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ABSTRACT

Although it is commonly assumed that increases in mathematical knowledge and problem-solving skill alter one's understanding of mathematical problems, little research exists to support this assumption. The present study investigated the relationship between mathematical background and problem perception in two experiments. Experiment 1 employed hierarchical clustering analysis to compare the way that experts (nine mathematics professors) sorted 32 mathematics problems typical of college mathematics courses with the way that novices (19 undergraduates) sorted the same problems. The experimenters assigned an a priori mathematical "surface structure" and a mathematical "deep structure" characterizations to each problem. "Surface structure" refers to the items described in the problems themselves. "Deep structure" refers to the mathematical principles necessary for solution. Results indicated that the two groups use different criteria for considering problems to be related. Experiment 2 compared changes in mathematics problem perception of students who took a computer course during the same time period. Training in problem solving resulted in the experimental group's problem perception being more differentiated and more like that of experts. Appendices contain the 32 problems used in the experiments and the mathematics pretest. (Author/RL)

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PROBLEM PERCEPTION AND KNOWLEDGE STRUCTURE IN EXPERT AND NOVICE MATHEMATICAL PROBLEM SOLVERS

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Running Head: Mathematical Problem Perception

Abstract

Although it is commonly assumed that increases in mathematical knowledge and problem-solving skill alter one's perception (i.e., understanding) of mathematical problems, little research exists to support this assumption. The present study investigated the relationship between mathematical background and problem perception in two experiments. Experiment 1 employed hierarchical clustering analysis to compare the way that experts (mathematics professors) sorted 32 mathematics problems typical of college mathematics courses with the way that novices (undergraduates) sorted the same problems. The results indicated that the two groups use different criteria for considering problems to be related. Experiment 2 compared changes in the mathematics problem perception of students who took an intensive mathematical problem-solving course with changes in students who took a computer course during the same time period. Training in problem solving resulted in the experimental group's problem perception being more differentiated and more like that of experts.

Problem Perception and Knowledge Structure in
Expert and Novice Mathematical Problem Solvers

Problem-solving skill in mathematics rests in part on the ability to accurately perceive or understand a problem. Accurate perception permits one to identify the appropriate means to a solution, whereas inaccurate perception precludes such identification and instead leads one on a "wild goose chase." Despite the importance of this phase of problem solving, little research has been done on the acquisition of mathematical problem perception. The purpose of our research was to investigate how problem perception varies with experience and training.

Outside of mathematics, problem perception has been found to develop with experience and to be directly related to increased success at solving problems. Expert chess players have been found to perceive board positions in terms of patterns or board arrangements, whereas novices do not (de Groot, 1965; Chase & Simon, 1973). Success at solving physics problems covaries with the relationships perceived among problems. Novices perceive similarities among problems according to their surface structure, e.g., wording, whereas experts perceive according to deep structure, e.g., the principles of physics (Chi, Feltovich, & Glaser, Note 1). Evidence that differences in problem perception in physics contribute to the superior problem solving of experts over novices has yet to be reported. However, when students were taught some principles of elementary physics either in a hierarchical or non-hierarchical fashion, those given the hierarchical organization exhibited superior problem-solving performance (Reif and Eylon, Note 2). Finally, there is a body of research regarding problem perception which shows that the mental representation of a problem affects the processes directed towards its solution (Larkin, McDermott, Simon, and Simon, 1980). Surprisingly, little work has been directed to ascertaining how problem perception varies with the mathematical proficiency of problem solvers. Moreover, to our knowledge there has been no research into the changes in mathematical problem perception due to training in problem solving.

Research on problem perception in mathematics has shown, like that in other disciplines, that subjects with similar backgrounds perceive problems in similar ways. For example, high school students were asked (Hinsley, Hayes, and Simon, 1977) to examine a collection of algebra problems and to sort similar problems (i.e., problems of a certain "type") into piles. Results from such studies reveal that students will cluster problems, often with substantial agreement (Hinsley, et al., 1977; Chartoff, 1977; and Silver, 1979).

Moreover, problem-solving performance was found to covary with the degree to which a student's sorting data agreed with the experimenter's perceptions of problem structure (Chartoff, 1977; Silver, 1979). Unlike research on problem solving in other disciplines, however, research on problem solving in mathematics has not examined how problem perception varies with experience across the range from novice to expert.

The present research investigated mathematical problem perception as a function of problem-solving expertise in two experiments. Experiment 1 investigated the problem perception of experts and novices in a sorting task comparable to that used in previous studies. Experiment 2 investigated whether problem perception of novices would come to resemble that of experts, after the novices received intensive training in problem-solving techniques. Thus, Experiment 1 attempted to extend findings regarding differences in problem perception between novices and experts to the domain of mathematics problems. Experiment 2 also may be viewed as an extension of research investigating the various effects of training on problem-solving ability (Lester, 1978; Schoenfeld, 1979, 1980). However, it should be noted that Experiment 2 differs from previous investigations of training effects in that it is the first to assess whether problem-solving training will alter novices' perceptions to be more like those of experts.

Finally, it should be observed that the method of data analysis employed here, Johnson's (1967) hierarchical clustering method, is superior to the methods used in the research cited above. In contrast to past studies, which report subsets of problems sorted together, this method represents sorting relationships for all problems simultaneously. Also the sorting results are represented in a hierarchical arrangement that makes explicit how frequently any particular pair of problems is perceived by subjects as related. Through the use of more explicit analyses, the present study permits a richer interpretation of mathematical problem perception than has been previously obtained.

