73	165
L1 → EL1	141	3	113	125
L1 → EL2	138	6	134	104
L2 → EL1	183	19	71	109
L2 → EL2	198	4	74	106
EL1 → EL2	242	12	30	98

Table 4

The Results of Computing Ratios 1 and 2

Dependency	% in (0,0)	Ratio 1	Ratio 2	Dependency Supported
L1 → L2	33.77	.8295	.9407	Yes
L1 → EL1	36.91	.9741	.9876	Yes
L1 → EL2	36.13	.9571	.9754	Yes
L2 → EL1	47.91	.7889	.9045	Yes
L2 → EL2	51.83	.9487	.9783	Yes
EL1 → EL2	63.37	.7143*	.9143	No
EL2 → EL1	63.37	.2857*	.7857**	No

* Ratio 1 < .75
** Ratio 2 < .90

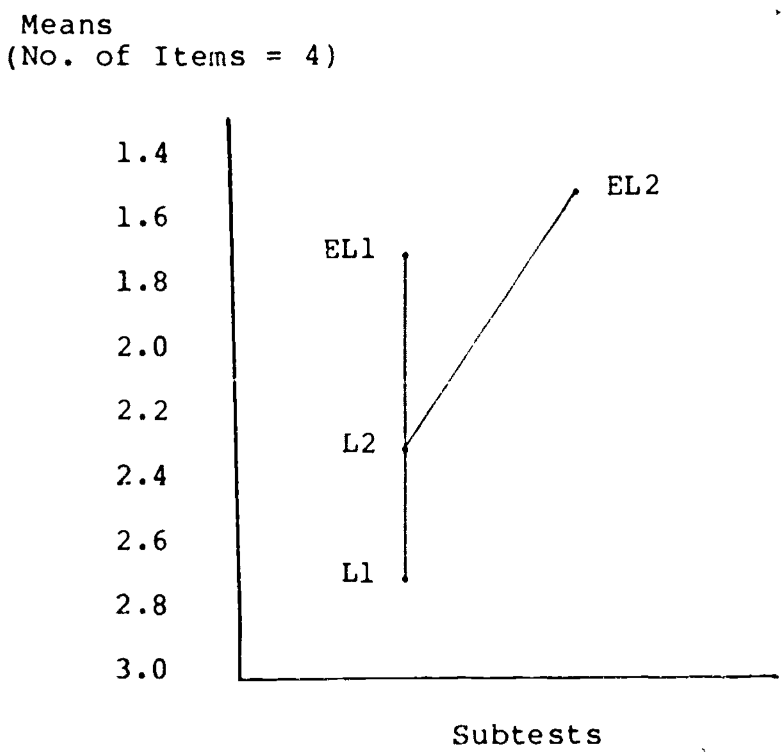


Figure 2. Hierarchy 1: Hierarchical Dependencies Supported with Subconcepts Ordered by Means.

Hierarchy 1 is not one that would have been hypothesized based on a logical ordering of the four subconcepts. It was expected that EL1 would be subordinate to EL2. Also, the relationship between L2 and EL 1 is surprising.

Table 5 lists the number of students in each of the 16 categories that are required for testing a hierarchy using Method 2, in which all dependencies among the four subconcepts in the proposed hierarchy are tested at once. Each category is indicated by an ordered quadruple of zeroes and ones in the order L1, L2, EL1, EL2.

Table 6 lists the number and percentage of students in each of the three Classes: Null, Intermediate, and Mastery. Table 7 lists the categories in the Intermediate Class that were classified as being positive (i.e., supporting Hierarchy 1), and the results of computing Ratio 1' and Ratio 2' using the data in Tables 6 and 7.

Table 5

Number of Students in Each of Sixteen Categories
(n = 382)

Category (L1,L2,EL1,EL2)	Number of Students in Each Category	% of Students in Each Category
(0,0,0,0)	125	32.72
(0,1,0,0)	12	3.14
(1,0,0,0)	54	14.14
(1,1,0,0)	51	13.35
(0,0,1,0)	1	.26
(0,1,1,0)	0	
(1,0,1,0)	18	4.71
(1,1,1,0)	11	2.88
(0,0,0,1)	3	.79
(0,1,0,1)	1	.26
(1,0,0,1)	1	.26
(1,1,0,1)	7	1.83
(0,0,1,1)	0	
(0,1,1,1)	2	.52
(1,0,1,1)	0	
(1,1,1,1)	96	25.13

Table 6

Number and Percent of Students by Class of Categories
(n = 382)

Class of Categories	Number of Students in Each Class	% of Students in Each Class
Null	125	32.72
Intermediate (I)	161	42.15
Mastery	96	25.13

Table 7

Testing of Hierarchy 1: L1 → L2 → EL1 and L1 → L2 → EL2
(n = 382)

Positive Categories in Intermediate Class (P)		Number of Students in Each Category
(1,0,0,0)		54
(1,1,0,0)		51
(1,1,1,0)		11
(1,1,0,1)		7
	Total	123

$$\text{Ratio } 1' = \frac{123}{161} = .7640 > .75$$

$$\text{Ratio } 2' = \frac{219}{257} = .8521^{*} < .90$$

*Hierarchy not supported.

