DOCUMENT RESUME

ED 204 360 TM 810 340

AUTHOR Blumberg, Carol Joyce: Porter, Andrew C.

TITLE Models of Continuous Growth and Their Implications for the Analysis of Nonequivalent Control Group Designs.

PUB DATE Apr 81

NOTE 27p.: Paper presented at the Annual Meeting of the American Educational Research Association (65th, Los Angeles, CA, April 13-17, 1981).

EDRS PRICE MF01/PC02 Plus Postage.

DESCRIPTORS *Analysis of Covariance: *Analysis of Variance: *Mathematical Models: Pretests Posttests: *Quasiexperimental Design

IDENTIFIERS *Change Scores: Fan Spread Hypothesis: Nonequivalent Control Groups: *Residual Scores

ABSTRACT

Analysis strategies are discussed for the nonequivalent control group design when three models of continuous natural growth are known. For Model I type natural growth it was shown that the fan spread hypothesis always holds, and Analysis of Covariance (ANCOVA), Analysis of Variance (ANOVA) of Residualized Gain Scores, and ANOVA of Standardized Change Scores all are potentially correct analysis strategies. For Model II and Model III type natural growth it was shown that the fan spread hypothesis holds and the ANCOVA, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores are potentially correct analysis strategies only when Model II and Model III type natural growth reduce to Model I type natural growth. Further, it was shown that given any natural growth situation, there is a value of K for which ANOVA of Index of Response is a potentially correct analysis strategy. In order that an index of response works, the exact form of natural growth must be known. Most growth models, and in particular most linear growth models, do not conform to the fan spread hypothesis nor are the usual analysis strategies correct for these models. (Author/RL)

Models of Continuous Growth and their Implications for the Analysis of Nonequivalent Control Group Designs

Carol Joyce Blumberg
Andrew C. Porter

Michigan State University

April, 1981

Paper presented at the 1981 annual meeting of the American Educational Research Association, Los Angeles, April, 1981.

The authors wish to thank Drs. James Stapleton, Robert Floden, William Schmidt, and Joseph Gardiner for their comments on earlier versions of this paper. We also wish to thank Ms. Linda Stiles for her excellent typing of the manuscript.

ABSTRACT

The paper discusses analysis strategies for the nonequivalent control group design when models of natural growth are known. Several models of continuous growth are shown to satisfy the fan spread hypothesis. For these growth models it is also shown that Analysis of Covariance, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores yield correctly defined treatment effects. On the one hand, these models are not restricted to linear growth, as past literature would suggest. On the other hand, these are the only models of growth shown to fit the fan spread hypothesis and for which the three analysis strategies yield correctly defined effects. Consequently, most growth models, and in particular most linear growth models, do not conform to the fan spread hypothesis nor are the usual analysis strategies correct for these models.

In many educational settings, a true-experimental design is not possible when a researcher wants to evaluate the effects of different treatments. Thus, quasi-experimental designs are employed. One of the more commonly used quasi-experimental designs is the nonequivalent control group design (Campbell & Stanley, 1966, pp. 47-50, 55-57, 61-64; Campbell, 1969). For this design, pre- and post-observations on the same measure are available for subjects in two non-randomly created comparison groups. The two groups may be either a treatment group and a control group or two different treatment groups. While the design allows for several pretest and posttest observations on each individual, in this paper consideration is restricted to designs with a single pretest and posttest.

There has been much discussion in the literature of the analysis strategies that are appropriate for use in connection with non-equivalent control group designs. The basic problem is to identify analysis strategies which will provide unbiased estimates of the treatment effects. This problem has come to be known as the problem of measuring change. Unless some assumptions are made, there is no knowably correct analysis strategy for use with any particular application of a nonequivalent control group design. A short and excellent discussion of this is given by Lord (1967). One of the possible approaches is to make some assumptions about the data that would result under natural growth. It is only within the context of a particular set of assumptions that the appropriateness and non-appropriateness of particular analysis strategies can be discussed.

One assumption frequently made in the literature is that data conform to the fan spread hypothesis (see e.g., Campbell, 1971; Kenny

& Cohen, 1980). The fan spread hypothesis states that the ratio of the difference of group means to the standard deviation common to the two populations (from which the two groups are drawn) is constant over time. Without loss of generality, throughout this paper it will be assumed that the pretest, X, is given at time t = 0. The posttest, Y(t), is given at some time, t > 0. Symbolically, then, the fan spread hypothesis can be expressed

$$\frac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\sigma_Y(t)} = \frac{\mu_{X_1} - \mu_{X_2}}{\sigma_X} \quad (1)$$

where μ_{X_i} = population mean for group i on the pretest; i = 1,2

$\mu_{Y_i}(t)$ = population mean for group i on the posttest at time t; i = 1,2

σ_X = standard deviation common to both populations on the pretest

and $\sigma_Y(t)$ = standard deviation common to both populations on the posttest at time t.

Assuming the fan spread hypothesis for natural growth, the appropriateness of some analysis strategies can be discussed. There is, however, a major problem with discussing analysis strategies for nonequivalent control group designs only in terms of the fan spread hypothesis. As will be shown later in this paper, it is only in rare cases that data conform to the fan spread hypothesis. Hence, the fan spread hypothesis should not be the focus of attention when discussing the problem of measuring change.

Following the lead of Bryk and Weisberg (1977) and in contrast to the fan spread hypothesis, this paper considers assumptions (models) about continuous natural growth. Even though making assumptions about continuous growth is further removed than the fan spread hypothesis from commonly employed analysis strategies, it is closer to the usual ways of thinking about growth and so should facilitate judgement about the reasonableness of the assumptions for actual data. Another reason for considering continuous growth models is that they reflect growth as a dynamic process changing over time in a continuous manner. By employing assumptions such as the fan spread hypothesis, much of the literature on the problem of measuring change has ignored this dynamic nature of growth, As will be shown later in this paper, an analysis strategy may be appropriate at certain time points for the posttest but not at other time points. Thus, dependencies on time must also be considered.