Experiment I

Method

Subjects

Nine professors of mathematics from Hamilton College and Colgate University (hereafter referred to as experts) participated in the study without pay. A total of 19 undergraduates from Hamilton College (called novices), all of whom had taken from 1 to 3 courses in college mathematics, also participated. Eleven served without pay as a condition of enrollment in a problem-solving course; eight were paid a total of $20 each for their participation.

Materials

Thirty-two problems were chosen for the study. Those problems were chosen to be typical of those taught in mathematics courses prior to calculus. Each is accessible to students with a high school background in mathematics; none requires calculus for its solution. The experimenters assigned an *a priori* mathematical "surface structure" and a mathematical "deep structure" characterizations to each problem. "Surface structure" refers to the items described in the problems themselves. "Deep structure" refers to the mathematical principles necessary for solution, as identified by the first author who is a mathematician. The problems are listed in Appendix A. The characterizations for the problems may be seen in the clustering diagrams (Figures 1, 2, and 3).

Each of the 32 problems was typed on a "3x5" card. Each subject read through the problems in a random order and decided which problems, if any, were "similar mathematically in that they would be solved the same way." A problem that was deemed dissimilar to others was to be placed in a "group" containing one card. Subjects were told that they might return from 1 to 32 "groups" to the experimenter. Novices performed the sorting task as a group. Due to insolvable scheduling problems, experts performed the task alone and at their convenience. All subjects finished the task in approximately 20 minutes.

Analysis

Similarity matrices were derived, one for the novices and one for the experts. Each matrix represented the 32 problems and contained as its i,j-th entry ($i<j$) the number of subjects who grouped together problems i and j. Thus, a similarity matrix consisted of 496 cells. First, a count was made of the number of problem pairs in the matrix receiving strong agreement, defined here as those pairs sorted by two-thirds or more of the sample. Second, each matrix was subjected to a clustering analysis with Johnson's (1967) hierarchical clustering routine.

Results

Initial inspection of the sorting data revealed that the experts sorted problems more consistently than novices. One example of this consistency is evident in the number of cells in the similarity matrix that represented strong agreement among subjects (at least 2/3 of those sorting the problems) about the similarity of a pair of problems. In the present study there were more strong agreement cells for experts (10) than for novices (4), $t(26) = 10.45$, $p<.01$.

A more detailed examination of differences between expert and novice sorts was obtained with clustering analysis (diameter method). The results are shown in Figures 1 and 2, for experts and novices, respectively.

Insert Figures 1 and 2 about here

The number of large branches for experts is seven, that for novices, five; this may indicate somewhat greater discrimination on the part of the experts. Far more important, however, are the problems deemed to be closely related, in this case those with a proximity level of at least 0.5. These strong clusters are bracketed together in Figures 1 and 2. It is immediately apparent from a comparison of the bracketed clusters that the experts sorted the mathematics problems differently from the novices.

Discussion

The results of Experiment 1 showed clear differences in the sorting of mathematics problems by experts and novices. This finding reproduces with mathematics problems the findings obtained previously with physics problems (Chi, et al., Note 1). Also consistent with this previous research, the sorting by novices appears to depend more on surface characteristics of problems, while the sorting by experts depends on the deep mathematical properties of the problems.

The different criteria for sorting by novices and experts may be discerned in Figures 1 and 2. For example, the first novice cluster in Figure 2, {1,32,9} includes three problems which deal with whole numbers, even though the means of solution for those problems (see the "deep structure" characterizations) are radically different. In contrast, experts clustered only problem 3 with problem 1: the solutions to these two problems, by induction, are almost identical. Similarly, three problems dealing with the roots of polynomials {10,29,24} were clustered by the novices although their deep structure characterizations were completely different and experts did not cluster them together. In further contrast, consider problems 9, 15, and 17, all of which have a deep characterization as "contradiction" problems. These were clustered together by experts and placed in three different clusters by novices.

In some cases, the same clusters were produced by novices and experts. We maintain that these cases occurred for one of two possible reasons. First, the _a priori_ knowledge of the students may have been sufficient to allow them to perceive the problems like experts: consider for example the cluster of problems {2,11,25, and 8}. Second, the surface and deep structures in some problem sets were coincident: for example, the pair {14,20} dealt with division (factors) and was solved by using number representations (modular arithmetic).

To ascertain if our interpretations of sorting differences between experts and novices were correct, we asked one of our experts (JTA) to give the characterizations of problems that he sorted. Table 1 presents JTA's characterizations of the problems in Experiment 1, along with the deep structure characterizations assigned by the first author.

Insert Table 1 about here

Inspection of Table 1 shows that of the 32 problems, there are only six discrepancies (problems 4,6,8,16,18,31) between th a priori classification of the problems and the classifications by J.T.A. Moreover, three of these discrepancies (4,6,31) appear to really involve surface characteristics of the problems. The expert's remarkably close agreement with the a priori characterizations supports the claim that these a priori characterizations reflect the deep structural properties of the problems. Given that the a priori characterizations represent deep structure, we can assert with some confidence that the problem perception of experts is far more determined by deep structure than that of novices.

