DOCUMENT RESUME

ED 209 077 SE 035 770

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TITLE Inequalities, Signum Functions and Wrinkles in Wiggle Graphs.
PUB DATE 81
NOTE 11p.; Contains occasional light and broken type.

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Calculus; *College Mathematics; *Graphs; Higher Education; Instructional Materials; Mathematical Concepts; *Mathematics Instruction; *Problem Solving; Secondary Education; Secondary School Mathematics; *Teaching Methods; Undergraduate Study
IDENTIFIERS *Equations (Mathematics); Functions (Mathematics)

ABSTRACT

Presented is a graphical approach to teaching higher degree, rational function, and absolute value inequalities that simplifies the solution of these inequalities and thereby reduces the amount of classroom time that has to be devoted to this topic. Applications are also given for signum functions, maximum-minimum, and points of inflection problems in calculus. The term "wiggle graph" is used to identify the shape of functions produced. Such methods for graphing functions are felt to provide students a way to have a feel for what is going on through seeing how the functions behave. (MP)

INEQUALITIES, SIGNUM FUNCTIONS AND WRINKLES IN WIGGLE GRAPHS

by

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Most college algebra and calculus teachers would agree that it is more difficult and time-consuming to teach higher degree, rational function and absolute value inequalities than it is to teach linear inequalities. One reason for this is that many textbooks today continue to use a "cases" approach to solve these inequalities [4, 5, 8]. In this paper, a graphical (wiggle graph) approach will be given which simplifies the solution of these inequalities and thereby reduces the amount of classroom time one has to devote to this topic.

Applications will also be given to signum functions, maximum-minimum, and points of inflection problems in calculus.

WIGGLE GRAPHS

Some graphical methods for solving higher degree inequalities were introduced by Brixey and Andree in [2]. These methods can be extended and improved by adapting some of the general graphical techniques of Boersig as found in [1]. Boersig gives methods for graphing functions which "enable the student to have a 'feel' for what is going on by seeing how functions behave." [1, p. 355]. For example, the general shape or "wiggle graph" of $f(x) = (x - 1)(x - 2)^2$ can be quickly sketched by noting that

(1) $f(x)$ is positive when x is large and

(2) $f(x)$ changes sign in the neighborhood of $x = 1$, but not in the neighborhood of $x = 2$ [1, p. 360]. See figure 1.

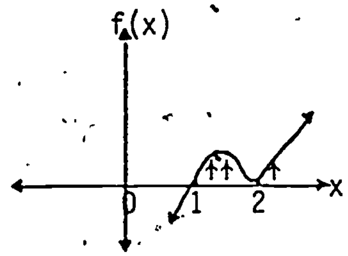

Fig. 1

HIGHER DEGREE INEQUALITIES

We now apply wiggle graphs to the solution of higher degree inequalities. For example, suppose that we wish to find

$$\{x \mid x(x-2)^2(x+3)^3(x-4)^4(x+5)^5 > 0\}.$$

We first let $f(x) = x(x-2)^2(x+3)^3(x-4)^4(x+5)^5$ and find its wiggle graph. See figure 2.

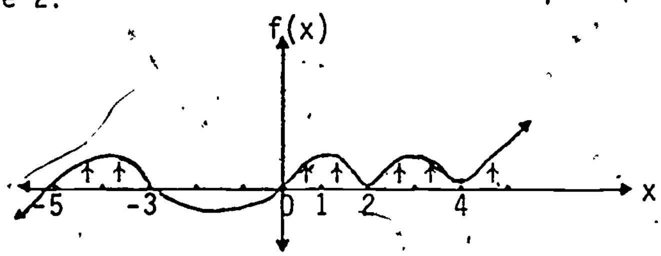

Fig. 2

The solution set is simply $\{x \mid f(x) > 0\}$, which is quickly seen from figure 2 to be $(-5, -3) \cup (0, 2) \cup (2, 4) \cup (4, \infty)$. Once the wiggle graph is sketched, it is just as easy to find $\{x \mid f(x) \geq 0\}$, $\{x \mid f(x) < 0\}$ or $\{x \mid f(x) \leq 0\}$.

