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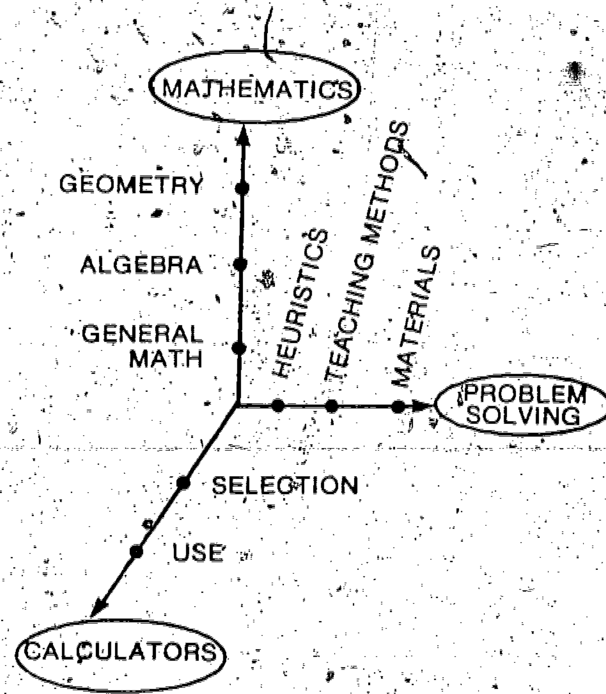
ABSTRACT

Six regional conferences designed to provide educational decision-makers and teacher-leaders with problem solving and calculator experiences, samples of materials, problem resources, strategies, guidelines, and summaries of research results, were held during 1980. The focus was on junior high and secondary schools, with a special emphasis on general mathematics. The purposes of the conferences were: (1) to describe and explore promising techniques for improving students' problem-solving abilities; (2) to provide direction on the effective use of the calculator in class; (3) to explore important curriculum options for general mathematics courses; and (4) to provide resources, describe techniques, and prepare participants for conducting in-service experiences in their own school systems on the above topics. A major portion of this document contains materials distributed to conference participants, and consists of papers compiled or written by the Conference Development Team. These papers were designed as resource materials for the participants to use when they conduct in-service activities for mathematics teachers at a later date and for participant use during the conferences. (MP)

1980 OHIO REGIONAL CONFERENCES

ON MATHEMATICS EDUCATION



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Athens, Ohio
1980

10/10/80

1980 Ohio Regional Conferences
on
Mathematics Education

Problem Solving and Handheld Calculators
in
Junior High and Secondary Schools

Packet Materials

1980 Ohio Regional Conferences
on
Mathematics Education

Preface

Background

Improving students' abilities to solve problems is the principal justification for the study of mathematics. In the past, major interest in mathematics education has focused on understanding mathematical ideas and on skill development. However, analyses of recent state and national assessment data indicate that computational skills are at acceptable levels, but problem solving abilities should be improved. The results of several research and development projects and the publication of resource materials in the past five years provide a wealth of ideas and recommendations for teaching students to solve problems.

While examining problem solving in grades 7-12, special considerations emerge. In particular, the role of hand-held calculators and curriculum developments in general mathematics are closely related to improving problem solving abilities. "Universal use of a calculator may shrink the curriculum in some ways but it will certainly expand the curriculum in a problem solving direction."* Developments in problem solving and the use of calculators lead to opportunities to enhance the curriculum of general mathematics courses in all secondary schools.

The six regional information dissemination conferences in mathematics education scheduled to be held in Ohio during 1980 are designed to provide educational decision-makers and teacher-leaders with problem solving and calculator experiences, samples of materials, problem resources, strategies, guidelines, and summaries of research results. The content of grades 7 through 12 are considered, but a special emphasis is placed on general mathematics.

Purposes

Regional conferences designed for leaders in secondary school mathematics will be conducted at six locations in Ohio during 1980.

The purposes of the conferences are:

- to describe and explore promising techniques for improving students' problem solving abilities
- to provide direction on the effective use of the hand-held calculator in junior and senior high school classes

*Jane Donnelly Gawronski and Dwight Coblentz. "Calculators and the Mathematics Curriculum." Arithmetic Teacher 23 (November, 1976): 510-512.

- to explore important curriculum options for general mathematics courses

- to provide resources, to describe techniques, and to prepare participants for conducting in-service experiences in their own school systems on the above topics.

Packet Materials

Conference participants receive a packet of material on problem solving and the use of calculators in secondary school mathematics. The twenty-four papers in the packet have been compiled or written by the Conference Development Team. These papers were designed as resource materials for the participants to use when they conduct in-service activities for mathematics teachers at a later date and for participant use during the conferences. Permission to copy any materials in the packet which are not copyrighted is accorded to educators for use in pre-service and in-service educational activities.

Individual members of the Conference Development Team prepared different papers for the packet. Thomas Butts of Case Western Reserve University, who was on leave to Ohio State University during the first semester of 1980, prepared "Posing and Re-posing Problems," "Problems, Problems, Problems!," "Review of the 1980 NCTM Yearbook," and "Using Calculators in Senior High School Mathematics." Also, he worked on a revision of "Teaching Problem Solving." David Kullman of Miami University prepared "Problem Solving Resources" in cooperation with Marilyn Suydam and "Real World Problems." Clyde Dilley of the University of Toledo developed "Conducting Problem Solving Workshops." Steven Meiring of the Ohio Department of Education prepared "Assessment Data and Implications for Problem Solving," "Using a Textbook for Teaching Problem Solving" and "General Mathematics." Marilyn Suydam of Ohio State University wrote several papers and provided others from the Calculator Information Center. These are included in the following sections: "Research in Problem Solving," "Research on Calculators," "Bulletins from the Calculator Information Center," "Selecting a Calculator," and "Mathematics Education Publications." Jason Brunk of Ohio University was asked to prepare "The House that Jack Built." Len Pikaart of Ohio University prepared "Basic Heuristics for Problem Solving," "Teaching Problem Solving," "Sharing Ideas with Colleagues," "Evaluation of In-service Activities," and "Conference Evaluation." All members of the team contributed reviews to the section called "Reviews of General Mathematics Textbooks."

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Assessment Data and Implications for Problem Solving

The 70's has been a turbulent period for mathematics education. Back to basics movements, minimum competency issues, funding initiatives, curriculum retrenchments, achivement test results, and calculator/computer advances have each contributed to the need to reassess the role that mathematics will continue to play in education. As we move into the 80's, one emerging response to some of the concerns of the 70's points toward more systematic attention to the area of problem solving.

During the past decade, we have seen a gradual shift in priorities away from content emphases and individual pupil considerations to the acquisition of meaningful skills. This shift, in large part, has been in response to "basics-related" issues. But at the same time, there has been an equal concern that reactions to "basics" and "minimum competencies" could become so narrow that overemphasis of low-level skills will preclude development of the requisite understandings to make those skills operational. There is no disagreement, however, on the key issue. Students should be able to apply the mathematics they learn to appropriate problem situations.

In 1973 and 1978, National Assessment of Educational Progress (NAEP), funded by the National Institute of Education, conducted two national studies of mathematics achievement at the 9-, 13-, and 17-year-old levels. From 1976 to 1978, the State of Ohio authorized the Ohio Department of Education to conduct statewide assessments in mathematics at grades 4, 8, and 12. These assessment studies provide us the means to contrast student performance on skill and application level items. This paper will examine selected findings from these assessments and discuss implications of the findings as they pertain to problem solving at the secondary level.

NAEP Findings

The NAEP studies measured achievement at four levels--knowledge, skill, understanding, and application. Generally, the results were favorable for items involving knowledge and skills. Contrasting performance between age levels, whole number performance shows the greatest growth from 9 to 13.

	Percent Correct		
	Age 9	Age 13	Age 17
4285 3273 +5125	51	85	90
3-digit number from 3-digit number, two regroupings	50	85	92
671 X402	3	66	76
28 ⟌3052		46	50

A sizable growth occurs in fraction and decimal performance from age 13 to age 17.

1/2 + 1/3 =	1	33	66
2-2/5 + 5 =	--	44	66
3-1/2 X 6-2/3 =	--	28	42
7.54 + 1.52	15	72	88
3.57 + 1.2	–	56	80

Graphical interpretation at ages 13 and 17 is generally good at the knowledge level, but is weak on the understanding and application levels. Estimation skills involving fractional and decimal computation are very weak

A.03

at all grade levels, possibly suggesting a mechanical rather than a conceptual approach is used in working with such numbers.

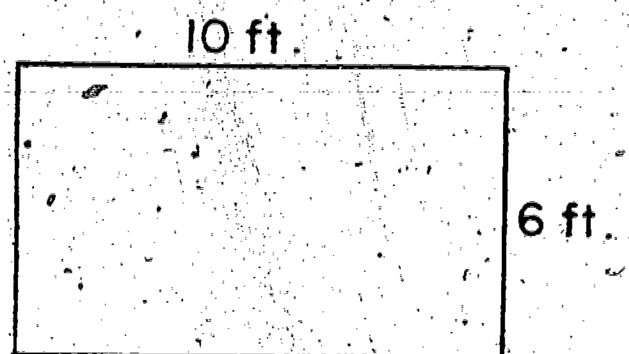
ESTIMATE the answer to 12/13 + 7/8. You will not have time to solve the problem using paper and pencil.

	Percent Responding Age 13	Age 17
1	7	8
2	24	37
19	28	21
21	26	15
I don't know.	14†	16†

30 ÷ 317

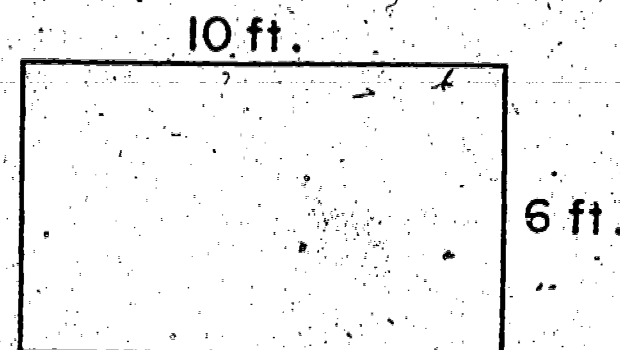
	Percent Responding Age 13	Age 17
.1	7	15
.01	22	30
1	6	8
10	59	41
I don't know.	5†	6

Performance in making measurements is generally adequate. But achievement involving other aspects of measurement, particularly perimeter, area, and volume is not. Moreover, vocabulary significantly influences performance on items for which different terminology is employed.



What is the PERIMETER of this rectangle?

	Percent Responding Age 9	Age 13
16 feet	66	25
30 feet	2	1
32 feet	8	49
36 feet	2	4
60 feet	3	17
I don't know.	16†	4



What is the DISTANCE ALL THE WAY AROUND this rectangle?

	Percent Responding Age 9	Age 13
16 feet	39	12
30 feet	4	1
32 feet	40	69
36 feet	4	4
60 feet	4	13
I don't know.	8†	1

Achievement on textbook-like, simple, one-step word problems is satisfactory. However, for more complex problems or on problems that deviate even slightly from a textbook format, performance drops significantly.

Percentages of Response on One-Step Consumer Item, Ages 9, 13 and 17

MENU			
HAMBURGER	$.85	MILK	$.20
HOT DOG	.70	SOFT DRINK	.15
GRILLED CHEESE SANDWICH	.55	MILK SHAKE	.45
		ICE CREAM	.40
FRENCH FRIES	.40		

Sue had a hot dog, french fries, and milk. How much did she spend?

	Percent Responding Age 9	Age 13	Age 17
$1.20	8	2	1
$1.30	57	92	95
$1.40	9	2	2
$1.50	16	3	1
I don't know.	7**	+**	+**

A.04

Percentages of Correct Response on a Nonroutine Application Problem, Ages 9, 13 and 17

Right now Bob has $12 in the bank and Carol has $26 in the bank. From now on each will save $1 every week. How much money will Bob have saved when he has half as much as Carol?

	Percent Correct Age 9	Age 13	Age 17
Respondents answering $14	12	20	32

EXHIBIT 32. Percentages of Correct Response on a Nonroutine Application Problem, Ages 9 and 13

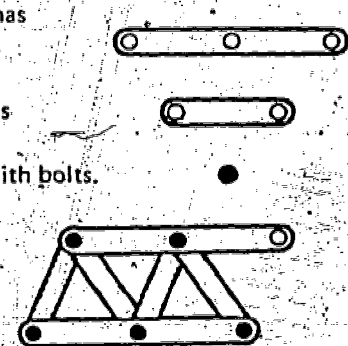
Mike's building set has

60 long pieces

60 short pieces

and 60 nuts with bolts.

How many of these can he make?



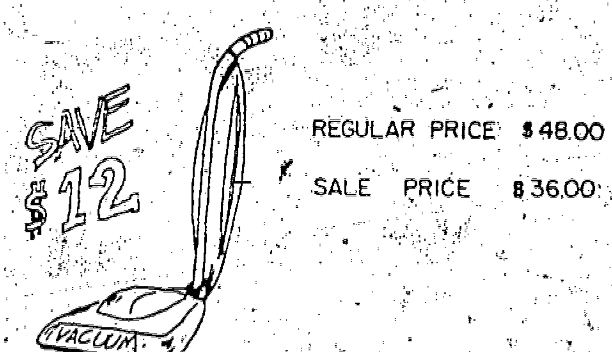
	Percent Correct Age 9	Age 13
Respondents answering 12	3	24

A panel of distinguished mathematics educators were asked to examine 1978 NAEP results and to make observations and recommendations. Reacting to the possible effect that the "back to basics" trend may be having on results, the panel noted that this movement has often resulted in a narrowing of the curriculum, stressing computational skills and knowledge of facts and definitions with less time available for instruction on other topics. James Wilson cited the effect on textbooks: "By the mid-1970's, textbooks in the schools emphasized computational skills. Word problems were streamlined to present students with the least possible amount of complexity."

Generally, the panel was satisfied with student performance on most computational and knowledge levels. Some concerns were expressed about the need to promote better conceptual understandings of topics such as fractions, decimals, and percents with less emphasis on rote, mechanical applications of algorithms. However, in the area of problem solving, the panel registered its greatest concerns. According to Thomas Carpenter, "Youngsters don't think about problems. They search for ways to apply the mechanical approaches that they have been taught, and if anything out of the ordinary appears in the problem, they are at a loss." The panel warned that insufficient attention to higher-level processes, including problem solving is a likely result, producing a situation in which skills become ends in themselves rather than being regarded as tools to be applied in problem-solving situations. Carpenter again cautioned, "People are teaching skills and expecting that facility with skills will automatically transfer to problem-solving ability. But the evidence shows that that's just not so."

The 1973 assessment by NAEP focused attention on the discrepancy between generally high achievement on computational topics and low performance on consumer-related applications. 1978 assessement results indicate that 17-year-olds have fared no better five years ~~later~~.

Percentages of Response on Consumer Problem Involving Percent, Ages 13 and 17



What is the percent of discount?

	Percent Responding Age 13	Percent Responding Age 17
12	52	34
20	14	14
25	18	36
33 1/3	4	4
75	3	3
I don't know	9	7

What is the price of a 12-foot by 15-foot piece of carpeting that sells for $7.00 per square yard?

Answer $140

13% of 17-year-olds answering correctly

A roast is to be cooked 20 minutes for each pound. If a roast weighing 11 pounds is to be done at 6:00 p.m., what time should it be put in the oven to cook?

Answer 2:20, 2:20 p.m.

25% of 17-year-olds answering correctly

The panel of reactors was also concerned with the weak ability of students to examine their results for reasonability or practicality. On an item asking how many cubic feet of concrete would be needed to pave an area 30 feet long and 20 feet wide with a layer four inches thick, only 9% of the 17-year-olds responded correctly. But a majority chose 30 x 20 x 4 = 2400 as their answer. Another problem required 13-year-olds to determine how many cars would transport 30 people, four to a car. Forty percent of the 13-year-olds would have sent two people off in half a car. As panelist Iris Carl noted: "Although students may have skill in using distinct algorithms, the depth of understanding of the concept required to effectively apply skills in various situations appears to be missing."

One of the objectives of NAEP in conducting mathematics assessments in 1973 and 1978 was to determine whether any changes in achievement occurred during that time. The results of a comparative study on achievement on the same items in 1973 and 1978 revealed: 9-year-olds' performance declined very slightly; the decline for 13-year-olds was slightly larger and the decline for 17-year-olds was appreciable. Considered by cognitive level, nine-year olds dropped significantly only in one area -- the application area of problem solving. Seventeen-year-olds performance declined significantly in every area except the knowledge area.

Performance on Application Items

	Number of Items	Average Performance 1973	Average Performance 1978	Change In Average Performance
Age 9	9	38%	32%	-6 *
Age 13	12	42	38	-3 * +
Age 17	25	33	29	-4 *

*Change is significant at .05 level
+Figures do not total because of rounding

Roy H. Forbes, Director of the NAEP, noted that the pattern found in the mathematics assessment--larger declines for the older ages and more tendency for declines to appear on higher-order cognitive skills--may be symptomatic that the "back to basics" movement is working too well, to the disadvantage of higher level skills. Panelists felt that many times students are not introduced to higher-level processes until they have completely mastered the requisite lower-level skills and knowledge. The panel warned that this could severely limit students' exposure to a wide range of mathematical topics and could result in students having very little idea of how the various things they had learned were related.

Responding to the NAEP findings, panelists offered a number of recommendations. The following four relate directly to problem solving.

1. An expanded definition of what is "basic" in mathematics is crucial to foster students' ability to cope with different types of mathematical problems. Students must be introduced to exercises involving higher-level as well as lower-level cognitive processes.
2. Teaching of problem solving should receive more emphasis in the schools. It appears that many teachers feel outside pressure to drill mastery of skills and not to teach problem solving.
3. Success on tests should not be the only criterion used to measure the effectiveness of mathematics programs. Too often the test then becomes the objective, and areas that require more time to teach or are not as easily subject to modification fall by the wayside.
4. Approaches to problem solving should not be of the type that encourage mechanistic, algorithmic methods. It should be emphasized that the ability to analyze a problem situation is equally as important as the correct solution.

Ohio Assessment Findings 1976-78

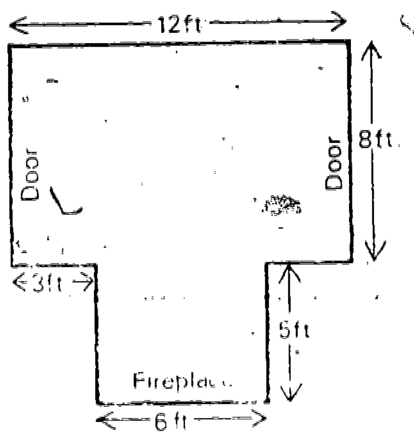
Ohio results reflect the same general trends as those from national assessment: proficiency with whole number computations, generally in place by grade 8; less proficiency with computations involving other kinds of numbers, but growing from grade 8 to grade 12; and significantly less proficiency with problem solving applications, generally satisfactory by grade 12 only for simple, one-step, textbook-like problems. (See Tables A and B in the Appendix.)

Of particular concern in Ohio assessment results are the very weak performances involving geometric applications. Both at the eighth and twelfth grade levels, geometry application items consistently rank among the lowest scoring items.

Eighth Grade Geometry Items

What is the perimeter of the room in this illustration?

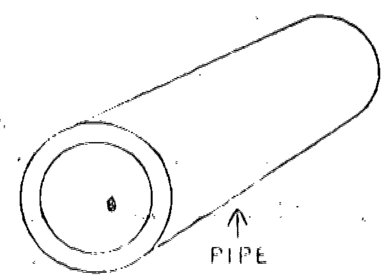
A. 34 feet
* B. 50 feet
C. 100 feet
D. 156 feet
E. I don't know.



(44%)

The radius of the inside of the pipe is 3 inches. The wall of the pipe is 1 inch thick. What is the diameter of the pipe?

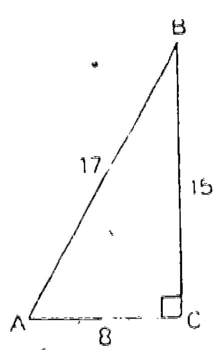
A. 4 inches
B. 5 inches
C. 7 inches
★ D. 8 inches
E. I don't know.



(20%)

What is the area, in square units, of ___ ABC?

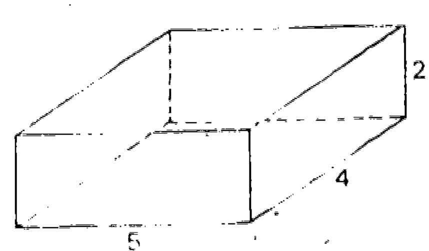
A. 40
* B. 60
C. 68
D. 120
E. I don't know



(16%)

What is the volume, in cubic units, of the rectangular solid shown below?

A. 11
B. 20
C. 22
★ D. 40
E. I don't know



(45%)

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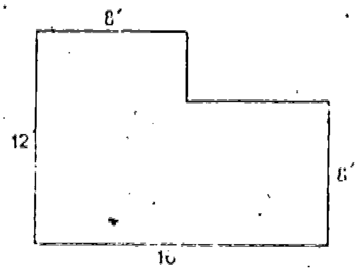
Twelfth Grade Geometry Items

How many [illegible] square fe[illegible] be ne[illegible] room pictured below?

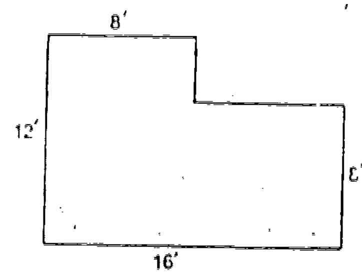
A 44
B 52
C 56
* D 160
E 192



(41%)

What [illegible] around the room pictured below?

A 44 ft
B 48 ft
C 52 ft
* D 56 ft
E 192 ft



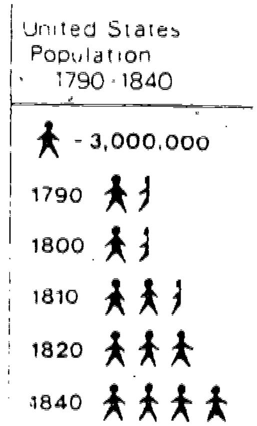
(30%)

Another area of interest among Ohio data is performance on interpreting and using graphical and tabular data. For items requiring a straight-forward entry selection from a graph or table, results are uniformly high. However, for questions requiring the reader to interpret a data entry or involving some slightly unusual feature, the performance plummets. On the following pictograph, only 35% of the 8th-graders were able to correctly interpret the correct table entry, possibly because the picture symbol represents a nonunit quantity. For another item in which each symbol did represent a unit quantity, performance rose to 95%.

What was the approximate population in the United States in 1810

A 1,500,000
B 2,500,000
C. 6,500,000
* D 7,500,000
E I don't know

(35%)



Generally, Ohio assessment results corroborate NAEP panel observations that students are not well-equipped to respond to situations which require them to perform slightly differently that the most simple, straight-forward exercise or application for which they have had previous practice.

The following table contrasts the average performance by problem type for 8th and 12th grade students on the 1977-78 assessments. For the 1978 12th grade assessment, a panel of secondary mathematics educators was requested to review assessment items prior to administration and to <u>predict</u> the average performance on each. The last column of the table includes these predictions for the 12th grade items. Note the consistently low prediction for each problem type. A similar trend was also observed for the 1976 12th grade assessment.

Average Performance by Grade Level and Problem Type (1977-78 Assessments)

Problem Type	Grade 8	Grade 12	Panel Predictions(12th)
whole number computations	92%	91%	81%
decimals and fractions	70%	77%	67%
one-step applications	55%	83%	67%
two- or more step applications	55%	66%	61%

Some questions are raised by the results in this table. Are pessimistic predictions of whole number, fraction, and decimal computation abilities indicative of misdirected instructional emphasis in grades 9-12? Is the amount of instructional time assigned to different problem type learning experiences consistent with the amount of growth in student performances from grades 8 to 12?

Problem Solving: A Direction for the Future

Assessment test results provide significant data pointing toward discrepancies between what is and what should be. During the 70's, such test results have frequently been misused--sometimes as means for special interest groups to forward their causes and at other times held up as threats to established patterns or policies. Part of this misuse is perhaps due to the tone of "back to basics"--an unspoken but assumed statement that mathematics education did, in the past, prepare students adequately for their responsibilities and that the same criteria are still appropriate. Whether or not performances today compare favorably with the past is really not the issue. The times have changed. The clientele has changed. But most importantly, the needs have changed.

There is much in present assessment data in which to take pride in student achievements. Mathematics education is doing some things exceedingly well. But the data do reflect significant discrepancies between skills and the application of those skills to relevant situations. The data also point to weak student abilities to transfer existing knowledge and abilities to even slightly novel situations. If we are to judge the goals for mathematics education by today's needs and project on that basis for the future, each of these areas of weakness appears to be an area where citizens will increasingly rely on personal capabilities.

Some have suggested that mathematics educators have rediscovered problem solving. But problem solving has always been the umbrella goal under which every individual content topic and skill exercise has been conducted. However, we are coming to realize that problem solving must be a more immediate goal as well--one that directs daily activities so that means do not become ends, so that unwarranted assumptions about automatic transfer do not rule out attention to higher-level cognitive skills.

We are recognizing that regular and specific instructional activities must be planned that pursue problem solving directly, not secondarily. Programs must be structured and monitored in such a way that regular growth toward problem solving subgoals occurs at each grade level.

A.12

Every student at every grade level must have the opportunity to grow in his ability to apply his existing skills and knowledge to situations that he finds relevant, but not necessarily rehashes of previous learning activities. Then mathematics education will be taking firm and steady steps toward meeting the challenge of the future.

Table A: OHIO 8TH GRADE ASSESSMENT DATA

Number of Items	Problem Type	Average Performance
9	whole number computations	92%
10	decimals and fractions	70%
15 (10)	one-step applications (without geometry)	55% (68%)
7	two- or more step applications	55%

Computation Items	Applications

Computation Items

$$\begin{array}{r} 956 \\ +\ 357 \\ \hline \end{array}$$ (97%)

$$\begin{array}{r} 64 \\ \times\ 35 \\ \hline \end{array}$$ (90%)

0.941 - 0.36 = (63%)

21) $14.91 (65%)

$\frac{1}{5}$ ([illegible])

(decimal answers)

$2\frac{5}{6}$ = (84%)

(improper fractions)

Applications

One-step:

If you had 285 pieces of candy and wanted to put 15 pieces into a package, how many packages could you make?

A. 17
* B. 19 (81%)
C. 270
D. 4,275
E. I don't know.

One-step:

If it takes you 37 minutes to get to school, what time must you leave home to be at school at 8:15?

A 7:35
* B 7:38 (72%)
C 7:45
D 7:50
E I don't know.

Two- or more step:

The phone rate increased from $5.00 a month to $6.00 a month. What was the percent of increase?

A. $16\frac{2}{3}$%
* B. 20% (59%)
C. $83\frac{1}{3}$%
D. 120%
E. I don't know.

Table B: OHIO 12TH GRADE ASSESSMENT DATA

Number of Items	Problem Type	Average Performance	Predicted Performance
9	whole number computations	91%	81%
16	decimals and fractions	77%	67%
12	one-step applications	83%	67%
9 (7)	two- or more step applications (without geometry)	66% (75%)	61 (62%)

Computation Items	Applications

Computation Items

2378 + 25 + 403 = (95%)

7) 2156 (77%)

8.4 + .03 + 109 = (90%)

2.06
× .14 ([illegible])

$\frac{3}{8} + \frac{1}{2} + \frac{3}{4}$ ([illegible])

Convert $\frac{44}{5}$ to a mixed number ([illegible])

Applications

One-step:

There are 206 tomato plants in each row and there are 14 rows. How many tomato plants are there all together?

A 1,030
B 2,684
C 2,864
* D 2,884 (94%)
E 21,424

Two- or more step:

If a coat originally sells for $120.00 and is on sale for 20% off, what is the sale price of the coat?

A $24.00
B $60.00
* C $96.00 ([illegible])
D $100.00
E $108.00

Two- or more step:

Record albums are on sale at 3 for $8.85. How much would 5 albums cost.

A [illegible]
B $13.85
* C $14.75 (75%)
D $15.00
E $44.25

1980 Ohio Regional Conferences on Mathematics Education

Basic Heuristics for Problem Solving

"Learning to solve problems is the principal reason for studying mathematics."* How do students solve problems? How do you solve problems? When presented with a problem what do you do? A reasonable reply is "it depends on the problem," or "I don't always do the same thing." The first point is that when people solve problems they do something. Some of these things may be productive, others may not be. The purpose of this paper is to list and exemplify one set of heuristics, very general strategies, which have been found helpful in solving problems. Good problem solvers probably have used these heuristics before, but not seen them listed. The goal is to improve problem solving abilities of students and one very promising approach is to help students learn to use heuristics.

Interested teachers may elect to teach their students how to use the heuristics presented here or to modify this list. Ultimately they wish to develop another list. There is no research evidence that any particular list of heuristics is best, but there is research evidence that teaching students to use heuristics does improve their problem solving ability.**

The list of heuristics and many of the sample problems are taken from a new publication: Problem Solving: A Basic Mathematics Goal, a two volume set prepared by Steven P. Meiring and published by the Ohio Department of Education (Columbus, 1980). Booklet one is entitled, "Becoming a Better Problem Solver" and the second booklet is entitled, "A Resource for Problem Solving." Although the booklets were prepared as an inservice tool for teachers of grades K-9, they are highly recommended to secondary teachers for three reasons: (1) The problem solving heuristics or strategies are applicable in elementary, middle, junior high, and senior high school; (2) many of the sample problems and other content can be used in secondary school; and (3) articulation between schools can be improved if secondary school teachers are familiar with the basic learnings taking place in earlier grades.

In the following section, more than twenty heuristics will be listed, described, and exemplified. The order of listing them is not important, but is the same order as they are referred to in the Problem Solving booklets. When solving a particular problem, a student might use a single heuristic but more often, several would be employed. The important point is that after a student is familiar with problem solving heuristics, they become a repertoire which he or she can use to attack problems

* National Council of Supervisors of Mathematics. Position Paper on Basic Mathematical Skills, 1977. Reprinted at the end of this paper.

** For example, a list of 44 "problem solving skills" (heuristics) developed in the Lane County Mathematics Project in Eugene, Oregon is attached at the end of this paper. Also, see Travers et al, Mathematics Teaching (Chapter 5). New York: Harper and Row, 1977.

B.02

Heuristics

1. Look for a Pattern

(1) The ten members of a chess club decide to hold a tournament and that each member must play every other one. How many games must be played? Observe that one player must play 9 others and another player plays 8 new games (his game with the first player is already accounted for.) A third player plays in 7 new games (9-2). We see a pattern emerging.

(2) Fill in the missing numbers in this sequence:
3.14, 3.11, 3.08, 3.05, ____, ____, ____, ____, 2.9.

2. Construct a Table

(1) The problem 1.(1) could lend itself to solution by a table construction as follows:

Player	New Games
A	9
B	8
C	7
D	6
.	.
.	.
.	.
H	2
I	1
J	0
	45

(2) A woman makes furniture as a hobby. If she uses 31 legs to make only 4 legged benches and 3 legged stools, how many of each did she make? Try a table like the following:

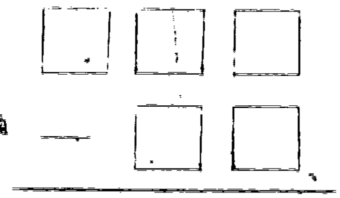
Benches Possible	Stools	Total Legs	Solution
8	0	32	No
7	1	31	Yes
6	2	30	
.	.		
.	.		
.	.		

3. Consider Possibilities Systematically

All the above examples have employed this heuristic. A table is a way of systematically considering possibilities.

B.03

(1) Use the digits 0, 2, 4, 6, and 8 to fill in the boxes below so that the difference is as small as possible.



4. <u>Act it Out</u>

(1) If six people meet and each shakes everyone else's hand, how many handshakes take place? A group of students was given this problem and 6 of them went to a corner to act it out!

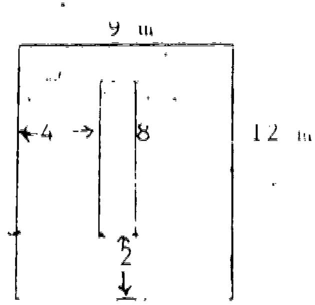
(2) Can you make change for 50¢ using (a) 6 coins; (b) 7 coins; (c) 8 coins?

(3) A man buys a horse for $60, sells it for $70, buys it back for $80 and sells it for $90. Does he make a profit or a loss? How much?

5. <u>Make a Model</u>

(1) Can you cut a circular shape into 2 <u>non</u>congruent shapes that have equal areas?

(2) A carpet 9m by 12m has a 1m by 8m piece cut out as pictured. A scale model might help you find a solution.



(3) Can you obtain [illegible] liters of water using only a [illegible] liter and a [illegible] liter container? Any two convenient jars could be marked appropriately and used in the model.

6. <u>Guess and Check</u>

We emphasize "guess and check" to [illegible] information which might be useful as he or she proceeds. ("Trial and error" implies simply that the solver tries a solution but does not use the information.) This heuristic is used often and can be productive.

(1) Can you find 2 numbers such that their sum is 15 and product 56? See also 2.(2), 3.(1) and 4.(2) above.

(2) If each letter is a code for a digit (0, 1, 2, . . ., 9), what is the following addition problem?

```
  SEND
+ MORE
 MONEY
```

7. Work Backwards

(1) Alice, Barbara, and Connie played a game with chips.
Alice lost the first round.
Barbara lost the second round.
Connie lost the third round.
If the loser must double each player's chips, and each had 8 chips at the end of the 3 rounds, how many chips did each start with?

(2) One of nine baseballs is heavier than the rest. How can you determine which is the odd ball with a balance scale in just two weighings? Try considering the last weighing first.

8. Make a Drawing, Figure or Graph

(1) Construct △ AXB isosceles with base angles of 15° and obtuse angle at X. Knowing that ABCD is a square with X in the interior, prove that X, C, and D form vertices of an equilateral triangle.

(2) When standing atop an object of 10 or 20 feet high, if you look down, it appears much higher than if you stand at the base and look up. Is there a physical explanation of this?

(3) What is the largest area you can fence in with 48m of fencing if a large barn can be used for one side?

9. Select Appropriate Notation

(1) Find all rectangles with integral sides such that the area equals the perimeter.
A = P doesn't help much, but
$A = \ell \cdot w$ and $p = 2\ell + 2w$ or
$\ell \cdot w = 2\ell + 2w$ might help.

10. Restate the Problem (in your own words)

(1) Can you find 3 positive integers such that the sum of the reciprocals is an integer? A useful restatement might be:

(2) Find positive (distinct) integers A, B, C, and D such that
$1/A + 1/B + 1/C = D$

11. Identify Wanted, Given, and Needed Information

(1) A bottle of cider costs 50 cents. If the cider costs 30¢ more than the bottle, how much does the bottle cost?

(2) See 8.(1) also.

12. Write an Open Sentence

(1) If the perimeter of a rectangle is 33cm and the length is twice as long as the width, what are its dimensions?

B.05

(2) Three Guernsey cows and two Holstein cows give as much milk in four days as two Guernsey and four Holstein cows give in three days. Which kind of cow is the better milk producer?

(3) A can with 40 marbles in it weighed 135 grams. The same can with 20 marbles weighed 75 g. What is the weight of the can?

13. Identify a Subgoal

Sometimes a problem may seem overwhelming but there may be a subgoal which is simpler.

(1) Fill in the missing numbers in this addition table:

+	Z		X	
	36			29
32			51	
		61		
Y	55		62	

The problem may appear difficult, but many people could find X, then Y, then Z, etc.

(2) Olson, Smith, Hayward, and Duncan had dinner together one evening. When they left the restaurant each of them, by mistake, was wearing the coat belonging to someone else in the party and the hat of yet someone else. Smith took Olson's hat, and the man who took Duncan's hat took Hayward's coat. Which hat and coat did each of the men take?

14. Solve a Simple (or similar) Problem

(1) Consider the problem 1.(1) about the number of games played by 10 members of a chess club. One strategy might be, could I solve the problem if there were 2, 3, or 4 people in the club.

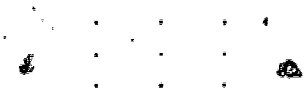
(2) How many lines can be drawn connecting 100 points on a circle? Again, it is simpler to find an answer if one considers the problem for 2, 3, or 4 points (problem (1) and (2) are very similar!)

15. Change Your Point of View

Sometimes a solver meets frustration and changing a point of view may open new alternatives.

(1) A test track for new cars is one kilometer around. For the first lap, the test driver averages 30 km/h (kilometers per hour). How fast does the car have to travel the second lap to average 60 km/h for the 2 laps? Many people might guess 90 km/h because $\frac{30 + 90}{2} = 60$. However, 90 is not correct. Try calculating the time to cover 2 laps at 60 km/h.

(2) Without lifting your pencil from the paper, draw 4 straight line segments which pass through all 9 points.



16. Check for Hidden Assumptions

Here's a little exercise in creativity:

(1) Coming down a hill covered with snow are two ski tracks. One track is on one side of a tree and the other is on the other side. Try to find at least 10 different physical explanations to fit these facts.

17. Use a Resource

Sometimes some specific information may be helpful. Adult problem solvers often look up a formula they need or ask someone for help. It may be important in working with youngsters to indicate that using a resource is often a viable heuristic. Another useful, and similar, heuristic is, "Draw from your cognitive background." In problem 8.(1) one might have to recall what he or she knows about isosceles and equilateral triangles to solve the problem. In 15.(1) one might need to recall $D = rt$.

Note. The final 5 Heuristics are often called "Looking Back Strategies" because they are typically employed after a solution has been found. However, mathematics teachers realize that a great deal of learning can still take place after one finds a solution to a problem. The looking back heuristics are probably the hardest to teach because students have been conditioned to stop thinking about a problem as soon as a solution is found. We urge a concerted effort to emphasize these heuristics to improve student learning.

18. Generalize

Often solving a problem permits one to make a general statement which covers a great many situations, including the problem just solved.

(1) Consider problem 1.(1) about the chess club. After finding that the situation calls for $9 + 8 + 7 + \ldots + 2 + 1 + 0 = 45$ games, it is not hard to see that if n people are in the club,

$$(n - 1) + (n - 2) + \ldots + 2 + 1 + 0 = \frac{n(n - 1)}{2}$$ games must be played.

(The sum of the first (n - 1) integers.)

(2) Similarly, if one has solved 8.(1) and found (via a simpler problem and/or working backwards) that one heavy ball of 3 can be determined in one weighing and one of 9 in two, he or she might determine that a heavy ball can be found in n weighings if there are 3^n balls.

19. Check the Solution

Students sometimes make errors. Checking the solution can be helpful in preventing one from thinking an incorrect answer is correct. For example, recall problem 15.(1). If a person thought 60 km/h was correct, he or she could check it as follows.

D = rt so the time for laps of 1 km each at 60 km/h would be:

$2 = 60t$

$5 = 1/20$ hour = 2 minutes.

But how long did the first lap take?

$D = rt$

$1 = 30t$

$t = 1/30$ hour = 2 minutes.

Thus, the driver cannot go fast enough to do 2 laps at 60 km/h if the first lap was at 30 km/h.

20. Find Another Way to Solve it

Often, after a problem has been solved, another, even "slicker" way can be found to solve the problem. Good problems are characterized as being solvable in more than one way--if there is only one way to solve a problem, students' efforts must follow a rigid path to achieve a solution.

21. Find Another Solution

Happily, some problems have more than one solution. Consider 2 (2) which has more than a single solution. If this heuristic is in a student's repertoire, additional solutions may be sought.

22. Study the Solution Process

Many important learnings can [illegible] studies the solution process after working on a problem. Some authors distinguish between product and process. The product in problem solving is the answer(s). Actually, any particular problem is of little value in itself; the answer is not the goal. Rather the students' ability to achieve one or more answers is of great value. The processes he or she employs are much more important and useful for future problem solving situations than the answer.

Summary

The following heuristics have been listed and exemplified in this paper.

1. Look for a pattern
2. Construct a table
3. Consider possibilities systematically
4. Act it out
5. Make a model
6. Guess and check
7. Work backwards
8. Make a drawing, figure or graph
9. Select appropriate notation
10. Restate the problem
11. Identify wanted, given, and needed information
12. Write an open sentence
13. Identify a subgoal
14. Solve a simpler problem
15. Change your point of view
16. Check for hidden assumptions
17. Use a resource
18. Generalize
19. Check the solution
20. Find another way to solve it
21. Find another solution
22. Study the solution process

NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS POSITION PAPER ON BASIC MATHEMATICAL SKILLS

INTRODUCTION

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that t.

RATIONALE FOR THE EXPANDED DEFINITION

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

DEFINING BASIC SKILLS

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

TEN BASIC SKILL AREAS

Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying Mathematics to Everyday Situations

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the Reasonableness of Results

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and Approximation

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate Computational Skills

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry

Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

Reading, Interpreting, and Constructing Tables, Charts, and Graphs

Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using Mathematics to Predict

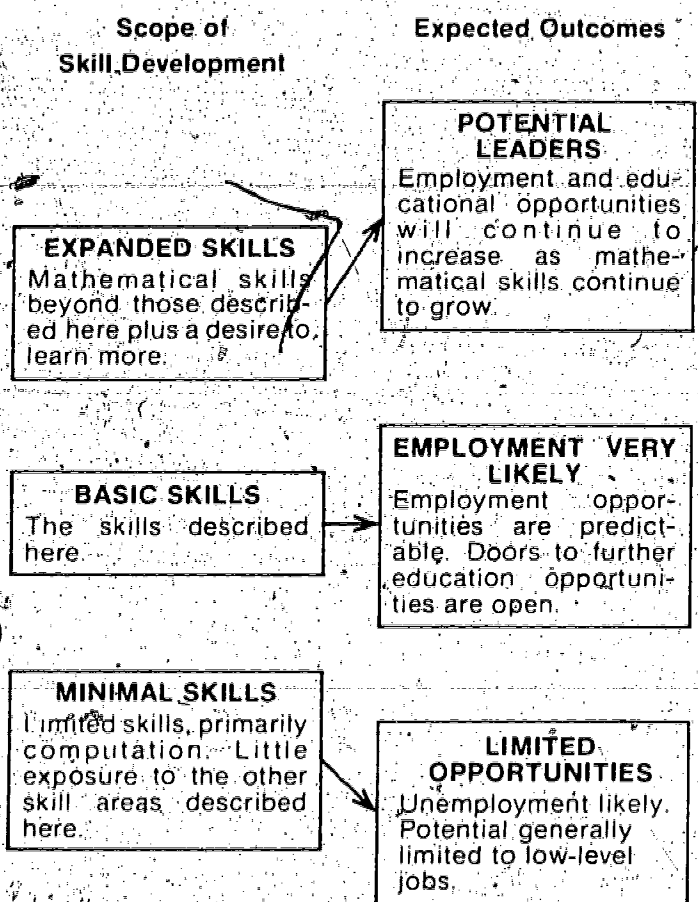
Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer Literacy

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

BASIC SKILLS AND THE STUDENT'S FUTURE

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.



MINIMUM ESSENTIALS FOR HIGH-SCHOOL GRADUATION

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

B.11

DEVELOPING THE BASIC SKILLS

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

EVALUATING AND REPORTING STUDENT PROGRESS

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.
b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

B.12

CONCLUSION

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but *forward* to the essential needs of adults in the present and future.

You are encouraged to make and distribute copies of this paper.

The NCSM position paper was prepared pursuant to a contract with the National Institute of Education, U.S. Department of Health, Education and Welfare. Contractors undertaking such projects under government sponsorship are encouraged to express freely their judgment in professional and technical matters. Points of view or opinions do not, therefore, necessarily represent official National Institute of Education position or policy.

NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS
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N C S M

January, 1977

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A CLASSIFIED LIST OF PROBLEM-SOLVING SKILLS*

A. Problem Discovery, Formulation
1. State the problem in your own words. (Read and restate.)
2. Clarify the problem by asking questions.
3. Visualize an object from its description, mathematical notation, or its drawing.
4. Follow written and/or oral directions.

B. Seeking Information
5. Determine and collect data needed to solve the problem.
6. Share data and results with other interested persons.
7. Listen to persons (including teacher) who have relevant knowledge and experiences to share.
8. Identify sources of information.
9. Search printed matter for needed information.
10. Make necessary measurements for obtaining a solution.
11. Record solution possibilities, solution attempts (including making a list or table).
12. Recall and list related information and knowledge.

C. Analyzing Information
13. Eliminate (or ignore date (or information) not needed.
14. Recognize attributes (properties) of an object or mathematical idea.
15. Find likenesses and differences.
16. Classify objects or mathematical ideas (sorting).
17. Make and use a drawing, diagram, or physical model.
18. Make a systematic listing, a table, chart, or graph.
19. Identify trends (patterns) suggested by data in lists, charts, tables, or graphs.
20. Select appropriate notation.
21. Translate the problem situation into mathematical notation.
22. Break a complicated problem into manageable parts. (Break problem into parts or steps.)

D. Solve--putting it together--Synthesis
23. Make predictions and conjectures (usually statements of relationships) based upon observed patterns.
24. Make decisions based upon data.
25. Make necessary measurements for checking a solution.
26. Make necessary computations needed for the solution.
27. Recognize limits and/or eliminate possibilities (including the use of contradictions).
28. Make reasonable estimates as answers.
29. Guess, check, and refine.
30. Solve an easier but related problem. Study solution process for clues.
31. Satisfy one condition at a time.

*Developed by the Lane County Mathematics Project. Oscar Schaaf, Director. Eugene, Oregon, 1978.

 B.14

32. Look at problem situation from varying points of view (including "Aha" phenomena).
33. Apply what you know (deductive reason--chain reasoning).
34. Work backwards (using undoing procedures).
35. Check calculated answers by making approximations.
36. Detect errors.

E. Looking Back - Consolidating Gains
37. Make explanations based upon data.
38. Solve a problem by using different procedures.
39. Search for or be aware of other solutions (including the use of symmetry).
40. Double check solutions by using some formal reasoning method (mathematical proof).
41. Study the solution process.
42. Find or invent a problem situation which can be solved by a set of calculators.
43. Generalize a problem solution so as to include other solutions.

F. Looking Ahead--Formulating New Problems
44. Invent new problems by varying an old one (including the use of analogy.

PROBLEM SOLVING RESOURCES

A. PROBLEM SOLVING STRATEGIES AND HOW TO TEACH PROBLEM SOLVING.

Botts, Truman. "Problem Solving in Mathematics," I, II. Mathematics Teacher, 58 (Oct., Nov. 1965), 496-500, 596-600.

Emphasizes the importance of problem solving in school mathematics and illustrates the way in which one good problem leads to a whole chain of problems.

*Butts, Thomas. Problem Solving in Mathematics. Glenview, Illinois. Scott Foresman & Company, 1973.

A collection of problems dealing with arithmetic and elementary number theory, includes some discussion of problem solving from both the student's and the teacher's point of view.

"Conquering Word Problems in Mathematics." Glen Ellyn, Illinois Math House, 1979.

Ten cassette/worksheet lessons on practical ways to approach word problems. Topics range from how to read the problem to checking the answer.

*Fremont, Herbert. Teaching Secondary Mathematics Through Applications (2nd Edition). Boston. Prindle, Weber and Schmidt, 1979.

A methods course textbook which takes a problem-solving approach to developing appreciation and enthusiasm for mathematics. It utilizes a wide variety of applications and puzzles to illustrate why mathematics is important.

*Gardner, Martin. The Aha! Box. New York. Scientific American, Inc. 1976.

Sudden hunches that lead to short, elegant, problem solutions are sometimes called "aha!" reactions. This is a set of six sets of mathematical problems presented in filmstrip/cassette format.

*Gardner, Martin. Aha! Insight. San Francisco. W. H. Freeman & Co. 1978.

Sixty-five puzzles which have quickie solutions, each presented with the aid of a series of cartoons. Discussion of each problem leads the reader toward a solution and into other related problems.

*Goldin, Gerald A. and McClintock, C. Edwin (editors). Task Variables in Mathematical Problem Solving. Columbus, ERIC/SMEAC, 1979.

A framework for research in problem solving is provided by categorizing and defining variables describing problem tasks. Teaching strategies are also described.

*Greenes, Carole; Gregory, John; and Seymour, Dale. Successful Problem-Solving Techniques. Palo Alto, California, Creative Publications, 1977.

Describes nine basic techniques for solving problems. The book is written for both students and teachers. Limited reproduction permission granted.

*Greenes, Carole; Willcutt, Robert; and Spikell, Mark. Problem Solving in the Math Lab: How To Do It. Boston. Prindle, Weber and Schmidt, 1972.

A resource book for pre-service or in-service mathematics education courses. The lessons, based on classroom experiences, are intended to develop problem-solving abilities via laboratory activities.

*Hatfield, Larry and Bradbard, David (editors). Mathematical Problem Solving: Papers from a Research Workshop. Columbus, ERIC/SMEAC, 1978.

Five papers discussing research and instruction in problem solving.

Henderson, Kenneth B., and Pingry, Robert E. "Problem Solving in Mathematics." In The Learning of Mathematics: Its Theory and Practice, edited by Howard F. Fehr.

Twenty-first yearbook of the National Council of Teachers of Mathematics. Washington, DC. 1953.

A useful reference for the teacher that discusses the theories of problem solving and their implications for instructional approaches in the classroom.

Higgins, Jon L. "A New Look at Heuristic Teaching." Mathematics Teacher 64 (Oct. 1971), 487-495.

A reformulation of the concept of heuristic teaching in terms of problem solving methods and instructional methods. Consequences include a more flexible approach to and more student participation in the teaching/learning process.

Hughes, Barnabas. Thinking Through Problems. Palo Alto, California, Creative Publications, 197 .

A manual which provides the teacher with a structure within which problem solving comes alive. Uses topics such as Pascal's triangle, Fibonacci numbers, and finite differences to arouse students' interest in exploring and analyzing patterns. For gifted students in grades 10-14.

*Judd, Wallace. Problem Solving Kit for Use with a Calculator. Chicago. Science Research Associates. 1977

This classroom kit includes 12 problem-solving sections, each of which stresses a different problem-solving technique or area. In addition to the student workcards and record folders, the kit contains spirit masters for evaluating student achievement and a teacher's guide.

*Kilpatrick, Jeremy and Wirszup, Izaak (editors). Soviet Studies in the Psychology of Learning and Teaching Mathematics. Volume 6, Instruction in Problem Solving. Stanford, California. School Mathematics Study Group, 1972. Available from: National Council of Teachers of Mathematics, Reston, Virginia.

Kinsella, J.J. "Problem Solving." In The Teaching of Secondary School Mathematics, edited by M.F. Rosskopf. Thirty-third yearbook of the National Council of Teachers of Mathematics. Washington D.C.

This discussion provides teachers with an extensive collection of techniques for teaching problem solving.

*Krulik, Stephen (editor). Problem Solving in School Mathematics, 1980 Yearbook. Reston, Va. National Council of Teachers of Mathematics, 1980.

A collection of essays on learning and teaching how to solve problems. Brand new!

Krulik, Stephen; and Rudnick, Jesse A. Problem Solving: A Handbook for Teachers. Rockleigh, N.Y. Longwood Division, Allyn and Bacon, 1980.

A collection of problem solving strategies and teaching suggestions, plus strategy games, over 200 unusual problems, and 25 blackline masters of selected problems ready for duplication.

*Lesh, Richard, et al (editors). Applied Mathematical Problem Solving. Columbus. ERIC/SMEAC. 1979.

Varied perspectives on applied problem solving, including research findings and models for instruction. Considers motivation, learning disabilities, information processing, and theories of problem solving.

*Meiring, Steven. Problem Solving: A Basic Mathematics Goal. Columbus, Ohio. Department of Education. 1979.

A resource manual for in-service education. Includes an analysis of problem-solving strategies, teaching considerations, and ways to make problem solving a part of any mathematics curriculum.

Niman, John (editor). "Problem Solving." Special issue of School Science and Mathematics, 78 (March 1978).

A dozen articles on the role of problem solving in the mathematics curriculum; from history and research findings to practical classroom techniques.

Pinker, Aron. "On the Teaching of Applications of Mathematics to Students Who Aspire to Major in the Biological, Behavioral and Social Sciences." School Science and Mathematics 78 (Dec. 1978), 635-642.

Description and examples of the direct quote word problem (DQWP). These involve a direct quotation from some publication; identification of the source; and a mathematical question to be answered on the basis of this quotation.

*Polya, George. How to Solve it. Princeton U. Press, Princeton, N.J., 1945, 1973.

A classic handbook of problem-solving techniques, organized around four main themes: understanding the problem, devising a plan, carrying out the plan and looking back. Includes a "dictionary of Heuristic."

*Polya, George. Mathematical Discovery, 2 vols. New York. John Wiley and Sons. 1962, 1965.

Presents problem solving from the heuristic point of view. By presenting model solutions which suggest a common pattern of solving problems, it gives the reader good examples for imitation, along with many opportunities for practice. The second part develops a general theory of understanding, learning, and teaching problem solving.

*"Problem Solving." Special issue of The Arithmetic Teacher, 25 (November, 1977).