Since Hierarchy 1 was not supported using Method 2, two other probable hierarchies were tested using this method. Hierarchy 2 (L1 → L2 → EL1 → EL2) is a linear ordering of the subconcepts based on the means of the four related subtests. Hierarchy 3 (L1 → L2 → EL2, and L1 → EL1 → EL2) is a logical ordering of the subconcepts based on an analysis of the characteristics of each subconcept.

Hierarchies 2 and 3 were not supported using the established decision rules.

Table 8

Testing Hierarchy 2: L1 L2 EL1 EL2
(n = 382)

Positive Categories in Intermediate Class (P)		Number of Students in Each Category
(1,0,0,0)		54
(1,1,0,0)		51
(1,1,1,0)		11
	Total	116

Ratio 1' = $\frac{116}{161}$ = .7205* < .75

Ratio 2' = $\frac{212}{257}$ = .8249* < .90

*Hierarchy not supported

Table 9

Testing Hierarchy 3: L1 L2 EL2 and L1 EL1 EL2

Positive Categories in Intermediate Class		Number of Students in Each Category
(1,0,0,0)		54
(1,1,0,0)		51
(1,0,1,0)		18
(1,1,1,0)		11
	Total	134

Ratio 1' = $\frac{134}{161}$ = .8323 > .75

Ratio 2' = $\frac{230}{257}$ = .8949* < .90

*Hierarchy not supported

Discussion

It would seem that, by seventh grade, associating proper fractions with points on number lines of length one and of length two would be of equal ease for students. That it wasn't raises the question: Are we using teaching strategies, sequences, and activities that foster concept formation, or isolated rule formation? If students had an understanding of number lines, a concept of proper fraction as naming a number of equivalent parts of a defined unit, and a concept of fractions as names for numbers, they should be able to associate 1/5 with a point on a number line regardless of its length. A number line of length one is very similar to the part-whole (area) model that is usually the first and most constant model used in developing fraction concepts. The students can disregard the scaling and respond correctly, as long as they begin counting at the left--at zero. They can still use the rule, count the number of parts in all (in this case equivalent segments) for the denominator, and count the number of equivalent line segments from zero to the marked point for the numerator. When the number line is of length two, this rule doesn't work. The students need to know that the line segment from 0 to 1 is one unit (one whole), the line segment from 1 to 2 is another unit. When they are dealing with a proper fraction, they need to consider only the number of equivalent line segments from 0 to 1, and they must realize that the segments from 1 to 2 are not relevant.

Responses to individual test items indicate that some of the students tested were confused with the scaling or disregarded it. When responding to three test items where the number line was of length two, 15 percent to 25 percent of the sample chose fractions that indicated that they considered the whole number line the unit and not just the segment from 0 to 1. For example, 25 percent of the students selected 2/12 as the correct response when the number line was of length two and each unit segment was separated into six equivalent line segments.

The data, as well as the questions asked by students during data collection, suggest that many students do not associate the name "1/3" with a point indicated by 2/6 on a number line. Do these students have as part of their fraction concept that a fraction represents a number that has many names and that each of those names can be associated with the same point on the number line regardless of the number of segments in each unit? Do they have the flexibility in their concept to allow them to associate the fraction 2/6 with a point on a number where each unit segment has been separated into nine equivalent line segments? A question asked many times concerning test items on Subtests EL1 and EL2 was, "Do you mean to reduce?" or "Should I reduce?" Test items contained the questions: "Which fraction can name this point?" and "On which number line can the point marked by X be named by the fraction 1/3?"

Fourth- and fifth-grade students construct sets of equivalent fractions and learn the rule that you can multiply or divide each part of the fraction by the same number. Do they integrate this as part of their fraction concept or have they, in the first case, memorized a rule for a pattern and, in the second case, a rule for an "algorithm?"