Experiment 2

The results of Experiment 1 indicate that experts perceive mathematics problems differently from novices. Since experts are distinguished from novices partly because of their problem-solving experience, it is reasonable to hypothesize that appropriate training in problem solving would lead the novices' problem perception to become more differentiated and more like that of experts. Experiment 2 tested this hypothesis in the following manner. An experimental group was given an almost-month-long course in mathematical problem solving. A control group, of comparable background, was given course of similar length in structured programming, studying nonmathematical problem solving techniques. Both groups sorted the problem set from Experiment 1 and took a mathematics test before and after their courses. We expected the hierarchical clustering analyses of the sorting data to i..icate that problem-solving training in mathematics alters problem perception, and that resulting patterns of problem solving for the mathematics group would mimic those of experts.

Subjects

The nineteen novices of Experiment 1 participated in the study. Eleven of these subjects served in an experimental group, explained below. These subjects had previously studied from one to three courses in college mathematics, including at least one course in calculus. The remaining eight subjects served in a control group. These subjects had also studied from one to three courses in college mathematics, including at least one course in calculus. Thus, at the outset of the study, the subjects in the two groups were approximately equal in experience in mathematical problem solving.

Materials

The deck of 32 problems from Experiment 1 was used also for this experiment. Additionally, two forms of a mathematics test, each in two sections, were constructed to assess mathematical problem-solving performance. One form of the first section is given in Appendix B. The second section was part of a different study concerned with the teaching of problem-solving strategies. (The results are reported in Schoenfeld, Note 3, and will not be discussed here.)

Procedure

The experiment was conducted during a three and one-half week "Winter Term" at Hamilton College. An experimental group (11 of the 19 subjects) took a course in "Techniques of Problem Solving," taught by the first author.

The course met two and a half hours a day for 18 class days, with daily homework assignments. The class stressed specific mathematical problem-solving strategies called "heuristics," and managerial techniques applicable to mathematical problem solving in general. (See Schoenfeld (1979; 1980) for details.)

The control group (8 of the 19 subjects) took a course in "Structured Programming" which made comparable demands on the time of its subjects. The course taught a structured, hierarchical, and orderly way to solve problems involving the computer. As such, these subjects served as a control to compare with the experimental group in assessing the amount of math-specific knowledge and skills that might be acquired by the latter.

Both the experimental and control groups followed an identical procedure at the outset and end of the Winter Term. Within the first three days of the term, subjects were given a three and a half hour exam consisting of the mathematics tests described above and the card sort task described in Experiment 1. Similarly, within the last three days of the term, the subjects took an alternate form of the mathematics exam, and sorted the same set of problems they had used in the first card sort. The problems in the test and in the card sort were directly related to the content of the course, in the following sense: the problems could be solved by heuristic strategies studied in the course, which had been exemplified with problems of the same type (Schoenfeld, Note 3). As in Experiment 1, the clustering data was analyzed with Johnson's (1967) hierarchical clustering routine.

Results

Each subject's performance on the mathematics tests was scored two ways. Strict scoring required a flawless solution per problem, giving 20 points for each of five problems solved correctly. Lenient scoring awarded 20 points for each problem solved flawlessly, and partial credit (according to a predetermined scheme) for solutions that were partially correct. Table 2 presents the mean strict scores, and in parentheses the lenient scores, for experimental and control subjects before and after the term.

Insert Table 2 about here

Inspection of Table 2 reveals that scores on the tests increased across the term (with strict scoring $F(1,17) = 32.2$, $p<.001$; with lenient scoring, $F(1,17) = 47.5$, $p<.001$), and were greater for experimental than for control subjects (with strict scoring, $F(1,17) = 92.4$, $p<.001$; with lenient scoring, $F(1,17) = 130.6$, $p<.001$). The increase across the term was essentially only evident in the scores of experimental subjects (strict scoring, $F(1,17) = 67.2$, $p<.001$; lenient scoring, $F(1,17) = 48.2$, $p<.001$). These measures indicate strongly that the experimental group's problem-solving performance was significantly improved, the control group's more or less unchanged.

One means of monitoring change in the experimental and control groups' performance is to correlate various of the novices' sorting matrices with the experts' sorting matrix from Experiment 1. Table 3 presents the correlations of the experts' matrix with each of the four matrices from Experiment 2.

Insert Table 3 about here

Inspection of Table 3 indicates the dramatic shift in the nature of the experimental group's problem perception, their post test sort correlating more closely with the experts' sort than any of the others. The nature of that change also emerged from the clustering analysis.

Two other correlations should be noted here. The sorts of control and experimental novices at the pretest were substantially correlated ($r = .650$, $df = 496$, $p<.001$). This correlation shows that, as discussed earlier, both groups began the term with a fair amount of knowledge about the kinds of problems studied in the course. As expected, the correlation between experimental and control groups dropped significantly after instruction ($4 = .492$, $df = 496$, $p<.01$)--a consequence of the fact that the experimental group's sorts came to resemble the experts'.

Finally, the post-instruction sorting data from the experimental subjects were subjected to hierarchical clustering analysis. The results (diameter method) are shown in Figure 3.