Aspects of teaching problem solving in the primary and middle school classroom. Includes psychological research, pedagogical ideas, and a diagnostic-prescriptive approach to teaching word problems.

Scandura, J. M. "Mathematical Problem Solving." American Mathematical Monthly 81 (March 1974), 273-280.

"A major portion ... of problem solving ability can be traced to the presence or absence of higher order capabilities which make it possible to combine the constituent parts of a problem solution into a coherent whole."

Seymour, Dale and Shedd, Margaret. Finite Differences: A Pattern-Discovery Approach to Problem Solving. Palo Alto, California. Creative Publications, 1973.

A self-teaching resource using a finite differences approach to solving problems. The book is appropriate for students or teachers possessing a wide range of problem solving skills and mathematics backgrounds.

*"Teaching via Problem Solving." In Didactics and Mathematics. Palo Alto, California. Creative Publications, 1978.

One of the didactics from the Mathematics Resource Project. Discusses how to use heuristics in teaching.

*Travers, Kenneth J.; Pikaart, Len; Suydam, Marilyn N. and Runion, Garth E.; Mathematics Teaching. New York. Harper and Row 1977.

A chapter is devoted to problem solving, discussing the characteristics of a mathematical problem and ways of teaching problem solving.

Troutman, Andria Price; and Lichtenger, Betty Plunkett. "Problem Solving in the General Mathematics Classroom." Mathematics Teacher 67 (Nov. 1974), 590-597.

Describes seven specific abilities related to solving problems and suggests appropriate problem-solving activities that help to develop these specific abilities.

*Wickelgren, Wayne A. How to Solve Problems. San Francisco. Freeman and Company, 1974.

Analyses the basic methods of problem solving and identifies seven major methods. Illustrative problems and their solutions are provided.

Wirtz, R. Drill and Practice at the Problem Solving Level. Washington, D.C. Curriculum Development Associates. 1974.

Provides problem-solving activities at the manipulative, representational and abstract levels.

Yeshurun, Shraga. The Cognitive Method: A Strategy for Teaching Word Problems. Reston, Va. National Council of Teachers of Mathematics. 1979.

Details a method for transforming word problems into equations. The author claims that it works equally well with all ability levels.

B. SOURCES OF PROBLEMS

Ball, W. W. Rouse; and Coxeter, H.S.M., Mathematical Recreations and Essays, twelfth edition. Toronto, University of Toronto Press, 1974.

The latest in a long line of revisions of this classic compendium of problems whose chief appeal is as games or puzzles, rather than as practical applications. While no calculus or analytic geometry is required, the solutions can be very challenging.

*Bell, Max S. Mathematical Uses and Models in Our Everyday World. Studies in Mathematics, Volume XX. SMSG, 1972.

A source book of mathematical problems taken from the real world. Emphasizes mathematical models, numerical information, measurement, and formulas. Many problems are open-ended, and few answers are included.

Bell, Max S. (editor). Some Uses of Mathematics. Studies in Mathematics, Volume XVI. SMSG, 1967.

A collection of 24 articles reprinted from The Mathematics Teacher, The American Mathematical Monthly and other sources. All deal with applications of mathematics in the real world.

*Charosh, Mannis (editor). Mathematical Challenges. Reston, Va. National Council of Teachers of Mathematics, 1965.

An annotated collection of 140 problems from the Mathematics Student Journal. Provides a variety of challenges not usually found in textbooks.

Dudeney, Henry E. Amusements in Mathematics. New York, Dover Publications, 1958, 1970.

A collection of problems and puzzles based on algebra, arithmetic, geometry, logic, combinatorics, and probability. Includes many "whimsical problems."

Dudeney, Henry E. 536 Puzzles & Curious Problems. New York. Charles Scribner's Sons. 1967.

536 puzzles and problems arranged by arithmetic, algebra, geometry, combinatorics, and topology and further classified by problem types. Settings are highly motivational, and most problems are within the ability of secondary students.

Engel, Arthur. "Geometrical Activities for the Upper Elementary School." Educational Studies in Mathematics. 3(1971), 353-394.

A selection of problem solving activities for grades 5-7 involving less familiar topics in geometry.

Engel, Arthur. "Teaching Probability in Intermediate Grades." International Journal of Mathematics Education in Science and Technology. 2(1971), 243-294.

Concerned with developing the intuitive background of probability. Many examples of problems.

Frohlichstein, Jack. Mathematical Fun, Games, and Puzzles. New York. Dover Publications. 1962, 1967.

Activities, puzzles, games, and problems are organized according to content headings that match most junior high and general mathematics texts permitting the user quick access to problem solving materials that reinforce particular topics under study.

Gardner, Martin. Mathematical Magic Show, New York. Alfred A. Knopf, 1977.

The latest in a long line of books based on the author's mathematical games column in Scientific American. Contains puzzles, games, tricks, and other mathematical diversions. The author has an uncanny ability to popularize recent and sophisticated mathematical discoveries, as well as illuminating old gems.

Greitzer, Samuel L. (editor). International Mathematical Olympiads, 1959-1977. New Mathematical Library, Volume 27. Washington. Mathematical Association of America, 1979.

Problems and their solutions from the first nineteen years of this international competition.

*Hill, Thomas (editor). Mathematical Challenges II - Plus Six. Reston, Va. National Council of Teachers of Mathematics. 1974.

Another 100 problems from the Mathematics Student Journal, plus six entertaining articles - three by high school students.

*Hlavaty, Julius H. (editor). Enrichment Mathematics for the Grades. 27th Yearbook, Reston, Va. National Council of Teachers of Mathematics. 1963.

Two dozen enrichment topics for use with academically talented students in elementary and junior high schools. High school students can also profit from some of them. Many challenging problems in each chapter.

*Hlavaty, Julius H. (editor). Enrichment Mathematics for the High School. 28th Yearbook. Reston, Va. National Council of Teachers of Mathematics. 1963.

Twenty-seven enrichment topics for academically talented students in grades 10-14. Each chapter provides a wealth of challenging problems.

*Honsberger, Ross. Ingenuity in Mathematics. New Mathematical Library, Volume 23. Washington. Mathematical Association of America. 1970.

Nineteen essays on enrichment topics in number theory, geometry, combinatorics, logic, and probability. Standard high school algebra and geometry courses furnish a sufficient basis for understanding each essay.

Hurley, James F. Litton's Problematical Recreations. New York. Van Nostrand Reinhold Co. 1971.

Nearly 600 problems that first appeared in "Problematical Recreations," a series of mathematical puzzlers in trade publications, sponsored by Litton Industries. Includes logic, probability, algebra, geometry, Diophantine equations, number theory and calculus.

Jacobs, Harold R. Mathematics: A Human Endeavor. San Francisco. W. H. Freeman & Company. 1970.

This is a general mathematics book that contains many good problematic activities.

*Kastner, Bernice. Applications of Secondary School Mathematics. Reston, Va. National Council of Teachers. 1978.

A dozen applications suitable for secondary school mathematics classes. Mathematical topics range from arithmetic to calculus, as they are used in physics, chemistry, biology, economics, and music.

*Kennedy, Joe and Thomas, Diane. A Tangle of Mathematical Yarns: 1979. Kennedy-Thomas, P.O. Box 132, Oxford, Ohio 45056.

Fifty exciting, imaginative, humorous story problems designed to encourage students to read. Math level is so low that middle school and high school students shouldn't be intimidated. Good for extra credit, class activity, small group work or whatever.

Kordemsky, Boris A. The Moscow Puzzles. New York, Charles Scribner's Sons. 1972.

359 mathematical recreations ranging from familiar whimsical puzzles to delightfully different problems. Most of these mathematical challenges are presented in intriguing story forms.

Kraitchik, Maurice. Mathematical Recreations. New York. Dover Publications, 1953.

250 problems ranging from ancient Greek and Roman sources right up to the present century. All can be done purely for amusement, but a good deal of mathematics can be learned from their solution. 25 positional and permutational games are also discussed.

Loyd, Sam. Mathematical Puzzles of Sam Loyd. New York, Dover Publications, 1959.

A collection of 117 "whimsical problems," first posed by America's greatest puzzlist near the turn of the century. Solutions involve arithmetic, algebra, probability, game theory, geometry, topology combinatorics and operations research.

*Mathematics Resource Project. Mathematics in Science and Society. Palo, Alto, California, Creative Publications, 1977.

Teacher commentary and classroom materials for teaching the application of mathematics to astronomy, biology, environment, music, physics and sports. Student activity pages are ready for duplicating. Can supplement any textbook. Emphasis on junior high math topics.

Mott-Smith, Geoffrey. Mathematical Puzzles for Beginners & Enthusiasts. New York. Dover Publications. 1946, 1954.

A collection of puzzles and problems ranging from easy to moderately difficult. Most of these problems can be solved with arithmetic, simple algebra, or elementary geometry.

Polya, George. Mathematical Methods in Science, New Mathematical Library, Vol. 26, Washington. Mathematical Association of America, 1963.

Lectures on simple physical problems, elementary calculus, and the relation between science and mathematics. Written from an historical perspective.

*Problem Solving. Philadelphia, The Franklin Press.

A monthly newsletter devoted to problem-solving research and instruction. Not all mathematics; mostly college-level problems.

Rapaport, Elvira (translator). Hungarian Problem Book I, II. New Mathematical Library, 11, 12. Washington, Mathematical Association of America, 1963,

Problems and solutions based on the Eötvös competitions held in Hungary from 1894-1928. These contests are famous for the simplicity of concepts employed, the mathematical depth reached and the diversity of elementary mathematical fields touched.

Read, Ronald C. Tangrams -- 330 Puzzles. New York, Dover Publications, 1965.

Tangram puzzles including seven-, fourteen-, and fifteen-piece challenges. The organization and historical references make this an entertaining as well as challenging book.

Salkind, Charles T. (editor). The Contest Problem Book, I, II, III. New Mathematical Library, Volumes 5, 17, 25. Washington, Mathematical Association of America, 1961, 1966, 1973.

Problems and solutions ~~from~~ the annual high school contests of the MAA since 1950.

Schuh, Fred. The Master Book of Mathematical Recreations. New York. Dover Publications, 1968.

Presents the mathematics behind various puzzles, games, card tricks, and other amusing problems. Emphasis on problem-solving strategies, with step-by-step analysis and alternate solutions given.

Sentlowitz, Michael and Thelen, James. M. Baseball: A Game of Numbers. Menlo Park, California. Addison Wesley, 1977.

A collection of "algorithmic exercises with a purpose," based on actual major league statistics. Readers will learn much ~~about~~ baseball as they get valuable practice in using ratio, percent, averages, and graphs.

*Sharron, Sidney (editor). Applications in School Mathematics. 1979 Yearbook. Reston, Va. National Council of Teachers of Mathematics, 1979.

A collection of mathematical models and applications in the areas of art, biology, ecology, everyday life, finance, music and zoology. A comprehensive, annotated bibliography is included.

Smith, Seaton E., Jr., and Backman, Carl A. (editors). Games and Puzzles for Elementary and Middle School Mathematics. Reston, Va. National Council of Teachers of Mathematics, 1975.

A collection of readings from The Arithmetic Teacher since 1954. Includes different topics in mathematics.

*Sobel, Max A., and Maletsky, E. M. Teaching Mathematics: A Sourcebook of Aids, Activities and Strategies. Englewood Cliffs, N.J. Prentice-Hall, 1975.

A source for problems and solutions of problems. Written for the teacher but has sources for students.

*Usiskin, Zalman. Algebra Through Applications. Reston, Va., National Council of Teachers of Mathematics, 1979.

A product of the First Year Algebra via Applications Development Project, this student text uses real-world problems to motivate the mathematics. Includes some probability and statistics.

Vilenkin, N.Y. Combinatorics. Translated by A. Shenitzer and S. Shenitzer. New York. Academic Press, 1971.

An introduction to combinatorics containing over 400 problems with solutions.

Wylie, C. R., Jr. 101 Puzzles in Thought & Logic. New York. Dover Publications, 1957.

A collection of who-dunits, who-held-what position, and cryptarithms that require logic and deduction to unravel. The introduction provides an interesting discussion of methods to approach such problems.

Yaglom, A.M, and Yaglom, I. M. Challenging Mathematical Problems with Elementary Solutions. 2 volumes. Translated by J. McCawley, Jr. San Francisco. Holden-Day, Inc., 1964.

Well-known Russian problem book designed for mathematics students in upper high school grades and early years of college. Problems from the School Mathematics Circle at Moscow State University and from the Moscow Mathematical Olympiads.

C. RECENT ARRIVALS AND NEW DISCOVERIES

Biggs, Edith E. and MacLean, James R. Freedom to Learn: An Active Learning Approach to Mathematics. Don Mills, Ont., Addison-Wesley (Canada) Ltd., 1969.

An excellent source of problem-solving activities for children in middle and junior high schools. Includes many helpful suggestions for the teacher who wants to adopt an activities-based approach to teaching mathematics.

*Burns, Marilyn and Weston, Martha. The Book of Think. Boston. Little Brown and Company 1976.

Subtitled "How to solve a problem twice your size," the book emphasizes problem solving strategies. Not all examples are mathematical.

Friesen, Charles D. "Problem Solving: Meeting the Needs of Mathematically Gifted Students." Sch. Sci. Math. 80 (Feb. 1980), 127-130.

Challenging gifted students to solve problems published in mathematics periodicals has proved to be successful in meeting some of these students' needs. Seven specific benefits are discussed.

Hughes, Carole; Immerzeel, George; Ockenga, Earl; Schulman, Linda; and Spungin, Rika. Techniques of Problem Solving. Palo Alto, California. Dale Seymour Publications, 1980.

A comprehensive program for the development of problem solving skills and strategies in grades K-12. Includes problem card decks, development workbooks, duplicating masters, and activity kits. Designed to complement existing mathematics programs. Problem decks are organized according to grade level.

*Harnadek, Anita. Mathematical Reasoning. Troy, Mich. Midwest Publications, 1969.

Materials for high school students on critical thinking - both in Mathematics and other forms of persuasive argument.

*Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics. A Sourcebook of Applications of School Mathematics. Reston, Va. National Council of Teachers of Mathematics, 1980.

A collection of nearly 500 problems with solutions requiring the application of arithmetic, algebra, geometry, trigonometry, or probability in real-world situations. The annotated bibliography and comprehensive index make it easy to supplement the textbook with genuine, real-world applications. This book belongs in every school library and on every mathematics teacher's professional bookshelf!

*Kespohl, Ruth C. Geometry Problems My Students Have Written. Reston, Va. National Council of Teachers of Mathematics, 1979.

Three dozen problems, cleverly written and solved by tenth grade geometry students.

Kulm, Gerald and Bussmann, Hans. A phase-ability model of mathematics problem solving." J.R.M.E. 11(1980), 179-189.

Presents a model involving eight problem-solving phases and the abilities required of students at each phase. The model has yet to be verified empirically.

*Polya, George and Kilpatrick, Jeremy. The Stanford Mathematics Problem Book. New York, Teachers College Press, 1974.

The complete set of problems from the Stanford University Competitive examinations in Mathematics, held annually from 1946 to 1975. Hints and solutions are given.

Rubinstein, Moshe F. and Pfeiffer, Kenneth R. Concepts in Problem Solving. Englewood Cliffs, N.J. Prentice-Hall 1980.

Presents a framework of concepts basic to training in problem solving. Main topics include thought processes, language, models, uncertainty, decision making, and conflict.

Shields, Joseph J. "Mini-calculators and problem solving." Sch. Sci. Math. 80 (March 1980), 211-217.

Suggestions for exploiting the natural combination of real world problem solving experience and the hand-held calculator.

*Denotes resources on display at these conferences

POSING AND REPOSING PROBLEMS*

1. Introduction

For me, and I suspect the same is true for many other people, the real joy in studying mathematics is the feeling of exhilaration one gets after solving a problem -- the tougher the problem, the greater the satisfaction. But what factor(s) initially motivate someone to want to solve a problem? Possible answers to this question can range from the fear of the consequences if the solution to the problem is not handed in tomorrow to a personal curiosity, but a prime consideration has to be the manner in which the problem is posed. Examine the following problems:

Problem 1 Let $d(n)$ denote the number of positive divisors of the integer n. Prove that $d(n)$ is odd if and only if n is a square.

Problem 2 Which positive integers have an odd number of factors? (Justify your answer.)

Problem 3 Imagine n lockers, all closed, and n men. Suppose the first man goes along and opens every locker. Then the second man goes along and closes every other locker beginning with #2. The third man then goes along and changes the state of every third locker beginning with #3 (i.e. if it's open, he closes it; and vice-versa). If this procedure is continued until all n men have passed by all the lockers, which lockers are then open?**

*This paper is adapted from an article which appeared in the Yearbook on Problem Solving, NCTM, 1980.

**For middle school, choose a specific value for n, say $n = 100$ or $n = 1000$.

These three problems are in fact, different formulations of the same problem*. The first version posed is typical of a dry mathematical style. Less ponderous, the second is given as a question to answer rather than as a statement to prove. The third conveys this mathematical question in a very picturesque manner. I would argue that the phrasing of the third version (and, to a lesser extent, the second) would provide a (significant) source of motivation for the potential solver to tackle the problem. This paper, then, will consist of several suggestions on ways to pose or "repose" a problem to maximize this source of motivation.

2. What is a Problem?

Before offering any suggestions on the posing of a problem, let us first clarify what we mean by a (mathematical) problem. The word problem is derived from the Greek problema which literally translated means "something thrown forward." More mundane is Webster's definition: "a question raised for inquiry, consideration, or solution... a source of perplexity." A problem, then, is a perplexing question or situation. It is not simply a question - it must be perplexing. For example the question "What is the area of a rectangle of length 6 cm and width 3 cm?" is not a problem, but an exercise, probably for anyone reading this paper. An exercise is given (usually) to provide practice in using algorithms. To solve a problem requires insight.

A question or situation can be judged perplexing, and hence be

*To see the equivalence of Problems 2 and 3, examine a particular locker, #12 for example. Locker #12 is touched by men #1,2,3,4,6,12, i.e. the factors of 12. Since a locker is alternately opened and closed, lockers whose numbers possess an odd number of factors will be open in the end.

a problem, only in relation to a person and a time. The determination of the area of a rectangle given its length and width could be a problem for a young child.

Finally any problem must be accepted by the student as a problem - he or she must be interested in the solution.

To summarize, the three basic characteristics of a problem are:

(1) It is a perplexing question or situation.

(2) It is accepted by the student.

(3) At the time it is posed, there is some challenge to the student so the solution is not immediate.

3. Some Suggestions on Posing Problems

Before we offer a number of suggestions concerning effective ways to pose various types of problems, we give one suggestion concerning exercises. The ability to discern a pattern from a series of examples is an essential quality of a good problem solver, so

► Pose a sequence of algorithmic exercises which are examples of a general pattern.

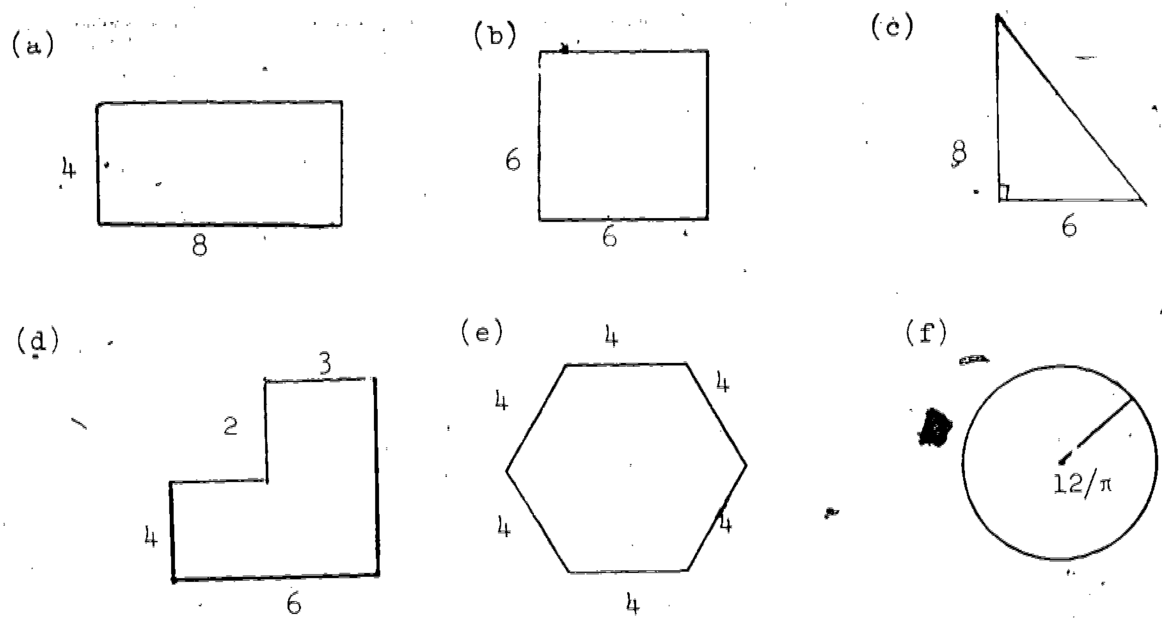
Example 1 Choose two whole numbers. Find their sum and non-negative difference. Add these results. Any observations?

Example 2 Compute (a) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$ (b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5}$

(c) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{99\cdot 100}$. Generalize.

Example 3 Compute (a) 25^2 (b) 35^2 (c) 45^2.
Do you see a pattern?

<u>Example 4</u> Find the areas of the following figures:



Do these figures have anything in common? Any observations?

Completing magic squares and magic triangles is another example illustrating this suggestion.

Now the reversal of an exercise is often a problem. Asking questions "backwards" creates problems which reinforce the knowledge of basic algorithms.

► Pose the <u>reversal</u> of a familiar question; ask the question the "opposite way".

Some classical examples of standard questions in algebra whose reversals are also standard include:

(a) Multiply polynomials -- Factor polynomials.

(b) Given a polynomial, find its roots -- Given a set of roots, find a polynomial with those roots.

Note that the reversal of a problem often has more than one solution. Some less familiar examples of reversal problems are:

<u>Example 5</u> Find three arithmetical problems whose solution is 13. (You can put as many boundary conditions as you care on such a problem; e.g., you must use at least 5 numbers and at least one +, ×, and ÷ sign).

Example 6 Find three solids whose surface area is 60.

Example 7 Write a) 45 b) 525 as the sum of consecutive integers in as many ways as possible. (You should clarify whether negative integers are allowed.)

Example 8 Cryptarithms A cryptarithm is an excellent vehicle for testing understanding of numerical algorithms. Three of my favorites are:

(a)
```
 DONALD
+GERALD
 ------
 ROBERT
```
D = 5. Other letters stand for a unique digit.

(b)
```
 HOCUS
+POCUS
 ------
PRESTO
```

(c)
```
   ABC
  XBAC
  ----
  ***C
  **A
***B
------
******
```

Another way to turn a question concerning a basic concept into a problem is:

► Pose a "Give an Example of" problem.

Example 9 Give, if possible, an example of:

(a) a proper fraction greater than 3/4.

(b) a polynomial of degree 5 with four terms.

(c) a triangle one of whose altitudes coincides with one of its medians.

(d) the equation of a circle which is tangent to both coordinate axes.

(e) a rectangle whose area is numerically equal to its perimeter.

This type of problem is often effective because of its non-specific answers. It is sometimes a remedy for "memorization without understanding."

► Pose a problem with realistic data.

Example 10 A roll of wallpaper is 20 1/2 inches wide and 42 feet long. How many rolls of wallpaper would you need to paper a room which is 9'6" wide, 12'3" long and 8'6" high; has woodwork which is 4" high; has a door which is 7'1" high and 3'4" wide; and has two windows which are 5'2" high and 3'3" wide? (Note: x'y" means x feet and y inches)

An example requiring algebra is:

Example 11* Normal seawater is about 3.5% solids with nearly 300 pounds of dissolved material per thousand gallons. This is about 33,000 parts per million, by weight (written ppm). Such water is highly toxic to drink because it contains barium and boron in addition to more common substances such as calcium, potassium, sodium, magnesium, and so on. Seawater reduced to 5000 ppm of salts is still unhealthy. While people can learn to tolerate up to 3000 ppm of salts, these solids should not exceed 500 to 1000 ppm for good drinking water. Water for irrigation is limited to 1500 ppm of dissolved solids. How many gallons of seawater (33,000 ppm) should be mixed with drinking water (500 ppm) to obtain 1000 gallons of irrigation water (1500 ppm)?

► Pose a problem in which the unknown quantity should reasonably be expected to be unknown in reality.

Example 12 You are in charge of buying food for the class picnic at which you are having ham sandwiches. Each sandwich will have 3 ounces of ham and you expect to need 250 sandwiches. Boneless ham costs $1.89 per pound and semi-boneless (which is 20% "waste") costs $1.53 per pound. Which type of ham should you buy? How much of it?

Another example requiring algebra is:

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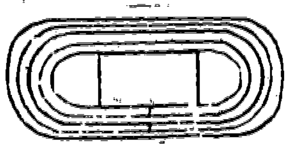
*"Direct Quote Word Problems", A. Pinker, Applications in School Mathematics, 1979 NCTM Yearbook, p. 34-35.



Example 13* A 6-lane track for running foot races is in the shape of a rectangle whose length is 1.5 times its width with a semi-circle on each end. Each lane is to be 1 meter wide. What is the length and width of a rectangle if the inside track is to be 1500 meters long? For a 1500 meter race, the inside runner would start at the finish line. Where should the runners in the other 5 lanes start?

► Pose a problem in which the unknown quantity is one someone might plausibly have a reason to seek.

Example 14 If an 8-inch pizza serves two, how many should two 12-inch pizzas serve?

And one needing algebra:

Example 15 Suppose you wish to buy a new car and you feel you can afford at most $100 per month. If used car loans are made for a maximum of 36 months at 15% interest, what price car should you consider?

► Pose a problem containing insufficient or extraneous data.

The ability to discern the data necessary to solve a problem is certainly crucial in "real world" problem solving. Consequently posing problems demanding a critical analysis of the data is to be commended.

Example 16 Two sides of a triangle have lengths 4 cm and 6 cm. Find the perimeter and area of the triangle.

While the data are clearly insufficient to determine a unique solution, one should seek a solution of the form: "any real numbers P, A satisfying $12 < P < 20$, $0 < A \leq 12$" rather than accepting "no solution". Problems possessing more than one solution are perfectly reasonable problems.

The remaining suggestions apply mainly to an open search problem, that is, a problem which requires the solver to search for a pattern,

*Sourcebook of Applications, MCTM - MAA

to select a method, or to discover a relationship which is not obvious from the statement of the problem. In most of the previous examples, the phrasing of the problem suggested an algorithm to apply, an equation to solve, etc. No such hints occur in the statement of an open search problem. Problem 3 in the introduction is an example of such a problem.

Especially for open search problems, but for others as well, the basic tenet concerning their posing is:

Fundamental Axiom of Problem Posing

Pose the problem in a manner which requires the solver first to guess the solution.

Though this axiom cannot always be followed, all too often a good problem is spoiled by including the answer in the statement of the problem. The example in the first section furnishes a nice illustration; others are:

Example 17

Form A For any positive integer $n \neq 2,3,5$ a square can be partitioned into n smaller squares.

Form B For which positive integers n can you partition a square into n smaller squares?

Example 18

Form A Prove that $(n-1)! \equiv 0 \bmod 4$ if and only if n is a composite > 4.

Form B For which positive integers n is n a factor of the product $(n-1)(n-2)\ldots3\cdot2\cdot1$? (For example, 5 is not a factor of $4\cdot3\cdot2\cdot1=24$, but 6 is a factor of $5\cdot4\cdot3\cdot2\cdot1=120$.)

Form A is typical of college number theory texts, while in Form B this problem could be posed immediately following the introduction of the concepts "factor" and "prime" (in fact, it is almost an alternate characterization of "prime") in middle school. Form B also illustrates the fact that brevity is not necessarily a virtue in posing a problem: Including illustrative examples, for instance, simply enlarges the set of potential solvers.

As a rough rule of thumb, any problem whose statement includes the words "prove that", "show that", etc., will not encourage guessing and often can be rephrased to do so. The use of the phrase "find all" is still another way to pose an open search problem which requires guessing and/or searching for patterns.

> Example 19 Find all primes p for which the integer $5p + 1$ is a square.

Even standard theorems can be posed as questions.

> Example 20 For which positive integers n does the fraction $\frac{1}{n}$ have a finite decimal expansion?

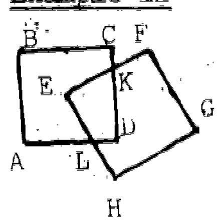
And a geometric example:

> Example 21 In which quadrilaterals do the diagonals bisect each other?

Such a problem could profitably be used to begin a unit on decimals or on quadrilaterals.

► Pose a problem having only one solution as an open question.

> Example 22



What values are possible for the are of quadrilateral EKDL if ABCH and EFGH are squares of side 12 and E is the center of square ABCD?

► Pose a problem having no solution as a question.

> Example 23 Which integers in the set {17, 27, 37, ...} can be written as the sum of two primes?

All too often students regard textbook problems as artificial and one remedy cited earlier was to make the problem as realistic as possible. Another alternative is to pose the problem in a totally unrealistic manner, i.e.,

► Pose a whimsical problem.

These problems often paint a vivid, sometimes ridiculous, picture in the mind of the potential solver. The Locker Problem of the first section is one example, (can't you just "see" someone walking down a row of 10,000 lockers, opening every other one?).

Another example is:

Example 24 Two riders on bicycles, 200 miles apart, begin traveling toward each other at the same time. The first cyclist travels at 10 mph and the second 15 mph. A fly begins flying between the bicycles at the same time, starting from the front wheel of the slower bicycle. If the fly travels at 20 mph flying back and forth between the bicycles, being able to reverse directions without losing any time, how far will the fly travel before the bicycles meet?

A whimsical problem can appeal to a student because he does not regard it as phony and it may pique his intellectual curiosity.

4. Reposing Problems

In this section we apply some of the suggestions of this paper to the "reposing" of problems.

Our fundamental axiom deserves reemphasizing.

- Pose the problem as a question if at all possible.

Example 25

Form A If any item is on sale for 20% off, most sales people will first deduct 20% of the price and then add the applicable sales tax (say 4%). Show that if the salesperson first added the 4% sales tax and then deducted 20% of the total, the price to the consumer would be the same.

Form B (Same assumptions as Form A about the sales price) Which method would you prefer if you were the a) consumer, b) merchant, c) tax collector? Why?

- Given any skill or concept, it is usually possible to pose a set of non-routine problems of varying type and difficulty involving that skill or concept.

An algebraic example is:

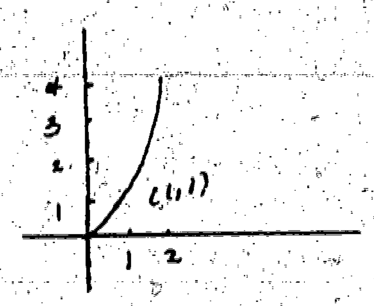
Example 26 The straight line.

Typical problems involve finding the equation of a line given two points on the line, graphing a line given its equation, determining the equation of a line given its graph, etc. Several non-routine problems are:

Problem A Give the equations of three lines containing the point (1,2).

Comment: The non-specific answers required should help those students who are merely memorizing the various formulae. Students who choose $x = 1$, $y = 2$, $y = 2x$ might be regarded as "clever", they are insightful problem solvers.

Problem B (Algorithmic) Take a point P other than (1,1) on the parabola $y = x^2$, $0 \leq x \leq 2$, and compute the slope of the line containing P and (1,1). Do the same for six other points on the curve. What do you think the slope of the line tangent to $y = x^2$ at (1,1) is?



Comment: Algorithmic problems with a purpose.

Problem C (Open Search) Find all possible values for the slope of a line which contains the origin and intersects the circle $(x-12)^2 + (y-5)^2 = 25$.

Problem D Write the equations of several line segments with $-2 \leq y \leq 2$ which will spell your first name.

- The problem of posing and reposing algebraic word problems is a tough one. The solution of word problems requires the translation of English words into mathematical terms -- a critically needed skill in any discipline which uses mathematics. One difficulty is that the quantity sought in a word problem requiring the solution of an equation is often known in "real life". When the problem is posed realistically, however, it can be solved using only arithmetic.

Example 27

A. The Bigtown Coliseum has 20,000 seats -- 8,000 reserved seats and 12,000 general admission seats. Reserved seats cost \$10.00 each while general admission seats cost \$5.00 each. The newest rock group sensation 'SMOOCH' is giving a concert there. If \$130,000 was collected from the sale of 18,500 tickets, how many reserved seats were sold?

Comment: Surely the person selling the tickets knows how many he sold. At best he would want to compute the amount of money received -- an arithmetic problem.

A somewhat more realistic version of this problem might be:

B. The Bigtown . . . concert there. The promoter estimates his expenses for ushers, programs, janitorial service, etc. at \$10,000. If SMOOCH demands 40% of the price of each ticket sold plus \$40,000, how many reserved seats must he sell to break even? to make a profit of \$50,000?

P.S. Attempts at humor (however feeble) in the statement of mathematical problems are usually worthwhile.

● In order to incorporate interesting open search problems into your class, you may consult one of the many problem books currently available. We close this section with three example of such problems with suggestions for reposing them in a more challenging manner:

Example 28

Problem 1 Prove that a positive integer greater than 9, all of whose digits are identical, cannot be a perfect square.

Problem 1A (Better) Which squares have identical digits?

Problem 1B (Best) What is the maximum number of identical non-zero digits in which a square can end?

Comment: Formulations 1A and 1B repose the problem as a question to encourage guessing. Version 1B has a positive answer (rather than "none").

Example 29

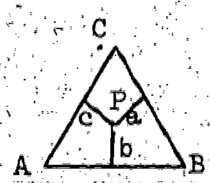
Problem 2 Show that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is never an integer.

Problem 2A The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is known to diverge and consequently the sum $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n}$ can become arbitrarily large. If we examine a few of its partial sums, we notice that $\sum_{k=1}^{4} \frac{1}{k} \approx 2.083$, $\sum_{k=1}^{11} \frac{1}{k} \approx 3.02$ and $\sum_{k=1}^{31} \frac{1}{k} \approx 4.0272$. Can $\sum_{k=1}^{n} \frac{1}{k}$ ever be an integer?

Comment: Giving some numerical examples often helps clarify the problem and makes it more tantalizing to the solver. A calculator can be used to compute these and other examples.

A geometry example:

Example 30



Problem 3 For a point P inside equilateral triangle ABC, the sum of the perpendiculars a, b, c from P to the sides equals the altitude of ABC. (Viviani's Theorem).

Problem 3A A man bought an estate enclosed by three straight roads each 1 mile long. He wished to build his house somewhere on the estate so that if he should have a straight driveway made from the front to each of the three roads, he might be put to the least expense. Where should he locate the house? (Dudeney, 535 Curious Problems and Puzzles, # 284)

Comment: The problem is posed as a question. By quantifying the problem, experimenting with several locations to help guess the solution is encouraged.

5. Summary

This paper is concerned with the improvement of the oft-neglected art of problem posing. The three key words in this sentence are "neglected", "art" and "improvement". One has only to examine the problem sections of most textbooks to see the "neglect". They often consist of lists of algorithmic exercises, unimaginative word problems, "prove that" problems, etc. (the Review Problems are usually worse). We have offered one axiom and ten suggestions for improving the manner in which problems are posed. They are:

(1) Pose a sequence of algorithmic exercises which are examples of a general pattern.

(2) Pose the reversal of a familiar question; ask the question in the opposite way.

(3) Pose a "Give an Example of" problem.

(4) Pose a problem with realistic data.

(5) Pose a problem in which the unknown quantity should reasonably be expected to be unknown in reality.

(6) Pose a problem in which the unknown quantity is one someone might plausibly want to seek.

(7) Pose a problem containing insufficient or extraneous data.

<u>Axiom</u> Pose the problem in a manner which requires the solver first to guess the solution.

(8) Pose a problem having only one solution as a question.

(9) Pose a problem having no solutions as a question.

(10) Pose a whimsical problem.

As for "art", to pose a problem so that the potential solver will

(1) be motivated to solve the problem,

(2) understand and retain the concept involved in the solution of the problem,

(3) learn something about the art of solving problems,

requires the creativity of an artist. The suggestions offered in this article will hopefully "improve" the state of the art.

The study of mathematics is solving problems. It is incumbent, therefore, upon teachers of mathematics at all levels to teach the art of problem solving. The first step in this process is to pose the problem properly.

Prepared by Thomas Butts, 1/80

REAL-WORLD PROBLEMS AND APPLICATIONS

Applying mathematics in the real world is a form of problem solving, in which the problems come from outside of mathematics itself. In this aspect of problem solving, mathematics is used as a tool to formulate and analyze a model of some real-world problem. This mathematical model is some bit of mathematics, such as numerical expression, equation, function, geometric figure, or algebraic structure, which expresses in abstract terms and symbols something about reality. Mathematical modeling is a cyclic process which takes the problem solver from the world of reality to the world of mathematics and back again.

One very illuminating way to describe the mathematical modeling process is the "dive-swim-climb" analogy, attributed to Professor J. Synge. According to Synge, solving a real-world problem mathematically involves three stages:

1) A dive from the world of reality into the world of mathematics.
   This first stage involves translating the problem from ordinary language into mathematical terminology and symbolism. At the end of this stage there will be a pure mathematical problem (the mathematical model) to be solved.

2) A swim in the world of mathematics.
   During this stage we find a solution to the mathematical problem that was formulated during the dive. This may consist of solving some equations or inequalities, proving a geometric theorem, evaluating a function, writing and running a computer program, or some other mathematical task. We perform this task without necessarily thinking about any connection with the real world.

3) A climb back into the world of reality, carrying a prediction in our teeth.
   At this point we translate the mathematical solution back into terms of the real world from whence the problem arose in the first place. The numbers or geometric figures must be interpreted so that they make sense to the person who asked the question. This answer is, in fact, a prediction of what will occur in a given real-world setting.

Such a prediction can then be tested to see whether or not it agrees with observed facts. If it does, then we can say that we have solved the problem. If not, then the process must be repeated with another dive into the world of mathematics, looking for a mathematical model that will do a better job.

As an example of this process, consider the problem of determining how high a punted football rises above the ground. We dive into the world of mathematics by thinking of the football as a projectile and interpreting its path as a parabola, whose y-coordinate is given by $y = -16t^2 + v_0 t$, where t is the time in seconds, and v_0 is the initial upward velocity in units of feet per second. This model has conveniently been provided to us by the physicists, who study similar problems.

Our swim in mathematics first requires that we re-formulate the mathematical problem to compensate for the fact that we don't know v_0. (We do know t, the "hang time" of the punt.) At the top of its path, the football has vertical velocity zero. If we make this our starting point, with y = 0, we will need to compute y after 2 seconds (half of the hang time). We find that

$$y = -16\ (2^2) + 0(2) = -64.$$

As we climb back into the real world, we reinterpret our answer of -64 by remembering that our origin was chosen at the top of the parabolic path. If the ground is -64 feet from that point, it means that the football must have actually been kicked to a height of 64 feet above the ground. This answer can be tested by using more sophisticated equipment to measure the height of a punt. In all probability we will find that our answer is too high, because our model did not take the air resistance or buoyancy into account, although football fans are well acquainted with the effect of wind on a team's kicking performance. At this point we might want to dive back into the world of mathematics and develop a model that includes air resistance, or we may stick with what we have, recognizing its shortcomings.

A remark about the work of pure and applied mathematicians is in order here. The pure mathematician can find satisfaction in swimming endlessly in the world of mathematics. A solution that is mathematically correct is his goal, and further testing is not required. On the other hand, the applied mathematician does not have this luxury of accepting any mathematical solution. Every potential solution must be checked against reality. The goal of an applied mathematician is a better understanding of reality, and many perfectly good mathematical solutions must be rejected because they simply do not make sense in the physical or social world. Students should be made aware of this distinction as they shift from routine exercises on mathematical skills to problems which ask questions about the real world.

An alternative way of describing the process of mathematical modeling is the MR/PV approach, attributed to Dr. Ralph Thomas. Here we consider four steps:

1) Measure. Obtain data from the real world. This data usually comes from observation and measurement, although some of it may be found in tables or books.

2) Relate. Look for mathematical relationships in the data, trying to tie the data together by means of an equation, function, or geometric figure. We search for patterns that will bring order to what started out as raw facts. These relationships constitute the mathematical model itself.

3) Predict. Once we have discovered a pattern to our data, we can use this pattern to predict an outcome that was not previously recorded. Such predictions often take the form of interpolation (finding values in between those already observed) or extrapolation (finding values beyond those previously noted).

4) Verify. We return to the real world with our prediction and attempt to test it against observable facts. As in the dive-swim-climb analogy, we may need to collect more data and go "back to the drawing board" to formulate a mathematical relation that does a better job.

As an example of MR/PV at work, it has been observed that crickets chirp faster when the air is warmer. If we want to understand this phenomenon mathematically, we may attempt to obtain an equation giving the rate of chirping as a function of temperature. Suppose that we obtain the following data by counting the average number of chirps per minute and measuring the temperature on two different evenings.

T = Temperature (Fahrenheit)	55°	62°
C = Chirps per minute	60	88

One simple relationship would be linear. A routine application of the two-point form of a linear equation leads to $C = 4(T - 40)$. This mathematical model now can be used to generate many predictions about the rate of chirping at other temperatures. One of these predictions can be verified (or contradicted) by listening to the crickets on an evening when the temperature is 68°. If the average number of chirps per minute on that evening turns out to be 112, we will have more confidence in our model. However, a rate significantly lower or higher than this would cause us to seek a different mathematical relationship, which could then be tested on still another evening.

While the notion of a mathematical model is central to applied mathematics, we should not let the formalities of constructing such a model interfere with the development of problem solving skills. Going through the complete modeling process for every application is just as pedantic as trying to make every proof rigorous. We do want our students to be acquainted with the notion of a mathematical model, but the actual modeling process can be done less formally.

In addition to being a good source of problems, real-world applications provide a number of important fringe benefits.

1) Motivation. Students are generally more willing and eager to learn those things which they perceive as useful. Utility is an intrinsic form of motivation, as opposed to external incentives, such as grades.

2) Enrichment. A good teacher seldom finds the textbook completely adequate to meet the needs of all students. The better students need more challenging problems, while the slower ones need more opportunities to practice basic skills. The whole class can benefit from interesting examples of mathematics in use. Applications can help meet all of these needs.

3) <u>Variety</u>. Applications can provide an alternative to the usual classroom routines -- especially when the students are involved in gathering their own data, such as in an outdoor measurement project or a statistical survey.

4) <u>Practice</u>. Applications can provide a vehicle whereby students have an opportunity to use their mathematical skills in new and interesting contexts. They can see some purpose to this type of drill.

5) <u>Unification</u>. By relating mathematics to other subjects in the curriculum, the teaching of applications can be an aid in integrating various disciplines. For example, many applications of mathematics deal with physical and biological sciences, social studies, industrial arts, home economics, art, and music. Some excellent opportunities arise for teachers in different departments to plan together so that the science, social studies, or art teacher will ask students to use concepts and skills newly learned in the mathematics classroom.

6) <u>Functional literacy</u>. Although by no means new, the notion that our schools must prepare students to function as wise consumers and enlightened citizens in an increasingly technological culture has received new impetus during the past decade. Functional literacy implies not only a proficiency in computational skills, but the ability to use those skills correctly in a variety of situations that may appear in everyday life. By including real-world applications in the curriculum, we provide such contexts and give the students important experience in putting their skills to work.

7) <u>Career Education</u>. The National Council of Teachers of Mathematics has called for cooperative efforts on the part of guidance counselors and mathematics teachers in helping students to make appropriate educational vocational, personal, and social choices. Today more and more jobs require mathematical skills, even as many of our high school students are graduating with weaker mathematical foundations. Through applications, we can spotlight some of the mathematics needed in various careers and emphasize the importance of mathematics as it relates to the future vocations of our students.

A good classroom application is a compromise between authenticity and usefulness. It should be close enough to reality to make students believe that it is important. On the other hand, the mathematics has to be at a level appropriate to the students' abilities. The statement of the problem must be simplified or streamlined to the extent that students can identify key facts and relations and not become lost in a tangle of irrelevant and redundant information. Some features which make an application particularly appropriate for classroom use include the following:

1) The application need not be immediately practical to the student, but it should be plausible, if not clear, that somebody has a good reason to seek an answer to the problem. The students should see the problem as related to their own experiences or those of somebody known to them. It is important to remember that merely phrasing a problem in terms real-world objects and activities does not make it a genuine application. Very few of the word problems in our textbooks ask questions that anyone outside of the mathematics classroom would care to answer. For example: "A 10-foot board is cut into two pieces, one of which is 6 inches short of being three times the length of the other. Find the length of each piece."

2) The problem should be realistic in terms of the data used as well as the unknown whose value is sought. If possible, data should be taken from real life, which means that prices, interest rates, speeds of airplanes, etc., may have to be updated periodically. The other half of this criterion rules out problems asking for Mary's age, given information about John's age and their ages together; or problems asking for the dimensions of a room, given its volume and area of each wall.

3) Calculations should be feasible with a modest hand-held calculator. Today we can pose problems with realistic data and not burden the students with an unreasonable computational load. The other side of this coin is that we need to teach students about precision and accuracy and how to interpret results obtained from inexact data. Most teachers are very unsure about these things, and there is a real need for better pre-service and in-service education in this area.

4) The solution should not be "intuitively obvious" to the students. The result should be convincing, testable, and a little surprising. Curiosity is another intrinsic source of motivation, and the students should appreciate the need for doing some mathematics in order to obtain the answer.

5) The application should illuminate and motivate some mathematical ideas. In other words, the solution to the problem should cause the students to ask further questions about the mathematics itself or see this mathematics in a new light. Historically mathematics and science have enjoyed a symbiotic relationship, with mathematics providing tools for solving tough problems and science suggesting new avenues of mathematical investigation. On a smaller scale, the theory and applications of mathematics should reinforce each other this way in the classroom.

6) The application should be interesting to the teacher. Enthusiasm (or the lack of it) is contagious. If we want our students to enjoy solving problems, then we must paint a picture of problem solving as a stimulating challenge, rather than as drudgery. This is surely easier to do when the teacher finds the application interesting. Every teacher should begin to develop his or her own personal file of interesting applications which can be tapped for classroom use as the occasion warrants.

Real-World Problem Solving: Bicycles

1. Examine the frame of a bicycle, and look for basic geometric shapes, such as triangles. Determine whether or not any parts of the frame form parallel lines.

2. Count the teeth on each gear wheel of a particular bicycle and make a table showing these numbers. (A gear attached to the rear axle is called a sprocket, while a gear attached to the pedals is called a chainwheel. The gear advantage is the ratio of wheel revolutions to pedal revolutions when the bike is in a particular gear. How can the gear advantage be determined from the numbers of teeth on the sprocket and chainwheel?

3. For a bike with 27-inch diameter wheels, 42 teeth on the chainwheel and 28 teeth on the sprocket, how far will it move along the street for each complete revolution of the pedals? Derive a general formula relating distance traveled to wheel diameter and gear advantage.

4. The gear ratio for a bicycle is defined to be

   $\frac{cd}{s}$, when c = # teeth on chainwheel
   s = # teeth on sprocket
   d = diameter of rear wheel in inches

   If a bike rider cranks the pedals at 72 r.p.m. and the gear ratio is 73.0, how fast is the bike traveling in miles per hour?

5. A bike rider's cadence is the number of revolutions per minute made by the pedals. Derive a formula relating cadence, gear ratio, and linear speed of a bicycle.

6. Gearing of a bicycle is often set up so that, for any three consecutive gears, the gear advantage of the middle one is approximately the geometric mean of the gear advantages of the other two. Check this out for a particular bicycle. Design a 5-speed gearing system that makes use of this principle. Assume that your chainwheel will have 48 teeth and the sprockets for high and low gear will have 14 and 28 teeth respectively.

7. Bicycle riders are affected by air resistance, known as wind drag. When riding in still air, the wind drag is equal to the speed of the bicycle but acts in the opposite direction. If a wind is blowing, it contributes to the total wind drag, which is then found by means of vector addition.

   A cyclist is riding due north at 18 miles per hour. The wind is blowing from 30 degrees south of east at 5 miles per hour. Find the total wind drag.

8. The distance needed to stop a bicycle is a function of the speed of the bike and the pressure applied to the brakes. Perform an experiment that measures stopping distances at different speeds. Try to use the same brake pressure each time.

   Make a graph of stopping distance versus speed and another graph of stopping distance versus the square of the speed. Try to formulate an equation [i.e. a mathematical model] relating these two variables.

9. If the bicycle to be used in problem 8 is not equipped with a speedometer, develop one or more methods of determining its speed.

References:

Pamela Ames. "Bring a bike to class." Arithmetic Teacher, 25 (Nov. 1977), 50-53.

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J.R. Metz. "Pedaling Mathematics." Mathematics Teacher, 68 (Oct. 1975), 495-498.

E 10

Real-World Problem Solving: Measurement at Home

Home Sweet Home

1. A pair of drapes to cover a double window will require 5 widths of material, $2\frac{1}{2}$ widths sewn together for each drape. The length of the drapes must be 41 inches, with 3 additional inches at the bottom for a hem and six additional inches at the top for pleats. How much will the material cost, at $3.50 per yard?

2. One gallon of paint is supposed to cover 400 square feet. A bedroom is 10 feet by 12 feet with a ceiling height of 7 feet, 9 inches. The hall door measures 30 inches by 79 inches, and the closet door measures 48 inches by 80 inches. There is one double window measuring 36 inches by 88 inches. The walls and ceiling are to be given a single coat of paint, all one color. How many gallons of paint will be required? This paint sells for $14.50 per gallon and $4.50 per quart. How much will the paint for this room cost?

3. If the homeowner decides to paper the walls of the room described in problem 2, how many single rolls of wallpaper will be needed, if each single roll is supposed to cover 35 square feet? Because of the need for cutting and fitting, do <u>not</u> subtract the area of the window this time, but do subtract the area of the doors.

4. A dining room table has a circular top with a 54-inch diameter. Four leaves, each 12 inches wide, can be added to extend the table. Two tablecloths are to be made, one for the circular table and the other for the table fully extended. A minimum overhang of 8 inches is desired, all around the table. The material is 45 inches wide, but each hem and seam will require an additional inch. How many yards of material must be purchased?

Lots of Problems

5. The lot is 85 feet by 150 feet. The house covers 1870 square feet and the driveway another 950 square feet. The rest of the lot (with the exception of a few small flower beds) is to be put in grass. Grass seed costs $13.95 for a 3-pound box, and the recommended seeding rate is $1\frac{1}{2}$ pounds per thousand square feet. How much will it cost to seed the yard? Fertilizer costs

E 11

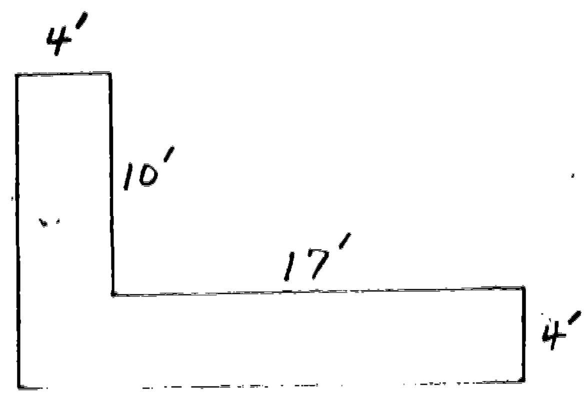
$8.69 for a bag that feeds 5000 square feet. How much will enough fertilizer for one feeding of this yard cost?

Sod costs $1.60 per square yard. How much would it cost to cover the yard with sod? (Note: This problem can be made more realistic by providing a plot plan from a magazine and having students take their data from this plan.)

6. A patio is to be poured in the shape of a quarter circle, with a radius of 21 feet. The concrete will be 4 inches deep. If concrete costs $36.40 per cubic yard, how much will the total concrete bill be?

7. The driveway measures 16 feet by 59 feet. What is the weight of snow that must be removed after a 5-inch snowfall? (The weight of snow varies with its moisture content. For this problem assume that it weighs 3.5 pounds per cubic foot.)

8. The region shown at the right is to be planted with shrubs. The nursery recommends that these be placed 3 to 4 feet apart (center to center). How many shrubs will be needed? Mulch, three inches deep, is to be placed around the shrubs. How many bags of mulch will be required if each bag contains [illegible] feet. Allow one square foot per shrub that will *not* be covered with mulch.



[illegible]

9. A certain cow [illegible] milk [illegible] to the farmer [illegible] this milk [illegible] $11 [illegible] How much income [illegible] this cow [illegible] in the year?

[illegible] milk weighs [illegible] pounds per gallon [illegible] per gallon [illegible] with what [illegible] in the [illegible]

E 12

10. A 14-inch by 16-inch by 38-inch rectangular bale of hay weighs 60 pounds. How much does this hay weigh per cubic foot?

11. Some farmers bale hay in round bales 5 feet in diameter and 5 feet long. How many rectangular bales of the size in problem 11 does it take to equal one of these round bales?