A question that needs answering is: What is the concept of equivalent fractions for students at the completion of elementary school? In discussing the results and implications of the NAEP Mathematics Assessment, Carpenter et al. (1975) state:

> If, in the upper elementary grades, the concept of equivalent fractions has been developed well, and it should have been, then the data imply that pupils have not mastered the application of equivalent fractions to the solution of problems. One suspects that 13-year-old pupils see fractional parts, equivalent fractions, and computational algorithms as separate, unrelated topics. (pp. 442-443)

The results of this study seem to indicate that some students do not have a very flexible concept of equivalent fractions. Payne (1976) and Steffe and Parr (1968) report evidence supporting this contention. Perhaps some students have a group of isolated, inflexible, specific rules that are not synthesized and which allow for very little transfer. Brownell and Hendrickson (1950) claimed that, as concepts develop, they move along various lines of change. They become more abstract, clearer, and more definite. They also change in their implications, relationships, and transferability. The process of learning concepts, then, is primarily one of snythesis. How can we structure situations so that the students will feel the need to synthesize and will attempt to do so? Payne (1976) and his colleagues have been experimenting with teaching sequences and strategies that will do this. So far they have met with limited success.

The attempt was made to establish a hierarchy of the four related subconcepts of the fraction concept with a number line as a mode. One definite hierarchy was not clearly established. What was indicated was that Subconcept L1 was acquired first by 34 percent of the students in the Intermediate Class, in comparison to 7 percent, less than 1 percent, and 2 percent of the students in the Intermediate Class who first acquired Subconcepts L2, WL1, and EL2, respectively. (The percentage of the total sample for each category is listed in Table 5.) The descending order of the means for the four subtests in the order L1, L2, EL1, and EL2 (see Table 1) is identical to the descending order of the subconcepts when the number of students who achieved criterion on each related subtest is considered (see Table 10).

Table 10

Number of Students Attaining Criterion on Each Subconcept (n = 382)

Subconcept	Number of Students Attaining Criterion
L1	238
L2	180
EL1	128
EL2	110

It is of interest that 33 percent of the 382 students tested did not attain criterion on any of the subtests, and only 25 percent attained criterion on all four subtests. Of the 42 percent in the Intermediate Class, 14 percent of the students tested attained criterion only on Subtest L1. The other 28 percent of the students attained criterion on other subsets of the subtests.

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THE RELATIONSHIP OF AREA MEASUREMENT AND LEARNING INITIAL FRACTION CONCEPTS BY CHILDREN IN GRADES THREE AND FOUR

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There are several physical models which are used in school mathematics to introduce fraction concepts. One of the earliest models used is a region such as a rectangular or a circular one which has been partitioned into n congruent parts. The entire region is defined as one whole unit and each of the parts is defined as 1/n of the region. Curriculum materials often show one or more of the n parts beside the region on the printed page. It apparently is assumed that the child can measure the whole region or certain parts of it in terms of these smaller parts with measure 1/n. On the other hand, some children in grades three and four do not have well-developed area concepts.

The purpose of this study was to determine the relationships between the child's area concept and the ability to learn fraction concepts using area models. If one's area concept is helpful in learning fractions, it appears that appropriate activities in area measurement should aid fraction learning and should precede fraction activities. A second purpose was to determine the effect of grade level on the ability of the children to learn fraction concepts at the third- and fourth-grade levels.

Procedures

Subjects

The 56 subjects were chosen from two third-grade and two fourth-grade classes in Greater Vancouver, British Columbia.

Area Concept Test

The Area Concept Test was composed of six items. The test included two conservation of area items and two measurement of area items similar to those used by Piaget (1960). Both kinds of items were included because it is not clear that ability to perform one of these tasks is necessarily prerequisite to performing the other kind of task (Taloumis, 1975). In the other two items the child was asked to measure a region in terms of a set of blue rectangular cards and a set of red cards. In one item it took the same number of blue as red cards and in the second item it took fewer blue than red units. The child was then asked to measure a second region using the red cards and predict if it would take the same number or more or fewer blue cards than red cards to cover the second region. The Area Concept Test was given in a one-to-one interview and audio recorded. In each item the child was asked to justify his or her response.

Unit of Instruction

The unit was based on a revision of the material used by Muangnapoe (1975). The instruction included identifying fractional parts of regions using oral names, written work names, and fraction symbol names. Fraction notation was used for unit fractions and for other numbers less than one, equal to one, and greater than one. However, mixed forms were not used for numbers greater than one. Order was included for some cases where the fractions had the same denominator or same numerator, and equivalnet fractions were not necessary.

The main instructional techniques were paper folding by teacher demonstration and by each child. The children folded paper rectangles which measured 28 cm by 5.5 cm and paper discs. Later the children completed worksheet exercises using their material kits and, finally, completed worksheets without the use of the materials. At first only oral language was used during the folding activity. This was followed by use of oral and written word names and finally fraction numerals were used.