Insert Figure 3 about here

General Discussion

The correlations given in Table 3 demonstrate a substantial change in the experimental group's perceptions of problem relatedness, in conjunction with their improved problem-solving performance. A comparison of the bracketed clusters in Figure 3 with those in Figure 1 (experts) and Figure 2 (combined novices before instruction)* indicates clearly that novices sorted the problems more like experts after the winter term more than they did initially. Further, it substantiates the hypothesis that the students' perceptions of problem

*To save space the cluster diagrams for the experimental and control groups' sorts prior to instruction have not been included. The matrix from which Figure 2 was derived was strongly correlated with both the experimental pretest matrix ($r = .918$, $df = 496$, $p<.001$) and the control pretest matrix ($r = .889$, $df = 496$, $p<.001$).

relatedness were less based on surface structure, and more based on deep structure, after instruction.

Observe that of ten bracketed clusters, only five of the novices' clusters (50%) were homogeneous with regard to their deep structure characterizations. Six of eight (75%) of the experimental group's clusters were homogeneous, as were eight of eleven (73%) of the experts' clusters. The experimental group's cluster {9,17,10} illustrates the change in perception. These three problems share the same deep structure (contradiction). However, as shown in Figure 2, these problems had appeared in three distinct clusters prior to the problem-solving instruction. Problem 10, moreover, appeared initially in a cluster homogeneous with regard to surface structure (polynomials, roots).

The experimental group's post-instruction sorting, while more closely approximating the experts' sort, cannot be truly called "expert-like." The experts' extended knowledge and experience allow them perceptions inaccessible to the novices. Consider, for example, the three bracketed clusters including problem 1: novice {1,32,9}; experimental {1,3,21}; expert {1,3}. The experimental group drops problems 32 and 9, which were similar to problem 1 only in that they deal with whole numbers. Problem 3, which shares the same deep structure as problem 1, is added. The mimicry of expert perceptions is not exact, however: problem 21 is added as well. The addition of problem 21 provides an indication of the "intermediate" status of the experimental group. Problems 12 and 21 were included in the card sort to see if the experts would cluster them together. Underlying the experts' perception of problem 21 is the observation that multiples of 9 and multiples of 4 both include multiples of 36 (their intersection), and that one must compensate for subtracting the first two sets by adding the third. This is structurally similar to the rule $N(A \cup B) = N(A)+N(B) - N(A \cap B)$ upon which problem 12 is based. This is a rather subtle observation.

While experts' experience with combinatorics problems might make such an observation readily accessible, novices even with training cannot be expected to see such subtleties. In the absence of such knowledge, it is plausible to think that "looking for patterns" will help to solve problem 21--and thus to sort it with two other "patterns" problems.

The results of Experiments 1 and 2 support and extend previous research on problem perception. The novices' card sort indicated that, in the broad domain of general mathematical problem solving, students with similar backgrounds will perceive problems in similar ways. This is consistent with previous research in mathematics, which had considered only word problems in algebra (Hinsley, et al., 1977; Chartoff, 1977; Silver, 1979). Like research in physics (Chi, et al., Note 1), it suggests that surface structure is a primary criterion used by novices in determining problem relatedness. Moreover, it makes the simple but important point that students' problem perceptions change as the students acquire problem-solving expertise. Not only their performance, but their perceptions, become more like experts'.

In general, questions regarding the deep structures in individual disciplines and the nature of experts' perceptions in those disciplines are more complex than those regarding surface structures and novices' perceptions in them. Physics, for example, is a strongly principle-driven domain; one can speak of the principles which apply to a problem (e.g., "conservation of momentum" or "F = ma") without difficulty. The research by Chi, et al. (1980) indicates that experts' problem perceptions and categorizations are strongly consistent with that structure. That is, the relevant principles of physics appear to form the basis for experts' considering problems to be related. In chess, it was held that both the deep structure of the discipline and experts' perceptions were largely principle-based (e.g., "protect the center"). However, research on chess perception (de Groot, 1965; Chase and Simon, 1973) served to indicate that experts' perceptions of routine problems (similar in a way to the routine physics and mathematics problems discussed above) may be based on the acquisition of a "vocabulary" of known situations which is not necessarily principle-based. Mathematics does not appear to be as principle-driven as physics. Yet, the results described above indicate that deep structure may serve in large part as the basis for experts' perceptions of problem relatedness. Further research might profitably be directed towards the elucidation of deep structures in particular disciplines, towards the elucidation of the nature of experts' perceptions in those disciplines, and of the relationships between them.