12. A combine has a 16-foot cutter bar. (i.e. it cuts a swath 16 feet wide.) How far must the combine travel to harvest one acre? (One acre equals 43,560 square feet.) If the machine travels at 3 miles per hour, how many acres can be harvested from 6:30 a.m. until 2:15 p.m., with a 45-minute break for lunch and two 15-minute rest periods? First solve this problem without allowing for the time needed to turn around at the ends of the field. Then make some realistic assumptions about turn-around time and get a second solution.

(See a related problem, 0-32, on page F09.)

Problems, Problems, Problems!

The 2xy problems in this collection have been partitioned into three categories:

I Open Search Problems

II Application Problems

III Miscellaneous Problems

These titles are explained in greater detail in each section and some of the decisions regarding placement of problems were arbitrary. The problems have been gathered from various sources, many of which are cited in section C - "Problem Solving Resources". Most of the problems have appeared in at least two sources.

Although most of the problems in this collection can be solved using ideas from arithmetic, informal geometry, and basic algebra, no hints concerning the level of difficulty or appropriate grade level have been given for any problem. This "omission" is intentional for (at least) three reasons. Most problems can appeal to students at various levels of sophistication depending on what is accepted as a solution. For example, the determination of a certain length could be done by solving an equation, by approximation using some trial and error method, or by making a physical or scale model and measuring. Secondly, when one encounters a problem in the real world, there is usually no indication of its difficulty. Finally, as the old philosopher says, "Difficult for you; easy for me."

Following some of the problems are,

1. VARIANT which indicates an alternate way to pose essentially the same problem.
2. COMMENT - which contains comments on ways to effectively use the problem

and

ETC. - which indicates that the reader is encouraged to pose a similar problem. (You might ask yourself: "What would happen if ...?")

And now, on to the problems:

III Open Search Problems

An open search problem is one which requires the solver to search for a pattern, to select a strategy, or to discover a relationship which is not obvious from the statement of the problem. In exercises or in many "word problems", the phrasing of the problem suggests an algorithm to apply, an equation to solve, etc. There are usually no such hints in an open search problem.

The problems are in random order except that certain classes of problems have been grouped together.

01. When all the whole numbers are listed in order: 12345678910111213... what digit is in the 206, 789th place? (Example: The digit 2 is in the 15th place.)

    VARIANT: It takes 206, 789 digits to number the pages of a very long book. How many pages are in the book?

02. Calculate: $\sqrt{\underbrace{(111\ldots1)}_{100\ 1\text{'s}}\ \underbrace{(100\ldots05)}_{99\ 0\text{'s}} + 1}$

    COMMENT: Calculator useful.

03. Suppose the scoring in football is simplified to touchdown - 7 points, field goal - 3 points, and no other scoring. What scores are impossible to achieve? (e.g. 2-1, 8-5)

    VARIANT: With a supply of 13¢ and 15¢ stamps, what amounts of postage could you put on a package?

04. Express a) 45 b) 30 c) 525 as the sum of two or more consecutive integers in as many ways as possible. In general, which positive integers can be expressed as the sum of two or more consecutive integers?

    COMMENT. Good problem to practice addition, subtraction, and division.

05. Imagine 1000 lockers, all closed, and 1000 people. Suppose the first person goes along and opens every locker. Then the second person goes along and closes every other locker beginning with #2. The third person then goes along and changes the state of every third locker beginning with #3 (i.e. if it's open, she closes it; and vice-versa). If this procedure is continued until all 1000 people have passed by all the lockers, a) which lockers have been touched by exactly two people, and b) which lockers are now open?

COMMENT: 1) Good problem to <u>introduce</u> a number theory unit in middle school. 2) A whimsical problem often appeals to a student because of the mental picture he gets.

06. A person 6 feet tall walks all the way around the earth at the equator. How much farther does his top of his head travel than the bottom of his feet?

VARIANT: A steel band is stretched tightly all the way around the earth at the equator. If this band is spliced and another 10 foot section is added, what is the tallest object which could now walk underneath it?

COMMENT: 1) Another whimsical problem with a (generally) unexpected result. 2) Good problem for students to first estimate or guess the answer before attempting to solve.

07. How can single digit integers be placed in the lower boxes so that whenever an integer is placed in the lower row, the integer above it must occur that many times in the lower boxes.

0	1	2	3	4	5	6	7	8	9

For example, if 2 is placed in the box below 4, then 4 must appear twice in the lower boxes.

ETC.: Try it for boxes 0 - 7 , 0 - 8

08. Louie the Gambler took his week's paycheck to the casino. He paid the $2 entrance fee, lost 1/2 of his remaining money, and then tipped the hatcheck girl $1 on his way out. This bad luck persisted; the same sequence of events happened each day for the next 4 days. At the end of the fifth day, he had $5 after he left the casino. What was his weekly paycheck?

VARIANT: Essentially the same strategy can be used to solve problems such as 1) How can boiling a pot of vegetables for 15 minutes be timed with a 11 minute "hourglass" and a 7 minute "hour glass"?

2) How can you bring exactly 6 liters of water from the pond to your house if you have only a 4 liter pail and a 9 liter pail?

09. In which rectangles with integer sides is the area numerically equal to the perimeter?

010. The set $\{1,2,3,4\}$ can be partitioned into two subsets $\{1,4\}$ and $\{2,3\}$ in which the sum of the elements in each subset is the same. The set $\{1,2,3,4,5\}$ cannot be so partitioned. For which positive integers n can the set S_n $\{1,2,\ldots,n\}$ be partitioned into two subsets (they need not have the same number of elements) in which the sum of the elements in each subset is the same? Describe a method of partitioning the set S_n when it is possible.

COMMENT: Good problem for finding a pattern and a discussion of the properties of a solution to a problem.

ETC.: Try 3,4,... subsets or sum of one subset is 1 more than sum of other subset.

011. Suppose a large wooden cube has been painted black. It is then cut up into a) 64 b) 1000 smaller cubes. How many cuts were required in each case? How many of the smaller cubes have 0,1,2,3 faces painted black?

ETC.: Find general pattern for cubes. Try cutting up into other shapes.

012. Which positive integers can be written as the difference of two squares?

VARIANT: Which positive integers can be the length of one leg of a right triangle with integer sides?

ETC.: 1) How many ways can an integer be written as the difference of two squares.
2) Which integers can be written as the difference of two cubes?

013. Roger Miller claimed on a TV show that he was 1/3 Cherokee. Discuss the validity of his claim.

COMMENT: Good introductory problem for several units. (See article

014. (Fermat) Find the remainder when [illegible]^{999,999} is divided by [illegible].

VARIANT: Find the a) one's b) last two digits of [illegible]^{999,999}.
Same for [illegible]^{999,999}.

015. Find the smallest positive integer with exactly [illegible]

016. Three dimensional tic tac toe is played on a 4 x 4 x 4 grid and the object is to get as many lines of 4 "marbles" in a row as possible. How many lines of 4 in a row are there in this 4 x 4 x 4 grid?

ETC.: Try larger grids.

017. Magic Squares

An additive magic square is a square array of numbers - all different - so that the sum of the numbers in each row, in each column, and along both diagonals is the same. For example,

5	3	10
11	6	1
2	9	7

is a 3 x 3 magic square whose common sum is 18.

a) Complete each of the following magic squares

16	15	20
		13
14		18

6		
13		
2		

8		
13		

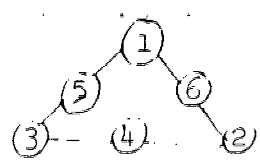
	14	

b) Can you find a 3 x 3 magic square all of whose entries are i) odd integers ii) end in 0 iii) end in 2 iv) leave a remainder of 3 when divided by 17?

c) What values are possible for the common sum?

ETC.: 1) Try larger additive squares 2) Try multiplicative magic squares 3) Use fractions instead of integers 4) Try magic triangles instead of squares

eg



Are there any other "magic" shapes? Try a 3 x 3 x 3 magic cube.

018. Decide whether the statement: "In any triangle, the sum of the lengths of the three ______ exceeds one half the perimeter" is true or false when the blank is replaced by a) altitudes b) medians, c) angle bisectors.

019. For which positive integers n is n a factor of the product $(n-1)(n-2)\ldots3\cdot2\cdot1$? (For example, 5 is not a factor of $4\cdot3\cdot2\cdot1=24$, but 6 is a factor of $5\cdot4\cdot3\cdot2\cdot1=120$.)

COMMENT: Good problem for middle school to get across the the difference between prime and composite integers.

VARIANT: For which positive integers n is

a) n a factor of (n-1)(n-3)... 2(or 1)?

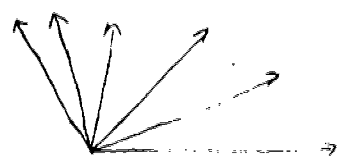
b) n^2 a factor of (n-1)(n-2)...3•2•1?

020. Given the line

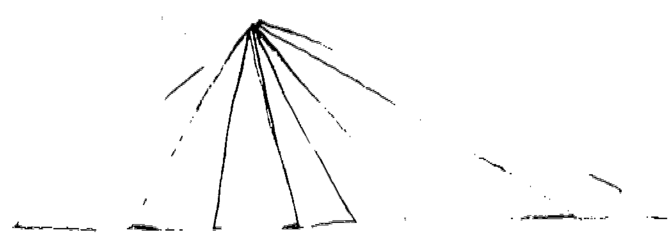
How many segments are determined by a) 2 b) 3 c) 4 d) 5 e) n points on this line?

VARIANT: The points need not be on a line.

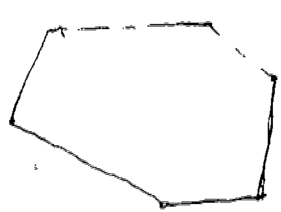
021. How many angles are in this picture? Generalize if more rays are added.



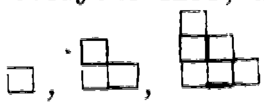
022. How many triangles are in this picture? Generalize



[illegible] How many diagonals does this hexagon have? Generalize



024. If 10 people are at a party and everyone shakes hands with everyone else, how many handshakes would there be? Generalize.

025.  Find the area and perimeter of each of these polygons. Generalize.

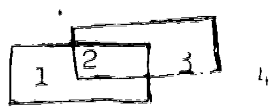
026. For which positive integers n is the sum of the first n positive integers a factor of the product of the first n integers? (Example: $n = 4$, $1+2+3+4 = 10$, $1\cdot2\cdot3\cdot4 = 24$ so false; $n = 5$, $1+2+3+4+5 = 15$, $1\cdot2\cdot3\cdot4\cdot5 = 120$, true.)

027. Find all pairs of integers (a,b) so that the sum of all the integers between a and b is 1000.

ETC.: Find (a, b) so the sum of every other integer between a and b is 1000.

028. You, a druggist, received 10 bottles of a certain drug, each with 100 capsules. One bottle contained capsules which were 10 milligrams too much. How can you find the faulty bottle as quickly as possible with a scale? Suppose more than one bottle (but you didn't know how many) contained heavy capsules. How could you find them efficiently with a scale?

029. a) Into how many regions can two rectangles partition the plane?

(Example: 

4 regions)

Consider all cases.

ETC.: Try other plane figures.

b) Into how many regions can a cube and a sphere partition space?

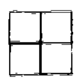
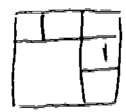
ETC.: Try other solids

COMMENT: Good problem for testing visualization.

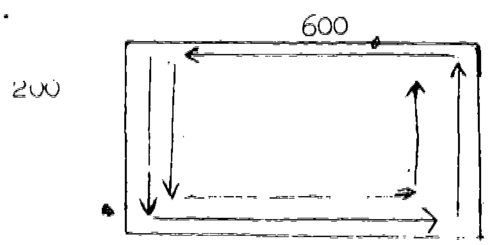
030. Using 16 unit squares (□) , many different polygons can be formed. What values are possible for the perimeter and area of these polygons? Answers the same question for 16 unit cubes, surface area, and volume. ETC.

COMMENT: Good problem to practice skill of finding perimeter.

031. For which positive integers n can you partition a square into n smaller squares? (Example: $n = 2$ impossible;

$n = 4$  $n = 6$ )

032. Given a hay field 600 m long and 200 m wide, and a hay rake 4 m wide, how many circuits of the field must the farmer make to have raked 1/2 the hay? He rakes as shown in the figure.



033. Consider the set of unitary fractions $F = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\}$.
a) Find i) 3 ii) 4 iii) 5 fractions from F whose sum is 1. Generalize if you can.

b) Find i) 2 ii) 3 iii) 4 iv) ... fractions from F whose sum is as close, but not equal, to 1 as possible.

034. Can $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ ever be an integer?

035. Calculate $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100}$

ETC.. Can you find other choices for the denominators which make a similar sum of fractions "easy" to calculate?

036. For which positive integers n is

a) $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n^2} > 1$

b) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} < 1$?

037. For which positive integers n does $\frac{1}{n}$ have a

a) finite decimal expansion (Example: $\frac{1}{4} = .25$)

b) repeating decimal expansion of period 2 (Example $\frac{1}{11} = .090909$)

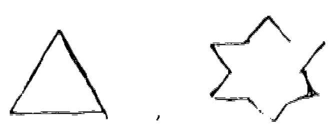
c) repeating decimal expansion of period 3

d) repeating decimal expansion of period 6 ?

ETC.

038. Find the prime factorization of the smallest positive integer which is 1/2 of a square, 1/3 of a cube, and 1/5 of a fifth power. ETC.

039. A infinite snowflake:



Each second a new equilateral triangle emerges in the middle third of each side. Discover what you can.

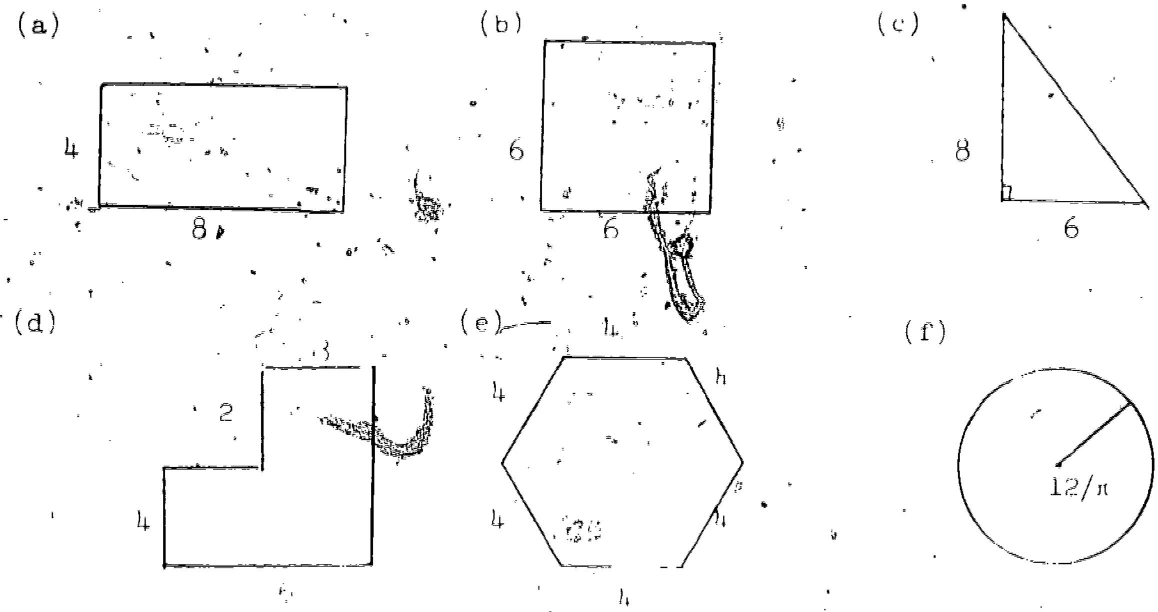
COMMENT. An open ended question could examine area, perimeter, length of time before a certain phenomenon occurs, etc. Encourages student to pose as well as solve problems.

040. Choose 3 distinct digits and write all possible 3 digit integers containing the chosen digits. Add these integers and then divide by the sum of the three digits. What values are possible

for this quotient? Do the same for a) 2 b) 4 or c) 5 digit integers. ETC.

COMMENT: Practice skills.

041. Find the areas of the following figures:



Do these figures have anything in common? Any observations?
COMMENT: More skill practice with a purpose.

042. Take a piece of string. Place it so it crosses itself a) 1 b) 2 c) 3 d) 4 e) n times. How many regions are formed?

043 Into how many pieces can you cut a pizza with a) 2 b) 3 c) 4 d) 5 e) n cuts? Consider all cases (Example, 2 cuts determine 3 or 4 pieces). The circular shape is maintained throughout the cuts.

ETC 1) What happens if you can move the pieces after each cut?
2) Consider other shapes for the "pizza," e.g. square, horseshoe, crescent (of moon).

044. Slice as a cylindrical piece of cheese into 8 identical pieces with as few cuts as possible.

ETC.: Consider other "highly symmetric" solids.

045. Into how many pieces can you cut an orange with a) 2 b) 3 c) 4 d) 5 e) n cuts. Consider all cases. The spherical shape is maintained throughout.

ETC.: 1) What happens if you can move the pieces after each cut?
2) Consider other shapes, e.g. cube, cylinder, doughnut.

046. How many triangles with integer sides have a longest side(s) of length a) 1 b) 2 c) 3 d) 4 e) 5 f) n units? In each case, how many of the triangles are isosceles?

047. How many different polygonal-shaped floor tiles could you make so you could tile a floor using tiles of only one shape?

COMMENT: Can rephrase this problem to suit different audiences; use only regular polygons, allow curved shapes.

048. a) How many ways can you write a) 12 b) 937 c) n as a sum of two whole numbers?
b) How many ways can you write a) 3 b) 4 c) 5 d) 12 e) n as the sum of one or more whole numbers if the order in which the numbers are written matters (Example: 4 : 4 = 3 + 1 = 1 + 3 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1 ; 8 ways)
c) In b) how many ways can you write these integers as a sum of one or more whole numbers if 1 cannot be used? (Example: 4 : 4; 2 + 2; 2 ways). ETC.

049. In how many ways can you pave a 2 x n road with 2 x 1 bricks?

ETC.: Change the road to something movable so symmetry must be considered (e.g.  is equivalent to ).

050. What is the maximum number of identical non-zero digits in which a square can end?

ETC.: Cube, fourth power.

051. a) How many different ways can you color the a) vertices of a square with 2 colors b) faces of a cube with 2 colors c) vertices of a cube with 2 colors?

COMMENT: Good problem after discussing symmetry.

ETC.: Different shapes, more colors.

052. A a) tetrahedron b) cube are to be colored so that no two adjacent faces (sharing an edge) have the same color. How many colors are required?

ETC.: Different shapes, more colors.

053. How many different necklaces are there with 3 red beads and 7 blue beads?

COMMENT: Circular symmetry.

054. How many squares are on a checkerboard? Generalize.

055. What is the fewest number of line segments necessary to construct a figure (like a checkerboard) which has exactly 100 squares? (18 segments determine a checkerboard.)

056. Find all ways of arranging the ten digits $0,1,2,\ldots,9$ into a ten digit integer which is divisible by each of the 17 integers $2,3,4,\ldots,17,18$.

057. Find the smallest positive integer n so the product $999n = 111\ldots1$.

058. Using the digits $1,4,6,7,9$, form a 3-digit integer and a 2-digit integer whose a) sum b) difference c) product d) quotient is as large as possible, as small as possible. ETC.

COMMENT: More good skill practice. Calculator to check results.

059. Some cryptarithms:

a) $\begin{array}{r} ABA \\ -CA \\ \hline AB \end{array}$ b) $\begin{array}{r} DONALD \\ +GERALD \\ \hline ROBERT \end{array}$ $D = 5$ c) $\begin{array}{r} HOCUS \\ +POCUS \\ \hline PRESTO \end{array}$ d) $\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array}$

e) $\begin{array}{r} FORTY \\ +TEN \\ +TEN \\ \hline SIXTY \end{array}$ f) $\begin{array}{r} EEO \\ \times OO \\ \hline EOEO \\ EOO \\ \hline OOOOO \end{array}$ E = any even digit, O = any odd digit g) $\begin{array}{r} ABCDE \\ +ABCDE \\ \hline FGHJK \end{array}$

ETC., ETC., ETC.

060. Let $a_1, a_2, a_3, \ldots, a_n$ by any rearrangement of $1,2,3,\ldots,n$. For which integers n is the product $(a_1-1)(a_2-2)(a_3-3)\ldots(a_n-n)$ always even?

061. For which positive integers n is

a) $1! + 2! + 3! + \ldots + n!$?

b) $1!\,2!\,3!\,4!\ldots n!$ a square?

062. a) A <u>pentomino</u> is a polygon formed by connecting 5 unit squares so any two are joined along an edge(e.g. yes, no)? How many pentominos are there? How many of them fold up into an open box?

b) a hexomino is such a polygon composed of 6 unit squares. How many hexominos are there? How many of them fold up into a closed box?

063. For a 2-digit integer such as 35, 11 is always a factor of the sum of that integer and its reversal (e.g. 11 is a factor of $35 + 53 = 88$). Why? Is this fact true for a) 3 b) 4 c) 5 d) 6 e) n digit integers?

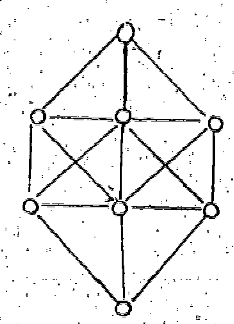
064. Fold a piece of paper twice and cut off the folded corner. How many holes are in the unfolded piece of paper? How many holes will there be if the paper is folded a) 3 b) 4 c) 5 d) n times?

COMMENT: Physical model is appropriate.

065. Find the smallest positive integer which (simultaneously) has a remainder of $n-1$ when divided by n for $n = 2,3,4...9,10$.

066. Find a set of 5 positive integers so no subset has a sum divisible by 5. Do the same for 6,7. Generalize.

067. Place the numbers $1,2,...,8$ into the 8 circles so that no two integers in consecutive order are joined by a line segment.



068. Find the smallest integer n for which the statement, "In any set of n different two digit integers, there are two disjoint subsets whose elements have the same sum" is true.

069. a) How many zeroes are at the end of $1000!$?
b) What is the last non-zero digit?

070. For what integers n is $(1\cdot2\cdot3\cdot\ldots\cdot n)^2 > n^n$?

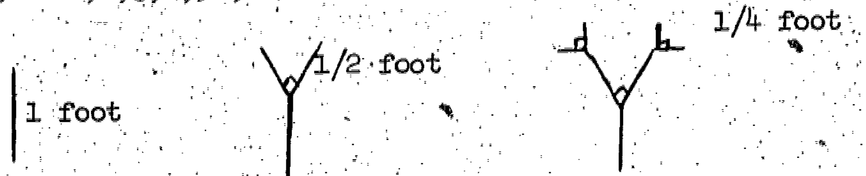
071. For what integers n is 2^n+1 divisible by 3?

072. What is the maximum value of the area of a triangle inscribed in a parallelogram of base 8 and height 6 ?

073. How many a) 4 b) 5 c) 6 d) n digit integers are there whose only digits are 1 and 2 if both digits must occur at least once. In each case, how many of the integers are divisible by 3? ETC.

074. Find the sum of all distinct 4 digit integers which contain the digits 1,2,3,4,5 ; each at most once. ETC.

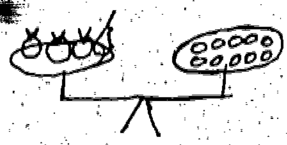
075.

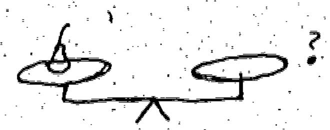


Discover what you can about this infinite tree.

076. N people are standing in a field so that no two people are the same distance apart. Then everybody shoots his nearest neighbor once.
a) For which N does at least one person survive (no matter where the people stand)?
b) What is the maximum number of bullets which can hit one person?

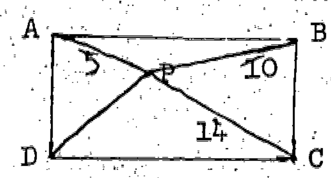
077.  1 apple + 6 plums = 1 pear

 3 apples + 1 pear = 10 plums



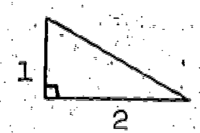
1 pear = ? plums

078. On a digital watch showing hours and minutes, how many times during a 12 hour period are at least two of the digits the same?

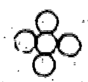


079. a) Find $\overline{PD}$

b) Find $\overline{AB}$ and $\overline{BC}$

080. Make a square out of 20 triangles shaped like 

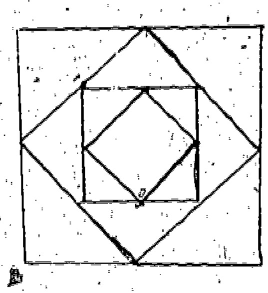
081. Pennies are arranged in a regular pattern on a large table.

a)  Suppose each penny touches four other pennies.

b)  Suppose each penny touches 6 other pennies.

In each case find the percentage of the tabletop covered by the pennies.

082. Each successive square is formed by joining the midpoints of the previous square. If the original square has side length 1, find the area and perimeter of the 20th square. ETC.



083. "I counted 50 steps that I made down the escalator" said Amy. "But I counted 75 steps, though I was walking three times as fast as you were" said Bob. If the escalator were stopped how many steps would be visible?

084. What percentage of all powers of 2 begin with 1 (e.g. 16, 1024)?

085. A security guard is responsible for the safety of three locations A,B,C. At what point P should he stand so that the largest of the distances PA, PB, PC is as small as possible?

086. Bjorn beat Jimmy in a tennis set 6-3. Five games were won by the player who did not serve. Who served first?

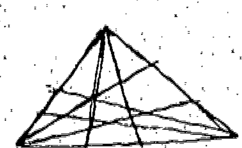
087. What is the smallest percentage of the points the winner of a tennis set can win if the winner of each game must win at least 4 points and beat his opponent by 2 points and the winner of a set must win at least 6 games and beat his opponent by 2 games.

088. Consider $S = \{1,2,3, \ldots ,n\}$.

a) Choose a,b,c at random from S with replacement. Find the probability that $a = b + c$.

b) Choose a, b at random from S with replacement. Find the probability that $x^2 + ax + b = 0$ has real roots.

c) Choose a,b,c,d at random from S with replacement. Find the probability that the system $\begin{matrix} ax + by = 20 \\ cx + dy = 35 \end{matrix}$ has a solution.

089. 

Suppose a) 2 b) 3 c) 5 d) n lines are drawn from each vertex. Into how many regions is the triangle partitioned?

090. Give, if possible, an example of:

(a) a proper fraction greater than 3/4.

(b) a polynomial of degree 5 with four terms.

(c) a triangle one of whose altitudes coincides with one of its medians.

(d) the equation of a circle which is tangent to both coordinate axes.

(e) a rectangle whose area is numerically equal to its perimeter.

(f) two real numbers a, b for which $a + b = a + b$.

(g) a rational function with two horizontal asymptotes.

COMMENT: Good way to test basic definitions and concepts in a way that the student may not merely regurgitate memorized facts.

(h) a right scalene triangle of area 4.

II Application Problems

Generally speaking, an application problem is one in which you apply appropriate algorithms to find a solution. The strategy for solving such a problem is often evident; the main task is to carry out the calculations, solve the equation, etc. Where many of the open search problems (and the miscellaneous problems in the next section as well) are in some sense unique, the problems in this section are intended to be prototypes or models which you can adapt to areas which will interest your students. In order to find data for "real world" applications, you may wish to consult mail order catalogs, almanacs, Guiness Book of World Records and other similar sources. Though such problems are not numerous in this section (due largely to space limitations) the following four types of application problems deserve emphasis in the classroom:

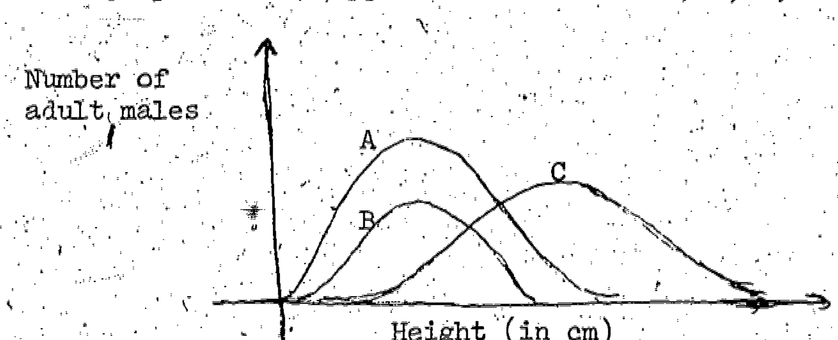
(1) Interpreting and constructing tables, graphs, charts, etc.

(2) Problems containing insufficient or extraneous data and/or problems where the data must be gathered by the students.

(3) Decision making problems; a problem which asks the solver to shoose from alternate courses of action.

(4) Simulation problems; using dice, telephone books (as a random number table), computers, etc., to simulate real world problems.

A1. Here is a graph of the heights of three tribes, A, B, C.



a) Which tribe has largest population of adult males?

b) In which tribe is the average height the a) largest b) smallest?

c) To which tribe does the a) tallest, b) smallest, male belong?

d) In which tribe is the variation of heights the a) greatest b) smallest?

A2. A new piece of equipment costs $10,000. The cost for maintenance and repair are:

Year	1	2	3	4	5	6	7	8	9	10
Cost	500	780	940	1100	1200	1430	1920	2300	3300	4400

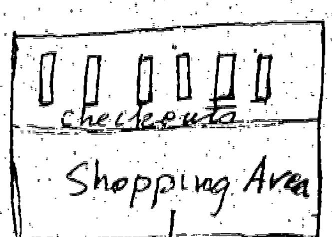
a) After how many years should this piece of equipment be replaced so the average yearly cost of purchasing and operating it is minimized?

b) Suppose the inflation rate is 10% per year for the next 10 years. Reconstruct this table and answer the question in part (a).

A3. You are in charge of buying food for the class picnic at which you are having ham sandwiches. Each sandwich will have 3 ounces of ham and you expect to need 250 sandwiches. Boneless ham costs $1.89 per pound and semi-boneless (which is 20% "waste") costs $1.53 per pound. Which type of ham should you buy? How much of it?

COMMENT: Decision problem.

A4.



Here is a store with a shopping area and six checkouts. Two checkouts are open. Roll a die to determine the number of poeple who leave the shopping area and go to the checkouts. Allow 1 person at each checkout to finish after each toss of the die. What is the smallest number of checkouts which should be open to avoid long lines?

COMMENT: (1) Simulation problem to study queuing theory (2) Computer useful.

A5. How many pennies are in a stack as tall as the Sears Tower (in Chicago)?

COMMENT: (1) No numbers used; students must look up height of Sears Tower and decide whether to measure 1 penny, 10 pennies, 100 pennies, etc. Discuss measurement error in each case.
(2) Calculator useful.

A6. If one million \$1 bills are placed end to end, how far would they reach? How many \$1 bills would it take to go all the way around the world at the equator (some may get wet)?

A7. (a) At the instant you finish reading this question, approximately how many seconds have you been alive? (b) How many times will your heart beat in 10 years?

A8. Some examples of Project Problems in which students gather or supply their own data:

a) How many cows would it take to supply the school cafeteria with milk for one month?

b) How much can you earn from a paper route in one year?

c) How much would it cost to buy and operate a 5 year old car for one year?

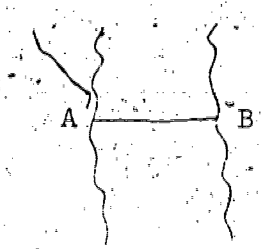
d) How much would it cost for you and your family to take a vacation to Disney World?

e) Design a parking lot for the nearby grocery store.

f) How much would it cost to wash all the windows in the Sears Tower?

g) How many column inches of your favorite newspaper would be needed to print the entire text of a typical half-hour television news show?

A9. Find the height of the flag pole in the front of the school in as many ways as possible.

A10. 

Measure the width $\overline{AB}$ of this river.

A11. (a) Two sides of a triangle are 9 cm and 12 cm long. What values are possible for the perimeter and area of the triangle?
(b) The sides of a quadrilateral have lengths 7,8,9, and 10. What values are possible for the area of the quadrilateral?

COMMENT: The data are insufficient to determine a unique answer, but we should seek solutions which include all possible values; not just say "no unique solution".

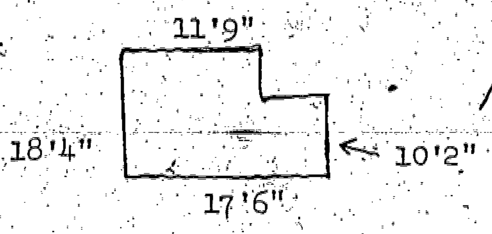
A12. One of the most popular methods for estimating the size of a population (of animals, say) is the <u>capture-recapture</u> method. A sample of the population is caputered (say 100), marked and released. Later 200 animals were captured and 15 marked ones were found a) What is a reasonable estimate for the size of the population? b) How would you estimate the amount of blood a monkey has using a similar procedure?

A13. The standard house current is 120 volts. Fuses or circuit breakers are used to keep wires from becoming overloaded. What is the minimum resistance (in the form of toasters, refrigerator, etc.) that could be handled with a 20 amp fuse before a fuse would blow out?

COMMENT: Ohm's Law needed; an example of constructing a problem based on familiar law, formula, etc.

A14. Your gas bill for last January was $67.74 for 28.8 thousand cubic feet of natural gas; $83.56 for 25.2 thousand cubic feet this January. a) What was the increase in the price per thousand cubic feet of natural gas for the year? What percentage increase is this? The Utilities Commission allowed the Gas Company an 8% increase during the year. In addition they were allowed to "pass through" any additional gasoline costs they had.
b) What was the increase per thousand cubic feet of this "pass-through" cost for the year?

c) Suppose the Gas Company is allowed a 9% increase and "pass-through" costs increase 37¢ per thousand cubic feet next year. How much will you pay for 29.3 thousand cubic feet of natural gas next January?

A15. 

You have decided to carpet or tile the living-dining area of your home. The carpet comes on rolls 12 feet wide and tiles come in 12 inch squares. How much of each material would you buy? How many feet of carpet? How many tiles?

COMMENT: Decision-making problem real data.

A16. You are in charge of refreshments for the football games this year. Last year 13,593 people attended six home games and they sold 9,854 hot dogs, 5612 hamburgers, and 21,618 soft drinks. This year the team is better but there are only 5 home games. They expect a 15% increase in attendance at each home game. Price hikes, however, will lower demand for food by 5%. How many of each item should you order for the first home game?

A17. Painters get paid by the job or by the hour. A painter is offered the following alternatives: either \$300 plus \$8 for each hour worked less than 40; or a straight hourly wage of \$8.50. Under what conditions should the painter choose the hourly rate?

A18. A tailor has the following materials available: 16 yds cotton, 11 sq. yds. silk, 15 sq. yds. wool.

A suit requires - 2 sq. yds. cotton, 1 sq. yd. silk, 1 sq. yd. wool

A gown requires - 1 sq. yd. cotton, 2 sq. yds. silk, 3 sq. yds. wool

A suit sells for \$30, a gown for \$50. How many of each should he make?

A19. Modern airplanes are equipped with transponders which allow Air Traffic Controllers to tell which airplane makes a given blip on the screen. The transponders have 4 dials calibrated from 0 to 7 and the controller asks each pilot to set his transponder to a given 4-digit number. How many different transponder settings are there? Suppose the controller uses the first digit to indicate direction eg. 0 means south, 1 means southwest, etc. How many planes can be assigned distinct settings?

A20. In book publishing it is common to print several pages on a large sheet and then fold the sheet in such a way it can be trimmed and stitched to give a book. There are "signatures" of 32 pages (16 leaves) (4 folds). How should the pages be printed on the sheet of paper so they will be in correct order and facing the right way when they are cut and stitched?

A21. Get a road map for the eastern U.S. What is your best route from Cleveland to St. Louis?

A22. A roll of wallpaper is 20 1/2 inches wide and 42 feet long. How many rolls of wallpaper would you need to paper a room which is 9'6" wide, 12'3" long and 3'6" high; has woodwork which is 4" high; has a door which is 7'1" high and 3'4" wide; and has two windows which are 5'2" high and 3'3" wide? (note: x'y" means x feet and y inches)

A23. A pendulum on a clock should swing once every 3 seconds but it doesn't. It swings 302 times when it should swing 300. How much time does the clock gain in one month?

A24. Super Saver fares offered by most airlines say that you save 40% on a long trip if you fly Monday-Thursday; 30% if you fly Friday-Sunday. The usual round trip fare from Cleveland to San Francisco is $582. a) What is the Super-Saver fare on Saturday? b) If the rates increase 3% next month, what will a Super-Saver ticket to leave on a Tuesday cost?

A25. The inflation rate was 9.4% and 12.3% in two recent years. a) If an item cost $100 two years ago, what does it cost now? b) If the inflation rate remains at 12.3% per year, how long will it take prices to double?

A26. If any item is on sale for 20% off, most sales people will first deduct 20% of the price and then add the applicable sales tax (say 4%). The sales clerk could first add the 4% sales tax and then deduct 20% of the total. Which method would you prefer if you were the a) consumer, b) merchant, c) tax collector? Why?

A27. A car depreciates 30% the first year, 18% the second year, 13% the third year and 12% the fourth year. How long does it take to lose half its value?

A28. Cleveland has a population of 694,750. Of these 68% are qualified to vote, 61% of the qualified voters were registered and 53% of those registered voted in the last election. How many people voted in the last election?

A29. The price of hamburger was $1.49 per pound in September. It rose 11% in October and dropped 11% in November. What was the price per pound at the end of November?

A30. How long would it take to spread a rumor (disease) in a town of 80,000 if each person who hears it tells 3 others within 15 minutes?

A31. 

A 6-lane track for running foot races is in the shape of a rectangle whose length is 1.5 times its width with a semi-circle on each end. Each lane is to be 1 meter wide. What is the length and width of a rectangle if the inside track is to be 1500 meters long? For a 1500 meter race, the inside runner would start at the finish line. Where whould the runners in the other 5 lanes start?

A32. The Bigtown Coliseum has 20,000 seats -- 8,000 reserved seats and 12,000 general admission seats. Reserved seats cost $10.00 each while general admission seats cost $5.00 each. The newest rock group sensation 'SMOOCH' is giving a concert there. The promoter estimates his expenses for ushers, programs, janitorial service, etc. at $10,000. If SMOOCH demands 40% of the price of each ticket sold plus $40,000, how many reserved seats must he sell to break even? to make a profit of $50,000?

A33. During the first three years of growth a tree grows only its trunk. During the 4th year the trunk divides and grows into two main branches. During the 5th year and every year thereafter, it grows two new branches on each old branch. How many new branches are grown during the 8th year? What is the total number of branches on the tree at the end of the 8th year?

A34. A tire has an outside diameter of 28 inches. When the radius of the tire is decreased 1/4 inch due to wear, how many more revolutions per mile does the tire make?

A35. A busline offers two package services. The length, width, and height sent the ordinary way must add up to 141 inches or less and the longest measurement cannot exceed 60 inches. A package sent by "next bus out" service must be 24" x 24" x 45" or less.
a) What is the largest possible volume of a package sent in the ordinary way?
b) How many times larger is this box than the largest one which can be sent by "next bus out" service?

A36. In 1979, Bell Telephone charged 50¢ for the first minute and 34¢ for each additional minute for a weekday direct dial call from Cleveland to Atlanta
a) How long can you talk for $10?
b) From 5pm - 11 pm, there is a 35% discount. How long can you talk for $10?
c) On Saturday, there is a 60% discount. How long can you talk for $10?

A37. In early 1980, there was a $500 discount for buying a big car. A small car averaging 20 mph sold for $5998. A big car averaging 15 mph sold for $6234. Including the discount, which car is the better buy?

A38. The U.S. Postal Regulations state that the length plus the girth of a package cannot exceed 100 inches. What is the largest package you could send by parcel post?

A39. What are the dimensions of the smallest rectangular paper which can be cut to make a book cover for your math book which is 2.5 cm thick, 21 cm tall, 26 cm long and a 6 cm overlap inside the book?

A40. License plates in Ohio consist of three letters followed by three digits. How many different license plates can be made?

A41. Baskin-Robbins Ice Cream advertises 31 flavors. How many different a) single b) double c) triple dip cones can be made?

A42. How much "string" does it take to a) stitch a baseball b) string a wooden tennis racket?

A43. Find the volume of a football without using water displacement. Check your answer.

A44. A ladder is leaning against a building 7 feet from the base. If the top of the ladder slips 4 feet down, then the foot of the ladder slides how far?

A45. You receive a chain letter with 5 names on it. You are to send $1 to the person at the top of the list, cross his name out, and place your name at the bottom of the list. You then send 5 copies of this letter to friends with the same instructions. How much money will you ultimately receive?

A46. On a grill which holds 2 steaks, how long would it take to cook 3 steaks 20 minutes - 10 minutes on a side?

A47. On a 120 mile trip, you averaged 30 mph going and 40 mph returning. What was your average speed for the entire trip?

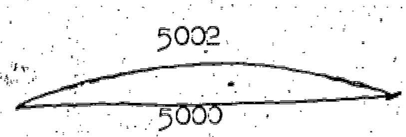
A48. A security guard is responsible for 3 locations A, B, C. He wants to place himself at a point P so the largest of the three distances PA, PB, PC is as small as possible?

A49.

What is the largest (area) rectangle you could slide down one corridor, around the corner and then down the other corridor?

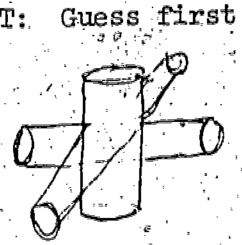
COMMENT: Model furniture moving - easy case of Sofa Problem.

A50. Suppose a 5000 foot long railroad rail is solidly anchored at both ends (with no expansion joints). When it is very hot it expands by 2 feet. How far off the "ground" does the rail get?



COMMENT: Guess first

A51.



Find the volume common to the three pipes of radius 2 cm if their axes of symmetry are mutually perpendicular.

COMMENT: Calculus unnecessary.

A52. Where should you sit in a theatre to get the best view?

PAGE F30 "COLORADO SAM" REMOVED DUE TO COPYRIGHT RESTRICTIONS

A54. Three soldiers are 6 miles from their barracks. It is 9:10 pm and they must be there by 10 pm. On foot they can cover the distance in 1 1/2 hours. One has a bicycle which can cover the distance in 1/2 hour. The bicycle can carry 2 people. Can they make it? How, or why not?

A55. The National Security Council of Freedonia has 5 members. The defense documents are in a safe. How many locks must the safe have and how should the key be allocated so any majority can open the safe but any minority cannot?

Miscellaneous Problems

The miscellaneous problems of this section consist of puzzles, physical problems, some problems which could have been classified as open search problems, and even a few games.

M1. What is the length of the shortest "segment" of unit squares (□□···□) which can be folded up into a cube?

M2. Two riders on bicycles, 200 miles apart, begin riding toward each other at the same time. The first cyclist travels at 10 mph; the second at 15 mph. A fly begins flying between the bicycles at the same time, starting from the front wheel of the slower bicycle. If the fly travels at 20 mph flying back and forth between the bicycles, how far will the fly travel before the bicycles meet?

M3. I am thinking of an integer between 1 and 1000. What is the maximum number of questions you need to ask me to determine my number? I may answer any question only "yes" or "no."

M4. You are standing at some point on the earth (assumed spherical). You walk 10 miles south, 10 miles east, and 10 miles north and return to the point from which you started. At what point(s) on the earth could you be?

M5. Make 100 using the digits 1, 2, ..., 9 in as many ways as you can using any arithmetic operations you wish. (Example: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$; $123 - 45 - 67 + 89 = 100$)

COMMENT: There are many puzzles of this kind, a classic being the Four 4's puzzle. Restricting yourself to only addition and subtraction in this puzzle is challenging.

M6. Find all arrangements of 4 points in the plane so that at most two distinct lengths are determined.

M7. What is the sum of all the digits used in writing down all the integers from 1 to 1 billion (inclusive)?

M8. Two ferry boats go back and forth across a river with constant speed. They leave opposite shores at the same instant, meet for the first time 700 feet from one shore, continue on their way to the banks, return and meet for a second time 400 feet from the opposite shore. How wide is the river?

M9. If you fold a piece of paper .003 cm thick 50 times each doubling the thickness each time, how thick is the folded paper?

COMMENT: In trying to fold a piece of paper, one finds you can keep doubling the thickness a maximum of 7 times.

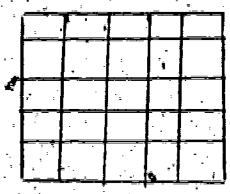
M10. Form 4 equilateral triangles with 6 toothpicks of equal length.

M11. Take a single strip of paper and fold it to make a region in the shape of a regular pentagon. ETC: Many paper folding problems

M12. Describe the shape of the solid formed by revolving a cube about one of its diagonals.

M13. Remove the two opposite corner squares of a checkerboard. Can you cover the remaining 62 squares with 31 1x2 dominoes? What if you remove one black square and one white square from anywhere on the checkerboard. Can you cover the remaining 62 squares with 31 1x2 dominoes?

ETC.: Do the results change if a different size "checkerboard" is used?

M14. 

Place 5 pennies on this "board" so that no 2 pennies are in the same row, in the same column, or along the same diagonal.

ETC.: How many ways can you do it? What happens with larger squares?

M15. How many games are played in a) single b) double elimination tournament with 649 teams?

M16. Take 3 empty cups and a) 11 b) 10 pennies. Can you place an odd number of pennies in each cup?

ETC.: How many ways? Use more pennies?

M17. FATHER
HERMAN
LOUISA
ISABEL

It is easy to put each name in a separate "compartment" using 3 lines; just draw a line under each name. Can you do it with 2 lines?

ETC.: Can you construct a similar puzzle? How about one with 8 lines--7 lines is easy, but can you do it with 4 lines?

M18. What is the smallest piece of square paper which can completely cover the surface of a unit cube?

M19. Place 4 identical Coke bottles on a tabletop so the distance between any two mouths is the same.

M20. A penny size circular hole is cut in a piece of paper. Can you get a quarter through the hole without tearing the paper?

M21. Do each of the following in order: 1) Take your age, 2) Multiply by 5, 3) Add 25 4) Multiply by 2 5) Add the number of brothers and sisters you have 6) Subtract 50. What is the significance of your result? Why? Does this puzzle work for people of all ages and all numbers of brothers and sisters?

COMMENT: There are many of these "Take-a-number" puzzles which are usually based on algebraic cancellation or the place value number system.

M22. You enter a store and spend half your money. When you come out you find you have just as many cents as you had dollars going in, but just half as many dollars as you had cents going in. How much money did you have when you entered the store?

M23. A friend told me the other day that his father died recently, at an age which was 1/29 of the year of his birth. How old was his father in 1970?

M24. Can you draw a hopscotch "board" with one continuous stroke so your pencil never leaves the paper and you never go along the same line twice?

M25. A telephone call interrupts you as you are dealing a bridge hand. When you return, no one remembers where you stopped. How can you complete the deal correctly without counting the number of cards in any hand?

M26. Take any three digit integer (say 638), reverse the digits and find the non-negative difference (836-638 = 198), reverse the digits and add (198 + 891 = 1089). (If a two digit difference is obtained, insert a 0 before reversing the digits and adding.) What values are possible for the final sum?

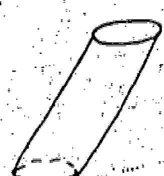
ETC.: Try it for 4,5,6,... digit integers.

M27. Given the four digits 2,3,5,8 arrange them to form the largest (8532) and smallest (2358) possible integers. Now compute the difference (8532-2358=6174). Repeat the process: 7641-1467=6174. We stop since we got the same number twice. If we make other choices for the 4 digits, with what 4 digit integers can this process stop?

M28. Begin with any positive integer, say 106, and determine the sum of the squares of the digits - 37. Repeat with each succeeding integer to obtain the sequence 106,37,58,89,45, 42,20,4,... Try some other examples. Do you notice any patterns in the sequences obtained?

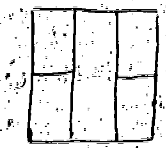
M29. What shapes can be used to make a square shadow on a wall with a candle or flashlight?

COMMENT: Try it.

M30  What flat shape can be rolled up to make this "glass"?

M31. How could you measure 1/2 glass (cylindrical) of water using no measuring instruments?

M32.  In the diagram, a heavy log is being moved by rolling it on the cylinders. If the circumference of each cylinder is 6 feet, how far will the log move for each revolution of the cylinders?

M33.  The top and bottom view of a solid is shown. Draw its side view.

M34. Suppose the sun is directly overhead. How should you hold a rectangular box over a horizontal table so its shadow has maximum area?

M35. Two people alternately place pennies (without overlap) on a tabletop. The person to place the last penny on the table is the winner. Is there a winning strategy for either player?

COMMENT: How does the shape of the table influence the strategies?

M36. There are n checkers on a table. Two players alternately remove an odd number of checkers (less than 10). The winner is the one who removes the last checker. For what values of n will the first player win?

M37. Two players alternately choose one of the integers 1,2, or 3 and keep track of the cumulative sum. The first person to reach a) 25 b) 28 is the winner. Is there a winning strategy for either player for either sum?

M38. Consider the sequence 1,2,3...,19,20. Two players take turns placing a + or - sign in front of one of the numbers. After all 20 signs have been placed, the second player wins the absolute value of the sum. Find the best strategies for both players. How much can the second player expect to win?

M39. An 8" x 6" rectangular piece of paper is folded so that two diagonally opposite vertices coincide. What is the length of the fold line? ETC.

M40. You purchased some one cent stamps, 3/4 as many 2¢ stamps, 3/4 as many 5¢ stamps as 2¢ stamps, and five 8¢ stamps. You pay for them with a single bill and receive no change. How many stamps of each kind did you buy?

M41. At what times during a 12 hour period is the minute hand of the clock directly over the hour hand?

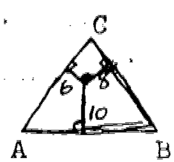
ETC.: When are the hour hand and minute hand opposite, perpendicular?

M42. How many ping pong balls will it take to fill up your classroom?

COMMENT: Good problem to introduce volume. Easy to estimate, but sphere packing is a complicated problem.

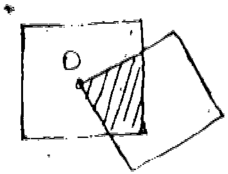
M43. You are one-fourth of the way across a railway bridge when you see a train one bridge length away. Which way should you run?

M44.



ΔABC is equilateral. Find its area.

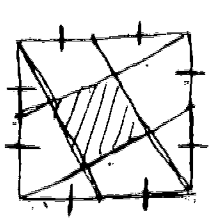
M45.



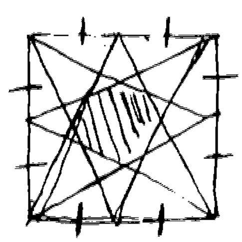
The squares are congruent. O is the center of the left square. Find all possible values for the shaded area.

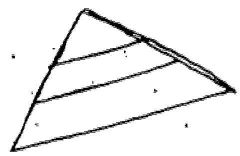
M46. Compute the shaded area:

a)

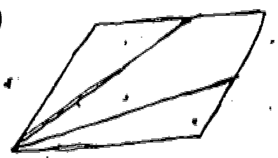


b)



M47. a) 

Find two line segments parallel to the base which will divide the triangle into three regions of equal area.

b) 

Find two line segments emanating from one vertex which divides the parallelogram into three regions of equal area.

ETC.: Other shapes; more lines.

M48. A a) triangular pyramid b) cube is cut by a plane. What different figures can be formed?

COMMENT: 1) Rephrase as cutting a cheese etc. if you wish. 2) Leads into the conic sections.

M49. How many ways can you make a die (assuming the numbers on the opposite faces add to 7)? How many ways if they need not add to 7?

M50. Given two positive integers m and n, what is the largest value of the smaller of the two numbers $\sqrt[m]{n}$, $\sqrt[n]{m}$?

COMMENT: Calculator useful.

M51. Calculate: $\sqrt{2 + \sqrt{2 + \sqrt{2 \ldots}}}$

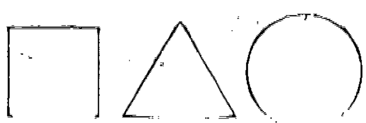
M52. Is it possible to "load" two dice (each one may be "loaded" differently) so that the eleven sums 2,3,4,...,12 are equally likely to occur when the dice are tossed?

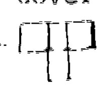
M53. A cliff is 300 m high. Consider two free falling raindrops such that the second leaves the top of the cliff after the first one has fallen .001 mm. What is the distance between them the instant the first one hits the ground?

COMMENT: 1) Need formula $S = \frac{1}{2} gt^2$. 2) Encourage student to guess first.

M54. a) Take any two integers, say 4 and 9, and form the sequence of eleven integers 4,9,13,22,35,51,...,350. Show that the sum of these terms is always eleven times the seventh term.
b) Suppose the sequence continues. Form the quotients 9/4, 13/9, 22/13,... . What is the limiting value of the quotients?