Certainly, this technique works equally well on quadratic inequalities and eliminates the need for the "cases" approach as found in the popular texts [4] and [5]. Moreover, it can be shown that this technique also eliminates the "cases" approach for solving rational function and absolute value inequalities as in [8].

RATIONAL FUNCTION INEQUALITIES

Suppose we wish to find

$$A = \{x \mid \frac{x(x-2)(x-1)^2}{x+2} > 0\}.$$

The usual procedure is to consider the case in which $x + 2 > 0$ and find the solution set for $x(x - 2)(x - 1)^2 > 0$; then consider $x + 2 < 0$ and find the solution set for $x(x - 2)(x - 1)^2 < 0$. The final solution set is taken to be the union of the solution sets in the two cases investigated. However, it suffices to find $B = \{x \mid x(x - 2)(x + 2)(x - 1)^2 > 0\}$, since $a/b > 0$ if and only if $ab > 0$. Considering the wiggle graph of $f(x) = x(x - 2)(x + 2)(x - 1)^2$ in figure 3, we see that the solution set is $(-2, 0) \cup (2, \infty)$.

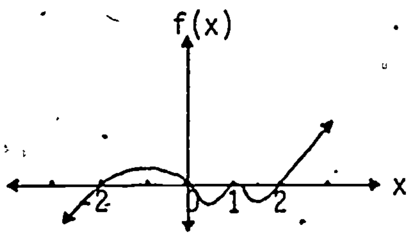

Fig. 3

One word of caution is in order.

$$A = \{x \mid \frac{x-1}{x+2} \geq 0\} \neq \{x \mid (x - 1)(x + 2) \geq 0\} = B$$

The number -2 is in set B but not in set A. Nevertheless, the wiggle graph of $f(x) = (x - 1)(x + 2)$ gives us an *almost* accurate picture of A and the zeros of the denominator or poles of the function can be treated individually.

ABSOLUTE VALUE INEQUALITIES

Since $|a| - |b| < 0$ (>0) if and only if $a^2 - b^2 < 0$ (>0), it suffices to apply the wiggle graph technique to $k(x) = (x^4 + 4x^3 - 17x^2 + 16x - 4)^2 - (x^4 + 11x^2 - 32x - 4)^2$ if one wishes to find

$$C = \{x \mid |x^4 + 4x^3 - 17x^2 + 16x - 4| < |x^4 + 11x^2 - 32x - 4|\}.$$

Since $k(x)$ is the difference of squares, it takes but a moment to see that $k(x) = 8x(x - 2)(x + 2)(x + 1)^2(x - 3)(x - 4)$. The wiggle graph for $k(x)$ is given in figure 4. Therefore, $C = (-\infty, -2) \cup (0, 2) \cup (3, 4)$.

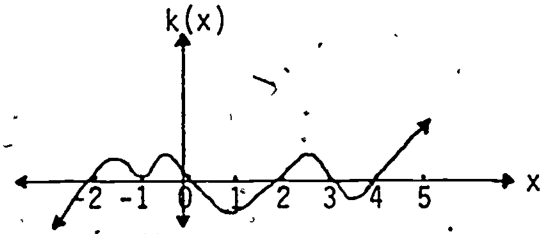

Fig. 4

SIGNUM FUNCTION

Recall that the signum function is defined as

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Its graph is given in figure 5.

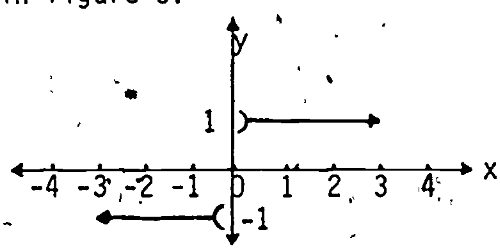

Fig. 5

With the aid of the wiggle graph, we are now able to easily visualize the graphs of some rather "exotic" and complicated functions.