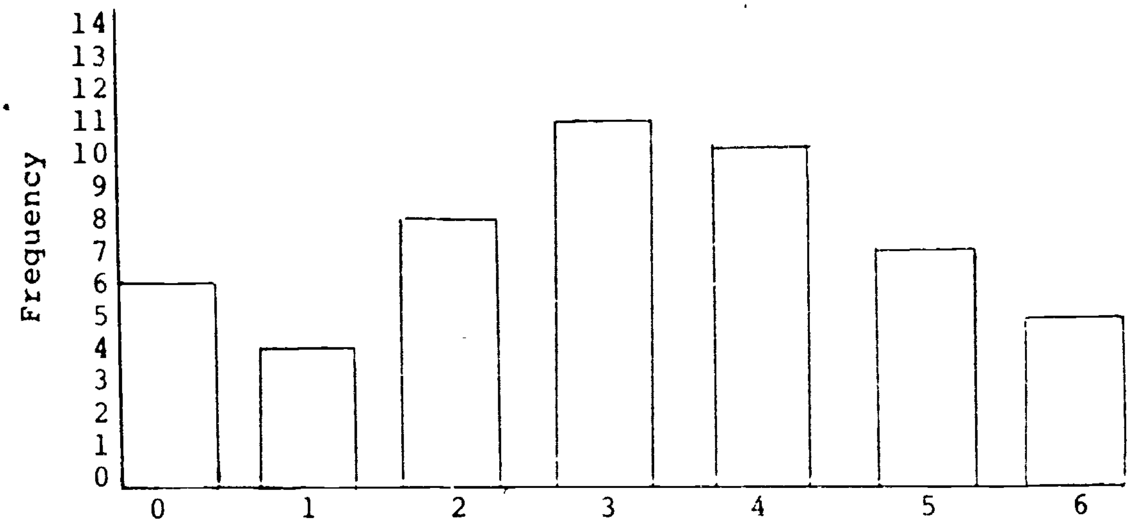
Posttests

The Fraction Concept Test of 51 items was similar in nature to those on the worksheets completed by the pupils. This included items on (a) identifying the larger of two fractions by word name, (b) identifying figures which have equal-sized parts, (c) translating from word names to fraction numerals and conversely, (d) deciding whether two diagrams show equal amounts shaded, (e) shading a previously drawn diagram to show a given graction, (f) writing a fraction to indicate the part shaded, and (g) identifying a diagram which shows a given fraction.

The Transfer Test was composed of eight items sampling extension to equivalence, mixed numerals, and the "part of a set" meaning of a fraction.

Procedure

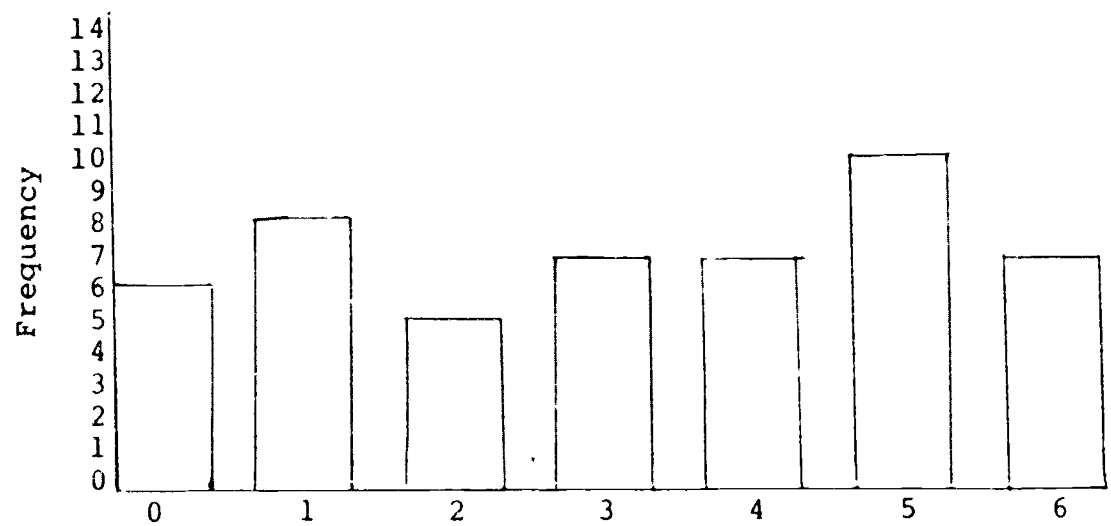
The Area Concept Test was administered to 101 pupils in grades three and four. Figures 1 and 2 show the frequency of pupils scoring zero to six for grades three and four, respectively. Pupils who scored two or less were classified as low and pupils who scored four or more were classified as



Score
N = 51

Figure 1

Frequency of Pupil Scores on Area Conservation and Measurement Test: Grade 3



Score
N = 50

Figure 2

Frequency of Pupil Scores on Area Conservation and Measurement Test: Grade 4 high. From these, the 56 subjects for further study were chosen by random selection.