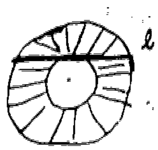
COMMENT: 1)(a) Makes a nice puzzle in that teacher can walk around the room and tell students their sum before they are finished writing down all the terms. (2)(b) Calculator useful. (3) These are only two of many problems concerning Fibonacci sequences and the golden ratio.

M55.  Describe, sketch, or construct a solid which fits snugly through each of these "holes". What is the volume of the solid?

M56. For which positive integers n can you cover an n x n checkerboard with pieces shaped like  ?

M57. Can you pack an 8 x 8 x 6 box with 1 x 2 x 4 bricks.

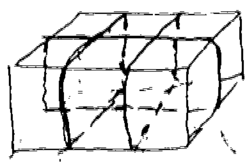
M58. A pilot starts flying due northeast from any point on the equator. What are possible destinations for his flight?

M59.  If the length of chord ℓ is 10 units, find the shaded area.

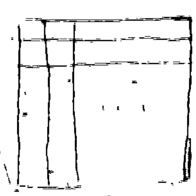
M60. a) What is the largest sum of money in coins (half-dollar or less) you could have and be unable to make change for a dollar, a half-dollar, a quarter, a dime, or a nickel?

b) In how many ways can you make change for a dollar?

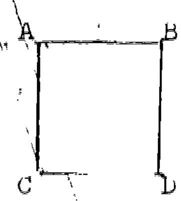
M61. Augustus de Morgan claimed he was x years old in the year x^2. He died in 1871. When was he born? Can anyone living in the twentieth century make that claim? ETC.

M62. 

What are the dimensions of the largest rectangular package which can be tied as shown with 12 feet string? Neglect any knots, etc.

M63. 

Cutting along the lines, what is the fewest number of squares into which this 13x13 square can be dissected?
(13x13 square with 169 unit squares)

M64. 

Fold a rectangular piece of paper so that corner D lies on side AC. What is the shortest possible length for the crease?

Prepared by Thomas Butts, 2/80

Teaching Problem Solving: A Tough but Rewarding Assignment

A dedicated and able teacher committed to improving his or her students' problem solving abilities is facing an uphill struggle. Although many educators agree that the ability to solve problems is the major justification for mathematics, relatively little has been done in the past three decades to make substantial improvements. Individual teachers have been concerned and some have accomplished a lot with their students. But national efforts have been less than substantial. However, it appears that mathematics education is on the threshold of a new frontier. We have seen a revolution in the 1960's place emphasis on the "why" of mathematics--understandings. Before 1950 and again recently in 1975-80 an emphasis was placed on skill learning ("Back to the Basics"). Now "basic" is acquiring a new definition and many of us expect a new emphasis on problem solving.

But students are in classrooms now. Many teachers do not want to wait five or ten years for instructional packages aimed at problem solving. What can a teacher do?

Commitment

As with so many other endeavors, a sense of commitment is the first priority in the teaching of problem solving. Such a commitment involves time and flexibility as crucial factors. Teachers are typically very busy, and finding time to take on additional responsibilities is not easy. A few school systems provide "development grants" for teachers. Sometimes groups of teachers can receive support or partial support during the summer for a special project, such as developing problem solving instructional units. More often, the extra time for a modification of the curriculum is simply something a teacher does! It's hard to say where the time comes from but the job gets done.

Another difficulty is finding time in the course curriculum to add something new--like problem solving. Over a period of more than three thousand years mathematics content has grown. More and more of this content has been placed on pre-college learning and seldom does anyone recommend deleting content! As one examines the scope and sequence of a course, like eighth grade mathematics, for example, it is difficult to say, "students don't need this topic" or "let's cut out ____." Some hard decisions may have to be made.

On the other hand, problem solving has always been a part of mathematical learning--so some time is available. We believe that problem solving is so important to students that it deserves more emphasis--and more time. The major point is that teachers committed to improving problem solving abilities will have to examine the total curriculum and carefully plan enough time to provide sufficient experience to make a measurable improvement.

In addition to finding the time, one has to be willing to be flexible when consciously teaching problem solving. Looking at different approaches to the same problem, encouraging students to guess (without fear), allowing students to pose problems, and being quiet are only four of several ways in which a teacher may have to deviate from established patterns of classroom behavior.

Making this commitment to teach problem solving raises several questions:

(1) What must I consider in planning problem solving experiences?

(2) How can I create a problem solving environment in my classroom?

(3) How can I effectively teach someone to solve problems?

(4) How can I evaluate problem solving ability?

(5) What can I do in tomorrow's class to begin to promote problem solving in my classroom?

Planning

The teacher who is committed to teaching problem solving with the same devotion as teaching skills and applications must be prepared. "The greatest expenditure of teaching time and energy should occur in planning for the classroom activities that are to occur."* Planning tasks include:

- Identify specific learning goals for the lesson or unit.
- Select problems or situations which are interesting to pupils.
- Think through solving problems to anticipate student strategies, alternatives, distractions, or confusions.
- Develop a lesson plan and schedule for activities and events.
- Devise ways to assist students if they encounter difficulty.

*Meiring, Steven P. Problem Solving: A Basic Mathematics Skill, Booklet 2. Columbus, Ohio: Ohio Department of Education, 1980. p. 61.

* Plan flexible time so you can adjust the plan, if necessary.
* Provide an opportunity to look back at the problem solving experiences.
* Develop a plan to evaluate student learning.
* Check plans to insure that you have provided motivation and reinforcement for students.
* Examine plan methodologies and instructional activites to see if another alternative would serve better.
* Compile a list of materials that will be needed, including instructional media such as an overhead projector, transparencies, chalkboard, and student handouts. Also, determine if manipulatives or related materials will be needed, such as string, rubber bands, straws, meter sticks, scissors, thumb tacks, glue, rubber cement, tape, scrap paper, poster board, pens, pencils, geometric shapes, graduated containers, calculators, drawing instruments, stop watch, spinners, or plastic grids.

Several of these tasks merit elaboration; the first two now, others later in this paper. It is important to identify clear goals for students--what do you want students to learn? Traditionally, it has been clear that teachers want students to learn to solve problems, but that is a vague specification. What must they learn to be better problem solvers? Many textbooks contained sections of specific kinds of problems, like distance/rate/time problems. The idea was that practice in solving these problems would transfer to other similar problems. The approach works to the extent that the student learns to solve that "type" of problem, but the concern of teachers is that we cannot predict all the types of problems student will face in the next 10, 20, or 30 years.

The heuristic approach offers hope of being more successful. Some problem types are so common that they will likely always be in the curriculum. But by teaching students to use heuristics, general methods of attacking problems, the student is armed with a repertoire of useful strategies. When faced with a new or novel problem, he or she has some approaches to try. The heuristic approach also helps to clarify the goals of instruction. (Please refer to the list of heuristics in the paper on "Basic Heuristics for Problem Solving.") Teachers can help students to "look for a pattern", to "make a table", or to identify and "solve a simpler problem." Research has shown that problem solving ability is enhanced by teaching heuristics and, when heuristics are taught, they are used. These heuristics then can be thought of as general skills to help students become better problem solvers. They are certainly more explicit than one general ability [illegible] solve problems better!

G04

Another major challenge in developing problem solving experiences is locating good problems. Little can be accomplished without the resource of problems. Some teachers make files of problems on 3" x 5" cards. Whenever they find an interesting and appropriate problem, they fill out a card and file it. A few teachers have set up such files by heuristic classification. If a heuristic is useful in solving a particular problem, they file it there. Usually several heuristics are applicable and they cross reference the problems.

Another useful technique is to develop some problems yourself. The paper on "Posing and Reposing Problems" presents helpful ideas to teachers for constructing problems.

Also, many problems are contained in the resource packet. Special attention is called to the following papers: "Problem Solving Resources", "Real World Problems", and "Problems, Problems, Problems!"

Creating a Problem Solving Environment

In addition to teaching heuristics, there are many ways of fostering a problem solving environment in the classroom. Some of the suggestions concern changes in the physical environment of the classroom while others involve altering the day-to-day instructional routine by including various problem solving activities.

Suggestions concerning the classroom include:

(1) Establish a problem file on 3" x 5" cards as indicated in the last section. Allow students to select a problem, write their names on the back of the card and try to solve the problem for an appropriate reward.

(2) Use the bulletin board by posting (a) a "Problem of the Week", (b) student solutions to problems or student "theorems" named after the student (e.g. Johnson's Solution or Theorom), (c) a list of heuristics for reference, (d) cartoons, newspaper and magazine articles, quotations, etc. related to problem solving.

(3) Construct a bookshelf for books of problems (see "Problem Solving Resources"), mathematical games and puzzles, magazines (e.g. Mathematics Student Journal (NCTM), Scientific American, Games etc.) and the like.

(4) Create a problem solving center in one corner of the classroom. It could be near the bookshelf (3) and contain various manipulative devices and other materials listed in the previous section. A catchy name like "Sleuth's Corner", "221 B Baker Street", or one chosen by the students is appropriate.

Some suggestions for ways to use problems to vary the instructional routine are:

(5) Begin a unit with a problem. Example: "How many pennies would it take to fill this classroom?" would be suitable to begin a unit on volume. (Paper E) are especially useful in this way.

(6) Close the class period with a problem. Example: After a lesson on removing common factors; assign: Factor (a) $3300 x^{90} + 2520 x^{81} + 2310 x^{72}$; (b) $13{,}672 x^{10} - 27679 x^{8} + 18{,}627 x^{6} - 30{,}000 x^{4} + 223$. (These "problems" are really exercises, but each has a moral).

(7) Solicit problems from students and/or ask them to construct problems based on a picture from a magazine, a short newspaper article, etc.

(8) Play a game.

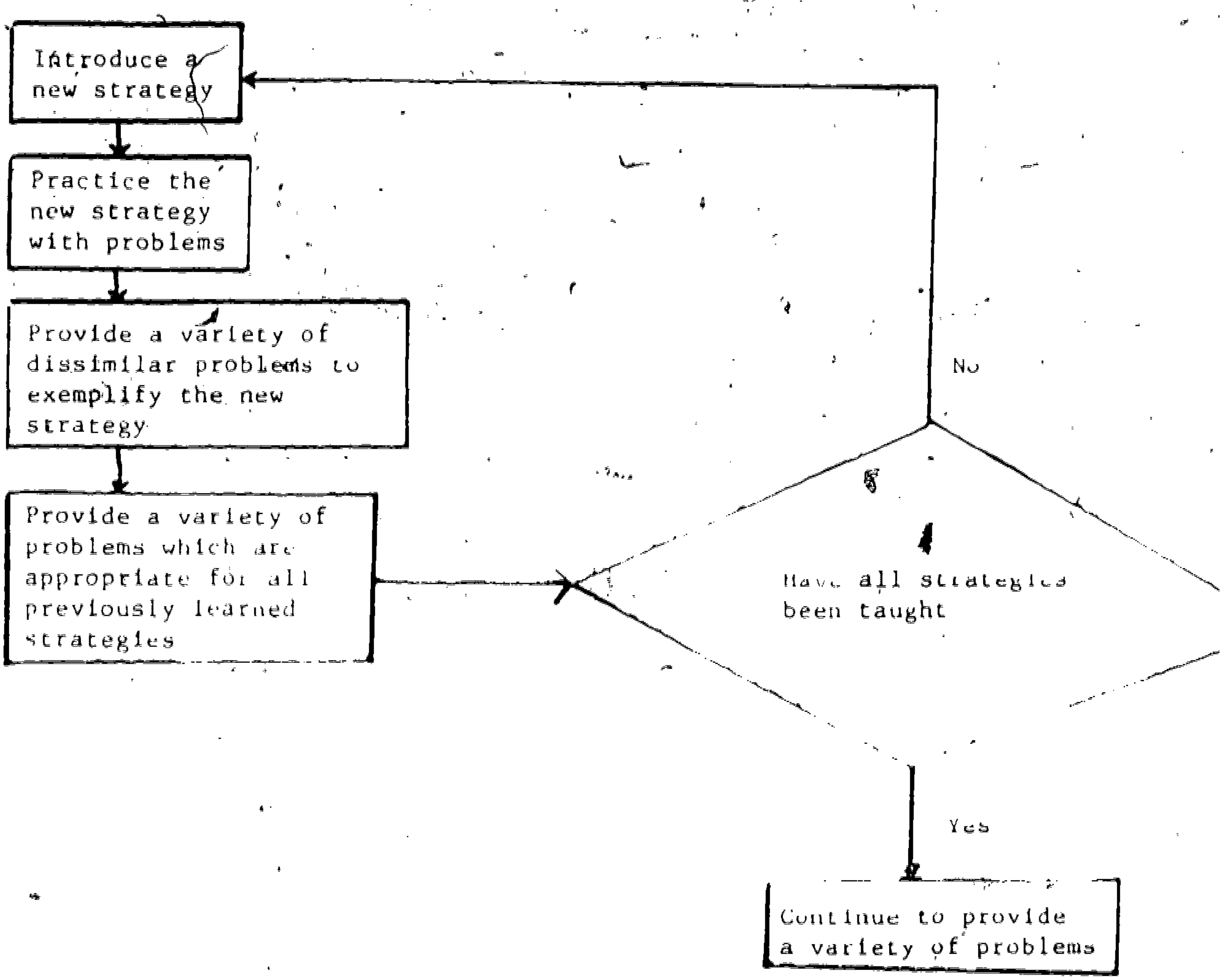
(9) Have a (mathematical) Scavenger Hunt. Sample list: "Find" (1) a parallelogram in the classroom, (2) a polygon whose area is numerically equal to its perimeter, (3) a number greater than 10,000 with exactly 8 factors, (4) four numbers whose sum exceeds their product, (5) a quadrilateral whose diagonals are perpendicular (6) a rectangle and a triangle having the same area, (7) two polygons which can partition the plane into 10 regions., etc.

You can probably think of other ways to bring problems and problem solving into <u>your</u> classroom.

Teaching Problem Solving

One cannot anticipate all the situations that could arise in any teaching situation - especially in problem solving! Our goal here is not to prescribe what should be done, but rather to offer some comments on some of the critical issues in the teaching of problem solving; especially teaching heuristics and asking questions.

Research attests to the value of teaching heuristics or strategies - the following flow chart offers one procedure.



After students have learned to use three or so heuristics, they should be provided with an opportunity to solve a variety of dissimilar problems, using heuristics of their choice. This step helps to demonstrate student selection of heuristics and the advantages of using them. As the students' backgrounds grow in terms of heuristics they can use effectively, their success in problem solving will show signs of significant improvement. You should decide on a reasonable order in which to introduce the heuristics. "Act it out" comes easier than "Look for a simpler problem". Special emphasis needs to be given to the looking back heuristics:

Generalize
Check the solution
Find another way to solve the problem
Find another result
Study the solution process

It appears that students do not use these heuristics on their own. The implication is that teachers should use them to provide a role model for students and to demonstrate how much can be learned from using the looking back heuristics. In applying these heuristics, teachers need to reserve time for them. Sometimes it is effective to follow a student selected strategy for a problem even if the teacher knows a "better" way. If an answer is obtained, the teacher is in an ideal position to ask questions to guide students to the alternate strategy. Such an experience may be valuable to student learning.

Even more fundamental in the teaching of problem solving is the art of <u>asking questions</u>. Not all of us can be as perceptive as Socrates, but we must try. A question, here, means posing a problem as a question; not the essentially rhetorical questions which are often overused in the classroom. For example, several familiar theorems in plane geometry can be studied by investigating the question, "In which quadrilaterals do the diagonals bisect each other?" Posed in this manner, the question suggests the use of several heuristics and, more importantly, <u>encourages guessing</u>. Polya says "Let us teach guessing". Because many students fear being wrong, you may wish to [illegible] into guessing by asking nonthreatening questions. I have been successful beginning each class with a "Question of the Day" (Examples: (1) What do the following have in common: ball, paper, blue?, (2) Which of the seven dwarfs has no beard?, (3) What U.S. city was named after a radio-TV show?) Students will normally respond eagerly to such trivia questions. A next step could be to play the game "In the Bag" where you place an object (real or imaginary) in a bag and challenge the students to guess its identity by asking questions which can be answered only "yes" or "no". Finally, mathematical questions can initially be posed to small groups of students so there is a collective guess. Gradually the individual student will develop the confidence to guess intelligently and without fear.

Once a question is posed, the next step for the teacher can be an unnatural one - <u>be quiet</u>. An important aspect of using questioning is a phenomenon called "wait time"--the time between the end of a teacher's question and either a student response or teacher talk of some sort, such as repeating, rephrasing, or redirecting the question. Incredibly, average wait times as measured in a classic study by Mary Budd Rowe, were less than <u>one second</u>.

> "(A teacher) must be prepared to meet often with the disconcerting silence of students." (George Polya)

A critical aspect of problem solving involves the opportunity to carefully weigh evidence, to compare given conditions and relationships to previously known information or problems, and to search for the untried and unthought. Instead of rapid-fire questioning, the problem solving teacher may even have to advise the class to slow down in their responses and to think them through more carefully.

A most significant aspect of the Rowe study involved the change in character of a class when the teacher consciously made an attempt to increase wait time substantially. The following effects were observed when wait times were increased to three seconds or more.

a. The length of student responses increased.
b. The number of unsolicited but appropriate responses increased.
c. Failures to respond decreased.
d. Confidence increased.
e. The incidence of speculative thinking [illegible]
f. Teacher-centered show-and-tell decreased and child-[illegible] increased.
g. Inferences supported with evidence increased.
h. The number of questions asked by children increased and the number of experiments they proposed increased.
i. Contributions by slow learners increased.
j. Disciplinary moves decreased.

The rapid questioning method represents a communication model in which information is constantly emanating and returning to the teacher from isolated individuals in the class. In a slower paced questioning atmosphere, the communication model becomes one of group participation. The teacher plays more of an equal role with more participation being assumed by individual class members and interaction occurring among students.

We close this section with several miscellaneous comments related to the teaching of problem solving.

(1) Small group (3-4 people) instruction is usually effective (see Meiring, op cit. pp 60-100, for a discussion of techniques advantages, and disadvantages of large group, small group, and individualized instruction.)

(2) A good deal of practice in using skills and algorithms occurs in the solution of most problems, so assigning one or two problems instead of ten or twenty exercises requires no more time.

(3) Encourage oral presentations of problem solutions by students. Ask such questions as "What prompted you to try this approach?" "Was this approach the first thing you tried?", "Does anyone else see another way to solve the problem?".

(4) Examine more than one way to solve a problem; ie employ the looking back heuristics. More often than not, a good problem situation is abandoned too quickly.

(5) To model being a problem solver, join the class in trying to solve a problem none of you have seen before. Seeing the teacher "struggle" should be beneficial for all. It may help change the authority-figure role of the teacher and bolster the confidence of the student.

(6) Emphasize the validity of trial and error procedures for solving a problem requiring a numerical answer. While such a solution may not have aesthetic appeal, it is usually based on sound reasoning and often can effectively make use of a calculator or computer. A group of secondary teachers were given the problem: "Find the maximum area of a quadrilateral whose side have lengths 7,8,9 and 10." While none of them rediscovered Brahmagupta's formula, all but one obtained a numerical estimate correct to within .5% using physical models and trial and error procedures.

Teaching Real World Problem Solving

On teaching real world problem solving [illegible], offer a few remarks to supplement those in "Real World Problems and Applications" (paper [illegible])

(1) Many real world problems appearing in textbooks are, in fact, [illegible] in the blank exercises requiring only arithmetic skills. Part of the task in solving a real world problem is to determine a reasonable mathematical model.

(2) Real world problems often contain extraneous or insufficient data. In fact, asking students to gather their own data is worthwhile.

(3) Simulation methods using dice, a random number table, a telephone book, or a computer are especially appropriate.

(4) Most real world problems are not purely mathematical, so their use offers a great opportunity for team teaching, or at least parallel lessons with another class.

(5) In solving a real world problem, one often seeks only an approximate answer, so estimation techniques and (measurement) error analysis are useful tools.

(6) Because of the emphasis on numerical answers, trial and error strategies are often appropriate.

(7) Real world problems frequently reoccur, so studying the solution process and finding alternate solutions are especially important.

Evaluation

Problem solving ability must be evaluated if students are to perceive its importance. If class time is spent on solving problems but tests are computational exercises taken from the teacher's guide, then the impact is lost. Evaluating student success in problem solving, however, is one of the most difficult tasks facing a teacher. If problem solving is viewed as "answer-getting" the task is simplified but maybe artificially so. Our overall goal is improving students' abilities to solve problems. If answer-getting is the only criteria, then it is easy to fall into a trap of providing students with problems that are quickly solved with narrow procedures, in order to have students solve a large number of problems (to increase the reliability of a test). Such a point of view can lead teachers to teaching students to solve "problem types". If the evidence you accept of success in problem solving is getting the answer, it is natural to show students how to do a type of problem and give them several problems of that type!

If the goal is to help students solve problems that are new to them, the task is more difficult but will mean more to the students. Solving novel problems requires more student exploration, thinking time, and "restarts" of the solving process. Fewer problems can be presented to students.

We suggest that teachers look at a two stage model of evaluating students. The first stage is to determine if students can use teacher selected heuristics, e.g., given an appropriate problem, can the student set up a table to organize data to solve it or can the student draw a picture which includes the key elements of the problem? Thus, we suggest evaluating student ability to use heuristics directly. The second stage is to evaluate student ability to select appropriate heuristics and use them to solve new problems. Simply checking answers is not enough. The process involves looking at how students tried to solve the problems. Sometimes it may be a matter of looking over the students' written work. Some teachers have tried problem solving interviews--asking a student to solve a problem in a conference setting by thinking aloud. The process is time consuming but may have a lot of payoff.

Since problem solving is more than answer-getting, a good evaluation scheme should give credit for

* attempted solutions using "reasonable" heuristics.
* the length of time spent before "giving up"
* attempting the problem at several different times (to capitalize on Poincare's notion of an "incubation period" for solving problems).
* willingness to employ looking back heuristics (e.g. solves a problem in more than one way, generalizes the problem).
* willingness to share a solution with classmates in an oral presentation (some teachers give extra credit points).

Two other notions which I have employed with some success are:

(1) Keep an anecdotal file (on a 4" x 6" card) on each student recording problems solved or attempted with notes on imaginative solutions, "good tries", etc. This suggestion may be impractical for a large number of students, but try it for one class to determine if it's worthwile for you.

(2) Give a four problem test with 50 points awarded for the first correct solution, 25 points for the second, 15 for the third, and 10 for the fourth. Basically you read the four solutions, decide their relative merit and award points accordingly.

Tomorrow

What can you do to start? First, we suggest that you do not plan to start tomorrow to devote 1/3 of your mathematics course to problem solving. We would like to see mathematics split into 1/3 for computational skill, 1/3 for understandings and 1/3 for problem solving, but it is unrealistic to make such a drastic change in a very short time. Examine various sources looking for suitable problems. Look at newspapers, a wide range of magazines (e.g. Mechanix Illustrated, Psychology Today, Scientific American) etc searching for problem situations. Try to develop a "question sense" - an ability to formulate problems based on all types of situations. Then, we suggest planning a single class period for problem solving and exploring what strategies (heuristics) your students use. Continue by choosing from 1 to 3 strategies and plan a small unit to develop them. Then try another period for open ended problem solving. In following this process a teacher carefully develops his or her instructional skills to help students become better problem solvers. An important danger to avoid is to try to do too much at once. Small steps, carefully planned, provide more promise of success than a grand commitment that may be impossible to achieve.

Coda

Our hope is that problem solving will eventually become a philosophy, rather than a method, of teaching. The ultimate reward is the "aha" expressed by a successful problem solver. The price for this moment is a wholehearted commitment, careful planning, imaginative teaching, and hard work. It's worth it!

Appendix

One of the most impressive projects currently near completion in Oregon is the Lane County Mathematics Project. Directed by Professor Oscar Schaff, the project has been developing materials for grades 4-8 for the past three years. These materials, which are supplements to a standard mathematics program, emphasize the improvement of problem solving abilities. One of the interesting features of the materials is the use of interesting problems, challenges and games to provide drill. Samples, used with permission of the director, are included at the end of this paper to demonstrate some of the problem presentations and the use of drill to teach students to use heuristics. This is one useful way to save time in the curriculum to provide for an emphasis in problem solving.

The following pages are taken from grade 8 of the Lane County Mathematical Problem Solving Project and used by permission of the Director, Oscar Schaaf. Two of the principle authors are Scott McFadden and Richard Brannan. Currently these materials, developed for grades 4 through 8, are in different stages of development, field testing, revision, and implementation. Inquires about the project and use of materials should be submitted to the director:

Oscar Schaaf, Director
Lane County Mathematics Project
Lane Education Service District
1200 Highway 99 North
Eugene, Oregon 97402

The few pages selected have been chosen to reflect some of the problems presented to students and the techniques used to have students practice computational skills while developing and sharpening problem solving strategies.

(Revised 3/80)

PROBLEM SOLVING PUPIL MATERIALS - GRADE 8

LESSON TITLES

I. Pre-Problem Solving Activities

- Guess and Check
- Looking for Patterns
- Make a Systematic List
- Make and/or Use a Diagram, Model, or Drawing

II. Challenges

- What's The Sum?
- Use All Nine
- Pool
- Logical Thinking
- A Game of Cards
- Pizza Puzzles
- Magic Triangles
- Dominoes
- A Game of Darts
- Happy Numbers
- The Last Digit
- Birthday Party
- Where Is The Monkey?
- A Row of Pennies
- Mr. Wizard's Marble Trick
- Chips Ahoy

III. Drill and Practice

- Complete the Problem
- More Than One Solution
- A Potpourri of Decimal Digit Draw Activities (Ideas for Teachers)
- Distributive Property
- Create a Problem
- Patterns
- Find a Path
- The Best Guess (Money)
- The Best Guess (Decimals)
- The Best Guess (Fractions)

IV. Variation (Algebra Readiness)

- Hamburgers for Slim
- Formulas
- Believe It Or Not Formulas
- Cars and Bikes

V. Integer Sense (Algebra Readiness)

- Thermometer Readings
- Negative Numbers
- What's The Sum?
- A Tricky Pattern

VI. Equation Solving (Algebra Readiness)

- Puzzle Problems
- The Hidden Solution
- Arthur's Dilemma
- Another Dilemma
- Unusual Equations

VII. Protractor Experiments

- Protractor Practice
- More Protractor Practice
- What's The Sum?
- Water-Wheel Geometry

VIII. Investigations in Geometry

- Patterns in Geometry
- Quadrilaterals
- Snipping Quadrilaterals
- Oil Wells
- Finding Distances
- Congruent Triangles
- Drawing Triangles
- Pythagoras

IX. Percent Estimation

- Meaning of Percent
- One-Percent Method
- Percents and Fractions
- Estimation
- Short Stories

X. Calculator

- A Number Times Itself
- Best Estimate
- Discovering Patterns
- Missing Parts
- Calculator Puzzlers
- I'm Thinking of a Number

XI. Probability

- Dice
- Roll That Cube
- Coin Toss
- The Gum Machine

ms

GUESS AND CHECK

Sometimes a problem can be solved by guessing. If at first you don't suceed, try again!

1. a. Who am I? If you multiply me by myself and then add 3, the result is 84.

   b. Who am I? If you subtract me from 30 and then double that answer, the result is 48.

   c. Who am I? I'm a 2-digit number. If you reverse my digits and add us together, the result is 110. (Give all possible answers.)

2. Julia and Ed were asked to add two numbers together. Julia, by mistake, subtracted the two numbers and gave the answer as 10. Instead of adding, Ed multiplied the two numbers and the result was 651. What was the correct total?

3. Use each of the digits 0, 1, 2, 3, 4, 5. Find the smallest answer possible.

$$\begin{array}{r} \square\,\square\,\square \\ -\ \square\,\square\,\square \\ \hline \end{array}$$

4. The Happy Holiday Hotel is blessed with cheerful bedbugs. In each single bed you can find 7 bedbugs and in each double bed there are 13 bedbugs. How many beds of each size are there in the hotel if there are 106 bedbugs?

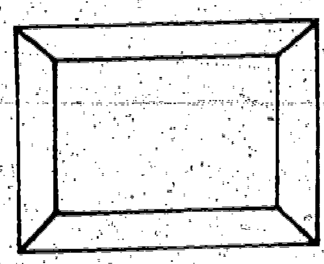
5. Patti has 74¢ in her purse. She does not have a half-dollar. She can't make change for a half-dollar either! What coins does Patti have?

MAKE A SYSTEMATIC LIST

All of the following problems can be solved by making a systematic list. Organize your work and look for patterns.

1. Mr. Young makes picture frames like the one illustrated. He makes them from pieces of wood with measurements of

   10 inches, 15 inches, 20 inches, and 25 inches.

   One possible frame is a 10 by 10; another is a 15 by 25. List all the other possible frames.

2. The corner store sells pencils for 5¢, 10¢, and 15¢. List List all the ways Kay can spend exactly 45¢ on pencils.

3. Slim was studying the menu. He had the following choices.

   1st course Soup or salad
   2nd course Chicken, beef, lamb, or ham
   3rd course Pie, cake, or sherbert

   List all possible combinations of food Slim could order.

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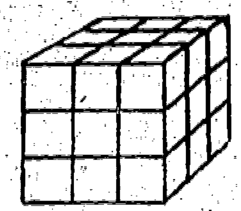
DIAGRAM, MODEL, OR DRAWING

Use a diagram, a model, or a drawing to help you solve these problems.

1. A frog is at the bottom of a 10-foot well. Every day he crawls up 3 feet. But at night he slips down 2 feet. How many days will it take for him to get out of the well? (The answer is <u>not</u> 10.)

2. It takes 27 sugar cubes to fill a cubical box. How many cubes will be touching the bottom of the box? the sides?



3. Six girls have just completed a 400-metre race. The girls' names are Chris, Debbi, Julie, Kay, Nancy, and Vicki. Use the following clues to help you determine in which order the girls finished.
   - Nancy beat Julie by 6 metres.
   - Chris finished 11 metres behind Vicki.
   - Nancy finished 2 metres ahead of Kay but 3 metres behind Vicki.
   - Debbi finished exactly halfway between the first and last person.

4. A service station has a large container of antifreeze. A customer needs exactly 2 quarts. The only containers available are a 5-quart can and a 4-quart can. How can the attendant use these containers to measure out the 2 quarts?

A GAME OF CARDS

Kim has a special deck of cards. After shuffling the cards she turned six of them face up on the table.

8	11	1	6	3	2

Kim explained the game this way:

- Use each of the first five numbers in any order.
- Use any operations you want.
- Your answer must be the last card turned up (in this case, 2).

Here are two different solutions. Study them to make sure they're correct.

$$\frac{11 + 1}{3} - (8 - 6) = 2$$

$$\left(8 - \frac{1 + 11}{6}\right) \div 3 = 2$$

See if you can find another solution.

Kim dealt out five more hands. See how many different solutions you can find for each of them. Remember to express the results in correct mathematical form.

1. 3, 7, 1, 8, 13 __________ Answer is 6
2. 10, 15, 2, 10, 7 __________ Answer is 11
3. 4, 16, 7, 8, 9 __________ Answer is 1
4. 12, 7, 9, 7, 1 __________ Answer is 13

*5. 4, 25, 10, 22, 4 __________ Answer is 4

A ROW OF PENNIES

Suppose that there are 200 pennies, all in a row. All of them are "heads" up.

Now, suppose that 200 people line up and –

- the first person turns over all the pennies.
- starting with the 2nd penny, the 2nd person turns over every other one.
- starting with the 3rd penny, the 3rd person turns over every 3rd penny.
- starting with the 4th penny, the 4th person turns over every 4th penny.

etc.

etc.

etc.

1. Suppose you're the 200th person. Will you find the 200th penny "heads" or "tails?"

2. After everyone has gone through the line, which coins will show "tails?"

3. After everyone has gone through the line, how many persons will have turned over the 200th penny?
   Who are they? (Give their numbers.)

COMPLETE THE PROBLEM

1.
```
   2 3 4 5
 + □ □ □ □
   6 7 8 9
```

2.
```
   1 5 7 9
 + □ □ □ □
 1 0 0 0 0
```

3.
```
   4 6 7 1
 - □ □ □ □
   2 8 8 8
```

4.
```
   □ 1 □ 2
 - 7 □ 9 □
   1 1 1 1
```

5.
```
   □ 5 3 □
 -   9 4 8
     □ □ 4
```

6.
```
     □ □ □
   x     7
   2 3 4 5
```

7.
```
     1 2 3
 x     □ □
     □ □ 5
   □ □ □
   5 5 3 □
```

8.
```
     8 □ □
 x       7
   □ □ 6 3
 □ □ 6 □
 □ 2 □ 9 □
```

9.
```
     2 6 □
 x     4 □
   1 3 4 □
 □ 0 □ □
 1 □ 0 6 0
```

10.
```
       2 6
 24 ) □ □ □
```

11.
```
      2 7
 □ ) 2 1 6
```

12.
```
           □ 9
 73 ) 2 1 1 □
      1 4 6
        □ □ □
        □ □ □
            2
```

13.
```
            5 □
 □ □ ) 2 9 7 1
       2 6 5
         □ □ □
         □ □ □
             3
```

14.
```
             1 □ □
 215 ) □ □ □ □ □
       □ □ □
       □ 5 □ 9
       □ 5 □ 5
           □ 4 □
           □ □ □
               0
```

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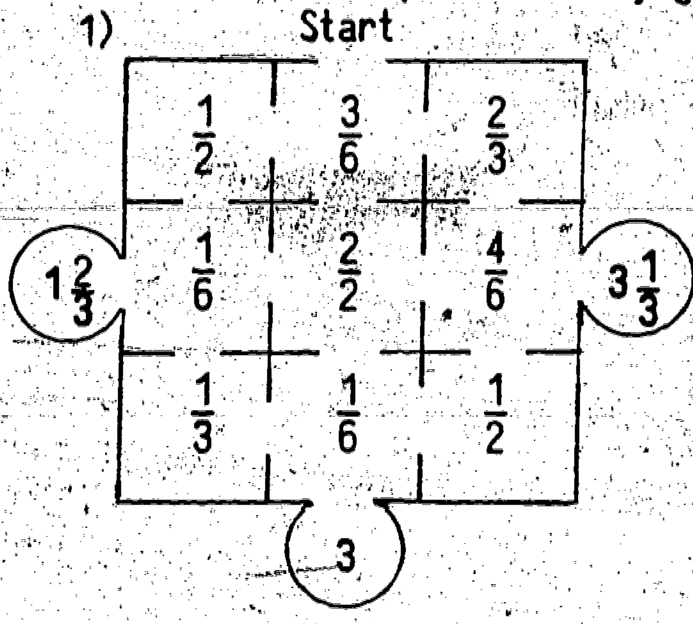
FIND A PATH

Draw a path so the numbers along the path add to the number in the circle. The path can only go through the open gates.

1)


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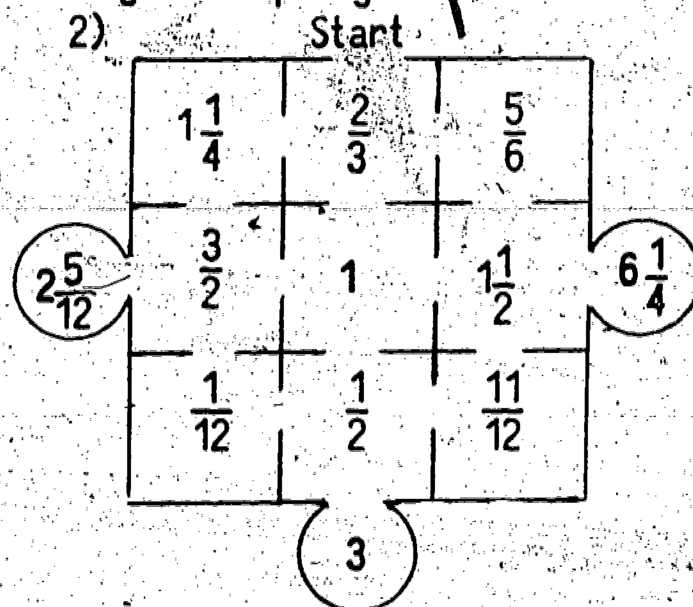

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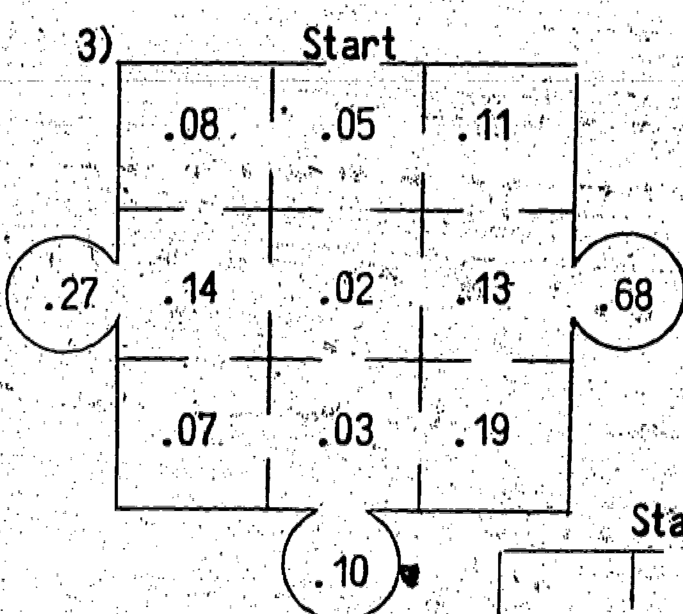
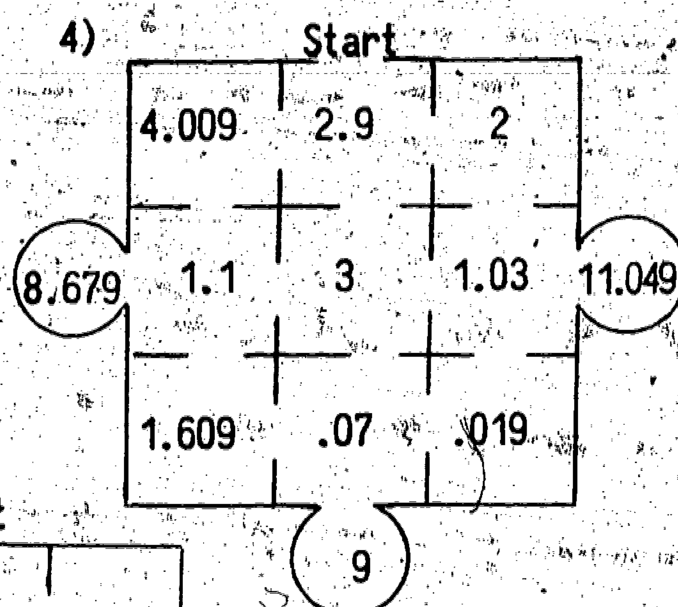
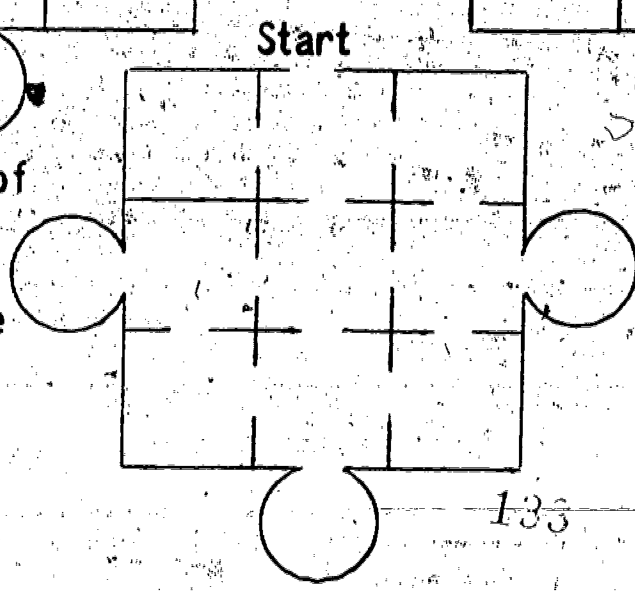


4)


5) Create one of your own. Give it to someone else solve.



IV-C:8
Variation
(Alg.Read.)

FORMULAS

Sammy was quite pleased with himself. He had just figured out a new formula. Sammy named it the

Ideal Weight For Everyone

$W = 5H - 190$ H is the person's height in inches.
W is the weight in pounds.

Is this a reasonable formula to use? Let's first see how it works for Sammy. Then we'll see how it works out for you.

Class Exercises

1. Sammy is 64 inches tall and weighs 128 pounds. According to the formula, how much should Sammy weigh?

2. What is your own height in inches? According to the formula, how much should you weigh?

3. Mr. Garcia is 6 feet tall and weighs 173 pounds. According to the formula, how much should he weigh?

4. Karen's little sister is 40" tall. According to the formula, how much should she weigh?

5. Do you think Sammy ought to change the name of the formula? Why?

BELIEVE IT OR NOT FORMULAS

Would you believe that a cricket can be used as a thermometer? The warmer it is, the faster a cricket chirps. This formula gives a close approximation:

$T = \frac{C}{7} + 3$ C is the number of chirps in one minute. T is the temperature in Celsius.

Formulas are statements of variation.
The weight-height formula suggests that
- a person's weight increases as he gets taller.

The sleep-age formula suggests that
- the number of hours of sleep a person needs decreases as she gets older.

According to the cricket chirps formula, how does the temperature vary as the cricket chirps faster? Let's find out.

1. Use the cricket formula to complete this table.

C	56	63	70	84	112	154
T						

   a. If the temperature is 14°, estimate the number of chirps a cricket would make. Then check your estimate by using the formula.

   b. Estimate the number of chirps if the temperature is 16°. Then check your estimate by using the formula.

2. A certain kind of ant also can be used as a thermometer. The warmer it is, the faster the ant runs.

$T = \frac{5R + 8}{2}$ R is the speed in centimetres per minute. T is the temperature in Celsius.

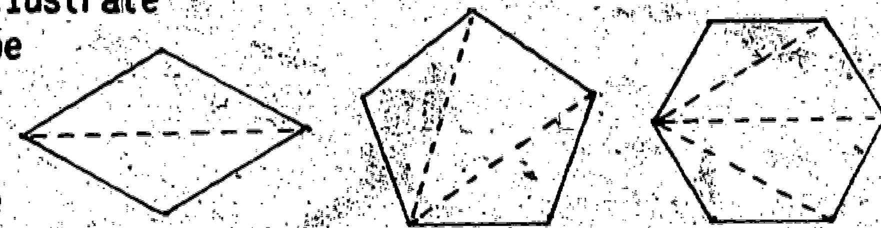
Use the formula to complete this table.

R	4	5	6	8	9	10
T						

   a. If the temperature is 21°, <u>about</u> how fast does the ant run?
   b. If the temperature is 35°, <u>about</u> how fast does the ant run?
   c. If a cricket chirps 112 times a minute, how fast does the ant run?

Patterns in Geometry (cont.)

2. These drawings illustrate how figures can be subdivided into triangles.

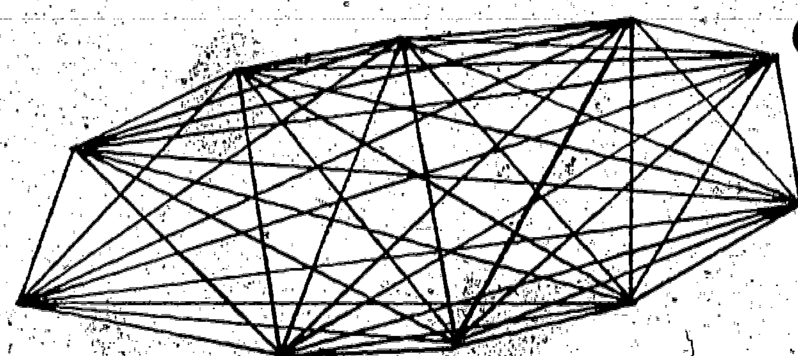


a. Use this same technique for other figures with 7, 8, and 9 sides. Complete the table.

Number of sides	4	5	6	7	8	9
Number of triangles	2	3	4			

b. Study the completed table. Look for patterns. If you used the same technique, how many triangles would you expect in a 100-sided figure?

3. Ten very talkative teenagers decided to install their own special telephone system. The system would connect each house with the other nine. How many wires would the system need?



Marty made a drawing. But that didn't help much. There were just too many lines to keep track of.

Kyle decided instead to try simpler cases and look for a pattern.

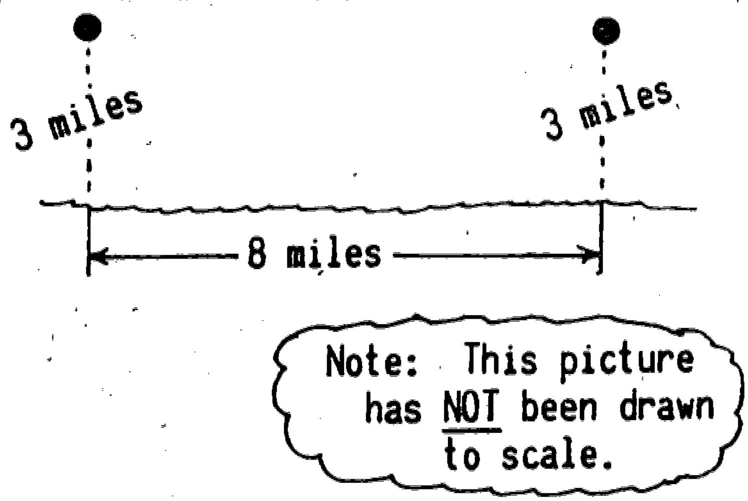
a. Make drawings to show the number of wires needed for

2 persons 3 persons 4 persons 5 persons.

b. Look for a pattern. How many wires would be needed for 10 persons?

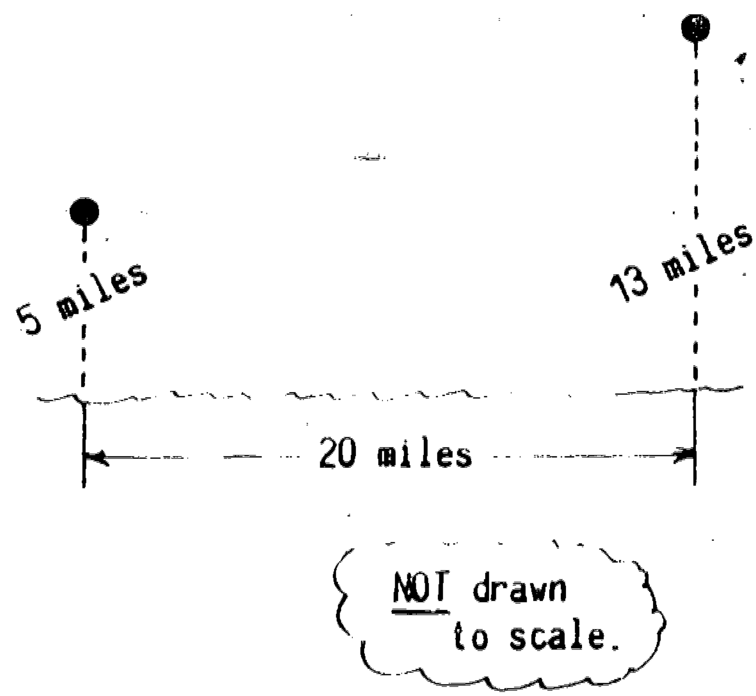
OIL WELLS

Two oil wells are located 3 miles from the shore of a river. A single storage tank is to be built on the shore. Oil is to be piped from each well directly to the storage tank. Of course, the company wants to use as little pipe as possible.



1. What is the best location for the storage tank?
2. How much pipe is needed?

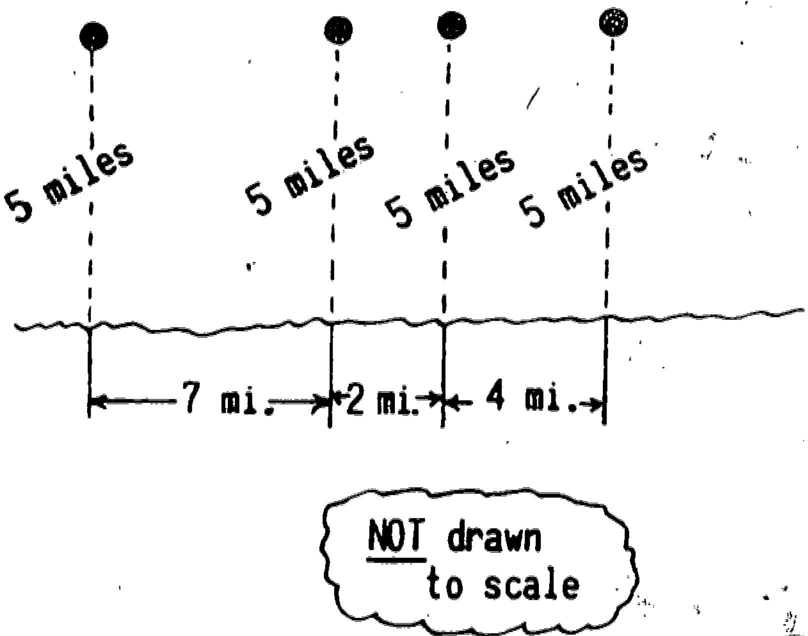
On another stretch of the river, two more oil wells have been drilled (see drawing). Another storage tank is to be built to service these two wells.



3. What is the best location for the storage tank?
4. How much pipe is needed?

Oil Wells (cont.)

On another stretch of the river there are 4 oil wells located according to the drawing. A single storage tank is to be built to service all 4 wells.



5. What is the best location for the storage unit?

6. How much pipe is needed?

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Using a Textbook for Teaching Problem Solving

The textbook is the resource around which most classroom activities are planned. Consequently, any decision to place more emphasis on problem solving instruction should consider how the textbook can be used more effectively to support problem solving objectives. The following suggestions are not intended to be exhaustive but rather to serve as a springboard to additional ideas.

1. Reexamine the role problem solving assumes in the textbook. Most textbook authors incorporate problem solving into their materials. The teacher's guide for your textbook likely discusses the role the author assigns to problem solving, indicating how the book is designed to support this role. Problem solving may be addressed in several ways: periodic, enrichment or challenge problems, starred sections in exercises, applications emphasis in story problems, special activity sections, special units on problem solving or problem solving components (e.g. solving equations), historical references to problem solvers, and famous or unsolved problems.

Authors also treat content topics in a manner which supports their problem solving philosophy and goals. A systematic 4-, 5-, or 6-step approach may be used in working through illustrative problems (we do not advocate rigid adherence to such a model); some applications may be arranged by topics (e.g. coin or mixture problems); highlighting and other graphic features of the book layout may suggest approaches to solving problems the author wishes to convey.

The important point is to become aware of the problem solving strengths the text possesses and to determine the impression about problem solving the materials are likely to make upon students. You will be in a better position to determine how to supplement the text with additional activities, whether to reorganize topics, what to emphasize or deemphasize, and whether you need to provide an expanded or alternate approach to problem solving than that taken by the book.

2. Identify precisely your daily objectives for using problem materials. There are different aspects of problem solving that can be emphasized at a particular time. Sometimes, the objective may be to practice a particular heuristic such as "write an equation." At another time, the emphasis may be on a specific technique such as solution by factoring. At still another time, prerequisite subskill practice, such as finding the LCM for two numbers, may be the central focus. Unless the purpose for study of a particular topic is clearly identified, students are not likely to appreciate how it is related to the more comprehensive process of problem solving. A further advantage to you is that such identification can provide you feedback about your classroom teaching priorities. Reviewing your list of daily problem solving objectives can help you ascertain the relative balance or imbalance among problem solving goals within your program.

3. Regroup problem materials to better achieve your purposes. Problems in most texts are grouped by type (e.g. time-rate-distance) or to practice a technique (e.g. solution by quadratic equation). This results in only one type of problem solving practice--applying the same or similar techniques to very similar problems. But according to Brownell[1] (1942, p. 439), "Practice in solving problems should not consist in repeated experiences in solving the same problems with the same techniques, but rather in the solution of different problems by the same techniques and in the application of different techniques to the same problem." To achieve this end, it may be

[1] William A. Brownell. "Problem Solving," The Psychology of Learning, Chicago: Part II of the Forty-first Yearbook of the National Society for the Study of Education.

necessary to regroup practice problems according to the specific problem solving goal to be addressed.

Other reasons to select problems from other sections of the book concern motivation and timing. Presenting a problem for which prerequisite skills have not yet been learned can establish the need to acquire more appropriate techniques and procedures to handle the problem difficulties posed by such situations, thus motivating the need for further content study. After introduction of particular, powerful means for dealing with certain situations, many potential "problems" become mere exercises. Confronting such problems before being aware of the existence of superior techniques is an aspect of real problem solving wtih which students should have some experience. The best techniques are not always known, accessible, or practical, but still the solver <u>must</u> somehow prevail over the problem.

4. <u>Sacrifice quantity for quality in problem solving learning</u>. Generally in problem solving, more time should be spent on indepth consideration of fewer selected problems. Good problems merit careful examination from many perspectives. Different approaches leading to solution, implications to other problems, generalization or extension of result(s), effect on the solution by variation of given conditions are but a few of the worthwhile activities that can accompany more deliberate and intensive study of fewer problems. More often than not, good problems are dropped too quickly than dealt with overlong

Students can hardly be blamed for developing a superficial attitude about problem solving and their responsibilities for persevering with problems when their experience typically consists of cranking through twenty very similar problems. By necessity, they come to expect to spend no more than a few minutes effort with each problem in order to complete their homework in a reasonable time. But assignment of fewer problems (accompanied by specific problem solving objectives) can change the focus for assignments from answer-getting to thinking and reasoning activities.