Example: Graph $y = \operatorname{sgn}[x(x - 2)(x + 2)(x - 1)^2] + x$. Let $g(x) = x(x - 2)(x + 2)(x - 1)^2$. Then

$$y = \operatorname{sgn}[g(x)] + x = \begin{cases} 1 + x & \text{if } g(x) > 0 \\ x & \text{if } g(x) = 0 \\ -1 + x & \text{if } g(x) < 0 \end{cases}$$

First, in figure 6, lightly dot in the graphs of $y = 1 + x$, $y = x$ and $y = -1 + x$. Then it is easy to look at the graph in figure 3 to see when

$g(x)$ is greater than, less than, or equal to, zero and sketch the graph of $y = \mathrm{sgn}[x(x - 2)(x + 2)(x - 1)^2] + x$ as shown in figure 6 (i.e., the final, heavily marked graph).

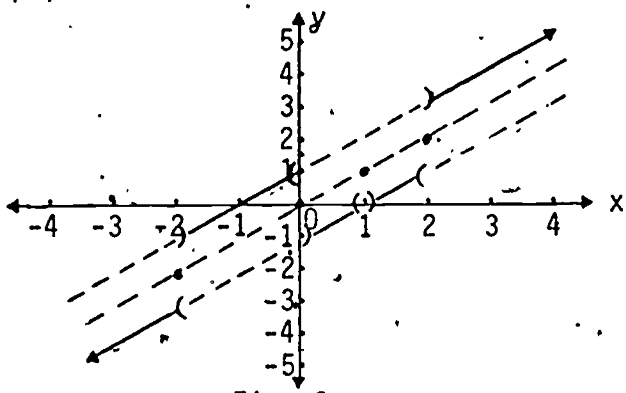

Fig. 6

Example: Graph

$$y = x\,\mathrm{sgn}\,\{|x^4 + 4x^3 - 17x^2 + 16x - 4| - |x^4 + 11x^2 - 32x - 4|\} + 2.$$

Let $h(x) = |x^4 + 4x^3 - 17x^2 + 16x - 4| - |x^4 + 11x^2 - 32x - 4|$. Since

$$y = x\,\mathrm{sgn}[h(x)] + 2 = \begin{cases} x + 2 & \text{if } h(x) > 0 \\ 2 & \text{if } h(x) = 0 \\ -x + 2 & \text{if } h(x) < 0, \end{cases}$$

we need those values of x for which $h(x)$ is positive, negative or zero. These can be read directly from the wiggle graph of $k(x)$ in figure 4.

Then, for the originally desired graph of $y = x\,\mathrm{sgn}\,h(x) + 2$, lightly dot the graph of $y = x + 2$, $y = 2$, and $y = -x + 2$ as guidelines, consult the wiggle graph in figure 4, and sketch the result as it appears in figure 7.

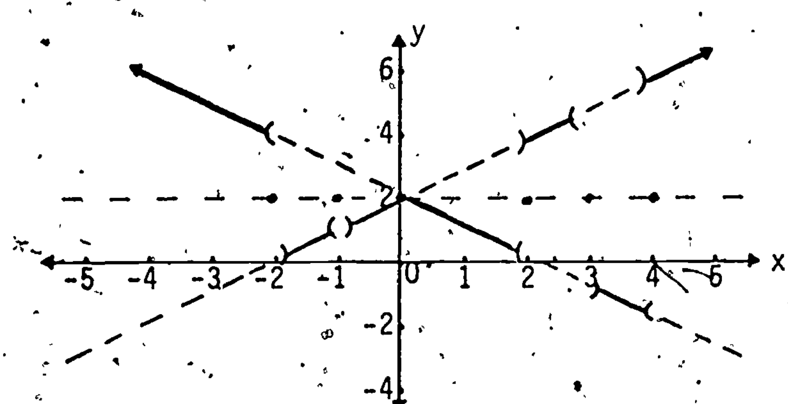

Fig. 7

The functions considered in this section are excellent examples of piecewise continuous functions which have both removable and non-removable points of discontinuity. (In preparing your own problems involving absolute values, note that if $P(x)$ and $Q(x)$ are polynomials such that $P(x) \neq Q(x)$ and a and b are nonzero real numbers, then $|aP(x) + bQ(x)| < |aP(x) - bQ(x)|$ if and only if $4ab\,P(x)Q(x) < 0$. Thus there are infinitely many problems with the same graph as the graph in figure 7.