The approach to investigating the problem of measuring change taken in this paper is to first posit a particular model of natural growth. The models of natural growth posited will be representative of growth models that have been suggested for various types of academic and/or physical growth. Given a particular model of natural growth, a description can then be given in terms of parameters estimable from a particular nonequivalent control group design. Against these parameters, the appropriateness of analysis strategies can then be investigated.

Analysis Strategies

Before proceeding further, it is convenient to introduce the various analysis strategies to be considered in this paper and to specify how treatment effects are defined under each. Throughout this paper, it is assumed that treatment effects are additive.

That is, that a treatment causes an increase or decrease of the same amount for everyone from their scores under natural growth. The null and alternate hypotheses for each analysis strategy can then be stated

$$\begin{array}{ll} & H_0: \ \alpha_{\text{analysis strategy}} = 0 \\ \text{vs.} & H_1: \ \alpha_{\text{analysis strategy}} \neq 0 . \end{array}$$

where $\alpha_{\text{analysis strategy}}$ defines the treatment effect under a particular analysis model.

The primary question investigated in this paper is which analysis strategies yield null treatment effects under each natural growth model considered. An analysis strategy is to be considered a potentially correct analysis strategy if under natural growth it gives null treatment effects. The reason for the word potential is that there remain questions of distributional assumptions and precision.

Analysis of Covariance

The linear model for ANCOVA can be stated as (Winer, 1971)

$$Y_{ij}(t) = \mu_Y(t) + (\alpha_{AC})_i + \beta_{Y(t)\cdot X}(X_{ij} - \mu_X) + (e_{AC})_{ij}$$

where X_{ij} and $Y_{ij}(t)$ are the pretest and posttest scores respectively for individual j in group i; i = 1,2

μ_X and $\mu_Y(t)$ are the population grand means for X and Y(t) respectively

$\beta_{Y(t)\cdot X}$ is the slope of the regression line of Y(t) on X for each group

AC denotes analysis of covariance

$$(\alpha_{AC})_i = \mu_{Y_i}(t) - \mu_Y(t) - \beta_{Y(t)\cdot X}\cdot(\mu_{X_i} - \mu_X) \qquad (2)$$

is the treatment effect for group i; i = 1,2

and $(e_{AC})_{ij}$ = the error term for an individual.

In this paper, all designs considered are two group designs. For two group designs, $(\alpha_{AC})_1 = -(\alpha_{AC})_2$ because $\sum_i (\alpha_{AC})_i = 0$.

For later convenience, let α_{AC} symbolize the quantity $2(\alpha_{AC})_1$.

$$\begin{aligned} 2(\alpha_{AC})_1 &= (\alpha_{AC})_1 + (\alpha_{AC})_1 \\ &= (\alpha_{AC})_1 + -(\alpha_{AC})_2 \qquad \text{since } (\alpha_{AC})_1 = -(\alpha_{AC})_2 \\ &= [\mu_{Y_1}(t) - \mu_Y(t) - \beta_{Y(t)\cdot X}\cdot(\mu_{X_1} - \mu_X)] \\ &\quad - [\mu_{Y_2}(t) - \mu_Y(t) - \beta_{Y(t)\cdot X}\cdot(\mu_{X_2} - \mu_X)] \qquad \text{by (2)} \\ &= \mu_{Y_1}(t) - \mu_{Y_2}(t) - \beta_{Y(t)\cdot X}\cdot(\mu_{X_1} - \mu_{X_2}) . \end{aligned}$$

Hence

$$\alpha_{AC} = [\mu_{Y_1}(t) - \mu_{Y_2}(t)] - \beta_{Y(t)\cdot X}\cdot(\mu_{X_1} - \mu_{X_2}) . \qquad (3)$$

Analysis of Variance of Index of Response

ANOVA of Index of Response (Cox, 1958) is actually a set of analysis strategies. ANOVA of Index of Response will first be discussed in general and then some specific cases will be discussed in further detail. Let K be some real constant. An index of response is then defined by $Z_{ij}(t)$, where $Z_{ij}(t) = Y_{ij}(t) - K\cdot X_{ij}$. An ANOVA of Index of Response is nothing more than an ANOVA performed on the $Z_{ij}(t)$'s. The linear model for ANOVA of Index of Response is then as for ANOVA

$$Z_{ij}(t) = \mu_Z(t) + (\alpha_{IR})_i + (e_{IR})_{ij}$$

where $\mu_Z(t)$ = population grand mean for Z

$= \mu_Y(t) - K\cdot\mu_X$

IR denotes Index of Response

$(\alpha_{IR})_i = \mu_{Y_i}(t) - \mu_Y(t) - K(\mu_{X_i} - \mu_X)$ is the treatment effect for group i; i = 1,2

and $(e_{IR})_{ij}$ = the error term for an individual.

Let $\alpha_{IR} = 2(\alpha_{IR})_1$. By a derivation analogous to that for α_{AC} it can be shown that

$$\alpha_{IR} = \mu_{Y_1}(t) - \mu_{Y_2}(t) - K\cdot(\mu_{X_1} - \mu_{X_2}) . \qquad (4)$$

Assuming that $\mu_{X_1} \neq \mu_{X_2}$, notice that $\alpha_{IR} = 0$ if and only if

$K = \dfrac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\mu_{X_1} - \mu_{X_2}}$. Here, and throughout the rest of this paper, the cases where $\mu_{X_1} = \mu_{X_2}$ will be ignored. So, a proper index of response, namely $Z_{ij}(t) = Y_{ij}(t) - \dfrac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\mu_{X_1} - \mu_{X_2}} \cdot X_{ij}$, always exists unless $\mu_{X_1} = \mu_{X_2}$. The problem is, of course, that the value of $\dfrac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\mu_{X_1} - \mu_{X_2}}$ is unknown in most situations. It should also be noted that $\dfrac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\mu_{X_1} - \mu_{X_2}}$ is a function of time, hence the indices of response will be different for different values of t.