5. Use discrimination in assigning problems. All students need not be expected to labor with the same problems. Most teachers practice this principle with respect to more difficult problems. However, such a practice can be extended to a broader frame of reference. Most problem sets are organized progressively from easy to difficult problems. Some students do not need to spend much time on beginning problems, but very much may benefit from the challenge of more difficult activities. Unfortunately, after these students spend most of their efforts on problems leading up to those challenges, they have little time or motivation left to really think through those problems which are more appropriate for them.

There are also some text problems, frequently enrichment, which emphasize or draw upon particular heuristics or emphasize certain problem solving skills. Usually, these problems are appropriate for everyone. And since they often permit responses on a variety of levels, the thinking associated with even partial solutions makes them worth the effort and serves to establish the need for acquiring additional problem solving skills.

Example: Find the number of squares on a checkerboard.

(Hint: there are more than 1 x 1 squares present.)

Students frequently "live up to the challenge" of difficult problems (if there are not too many at one time) exceeding our expectations and using techniques and thinking that we may not even have considered.

How then do you discriminate without labeling students as "better" or "weaker"? One technique used with some success is to make a common assignment, but later while moving about the room to adjust the assignment for individual students by referring them to the particular parts of the assignment they should concentrate upon. Another approach is to have a frank discussion with the class about the differences in individual learning needs, pointing out that some students require more practice with easier problems before feeling confident to progress to more difficult situations. Making a comprehensive assignment ranging from easy to difficult problems but starring those to be graded will permit students to adjust their own learning styles to fit the situation they are expected to master. (This, of course, requires careful initial monitoring to avoid the entire class tackling <u>only</u> the starred problems.)

6. <u>Discuss with students the purposes for problem materials.</u> Many students have negative attitudes about written problems and about their own abilities to engage in any problem solving activity that requires indepth thinking. They perceive their roles primarily limited to exercise practice, mostly manipulating numbers and symbols according to a procedure recently demonstrated in class. Such attitudes demonstrate the need to talk with students about the role that various components of mathematics instruction assume in their mathematical development. They need to understand how skill development and practice, applications, and problem solving interrelate. Specifically, they need to talk over why they hold negative views about story problems. They need to find out why they lack confidence in applying their skills to challenging situations and to learn what it takes to develop such expertise. Having a frank and earnest discussion enables students to acquire a more accurate understanding of the nature of mathematical learning and application of personal skills.

7. <u>Use text materials as models for classroom development of questions.</u> Having students critique story problems as to motivation, content, and structure can be beneficial to the teacher in providing information for designing more appropriate learning activities. This exercise can also be coupled with the task of rewriting the problems after these critiques. Students can grow in their understanding about how questions are phrased and the relationships of given, needed, and requested information in a problem.

And they will appreciate better the role of insufficient or extraneous data in a problem. By observing how student questions differ in style and format from text questions, the teacher can obtain useful information about students' thinking and how they perceive such problems -- what it takes to make problems more interesting, challenging, understandable, and realistic.

8. Extend textbook problem backgrounds through self-developed materials. Since most textbooks do not have strong explicit problem solving strands, you will wish to supplement text problems with self-developed materials. The textbook can still be a primary tool in your efforts. Many times an exercise-oriented problem can be transformed into a problem solving activity with a slight modification or extension.

Grade 7: Following is the won-lost record for the National League on July 14, 1965. Arrange the teams in order of their performance (best to worst) by computing percentages.

	W	L		W	L
Chicago	41	46	New York	29	56
Cincinnati	49	36	Philadelphia	45	39
Houston	39	45	St. Louis	41	45
Los Angeles	51	38	Pittsburgh	44	43
Milwaukee	42	40	San Francisco	45	38

A slight modification to enhance the problem as a problem solving activity:

> Following is the won-lost record for the National League on July 14, 1965. Arrange the teams in order of their performance (best to worst) <u>without</u> computing percentages.

Text materials can also serve as backgrounds or data sources for self developed materials. Most tabular and graphical lessons in books are reading, question and response exercises -- calling for little insight or analysis by students. However, with supplementary questions, the same graphical or tabular material can be transformed into bonafide problems requiring students to do some critical reasoning.

However, the most useful purpose the textbook may serve is stimulus for creative development of problems similar to textbook exercises but requiring better problem solving skills. An algebra unit on linear equations might prompt this question.

Which of the following equations could represent the graph at the right?

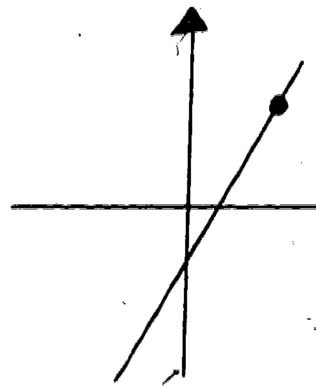
A. $x + 2y = 10$

B. $y = x^2$

C. $y = 2^x$

D. $3x - y = 2$

E. any of the above



Students typically can produce the graph for a linear equation, but many have not yet realized the unique 1-1 correspondence underlying the reverse question (E. is the favorite student response).

Using a Text Problem Fully

The issue of making better teaching use of fewer problems is so important that an example illustrating various uses and extensions is worth while. The following discussion is centered around a geometry lesson involving the content topic: the sum of the interior angles of a convex polygon is given by $S = (n-2) \cdot 180°$

Illustrating a problem solving methodology.

Each student in the class is asked to use a straight edge to draw a hexagon (not necessarily regular) and to find the sum of its angles with a protractor. Each student is then asked to draw a nine sided polygon and to find the sum of its angles in the same manner.

The teacher uses a table to tabulate individual results for each polygon, helping students to discover that, within measurement error, the

 H.08

sum of the interior angles of convex 6-sided and 9-sided figures appears to be constant and independent of other shape considerations (a side discussion might ensue later about discrepancies involving polygons with one or more concave vertices).

Students are then asked to consider, in small groups, the implications of these experimental results and are encouraged to formulate hypotheses that might be tested in some manner and to decide the nature of such tests.

Illustrating heuristics:

Students are told that the objective is to study the sums of the interior angles of convex polygons. From students, the teacher elicits the information that the simplest polygon is a triangle and the sum of its angles is known to be 180°. Students volunteer names for the next simplest polygons through 8-sided figures. They are asked what they know of the sums for these polygons (the quadrilateral will be the only one known). It might be wise to stress at this point the invariance of the sum of the interior angles for triangles and quadrilaterals regardless of size or shape. Then with careful questioning, the teacher should help the class discover that they can subdivide any polygon into triangles to obtain the angle sum.

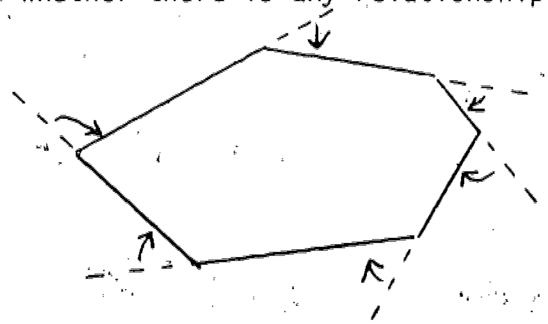
Using a table, students are encouraged to look for a pattern among their results and to develop a formula for the angle sum in terms of the number of sides. Succeeding columns in the following table could be added a step at a time as students appreciate the need to rewrite the data in a different form to recognize the pattern required to generalize for the case with n sides.

No. of sides	Sum of interior $\angle$'s	No. of Δ's	Alternate form
3	180°	1	$1 \cdot 180° = (3-2) \cdot 180°$
4	360°	2	$2 \cdot 180° = (4-2) \cdot 180°$
5	540°	3	$3 \cdot 180° = (5-2) \cdot 180°$
.	.	.	.
.	.	.	.
n		n-2	$(n-2) \cdot 180°$

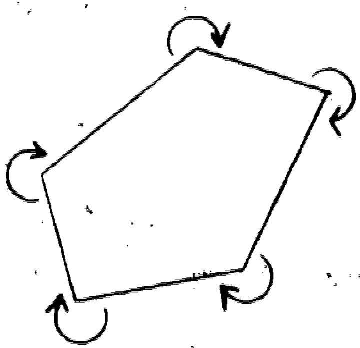
 H.09

Extending the problem:

Following this period of instruction, a problem for homework might review the notion of exterior angles and ask whether there is any relationship between the sum of the exterior angles and the number of sides. Students could be given the hint that the exterior angle sum might be related to the interior angle sum.



$S_e = 360°$



After discussing the above result in class, a subsequent extension might ask students to determine a relationship for the sum of the "total exterior angles" at each vertex in terms of the number of sides.

$S_{te} = 360° + n \cdot 180° = (n+2) \cdot 180°$

The richness of resulting discussions and explorations associated with staying with one problem long enough to explore thoroughly its implications far outweigh the disadvantages that students are exposed to fewer exercises. The insight and penetrating appraisals associated with such extensions provide a much more accurate glimpse of true problem solving than the more traditional exercise list that practices the equation $S = (n-2) \cdot 180°$ for various numbers of sides or angle sums and then moves to the next lesson with scarcely a backward look.

However, these kinds of approaches take time and, like most techniques associated with problem solving, should not be used exclusively. Students need a wide variety of different kinds of challenges, techniques, approaches,

H.10

and problems to round out their problem solving development. In this paper, we have tried to suggest how one instructional resource, the textbook, can be used more effectively to accomplish that end.

Research on Problem Solving at the Secondary School Level

For many years, research on problem solving in mathematics was focused on the elementary school level (and on word problems in particular). During the past 15 years, however, attention has increasingly also been directed toward the secondary school level. Moreover, concern over the process of problem solving (especially the effects of heuristics and strategies) has been reflected in the research.

This report highlights some of the findings which seem relevant to teachers. It is divided into three parts: (1) findings about students as problem solvers; (2) findings on the structure of problems; and (3) findings on problem-solving heuristics or strategies.

Findings About Students as Problem Solvers

- As you might expect, good problem solvers tend to have:
  - relatively high IQ scores
  - good reasoning ability
  - high reading comprehension scores
  - high quantitative ability or computation scores (contributing to success on numerical problems) and/or high spatial aptitude scores (contributing to success on geometric problems)

  (Dodson, 1971; Heseman, 1976; Hollander, 1974; Kilpatrick, 1968; Moses, 1978; Robinson, 1973; Talton, 1973)

- Other characteristics of good problem solvers include:
  - ability to understand mathematical concepts and terms
  - ability to note likenesses, difference, and analogies
  - ability to identify critical elements, and to select correct procedures and data
  - ability to note irrelevant detail
  - ability to estimate and analyze
  - ability to visualize and interpret quantitative or spatial facts and relationships
  - ability to generalize on the basis of few examples
  - ability to switch methods readily
  - higher scores for self-esteem and confidence, with good relationships with other students
  - lower scores for test anxiety

  (Dodson, 1971; Hollander, 1974; Krutetskii, 1976; Robinson, 1973; Suydam and Weaver, 1975, 1977; Talton, 1973)

- Capable students also skipped steps, had a feeling for elegant solutions, and reversed steps easily. Good problem solvers tended to forget the details of a problem and recall its structural features, whereas poor problem solvers tended to recall the specific details. (Krutetskii, 1976; Silver, 1979)

- Positive attitudes toward mathematics are correlated with success in some studies (e.g., Dodson, 1971) and not correlated in others (e.g., Kilpatrick, 1968).

- Lack of concern about messiness or neatness has also been noted to be related to success on problem solving! (Dodson, 1971)

- The strategies or methods by which students approached problems were not consistent for individuals from problem to problem, nor were the strategies used similar from problem to problem. Good problem solvers used a formal strategy more often than poor problem solvers, who tended to rely more often on a random trial-and-error strategy. (Robinson, 1973)

- Good problem solvers took more time to solve novel problems than poor problem solvers did. (Kalmykova, 1975; Robinson, 1973)

- There appears to be a difference in the way in which students tackle simple and difficult problems:
  - On complex tasks, random steps seem to appear in sequence.
  - On simple tasks, a gestalt seems to operate as an organizing principle for solving problems. (Peluffo, 1969)

Findings on the Structure of Problems

- Almost any problems can be of interest to some students. Studies of the types of problems students prefer indicate no relationship of these preferences to problem-solving success. (Cohen, 1977; Travers, 1967)

- Many persons have studied the difficulty level of problems by using such techniques as factor analysis, in which the computer manipulates data on students' answers until the data cluster into a number of groupings, or by ascertaining the relationship of the data from any one factor to overall success. In general, such factors can be classified as mathematical, reading, and reasoning factors; another categorization is by order, digital, and process variables. (Cromer, 1971; Jerman, 1972; Loftus, 1970; Webb, 1979)

- Following a series of explorations of the difficulty level of a large number of variables, it was reported that no one of them accounted for a significant portion of the variance, but the syntactic complexity of word problems is a definite contributor (e.g., Beardslee and Jerman, 1973). The studies suggest that mathematical achievement factors account for the most variance. (Sagiv, 1978; Webb, 1979)

- As most of us recognize, problems containing extraneous data or information are more difficult and take more time to solve than problems with no extraneous data. (Biegen, 1972; Blankenship and Lovitt, 1976; Fafard, 1977)

- It is also not surprising that the largest proportion of correct responses occurs when key words act as cues to the procedure or operation that can be used for solution. The proportion of correct answers was much smaller when key words appeared as distractors, indicating an incorrect operation or procedure. (Nesher, 1976; Wright, 1968)

- When pre-algebra students were asked to sort a collection of word problems into groups which they thought were mathematically related, four dimensions were identified:

  - mathematical structure
  - contextual details
  - question form
  - pseudostructure (problem similarity based on the presence of common measurable quantity, such as age or weight)

  The perceived salience of mathematical structure was significantly related to problem-solving competence. (Silver, 1979)

- The use of manipulative materials, pictures, diagrams, and similar aids enhances the probability of a problem being understood and solved correctly. (Caldwell, 1978; Nelson, 1975; Sherrill, 1973; Suydam and Higgins, 1977)

- By the upper elementary level, studies do not indicate that reading is as big a deterrent as is commonly believed. There are, of course, some studies which report a positive relationship between reading and success on problem solving, but it may not be of sufficient magnitude to be an accurate predictor of problem-solving success (e.g., Harvin and Gilchrist, 1970). In one study, poor reading was not a factor in the case of half of the standardized test problems solved incorrectly; on further analysis, it appeared that reading difficulties may have accounted for no more than 10 percent of the errors. Students who could read the problems simply could not solve them. (Knifong and Holtan, 1976, 1977)

- Little success has been found in developing instructional sequences to improve mathematical reading ability (e.g., Henney, 1971).

Findings on Problem-solving Heuristics or Strategies

- Research provides strong evidence that problem-solving performance is enhanced by teaching students to use a variety of strategies or heuristics, both general and specific. That is, students using a wide range of strategies were able to solve more problems. (Blake, 1977; Graham, 1978; Pennington, 1970; Webb, 1979; Wilson, 1968)

- When heuristics are specifically taught, they are then used more, and students achieve correct solutions more frequently. (Fowler, 1978; Hall, 1976; Kantowski, 1977; Lee, 1978; Vos, 1976)

- Training on a variety of heuristics is necessary so that students have a repertoire from which they can draw as they meet the wide variety of problems that exist; different mathematical content evokes different strategies. Certain ones (e.g., analysis and synthesis) are used more frequently than others. (Brandau and Dossey, 1979)

- Various strategies are used at different stages in solving problems. Thus, training on both integration (the capacity to integrate remaining components in a sequence of operations required for problem solution) and evaluation seem to be required for solving problems; when either was absent, the solution rate did not exceed chance. (Gallo, 1975)

- Flexibility in problem solving is a type of learned behavior. Students exposed to a variety of problems are able to make a smoother transition to new problems than those who are given practice only on many similar problems. Questioning may also contribute to the development of flexibility in problem-solving behaviors. (Cunningham, 1966)

- The use of regular patterns of analysis and synthesis was noted in the solutions of higher-scoring students. In many cases, these regular patterns were immediately preceded by a goal-oriented heuristic. Students who had no direction tended to establish as many facts as possible whether they were necessary for the solution or not. (Kantowski, 1977)

- Deduction and trial-and-error patterns were frequently found in a study with general mathematics students. (Dalton, 1975)

- A sequence in which problem types were moderately varied was better than either a highly varied or a non-varied sequence (Sumagaysay, 1972). It has been suggested that problems involving related theorems should be alternated so that students are made aware of the essential elements of a theorem and do not develop a "set" to use a particular theorem (Smith, 1973).

- The number of times a student attempted to solve a problem was unrelated to obtaining a correct solution. Changing the mode of attack in solving a problem was, however, significantly related to obtaining a correct solution. (Blake, 1977)

- Students at different developmental levels tend to differ in the extent to which they use particular strategies; for instance, students at a formal operational level (as defined by Piaget) used more means-ends heuristics than did concrete operational students (Grady, 1976). Formal operational students also used a larger variety of heuristics (Days, 1978). They used deduction, evaluation, and systematic trial-and-error strategies on significantly more problems.

- Students who scored high on divergent-type problems made fewer generalizations and used trial-and-error strategies more often. (Maxwell, 1975)

- Problem-solving skills are improved by incorporating them throughout the curriculum: that is, organizing the curriculum as a sequence of problems in which students induce organizational rules from examples. (Roman, 1975)

- Flener (1978) suggested:

  - Whenever possible, embed a teaching/learning experience in a problem-solving format.

  - Think in terms of hints or suggestions rather than absolute procedures to be followed.

- Experiment by giving students less help than usual.
- Do not be misled by the immediate benefits of structural [expository] teaching -- the long-range benefits of teaching through problem solving may be higher.

- There is no one optimal strategy or heuristic for problem solving. But in his review which focused on the relationship of instructional method, internal cognitive activity, and performance measures, Mayer (1974) concluded that little progress will be made until the emphasis on "which method is best" gives way to an attempt to define, and relate to one another, (1) external features of instruction, (2) internal features of subject characteristics, (3) activity during learning, and (4) outcome performance measures.

- It appears that techniques which attempt to "program" the solver to follow a fixed sequence of steps are not very effective. Experience and research suggest, however, that certain heuristic procedures which will improve mathematical problem-solving performance can be learned -- provided the teacher illustrates how the procedures work; gives ample opportunity for discussion, practice, and reflection; and supports and encourages the learner's efforts. (Kilpatrick, 1978, p. 191)

- Almost 40 years ago, Brownell (1942) collated from the research a list of suggestions for teachers. Three of them seem particularly pertinent:
  - To be most fruitful, practice in problem solving should not consist in repeated experiences in solving the same problems with the same techniques, but should consist of the solution of different problems by the same techniques and the application of different techniques to the same problem.
  - A problem is not necessarily "solved" because the correct response has been made. A problem is not truly solved unless the learner understands what he or she has done and knows why his or her actions were appropriate.
  - Instead of being "protected" from error, the student should many times be exposed to error and be encouraged to detect and to demonstrate what is wrong, and why.

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SECTION J.01-J.05 "PROBLEM SOLVING IN SCHOOL MATHEMATICS: NCTM YEARBOOK - A BRIEF REVIEW" DELETED DUE TO MARGINAL LEGIBILITY.

K-01

CALCULATOR INFORMATION CENTER

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Special Bulletin
December 1979

Research on Calculator Uses in Secondary School Mathematics

Research on the use of the calculator as a tool to aid in mathematics instruction at the secondary school level began with a few studies using rotary or electric calculators, and suddenly accelerated as hand-held calculators came into classrooms. On the following pages, 46 studies are listed. Five were conducted with desk calculators [Aldridge, Cech, Gaslin, Keough and Burke, Ladd]; they are included since their findings seem relevant. The remainder pertain specifically to hand-held calculators.

Of the 46 studies, 35 were experimental, 10 were surveys, and 1 (Jewell) was a curriculum analysis. The variance in quality is great, as with most sets of studies conducted under varying conditions. The majority focused on the effects of using calculators for computational needs as they arose; for only 12 were calculator-specific materials developed.

Eight of the experimental studies were conducted with general mathematics classes, while only three involved algebra 1 classes. The use of calculators with low achievers or for remedial work appears to have evoked more research attention than their use with the college-bound. A count indicates that there were the following number of studies at each level:

grade 7 - 12

grade 8 - 8

grade 9 - 20

grade 10 - 6

grade 11 - 6

grade 12 - 5

Almost no attention was given to problem solving except as a component of a standardized achievement test. Only two studies were focused on analysis of the learning process; the rest were concerned with the variable of achievement.

Researcher	Course/topic	Level	Materials
Aldridge	remedial	7-8	no
Anderson	varied	7	no
Blume/Mitchell	varied	8	(learning)
Boling	consumer mathematics	12	no
Cech	general mathematics	9	no
Chang	remedial	7-8	no
Fesharaki	decimals, percent, estimation	7-8	yes
Fischman	business mathematics	9-10	no
Gaslin	general mathematics	9	yes
Gallery	handicapped	7-9	yes
Hopkins	basic mathematics, problem solving	9	yes
Hutton	algebra	9	yes
Jamski	rational numbers	7	yes
Keough/Burke	varied	11-12	no
Kolpas	remedial	9	no
Ladd	low achievers	9	no
Laursen	general mathematics	9	no
Lawson	estimation	7	no
Lenhard	varied	7-12	no
Lunder	consumer mathematics	9-12	no
Peterson	pre-algebra, algebra 1,2	10-11	no
Quinn	algebra	8-9	no
Rudnick	varied	7	no
Schnur/Lang	varied	7-9	no
Shirey	consumer, business mathematics	10-12	no
Szetela	ratios, problem solving	7	yes
Szetela	trigonometry	9-10	yes
Toole	general mathematics	9	yes
Townsend	algebra 2	11	no
Vaughn	general mathematics	9	yes
Wajeeh	general mathematics	9	yes
Ward	general mathematics	9	no
Weaver et al.	varied	7	(learning)
Williams	general mathematics	9	yes
Zepp	proportions	9	no

Aldridge, Wanda Scott. Effects of Electronic Calculators on Achievement of Middle School Remedial Mathematics Students. (University of Georgia, 1976.) Dissertation Abstracts International 37A: 4078; January 1977.

Eighty-three middle school remedial math students used calculators as they worked through lessons in the Computational Skills Development Kit on an individualized basis for 4 weeks, while 90 students did all the required calculation with paper and pencil. Results on a standardized arithmetic skills posttest showed that the non-calculator groups scored significantly higher ($p < .001$) than the calculator group. Eighth graders in the calculator group scored higher than eighth graders in the non-calculator group, however.

[Achievement, Desk calculators, Elementary (grade 6), Junior high (grades 7-8), Remedial, Research]

Anderson, Lyle Eugene. The Effects of Using Restricted and Unrestricted Modes of Presentation with Electronic Calculators on the Achievement and Attitude of Seventh Grade Pupils. (University of Denver, 1976.) Dissertation Abstracts International 37A: 6321-6322; April 1977.

Three seventh-grade mathematics classes taught by the same teacher were randomly selected at each of four schools for the 20-week study. One class in each school was permitted restricted use of calculators (for checking paper-and-pencil computation and as an aid in problem solving), a second class was permitted unrestricted calculator use, and a third class was not permitted to use calculators. Pupils using calculators showed improved attitudes toward mathematics but no change in achievement, understanding of mathematical concepts, or computational skill. On an untimed problem-solving test, pupils using calculators solved problems correctly at almost twice the rate of pupils not using calculators.

[Achievement, Attitudes, Junior high (grade 7), Research]

Blume, Glendon W. and Mitchell, Charles E. The Calculator's Effect on Children's Solution Processes. Paper presented at NCTM 57th Annual Meeting, April 1979. ERIC: ED 170 166.

Three preliminary studies and one exploratory main study investigated the effect of calculator use on students at a variety of grade levels. For the 83 eighth-grade students in the main study, there was a significant association between calculator use and choice of the longer solution method on 5 of 7 items. The heuristics of simplication and application of a structure property such as distributivity appear to be calculator-inhibited rather than calculator-enchanced in the absence of specific instruction designed to encourage their use in conjunction with calculator computations. The number of errors in conceptualization of the problem does not appear to be affected by calculator use.

[Elementary, Research, Secondary, Solution methods]

Boling, Mary Ann Neaves. Some Cognitive and Affective Aspects of the Use of Hand-Held Calculators in High School Consumer Mathematics Classes. (The Louisiana State University and Agricultural and Mechanical College, 1977.) Dissertation Abstracts International 38A: 2623-2624; November 1977.

Over a period of 19 weeks, a group of 51 twelfth-grade consumer mathematics students used calculators to perform all computations, while a group of 43 students used traditional paper-and-pencil methods. No significant differences in mathematical problem-solving achievement or in attitude toward mathematics were found between the two groups. Strong positive attitudes toward the use of calculators in the classroom were found.

[Achievement, Attitudes, Consumer mathematics, Research, Secondary (grade 12)]

Cech, Joseph Philip. The Effect the Use of Desk Calculators Has on Attitude and Achievement in Ninth-Grade General Mathematics Classes. (Indiana University, 1970.) Dissertation Abstracts International 31A: 2784; December 1970.
(See also: Cech, Joseph P. The Effect of the Use of Desk Calculators on Attitude and Achievement with Low-Achieving Ninth Graders. Mathematics Teacher 65: 183-186; February 1972.)

The two main reasons for using calculators with low achievers in mathematics classes are motivation and achievement. This study of calculator effectiveness involved two teachers each teaching a calculator section and a regular section of general mathematics for seven weeks. Students in the experimental group were encouraged, but not forced, to check answers with the calculators. All classes were given pre- and posttests of attitude and achievement. Results did not support the hypothesis that students using calculators would show positive gains in attitude toward mathematics, or increased paper-and-pencil computational skill. Students could compute better with the calculator than without it, however.

[Attitudes, Desk calculators, General mathematics, Low achievers, Research, Secondary (grade 9)]

Chang, Lisa Li-Tze. An Examination into the Effects of Calculator-Assisted Instruction on the Mathematics Achievement and Attitude of Seventh and Eighth Grade Disadvantaged Students. (Cornell University, 1979.) Dissertation Abstracts International 40A: 1323-1324; September 1979.

Students (n = 126) in grades 7 and 8 were randomly divided into two groups. For 24 weeks, one group had calculators available during lessons, but not on tests or for taking home; the other group used only paper and pencil. No significant differences between groups were found on computation, concepts, or attitudes; a highly significant difference on problem solving favored the calculator group. Use of calculators did not seem to affect the seventh graders in improving their areas of learning difficulties, but eighth graders seemed to benefit on some topics.

[Achievement, Attitudes, Junior high (grades 7-8), Remedial, Research]

Fesharaki, Mahammad. A Study of the Effect of Hand-Calculators on Achievement, Estimation and Retention of Seventh and Eighth Graders on Decimals and Percent. (University of Missouri-Columbia, 1978.) Dissertation Abstracts International 39A: 6004; April 1979.

Students from four mixed seventh- and eighth-grade classes participated. After categorization into high, middle, and low achievement levels, classes were assigned to two teachers. Each teacher taught one calculator and one non-calculator class for 30 class periods. The findings indicate that students using calculators and developed calculator materials demonstrated significantly greater gains than did non-calculator students, on measures of decimals and percent. No differences in retention were found in achievement or estimation skills.

[Achievement, Decimals, Estimation, Junior high (grades 7-8), Percent, Research]

Fischman, Myrna Leah. New York City High School Students' Attitudes and Concept Learnings in Business Arithmetic When Using Electronic Calculators as Contrasted with Hand Calculation. (New York University, 1976.) Dissertation Abstracts International 37A: 774-775; August 1976.

Gaslin, William Lee. A Comparison of Achievement and Attitudes of Students Using Conventional or Calculator-Based Algorithms for Operations on Positive Rational Numbers in Ninth-Grade General Mathematics. Dissertation Abstracts International 33A: 2217; November 1972.
(See also: Journal for Research in Mathematics Education 6: 95-108; March 1975.

Use of units in which fractional numbers were converted to decimals and examples then solved on a calculator was found to be a "viable alternative" to use of conventional textbooks (including fractions) with or without a calculator, for low-ability or low-achieving students.

[Algorithms, Desk calculators, General mathematics, Low achievers, Rational numbers, Research, Secondary (grade 9), Units]

Gallery, Michael Edward. Teaching Calculator Use and Checking Account Skills to the Mildly Handicapped. (Utah State University, 1978.) Dissertation Abstracts International 39A: 2866; November 1978.

Two "packages" on calculator skills and on checking account skills were evaluated with 38 students from six high schools and two junior high schools. Eighteen students received instruction on the packages, while 20 students were in the control group. A one-way analysis of variance indicated that the difference between group posttest means for both packages significantly favored the experimental group. However, the mastery criterion was not achieved on either package.

[Handicapped, Junior high, Research, Secondary, Units]

Graeber, Anna O.; Rim, Eui-Do; and Unks, Nancy J. A Survey of Classroom Practices in Mathematics: Reports of First, Third, Fifth and Seventh Grade Teachers in Delaware, New Jersey, and Pennsylvania. Philadelphia: Research for Better Schools, Inc., 1977.

In a survey in 1977 of 1,343 teachers in grades 1, 3, 5, and 7, questions on calculator use were included. The percentage of teachers who had used calculators was: 3.9 percent at grade 1, 8.4 percent at grade 3, 19.4 percent at grade 5, and 25.6 percent at grade 7. In the first grade, calculators were used most frequently for drill; the next three most frequent usages were for checking, motivation, and remediation. Use of the calculator for drill decreased with grade level. Above first grade, the most frequent use was for checking, with motivation and word problems next most frequently reported used.

[Attitudes, Research (survey), Roles, Teachers (grades 1, 3, 5, 7)]

Hopkins, Billy Lynn. The Effect of a Hand-Held Calculator Curriculum in Selected Fundamentals of Mathematics Classes. (The University of Texas at Austin, 1978.) ERIC: ED 156 496.

This study investigated the effects on achievement and attitude resulting from use of a calculator-based curriculum and calculators in ninth-grade basic mathematics (in which students were at least two grade levels behind in mathematics achievement). Twelve classes from three schools were randomly assigned to either calculator (n = 83) or non-calculator groups (n = 84). For four weeks, teachers used guides prepared by the investigator to teach estimation, computation, and problem solving using the four operations with whole numbers. One-half of each group was randomly selected to take the posttest with calculators available, while the other half did not have calculators. Data were analyzed by analysis of covariance. Students using calculators in instruction scored as well in computation and significantly better in problem solving as their peers not using calculators. Attitudes were not significantly different. Students using calculators on the posttest did significantly better in both computation and problem solving than students not using calculators.

[Achievement, Attitudes, Basic mathematics, Estimation, Mixed operations, Problem solving, Research, Secondary (grade 9), Units]

Hutton, Lucreda Ann Williams. The Effects of the Use of Mini-Calculators on Attitude and Achievement in Mathematics. (Indiana University, 1976.) Dissertation Abstracts International 37A: 4934; February 1977.

A 4-week unit on powers, roots, and radicals was studied by one group of ninth-grade algebra students who had traditional instruction with no calculators, a second group who had traditional instruction but could use calculators during class, and a third group who had special calculator instruction plus access to calculators during class. No differences were found when groups were compared on achievement or attitude.

[Achievement, Algebra, Attitudes, Powers, Research, Roots, Secondary (grade 9), Units]

Jamski, William Donald. The Effect of Hand Calculator Use on the Achievement of Seventh Graders Learning Rational Number-Decimal-Percent Conversion Algorithms. (Indiana University, 1976) Dissertation Abstracts International 37A: 4934-4935; February 1977.
(See also: Jamski, William D. The Effect of Calculators on Achievement. MATYC Journal 13: 52-56; Winter 1979.)

Three classes of students at the seventh-grade level used calculators during a four-week unit on finding equivalent forms for fractions, decimals, and percents, while three other classes did not use calculators. On an immediate posttest a significant difference was identified between groups on items involving conversion from a simplified fraction to a decimal. No differences were found between groups when a retention test was given.

[Achievement, Junior high (grade 7), Rational numbers, Research]

Jewell, Wallace F., Jr. Hand Calculators in Secondary Education: Evaluation, Analysis and Direction. Unpublished doctoral dissertation, State University of New York at Buffalo, 1979.)

The purpose of this study was to analyze and evaluate present hand-held calculator uses and materials and propose possible directions for their utilization in the near future. Also included is an analysis of the usual high school mathematics curriculum of elementary algebra, geometry, intermediate algebra-trigonometry, and elementary functions with regard to calculator implementation.

[Curriculum, Research (review), Secondary]

Keough, John J. and Burke, Gerald W. Utilizing an Electronic Calculator to Facilitate Instruction in Mathematics in the 11th and 12th Grades. Final Report. July 1969. ERIC: ED 037 345.

The group using calculators achieved significantly more on a standardized test than did a group not using them.

[Achievement, Desk calculators, Research, Secondary (grades 11-12)]

Kolpas, Sidney J. The Use of Electronic Calculators as In-Class Instructional Aids in a Ninth-Grade Arithmetic Program. Unpublished doctoral dissertation, University of Southern California, 1978.

Eighty ninth-grade low- and under-achievers in four arithmetic classes participated in this 20-week study. The two experimental classes used calculators as instructional aids and as the sole solution-source for 50 percent or for 75 percent of each in-class assignment, while control classes (from another school) did not use calculators. The Comprehensive Test of Basic Skills was given as the pre- and posttest, and a researcher-constructed test was completed twice (first by hand, then with calculators to correct). No significant differences were found on the standardized test; significant differences on the researcher-constructed test favored the experimental groups both with and without calculators.

[Achievement, Low Achievers, Remedial, Research, Secondary (grade 9), Testing]

Ladd, Norman Elmer. The Effects of Electronic Calculators on Attitude and Achievement of Ninth Grade Low Achievers in Mathematics. (Southern Illinois University, 1973.) Dissertation Abstracts International 34A: 5589; March 1974.

Two-hundred-one low achievers were randomly scheduled into one of five control sections or one of five experimental sections. All groups followed the same lesson sequence, with control groups using only paper-and-pencil for all calculations and experimental sections using electronic calculators. Significant differences were found on both attitude and achievement tests from pre- to post-treatment for both groups, but no significant differences in posttest mean scores were found between groups.

[Achievement, Attitudes, Desk calculators, Low achievers, Research, Secondary (grade 9)]

Laursen, Kay William. Use of Calculators in High School General Mathematics: A Study Comparing Achievement, Attitude, and Attendance of General Mathematics Students Who Used Calculators with Students Who Did Not. (Brigham Young University, 1978.) Dissertation Abstracts International 39A: 733; August 1978.

From five schools with 18 general mathematics classes which had classroom sets of calculators, nine classes were randomly assigned to use calculators while nine other classes used paper-and-pencil methods of computation. Analyses of variance and covariance using pre-posttest data indicated that greater gains were made by students using calculators. However, no significant differences were found in attitudes or attendance.

[Achievement, Attitudes, General mathematics, Research, Secondary (grade 9)]

Lawson, Thomas James. A Study of the Calculator's and Altered Calculator's Effect upon Student Perception and Utilization of an Estimation Algorithm. (State University of New York at Buffalo, 1977.) Dissertation Abstracts International 39A: 647; August 1978.

Seventh-grade students from one junior high school were randomly assigned to use paper and pencil, a four-function calculator, or an altered calculator (with the function of the four operation keys masked). The four-day "interim work experience" was a 16 item computational task. Using analysis of variance, it was found that (1) computational ability correlated positively with ability to estimate, (2) the treatment had no effect on estimation ability, (3) students with low computational ability made the greatest number of errors when using calculators, and (4) altering the keyboard had no significant overall effect on ability to use calculators. Videotape analysis of students working indicated that many did not use estimation to verify answers. Interview information is also provided.

[Estimation, Four function calculators, Research, Secondary (grade 7)]

Lenhard, Rodger William. Hand-Held Calculators in the Mathematics Classroom at Stuart Public School, Stuart, Nebraska. (Montana State University, 1976.) Dissertation Abstracts International 37A: 5661; March 1977.

Analysis of at least eight tests taken by 125 students in grades 7 through 12 showed no differences in performance between those using and those not using calculators during the test on test scores, concept and computation errors, attitudes, time, and rank.

[Achievement, Attitudes, Junior high, Research, Secondary]

Lunder, Dennis A. The Impact of the Calculator on Test Anxiety for High School Consumer Mathematics Students. (University of Denver, 1978.) Dissertation Abstracts International 39A: 6607; May 1978.

Consumer mathematics students (n = 159) in grades 9-12 in three schools were randomly scheduled into two experimental groups (using calculators) or two control groups (not using calculators). One experimental and one control group used calculators on the posttest, while the other two groups did not. Analysis of covariance indicated that allowing use of calculators on the posttest improved achievement. The group using calculators for instruction and the test scored higher than the group not using calculators for instruction but using them on the test. Test anxiety was also explored; use of the calculator did not significantly reduce anxiety.

[Anxiety, Research, Secondary (grades 9-12), Testing]

Parks, Terry E. Calculator Survey, Shawnee Mission Public Schools. Shawnee Mission, Kansas, 1977.

Results of a calculator survey on availability of calculators to K-12 students and on parent and teacher opinion toward the use of calculators in the classroom is given. Data from both 1975 and 1977 are cited.

[Attitudes, Elementary (grades K-6), Research (survey), Secondary (grades 7-12)]

Peterson, Robert E. and Capoferi, Alfred. 1975-76 Fraser High School, Macomb Intermediate School District Project: Study on the Use of Hand-Held Calculators in Pre-Algebra (Grade 10), Algebra I (Grade 10), and Intermediate Algebra (Grade 11). Fraser, Michigan: Fraser Public School District, 1976.

A general project description is provided. Data, observations, and calculator activities and problems are reported for each of the mathematics groups.

[Algebra, Research, Secondary (grades 10-11)]

Quinn, Donald Ray. The Effect of the Usage of a Programmable Calculator upon Achievement and Attitude of Eighth and Ninth Grade Algebra Students. (Saint Louis University, 1975.) Dissertation Abstracts International 36A: 4234-4235; January 1976.

The programmable calculator was used in eighth- and ninth-grade algebra classes for evaluating algebraic expressions and for solving linear, quadratic, and systems of equations. Findings showed no significant differences in achievement when performance of students in the calculator classes was compared to performance of those in non-calculator algebra classes. However, students in the calculator classes showed less "anxiety toward mathematics" and had better "self-concept in mathematics" than students in non-calculator classes.

[Achievement, Algebra, Anxiety, Junior high (grades 8-9), Programmable calculators, Research]

Royce, George and Shank, James. Calculators in the Classroom? Science Teacher 44: 23-25; October 1977.

The results of a student attitude survey among junior high school students who had been allowed to use calculators to check mathematics computations are reported. Significant preference for using calculators in the classroom was displayed.

[Attitudes, Junior high, Research (survey)]

Rudnick, Jesse. Pocket Sized Calculators Versus Seventh Grade Math Students. Philadelphia: Temple University, May 1978. Multilith copy.

Approximately 700 seventh-grade students in two schools were randomly assigned either to calculator or control groups. Each of six teachers taught two experimental and two control classes, using the regular textbook. Students were "on their own" as to how and when they used calculators; they kept logs of when and for what operations calculators were used. No significant difference in computational skills was found between groups. Attitudes of students in both groups varied little. Parental attitudes, however, changed: while 50 percent opposed the use of calculators at the outset, only 33 percent were opposed at the end. At the start, 49 percent felt that their children would become highly dependent on the machine, while at the end this number dropped to 22 percent.

[Attitudes, Junior high (grade 7), Research]

Rudnick, Jesse A. and Krulik, Stephen. The Minicalculator: Friend or Foe? Arithmetic Teacher 23: 654-656; December 1976

The effect of the availability and use of a calculator on seventh graders' mathematics achievement was studied. Preliminary findings on parental attitude toward the use of calculators and on student achievement are discussed.

[Achievement, Attitudes, Junior high (grade 7), Research]

Schnur, James O. and Lang, Jerry W. Just Pushing Buttons or Learning? —A Case for Minicalculators. Arithmetic Teacher 23: 559-562; November 1976.

A study of the effect of the use of calculators on student computational ability is reported. Results showed that the treatment groups using calculators gained significantly more whole number computational ability than control groups not using the calculator. Sex of student and calculator usage interaction was not significant, nor was the interaction between ethnic/economic background and gain in computational ability.

[Achievement, Elementary (grades 4-6), Junior high (grades 7-9), Research]

Shirey, John Reginald. The Effects of Computer-Augmented Instruction on Students' Achievement and Attitudes. (University of Oregon, 1976.) Dissertation Abstracts International 37A: 3386-3387; December 1976.

Tenth, eleventh, and twelfth graders in consumer and business mathematics classes were randomly assigned to receive computer-augmented instruction or a low-cost alternative using tables and calculators to complete inquiry exercises. The instructional unit covered nine days. Results showed that the calculator group did significantly more inquiry beyond the minimum required than did the computer group.

[Business mathematics, Consumer applications, Research, Secondary (grades 10-12)]

Suydam, Marilyn N. Electronic Hand Calculators: The Implications for Pre-College Education. Final Report, Grant No. EPP 75-16157, National Science Foundation, February 1976. ERIC: ED 127 205 (full report); ED 127 206 (50-page form).

Szetela, Walter. A Study of Ratio Concepts, Skills, and Problem Solving Using Calculators in Grade 7. 1978. Xerox copy.

This study was concerned with the use or non-use of calculators by 39 students in grade 7 who used specially designed lessons to increase involvement in measuring activities. There were 11 days of instruction. On two tests on ratio skills, concepts, and problems and on a test of attitude toward learning ratios, no significant differences were found. On a test with unfamiliar ratio problems on which the calculator class used calculators, they scored significantly higher than the non-calculator class.

[Junior high (grade 7), Ratios, Research]

Szetela, Walter. Hand-Held Calculators and the Learning of Trigonometric Ratios. Journal for Research in Mathematics Education 10: 111-118; March 1979.

The investigator and the regular teacher used specially designed materials for teaching trigonometric ratios to 131 students in grades 9 and 10 randomly assigned to groups using calculator-based instruction or not using calculators for 13 days. Students built their own short trigonometric tables using measurements of specially constructed right triangles. On a quiz one week before the final achievement test, the calculator groups scored significantly higher than the non-calculator groups (.02). There were no significant differences on the final achievement test or in attitude toward learning ratios.

[Research, Secondary (grades 9-10), Trigonometry]

Toole, Betty A. Evaluation of the Effectiveness of Calculator Assisted Curriculum and Instruction in Ninth Grade General Mathematics Classes. Paper prepared for AERA Meeting, 1979.

The use of a supplementary calculator-assisted program, used one day a week in six different junior high schools (N = 400) for ninth-grade general mathematics was compared with non-use of calculators in that course. In the six-month period between pre- and posttesting, the experimental group gained 8 months more on the total test than the control group did. On subtests, the experimental group gained more than the control group in computation (7 months), concepts (5 months), and applications (1 year).

[Achievement, General mathematics, Research, Secondary (grade 9)]

Townsend, Gloria Childress. The Effect of Programmable Calculator Use on Probability Estimation Achievement and Attitude Toward Estimation of Students in Second Year Algebra. (Indiana University, 1979.) Dissertation Abstracts International 40A: 1936; October 1979.

Three second-year algebra classes used 10 days to investigate a series of probability exercises. The student-programming group estimated answers and then wrote their own programs to verify their estimates. The teacher-programming section used the teacher's programs to verify estimates, while the control group received the results of a hypothetical experiment to verify estimates. Some difference in estimation achievement was noted for the student-programming group; attitudes were significantly better than in the control group.

[Achievement, Algebra, Attitudes, Estimation, Probability, Programmable calculators, Research, Secondary (grade 11)]

Vaughn, Larry Richard. A Problem of the Effects on Hand-Held Calculators and a Specially Designed Curriculum on Attitude Toward Mathematics, Achievement in Mathematics, and Retention of Mathematical Skills. (University of Houston, 1976.) Dissertation Abstracts International 37A: 4938-4939; February 1977.
(See also: Creswell, John L. and Vaughn, Larry R. Hand-held Calculator Curriculum and Mathematical Achievement and Retention. Journal for Research in Mathematics Education 10: 364-367; November 1979.)

Four ninth-grade general mathematics classes used calculators as they studied decimals and percents for eight weeks in a specially-designed curriculum, while four other classes received traditional instruction with no calculators. Results showed that the calculator group scored significantly higher than the no-calculator group on an achievement test, but no differences between groups were found with respect to attitude or retention of mathematical skills.

[Achievement, Attitudes, Decimals, General mathematics, Percent, Research, Secondary (grade 9)]

Wajeeh, Abdullah. The Effect of a Program of Meaningful and Relevant Mathematics on the Achievement of the Ninth Grade General Mathematics Student. (Wayne State University, 1976.) Dissertation Abstracts International 37A: 2801-2802; November 1976.

The group of ninth-grade general mathematics students using an investigator-developed unit plus calculators for 15 weeks scored significantly higher ($p < .05$) on a standardized computation test than the group using only the developed unit, but there were no significant differences in attitudes.

[Achievement, Attitudes, General mathematics, Research, Secondary (grade 9)]

Ward, Dennis Elliott. The Effect of the Electronic Calculator on Problem-Solving Achievement and Attitudes Toward Mathematics of General Mathematics Students. (University of Southern California, 1978.) Dissertation Abstracts International 39A: 4038; January 1979.

In the first semester of 1977/78, 92 general mathematics students were randomly assigned to two experimental classes (using calculators) and two control classes (not using calculators). No significant differences in problem-solving achievement or attitudes were found; no gender effect was found. Better readers tended to score higher on problem solving both with and without calculators.

[Achievement, Attitudes, Gender, General mathematics, Problem solving, Research, Secondary (grade 9)]

Weaver, J. Fred; Blume, Glendon W.; and Mitchell, Charles E. Calculator Explorations with Seventh Grade Students: Some Calculator Inspired Instructional Materials, Observations, and Investigations. Technical Report No. 497. Madison: Wisconsin Research and Development Center for Individualized Schooling, July 1979.

Weiss, Iris R. Report of the 1977 National Survey of Science, Mathematics, and Social Studies Education. Final Report, National Science Foundation Contract No. C7619848. Research Triangle Park, North Carolina: Research Triangle Institute, Center for Educational Research and Evaluation, March 1978.

A national survey conducted for the National Science Foundation included questions about the extent of use of calculators in schools. For four grade ranges, the percentage of schools having calculators was: K-3, 28 percent; 4-6, 36 percent; 7-9, 49 percent; 10-12, 77 percent. Rural schools are as likely as suburban schools to have calculators, and both are significantly more likely to have them than schools in small cities or urban areas. The percentage of mathematics classes using calculators increased with grade level: K-3, 6 percent; 4-6, 14 percent; 7-9, 30 percent; 10-12, 48 percent. Most K-3 teachers indicated that calculators are not needed; in 4-6, 44 percent indicated they were not needed, while 39 percent needed them but did not have them; for 7-9, the comparable percentages were 42 percent and 28 percent; for 10-12, 33 percent and 18 percent.

[Elementary, Research (survey), Secondary]

Williams, David E. The Effect of the Use of the Mini-Calculator and an Associated Curriculum Supplement on Computational Skills and Attitudes Toward Arithmetic of Ninth Grade Non-College Bound Students. (Temple University, 1979.) Dissertation Abstracts International 39A: 6610-6611; May 1979.

This year-long study was conducted in five junior high schools in Philadelphia, with three teachers each teaching one class using calculators with the regular general mathematics curriculum, one class using the regular curriculum with calculators and supplementary calculator materials, and a comparison class not using calculators. Calculators were not used on the pre- or posttest (California Achievement Test). Analysis of covariance and t-tests of correlated means were applied. No significant differences were found in computational skill, while attitude differences favored the calculator groups.

[Achievement, Attitudes, General mathematics, Research, Secondary (grade 9)]

Williams, David and Tobin, Alexander. Philadelphia Mini-Calculator Program. Mathematics Teacher 71: 471-472; May 1978.

Results from a survey of 415 secondary teachers are briefly summarized. Guidelines were developed and a booklet for teachers prepared. The effect of using calculators and the curriculum supplement has been studied in five schools.

[Attitudes, Research (survey), Secondary, Teachers]

Williams, S. Irene and Jones, Chancey O. A Survey of the Use of Hand-Held Calculators in Advanced Placement Calculus Courses. Princeton, New Jersey: Educational Testing Service, 1979.

The questionnaire was sent to mathematics department chairpersons at the 2402 secondary schools that had five or more students taking an Advanced Placement examination in May 1977; returns were received from 1547 schools (64.4 percent), with 1403 scorable. About 89 percent indicated that they permitted some use of calculators in calculus courses. Slightly more than 10 percent had modified their courses to include special techniques particularly suited to the use of the calculator. In only 20 percent of the schools are calculators supplied by the school districts. An overwhelming majority think that student performance (on AP examinations) would not be significantly affected if calculator use was permitted.

[Calculus, Research (survey), Secondary, Teachers, Testing]

Wyatt, J. Wendell; Rybolt, James F.; Reys, Robert E.; and Bestgen, Barbara J. The Status of the Hand-Held Calculator in School--Implications for Parents, Teachers and Administrators. Phi Delta Kappan 61: 217-218; November 1979.

Yvon, Bernard R. and Downing, Davis A. Attitudes Toward Calculator Usage in Schools: A Survey of Parents and Teachers. School Science and Mathematics 78: 410-416; May-June 1978.

Two-hundred-fifty parents and teachers of grades K-9 replied to a 12-item questionnaire about the use of calculators. They were increasingly more accepting of calculator use as grade level (3-6, 7-8, 9-12) increased, with teachers significantly more favorable than parents at the two lower levels. Both groups were moderately negative about the use of calculators for homework and about whether skills with calculators will be essential to future success. They were accepting of the use of calculators for enrichment, motivation, and games. While moderately negative replies were given about the use of calculator to replace paper-and-pencil skills, they were very accepting of the use of calculators along with paper-and-pencil computation.

[Attitudes, Elementary, Research (survey), Secondary]

Zepp, Raymond Andrew. Reasoning Patterns and Computation on Proportions Problems, and Their Interaction with the Use of Pocket Calculators in Ninth Grade and College. (The Ohio State University, 1975.) Dissertation Abstracts International 36A: 5181; February 1976.

One-hundred-seventy ninth graders and 198 college freshmen were classified as having high, middle, or low ability in solving proportions. Half the students in each ability group were given calculators to use while working on a programmed unit in linear interpolation, while the rest of the students could only use paper and pencil for their computations. No significant differences were found between performances of students using calculators compared to those not using calculators, nor was there any significant interaction of use of calculators with ability to solve proportions. The hypothesis that students could understand a proportional train of thought better if the barrier of computation were removed was not borne out.

[Achievement, College, Ratios, Research, Secondary (grade 9)]

CALCULATOR INFORMATION CENTER

1200 Chambers Rd.
Columbus, Ohio 43212

(614) 422-8509

Special Bulletin
December 1979

Selected References on Calculator Uses in Secondary School Mathematics Instruction*

Aidala, Gregory. Calculators: Their Use in the Classroom. School Science and Mathematics 78: 307-311; April 1978.

The use of 15 calculators to promote problem-solving skills in an eighth-grade class is described. Suggestions and recommendations are included.

[Junior high (grade 8), Problem solving, Recommendations]

* Albrecht, Bob. Calculators for Beginners. Calculators/Computers 1: 21-26; May 1977. 1: 75-82; October 1977. 1: - ; December 1977. 2: 61-66; January 1978. 2: 91-96; March 1978. 2: 23-28; April 1978. 2: 87-95; May 1978.

This set of articles is a serialization of a self-teaching student workbook about four-function calculators and elementary computer programming. The articles cover: (1) Basic addition, subtraction, and multiplication; some computer-like notation for writing programs to solve problems; and directions for playing NIM. (2) Multiplication, powers, and multiplication game. (3) Division. (4) Mixed operations. (5) The change sign key. (6) A calculator version of the game Krypto. (7) The concept of memory and keys for manipulating numbers in memory.

[Calculator keys, Calculator memory, Four-function calculators, Games, Mixed operations, Powers, Problem solving, Programming, Worksheets]

* Averett, Dorothy M., et al. Using the Mini-Calculator to Teach Mathematics. Philadelphia: Curriculum Office, Instructional Services, School District of Philadelphia, 1977.

Calculator activities are presented for the following topics: place value, rounding off numbers and estimating answers, whole numbers, decimal fractions, common fractions, number patterns, powers and roots, algebra, geometry, and advanced topics. Other sections cover selecting a calculator for classroom use, preparing to use the calculator, types of use, the keyboard, "word-number" displays and problems, and additional source materials.

[Activities, Elementary, Secondary]

* These references are from: Suydam, Marilyn N. Calculators: A Categorized Compilation of References. Columbus: ERIC/SMEAC, 1979. They also appear on bulletins developed for the Calculator Information Center.

* Aviv, Cherie Adler. Pattern Gazing. Mathematics Teacher 72: 39-43; January 1979.

  Worksheets on pattern searches are provided.