WRINKLES IN WIGGLE GRAPHS

Many times in calculus a "cases" approach is used in analyzing the first and second derivatives of a function to see when the original function is increasing, decreasing, concave up or concave down [3], [9]. Since we are simply checking for those values of x for which $f'(x) > 0$ (<0) and $f''(x) > 0$ (<0), we can avoid the cases approach by quickly sketching the wiggle graphs of $f'(x)$ and $f''(x)$ and gleaning the information we need from the graphs. Care must be taken that one does not confuse the sketching of $f'(x)$ and $f''(x)$ with the graphing of $f(x)$.

Example: [8, p. 172] Determine the intervals in which the following function is increasing or decreasing, find the relative maxima and minima and then graph the function

$$f(x) = x^4 - \frac{4}{3}x^3 - 4x^2 + \frac{2}{3}.$$

Since $f'(x) = 4x^3 - 4x^2 - 8x = 4x(x + 1)(x - 2)$ it is easy to see that its wiggle graph is as follows

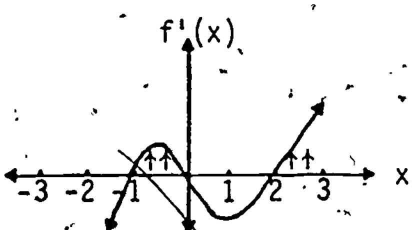

Fig. 8

It is helpful to the beginning student if he is also required to erect light vertical lines at $x = -1$, 0 and 2 and then either place the abbreviation "inc." or a straight mark with positive slope over those intervals for which $f'(x) > 0$ and similar remarks or marks over those intervals for which $f'(x) < 0$. See figure 9.

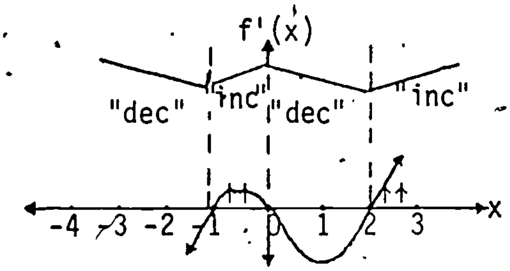

Fig. 9

The straight marks used in conjunction with the wiggle graph of $f'(x)$ make it clear to the student that he has relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 0$. Points of inflection can be found by finding the wiggle graph of $f''(x)$. This will be discussed in the next example. A rather accurate graph of $f(x)$ is given in figure 10.

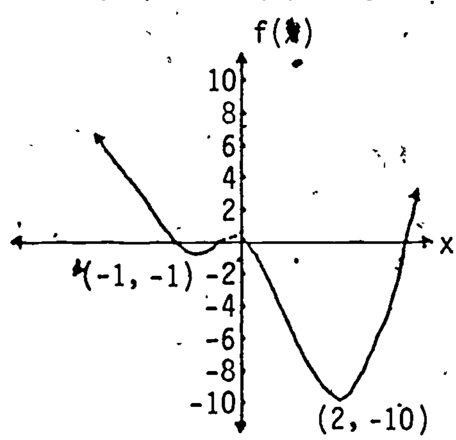

Fig. 10

Example: [8, p. 179] Determine all maxima, minima and points of inflection for the following function and then sketch an *accurate* graph.

$$f(x) = 5x^{2/3} - x^{5/3}.$$

It is interesting to note that $f(x) = x^{2/3}(5 - x)$ and that the techniques of Boersig [1] can be extended to give us a general graph for $f(x)$ before analyzing its derivatives. The factor $x^{2/3}$ contributes $x = 0$ as a root of $f(x)$ but $f(x)$ will not change sign in a neighborhood of $x = 0$; i.e., $x^{2/3}$ behaves essentially as x^2 in the neighborhood of $x = 0$. This is in total agreement with [1] and a wiggle graph of $f(x)$ is given in figure 11.