Two instructional groups were formed by having children from one third-grade class and one fourth-grade class combined, without regard for level. Thus, each group contained a mixture of third-grade and fourth-grade pupils, and high-level and low-level children. These groups were of approximately equal size. The investigator, using the same treatment, instructed all groups for seven 46-minute periods. Instruction took place for all groups between morning recess and lunch time. The posttests were administered on the eighth day in the pupils' regular class with no time limit.

Analysis

Item analyses were performed and Hoyt (1941) reliability estimates obtained. The Fraction Concept Test and Transfer Test data were analyzed using separate univariate analyses of variance. The two factors, Area Concept and Grade, had two levels each. Correlations were computed among the scores on Area, Fraction Concept, and Age in months.

Results and Conclusions

Test Analysis

The item difficulties by grade level of the Area Concept Test are shown in Table 1. The Hoyt reliability estimate of the Area Concept Test was .75. Item difficulties of the 51-item Fraction Concept Test ranged from .41 to .90 except for three items (0, .03, .17). Item difficulties for the eight-item Transfer Test ranged from .16 to .52. The Hoyt reliability estimates were .96 and .70 for the Fraction Concept Test and Transfer Test, respectively.

Table 1

Item Data: Concept Test

Item Number		1	2	3	4	5	6
Grade 3 (51 subjects)	Number correct	35	29	24	17	34	19
	P(Item difficulty)	.69	.57	.47	.33	.67	.37
Grade 4 (50 subjects)	Number correct	34	28	30	21	28	18
	P(Item difficulty)	.68	.56	.60	.42	.56	.36
Total (N = 101)	Number correct	69	57	54	38	62	37
	P(Item difficulty)	.69	.56	.53	.38	61	.37

Analysis of Variance

. A summary of the ANOVA's performed on the variables Fraction Concept and Transfer are contained in Tables 2 and 3, respectively. In both cases Area Concept was a significant factor, but in neither case was Grade significant.

The means on Fraction Concept and Transfer are given for High and Low levels in Table 4. It will be observed that the achievement test means of 69 percent for the Low level and 82 percent for the High level were reasonably high, whereas the Transfer Test means were considerably lower.

The correlations of age with other variables are given in Table 5. Only the correlation of .50 between Area Concept and Fraction Concepts was significant ($p < .01$). This, of course, is consistent with the results of the Analysis of Variance.

Table 2

Analysis of Variance for Fraction Concept

Source	df	MS	F
Area	1	743.14	18.24*
Grade	1	52.07	1.28
A X G	1	1.1	< 1
Error	52	40.75	

* $p < .01$

Table 3

Analysis of Variance for Transfer

Source	df	MS	F
Area	1	44.64	16.70*
Grade	1	4.57	1.71
A X G	1	3.51	1.31
Error	52	2.67	

* $p < .01$

Table 4

Group Means by Level for Fraction Concept and Transfer

	Low Level	High Level	Total	Number of Items
Fraction Concept	35.0	42.3	38.6	51
Transfer	1.5	3.3	2.4	8

Table 5

Correlation Matrix

	Age	Fraction Concept
Area Concept	.05	.50*
Age		.07

* $p < .01$

It was a surprising result that grade or age had no detectable relationship to either area measurement or fraction learning. Perhaps one year's difference at this particular age is not a great enough age span to expect differences in performance. For the children in the sample at least, it appears that children in grades three and four are equally capable of informal area measurement. Children in the third grade also appear to learn an initial fraction concept as well as those in the fourth grade. Again, the sample was from one school and it remains to be seen how general the results are.

It does appear that area concept level is related to fraction task achievement, at least when the fraction work is based on an area model. This is not to say that the children in the low area group cannot learn the fraction work. In the present study the low group mean was at about 70 percent.

While there were significant differences in the high and low area groups on the transfer measures, the means were fairly low. It is possible that more evidence of transfer could be shown by a different criterion. Perhaps there is a particular kind of transfer not detected by the test of a general nature. Time to criterion on the new task might be a measure of transfer worthy of consideration.

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