  [Integers, Pattern searches, Secondary (grades 7-12)]

* Bell, Max. Calculators in Secondary School Mathematics. Mathematics Teacher 71: 405-410; May 1978.

  The effects of calculators on education, including discussion of potential strengths and weaknesses, are described. Specific suggestions for needed curriculum development are presented.

  [Pros/cons, Roles]

* Bell, Max; Esty, Edward; Payne, Joseph N.; and Suydam, Marilyn N. Hand-held Calculators: Past, Present, and Future. In Organizing for Mathematics Instruction (F. Joe Crosswhite, editor). 1977 NCTM Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1977.

  An overview is provided for each of four reports dealing in whole or in part with calculators (the NACOME Report, the Euclid Conference Report, the status report to the NSF on calculators in precollege education, and the report on the Conference on Needed Research and Development on Hand-held Calculators in School Mathematics). Activities in several school systems are also cited.

  [Curriculum, Elementary, Pros/cons, Research (needed), Secondary, Status report]

* Billings, Karen and Moursund, David. Problem Solving with Calculators. Salem, Oregon: The Math Learning Center, University of Oregon, 1978.

  [Curriculum, Problem solving, Worksheets]

* Billstein, Rick and Lott, Johnny W. When Does a Fraction Yield a Terminating Decimal? Calculators/Computers 2: 15-19; January 1978.

  A two-hour module used in a Calculator Usage in Elementary Schools class for teachers is presented; it is appropriate for use in grades 7 and 8. Using worksheets, students convert fractions to decimals using any four-function calculator, do the prime factorization of the denominator, and find patterns formed by the denominator of the fractions that are terminating decimals.

  [Junior high (grades 7-8), Rational numbers, Teachers (preservice), Units, Worksheets]

* Bitter, Gary G. Curriculum Considerations for Use of the Hand-Held Calculator. Calculators/Computers 2: 15-16; September/October 1978.

  Different logic systems, rounding, converting fractions to decimals, decimal representation of a fraction, overflow, underflow, truncation, order of operations, "add-on" percent key, constant key, internal counting, division by zero, the negative sign, and consecutive operations are considered.

  [Selection]

Blume, Glendon W. A Calculator-Based Unit on Exponential Functions and Logarithms. The Computing Teacher 6: 47-48; May 1979.

Calculator-based activities are provided, with worksheets on population growth and inflation.

[Activities, Algebra, Fnnctions, Secondary, Units, Worksheets]

* Bolduc, E. J. Using a Minicalculator to Find an Approximate Value for Pi. School Science and Mathematics 77: 689-691; December 1977.

Geometry and a hand-held calculator are used to find a value of pi accurate to five decimal places.

[Geometry]

Bourjaily, Bill and Radachy, Marc. Mathematics by Calculator. Berea, Ohio: Berea City School District, 1977.

This year-long course for Mathematics 9-10 is designed to assist students who have had difficulty in mathematics as well as those who are unmotivated. Computational skills are applied in practical situations. Number, operation, space, symbolism, relation, proof, and approximation concepts are included as well as skills in computation with whole numbers, fractions, and decimals.

[Course description, Mixed operations, Remedial, Secondary (grades 9-10]

Burt, Burce C. (Editor). Calculators: Readings from the Arithmetic Teacher and the Mathematics Teacher. Reston, Virginia: National Council of Teachers of Mathematics, 1979.

* Caravella, Joseph R. Minicalculators in the Classroom. Washington: National Education Association, 1977. ERIC: ED 134 474.

The positive contributions of the calculator to basic education are explored. The introduction discusses questions educators ask about the use of the calculator. The first section briefly describes uses of the calculator in the classroom, discusses research on calculators, describes NCTM involvement, and presents the NACOME recommendations concerning calculators. The second section covers the implications of the use of the calculator in terms of curriculum, teacher in-service education, classroom management, instruction, and testing and evaluation. The third section gives guidelines for selecting and using calculators. The final section includes classroom activities keyed to the various functions of the calculator.

[Activities, Classroom management, Roles, Selection, Status report]

Chinn, William G.; Dean, Richard A.; and Tradewell, Theodore N. Arithmetic and Calculators: How to Deal with Arithmetic in the Calculator Age. San Francisco: W. H. Freeman & Co., 1978.

The calculator is used as a teaching tool, with arithmetic processes developed through the use of the calculator. A variety of real-world applications and mathematical recreations are included.

[Activities, Mixed operations]

★ Dubisch, Roy and Hood, Vernon R. Basic Mathematics with Hand-held Calculator: A Work-Text. Menlo Park, California: Benjamin/Cummings Publishing Company, 1979.

This worktext is for students at either secondary or college level.

[Basic mathematics, College, Secondary]

Ercolano, Joseph. Fractional Computations on a Calculator. Mathematics Teacher 72: 591-592; November 1979.

French, Doug. A Square Root Algorithm. Mathematics Teacher 79: 35; June 1977.

An algorithm for calculating square roots on the Casio Memory 8 calculator is given.

[Roots]

★ Friedlander, Richard J. Efficient Algorithms for the Calculator. Mathematics Teacher 71: 614-618; October 1978.

A brief discussion of calculator logic is followed by descriptions of efficient algorithms for reciprocals, powers, and parentheses.

[Algorithms, Calculator logic, Powers, Reciprocals]

★ Goodson, Bobby. Let's Put the Calculators Where They Belong. Calculators/Computers 2: 20-22; November-December 1978.

Use of the calculator in work with square roots, geometry, factoring, and other junior high topics is discussed. A worksheet on finding a square root and a crossword puzzle developed by a student are given.

[Activities, Games, Junior high, Worksheets]

★ Greenwood, Jay. A Product of Our Times. Mathematics Teacher 70: 234-238; March 1977.

Activities are described in which the calculator is used in a discovery learning situation for exploring number patterns in multiplication exercises.

[Pattern searches]

★ Henry, Loren L. An Invaluable Aid to Understanding Mathematics: The Hand-Held Calculator. School Science and Mathematics 77: 585-591; November 1977.

Instructional techniques are given for the use of calculators in finding prime numbers, solving trigonometric equations, and solving polynomial equations.

[Activities, Secondary]

* Hiatt, Art. A Geometry Problem for Hand-Held Calculators or Computers. Calculators/Computers 1: 37-38; May 1977.

A mathematical inquiry method (making and organizing observations, generalizing, specializing, inventing symbolism, and proving conjectures) and the calculator's role as an important tool for this process are discussed. An example is given in which the concept of area of a circle can be investigated with the calculator, and a worksheet for students if provided.

[Geometry, Roles, Secondary, Worksheets]

* Hiatt, Arthur A. Finding Areas Under Curves with Hand-Held Calculators. Mathematics Teacher 71: 420-423; May 1978.

The method of inquiry is applied to find the area of a circle using the calculator.

[Geometry]

* Hobbs, Billy F. and Burris, Charles H. Minicalculators and Repeating Decimals. Arithmetic Teacher 25: 18-20; April 1978.

An algorithm for generating on any calculator as many digits as desired in the decimal representation of the rational number N/D is given in detail.

[Algorithms, Rational numbers]

* Humphreys, Casey, et al. Calculator Cookery. Minneapolis: Minneapolis Public Schools, East Area Curriculum Office, 1977. ERIC: ED 146 009.

Worksheets were designed for use with low-achieving ninth and tenth graders; they are also appropriate for seventh and eighth graders. Activities are included on: introduction to the calculator, games, exploring algorithms, pattern search, estimation and reinforcement of basic computations, consumer applications, and societal applications. Notes for teachers are included with each of the lessons.

[Activities, Low achievers, Secondary (grades 7-10), Worksheets]

* Iowa Problem-Solving Project (ESEA Title IV-C). (Joan Duea, George Immerzeel, Earl Ockenga, John Tarr, and Jack Wilkinson) Cedar Falls: University of Northern Iowa, 1978.

The purpose of this project is to develop materials to improve the mathematical problem-solving abilities (using calculators) of students in grades 5-8. The materials are available in classroom sets for experimentation: 1 teacher's edition with teacher's notes, record sheets, answers, and tests; 15 pupil copies of skill booklets (approximately 30 pages); and 100-card problem deck. Modules include: Problem Solving . . . Using the Calculator, Using Tables, Using Guesses, Using Resources, Using the Calculator (Book II), Using Calculator Codes (Books I and II).

[Elementary (grades 5-6), Junior high (grades 7-8), Problem solving, Units].

Jacobs, Russell F. Problem Solving with the Calculator. Phoenix: Jacobs Publishing Co., Inc. 1977.

This booklet includes calculator exercises for whole numbers, decimals and fractions, percent, and supplementary activities (magic squares, number tricks, polygonal numbers, square root, and repeating decimals).

[Activities, Elementary (grade 6), Junior high, Problem solving]

Jamski, William D. Compound Interest a la Simple Calculator. Mathematics Teacher 72: 359-362; May 1979.

Worksheets provide explorations of the relationship between simple and compound interest.

[Activities, Consumer applications, Worksheets]

* Johnson, David C. Calculators: Abuses and Uses. Mathematics Teaching 85: 50-56; December 1978.

Four common abuses of the calculator are noted: calculations for no apparent purpose other than to use the calculator; games and puzzles with no apparent mathematical objective; mystical button pushing; and checking answers. A variety of examples in which the calculator can be used effectively are extensively discussed.

[Activities, Pros/cons, Roles]

* Judd, Wallace. Games, Tricks, and Puzzles for a Hand Calculator. Menlo Park, California: Dymax, 1974.

* Judd, Wallace. Games Calculators Play. New York: Warner Books, 1975.

* Judd, W. Dogfight and More Games Calculators Play. New York: Warner Books, 1977.

Games and other activities with calculators are provided.

[Games]

* Judd, Wallace. Problem Solving Kit for Use with a Calculator. Chicago: Science Research Associates, 1977.

This classroom kit includes 12 problem-solving sections, each of which stresses a different problem-solving technique or area. In addition to the student workcards and record folders, the kit contains spirit masters for evaluating student achievement and a teacher's guide.

[Kit, Problem solving]

Keller, Clifton. Using Tables to Teach Mathematics. Mathematics Teacher 71: 655-656; November 1978.

Use of calculators is suggested as an aid in performing the calculations involved in pattern searches.

[Integers (reciprocals), Pattern searches, Secondary]

* Ladd, Norman E. Working with the Calculator in Beginning Algebra. Skokie, Illinois: National Textbook Co., 1976.

This workbook includes exercises on formulas, exponents, evaluating expressions, operations with integers and fractions, solving equations, prime factors, GCD and LCM, multiplying and factoring polynomials, linear equations, irrational numbers, Pythagorean Theorem, distance and midpoint formulas, and the quadratic formula. A teacher's guide with answer key is included.

[Algebra, Secondary, Worksheets]

Lichtenberg, Donovan R. Minicalculators and Repeating Decimals. Mathematics Teacher 71: 524-530; September 1978.

The calculator is used to determine repeating decimals; several theorems and examples are provided.

[Decimals]

McHale, Thomas J. and Witzke, Paul T. Calculation and Calculators. Reading, Massachusetts: Addison-Wesley Publishing Co., 1978.

[Activities]

McWhorter, Eugene W. The Small Electronic Calculator. Scientific American 234: 88-98; March 1976.

Details are given concerning the inner workings of the electronic calculator.

[Building]

Michelow, Jaime S. and Vogeli, Bruce R. The New World of Calculator Functions. School Science and Mathematics 78: 248-254; March 1978.

The importance of functions is discussed and specific illustrations of the calculator as a function generator are presented.

[Functions]

* Michigan. Uses of the Calculator in School Mathematics, K-12. Monograph No. 12. Lansing: Michigan Council of Teachers of Mathematics, March 1977.

Part I covers desirable features of calculators for classroom use, basic assumptions on classroom use, and curriculum concerns. Part II is a collection of activities, organized under the following categories: Grades K-3, Grades 4-8 (number-numeration, whole number basic operations, common fractions and decimal fractions, integers, and supplementary materials), and Grades 9-12 (pre-algebra and algebra, geometry through higher mathematics). A bibliography is included.

[Activities, Elementary (grades k-6), Roles, Secondary (grades 7-12)]

* Miller, Don. Calculator Explorations and Problems. New Rochelle, New York: Cuisenaire Company of America Inc., 1979.
(See also: Miller, Don. Calculator Explorations. St. Cloud, Minnesota: Don Miller, 1977.)

The activities in this book were designed to make the calculator an integral part of the mathematical ideas being explored. Many use the calculator to generate data which suggest interesting number patterns that lead to further discoveries and generalizations. Others use the calculator to provide immediate feedback to improve estimation and mental computational skills.

[Activities, Elementary (grades 4-6), Secondary (grades 7-12)]

Miller, William A. and Hazekamp, Donald W. Calculator Graphing. Mathematics Teacher 71: 759-762; December 1978.

Three worksheets provide experiences in using calculators to compute squares, reciprocals, and square roots, plus rounding numbers to the nearest hundredth, plotting points, and drawing graphs.

[Junior high (grades 7-9), Reciprocals, Roots, Secondary, Worksheets]

Mitchell, Charles E. Problem Solving & RPN Logic. The Computing Teacher 6: 35-36; May 1979.

A calculator using RPN logic is integrated into instruction on problem solving. Specific illustrations are included.

[Calculator logic, Junior high (grade 7), Problem solving]

* Morris, Janet Parker. Problem Solving with Calculators. Arithmetic Teacher 25: 24-26; April 1978.

Activities using four-function calculators are classified by purpose: to explore number patterns, to discover relationships and develop concepts, to practice mental estimation, to reinforce inverse relationships, to solve application problems, to develop the "guess-then-check" technique, and for individual exploration and enrichment.

[Activities, Four-function calculators, Problem solving]

NCTM. Minicalculators in Schools. Arithmetic Teacher 23: 72-74; January 1976. Mathematics Teacher 69: 92-94; January 1976.

This report from the NCTM Instructional Affairs Committee presents nine justifications for using the hand-held calculator in classrooms, with some specific examples of curricular applications.

[Activities, Recommendations, Roles]

* NCTM. Position Statements: Use of Minicalculators. Mathematics Teacher 71: 468; May 1978.

The 1974 NCTM Board of Directors' position statement on calculators is reprinted.

[Recommendations]

★ NCTM. Minicalculator Information Resources. Reston, Virginia: National Council of Teachers of Mathematics.

This free resource list cites articles, books, newsletters, and media, plus information on calculator models appropriate for school use. It is updated at intervals.

[References, Selection]

★ NCTM. Mathematics Teacher: New Products, Programs, Publications.

In this monthly feature of the journal, calculators and materials for use with calculators in the classroom are frequently reviewed.

[Selection]

Neufeld, K. Allen. Calculators in the Classroom. Monograph No. 5. Edmonton: Alberta Teachers' Association, 1977.

[Activities]

★ NIE/NSF. Report of the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics. National Institute of Education and National Science Foundation, 1977. ERIC: ED 139 665.

Results of the conference, intended to provide a well-defined framework for future research and development efforts, are reported. Twenty-one recommendations covering the development of an information base, curriculum development, research and evaluation, teacher education, and dissemination are discussed.

[Recommendations, Research (needed), Roles]

★ North Carolina. Calculator Inservice Kit. Prepared by Mathematics Division, North Carolina State Department of Public Instruction, 1978.

In this notebook kit are transparency masters, calculator games, calculator activity cards for grades 5 through 9, and calculator problem card decks for teachers.

[Elementary (grades 5-6), Junior high (grades 7-9), Kit, Problem solving, Teachers (in-service)]

★ Ockenga, Earl. Calculator Ideas for the Junior High Classroom. Arithmetic Teacher 23: 519-522; November 1976.

Activities and games for computation and estimation, measurement and geometry, functions, and problem solving and applications are described.

[Activities, Junior high]

★ Ockenga, Earl and Duea, Joan. Ideas. Arithmetic Teacher 27: 28-32; January 1979.

Games for addition, subtraction, multiplication, and division are presented; the objective is to provide practice in estimation.

[Elementary (grades 1-6), Estimation, Games, Junior high (grades 7-8), Mixed operations]

* Olson, Melfried. Using Calculators to Stimulate Conjectures and Algebraic Proofs. Mathematics Teacher 72: 288-289; April 1979.

  Three examples are given that can be used to stimulate conjectures and that lend themselves to an algebraic proof. The calculator can enable students to test cases which would have been fruitless, if not impossible, without the calculator.

  [Algebra]

Pagni, David L. Number Theory for Secondary Schools? Mathematics Teacher 72: 20-22; January 1979.

* Pollak, Henry O. Hand-Held Calculators and Potential Redesign of the School Mathematics Curriculum. Mathematics Teacher 70: 293-296; April 1977.

  The content and teaching of secondary-school mathematics in the calculator era are discussed.

  [Curriculum, Roles, Secondary]

Råde, Lennart and Kaufman, Burt A. Adventures with Your Hand Calculator. Stockholm, Sweden: Biblioteksforlaget, 1976. St. Louis, Missouri: CEMREL, Inc., 1977.

  Part I of this book contains 20 problems which can be solved or investigated with a calculator. Part II provides a commentary on each of these problems. Although solutions are not always provided, the commentaries often present new situations which extend the ideas of the original problem.

  [Activities, Mixed operations, Problem solving, Secondary]

* Rising, Gerald R.; Krist, Betty J.; and Roesch, Carl. Using Calculators in Mathematics. National Institute of Education Contract No. 400-78-0013. Buffalo: State University of New York at Buffalo, 1978. Xerox copy.

  This preliminary set of materials incorporates the use of programmable calculators in the standard mathematics curriculum in grades 11 and 12 (following the New York State syllabus). The first draft (chapters 1-8) is available for experimental use.

  [Course description, Programmable calculators, Secondary (grades 11-12]

Roberts, Edward M. Fingertip Math. Dallas: Texas Instruments, 1974.

  This book explains how to use a hand-held calculator effectively.

  [Activities]

Rouse, William. The Power Box: A Solution to Calculator Logistics. Mathematics Teacher 72: 516-517; October 1979.

* Rudolph, William B. and Claassen, A. D. The Calculator Book. Boston: Houghton Mifflin, 1976.

[Activities]

Russakoff, Andrew. Calculator Calculations. Mathematics Teacher 72: 188; March 1979.

Using the calculator for graphing the relations for equations like $x^a + y^a = 5^a$ is presented.

[Algebra, Secondary]

Schlossberg, E. and Brockman, J. The Pocket Calculator Game Book. New York: William Morrow, 1975.

Schlossberg, Edwin and Brockman, John. The Kid's Pocket Calculator Game Book. New York: William Morrow, 1977.

Schlossberg, Edwin and Brockman, John. The Pocket Calculator Game Book #2. New York: William Morrow, 1977.

[Activities, Elementary (grades 4-6), Games, Junior high (grades 7-8)]

* Schultz, James E. How Calculators Give Rise to a New Need for Skills in Algebra. School Science and Mathematics 78: 131-134; February 1978.

Illustrations are given of how doing certain algebraic manipulations prior to doing calculations can eliminate unnecessary storage of data, reduce the number of steps required to obtain an answer, and avoid data overflow.

[Algebra]

* Scott, Douglas E. Finding Roots with a Four-Function Calculator. Calculators/Computers 2: 77-81; January 1978.

The "repeat" capability of most four-function calculators is used in an algorithm to find any integral root of any number.

[Four-function calculators, Roots]

* Sharp, J. Norman C. The Calculator Workbox. Don Mills, Ontario: Addison-Wesley Publishers, 1977.

[Activities, Junior high (grades 7-8), Kit]

* Shumway, Richard J. Hand Calculators: Where Do You Stand? Arithmetic Teacher 23: 569-572; November 1976.

Arguments for and against the use of hand-held calculators in school mathematics are presented.

[Pros/cons]

Smith, Gerald R. Repeating Decimals. Calculators/Computers 2: 57-61; March 1978.

Determining repeating decimals is examined in this article, with 17ths, 19ths, 23rds, 31sts, and 41sts suggested for exploration.

[Decimals]

Smith, Susan. Calculating Order. Mathematics Teacher 71: 519-522; September 1978.

Worksheets provide a calculator experience developing and reinforcing the concept of order of operations, for students in grades 7-10.

[Mixed operations, Secondary (grades 7-10), Worksheets]

Snover, Stephen L. and Spikell, Mark A. Programmable Calculators Facilitate Simple Solutions to Mathematical Problems. January 1979. ERIC: ED 170 115.
Problems which can be used with calculators are presented.

[Problem solving, Programmable calculators]

* Snover, Stephen L. and Spikell, Mark A. The Role of Programmable Calculators and Computers in Mathematical Proofs. Mathematics Teacher 71: 745-750; December 1978.

Since numerous possibilities can be examined in a relatively short period of time, calculators and computers are powerful data-gathering instruments. How they can be used in proofs is explored with a simple problem from number theory.

[College, Programmable calculators, Proofs, Secondary]

* Snover, Stephen L. and Spikell, Mark A. Generally, How Do You Solve Equations? Mathematics Teacher 72: 326-336; May 1979.

Iterative techniques to solve various types of difficult problems are presented, with specific examples, flowcharts, and programs.

[Flow charts, Iteration, Numerical analysis, Programming, Programmable calculators, Secondary]

* Snover, Stephen L. and Spikell, Mark A. How to Program Your Programmable Calculator. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1979.

Procedures for programming calculators are explained in detail, with many specific examples for two types of calculator logic.

[Calculator logic, College, Programmable calculators, Secondary]

Stephen, J. Sydney and Trueman, Robert W. Calcu-Math. Buffalo, New York: Scholar's Choice, 1977. Stratford, Ontario: Scholar's Choice Limited, 1977.

This kit is a set of activity cards and worksheets designed to apply basic skills and stress estimation and problem solving while using a simple four-function calculator.

[Activities, Four-function calculators, Kit, Worksheets]

* Suydam, Marilyn N. Electronic Hand Calculators: The Implications for Pre-College Education. Final Report, Grant No. EPP 75-16157, National Science Foundation, February 1976. ERIC: ED 127 205 (full report); ED 127 206 (50-page form).

  Research is reported which was conducted to provide the NSF with information on the existing range of beliefs and opinions about the impact of the calculator on pre-college education. A literature search and several surveys of groups involved in calculator manufacture or in education were conducted. In addition, four position papers by prominent educators were obtained. The body of the report presents summaries of information collected relevant to availability of calculators, arguments for and against their use in schools, ways in which they are now used in schools, and research findings. Five recommendations related to study of and preparation for implementation of calculator use are made. Appendices to the report present an annotated list of references, the detailed findings of the surveys, and the position papers: Immerzeel et al. discuss plausible instances in which to use calculators; Pollak proposes criteria for redesigning the curriculum; Weaver makes suggestions for needed research; and Usiskin and Bell provide some perspectives on curriculum revision.

  [Curriculum, Pros/cons, Research (survey)]

Suydam, Marilyn N. Reference Bulletins from the Calculator Information Center. ERIC: ED 167 426.

Suydam, Marilyn N. (Editor). Information Bulletins from the Calculator Information Center. ERIC: ED 171 574.

Suydam, Marilyn N. State of the Art Reviews on the Use of Calculators. ERIC: ED 171 572.

Suydam, Marilyn N. Calculators: A Categorized Compilation of References. ERIC: ED 170 134. Supplement. ERIC: ED 171 585.

Suydam, Marilyn N.; Cummins, Kenneth; Gibney, Thomas; Hill, Johnny; Meiring, Steven; and Pikaart, Len. Ohio Regional Conferences on Mathematics Education: Final Report. (NSF-funded project.) Columbus: The Ohio State University, 1978. ERIC: ED 164 328.

  The report includes materials and problems developed for the conferences, designed for elementary school teachers and supervisors.

  [Activities, Four-function calculators, Problem solving]

* Teitelbaum, Eli. Calculators for Classroom Use? Arithmetic Teacher 26: 18-20; November 1978.

  Letting instruction focus on problem solving rather than computation alone is the point of this article. How calculators can be used to advantage is described through four illustrations, with comments on checking and on cheating.

  [Problem solving, Roles]

 L1.14

* Texas Instruments. The Great International Math on Keys Book. Dallas: Texas Instruments Learning Center, 1976.

Sections of this book cover the basic keys on a scientific calculator; how to use the calculator for converting units of measure; using the calculator for practical everyday problems, algebra, business and finance, trigonometry, probability and statistics, and physics and chemistry; and puzzles and games for the calculator.

[Activities, Scientific calculators, Secondary]

Texas Instruments. Understanding Calculator Math. Dallas: Texas Instruments, Inc., 1976.

This guide contains basic information, formulas, facts, and mathematical tools to "unleash the real power of your calculator."

[Activities]

Texas Instruments. Mathematics Learning with Calculators: Fundamental Mathematics. Dallas: Texas Instruments, Inc., 1977.

Supplementary materials in kit form are intended to teach certain mathematics concepts using the calculator. Signed numbers, algebraic logic, functions, graphing, statistics, and probability are included.

[Activities, Kit, Secondary, Units]

Texas Instruments. Mathematics Learning with Calculators: Introductory Algebra. Dallas: Texas Instruments, Inc., 1977.

Supplementary materials in kit form are intended to teach certain mathematics concepts using the calculator. Evaluating expressions, solving equations, signed numbers, formulas, and algebraic principles are included.

[Activities, Algebra, Kit, Secondary, Units]

Toth, Frank S., Jr. Calculator Experiments for Junior High. Calculators/Computers 2: 69-73; April 1978.

Three experiments used with seventh graders are presented: finding quotient and remainder, listing 10 non-zero multiples of a given number, and stating which of the numbers 2 to 20 are factors of a given number.

[Division, Junior high (grade 7), Multiplication]

Toth, Frank S., Jr. Calculator Experiments for Junior High (continued). Calculators/Computers 2: 69-73; May 1978.

This article, a continuation of one in the April 1978 issue, presents one experiment for determining the GCF and one for determining the LCM, using calculators.

[Division, Junior high (grade 7), Multiplication]

L1.15

Toth, Frank S., Jr. Calculator Experiments for Junior High. The Computing Teacher 6: 37-38; May 1979.

An experiment on prime factorization is presented; it could be used as as basic lesson plan with seventh graders.

[Activities, Junior high (grade 7), Units]

* Usiskin, Zalman. Are Calculators a Crutch? Mathematics Teacher 71: 412-413; May 1978.

That a crutch is a bad thing is questioned; use of the calculator as a useful crutch is illustrated.

[Roles]

* Vervoort, Gerardus and Mason, Dale. Calculator Activities for the Classroom and Teacher's Resource Book. Toronto: Copp-Clark Publishing Co., 1977.

A three-part series of ditto masters in which activities facilitate the use of calculators in mathematics. The series familiarizes students with the calculator as well as its use in problem solving. The teacher's guide provides a rationale.

[Activities, Problem solving, Worksheets]

Weaver, J. F. A Monadic Module Alias a Unary Unit. Project Paper 77-5. Madison: Wisconsin Research and Development Center for Cognitive Learning, University of Wisconsin-Madison, December 1977. ERIC: ED 161 670.
(See also: Weaver, J. F. A Monadic Module Alias a Unary Unit. Calculators/Computers 2: 29-36; April 1978.)

This module is excerpted from material prepared for ongoing calculator explorations with a class of accelerated seventh-grade students. Greater attention is given to unary or monadic operations. Relationships and properties to be identified, suggestions for the teacher, calculator algorithms, and record sheets are provided.

[Activities, Junior high (grade 7), Mixed operations, Units]

Weaver, J. F. Some Monadic/Dyadic Combos. Project Paper 77-6. Madison, Wisconsin: Wisconsin Research and Development Center for Cognitive Learning, December 1977. ERIC: ED 161 671.
(See also: Weaver, J. F. Some Monadic/Dyadic Combos. Calculators/Computers 2: 79-83; May 1978.)

This module is an extension of Project Paper 77-5 (and the article in the April issue of Calculators/Computers). It presents materials to extend students' understanding of algebraic operations with the use of calculators. Reference sheets, record sheets, and directions are included to aid students in focusing on monadic/dyadic operations.

[Activities, Junior high (grade 7), Mixed operations, Units]

 L1.16

Weaver, J. F. Calculators and Polynomial Evaluation. Madison: Wisconsin Research and Development Center for Individualized Schooling, The University of Wisconsin, July 1978. ERIC: ED 156 475.

The intent of this paper is to suggest and illustrate how hand-held calculators, especially nonprogrammable ones with limited data-storage capacity, can be used to advantage by students in one particular aspect of work with polynomial functions. The basic mathematical background upon which calculator application is built is summarized. Along with one calculator algorithm for polynomial evaluation that is relatively independent of calculator type, eight other algorithms that illustrate a variety of calculator types, characteristics, and features are given.

[Algebra, Algorithms]

* Woodburn, Douglas. Can You Predict the Repetend? Mathematics Teacher 69: 675-678; December 1976.

Worksheets are provided for an activity involving the use of the calculator in discovering patterns when a number is divided by 9, 99, and 999.

[Pattern searches, Secondary, Worksheets]

Woodward, Ernest and Hamel, Thomas. Calculator Lessons Involving Population, Inflation, and Energy. Mathematics Teacher 72: 450-457; September 1979.

Two lessons showing how a calculator can be used to help students discover the "rule of 72" and use it to investigate problems involving population, inflation, and energy are presented.

Yvon, Bernard R. and Downing, Davis R. Math Explorations with the Simple Calculator. Portland, Maine: J. Weston Walch, 1978.

This text, appropriate for the elementary and junior high school student, presents games, puzzles, problems, and exercises for the four-function calculator.

[Activities, Four-function calculator, Games]

* Calculators/Computers: The First Three Issues in Book Form. Menlo Park, California: Addison-Wesley, 1978.

[Activities]

CALCULATOR INFORMATION CENTER

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Information Bulletin No. 5
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LEADING A CALCULATOR WORKSHOP

Prepared by Jane Donnelly Gawronski

The use of calculators in the mathematics classroom is of great interest to school staffs and school community groups. Workshops and in-service programs on calculator use are often planned and conducted in response to this interest. If you are responsible for conducting or leading these activities, there are several things you can do to help to insure their success.

BEFORE THE WORKSHOP

First of all, select appropriate objectives for the workshop. These are not really intended as rigorous behavioral objectives for the workshop participants to master or to be pretested and posttested, but rather these objectives are to help you in selecting activities and materials for workshop participants to use. These objectives may include:

A) To have the participants become familiar with how to use a calculator.

B) To demonstrate calculator activities for specific topics, such as computational skills, geometry, order of operations, and problem solving.

C) To identify how to use the calculator in the classroom: management, storage, maintenance, record keeping, etc.

D) To become familiar with how to plan and conduct in-service programs on calculator use.

Once you've selected your objectives, you can consider ideas, activities, and materials that will help you to accomplish them. Fortunately, there are many books, pamphlets, and task cards commercially available, as well as articles in the literature. See the Calculator Information Center Reference Bulletins for comprehensive lists of these resources.

GETTING STARTED

Share your workshop objectives with the participants. This is helpful for the participants and lets them know what to expect to have an opportunity to learn. It is critical, however, to begin with activities that are non-threatening, low-risk for participants and that have a high potential for success. The calculator is still recent technology and must be introduced to many teachers. It's the children who are growing up with this technology in the world around them who accept it as a matter of course. Adults may sometimes be reluctant or inhibited. Some of the letter-number activities are appropriate as introductory vehicles for adults. For example:

1. Did you know your calculator was bilingual?

   Key in 15, and turn your calculator upside down.

Or, 2. Did you know your calculator can display the name of a best seller?

   Key in 37818, and turn your calculator upside down.

Activities like this one can give the novice calculator user some non-threatening practice with keying in numbers, reading calculator numerals, and checking the display. A quick introduction to addition, subtraction, multiplication, and division can follow immediately.

INTRODUCING THE KEYS AND ACTIVITIES

When an operation such as addition is introduced, "how to key in" addition examples should be discussed. The need for using the CLEAR and/or CLEAR/CLEAR ENTRY keys before entering a problem should be demonstrated. It is also helpful to start with an example for which participants already know the answer. This way they can verify for themselves that they are using the calculator correctly. An example such as the following can be used:

Example: $7 + 8 = ?$

Calculator Solution: $\boxed{C}$ 7 $\boxed{+}$ 8 $\boxed{=}$ DISPLAY 15

After introducing how to add using a calculator, lead participants in problem solving or exploratory activity requiring addition. For example:

1. Make your calculator display 895 using only the 1, 0, $\boxed{+}$, and $\boxed{=}$ keys.

Or, 2. How many different ways can you find to express 763 as the sum of primes?

The grade level interests and mathematical backgrounds of the workshop participants should help you to select appropriate activities for this.

This same general procedure can be used to introduce the operations of subtraction, multiplication, and division, as well as how to use special keys such as % or √. First, introduce the use of the calculator with a problem participants are apt to know the answer to (you may want to do more than one of these). Second, follow this with an activity where the use of the calculator is obviously an important asset for finding the solution.

Additional ideas and activities should be selected so they illustrate your objectives. A review of the materials cited in Calculator Information Center Reference Bulletins on "Instruction with Calculators" and "Books" will provide you with a variety of place value, computational, and problem-solving activities to use.

SUPPLIES

Participants should be provided with a copy of appropriate visuals as well as a "handout" for them to use. This way they can closely follow your presentation and practice or try out the ideas and activities.

It is also desirable if participants have their own individual calculators. If not, a calculator can be shared by two or more people. In fact, you may want to use activities that are designed so only one calculator is required for a group of two to four participants. However, if you have access to a classroom supply of calculators, by all means use them. But do not be surprised if participants bring their own calculators to use. This is all right and, in fact, may even be better. If they know how to use their own calculator and all its keys, then they are probably more apt to make effective use of the calculator in the classroom. In addition, this can provide a bit of "show and tell" atmosphere and more participant involvement in the workshop. Also, it provides an opportunity to identify characteristics of different calculators, since some calculators have Liquid Crystal Displays (LCD) and others will have Light Emitting Diode (LED) displays. Some calculators have special keys that do not appear on most simple four-function calculators. If your workshop objectives include one on calculator technology, you may want to go into some detail and indicate advantages and disadvantages of particular features of calculators.

ENDING YOUR WORKSHOP

At the conclusion of your workshop you might want to give participants a "Certificate of Competence with the Calculator" or a "License to Operate a Calculator." This is particularly appropriate for participants who have come to their first calculator workshop. Remember, too, that throughout the workshop you have been a model for how to introduce or use a calculator appropriately. The certificate is another idea that participants may want to use in the classroom. At this point you should also provide sources for more information or ideas about calculator use. These might include:

1. How to get on the Calculator Information Center mailing list.

2. Announcement of additional workshops or National Council of Teachers of Mathematics and Affiliated Group services.

Participants should also have an opportunity to evaluate the workshop. The following is a sample evaluation form you may want to use:

SAMPLE EVALUATION FORM

Topic/Title Speaker/Consultant Date

(Circle appropriate response)

1. Were the objectives for this session clear?	Clear	4	3	2	1	Vague
2. To what extent were the objectives met?	Fully	4	3	2	1	Little
3. How well was the presentation organized?	Very Well	4	3	2	1	Very Poor

 L2.04

4. How helpful do you think the presentation will be to your work? Very Much 4 3 2 1 Very Little

5. Did the physical arrangements help or hinder the meeting of the objectives? Help 4 3 2 1 Hinder

6. Were the related media appropriate? Very 4 3 2 1 Inappropriate

7. Was there enough time allowed to meet the objectives? Sufficient 4 3 2 1 Insufficient

RECOMMENDATIONS/CONCERNS/REMARKS

What I found most useful was:

What I would like more of is:

Additional remarks:

AFTER THE WORKSHOP

Read the evaluation form and compile the results. Use these to plan your next workshop! Modify or make changes and incorporate the evaluation results where appropriate. And GOOD LUCK!

The following checklist may help you in your planning.

Calculator Workshop Checklist

1. Invitation to potential workshop participants
2. Pleasant room environment -- refreshments help
3. Calculators
4. Handouts
5. Visuals
6. Media
7. Display or sample copies of materials available
8. Evaluation forms

This bulletin was prepared by Jane Donnelly Gawronski, Department of Education, San Diego County, San Diego, California.

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August 1980

Using Calculators in Pre-College Education: Third Annual State-of-the-Art Review*

Marilyn N. Suydam

Over the past several years, the cost of calculators has declined to a relatively stable level. Concurrently, calculator availability has become less and less an issue. Technology has provided prolonged battery life, and some calculators are so small and light they can be carried or worn easily, nullifying additional arguments about their availability. While resistance to their use in schools is still apparent, awareness of potential instructional applications has slowly continued to increase. Heightening this awareness is a significant recommendation from a national association.

A Recommendation for the 1980s

In April 1980, the National Council of Teachers of Mathematics released An Agenda for Action: Recommendations for School Mathematics of the 1980s. One of the eight recommendations addresses concerns presented by computing technology: "Mathematics programs must take full advantage of the power of calculators and computers at all grade levels." The introductory comments present a rationale for this stance:

> Beyond an acquaintance with the role of computers and calculators in society, most students must obtain a working knowledge of how to use them . . . .
>
> The availability of computing aids, including computers and calculators, requires a reexamination of the computational skills needed by every citizen. Some of these computational skills will no longer retain their same importance, whereas others will become more important.
>
> It is recognized that a significant portion of instruction in the early grades must be devoted to the direct acquisition of number

* The first annual review was prepared in April 1978; the second appeared in May 1979.

concepts and skills without the use of calculators. However, when the burden of lengthy computations outweighs the educational contribution of the process, the calculator should become readily available. (p. 8)

Recommended actions to accomplish the goal include the following:

3.1 All students should have access to calculators and increasingly to computers throughout their school mathematics program. (p. 9)

3.2 The use of electronic tools such as calculators and computers should be integrated into the core mathematics curriculum. (p. 9)

- Calculators should be available for appropriate use in all mathematics classrooms, and instructional objectives should include the ability to determine sensible and appropriate uses. (p. 9)

- Calculators and computers should be used in imaginative ways for exploring, discovering, and developing mathematical concepts and not merely for checking computational values or for drill and practice. (p. 9)

3.3 Curriculum materials that integrate and require the use of the calculator and computer in diverse and imaginative ways should be developed and made available. (p. 9)

- Schools should insist that materials truly take full advantage of the immense and vastly diverse potential of the new media . . . (p. 9)

- Educators should take care to choose software that fits the goals or objectives of the program and not twist the goals and developmental sequence to fit the technology and available software. (p. 9)

Teachers of other subjects in which mathematics is applied "should make appropriate use of calculators and computers" (p. 11). Furthermore, teachers and administrators are urged to "initiate interaction with the home to achieve maximum benefit to the student from coordinated home and school use of computers and calculators" (p. 10).

Other recommended actions address the needs of teachers, pointing out that colleges need to offer courses on instructional uses of calculators for both preservice and in-service teachers and that certification standards should require such preparation. Professional organizations should provide information through media and meetings of various types.

Thus, the NCTM acknowledges that computational skills are still necessary, but stresses the need to integrate calculator use at all levels, reinforces their usefulness in problem solving, notes the need for imaginative materials, and emphasizes the key component of teacher education.

Evidence on Availability and Uses of Calculators

The NCTM recommendation accepts the reality of the existence of calculators and computers. Data from the Second Mathematics Assessment of the National Assessment of Educational Progress (reported in Reys et al., 1980) support the fact that many children have access to calculators outside of the classroom: 75% of 9-year-olds, 80% of 13-year-olds, and 85% of 17-year-olds either own their own calculators or have one available to use. Other studies indicate that in some locations this percentage may be even higher; for instance, over 90% of the 220 households surveyed in Florida had at least one calculator (Conner, 1980), and in Indiana a survey of 417 students indicated that ownership or access ranged from 79% for first graders to 100% for sixth graders (Ewbank, 1979). Naturally, however, some studies report lower percentages; for example, only 68% of the Missouri children queried by Reys et al. (1980) had access to calculators.

Data from the many studies* still seeking an answer to the question, "Does use of calculators hurt achievement scores?", continue to support the fact that students who use calculators for instruction achieve at least as high or higher scores than students not using calculators, even though the calculator is not used on the test. (In the majority of studies during the past year, no significant differences were reported.) The decrease in time spent on paper-and-pencil practice did not appear to harm the achievement of students who used calculators.

Data from studies on learning mathematics with calculators, as well as evidence from the practical experiences of teachers, are slowly accumulating, indi-

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* This type of study on achievement comprises about two-thirds of all studies reported. Studies focusing on the development of specific mathematical ideas account for about one-sixth of the studies, while the remainder are surveys. While doctoral students continue to produce at least 50% of the research, ongoing investigations are being conducted by researchers in schools and colleges.

cating that calculators are useful in teaching a variety of mathematical ideas. Reports from Conner (1980) and Moser (1979), for instance, detail some specific ways in which calculators are useful instructional tools.

Surveys on Beliefs and Attitudes

When beliefs and attitudes are surveyed, however, it becomes obvious that many persons ignore the evidence from research on achievement and learning. Perceptions of the uses and importance of calculators in the mathematics curriculum depend primarily on the audience surveyed. The Priorities in School Mathematics Project (PRISM), conducted in 1979, devoted about 20% of its items to ascertain ways in which educators at all levels from primary through college, parents, and school board members feel about the use of calculators. Educators were much more supportive of increased use of calculators than were lay persons: 54% of the professional samples but only 36% of the lay samples would increase emphasis on them during the 1980s. Strongest support came from supervisors and teacher educators (85% and 74%, respectively); teachers at all levels had more reservationa (support averaged 50%); and parents and school board members gave weak support to increased emphasis -- and to almost all uses of calculators except checking answers. The percentage agreeing with various uses of calculators were:

	Professional Samples	Lay Samples
checking answers	93%	89%
doing a chain of calculations	89%	-
computing area	78%	-
making graphs	71%	-
solving word problems	70%	38%
solving equations	70%	-
learning why algorithms work	68%	-
doing homework	66%	37%
developing ideas and concepts	59%	49%
learning basic facts	51%	-
taking a test	50%	22%

Over 70% of the teachers at all levels endorsed having four-function calculators available. However, 67% of the professional samples and 88% of the lay samples believe that calculator use should be postponed until after paper-and-pencil algorithms are learned. Only 40% of the professional samples and 19% of the lay samples would let slower students use calculators, and putting students who have not learned paper-and-pencil computation by grade 8 into a calculator course was supported by only 34% (45% of the professional samples and 30% of the lay samples).

Other studies provide data which both compare and contrast with the PRISM data. Cohen and Fliess (1979) reported that over 63% of the teachers they queried favored the use of calculators. In a survey conducted in 1979, Reys and some colleagues interviewed a random sample of 194 classroom teachers in ten school districts in Missouri. The researchers reported that:

> The overwhelming feeling was that calculators exist, that there are many appropriate places for using them at all levels of the mathematics curriculum, and that the type and extent of this usage should be left up to the discretion of the individual classroom teacher. (Reys et al., 1980, p. 41)

While 84% of the teachers said that calculators should be available to children in school, only 35% had actually used calculators in mathematics classes (the data ranged from 14% at the primary level to 62% at the senior high level). Another 42% said they would like to use calculators. Teachers who had used them commented that:

> not only could they work more problems if a calculator was available, but also they actually covered more topics. They also reported dealing more with concept development and less with computation during their mathematics class. (Reys et al., 1980, p. 41)

It was also reported that

> most of the teachers who had not used a calculator in the classroom seemed aware of primarily two uses. One was as a computational device which they saw as defeating the major goals of school mathematics and the other as a tool for students to check the paper-and-pencil computations. . . . the majority of the teachers were unaware of the instructional potential of the calculator. (Wyatt et al., 1979, p. 218)

An average of 80% of the teachers felt children should master the four basic arithmetic operations before using calculators. (Interestingly, 76% of the primary teachers held this view; while 89% of the senior high school teachers did.) Indeed, 43% felt that using a calculator would cause students' ability to compute to decline. Teachers generally agreed, however, that slow students or senior high students who had never learned to compute should use a calculator because they would probably never be able to compute otherwise.

Slightly over 50% of the teachers wanted textbooks with activities using calculators. Forty-three percent favored use of calculators on problem-solving portions of standardized tests.

Four implications were drawn from the study (Wyatt et al., 1979):

(1) There is a need for leadership and direction for teachers regarding calculator use in schools.

(2) Training in the use of calculators as an instructional tool is needed.

(3) Dissemination of current information about calculator usage is needed to dispel many false conceptions.

(4) Materials which integrate calculators into the regular mathematics curriculum should be developed and disseminated.

As part of an investigation in which calculators were used in elementary school mathematics instruction, Conner (1980) surveyed parents of children in kindergarten through grade 5. Percentages favoring what she called "unrestricted" use of calculators as an instructional aid ranged from 13% for the elementary level and 16% for the middle school level to 29% for the high school level. When she asked about "regulated" use, the percentages rose to 83% for the elementary level, 80% for the middle school level, and 81% for the high school level.

Balka (1979) also found that parents were skeptical about the use of calculators in elementary grades. They agreed that calculators could be used along with paper-and-pencil computation, but strongly objected to using calculators in place of paper-and-pencil computation.

Successful integration of calculator uses in the mathematics curriculum will require careful and thorough communication among all concerned groups. Efforts to provide information on how calculators can be used successfully in teaching mathematics without harm to achievement must continue. And parents and other members of the public must receive assurance that necessary computational skills will still be taught. This point is clearly made in the NCTM Agenda for Action.

Development of Instructional Materials

Materials which integrate the use of calculators to teach mathematical ideas are still comparatively scarce. Most of the published articles, however, do present ideas for using calculators to promote learning on specific topics, including work with operations, functions, exponents, polynomials, square roots, and problem solving. There appears to be a decrease in the number of books focused solely on games, and an increase in the number of books which could be used to supplement on-going instruction.

Two compilations of materials may prove useful to teachers. One is a collection of articles from the Arithmetic Teacher and the Mathematics Teacher (Burt, 1979); the other is a categorized listing of references on calculators (Suydam, 1979). As has been true ever since calculators appeared in schools, however, there is a continuing need for materials which develop mathematical ideas.

Concluding Comment

While support from some groups for the use of calculators in schools is low, it is nevertheless changing as people accept the existence of calculators in their lives and in their children's lives. Concern continues to revolve around the issue of when the calculator should be used in relation to instruction on basic facts and algorithms: there is fear that paper-and-pencil computational skills will be lost and achievement scores will decline, despite the continuing reassuring research evidence on this point. Educators need to consider carefully ways of assuring parents that calculators can be used in developing a wide range of mathematical ideas which will promote mathematical achievement.

References

An Agenda for Action: Recommendations for School Mathematics of the 1980's. Reston, Virginia: National Council of Teachers of Mathematics, 1980.

Balka, Don A. A Survey of Parents' Attitudes Toward Calculator Usage in Elementary Schools. South Bend, Indiana: University of Notre Dame, 1979.

Burt, Bruce, C. (editor). Calculators: Readings from the Arithmetic Teacher and the Mathematics Teacher. Reston, Virginia: National Council of Teachers of Mathematics, 1979.

Cohen, Martin P. and Fliess, Robert F. Minicalculators and Instructional Impact: A Teacher Survey. Pittsburgh: University of Pittsburgh, 1979. ERIC: ED 178 360.

Conner, Totsye J. An Investigation of the Use of Hand-Held Calculators by Students in Elementary School. Research Monograph #32. Gainesville, Florida: University of Florida, Winter 1980.

Ewbank, W. A. Results of a Survey Carried Out in October, 1979, of Children in Grades 1-6, Upland Elementary and Middle Schools and Matthews Elementary School (Eastbrook Community Schools, Indiana) Relating to the Ownership and Access to a Pocket Calculator. Xerox copy, November 1979.

Moser, James M. The Effect of Calculator Supplemented Instruction Upon the Arithmetic Achievement of Second and Third Graders. Technical Report No. 502. Madison: Wisconsin Research and Development Center for Individualized Schooling, September 1979. ERIC: ED 180 764.

Priorities in School Mathematics (PRISM). Final Report, April 1980. ERIC: SE 030 577, SE 030 578.

Reys, Robert E.; Bestgen, Barbara J.; Rybolt, James F.; and Wyatt, J. Wendell. Hand Calculators: What's Happening in Schools Today? Arithmetic Teacher 27: 38-43; February 1980.

Suydam, Marilyn N. Calculators: A Categorized Compilation of References. Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, June 1979. ERIC: ED 171 152.

Wyatt, J. Wendell; Rybolt, James F.; Reys, Robert E.; and Bestgen, Barbara J. The Status of the Hand-Held Calculator in School--Implications for Parents, Teachers and Administrators. Phi Delta Kappan 61: 217-218; November 1979.

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Reference Bulletin No. 23
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References on Instructional Activities, Research Reports, and Other Topics Related to Calculator Use, K-12

In this bulletin, references are grouped to aid you in locating materials which you might find particularly useful.

ACTIVITIES FOR STUDENTS

Aviv, Cherie Adler. Pattern Gazing. Mathematics Teacher 72: 39-43; January 1979. Worksheets on pattern searches are provided.

Billings, Karen and Moursund, David. Problem Solving. Calculators/Computers 2: 73-94; November-December 1978. Chapter 2 from Problem Solving with Calculators is given. How to tackle a variety of problems is explained, using a worksheet-like format.

Dubisch, Roy and Hood, Vernon R. Basic Mathematics with Hand-held Calculator: A Work-Text. Menlo Park, California: Benjamin/Cummings Publishing Company, 1979. This worktext is for students at either secondary or college level.

Goodson, Bobby. Let's Put the Calculators Where They Belong. Calculators/Computers 2: 20-22; November-December 1978. Use of the calculator in work with square roots, geometry, factoring, and other junior high topics is discussed. A worksheet on finding a square root and a crossword puzzle developed by a student are given.

Goodson, Bobby. Calculator Resource Review. Calculators/Computers 2: 32-33; November-December 1978. Some books of calculator activities which the author considers particularly useful for junior high classes are cited.

Hiatt, Arthur A. Basic Skills: What Are They? Mathematics Teacher 72: 141-144; February 1979. With the calculator, students' mathematical thinking should be improved as they handle non-routine problems in which a number of accurate calculations are necessary. Specific examples involving use of the calculator in applying process skills are given.

Jurgensen, Marlinde. Measuring Triangles. Calculators/Computers 2: 50-52; November-December 1978. A teaching plan using calculators with the Pythagorean Theorem is presented, with a worksheet on measuring triangles.

Keller, Clifton. Using Tables to Teach Mathematics. Mathematics Teacher 71: 655-656; November 1978. Use of calculators is suggested as an aid in performing the calculations involved in pattern searches.

Kerr, Steven D. Who is Quickest with Mind and Fingers? Mathematics Teacher 72: 123-126; February 1979. A "thinking person's calculator race" is described, with sample questions.

Ockenga, Earl and Duea, Joan. Ideas. Arithmetic Teacher 26: 28-32; January 1979. Games for addition, subtraction, multiplication, and division are presented, with the objective of providing practice in estimation.

Olson, Melfried. Using Calculators to Stimulate Conjectures and Algebraic Proofs. Mathematics Teacher 72: 288-289; April 1979. Three examples are given that can be used to stimulate conjectures and that lend themselves to an algebraic proof. The calculator can enable students to test cases which would have been fruitless, if not impossible, without the calculator.

Reys, Robert E. et al. Keystrokes: Calculator Activities for Young Students: Addition and Subtraction. Palo Alto, California: Creative Publications, Inc., 1979. This workbook contains a variety of activities in which calculator use is integrated with mathematical instruction.

Russakoff, Andrew. Calculator Calculations. Mathematics Teacher 72: 188; March 1979. Using the calculator for graphing the relations for equations like $x^a + y^a = 5^a$ is presented.

Scott, Douglas E. What Day of the Week . . .? Calculators/Computers 2: 26-29; November-December 1978. An algorithm for determining the day of the week for any date is given for calculators (a BASIC program for computers is also included).

Snover, Stephen L. and Spikell, Mark A. How to Program Your Programmable Calculator. Englewood Cliffs, New Jersey: Prentice-Hall, 1979. Procedures for programming calculators are explained in detail, with many specific examples, for two types of calculator logic.

Wavrik, John J. A Short Presentation in "Computer Literacy" using Programmable Calculators. Calculators/Computers 2: 9-11; November-December 1978. This article outlines a one-lesson "computer literacy" presentation for grades 5 and 6 using the HP-25 programmable calculator. Following a ten-minute introduction in which nine points are cited, students are given experience entering and running programs.

Wavrik, John J. Programmable Calculators for Elementary Schools. Calculators/Computers 2: 53-55; November-December 1978. A lesson on simple programming for students in grades 4-6 is presented, with several programs included.

MORE MATERIALS . . . FROM PROJECTS (* indicates contact person)

Columbus Calculator Project (ESEA Title IV-C). Catherine Gilbreath, James Huber*, and Alberta Myers, Columbus (Ohio) Public Schools, 1978. Modules for each of these topics have been developed: Know Your Calculator, Basic Facts, Place Value, Decimal Computation, Properties, Rounding, Estimation, and Applications.

Calculating Devices. Don Fritz, L.B.J. High School, Austin, Texas, 1975. This is a one-quarter course for remedial ninth-grade students who have not mastered the fundamental operations. The course has been in the curriculum for five years.