Some specific values of K which are of interest are

(1) $K = 1$. When $K = 1$, the analysis strategy is more commonly known as ANOVA of Gain Scores. In this case, equation (4) reduces to

$$\alpha_{GS} = \mu_{Y_1}(t) - \mu_{Y_2}(t) - (\mu_{X_1} - \mu_{X_2}) \quad (5)$$

(2) $K = \frac{\sigma_Y(t)}{\sigma_X}$. When $K = \frac{\sigma_Y(t)}{\sigma_X}$, the analysis strategy is sometimes called ANOVA of Standardized Change Scores (Kenny & Cohen, 1980). In this case, equation (4) reduces to

$$\alpha_{SCS} = \mu_{Y_1}(t) - \mu_{Y_2}(t) - \frac{\sigma_Y(t)}{\sigma_X}(\mu_{X_1} - \mu_{X_2}) \quad (6)$$

(3) $K = \beta_{Y(t)\cdot X}$. When $K = \beta_{Y(t)\cdot X}$, the analysis strategy is sometimes called ANOVA of Residualized Gain Scores. In this case, equation (4) reduces to

$$\alpha_{RG} = \mu_{Y_1}(t) - \mu_{Y_2}(t) - \beta_{Y(t)\cdot X}(\mu_{X_1} - \mu_{X_2}) \quad (7)$$

Sometimes $\beta_{Y(t)\cdot X}$ is estimated when setting a value for K. In the literature on the problem of measuring change, the distinction between knowing versus estimating $\beta_{Y(t)\cdot X}$ for use in an ANOVA of Index of Response is not made clear. As will be seen, the distinction is not important for this paper which focuses on null treatment effects defined by each of several analysis strategies. Nevertheless, there are important differences between these two procedures when the topics of hypothesis testing or confidence interval estimation are considered (Olejnik & Porter, 1981).

Some Particular Natural Growth Models

Three models of individual natural growth will be considered.

MODEL I: $Y_{ij}(t) = g(t) \cdot X_{ij} + h(t)$

where $g(t)$ and $h(t)$ are continuous functions in t, $g(0) = 1$ and $h(0) = 0$. Further $X_{ij} > 0$ and $g(t) > 0$ for $0 < t < t_o$, where t_o is a time past which no observations on $Y(t)$ will be taken.

MODEL II: $Y_{1j}(t) = g(t)X_{1j} + h_1(t)$

and

$Y_{2j}(t) = g(t)X_{2j} + h_2(t)$

where $g(t)$, $h_1(t)$, and $h_2(t)$ are continuous functions in t, $g(0) = 1$, $h_1(0) = h_2(0) = 0$, $X_{ij} > 0$, and $g(t) > 0$ for $0 < t < t_o^*$; $i = 1,2$.

MODEL III: $Y_{1j}(t) = g_1(t)X_{1j} + h_1(t)$

and

$Y_{2j}(t) = g_2(t)X_{2j} + h_2(t)$

where $g_1(t)$, $g_2(t)$, $h_1(t)$, and $h_2(t)$ are continuous functions in t, $g_1(0) = g_2(0) = 1$, $h_1(0) = h_2(0) = 0$, $X_{ij} > 0$, and $g_i(t) > 0$ for $0 < t < t_o$; $i = 1,2$.

Model III is the functional representation of those growth models where there is a perfect correlation between the pretest and the posttest within each group. Models I and II are special important sub-models of Model III. In Model I, if $g(t) = 0$ for some time t, then the correlation is undefined for that point in time. If $g(t)$ is negative then the correlation is -1. Negative correlations, and in particular correlations of -1, between the pretest and posttest are not likely in actual situations. Hence, the restriction $g(t) > 0$ was made. The restriction $X_{ij} > 0$ was made solely for convenience. When $g(t) < 0$ or when $X_{ij} < 0$ for one or more individuals the theory becomes much more complicated. It was decided that these complications were beyond the scope of this paper. The reason for the restrictions $g(0) = 1$ and $h(0) = 0$ in Model I is consistency. By definition $Y_{ij}(0) = X_{ij}$. But under Model I, $Y_{ij}(0) = g(0)X_{ij} + h(0)$. Hence, $X_{ij} = g(0)X_{ij} + h(0)$ for each individual. Consequently, $g(0) = 1$ and $h(0) = 0$. The restrictions for Models II and III were placed there for analogous reasons.

Model I Results

Examples of Model I Type Growth

(1) Parallel growth. Parallel growth is defined by $g(t) \equiv 1$. That is, $Y_{ij}(t) = X_{ij} + h(t)$, where $h(t)$ is any continuous function. See Figure 1 for a pictorial representation of parallel growth.