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FOOTNOTE

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Table 1

Two Experts' Characterizations of the Problems

	A Priori Characterization*	Expert's Characterization (JTA)
1.	Patterns; induction	Induction
2.	Linear diophantine equation	Diophantine equation
3.	Patterns; induction	Induction
4.	Analogy (fewer variables)	Note x=y=z; use $(x+\frac{1}{x} \geqslant 2)$[a]
5.	Diagram; analytic geometry	Analytic geometry
6.	Unclear; Herron's Formula or analytic geometry	Analytic geometry
7.	Special cases	Factor out a[b]
8.	Special diophantine equation	Trial and error[c]
9.	Contradiction	Contradiction
10.	Contradiction	Contradiction; $x^2 \geqslant 0$.
11.	Linear diophantine equation	Diophantine equation
12.	Patterns; DeMorgan's Law	Iterate (patterns)
13.	Special cases	Cases; max 4 solutions
14.	Patterns; number representations	Mod 8[d]
15.	Uniqueness; contradiction	Contradiction
16.	Special cases; diagram	Symmetry; analytic geometry
17.	Uniqueness; contradiction	Contradiction
18.	Auxiliary elements	Analytic geometry[e]
19.	Patterns; combinatorics	Combinatorics
20.	Number representations	Modular arithmetic[d]
21.	DeMorgan's Law	Combinatorics
22.	Easier related problem; patterns	Iterate (patterns)
23.	Diagram	1,2,3 collapses[f]
24.	Special cases	Factor x^5
25.	Linear diophantine equation	Diophantine equation
26.	Patterns; induction; number rep.	Iterate (patterns)
27.	Patterns; induction	Combinatorics; induction
28.	Diagram; analytic geometry	Analytic geometry
29.	Diagram	Dumb problem (do it graphically)
30.	Analogy	Consider 2 dimensional case (i.e., analogy)
31.	Analogy	Contradiction[g]
32.	Number representations	Mod 10[d]

*Generated by the first author, AHS, who is a mathematician.

NOTES FOR TABLE 1

[a]the solution described by JTA is that suggested by the "fewer variables" technique.

[b]JTA's suggestion reduces the problem to the "special case" suggested by AHS.

[c]here JTA discriminates between this and the other three diophantine equations. JTA is a number theorist, and makes such distinctions.

[d]what AHS calls "number representations" JTA calls "modular arithmetic."

[e]implicit in AHS's assertion is a geometric approach.

[f]AHS's suggestion was that upon drawing the 25,50,75 triangle, one realizes that it collapses.

[g]after gaining inspiration from the 2-dimensional case, AHS makes his argument by contradiction.

Table 2

Mean Solution Scores on

Problems Before and After Term

	Before	After
Control	0(14)	0(24)
Experimental	2(21)	25(73)

Note: Parentheses contain scores obtained with a lenient scoring procedure (see text).

Table 3

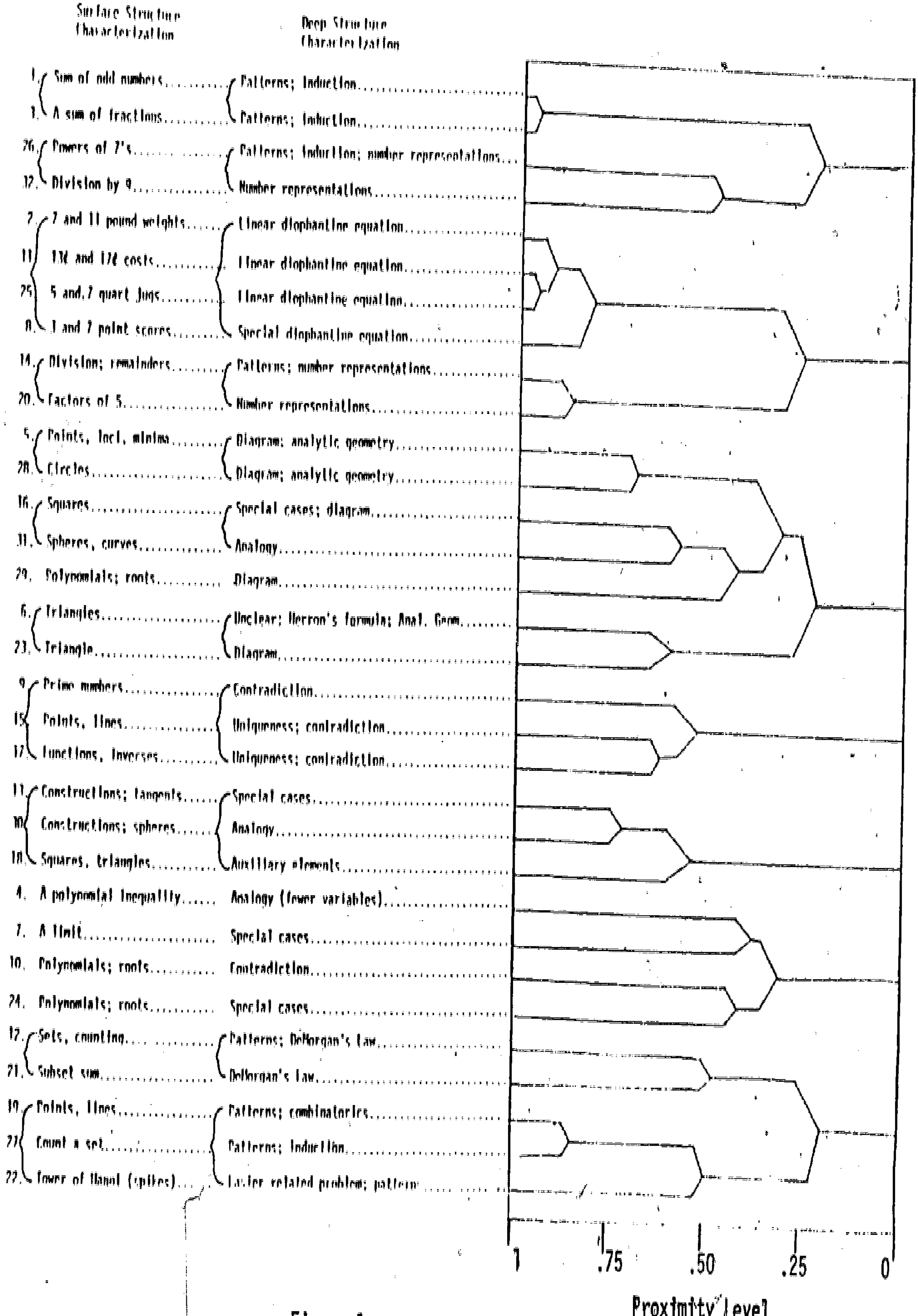
Correlations Between Sorting Matrices

of Novices (Given at Left) with Expert Sort

Control, Pretest	.551
Experimental, Pretest	.540
Control, Post Test	.423
Experimental, Post Test	.723