Iowa Problem-Solving Project (ESEA Title IV-C). Joan Duea, George Immerzeel*, Earl Ockenga, John Tarr, and Jack Wilkinson, Price Laboratory School, University of Northern Iowa, Cedar Falls, 1978. This project focused on developing materials to improve the mathematical problem-solving abilities of students in grades 5-8, using calculators. The materials are available in classroom sets for experimentation.

Kolpas, Sidney J. The Use of Electronic Calculators as In-Class Instructional Aids in a Ninth-Grade Arithmetic Program. Unpublished doctoral dissertation, University of Southern California, 1978. Eighty ninth-grade low- and underachievers in four arithmetic classes participated in this 20-week study. The two experimental classes used calculators as instructional aids and as the sole solution-source for 50% or for 75% of each in-class assignment, while control classes (from another school) did not use calculators. The Comprehensive Test of Basic Skills was given as the pre- and posttest, and a researcher-constructed test was completed twice (first by hand, then with calculators to correct). No significant differences were found on the standardized test; significant differences on the researcher's test favored the experimental groups both with and without calculators.

Lunder, Dennis A. The Impact of the Calculator on Test Anxiety for High School Consumer Mathematics Students. Unpublished doctoral dissertation, University of Denver, 1978. Consumer mathematics students (n = 159) in grades 9-12 in three schools were randomly scheduled into two experimental groups (using calculators) or two control groups (not using calculators). One experimental and one control group used calculators on the posttest, while the other two groups did not. Using calculators for instruction and testing resulted in higher scores than not using calculators for instruction but using them on the test. Test anxiety was also explored; use of the calculator did not significantly reduce anxiety.

Mason, Marguerite. The Hand-held Calculator in the Elementary School, An Exploratory Study of Two Issues: Dependency and the Effect on the Problem-solving Process. In Research Reporting Sessions, NCTM 57th Annual Meeting. Columbus: ERIC/SMEAC, March 1979. Pp. 49-51. Two pairs of students, matched on mathematics pretests, were chosen at each grade level from 2-6. One student from each pair was from an experimental class which used calculators, and the other from the control class which did not use calculators. Each student was given a set of computational exercises and five word problems to solve in once-a-month interviews. All students had a calculator available. The mode used to solve exercises and problems was recorded. There was no evidence that students became calculator dependent. Follow-up tests showed that it was advantageous to have calculators available for certain types of exercises (such as division for third graders).

Moser, James M. The Effect of Calculator Supplemented Instruction on the Arithmetic Achievement of Second and Third Graders. In Research Reporting Sessions, NCTM 57th Annual Meeting. Columbus: ERIC/SMEAC, March 1979. Pp. 29-31. Participants were children in grades 2 and 3 in eight classes in two schools. One school was implementing a traditional curriculum and the other used the DMP program. The calculator was used with the ongoing programs; teachers were given suggestions on how the calculator could be used, and kept logs which were validated by unannounced visitations. Amount of use of calculators ranged from 36% to 79% of instructional days, and from 16% to 48% of instructional minutes.

Shin, Joseph. A Survey on the Attitude of Schoolchildren Towards the Use of Calculators in Schools. Calculators/Computers 2: 39-41; November-December 1978. A questionnaire given to four classes (n = 152) of fourth formers in a Hong Kong middle school is presented. More than 77% of the children had some experience using calculators; 60% of them or their families owned a calculator. Over 76% thought that calculators should be used on tests. Other findings are also given, and the value of calculator use is discussed.

Rising, Gerald R.*; Krist, Betty J.; Roesch, Carl; and Jewell, Wallace. Using Calculators in Mathematics. National Institute of Education Contract No. 400-78-0013. State University of New York at Buffalo, 1978. This preliminary set of materials incorporates the use of programmable calculators in the standard mathematics curriculum in grades 11 and 12 (following the New York State syllabus). Chapters 1 through 7 are available to those especially interested in trying them out.

Calculator Inservice Kit. Prepared by Mathematics Division, North Carolina State Department of Public Instruction, 1978. In this notebook kit are transparency masters, calculator games, calculator activity cards for grades 5 through 9, and a calculator problem card deck for teachers. (For information, contact George Immerzeel, Iowa Problem-Solving Project, page 2.)

RESEARCH REPORTS (WHICH MAY ALSO CONTAIN IDEAS FOR INSTRUCTION)

Blume, Glendon W. The Calculator's Effect on Children's Solution Processes. In Research Reporting Sessions, NCTM 57th Annual Meeting. Columbus: ERIC/SMEAC, March 1979. Pp. 59-61. In this exploratory study, the 21 seventh-grade pre-algebra students tended to use different solution methods when working with calculators than when restricted to use of paper and pencil.

Downes, John P.; Jensen, Rosalie S.; and Johnson, Hiram D. 76 Questions: A Synthesis of the Research on Teaching and Learning Mathematics. Atlanta: Georgia Department of Education, December 1977. ERIC: SE 025 422. One brief section considers research on calculators.

Gallery, Michael Edward. Teaching Calculator Use and Checking Account Skills to the Mildly Handicapped. (Utah State University, 1978.) Dissertation Abstracts International 39A: 2866; November 1978. Two "packages" on calculator skills and on checking account skills were evaluated with 38 students from six high schools and two junior high schools. Eighteen students received instruction on the packages, while 20 students were in the control group. The difference between group posttest means for both packages significantly favored the experimental group. However, the mastery criterion was not achieved on either package.

Jamski, William D. The Effect of Calculators on Achievement. MATYC Journal 13: 52-56; Winter 1979. This is a report of Jamski's dissertation research (completed in 1976) with seventh graders. He concludes that the calculator appears likely to be successful with some topics, such as fraction-decimal conversion, but not with others.

Jewell, Wallace F., Jr. Hand Calculators in Secondary Education: Evaluation, Analysis and Direction. Unpublished doctoral dissertation, State University of New York at Buffalo, 1979. The purpose of this study was to analyze and evaluate present hand-held calculator uses and materials and propose possible directions for their utilization in the near future. Also included is an analysis of the usual high school mathematics curriculum of elementary algebra, geometry, intermediate algebra-trigonometry, and elementary functions with regard to calculator implementation.

Kobrin, Beverly. The Hand-held Calculator: Effects on Intermediate Grade Mathematics Achievement. (Brigham Young University, 1978.) Dissertation Abstracts International 39A: 3354; December 1978. In grades 4-6, the experimental group (n = 75) used calculators for a year at least one hour per week on selected calculator activities, while the control group (n = 141) did not use calculators. No significant differences in achievement were found between the two groups.

Suydam, Marilyn N. (Editor). Investigations with Calculators: Abstracts and Critical Analyses of Research. Columbus: Calculator Information Center, January 1979. This document contains expanded abstracts of 36 research reports in which the use of calculators was explored, with comments on each report by the abstractor.

Szetela, Walter. Hand-Held Calculators and the Learning of Trigonometric Ratios. Journal for Research in Mathematics Education 10: 111-118; March 1979. The investigator and the regular teacher used specially designed materials for teaching trigonometric ratios to 131 students in grades 9 and 10 randomly assigned to groups using calculator-based instruction or not using calculators for 13 days. Students built their own short trigonometric tables using measurements of specially constructed right triangles. On a quiz one week before the final test, the calculator groups scored significantly higher than the non-calculator groups. There were no significant differences on the final test or in attitude toward learning ratios.

Ward, Dennis Elliott. The Effect of the Electronic Calculator on Problem-Solving Achievement and Attitudes Toward Mathematics of General Mathematics Students. (University of Southern California, 1978.) Dissertation Abstracts International 39A: 4038; January 1979. In the first semester of 1977-78, 92 general mathematics students were randomly assigned to two experimental classes (using calculators) and two control classes (not using calculators). No significant differences in problem-solving achievement or attitudes were found; no gender effect was found. Better readers tended to score higher on problem solving both with and without calculators.

Wheatley, Charlotte L. The Effect of Calculator Use on the Problem Solving Strategies of Elementary School Pupils. In Research Reporting Sessions, NCTM 57th Annual Meeting. Columbus: ERIC/SMEAC, March 1979. Pp. 25-27. Fifty fifth graders were involved in this study; 25 were from classes that had used calculators for three months, while the other 25 were randomly selected from two classes with no previous in-class calculator experience. Each student responded to five selected problems during 40-minute sessions which were tape-recorded, transcribed, and coded. The coding data were tallied to obtain frequencies of strategy use and performance scores. The two groups did not differ significantly on the range of processes used, time to solution, or performance.

Williams, David E. The Effect of the Use of the Mini Calculator and an Associated Curriculum Supplement on Computational Skills and Attitudes Toward Arithmetic of Ninth-Grade Non-College Bound Students. Unpublished doctoral dissertation, Temple University, 1978. This year-long study was conducted in five junior high schools in Philadelphia, with three teachers each teaching one class using calculators with the regular general mathematics curriculum, one class using the regular curriculum with calculators and supplementary calculator materials, and a comparison class not using calculators. Calculators were not used on the pre- or posttest (California Achievement Test). No significant differences were found in computational skill, while attitude differences favored the calculator groups.

Williams, S. Irene and Jones, Chancey O. A Survey of the Use of Hand Held Calculators in Advanced Placement Calculus Courses. Princeton, New Jersey: Educational Testing Service, 1979. The questionnaire was sent to mathematics department chairpersons at the 2042 secondary schools that had five or more students taking an Advanced Placement examination in May 1977; returns were received from 1547 schools, with 1403 scorable. About 89% indicated they permitted some use of calculators in calculus courses. Slightly more than 10% had modified their courses to include special techniques particularly suited to the use of the calculator. In only 20% of the schools were calculators supplied by the school districts. An overwhelming majority think that student performance (on AP examinations) would not significantly be affected if calculator use was permitted.

MISCELLANEOUS ISSUES AND CONCERNS

D'Ambrosio, Ubiratan. Issues Arising on the Use of Hand-held Calculators in Schools. International Journal of Mathematical Education in Science and Technology 9: 383-388; 1978. The use of calculators is considered from pedagogical and sociological viewpoints. The effects of adoption of calculators in developing countries and for deprived minorities is noted and the role of calculators in problem solving is discussed.

Free, John. Pocket Calculators for More Than Math. Popular Science 214: 22ff; April 1979. Some features and some weaknesses of liquid display calculators are discussed.

Huff, Darrell. Calcu-Letter. Popular Science 214: 58; January 1979; 214: 23; March 1979. This bi-monthly column continues to present problems and discuss points related to calculators.

Usiskin, Zalman P. The Future of Fractions. Arithmetic Teacher 27: 18-20; January 1979. The effect of calculator use on the teaching of fractions is among the factors discussed.

Calculators. Business Education Forum 33: 28-31; January 1979. Various types of calculators available currently are described briefly.

Flow Charts and Calculators. In General High School Mathematics. Albany, New York: State Education Department, August 1978. Pp. 58-64. ERIC: SE 025 300. The use of calculators is encouraged, along with the use of flow charts. Reasons for using calculators are given.

Additional manuscripts have been received from Hutton and from Shumway et al. (on research) and from Snover and Spikell and from Szetela (on instructional activities). These and other references from newsletters of NCTM Affiliated Groups and from non-US journals will be listed, along with references from Calculator Information Center bulletins, in a forthcoming publication, Calculators: A Categorized Compilation of References (Marilyn N. Suydam, editor), which will be available later in the year from ERIC.

This bulletin (and all other reference bulletins from the Center) was prepared by Marilyn N. Suydam, The Ohio State University -- Director of the Calculator Information Center.

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Special
Information Bulletin
February 1980

Suggestions for Calculator Selection

This is a synthesis of considerations and appropriate suggestions for selecting a calculator for student use. Note that it is important that a calculator be selected in relation to anticipated curricular applications. It is strongly suggested that the way a particular calculator operates should be checked carefully: test the calculator before you buy to be sure it will serve your needs.

Things to Consider	Suggestions
Type of logic	The type commercially known as "algebraic", which allows data to be entered as mathematical sentences are usually written, is most often recommended, although RPN (Reverse Polish Notation) is advocated by some for secondary school use.
Number of functions	At least +, −, x, and ÷ for elementary school; those plus others as appropriate for secondary school courses. If trignometric, logarithmic, degree, radian, hyperbolic, or other functions would be useful in instruction for a particular mathematical sequence, then they should obviously be considered.
Type of decimal notation	Floating decimal point; negative sign that immediately precedes a negative number; check the way the calculator rounds numbers.
Overflow or error indicator	Clear indication of when display, input, or processing limit is reached, or when "illegal" operation is used.
Type of display	(At least) 8-10 digits; easily readable; acceptable viewing angle.
Role of keys	In general, each key should have only one purpose for elementary school use; for secondary school, keys may have more than one purpose (depending on level of student).
Keyboard format	Configuration and size of keys should facilitate accurate entry; on-off switch

Things to Consider	Suggestions
	should be easily accessible; keys should give some response when pressed. Note the position of the numeral in relation to the keys.
Size and weight	Appropriate for the user
Power source	Should provide long service, conserve energy. One opinion: "Consider the number of operating hours per battery replacement or charging. Automatic power-down displays and delayed power-off features insure the maximization of battery life. Long-life replaceable batteries seem to be the most cost- and time-efficient. Charging batteries and contending with electrical cords can be tedious." (Caravella, 1976, p. 548)
Special keys constant (K) change sign (+/-) parentheses square root percent fraction x factorial (x!) x squared (x^2) reciprocal (1/x) y to the xth power (y^x) xth root of y ($\sqrt[x]{y}$) x exchange y ($x \leftrightarrow y$)	Analysis of the curriculum in which the calculator is to be used will aid in deciding how important these keys are to the user (for example, the +/- key is important if you want convenient manipulation of integers); generally, you will have to "trade" some features for others you consider more desirable. Note how the keys handle the procedures.
Memory: two-key	Stores (STO) the displayed number for later recall (RECL): a useful feature for users even at early levels.
four key	Allows functions, usually addition (M+) and subtraction (M-), to be performed on the content of a memory register; with retention for later recall (MR); includes "memory clear" (MC), which could be useful at upper levels.
Memory indicator	Helpful, make sure that it is easy to interpret the symbol (for example, an "M" is easier than a ".").
Scientific notation	Note when and how it works (it may "cover up" repeating decimals).
Automatic constant	Allows calculator to count; note for which operations a constant applies,

Things to Consider	Suggestions
	and the position of the number treated as the constant--note also that it may operate differently with different functions.
Programming	Could be useful--but generally costly.
Printout	Not worth the current cost--but could be helpful to some users if cost dropped. Note that a printout may take an unexpected form--check how symbols appear.
Durability	Check on droppage, malfunctioning incidents, etc., and weigh this in relation to cost.
Cost	Within the budget . . .
Reliability of manufacturer	Adequate (12 month) warranty; repair service
Reliability of vendor	Prompt, responsive service

Types of Display

Two different types of display are available: LED (Light Emitting Diode) and LCD (Liquid Crystal Display). Each has advantages and disadvantages.

LED	LCD
less expensive	slightly more expensive
most use 9-volt battery: relatively short life	uses silver oxide battery (or AAA batteries): hundreds to thousands of hours of life
durable (depending on the particular calculator)	may be less stable (e.g., dropping may cause display to shift or lose part of a symbol)
"flashing" of symbols	"immediate" display of symbols
can be read in dark	depends on good reflected light
red or blue/green numerals: higher battery drain for blue/green than for red	black numerals on gray or yellow: low battery drain
red numerals not readable from wide angle; blue/green generally readable from wider angle	readable from wide angle (in good light)

Section M2.1-M2.2 "USING CALCULATORS: HOW--NOT SHOULD" DELETED
DUE TO MARGINAL LEGIBILITY.

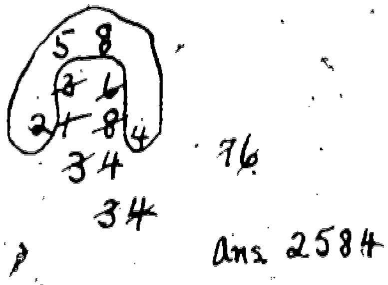
Where Can the Use of the Calculator Lead ?

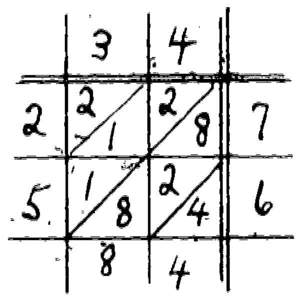
The hand-held calculator can be regarded as the

GREAT EQUALIZER

in the sense that now A L L can calculate.

Reluctance to use the hand-held calculator may well be analogous to that of using any of the "new" calculators or algorithms as they were introduced throughout history. Calculations were in earliest times with the aid of pebbles (stones, "calculi" (Latin)) in grooves in the ground or in a tray of sand or soil. Later came the various kinds of abaci and then algorithms on numerals. Even algorithms have changed through the years, for there were times during which 34 x 76 was done in the ways shown below.





The one at the left is called the "scratchout method" and the one at the right the "gelosia" or "lattice method." The latter led to the development of "Napier's Rods" in 1617. Later came the slide rule and the computer and now the calculator. Would it be natural that there was some resistance to some of these ? Yet each offered an improvement in some way over the preceding-- and our minds and skills did not cease to grow ! Indeed, through their use came new opportunities in mathematics and in problem-solving.

Problem: How can the calculator be used effectively and not compromise with understanding and competency.

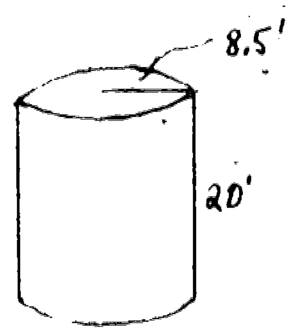
Where can the calculator lead ?

a) to increased facility in the use of [illegible] Problems need not now be made "to order" -- more "real life" problems now !

Students can also be encouraged to "set up" entire problems and to make simplifications before using the calculator --e.g. the value of the fluid in the tank at 63¢/gallon is given by

8.5 x 8.5 x 3.1416 x 20 x 15/2 x 63/100

Note that the expression can be simplified and then calculated.



An example of a "real life" problem whose solution is made easier now is to study the problem of how far to cut in at each corner of a rectangle in order to make a box of maximum volume ---- the use of a table and a calculator !

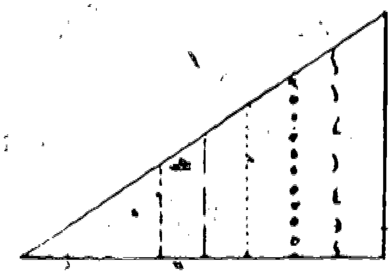
b) to better understanding of different kinds of numbers-- the rationals as repeating decimals (perhaps an invitation to study periodicity of the repeating decimal); the idea of square root and cube root but here to explore: 5 x 5 = 25. $\sqrt{25}$ = 5 but what two numbers (alike) multiply to give 26 ? -- problems from our environment too !

c) to greater efficiency in estimation -- if used in this direction

12.6 x 40 equals about what ? Let us see !
.13 x 457 equals about what ? Let us see !
What number x 25 lies between 290 and 310 ? Estimate !
Check !

d) to gain early ideas on ratio and the concept of (later to be called) trigonometric ratio.

A right triangle drawn as shown and with the calculator one can divide each height by the corresponding base --- ratio always about the same ! Useful ---- or that can arise from attempting to find the height of a pole and the previous activity become a part of the problem-solving process! Students can make their own "ratio tables" for different angles.



e) to become more proficient in mental arithmetic. Why not "pit one's self" against the calculator !
E.g.

[illegible]	[illegible]	438	=
20	x	734	=
98	÷	34	=
1328	[illegible]	98	=
50	x	34	

[illegible] calculator can help us.

f) to explorations which lead to [illegible] algebra [illegible] substantiated by algebra.

45 x 45	81 x 89	$(13)^2$
55 x 55	9[illegible] x 96	$(18)^2$
72 x 78	93 x 97	$(14)^2$

Is there a "quick way" to get the answers just above ? Calculator answers are correct but can we devise a way (algorithm) to perform on the digits to get the answer quicker ?

Another interesting exercise is to ask the student to calculate simultaneously $(13)^2$ and 14×12 ; $(15)^2$ and 14×16 ; $(17)^2$ and 16×18 ; $(16)^2$ and 15×17. Hence how can we quickly calculate 23×25 ? 29×31 ? et cetera.

Later, can the above "quick methods" be substantiated by Algebra ?

g) to facilitate the estimation of roots as $\sqrt[3]{1632}$

What whole number cubed seems to be just less than 1632? just greater? Hence we have a <u>lower bound</u> and we have an <u>upper bound</u>. Can we find lower and upper bounds in tenths ? then in hundredths ? Indeed here are some important mathematical terms growing from simple things: upper bound, lower bound, sequence, limit-- and students gain a "feel" for these terms. Calculators can help us learn good mathematics vocabulary.

h) to deepen ideas of number-theoretic concepts: prime number, factor, factorization. There are examples of this in various sources of suggested uses of the calculator.

i) to lead to the idea of solving equations.

What number "works" in this expression -- that is, what number makes it valid (or makes up the solution set !) ?

$$3\square^2 + 4\square = 39$$

j) to employ "algebra" in a new setting.

Although the discussion should perhaps concern the elementary school, it might be pointed out here that the attempt to solve

$$x^2 + 6x - 43 = 0$$

by the use of the calculator is made easier by rewriting the above as

$$x(x + 6) - 43 = 0$$

Easier in the sense that the number of steps is reduced.

Although the calculator is the GREAT EQUALIZER one should use it also in school to develop <u>more</u> <u>mental</u> <u>powers</u> in arithmetic. Indeed one use of the calculator in the classroom can well be to help devise ways and skills so we use it <u>less</u> !

Computers may be of great value in problem solving, but apparently the human brain alone is able to tackle the subtler aspects of creating an effective correspondence between the mathematical world and the world of experiment and observation.

One the one hand, mathematics teaching should be permeated with concrete examples which give an impression of how widely and diversely mathematical ideas penetrate into human problems generally, including everyday, technical and scientific matters. On the other hand, it is necessary to tell at least one lengthy connected story of the application of mathematics in real depth. This will amongst other things communicate the message that no-one can expect to solve the whole of any problem mathematically. There must be an integration of experiment and theory; there must be a combination of mathematical investigation with inferences from observation and experiment and from non-mathematical modes of reasoning. The best primary-school teaching is a good reminder of how effectively such integration can be carried out, and can be an inspiration to those of us attempting the same at other levels of education.

--Taken from the Presidential Address
of Sir James Lighthill, F.R.S.
as recorded in Development in Mahtematical Education, Proceedings of the Second International Congress on Mathematical Education (Edited by A.G. Howson). Cambridge at the University Press, 1973. pp. 95, 98.

Using a Calculator in Secondary School Mathematics

troduction

The third recommendation in the An Agenda for Action: Recommendations for hool Mathematics of the 1980's said "Mathematics programs must take full advantage the power of calculators and computers at all grade-levels". Five years earlier e NACOME report stated that all students should have access to a calculator by the me they reach the ninth grade. While there is some debate on the extent to which lculators should be used in elementary school, there is near unanimous agreement that ere are no real limits on calculator usage in the secondary school. The question is t whether, but how.

In this article we explore some uses of a simple scientific hand calculator (such the TI-30) in secondary school mathematics: algebra, geometry, trigonometry, analysis, d a taste of calculus. While a few of the uses will be examined in detail, most will in the form of suggestion which the reader is encouraged to pursue and to adapt to s own situation.

The principal reason for using a calculator is to foster realism: to solve realistic problems with realistic data using realistic algorithms to obtain realistic answers.

The ways in which a calculator can be used in secondary school mathematics have en arbitrarily partitioned into six categories:

I. Arithmetic calculations
II. Data generation
III. Concept motivation and reinforcement
IV. Logarithmic and Trigonometric calculations
V Algorithmic computations (you can perform more effectively with a calculator).
VI. Algorithmic computations (you would never perform without a calculator).

There is clearly a gray area for each category.

Arithmetical Calculations

The most obvious use of the hand calculator is to perform arithmetical calculations. e must be sure, however, that there is a real purpose in doing the calculation. Solving e equation $3x = 15$ illustrates the same principles as solving the equation $3,239x = 15.01$ less the latter equation models some real situation.

With a calculator we can deal with small numbers and large numbers (most calculators splay numbers between 10^{-100} and 10^{100}). We can deal with more accurate physical values.

<u>Example</u>. The acceleration due to gravity, g, is approximately 9.8 m/sec^2, but a more accurate value is given by $g = 9.70849\ (1 + .005288 \sin\phi - .000006 \sin^2 2\phi)$ where ϕ is the latitude in degrees.

We can find prime numbers and Pythagorean triples. We can compute the area of a triangle using Heron's formula or $A = .5\ xy \sin\alpha$. We can estimate binomial coefficients.

<u>Example</u>. How many different a) poker (P), b) bridge (B) hands are possible?

<u>Solution</u>:

a) $P = \binom{52}{5} = \dfrac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960$

b) $B = \binom{52}{13} = \dfrac{52 \times 51 \times \ldots \times 40 \times 39}{13 \times 12 \times 11 \times \ldots \times 2 \times 1} = 6.3501 \times 10^{11}$

by calculator using 52 $\boxed{\div}$ 13 $\boxed{\times}$ 51 $\boxed{\div}$ 12 ··· $\boxed{\times}$ 39.

(actual value: 635013559600).

We can solve a puzzle with an unexpected answer (see F-33 # M9). We can solve a realistic problem:

<u>Example</u>. Given a large (right) circular cylindrical storage tank, determine where to put the graduation marks.

We can even solve a classroom favorite.

<u>Example</u>. (Birthday Problem)

In a room with r people, find the probability that at least two of them have the same birthday.

<u>Solution</u>. We find 1 - Pr (no two people have the same birthday) =

$$1 - \frac{365 \times 364 \times \ldots \times (365 - r + 1)}{365 \times 365 \times \ldots \times 365}.$$

The values for various r are

r	5	10	20	23	30	40
Prob	.027	.117	.411	.507	.706	.891

calculating 364 $\boxed{\div}$ 365 $\boxed{\times}$ 363 $\boxed{\div}$ 365 ...

The hand calculator (especially a programmable one) is useful for performing repetitive calculations. In designing algorithms, we often try to exploit this property.

<u>Example</u>. A rumor (disease) is started by one person who tells three others within 15 minutes. These three people each tell three others within the next 15 minutes and so on. How many people know the rumor at the end of 2.5 hours?

<u>Solution</u>. A little reflection shows the answer to be $S = 1 + 3 + 3^2 + \ldots + 3^{10}$.

If we did not know the formula for S, we could calculate each term in turn and add it to the memory using the $\boxed{SUM}$ key, but a repetitive (recursive) procedure is easier. Notice that

$S_1 = 1$

$S_2 = 1 + 3 = 3\, S_1 + 1$

$S_3 = 1 + 3 + 3^2 = 3\, S_2 + 1$

$S = S_{10} = 3\, S_9 + 1$

so that we may enter 1 and then multiply by 3 and add 1 ten times, that is

$1 \boxed{\times} 3 \boxed{+} 1 \boxed{=} \boxed{\times} 3 \boxed{+} 1 \boxed{+} \cdots \boxed{\times} 3 \boxed{+} 1$

and obtain 88,573.

This recursive multiplication is discussed again in V under polynomial evaluation.

II. Data Generation

One can perform calculations to generate data in solving a problem with the hope of finding a pattern. There are thousands of problems which could serve as examples here; we list just a few.

(1) For which real numbers x is the quantity x (x+1) (x+2) (x+3) + 1 a square?

(2) For which integers n is n a factor of (n-1)!?

(3) Can you find a rule to calculate these products mentally:
43 x 47, 25 x 25, 81 x 89, etc?

(4) Find all primes p for which 105 p + 1 is a square.

(5) Generate the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... (Calculator routine: 1 $\boxed{SUM}$ $\boxed{EXC}$ $\boxed{SUM}$ $\boxed{EXC}$ ...). What happens to the ratios of successive terms: 2/1, 3/2, 5/3, 8/5, ...?

(6) Investigate properties of fractions represented by repeating decimals.

(7) (An unsolved problem) Take any integer n. If n is even, divide by 2; if n is odd, compute $\frac{3n+1}{2}$. Repeat. Is 1 always obtained? How many trials are needed to obtain 1 for a given integer n?

(8) Find the pattern:
$1^3,\ 1^3 + 2^3,\ 1^3 + 2^3 + 3^3,\ 1^3 + 2^3 + 3^3 + 4^3, \ldots$ Justify your answer.

Comment. This formulation of a typical problem requiring mathematical induction is far better than the usual: "Show A(n) = B(n) for n ≥ 1 using mathematical induction". Guessing the result first is a worthwhile experience.

III. Concept Motivation and Reinforcement

In addition to generating data to aid in a search for patterns, the calculator can also be used to motivate and to reinforce concepts, especially those involving functions.

(1) The fact that the domain of an inverse trigonometric function is restricted is often forgotten. By checking some values on a calculator, we find that if $f(x) = \arcsin x$, then
$f(2\pi/3)$ results in "Error" and
$f(\sin(2\pi/3)) = \pi/3$ since $2\pi/3$ is not in the domain of
$f(x)$, $-\pi/2 \leq x \leq \pi/2$. Error messages result in trying to evaluate other functions outside their domains.

(2) One way to reinforce the concept that $a^0 = 1$ is to enter any positive number a, then repeatedly calculate the square root; that is

$a \boxed{\sqrt{\ }} \boxed{\sqrt{\ }} \boxed{\sqrt{\ }} \cdots \to a^0 = 1$

To study the process more closely to learn something about the rate at which powers of a approach 1 and to learn of the limitations of the calculator, we could ask

(a) for which a does it require 20 steps to reach 1?

(b) what is the maximum number of steps for any a you can display on your calculator?

(c) what happens if we choose $a < 0$? Can we illustrate $a^0 = 1$ in some other way if $a < 0$.?

(3) The algorithm in (2) also looked at the ideas of function composition and rate of growth. You can investigate these ideas for functions other than $f(x) = \sqrt{x}$.

(a) Enter x $\boxed{x^2}$ $\boxed{x^2}$ $\boxed{x^2}$ ...
What happens for different values of x? Why?

(b) What is the largest x for which x $\boxed{\text{exp}}$ $\boxed{\text{exp}}$ $\boxed{\text{exp}}$ does not result in "Error"? (Note: $\boxed{\text{exp}}$ = $\boxed{\text{INV}}$ $\boxed{\text{ln}}$)

(c) What happens if we enter x and calculate x $\boxed{\text{COS}}$ $\boxed{\text{COS}}$ $\boxed{\text{COS}}$....?

(4) To introduce a concept, it is often effective to begin with a problem. Such a problem to introduce ratio and to use the calculator as well is:

"Suppose Earth is represented by a cherry-stone 3 mm in diameter. Locate and find the appropriate size of the Sun and the nearest star (4.3 light years away)".

IV. Logarithmic and Trigonometric Calculations

There is no longer a need for tables of logarithms and trigonometric functions in the back of the book. The calculator contains them to greater accuracy. Most calculators can handle angle measurements in degrees, radians, and grads and can employ common or natural logarithms. Interpolation becomes an estimation tool to check reasonableness of answers. Round-off error is a topic for discussion.

Trigonometry

In trigonometry we can solve more realistic problems (e.g., surveying) which require the determination of lengths of segments or measures of angles. We can plot more accurate graphs. The addition formulas, the half and double angle formulas and the like are then used primarily in solving trigonometric equations. The sine law and the cosine law can be used effectively. We can verify Taylor approximations.

Example. If $|x| < .1$, approximate the error $|\sin x - x|$ and $|\cos x - (1 - x^2/2)|$

Trigonometric identities can be reposed as trigonometric equations to solve allowing the student to guess that the equation is an identity rather than telling him/her.

Example. Solve for x:

(a) $\sin 2x - \sin x = 0$

(b) $\dfrac{\sin 2x}{\cos x} - 2 \sin x = 0$

(c) $\arcsin x + \arccos x = \pi/2$.

Logarithms and Exponentials

We can, if we wish, use only common and natural logarithms and show that in most applications the base we choose does not matter.

Example. The world's population in 1975 was 4 billion and doubling every 35 years. When will the population reach 11.2 billion?

Solution. $P(t) = 4.2^{t/35}$ where t = number of years after 1975. Thus we must solve

$11.2 = 4.2^{t/35}$

or $2.8 = 2^{t/35}$.

$\log 2.8 = \frac{t}{35} \log 2$	$\ln 2.8 = \frac{t}{35} \ln 2$
$t = \dfrac{35 \log 2.8}{\log 2}$	$t = \dfrac{35 \ln 2.8}{\ln 2}$
$t = \dfrac{35\,(.44715803)}{.30103}$	$t = \dfrac{35\,(1.0296194)}{.69314718}$
$t = 51.989939 \simeq 52$	$t = 51.989939 \simeq 52$
Year: 2027	Year: 2027

So the logarithmic base used does not matter. We can show why e is the "natural" base by showing how the quantity $(1 + 1/n)^n$ arises in a compound interest problem and then finding values of $(1 + 1/n)^n$ as n increases.

We can also estimate large numbers.

Example. The largest known prime (1980) is $2^{44,497} - 1$. Approximate this prime.

Solution. If $2^{44,497} = m\ 10^c$, then

$$44{,}497 \log 2 = \log m + c$$
$$13{,}394.932 = \log m + c$$
$$c = 13.394, \quad \log m = .931702$$
$$m = 8.5448019.$$

So $2^{44,497} - 1 \doteq 8.5448019 \times 10^{13,394}$, a number with 13,395 digits.

We can study the growth rate of exponential functions (logarithmic functions) and see why they eventually grow faster (slower) than powers of x.

Example. Find N so that

a) $(1.01)^x > x$ b) $(1.1)^x > x^3$ c) $\log x < \sqrt[4]{x}$ for all real $x > N$.

Example. It is easy to check that $2^3 < 3^2$, but $4^5 > 5^4$. For what values of a,b is $a^b < b^a$?

Studying exponential growth also exposes some of the limitations of the calculator.

Example. What is the largest power of a) 3 b) e c) 12 your calculator will display (in scientific notation)?

Example. Find the largest x so that

x $\boxed{\ln}$ $\boxed{\ln}$ $\boxed{\ln}$ does not result in "Error"

V. Algorithmic Computations (you can perform more effectively with a calculator)

Again, there are many examples; we give three in detail and mention a few others.

(1) Polynomial Evaluation

The Remainder Theorem gives us a convenient short cut for finding the value P (b), say, for it tells us that P (b) = R and R is easy to compute. To perform the division called for in the Remainder Theorem, we use synthetic division

Illustration 1. Divide $x^4 - 3x^3 + x + 3$ by $x - 2$ synthetically.

Solution. Form the array, noting that the coefficient of x^2 is zero. (We normally place the "2" associated with the divisor x = 2 on the right).

$$\begin{array}{rrrrrl} 1 & -3 & 0 & 1 & 3 & \underline{|2} \\ & 2 & -2 & -4 & -6 & \\ \hline 1 & -1 & -2 & -3 & -3 & = R \end{array}$$

The quotient $Q(x) = x^3 - x^2 - 2x - 3$; the remainder $R = -3$. By direct computation we also find that $P(2) = -3$.

Synthetic division is also known as the nested multiplication method of evaluating polynomials. Consider the fifth-degree polynomial, evaluated at $x = x_1$;

$$a_1x_1^5 + a_2x_1^4 + a_3x_1^3 + a_4x_1^2 + a_5x_1 + a_6.$$

We can rewrite this as

$$(\{[(a_1x_1 + a_2)x_1 + a_3]\ x_1 + a_4\}\ x_1 + a_5)x_1 + a_6.$$

In the original form, 5 + 4 + 3 + 2 + 1 = 15 multiplications are required, plus five additions. In the nested form, only five multiplications are required, plus five additions; it is obviously the more efficient method.

Comparing with the equations $b_2 = a_2 + b_1x_1 = a_1x_1 + a_2$ and $b_i = a_i + b_{i-1}x_1$ for synthetic division, we see that the successive terms are formed in exactly the same way so that synthetic division and nested multiplication are two names for the same thing.

Illustration 2. Using nested multiplication, the polynomial in Illustration 1 becomes

$P(x) = x^4 - 3x^3 + x + 3 =$

$x\ (x^3 - 3x^2 + 1) + 3 =$

$x\ [x\ (x\ (x-3)) + 1] + 3$ so that $P(2) =$

$2[2\ (2\ (2-3)) + 1] + 3 = -3$

which can easily be computed on a calculator by first storing 2 and then working from the inside out - the calculations are identical to those in Ill. 1.

Notice that summing the geometric series $1 + 3 + 3^2 + \ldots + 3^{10}$ (see Section I) is equivalent to evaluating the polynomial $P(x) = 1 + x + x^2 + \ldots + x^{10}$ when $x = 3$.

(2) Quadratic Formula

To solve $ax^2 + bx + c = 0$

on a calculator, we first let $D = b^2 - 4ac$.

If D 0, the solutions are given by

$$r_1 = \frac{-b-\sqrt{D}}{2a} = \frac{-b}{2a} - \frac{\sqrt{D}}{2a}$$

$$\text{and } r_2 = \frac{-b+\sqrt{D}}{2a} = \frac{-b}{2a} + \frac{\sqrt{D}}{2a}.$$

Note that

$$r_2 = r_1 + 2\left(\frac{\sqrt{D}}{2a}\right).$$

This relation suggests a program,

$\boxed{\text{Compute } \sqrt{D}/2a \text{ and store it.}}$

↓

$\boxed{\text{Compute } -b/2a.}$

↓

$\boxed{\text{Subtract the stored number: } r_1 = -\frac{b}{2a} - \frac{\sqrt{D}}{2a}.}$

↓

$\boxed{\text{Add the stored number twice: } r_2 = r_1 + \frac{\sqrt{D}}{2a} + \frac{\sqrt{D}}{2a} = -\frac{b}{a} + \frac{\sqrt{D}}{a}.}$

We can check the solution r_2 using nested multiplication to show $P(r_2) \approx 0$, that is,

calculate $(ar_2 + b) r_2 + c$ and hope it is essentially 0.

Check yourself on

$3.42 x^2 - 2.57x + .23 = 0$. We find

$D = 3.4585$, $\sqrt{D} \approx 1.8597042$

$\frac{\sqrt{D}}{2a} = .2718865$, $r_1 = .1038444$, $r_2 = .6476175$

and $(ar_2 + b) r_2 + c = -4E{-}10 = -4 \times 10^{-10}$.

A few comments about the program. Storing numbers that are used several times in a program is good practice; it cuts the risk of entry errors. With just one memory available, we first store $\sqrt{D}/2a$, generally a number with many digits. After finding r_2, exchange $\sqrt{D}/2a$ and r_2 so that r_2 is now in the memory and $P(r_2)$ can be calculated.

(3) Random digits

One procedure for generating 5 digit random numbers is:

(1) Enter .abcde (e.g. .71632)

(2) Multiply by 137 (98.13584)

(3) Subtract the integer part of the result in b) (98.13584) yielding another random number (.13584). Repeat.

Though this procedure occasionally fails, it is instructive to think about why it works when it does.

Other possibilities include:

- sketching graphs by calculating $f(a + .1n)$, $n = 0, 1, 2, 3, \ldots$
- calculating limits
- finding sums of finite series
- finding the equation of a "line of best fit" or least square" line. This can be done, even in Algebra I, if we plot points carefully and simply "eyeball" the answer.

One must exercise caution, however, because of round-off error.

Example. Graph $f(x) = \sin 20\pi x$, $-1 \le x \le 1$. We would get a mistaken impression of this graph if we calculated $f(-1 + .1n)$, $n = 0, 1, 2, \ldots$

Example. Find $\lim_{x \to \infty} (\sqrt{x^2 + x} - x)$

Calculation gives

x	10^2	10^5	10^6	10^7	10^8	10^9
L	.99995	.9999	.999	.99	.9	0

The limit is 1, round-off error affected the calculations for $x > 10^5$.

VI. Algorithmic Computations (you would never perform without a calculator)

We could give a mini-course in numerical analysis in this section, but we restrict ourselves to two notions: (1) increased use of trial and error, and (2) iterative methods to solve equations and linear systems.

(1) Trial and Error

Because the calculator allows us to perform a great many computations in a short time, a basic trial and error strategy is often practical and efficient.

Example. If you invest \$1000 at an annual percentage rate of 9.23%, how long will it take you to triple your money?

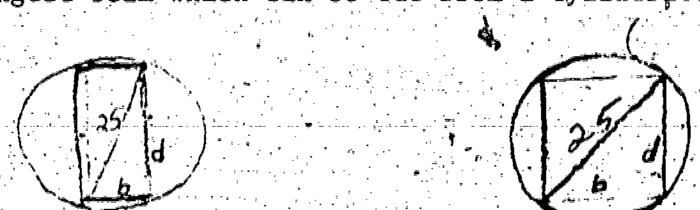
Solution. The problem is equivalent to solving the equation $(1.0923)^x = 3$, which can be done using logarithms. However, a 7th grader can solve the problem using trial and error as follows:

1.0923 $\boxed{\text{STO}}$ $\boxed{x}$ $\boxed{\text{RCL}}$ $\boxed{x}$ $\boxed{\text{RCL}}$ ···

shows $12 \leq x \leq 13$. Further checking shows $(1.0923)^{12.5} = 3.0149$; it takes approximately 12.5 years.

Example. The safe load for a beam of given length varies jointly as the breadth b and the square of the depth d (see Figure . What is the approximate breadth and depth of the strongest beam which can be cut from a cylindrical log of diameter 25 cm?

Solution.



Figure

Since s varies jointly as b and d, we know $s = kbd^2$. The problem, now, is to estimate the values of b and d for which s is the largest. We make a table and proceed by trial and error.

b	d^2	$s = kbd^2$
10	525	5250k
11	504	5544k
12	481	5772k
13	456	5928k
14	429	6006k
15	400	6000k
16	369	5904k
17	336	5712k

The Pythagorean Theorem tells us

$$d^2 = 25^2 - b^2 = 625 - b^2 .$$

Looking at the table, we see that the largest value of s occurs when $14 < b < 15$. To get a more accurate estimate we make another table.

b	d^2	$s \approx kbd^2$
14.0	429.00	6006.0k
14.1	426.19	6009.3k
14.2	423.36	6011.7k
14.3	420.51	6013.3k
14.4	417.64	6014.0k
14.5	414.75	6013.9k
14.6	411.84	6012.7k
14.7	408.91	6011.0k
14.8	405.96	6008.2k
14.9	402.99	6004.6k
15.0	400.00	6000.0k

By examining this table, we see that the maximum value for s occurs when $b \approx 14.4$ and $d \approx 417.64 \approx 20.4$.

Example. A cylindrical can holds 50 in^3. Find the dimensions of the can with minimum surface area.

Solution. This problem appears in every calculus book, but can, in fact, be solved by trial and error.

$$V = \pi r^2 h = 50 \qquad (V = \text{volume})$$

$$SA = 2\pi rh + 2\pi r^2 \qquad (SA = \text{surface area})$$

$$SA(r) = \frac{100}{r} + 2\pi r^2 \qquad \text{since } h = \frac{50}{\pi r^2}$$

When r is small, SA(r) is large and when r is large, SA(r) is also large. Thus SA(r) has a minimum and its graph resembles Figure 1. After a little checking, we make a table of values for $1.5 \le r \le 2.5$.

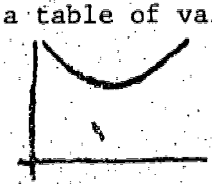


Figure 1

r	1.5	1.6	1.7	1.8	1.9	2.0
SA(r)	80.80	79.59	76.98	75.91	75.31	75.13

r	2.1	2.2
SA(r)	75.32	75.87

Thus the minimum occurs when $1.9 < r < 2.1$. We make another table of values for $1.9 < r < 2.1$ letting r increase by .01 each time to obtain a more accurate answer.

(2) Iteration

Our discussion of iteration is given in portions of several articles concerning iterative methods in solving equations (in one variable) and linear and non-linear systems. We interject some personal comments from time to time.

Fixed Point Iteration—An Interesting Way to Begin a Calculus Course

Thomas Butts

How do you begin a calculus course? A precalculus review? A discussion of the slope of secant and tangent lines? An investigation of limits? In this article we offer an alternative, a discussion of Fixed Point Iteration which combines the best features of each of these three traditional approaches and which is ideal for the use of the hand calculator as well. Fixed Point Iteration (FPI, sometimes called Picard Iteration) is a root-finding algorithm discussed in most elementary numerical analysis books (e.g., [1], [3]) and in at least two newer calculus texts ([2], [5]). To illustrate our approach we examine, in detail, one example which could cover the first few days of the course. We conclude with a discussion of the advantages of this method.

An Example

The following outline of a classroom scenario covers the first few days of the course. Some paraphrasing and condensing have obviously been done.

TEACHER: Now that the mechanics of the course are clear, solve the following problem:

Problem. Find the roots of $x^3 - 3x + 1 = 0$. How many roots are there? How could you find them?

CLASS: Three roots. Sketch the graph. The roots are

$$0 < r_1 < 1, \quad 1 < r_2 < 2, \quad -2 < r_3 < -1.$$

T: How could we find r_1 to three decimal place accuracy?
C: Try the bisection method [or words to that effect].
T: Use your calculator to find r_1.

Each student performs computations to determine that $r_1 = .347$. One such set of computations is
11

x	0	.5	.3	.4	.35	.345	.347	.348	.3475
$x^3 - 3x + 1$	1	$-.375$	.127	$-.136$	$-.007$	.006	.0008	$-.0018$	$-.0005$

T: What if we wanted six-place accuracy? Is there a more accurate and efficient method? Let me suggest one possibility: To find an x satisfying $x^3 - 3x + 1 = 0$ is equivalent to finding an x satisfying $x = (x^3 + 1)/3$. With your calculators, choose an initial guess x in $(0, 1)$, iterate using $x = (x^3 + 1)/3$ and see what happens.

With $x = .5$, for example, we obtain $x_0 = .5$, $x_1 = .375$, $x_2 = .350911$, $x_3 = .347737$, $x_4 = .347350$, $x_5 = .347303$, $x_6 = .347297$, $x_7 = .347296$, $x_8 = .347296$.

T: Notice that we can stop when two consecutive iterates are identical. What values of x_0 could be chosen as the initial guess?

The students check several possibilities on their calculators. Table 1 illustrates some of the cases.

TABLE 1

$x_0 = 0$	$x_0 = 1$
.333333	.666666
.345689	.432000
.347102	.360225
.347273	.348915
.347293	.347492
.347296	.347320
.347296	.347299
	.347297
	.347296
	.347296

$x_0 = 1.5$	$x_0 = 1.55$
1.45833	1.57463
1.36716	1.63473
1.18514	1.78952
.888196	2.24359
.566897	4.09783
.349062	23.27051
.353731	4200.78736
.348087	
.347392	
.347308	
.347298	
.347297	
.347296	
.347296	

$x_0 = -.9$	$x_0 = -1.5$
.090333	$-.791666$
.333579	.167945
.345706	.334912
.347105	.345855
.347273	.347123
.347294	.347275
.347296	.347294
.347296	.347296
	.347296

$x_0 = -1.85$	$x_0 = -1.9$
-1.77720	-1.953
-1.53775	-2.14972
$-.878766$	-2.97815
.107130	-8.47144
.333743	-202.31827
.345725	
.347108	
.347274	
.347294	
.347296	
.347296	

T: Thus for $-1.85 < x_0 < 1.50$ (approximately), the iterates converge to the root; otherwise they diverge to $+\infty$ or $-\infty$. These results raise (at least) two questions:
(1) Under what condition(s) will this algorithm work?
(2) How can we find the other two roots?

First things first. To try and explain this behavior, let us see what happens when we apply $g(x) = (x^3 + 1)/3$ to small intervals about x for several different values of x. To make comparisons easier, we take symmetric intervals of length .1 about successive iterates. For example, if we take a symmetric interval of length .1 about $x = .5$, i.e., [.45, .55], then

$$g([.45, .55]) = [.3637, .3888],$$

an interval of length .025— one fourth the length of the original interval (Figure 1).

If we now take an interval of length .1 about the new guess, .375, ([.325, .425]) and apply g we obtain

$$g([.325, .425]) = [.3448, .3589],$$

an interval of length .014 or approximately one-seventh that of the original interval (Figure 1).

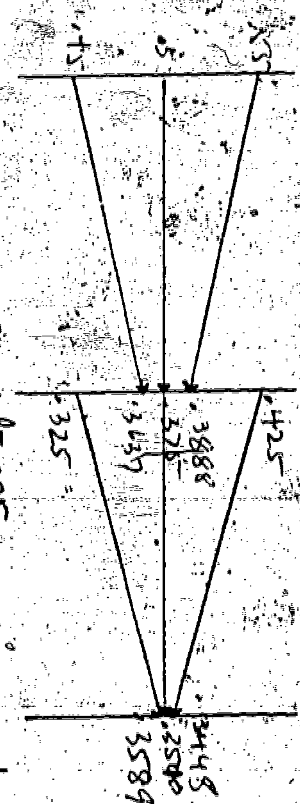


Figure 1

If, on the other hand, we perform a similar set of computations with the initial guess $x_0 = 2$, we obtain the results in Figure 2.

Interval I_1 is four times as long as the interval of length .1 about 2; I_2 is nine times as long. We observed that an initial guess of $x_0 = .5$ led to the root r_1 while the initial guess of $x_0 = 2$ did not.

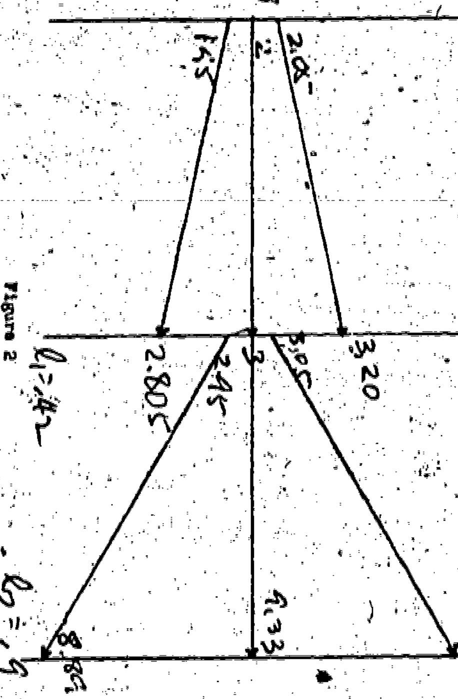


Figure 2

The students now perform similar computations for intervals around other values of x_0.

T: It seems plausible, therefore, that convergence occurs if these intervals get successively smaller (Figure 3).

Formally, if x_0 is an initial guess for the root of $x = g(x)$ and we take a symmetric interval of length 2ϵ about x_0, then we require

$$|g(x_0 + \epsilon) - g(x_0 - \epsilon)| < 2\epsilon$$

or

$$\left| \frac{g(x_0 + \epsilon) - g(x_0 - \epsilon)}{2\epsilon} \right| < 1.$$

After a brief intuitive discussion of the significance of lim $\epsilon \to 0$, we make the

Definition. The *magnification factor* of $g(x)$ at $x = x_0$ is

$$\mathrm{MF}(g(x_0)) = \lim_{\epsilon \to 0} \frac{g(x_0 + \epsilon) - g(x_0 - \epsilon)}{2\epsilon}.$$

This definition leads naturally to the

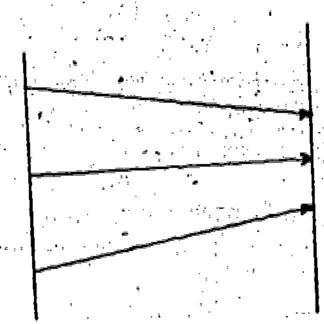
Theorem. *The Fixed Point Iteration Algorithm converges if* $|\mathrm{MF}(g(x))| < 1$ *for values of* x *near the initial guess* x_0.

T: For $g(x) = (x^3 + 1)/3$, we find

$$\mathrm{MF}(g(x_0)) = \lim_{\epsilon \to 0} \frac{g(x_0 + \epsilon) - g(x_0 - \epsilon)}{2\epsilon} = x_0^2$$

so that convergence is guaranteed if $x_0^2 < 1$, i.e., $-1 < x_0 < 1$. Experimentally we saw that convergence occurred for the longer interval, $(-1.85, 1.5)$.

Obviously the magnification factor is one interpretation of the first derivative and we can make the formal definition any time after this point. I prefer to exploit the problem a bit more before so doing.



[It may happen that $g(x_0 - \epsilon) > g(x_0)$ and $g(x_0 + \epsilon) < g(x_0)$.]

Figure 3

T: How about finding the other two roots?

C: Divide $x^3 - 3x + 1$ by $x - .347296$ and solve the resulting equation using the quadratic formula.

We obtain

$$x^3 - 3x + 1 = (x - .347296)(x^2 + .347296\,x - 2.87939)$$

so that

$$r_2 = 1.53209, \qquad r_3 = -1.87939.$$

T: Can we obtain these roots using Fixed Point Iteration?

C: We must rewrite $x^3 - 3x + 1 = 0$ using other choice(s) for $g(x)$ so that $f(x) = x - g(x)$ and $MF(g(x)) < 1$ for x near 1.5 and x near -2.