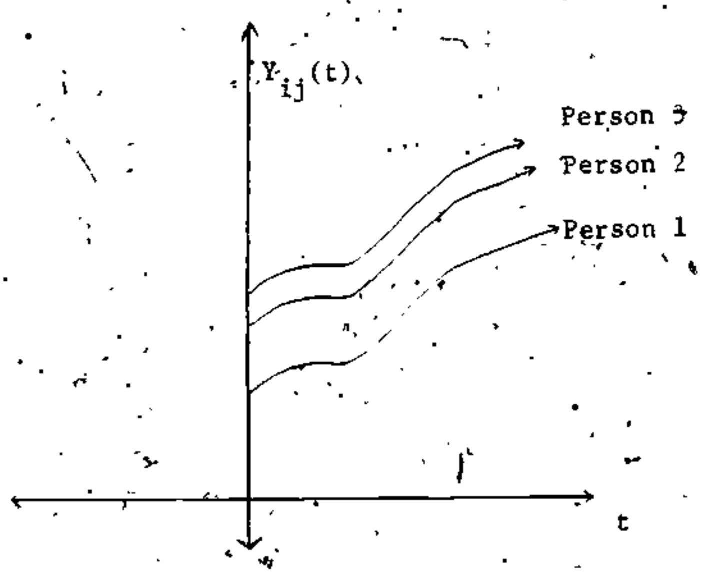

Figure 1: An Example of Parallel Growth

(2) Differential linear growth. Differential linear growth is defined by $g(t) = m \cdot t + 1$ and $h(t) \equiv 0$, where m is some nonzero

constant. That is, $Y_{ij}(t) = (m \cdot t + 1)X_{ij}$. See Figures 2 and 3 for pictorial representations of differential linear growth. The solid portions of each curve in these figures and in all the remaining figures indicate that part of the natural growth curves under consideration in this paper.

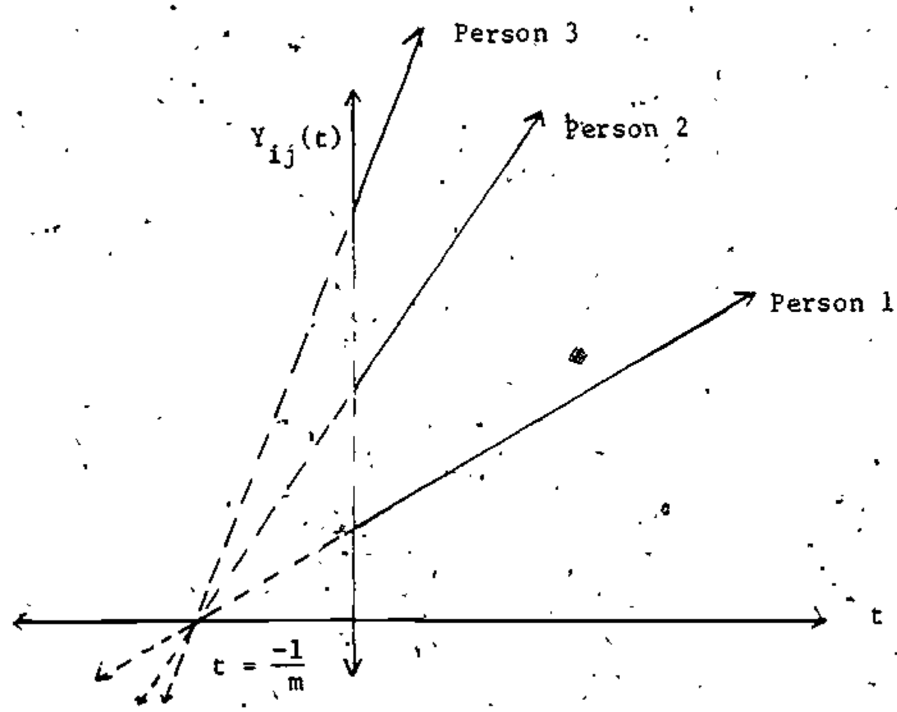

Figure 2: Differential Linear Growth under Model I when $m > 0$

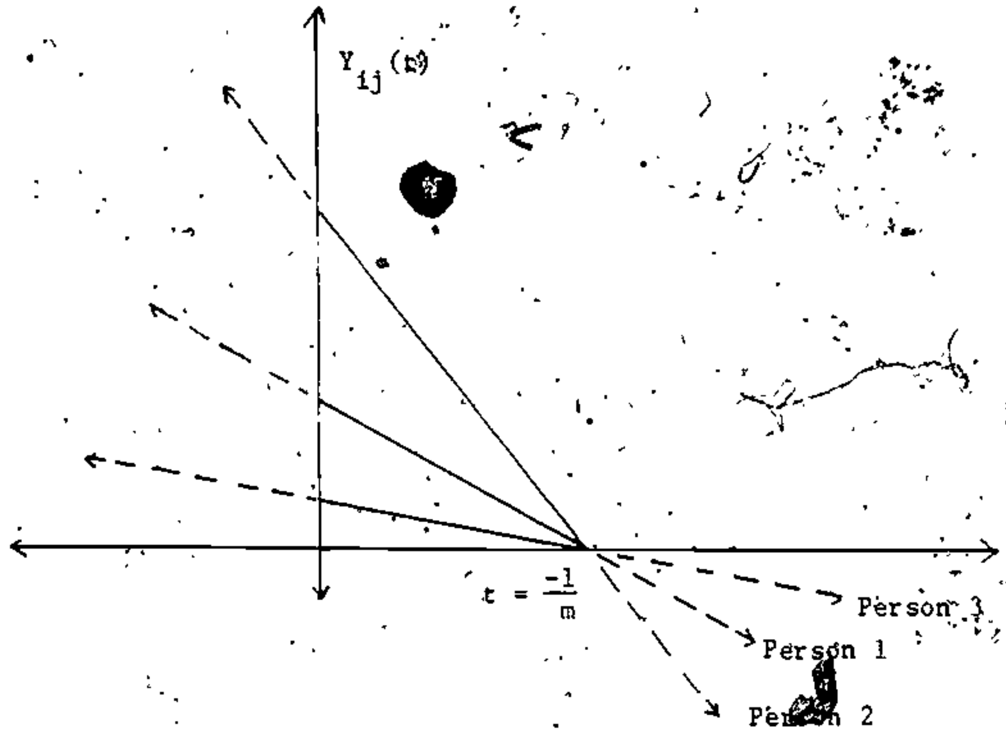

Figure 3: Differential Linear Growth under Model I when $m < 0$

(3) Exponential type growth. Exponential type growth is defined by $g(t) = a \cdot b^t + c$ and $h(t) \equiv 0$, where a, b, and c are constants with $b > 0$. The reason for the restriction $b > 0$ is to make b^t real valued. For values of $b < 0$, b^t takes on complex values. Hence, $Y_{ij}(t) = (a \cdot b^t + c) \cdot X_{ij}$. See Figures 4 and 5 for pictorical representations of exponential type growth.