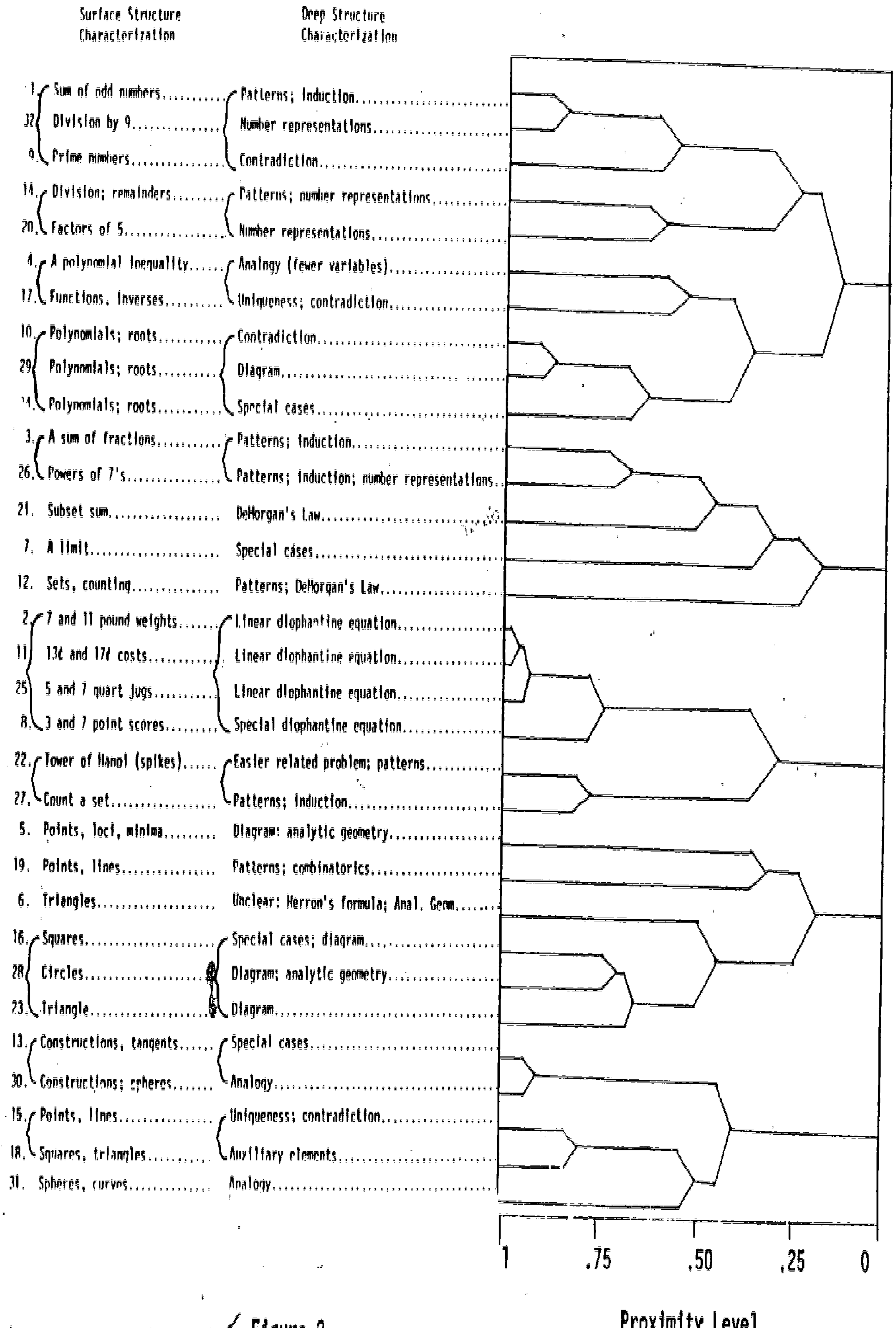
Note: With df = 496, all correlations are significant. All pretest correlations and the control post test correlation are significantly less ($p<.01$) than the Experimental Novice Post Test correlation.