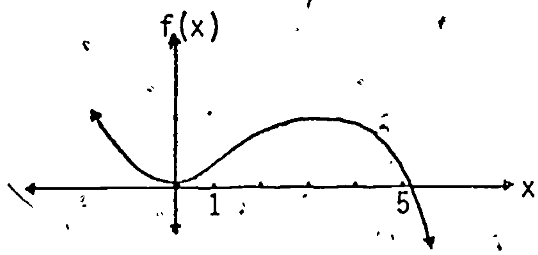

Fig. 11

We now need to analyze the derivatives so that we can determine all of the "wrinkles" in $f(x)$.

$$f'(x) = 5(2 - x)/(3x^{1/3}) \text{ and}$$
$$f''(x) = -10(1 + x)/(9x^{4/3}).$$

We wish to find all $x \in \mathbb{R}$ such that $f'(x) > 0$ (<0) and $f''(x) > 0$ (<0). Just as above, in the rational function section, it suffices to find all $x \in \mathbb{R}$ such that $P[f'(x)] = \frac{5}{3}x^{1/3}(2 - x) > 0$ (<0) and $P[f''(x)] = \frac{-10}{9}x^{4/3}(1 + x) > 0$ (<0). The "P" is used in front of $f'(x)$ and $f''(x)$ to indicate we are considering a "product" of the variable terms of $f'(x)$ and $f''(x)$.

The wiggle graphs of $P[f'(x)]$ and $P[f''(x)]$ are in figure 12. Rather than a light vertical line erected on the zero $x = 0$, it is recommended that a light wavy line be erected. This is to indicate that $f'(x)$ and $f''(x)$ do not exist there. Also, it is helpful to either make the symbol $\smile$ or $\frown$ over those intervals for which $P[f''(x)]$ is positive or negative indicating that $f(x)$ is concave up or concave down in those intervals.

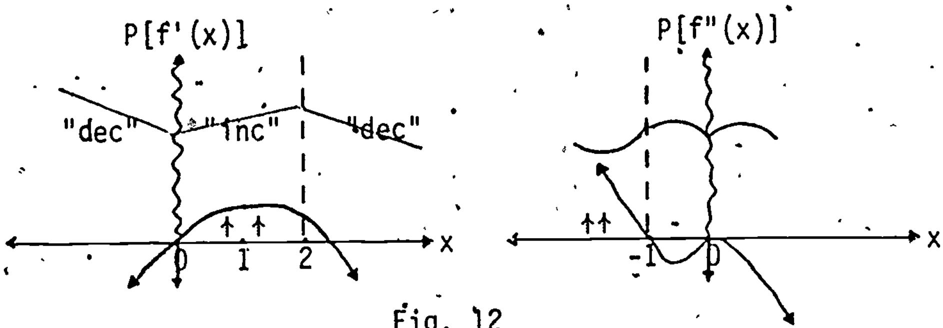

Fig. 12

The wavy line at $x = 0$ along with the back to back downward concavities for P[f'(x)] helps us to ascertain a cusp at $x = 0$. The graph of the function $f(x) = 5x^{2/3} - x^{5/3}$ with all of its "wrinkles" is as follows

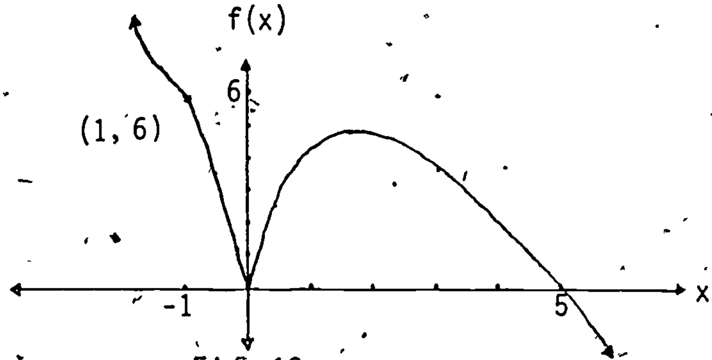

Fig. 13

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