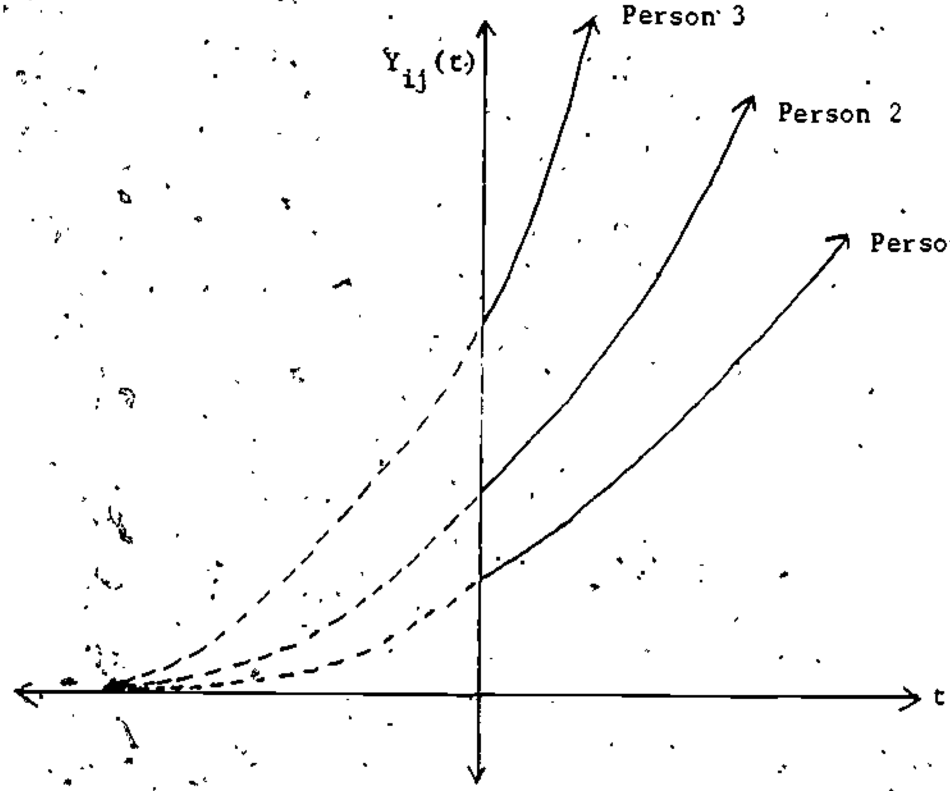

Figure 4: Exponential Type Growth

when a = 1, b > 1, and c = 0

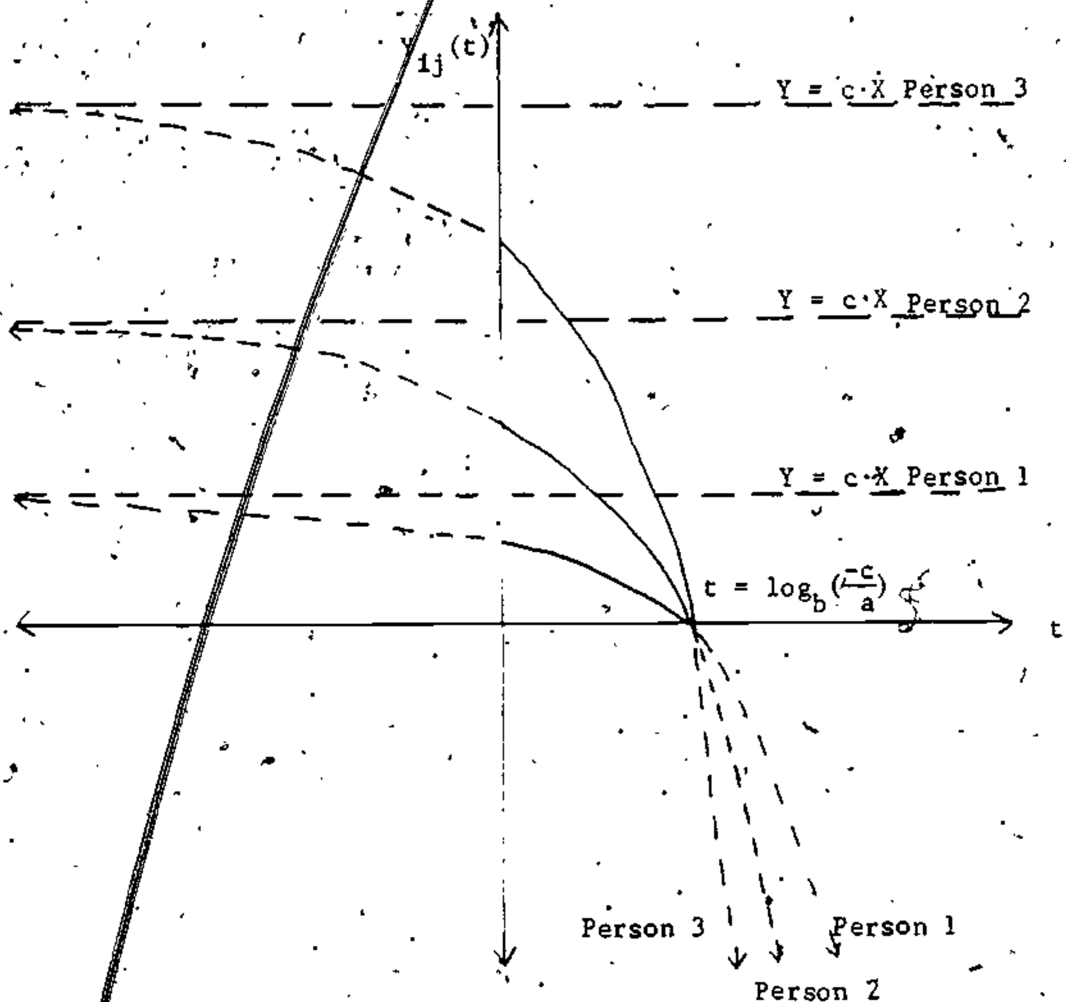

Figure 5: Exponential Type Growth

when $a < 0$, $b > 1$, $c > 1$, and $a + c = 1$

The inclusion of monotone decreasing functions (see e.g., Figures 3 and 5) as representatives of natural growth may seem strange. One topic which has received attention from learning theorists is forgetting curves. Forgetting curves are, of course, decreasing functions and hence the decision was made to include monotone decreasing functions as representatives of natural growth.

Some other particular examples of Model I type natural growth are:

(4) Logarithmic type growth

$$Y_{ij}(t) = [\log_a (b \cdot t + a)] \cdot X_{ij} ,$$

where a and b are constants with $a > 0$ and $a \neq 1$.

(5) Cumulative normal (Normal Ogive) type growth

$$Y_{ij}(t) = 2 \left[\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}v^2)\,dv\right] \cdot X_{ij} .$$

(6) Logistic type growth

$$Y_{ij}(t) = \frac{(1 + c) \cdot d^t}{1 + c \cdot d^t} \cdot X_{ij}$$

where c and d are constants with $c > 0$ and $d > 1$.

Fan Spread Hypothesis Under Model I

Recall that Model I type growth is defined in general by

$$Y_{ij}(t) = g(t) \cdot X_{ij} + h(t) .$$

First, notice that

$$\mu_{Y_1}(t) = g(t)\, \mu_{X_1}(t) + h(t)$$

and

$$\mu_{Y_2}(t) = g(t)\, \mu_{X_2}(t) + h(t) \tag{8}$$

Second, notice that

$$\sigma^2_{Y_1}(t) = g^2(t)\sigma^2_{X_1}(t)$$

and (9)

$$\sigma^2_{Y_2}(t) = g^2(t)\sigma^2_{X_2}(t) .$$

where $\sigma^2_{X_1}$ and $\sigma^2_{X_2}$ are the variances on the pretest for group 1 and group 2, respectively

and $\sigma^2_{Y_1}(t)$ and $\sigma^2_{Y_2}(t)$ are the variances on the posttest for group 1 and group 2, respectively.