CLUSTER ANALYSIS OF EXPERTS' CARD SORT



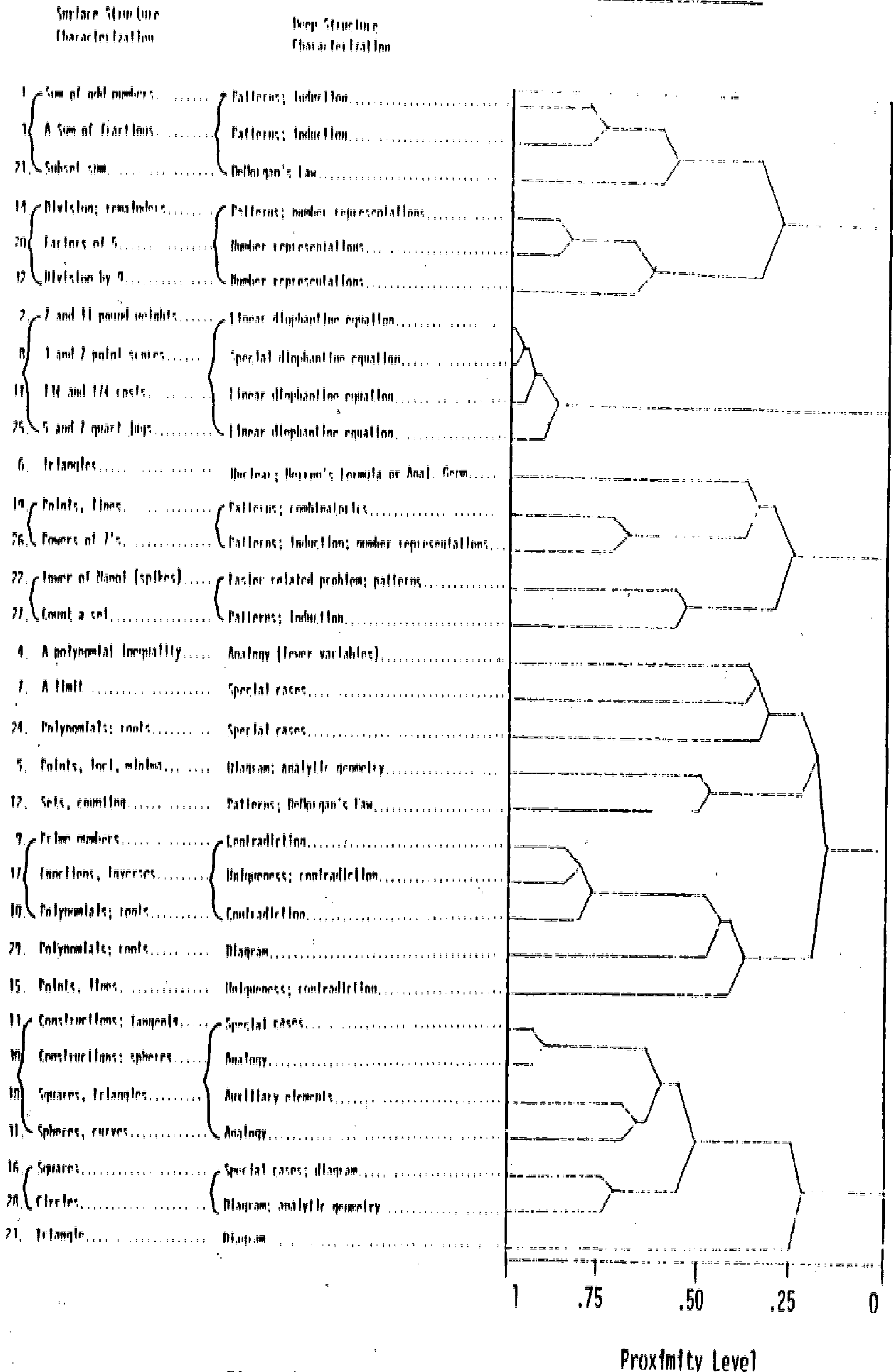
--Figure 1--

CLUSTER ANALYSIS OF COMBINED NOVICES' CARD SORT



--Figure 2--

CLUSTER ANALYSIS OF EXPERIMENTAL GROUP'S CARD SORT, AFTER INSTRUCTION



--Figure 3--

APPENDIX A

PROBLEMS USED IN CARD SORT

1. Show that the sum of consecutive odd numbers, starting with 1, is always a square. For example,

$$1+3+5+7 = 16 = 4^2.$$

2. You have an unlimited supply of 7 pound weights, 11 pound weights, and a potato which weighs 5 pounds. Can you weigh the potato on a balance scale? A 9 pound potato?

3. Find and verify the sum

$$\frac{1}{1.2} + \frac{2}{1.2.3} + \frac{3}{1.2.3.4.} + \ldots + \frac{n}{1.2.3\ldots(n+1).}$$

4. Show that if x,y, and z are greater than 0,

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8.$$

5. Find the smallest positive number m such that the intersection of the set of all points $\{(x,mx)\}$ in the plane, with the set of all points at distance 3 from (0,6), is non-empty.

6. The lengths of the sides of a triangle form an arithmetic progression with difference d. (That is, the sides are a, a+d, a+2d.) The area of the triangle is t. Find the sides and angles of this triangle. In particular, solve this problem for the case d = 1 and t = 6.

7. Given positive numbers a and b, what is

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} \quad ?$$

8. In a game of "simplified football," a team can score 3 points for a field goal and 7 points for a touchdown. Notice a team can score 7 but not 8 points. What is the largest score a team <u>cannot</u> have?

9. Let n be a given whole number. Prove that if the number (2^n-1) is a prime, then n is also a prime number.

10. Prove that there are no real solutions to the equation

$$x^{10}+x^8+x^6+x^4+x^2+1 = 0$$

11. If Czech. currency consists of coins valued 13 cents and 17 cents, can you buy a 20-cent newspaper and receive exact change?