Third, notice that while $Y_{ij}(t)$ and t can take any one of an infinity of possible relationships, some of which have just been illustrated, for any time t, X and Y have a linear relationship. Hence, $\rho_{XY(t)} \equiv 1$ for each of the two populations.

For the remainder of the Model I results section it will be assumed that there are equal variances across the two populations on the pretest and consequently, by equation (9), on the posttest. Hence,

$$\frac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\sigma_Y(t)}$$

$$= \frac{[g(t)\ \mu_{X_1} + h(t)] - [g(t)\ \mu_{X_2} + h(t)]}{g(t)\ \sigma_X}$$ by equations (8) and (9)

$$= \frac{g(t)\cdot(\mu_{X_1} - \mu_{X_2})}{g(t)\ \sigma_X}$$

$$= \frac{\mu_{X_1} - \mu_{X_2}}{\sigma_X} .$$

So,

$$\frac{\mu_{Y_1}(t) - \mu_{Y_2}(t)}{\sigma_Y(t)} = \frac{\mu_{X_1} - \mu_{X_2}}{\sigma_X} \qquad (10)$$

Hence the fan spread hypothesis holds when $Y_{ij}(t) = g(t)\,X_{ij} + h(t)$ under natural growth. Equation (10) can be rewritten in the form

$$0 = \mu_{Y_1}(t) - \mu_{Y_2}(t) - \frac{\sigma_Y(t)}{\sigma_X}\,(\mu_{X_1} - \mu_{X_2}) \ . \qquad (11)$$

This form is more convenient when discussing potentially correct analysis strategies.

Analysis of covariance. A treatment effect under ANCOVA is given by equation (3)

$$\alpha_{AC} = (\mu_{Y_1}(t) - \mu_{Y_2}(t)) - \beta_{Y(t)\cdot X}\,(\mu_{X_1} - \mu_{X_2}) \ . \qquad (3)$$

But, $\beta_{Y(t)\cdot X} = \rho_{XY(t)} \cdot \frac{\sigma_Y(t)}{\sigma_X}$

$= \frac{\sigma_Y(t)}{\sigma_X}$ under Model I type growth because $\rho_{XY(t)} \equiv 1$.

So,

$$\alpha_{AC} = \mu_{Y_1}(t) - \mu_{Y_2}(t) - \frac{\sigma_Y(t)}{\sigma_X}\cdot(\mu_{X_1} - \mu_{X_2}) \ . \qquad (12)$$

Hence, by comparing equations (11) and (12), ANCOVA is always a potentially correct analysis strategy under Model I type natural growth.

ANOVA of Index of Response. A treatment effect under ANOVA of Index of Response is given by equation (4)

$$\alpha_{IR} = (\mu_{Y_1}(t) - \mu_{Y_2}(t)) - K\cdot(\mu_{X_1} - \mu_{X_2}) \ . \qquad (4)$$

Hence for ANOVA of Index of Response to be a potentially correct analysis strategy it is necessary to have under natural growth

$$0 = (\mu_{Y_1}(t) - \mu_{Y_2}(t)) - K(\mu_{X_1} - \mu_{X_2}) . \qquad (13)$$

But, by equation (11), for Model I type natural growth

$$0 = (\mu_{Y_1}(t) - \mu_{Y_2}(t)) - \frac{\sigma_Y(t)}{\sigma_X}(\mu_{X_1} - \mu_{X_2}) . \qquad (11)$$

Hence, by comparing equations (13) and (11), ANOVA of Index of Response is a potentially correct analysis strategy if and only if $K = \frac{\sigma_Y(t)}{\sigma_X}$. For ANOVA of Standardized Change Scores, $K = \frac{\sigma_Y(t)}{\sigma_X}$. For ANOVA of Residualized Gain Scores, $K = \beta_{Y(t)\cdot X} = \rho_{XY(t)} \cdot \frac{\sigma_Y(t)}{\sigma_X}$. But, $\rho_{XY(t)} \equiv 1$ under Model I type growth. Hence, $K = \frac{\sigma_Y(t)}{\sigma_X}$ for ANOVA of Residualized Gain Scores. Consequently, ANOVA of Standardized Change Scores and ANOVA of Residualized Gain Scores are both potentially correct analysis strategies for all Model I type natural growth situations with the additional assumption that $\sigma_{X_1} = \sigma_{X_2}$.

For ANOVA of Gain Scores, $K = 1$. Hence, ANOVA of Gain Scores is a potentially correct analysis strategy only for any time, t, when $\frac{\sigma_Y(t)}{\sigma_X} = 1$. Recall that under parallel growth, $g(t) \equiv 1$. Also, by equation (9), $\sigma_Y(t) = g(t)\,\sigma_X$ for all Model I type natural growth. Hence, under parallel growth, $\sigma_Y(t) = \sigma_X$. So, ANOVA of Gain Scores is always a potentially correct analysis strategy under parallel growth.

Model II Results

Recall that Model II type natural growth was defined by

$$Y_{1j}(t) = g(t)\ X_{1j} + h_1(t)$$

and

$$Y_{2j}(t) = g(t)\ X_{2j} + h_2(t)$$

where $g(t)$, $h_1(t)$ and $h_2(t)$ are continuous functions in t, $g(0) = 1$, $h_1(0) = h_2(0) = 0$, $X_{ij} > 0$, and $g(t) > 0$ for $0 < t < t_o$. Hence,

$$\mu_{Y_1}(t) = g(t)\cdot\mu_{X_1} + h_1(t)$$

and (14)

$$\mu_{Y_2}(t) = g(t)\cdot\mu_{X_2} + h_2(t)\ .$$

Further,

$$\sigma_{Y_1}(t) = g(t)\cdot\sigma_{X_1}$$

and (15)

$$\sigma_{Y_2}(t) = g(t)\cdot\sigma_{X_2}\ .$$

As was done in the Model I results section, assume for the remainder of the Model II results section that $\sigma_{X_1} = \sigma_{X_2} = \sigma_X$. Hence, by the set of equations (15), $\sigma_{Y_1}(t) = \sigma_{Y_2}(t) = \sigma_Y(t)$. Further, $\rho_{XY(t)} \equiv 1$ for each group. So, $\beta_{Y(t)\cdot X}$ is the same for both groups and this common value is $\dfrac{\sigma_Y(t)}{\sigma_X}$. Thus, by the same derivations as given in the Model I results section, ANCOVA, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores are all potentially correct analysis strategies and the fan spread hypothesis holds when

$$0 = \mu_{Y_1}(t) - \mu_{Y_2}(t) - \frac{\sigma_Y(t)}{\sigma_X}(\mu_{X_1} - \mu_{X_2}) . \qquad (16)$$

Plugging equations (14) and (15) into equation (16) yields

$$0 = [g(t)\mu_{X_1} + h_1(t)] - [g(t)\mu_{X_2} + h_2(t)] - \frac{g(t)\sigma_X}{\sigma_X}(\mu_{X_1} - \mu_{X_2}) .$$

This equation simplifies to

$$0 = h_1(t) - h_2(t) .$$

Hence, under Model II type natural growth ANCOVA, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores are all potentially correct analysis strategies and the fan spread hypothesis holds if and only if $h_1(t) = h_2(t)$. Consequently, the three analysis strategies are potentially correct and the fan spread hypothesis holds under Model II type natural growth only when it reduces to Model I type natural growth.

Model III Results

Recall that Model III type natural growth was defined by

$$Y_{1j}(t) = g_1(t) X_{1j} + h_1(t)$$

and

$$Y_{2j}(t) = g_2(t) X_{2j} + h_2(t) .$$

where $g_1(t)$, $g_2(t)$, $h_1(t)$ and $h_2(t)$ are continuous functions in t, $g_1(0) = g_2(0) = 1$, $h_1(0) = h_2(0) = 0$, $X_{ij} > 0$, and $g_i(t) > 0$ for $0 < t < t_0$; $i = 1,2$. All of the specific classes of natural growth curves listed under Model I (e.g., linear, exponential, and logarithmic) are also possible under Model III, but under Model III each group's growth may be defined by a different set of values for the constants.