12. If N(A) means "The number of elements in A," then $N(A \cup B) = N(A)+N(B)-N(A \cap B)$. Find a formula for $N(A \cup B \cup C)$.

13. Construct, using straightedge and compass, a line tangent to two given circles.

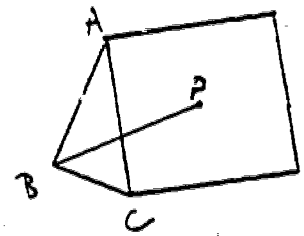
14. Take any odd number; square it; divide by 8. Can the remainder be 3? or 7?

15. You are given the following assumptions:
    i) Parallel lines do not intersect; non-parallel lines intersect.
    ii) Any two points P and Q in the plane determine an unique line which passes between them.

Prove: Any two distinct non-parallel lines L_1 and L_2 must intersect in an unique point P.

16. Two squares "s" on a side overlap, with the corner of one on the center of the other. What is the maximum area of possible overlap?

17. Show that if a function has an inverse, it has only one.



18. Let P be the center of the square constructed on the hypotenuse AC of the right triangle ABC. Prove that BP bisects angle ABC. [See figure at right.]

19. How many straight lines can be drawn through 37 points in the plane, if no 3 of them lie on any one straight line?

20. If you add any 5 consecutive whole numbers, must the result have a factor of 5?

21. What is the sum of all #'s from 1 to 200, which are not multiples of 4 and 9? You may use the fact that

$$(1+2+\ldots+n) = 1/2\ (n)(n+1).$$

fig. 1 fig. 2

22. Your goal is to convert figure 1 to figure 2. You may move only one disk at a time from one spike to another, and you may never put a larger disk on top of a smaller one. How to?

23. Determine the area of a triangle whose sides are given as 25, 50, and 75.

24. If P(x) and Q(x) have "reversed" coefficients, for example

$$P(x) = x^5+3x^4+9x^3+11x^2+6x+2,$$

$$Q(x) = 2x^5+6x^4+11x^3+9x^2+3x+1,$$

What can you say about the roots of P(x) and Q(x)?

25. You have 2 unmarked jugs, one whose capacity you know to be 5 quarts, the other 7 quarts. You walk down to the river and hope to come back with precisely 1 quart of water. Can you do it?

26. What is the last digit of $(\ldots((7^7)^7)^7\ldots)^7$, where the 7th power is taken 1,000 times?

27. Consider the magical configuration show at right. In how many ways can you read the word "ABRACADABRA?"

```
     A
    B B
   R R R
  A A A A
 C C C C C
A A A A A A
 D D D D D
  A A A A
   B B B
    R R
     A
```

28. A circular table rests in a corner, touching both walls of a room. A point on the rim of the table is eight inches from one wall, nine from the other. Find the diameter of the table.

29. Let a and b be given real numbers. Suppose that for all positive values of c, the roots of the equation

$$ax^2+bx+c = 0$$

are both real, positive numbers. Present an argument to show that a must equal zero.

30. Describe how to construct a sphere which circumscribes a tetrahedron (the 4 corners of the pyramid touch the sphere.)

31. Let S be a sphere of radius 1, A an arc of length less than 2 whose endpoints are on the boundary of S. (The interior of A can be in the interior of S.) Show there is a hemisphere H which does not intersect A.

32. Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. For example, consider 12345678: $1+2+3+4+5+6+7+8 = 36 = 4 \times 9$,

So 12345678 is divisible by 9.

Appendix B

Mathematics Pretest

1. If S is any set, we define O(S) to be the number of subsets of S which contain an odd number of elements. For example: the "odd" subsets of {A, B, C} are {A}, {B}, {C}, and {A, B, C}; thus O({A,B,C}) = 4. Determine O(S) if S is a set of 26 objects.

2. Suppose you are given the positive numbers p,q,r, and s. Prove that
$$\frac{(p^2+1)(q^2+1)(r^2+1)(s^2+1)}{pqrs} \geq 16.$$

3. Suppose T is the triangle given in figure 1. Give a mathematical argument to demonstrate that there is a square, S, such that the 4 corners of S lie on the sides of T, as in figure 2.

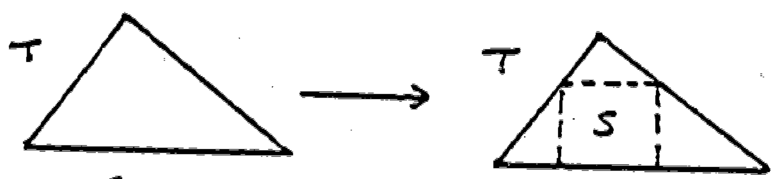


fig. 1 fig. 2

4. Consider the set of equations
$$\begin{Bmatrix} ax + y = a^2 \\ x + ay = 1 \end{Bmatrix}$$
For what values of "a" does this system fail to have solutions, and for what values of "a" are there infinitely many solutions?

5. Let G be a (9 x 12) rectangular grid, as illustrated to the right. How many different rectangles can be drawn on G, if the sides of the rectangles must be grid lines? (Squares are included, as are rectangles whose sides are on the boundaries of G.)