Even more generally, each group's natural growth may be from a different class of growth curves. For example, group 1 may follow logarithmic type growth while group 2 follows exponential type growth.

As was done under the Model I results section, assume for the remainder of this section that $\sigma_{X_1} = \sigma_{X_2} = \sigma_X$. Then

$$\sigma_{Y_1}(t) = g_1(t)\sigma_{X_1} = g_1(t)\sigma_X$$

and (17)

$$\sigma_{Y_2}(t) = g_2(t)\sigma_{X_2} = g_2(t)\sigma_X .$$

As seen earlier when $\rho_{XY(t)} \equiv 1$,

$$\mu_{Y_1}(t) - \mu_{Y_2}(t) - \frac{\sigma_Y(t)}{\sigma_X}(\mu_{X_1} - \mu_{X_2})$$

is the expression for a treatment effect for ANCOVA, ANOVA of Standardized Gain Scores, and ANOVA of Standardized Change Scores. Notice that the presence of $\sigma_Y(t)$ in the expression above implies that a common variance is assumed for the two populations, which is, in general, not the case for Model III type natural growth (see the set of equations (17)). Nevertheless, any one of the three analysis strategies can be used with data from Model III and so yield estimated treatment effects. The question can then be raised as to whether these estimated effects have an expected value of zero under Model III type natural growth. This question is beyond the scope of the present paper. Thus, we stop with the conclusion that Model III type growth is inconsistent with the parametric definition for each of the three atrategies. The fan spread hypothesis also makes the assumption of common variances for both groups at any time t. Hence, the fan spread hypothesis is also inconsistent with Model III type growth.

Differential Linear Growth

It seems that several authors have either believed or else have by omission led their readers to believe that all (or at least most) differential linear growth is equivalent to the fan spread hypothesis (e.g., Bryk & Weisberg, 1977; Kenny, 1975; Olejnik, 1977). The concepts of differential linear growth and of the fan spread hypothesis are, however, distinct concepts. In the Model I results section it was shown that many forms of natural growth other than differential linear growth conform to the fan spread hypothesis. Hence, differential linear growth that conforms to the fan spread hypothesis is a subset of all natural growth that conforms to the fan spread hypothesis. Further, as will be shown below, differential linear growth conforms to the fan spread hypothesis only in rare cases.

Differential linear growth under Model III is defined by

$$Y_{ij}(t) = (b_i \cdot t + 1) \cdot X_{ij} .$$

That is, differential linear growth under Model III is Model III type natural growth with

$$g_1(t) = b_1 \cdot t$$

$$g_2(t) = b_2 \cdot t$$

and $\quad h_1(t) \equiv h_2(t) \equiv 0 .$

In the previous section it was shown that Model III type natural growth is inconsistent with the fan spread hypothesis unless it reduces to Model I type growth. Differential linear growth under Model III reduces to Model I type growth if and only if $b_1 = b_2$. Hence, differential linear growth under Model III conforms to the fan spread hypothesis only in those rare cases where $b_1 = b_2$. This same argument also shows that ANCOVA, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores are potentially correct analysis strategies under Model III type differential linear growth if and only if $b_1 = b_2$.

Summary

In this paper three models of continuous natural growth were considered. For Model I type natural growth it was shown that the fan spread hypothesis always holds. Additionally, for Model I type growth, it was shown that ANCOVA, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores all are potentially correct analysis strategies. For Model II and Model III type natural growth it was shown that the fan spread hypothesis holds and that ANCOVA, ANOVA of Residualized Gain Scores, and ANOVA of Standardized Change Scores are potentially correct analysis strategies only when Model II and Model III type natural growth reduce to Model I type natural growth. Further, it was shown that given any natural growth situation, there is a value of K for which ANOVA of Index of Response is a potentially correct analysis strategy. But the efficacy of this strategy is more apparent than real. In order that an index of response works, the exact form of natural growth must be known. This is, of course, rarely the case for empirical research.

References

Bryk, A. S., & Weisberg, H. I. Use of the nonequivalent control group design when subjects are growing. Psychological Bulletin, 1977, 84, 950-962.

Campbell, D. T. Reforms as experiments. American Psychologist, 1969, 24, 409-429.

Campbell, D. T. Temporal changes in treatment-effect correlations: A quasi-experimental model for institutional records and longitudinal studies. In G. V. Glass(Ed.), Proceedings of the 1970 invitational conference on testing problems: the promise and perils of educational information systems. New York: Educational Testing Service, 1971.

Campbell, D.T., & Stanley, J. C. Experimental and quasi-experimental designs for research. Chicago: Rand McNally, 1966.

Cox, D. R. Planning of experiments. New York: John Wiley & Sons, Inc., 1958.

Kenny, D. A quasi-experimental approach to assessing treatment effects in the non-equivalent control group design. Psychological Bulletin, 1975, 82, 345-362.

Kenny, D. A., & Cohen, S. H. A reexamination of selection and growth processes in the nonequivalent control group design. In K. F. Schuessler(Ed.), Sociological Methodology 1980. Jossey-Bass, 1980.

Lord, F. M. A paradox in the interpretation of group comparisons. Psychological Bulletin, 1967, 68, 304-305.

Olejnik, S. F. Data analysis strategies for quasi-experimental studies where differential group and individual growth rates are assumed. Unpublished Ph.D. dissertation, East Lansing: Michigan State University, 1977.

Olejnik, S.F., & Porter, A.C. Bias and mean square errors of estimators as criteria for evaluating competing analysis strategies in quasi-experiments. Journal of Educational Statistics, 1981, 6, 33-53.

Winer, B. J. Statistical principles in experimental design (2nd ed.). New York: McGraw-Hill, 1971.