After some discussion, the alternatives

$$x = \sqrt[3]{3x - 1}, \quad x = -\frac{1}{x^2 - 3}, \quad x = \frac{3}{x} - \frac{1}{x^2}$$

(and possibly others) are analyzed and the roots r_2 and r_3 are determined using appropriate choices for $g(x)$.

Comment: In practice, it is not absolutely necessary to find the magnification factor before iterating. Simply rewrite the equation $f(x)=0$ in the form $x=g(x)$, choose an initial guess x_0, and iterate. If it works, it works. If not, find another way to rewrite $f(x)=0$ in the form $x=g(x)$. Another description of iteration is given in the next short section.

USE OF $x = g(x)$ FORM*

We now discuss another method that is of general applicability, and which also lets us develop some necessary theory. We begin with the equation $f(x) = 0$, and rearrange it into an equivalent expression of the form

$$x = g(x), \quad \text{such that if } f(r) = 0, \quad r = g(r).$$

Under suitable conditions, which we develop below, the algorithm

$$x_{n+1} = g(x_n), \quad n = 1, 2, 3, \ldots$$

will converge to a zero of $f(x)$. Consider a simple example:

$$f(x) = x^2 - 2x - 3 = 0,$$

which has obvious roots at $x = 3$, $x = -1$.

Rearranging yields:

$$x = \sqrt{2x + 3},$$

so $g(x) = \sqrt{2x + 3}$. Starting with $x_1 = 4$, we get

$$\begin{aligned} x_2 &= \sqrt{11} = 3.316, \\ x_3 &= \sqrt{9.632} = 3.104, \\ x_4 &= \sqrt{9.208} = 3.034, \\ x_5 &= \sqrt{9.068} = 3.011, \\ x_6 &= \sqrt{9.022} = 3.004. \end{aligned}$$

(This is only one of many possible rearrangements.) The various iterates appear to converge to $x_\infty = 3$.

The equation $f(x) = x^2 - 2x - 3 = 0$ can be rearranged in other ways also. For example, $x = 3/(x - 2)$ is an alternative rearrangement of form $x = g(x)$. If $x_1 = 4$,

$$\begin{aligned} x_2 &= 1.5 \\ x_3 &= -6, \\ x_4 &= -0.375, \\ x_5 &= -1.263, \\ x_6 &= -0.919, \\ x_7 &= -1.028, \\ x_8 &= -0.991, \\ x_9 &= -1.003. \end{aligned}$$

Note that this converges, but to the root at $x = -1$, and that the iterates oscillate rather than converge monotonically.

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* The method is called by some authors simply "the method of iteration."

Consider a third rearrangement:

$$x = \frac{x^2 - 3}{2}.$$

From $x_1 = 4$ we get

$$x_2 = 6.5,$$
$$x_3 = 19.635,$$
$$x_4 = 191.0,$$

which obviously is diverging.

Figure 1.11 illustrates the several cases; (a) shows monotonic convergence, (b) shows oscillatory convergence, and (c) shows divergence. For a function $x = g(x)$, the solution is at the intersection of the line $y_1 = x$ with the curve $y_2 = g(x)$. In every case, we move vertically to the curve and then horizontally to the line, and repeat.

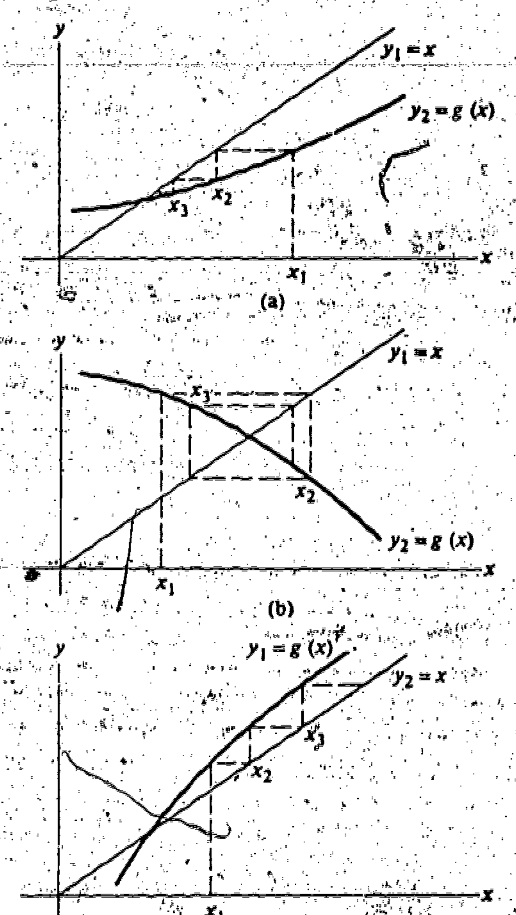


Figure 1.11

An algorithm for this method is shown below.

Iteration with the form $x = g(x)$

To determine a root of $f(x) = 0$, given a value x_1, reasonably close to the root.

Rearrange the equation to an equivalent form $x = g(x)$.
DO WHILE $|x_2 - x_1| \geq$ tolerance value,
Set $x_2 = g(x_1)$.
Set $x_1 = x_2$.
ENDDO.

Note. The method may converge to a root different from the expected one, or it may diverge. Different rearrangements will converge at different rates.

Comment: A few interesting problems whose solutions can involve iteration are:

(1) Solve for x: $2 \cos x = x$; $x = \tan x$; $2x^4 = e^x$.

(2) Railroad track problem (see paper F) Suppose a 5000 foot long railroad rail is solidly anchored at both ends (no expansion joints). When the rail is hot, it expands by 2 feet. How high off the ground is the midpoint of the rail?

(3) For which real numbers b does the equation $b^x = x$ have a) 0 b) 1 c) 2 solutions?

(4) Give an example of two parabolas whose graphs intersect in exactly a) 0 b) 1 c) 2 d) 3 e) 4 points.

We now look at iterative methods for solving systems of equations.

We have already discussed the Gaussian elimination method for solving systems of equations. We shall now discuss iterative methods. At the close of the section we shall compare the merits of the Gaussian elimination method and the Gauss-Seidel iterative method.

We introduce an iterative method by means of an example.

Solve the system of equations

$$\begin{aligned} 6x + 2y - z &= 4 \\ x + 5y + z &= 3 \\ 2x + y + 4z &= 27 \end{aligned} \qquad (1)$$

Rewriting the system in the form

$$\begin{aligned} x &= \frac{4 - 2y + z}{6} \\ y &= \frac{3 - x - z}{5} \\ z &= \frac{27 - 2x - y}{4} \end{aligned} \qquad (2)$$

we have isolated x in the first equation, y in the second, and z in the third.

Now make a guess at the solution, say $x = 1$, $y = 1$, $z = 1$. The accuracy of the guess affects only the speed with which we get a good approximation. We call these values $x^{(0)} = 1, y^{(0)} = 1, z^{(0)} = 1$. We substitute these values into the right side of system (2) to get values of x, y, and z that we denote $x^{(1)}$, $y^{(1)}$, and $z^{(1)}$.

$$x^{(1)} = \tfrac{1}{2}, \quad y^{(1)} = \tfrac{1}{5}, \quad z^{(1)} = 6$$

These values are then substituted again into system (2) to get

$$x^{(2)} = 1.6, \quad y^{(2)} = -0.7, \quad z^{(2)} = 6.45$$

The process is repeated to get $x^{(3)}$, $y^{(3)}$, and $z^{(3)}$, etc. Repeating the iteration will, under certain conditions, give us a better approximation of

the exact solution each time. For this simple system, the solution is easily seen to be $x = 2$, $y = -1$, $z = 6$, so that after the second iteration one has quite a long way to go.

The following theorem gives one set of conditions under which this iterative method can be used.

THEOREM 4-1. *Let A be the matrix of coefficients of a system of n equations in n variables. If*

$$\underbrace{|a_{ii}|}_{\substack{\text{diagonal element} \\ \text{of } i\text{th row}}} > \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|}_{\substack{\text{sum of absolute values of} \\ \text{all other elements in } i\text{th row}}}, \qquad \text{where } i = 1, 2, \ldots, n$$

then the system of equations has a unique solution and the iterative method converges to this unique solution no matter what values are selected for the initial guess.

Thus this method can be used when the diagonal elements dominate the rows. We see that the system in our example satisfies these conditions.

Of course, the better the initial guess the sooner one gets a result to within a required degree of accuracy. Note that the method has the advantage that if an error is made at any stage, this merely means that a new initial guess is instituted at that stage.

There are many theorems similar to the one above that guarantee convergence under varying conditions. Investigations have made available various swiftly convergent methods for various special systems of equations.

The Gauss-Seidel method, a refinement of the one above, leads to a more rapid convergence. The latest value of each variable is substituted into system (2) at each stage.

As before, let $x^{(0)} = 1$, $y^{(0)} = 1$, $z^{(0)} = 1$ be the initial guess. Substituting the latest value of each variable into (2) every time,

$$x^{(1)} = \frac{4 - 2y^{(0)} + z^{(0)}}{6} = \frac{1}{2} = 0.5$$

$$y^{(1)} = \frac{3 - x^{(1)} - z^{(0)}}{5} = \frac{3}{10} = 0.3$$

$$z^{(1)} = \frac{27 - 2x^{(1)} - y^{(1)}}{4} = \frac{25.7}{4} = 6.4250$$

Thus after one iteration, $x^{(1)} = 0.5$, $y^{(1)} = 0.3$, $z^{(1)} = 6.4250$. Notice that we have used $x^{(1)}$, the most up-to-date value of x, to get $y^{(1)}$, and we have used $x^{(1)}$ and $y^{(1)}$ to get $z^{(1)}$.

Continuing,

$$x^{(2)} = \frac{4 - 2y^{(1)} + z^{(1)}}{6} = 1.6375$$

$$y^{(2)} = \frac{3 - x^{(2)} - z^{(1)}}{5} = -1.0125$$

$$z^{(2)} = \frac{27 - 2x^{(2)} - y^{(2)}}{4} = 6.1844$$

Note how, after only two iterations, this set is much closer than the set $x^{(2)} = 1.6$, $y^{(2)} = -0.7$, $z^{(2)} = 6.45$ to the exact solution of $x = 2$, $y = -1$, $z = 6$. Both methods were programmed for the computer. Tables 4-1 and 4-2 give the results obtained for this particular system. They illustrate the Gauss-Seidel method's more rapid convergence to the exact solution.

Table 4-1. First Method

Iteration	x	y	z
Initial Guess	1	1	1
1	0.5	0.2	6
2	1.6	−0.7	6.45
3	1.975	−1.01	6.125
4	2.024167	−1.02	6.015
5	2.009167	−1.007833	5.992917
6	2.001431	−1.000417	5.997375

Table 4-2. Gauss-Seidel Method

Iteration	x	y	z
Initial Guess	1	1	1
1	0.5	0.3	6.425
2	1.6375	−1.0125	6.184375
3	2.034896	−1.043854	5.993516
4	2.013537	−1.001411	5.993584
5	1.999401	−0.998597	5.999949
6	1.999524	−0.9998945	6.000212

Table 4-3

	x	y	z
First Method	0.001431	0.000417	0.002625
Gauss-Seidel Method	0.000476	0.0001055	0.000212

Table 4-3 gives the differences between the solutions obtained in the two methods and the actual solution after six iterations. The Gauss-Seidel method converges much more rapidly.

Comments: (1) Iteration works equally well for 2 by 2 linear systems.

Example

$2x+y=4$
$x+2y=5$

Initial guess: (0,0) $x_{n+1} = \dfrac{4 - y_n}{2}$; $y_{n+1} = \dfrac{5 - x_{n+1}}{2}$

gives

x		2	1.25	1.0625	1.0156	1.0039
y	0	1.5	1.875	1.9688	1.9922	1.9980

(2) A flow chart for solving a 3 by 3 linear system using iteration (if applicable) on a calculator with a single memory and an exchange key is

Enter z_0 the initial guess for z

$\boxed{STO}$

Compute x

$\boxed{EXC}$

Compute y

$\boxed{EXC}$ Repeat

Compute z

$\boxed{EXC}$

Let us now compare the Gaussian elimination method with the Gauss-Seidel iterative method.

The Gaussian elimination method is finite and leads to a solution for any system of linear equations. The Gauss-Seidel method converges only for special systems of equations; thus it can only be used for such systems.

A second factor of comparison must be the efficiency of the two methods, a function of the number of arithmetic operations (addition, subtraction, multiplication, and division) involved in each method. For a system of n equations in n variables where the solution is unique, Gaussian elimination involves $(4n^3 + 9n^2 - 7n)/6$ arithmetic operations. The Gauss-Seidel method requires $2n^2 - n$ arithmetic operations per iteration. For large values of n, the number of arithmetic operations required by each method is, respectively, approximately $2n^3/3$ and $2n^2$ per iteration. Therefore, if the number of iterations is less than or equal to $n/3$, then the iterative method requires fewer arithmetic operations. As a specific example, consider a system of 300 equations in 300 variables. Elimination requires 18,000,000 operations, whereas iteration requires 180,000 operations per iteration. For 100 or fewer iterations the Gauss-Seidel method involves less arithmetic, it is more efficient. It should be stated that the Gaussian elimination method involves movement of data, for example, several rows may need to be interchanged. This is time consuming and costly on computers. Iterative processes suffer much less from this factor. Thus, even if the number of iterations is more than $n/3$, iteration may require less computer time.

A final factor in the comparison of the two methods is the accuracy of the methods. Round-off errors are minimized in the Gaussian elimination method by using the pivoting technique. However they can still be sizeable. The errors in the Gauss Seidel method, on the other hand, are

SETS OF NONLINEAR EQUATIONS

As mentioned previously, the problem of finding the solution of a set of nonlinear equations is much more difficult than for linear equations. (In fact, some sets have no real solutions.) Consider the example of a pair of nonlinear equations:

$$\begin{aligned} x^2 + y^2 &= 4, \\ e^x + y &= 1. \end{aligned} \tag{2.24}$$

Graphically, the solution to this system is represented by the intersections of the circle $x^2 + y^2 = 4$ with the curve $y = 1 - e^x$. Figure 2.4 shows that these are near $(-1.8, 0.8)$ and $(1, -1.7)$. We can use the method of iteration to improve these approximations. Just as in Section 5, Chapter 1, we rearrange both equations to a form of the pattern $x = f(x, y)$, $y = g(x, y)$, and use the method of iteration on each equation in turn. Under proper conditions, these will converge. For example, if we rearrange Eqs. (2.24) in the form

$$x = \pm\sqrt{4 - y^2}, \quad (- \text{ sign for leftmost root}),$$

$$y = 1 - e^x,$$

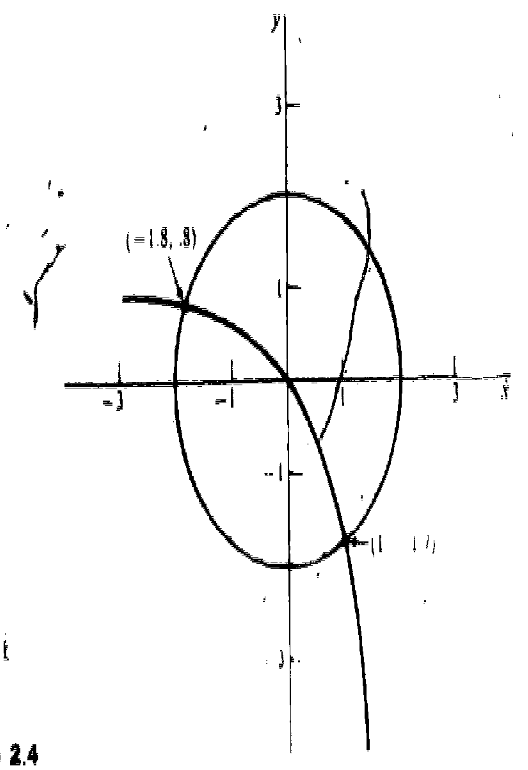


Figure 2.4

we get the following successive values, beginning with $y_1 = 0.8$ in the first equation:

x-values:		−1.83		−1.815		−1.8163		−1.8162		
y-values:	0.8		0.84		0.8372		0.8374		0.8374	

When we begin at $y = -1.7$ to find the root to the right of the origin, we get

x:		1.05		0.743		1.669		Imaginary value
y:	−1.7		−1.857		−1.102		−4.307	

The equations diverge! (Beginning with $x = 1.0$ in the second equation is no help. This also diverges.) However, with a different rearrangement of the original equations, such as

$$\begin{aligned} x &= \ell n(1 - y), \\ y &= \pm\sqrt{4 - x^2}, \quad (- \text{ sign for rightmost root}), \end{aligned} \tag{2.25}$$

we get

x:		0.993		1.006		1.0038		1.0042		1.0042
y:	−1.7		−1.736		−1.7286		−1.7299		−1.7296	

The pair of rearranged equations in (2.25) converges.

Some of the difficulties with sets of nonlinear equations are apparent fro this simple example. If there are more than two equations in the system, finding convergent form of the equations is increasingly difficult. A criterion for co vergence (sufficiency condition only) is as follows:

The set of equations

$$x = f(x, y, z, \ldots), \qquad y = g(x, y, z, \ldots), \qquad z = h(x, y, z, \ldots), \qquad \ldots$$

will converge if, in an interval about the root,

$$\begin{aligned} |f_x| + |f_y| + |f_z| + \cdots &< 1, \\ |g_x| + |g_y| + |g_z| + \cdots &< 1, \\ |h_x| + |h_y| + |h_z| + \cdots &< 1, \\ \cdots\cdots &< 1. \end{aligned}$$

In the above inequalities, the subscript notation designates partial deriv tives. Computing all the partial derivatives and knowing where the root is a major problems. Getting starting values for the multidimensioned system is al correspondingly difficult.

The House That Jack Built

Jason W. Brunk
Ohio University

On any given job the tool you are most likely to need is the one you left at the shop (Brunk's Fourth Law). Of course, if all you own are a hammer and a screwdriver you can forget Brunk's Law. Just bring your whole shop. Unfortunately, that won't work for me. I am so intrigued by man's ingenuity in devising tools that extend his capabilities that I'll buy a new tool at the drop of a plumb bob. My wife says I'm a tool addict. It's true. I get withdrawal symptoms if I can't find my retracting tape measure.

As you acquire more tools you make any job potentially easier, but you also increase the risk that you will fall victim to Brunk's Law. Like Murphy's laws, Brunk's laws are natural laws; they can't be repealed. All you can do is accommodate yourself to them. This isn't really so difficult. The trick is to develop a certain flexibility of attitude--a problem solving orientation. You stop thinking narrowly of a hammer as something to drive a nail or of a screwdriver as something to turn a screw. Rather, you concentrate on the demands of the task and ask, "Given the resources at hand, how can I best solve this problem?" Of course, as any craftsman will tell you, it is better to use the tool designed for the job. But we are talking here about emergencies.

Three years ago I decided to design and build an energy efficient house. I am neither an architect nor a builder. But I discovered early on that you can do almost anything if you are willing to work at it (Brunk's Third Law).

Incidentally, it seems to me that the world is full of people who are intimidated by machines, mathematics, new ventures, and anything mysterious or unknown. They seem to develop little anxieties about these things and build themselves into psychological playpens. Playpens (underline pens) are fine for toys, but they offer a seriously restricted environment and limited opportunity for exploration and growth, as any ten-month old infant will effectively inform you.

Anyway, I set out to build my dream house, and one of my first efforts was to add a few new tools to my shop. Now my wife was not too disturbed when she learned that I wanted to purchase a new hammer, a crowbar, and a 50 foot tape measure. Nor was it very difficult to convince her that I needed a new portable power saw. But when I mentioned needing a calculator- with square root, yet--she objected. Nobody builds houses with square roots.

It's hard to rebut the point. I have observed many carpenters at work, and I've never seen one employ the square root algorithm. Yet they still build houses because they know little tricks. Such tricks work (though the carpenter may not know why), but the tricks are limited and inflexible. If the carpenter faces a problem he has never faced before he may be defeated. I'll tell you more about that later. The problem at the moment is not square root but square attitudes.

Of course, I can build a house without a calculator. I can drive a nail with a piece of one inch pipe if no hammer is available, but a hammer makes it easier, and I can do a better job. A calculator makes the job easier also, and you never hit your thumb with it.

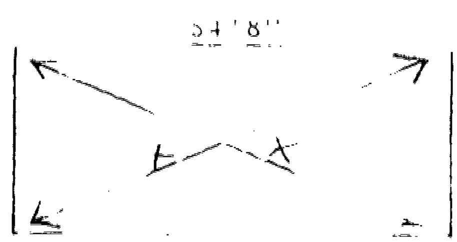
Well, I got the calculator--square root and all, and, as it turned out, the calculator was probably the most valuable addition to my shop that I could have made. I used it in every phase of planning, building, and accounting.

In determining needed materials, for example, I had to calculate (1) quantitites, such as number of bricks and concrete blocks, (2) linear measures, such as perimeter (e.g., drain tile around base of house and termite shield), length of electrical wire, and length of water pipe, (3) area, such as roof, floor, and walls (for plywood, wallboard, and paint), (4) volume, such as concrete and insulation (concrete is sold by the cubic yard but is usually placed as covering an area to a specified thickness).

I also had to make other kinds of calculations. There were problems involving (5) conversions, such as in buying lumber. Lumber in quantity is sold as board feet (a piece of wood one inch thick and twelve inches square or the equivalent).* (6) lay-outs for foundations, rooms, staircases and roof angles. (7) formulas, for example, calculating heat loss through walls of different insulating values and amount of insulation needed. (8) financial matters, including total costs, unit costs, discounts, taxes, and comparison shopping. I used all four basic operations, algebraic equations, the Pythagorean theorem, square root, decimals, reciprocals, percentage, ratio, and conversions.

In the following paragraphs I'll describe in detail a number of examples of the above applications and attempt to illustrate the versatility of the calculator in building as I used it. No doubt the reader will think of numerous adaptations of these examples.

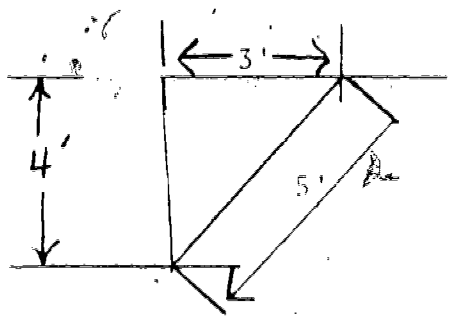
One of the very first problems in the actual construction of a new building is laying out the foundation. In designing the house the square corners are achieved with a T-square and a right angle triangle on a piece of drawing paper on your kitchen table. In making small projects in the workshop an 18" x 24" carpenter's square is a good tool. But when you walk out onto that big grassy field to stake out the corners of your dream house--a house that you fully intend to be perfect--you want the rectangles to be true and the corners to be square. But how? There is no giant T-square available. I could have measured off the rectangle approximately and then measured the corners diagonally until the two diagonals were equal--a very cumbersome trial and error procedure. Moreover, in addition to the main foundation, I would



have to lay out offsets, porches, and later, [illegible] I needed [illegible] better. Of course, I knew of the 3 4 5 rule that carpenters use. This rule or trick is

*An eight foot long board two inches thick and four inches wide would contain 5-1/3 board feet. $\frac{2'' \times 4'' \times 96''}{144''} = 5.33$ feet.

simply a particular application of the Pythagorean theorem.



$3^2 + 4^2 = 5^2$ or

$30^2 + 40^2 = 50^2$

Having established the correct directions of two sides of the foundation, you merely extend the sides to the correct length. Then you continue with the 3-4-5 rule at each of the two other corners and thereby complete the four sides with 90° corners.

This procedure is generally employed and the small error--probably several inches--is ignored. I was not willing to accept the probable error of this method. Therefore, I chose to use the Pythagoream theorem as calculated in figure 3a.

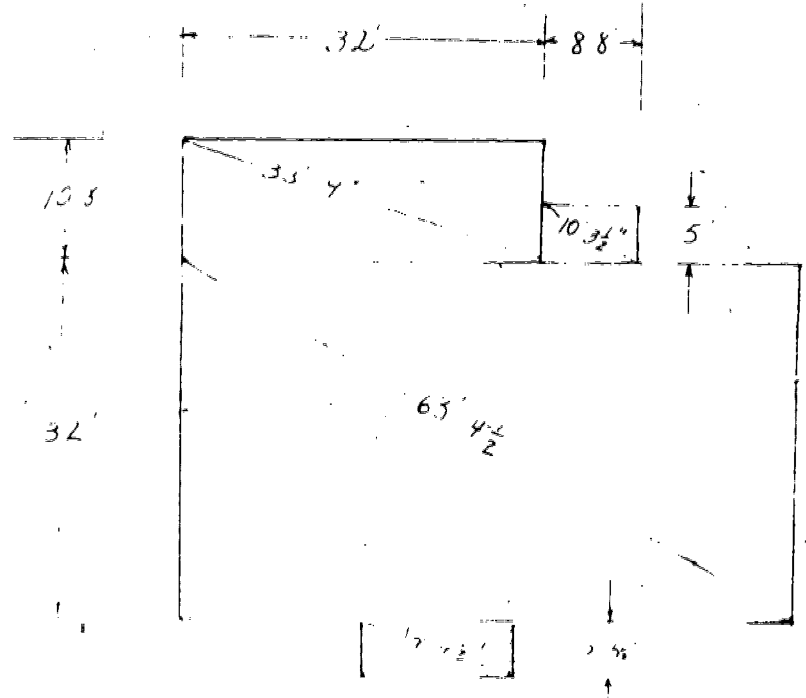
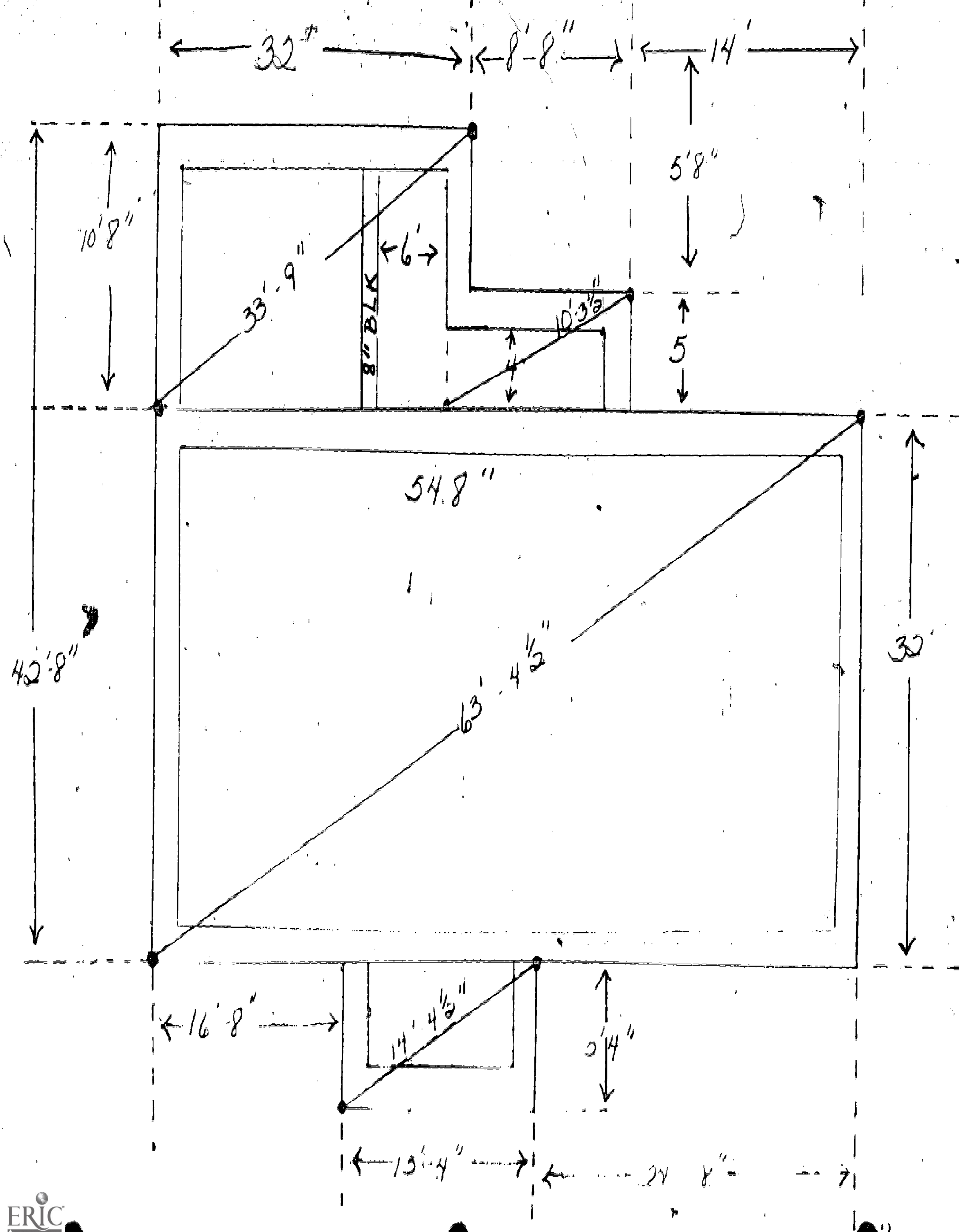


FIGURE 3a

2988 40 + 10

4012.4809

... lative. There is always s me erro. The question is simply h w much erro.


32'
8'8"
14'
5'8"
10'8"
33'-9"
6'
8" BLK
10'3½"
4'
5
54.8"
42'8"
63'-4½"
32'
16'8"
14'-4½"
2'4"
13'-4"
24'8"

is tolerable. I have lived in houses where the corners were so crooked that kitchen counter tops would not fit without making them crooked also. I've had houses where, because the floor was not level, the file cabinet drawers would not stay closed. In one house I put up a shelf in my son's bedroom for his toys. The shelf was level but the ceiling was not. Since we tend to assume that a room is square, plumb, and level, it becomes the frame of reference. Consequently, the shelf appeared to be going up hill. It looked bad. I had to readjust it--make it unlevel--so it appeared level. And his toys rolled off the shelf.

When planning a concrete block foundation or a brick veneer wall you have to calculate the number of blocks or bricks you need. You don't want to run out because, at least with bricks, you may not be able to match them later. You don't want to order too many either. The supplier doesn't want them back, so you have wasted materials (lost money) plus the problem of disposing of the surplus.

It's a straightforward problem, however. Calculate the area in square feet. Subtract areas of windows, doors, or walls of other material such as wood. Take the remaining area figure and multiply by seven (there are roughly seven bricks in a square foot of wall). Round up to the nearest thousand and order the bricks. There's a little more to it if you have a brick chimney or some other feature, but the principle is the same. Some masons are more wasteful than others. I decided to order an extra thousand bricks to be safe. I figured I'd build an outdoor fireplace with the excess. As it turns out my original estimate was almost perfect. But I haven't had time to build the fireplace. Calculators don't help you much with that problem.

One of my most interesting problems was building the staircase. One of the carpenters recommended that I get a local lumber company to prefabricate it. I called one lumber company and found that it would cost about $800 for the staircase (1978 estimate). Maybe they planned to use silver nails, but I couldn't see $800 for one staircase--especially when I needed two of them. I decided to do it myself. I had never made one and didn't know how. Nevertheless, I purchased three 2 x 10's, each fourteen feet long, for about $27. I laid one 2 x 10 across two saw horses and plotted the appropriate cuts.

Now that is an interesting problem. How many steps are there to be? How high is each step to be? How deep should each step be (front to back)? And how do you get the whole thing to come out even and level?

First, I figured the angle that the carriage (the 2 x 10 that holds the individual steps) should be. Then I figured each step--the riser and the tread dimensions (see figure 4). There is an old rule that says the rise of the step plus the depth of the step (the tread) should equal about 17 inches. Considering the dimensions I had to work with--floor-to-floor is fixed--I determined that each riser would be 7-11/16" with 14 steps, which is close to optimal, and that would give a tread of 9-1/2".

I began the layout using the Pythagorean theorem to get the angles at 90°, but in checking my work I discovered how to use the carpenter's square to do the job--it is much simpler. So I didn't need the calculator for that job after all, but I didn't tell my wife. When you don't need something you've bought after you told your wife you needed it, don't tell your wife (Brunk's Second Law).

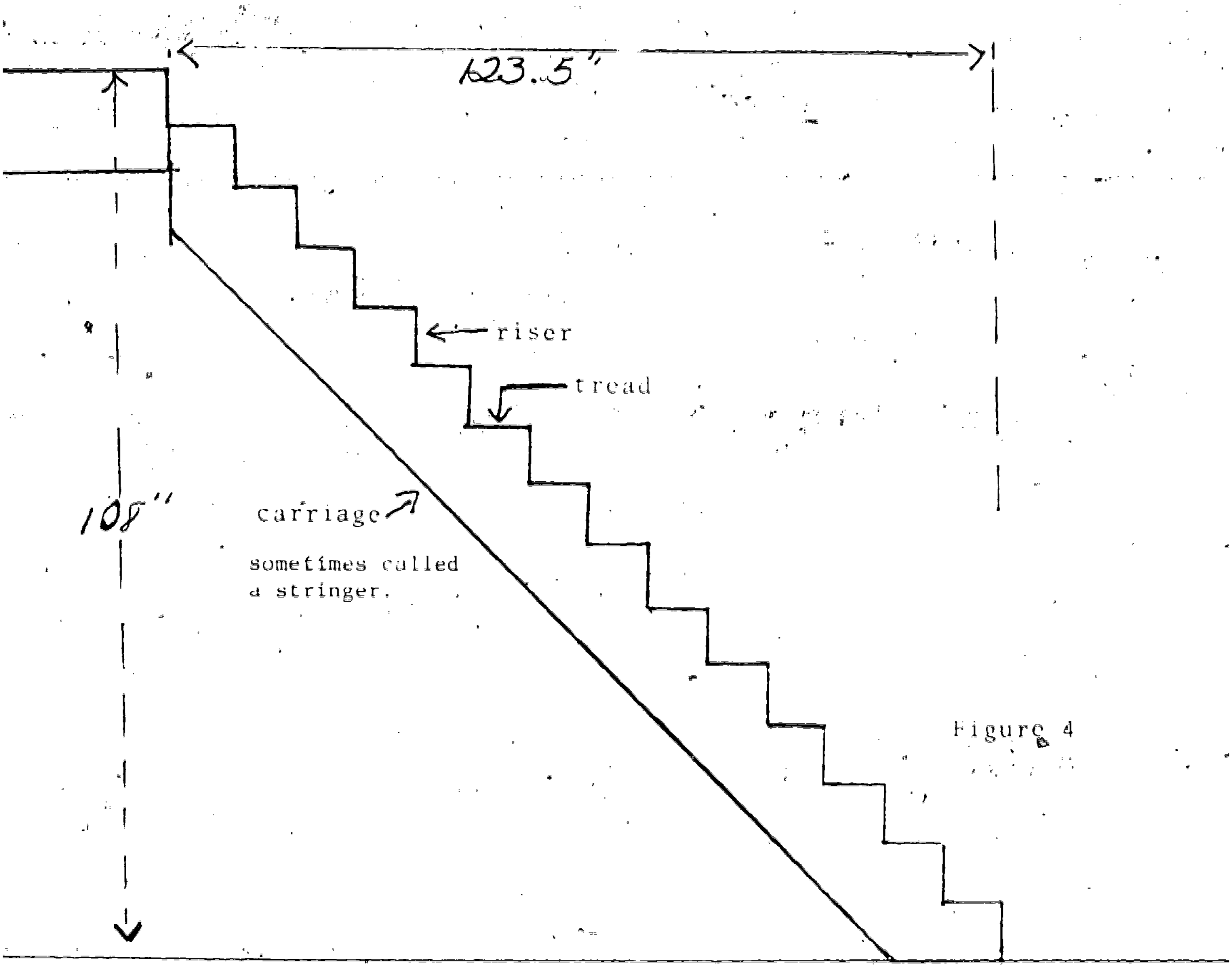


Figure 4

[illegible] $\frac{108"}{14}$ [illegible] about [illegible]

[illegible] $\frac{123.5}{13}$ [illegible]

[illegible]

the last tread

The problem I had the most fun solving was that of determining how to insulate the walls. There are really several problems involved in this decision. Every kind of material--bricks, wood, plasterboard, air, and fiber glass, for example--has some insulating value, usually expressed as an R value. R stands for resistance to heat loss. Therefore, the higher the R value the better the insulating quality of the material. R values are usually expressed in R per one inch thickness per square foot of area, except that manufacturers of batts and roll type insulation usually give the total R value for the particular thickness being sold (e.g., six inch fiber glass batts are rated R-19. The associated R value per inch of thickness is therefore slightly over three.)

There are many types of commercial insulation available. They have different R values, characteristics, costs, and installation requirements. Selecting the "best" insulation therefore is very complex. For example, urethane--which has an R value of 6.5 (twice that of fiber glass)--is not only flammable but gives off a deadly gas when burned. Some insulation may corrode copper water pipes and some insulation shrinks, leaving uninsulated space. Standard walls have a 3-1/4" hollow space where insulation is usually placed, but there is nothing except expense to prevent you from making this space bigger. The problem boils down to what kind and how much.

I first narrowed my choice of insulation to two kinds: fiber glass and cellulose. Then I calculated for walls of different thickness (1) the cost of materials and (2) the cost of the heat loss.

The standard practice is to calculate the cost of the insulation installed and the time period over which savings in fuel bills will repay that cost. A home owner might say, for example, "If it takes more than five years to get my money back in fuel savings, I won't insulate because I plan to move before that time."

When you build a dream house, you don't plan to move, so I used a different approach. I simply calculated the cost of money (interest) spent as though the loan were never to be repaid. As long as I own the house, it is costing me the interest (either in money borrowed or in lost savings). This approach works best on items that are durable and that generally require no maintenance, such as insulation. I used it because it simplified the calculation of cost. For example, if I spend $100 on insulation and borrow money at 10 per cent, the insulation is really costing me $10 a year for as long as I own the house. If I repay the loan then I can't invest the $100 I used to repay it and am losing the interest I would have gotten.

The first step, then, is to calculate the separate areas of the walls, ceilings, floors, windows, and doors. Then you must calculate the heat loss for each of these areas. Added up they constitute the total heat loss for that room. I calculate only exterior walls or walls adjacent to an unheated space since heat lost to an adjacent heated space is not really a loss.

Heat loss through a wall depends on three factors: the area of the wall, the resistance to heat loss of the wall, and the difference between the temperatures on the inside and outside of the wall. Here is the formula:

Heat loss per hour (BTUH) =

(1/R) x [temperature differential between inside & outside of wall] x area

For example, if your walls have a heat loss area of 300 square feet and a total R value of 14, and if it is zero degrees outside and you want to maintain 70° inside, the equation is

$$(1/14) \times 70 \times 300 = 1500 \text{ BTUH}.$$

Since one kilowatt hour (KWH) produces approximately 3415 BTU's it would cost $\frac{1500}{3415}$ of whatever the current cost of electricity is--approximately 5.6¢ in Athens, Ohio August, 1980--to replace that loss using an efficient electrical heating device. That is about 2.5¢ per hour in the above example.

The formula can be set up several ways, depending on what you want. I set it up on a monthly basis like this:

$$\text{cost per month} = \frac{\text{BTUH}}{3415} \times 5.6¢ \text{ per KWH} \times 24 \text{ hrs.} \times 30 \text{ days}$$

Obviously, this can give only an approximation since the temperature outside will vary considerably. Also, opening and closing doors and windows, as well as the number of people and heat producing equipment (e.g., lights or dishwasher) in the room will affect the outcome. For example, I built a pantry off the kitchen and put a small baseboard heating unit in it. I also put an auxiliary refrigerator in the pantry intending to use it only during the summer months. But I found that as long as the refrigerator was operating, I didn't need the heating unit on. So we had the use of the refrigerator the year round.

In planning the heating equipment requirements you would probably use the temperature differential for the coldest expected weather so as to be sure to have sufficient heating reserve power to maintain your desired inside temperature. But in calculating heat loss costs (using my perpetual interest charge method) I used an average differential temperature because the coldest day differential would cause distorted figures. Using this average temperature differential I compared two types of insulation and three thickness of wall cavity, 2 x 4, 4 x 6, and 2 x 8. Although the differences were sometimes only a few cents per month, I chose the 2 x 8 construction with cellulose blown in under pressure (to prevent settling). Extra insulation has several advantages other than simple fuel cost savings. One advantage is that the overall temperature of the room is more uniform with good insulation. In a poorly insulated room the difference between the temperature at the ceiling and at the floor might be as much as 10°--say 75° at the ceiling and 65° on the floor. When people have cold feet, they feel cold all over and turn up the thermostat. So even though the average temperature is say 70°, the effect is that of a lower temperature. A second advantage is that mortgage interest is usually higher than savings account interest. In other words, the cost of insulation money will probably decrease when the mortgage is paid off, while the cost of fuel will likely increase. My conclusion is that spending more on insulation now will be repaid on a rising curve basis over time and not on a straight line basis. A final advantage is that some fuels may become exhausted or scarce. It may therefore become necessary to get the most efficient use possible from what fuels are available regardless of cost.

Has this heavy insulation paid off. I have no doubt that it will. At present, for example, I am heating three floors- a 1500 square foot basement, a 1500 square foot first floor, and a finished attic of about 1000 square feet all with one wood stove in the basement using about six cords of wood a season, the 1979 cost of which was about $350. I had put the wood stove in for a back-up heating source. It has turned out to be entirely sufficient in itself.

One final situation where the calculator is very helpful is in determining the amount of concrete to order. Concrete is sold by the cubic yard, and for basement floors, patios, sidewalks, and driveways it is usually placed about four inches thick. Twenty seven cubic yards will cover 81 square feet at four inches thick.

27 cubic feet = 27 x 12" x 12" x 12" = 46656 cubic inches.

A cubic foot of concrete four inches thick is 12" x 12" x 4" = 576 cubic inches.

Therefore, $\frac{46656}{576}$ = 81 square feet at 4" thick.

Divide the total area of concrete by 81 to find number of yards needed. A basement 30' by 52'8" would require about 20 yards of concrete.

$$\frac{30' \times 52'\,8''}{81} = \frac{30' \times 52.667'}{81} = \frac{1580}{81} = 19.5$$

But you would probably need to order at <u>least</u> 20 yards--the ground is never even, and the calculations are usually approximations. Also, it is better to have too much than not enough. It is a good idea to have a place ready for any excess concrete, such as a sidewalk or trash can area.

There is no doubt that the calculator saved me a great amount of time in making many, many computations. It also surely reduced the number of errors I would have made by having to compute everything with pencil and paper.

But there is another bonus. Some of the computations would have been so cumbersome without the calculator that they might not have been made at all. I've forgotten how to do square root and would probably have fallen back on the 3-4-5 rule without the calculator. Therefore, the layouts of the foundation and of the various rooms of the house are no doubt more accurate than they would have been without use of the calculator.

Similarly, the insulation problems would have been rather staggering without the calculator, if only because of the many times I had to solve the two equations. Certainly, I would have been tempted to investigate fewer combinations.

But there is one important result that is quite different from all the others. Being able to carry out otherwise tedious computations quickly and accurately allowed me to adopt a more open, searching, problem solving attitude. I began to look for problems and novel solutions. It became a sort of game to challenge the old conventions and to seek better solutions.

Sometimes the old ways worked best, such as using the carpenter's square in laying out cuts for the staircase carriages. But those situations proved to be the exceptions. Generally, I was able to find better solutions--better for me, at least--and I had fun doing it.

This result, therefore, may well have been the most important one of all.

Good tools make any job easier. If they can also change the job from a chore to a challenge, they open the way for more creative behavior. In addition to the possibility of finding innovative solutions to old problems, they may lead to a personal change toward a more constructive, problem solving attitude.

The old ways may sometimes be best. But if no one had ever been challenged to strike out in a new direction, to search for a better world, we'd all still be writing on the cave walls.

Somewhere in all that there must be Brunk's First Law.

GENERAL MATHEMATICS

General mathematics, as used in this paper, refers to mathematics courses in grades seven and eight and to nonalgebra alternatives in grades nine through twelve. This set of mathematical experiences is expected to perform a variety of functions in the mathematics curriculum:

(1) broadening, enriching, and finalizing the principally arithmetic program of the elementary;

(2) preparation for individual's social responsibilities in society;

(3) extension of capabilities for meeting personal and vocational needs; and

(4) formalization of arithmetic and nonarithmetic understandings as preparation for traditional academic coursework associated with professional opportunities.

Functions (2) and (3) from this list primarily entail application of mathematics to real-life needs--an area of major interest at the present. These functions can be further translated into the following student goals:

(a) to recognize and use the mathematics necessary for one's own consumer affairs, selecting and using goods and services wisely--choosing and buying personal items, products for a home, for a family, or for a small business;

(b) to recognize and use the mathematics necessary for other quantitative aspect of one's daily life--mathematics related to cooking, sewing, gardening, minor home repair, personal projects, money management, and recreation;

(c) to possess requisite mathematics to enter a job situation--appropriate background, learning skills, and attitudes that additional job-related applications of mathematics are learnable with relative ease; and

(d) to use logical reasoning and problem solving in reaching decisions, weighing evidence, and interpreting data to arrive at sound conclusions.

Achievement of these student goals requires understanding and reasonable competence in arithmetic computation, including oral, paper-pencil, and calculator components--supported by estimation and approximation skills; understanding and use of simple algebraic formulas; abilities to interpret and analyze graphs and statistical information; knowledge of geometric forms and spatial relationships; understanding and use of measuring instruments, including the relation of geometric and numerical concepts as the mathematics of measuring instruments; and problem solving and logical abilities, ranging from skill with simple word problems to more complicated analyses of situations requiring an awareness of productive reasoning techniques.

General mathematics has generally not realized these student outcomes to the extent that most educators and some others desire. The purpose of this paper is to examine some underlying causes for this lack of success and to offer some recommendations for strengthening this area of the mathematics curriculum.

A Question of Direction and Purpose

Historically, general mathematics began to emerge around the turn of the century as arithmetic moved down from the academies and high schools to become an elementary school subject. Mathematics was gradually being perceived less as an instrument to develop the mental faculties of future professional people and more as a utilitarian tool for the increasing numbers of individuals going on to secondary schooling. During this time, instruction was changing from lengthy examinations of involved problems requiring complex manipulations, memorizations, and theorem-proof approaches to a more fragmented fact and subskill approach.

General mathematics was given its first national impetus in 1922 when the National Committee on Mathematical Requirements advocated a general mathematics program for grades 7-9 which would include topics from arithmetic, algebra, intuitive geometry, numerical trigonometry, graphs, and descriptive statistics.[1] The evolving junior high school with its philosophy on exploration provided a ready means for this recommendation to be implemented in grades seven and eight. However, the more conservative posture of high schools prevented general mathematics from making firm inroads into the ninth grade as an extension of ideas in grades seven and eight. Algebra remained the most frequent offering with general mathematics planned and taught as a lower-level alternative to algebra rather than as a continuation of a broadly planned general mathematics program.

The influences of John Dewey and the depression brought more pressure for change in the high school curriculum. An increasing demand for mathematics for vocational and personal needs led to some deemphasis of sequential course work. This was accompanied by drops in enrollment in academic subjects amid criticisms that many topics had little or no utility for the general student.

By the end of World War II, concerns which surfaced during the induction and training of draftees provided further support for mathematics to play a more important role in general education and in the vocational and personal needs of students. Teachers responded by rallying about the notion of *functional competence* for graduates. Such competence was generally gauged by 29 items identified by the 1945 NCTM Commission on Post-War Plans. Mathematical literacy was cited as a necessary skill for effective citizenship in a democracy.

In Thesis 12 of its recommendations, the Post-War Committee took great pains to emphasize the importance of general mathematics at grade nine as a worthwhile course that focused on competence in applying mathematics to a variety of topics. It warned teachers to avoid stigmatizing general mathematics as the penalty for failure in algebra and to avoid propagandizing unduly for algebra.

However, efforts to place general mathematics on an equal and parallel plane with algebra went largely unrealized as the number of students continuing their education into high school swelled dramatically, widening already existing ability range differences. The division between algebra and general mathematics was further heightened by the post-Sputnik era of the sixties when tremendous resources and attention were given to structure, formalism, and insertion of upper level mathematical ideas into lower grades. These curriculum innovations were associated with strengthening those aspects of the program which led to academic coursework at the high school level. Similar effort and resulting prestige were not given to general mathematics nor to any resulting expansion of such alternatives beyond grade nine.

Since the sixties, school consolidations and increasing awareness of the needs of students who will likely terminate their education with high school stimuated some programs to expand their offerings in general mathematics to include business mathematics, consumer mathematics, vocational mathematics, and similarly named courses. However, an integrated general mathematics program has still not emerged. New courses in general mathematics generally continue to be regarded as those for the less successful and less willing. Emergence of pre-algebra courses and continued experimentation with algebra as an eighth-grade elective lend further support to the notion that "algebra-like" courses are still the best choices for those students who "can achieve."

Despite the good intentions and far-sighted recommendations of prominent groups of educators throughout the history of general mathematics development, it has never achieved a well-defined, independent role in the mathematics curriculum. Rather, general mathematics programs continue to be hodgepodge collections of remedial alternatives generating little student interest, professional commitment, and enthusiasm.

A Question of Instruction

The Thorndike influence on mathematics instruction (segmented drill-work on subskills) continues to characterize many general mathematics programs. Most topics are presented as tasks by example. Teachers and students alike seem to view mathematics learning as small, incremental steps in which it is the teacher's responsibility to present a new technique to be practiced and the student's to parrot these actions in nearly identical situations. "Show me how to do this set of problems, don't assign too many, and then give me class time to do them." Furthermore, text materials are organized as single concept lessons requiring little more than rote-like responses to generate satisfactory answers. And few text materials consist of problem backgrounds or applications that students can identify with.

This instructional picture, though not universal, does unfortunately characterize too many programs. But reasons behind acceptance of such low-level teaching goals are mixed:

(1) teachers are well-meaning, but are frustrated by unsuccessful attempts to provide alternative approaches;

(2) teacher preparation programs have largely ignored or not had time to prepare prospective teachers with the skills to handle general mathematics courses or student needs;

(3) ability ranges in such courses are so wide that lower level instructional outcomes are easiest to achieve, and perhaps, the only ones possible for some students;

(4) students have developed attitudes about mathematics and themselves in previous courses that strongly influence the kind of instruction they expect and will cooperate with;

(5) laboratory experiences necessary to supply conceptual understandings underlying many applications are difficult or impossible to organize due to limitations of space, time, student load, resources, and teaching preparation;

(6) teachers are not always aware of students' true abilities and may lack the evaluation techniques and instruments to accurately assess the extent to which students can use their mathematical skills;

(7) due to their preparation, some teachers are predisposed away from general mathematics and are, consequently, not highly motivated to commit the required teaching energies for coursework they deem to be remedial;

(8) conversely, others make an honest attempt to salvage students for the "good" academic courses by stressing prerequisite computational skills;

(9) some teachers lack a full understanding of the interrelationships of knowledge, concepts, applications and problem solving in enabling students to be able to use the mathematics they possess.

Instructional methodologies employed in most general mathematics classes, no matter how well-meaning, create an overly-simplistic and artificial view of how mathematics is used to meet real problems. The student has no exposure to the comprehensive process of addressing a real problem, analyzing it mathematically, employing problem solving skills, and interpreting the results into personally meaningful decisions. He therefore is able to draw little association between classroom activities and the personal needs he faces outside the classroom.

A Question of Outcomes

General mathematics students usually know more computationally than they are credited with knowing or that they realize, almost always know less about using their knowledge and skills than is presumed by their teachers, and perceive mathematics as an activity you do in a classroom--in the real world, you learn and use other means to cope with problems.

These observations, supported by assessment information and the experiences of veteran master teachers, highlight the need to place more emphasis on making mathematics knowledge operational. The failure of many existing programs to assign this goal a higher priority is quite possibly a product of a belief in a linear hierarchy of student learning outcomes.

SKILLS & CONCEPTS	APPLICATIONS	PROBLEM SOLVING

Understanding of concepts and mastery with skills, mostly computational, are perceived to be the requisite for any further application of mathematics. Therefore, a majority of time is presumed to be justifiably spent on development of skills which are deemed unsatisfactory. Applications are assigned some importance, but primarily, as vehicles to practice skills. Therefore, relevance of application situations to students' real-life problems is not considered as essential as packaging the application situations in small enough tasks to assure computational attack.

Problem solving is perceived as an intellectual activity that many general mathematics students are not able to undertake in any formal sense, for want of prerequisite background, weak learning skills, and poor attitudes. Hopefully, development of a strong skill background and work with some, admittedly contrived, applications will permit students to transfer their knowledge and skills to later-life needs as they develop. And perhaps, problem solving development can be undertaken by other teachers in other courses.

The lack of success with this approach to general mathematics instruction is evidenced in concerns voiced over assessment results, by back to basics critics, and by concerned employers. In place of the linear model for student outcomes, a more comprehensive view is needed to assure student <u>use</u> of what they learn. It is essential to have reliable assessment information which accurately reflects student capabilities, so that instruction is not misdirected nor overestimated. It is necessary to have parental and community input as to their concerns, expectations, and needs for mathematically prepared graduates. And student feedback merits careful review concerning attitudes, perceptions, and expectations. This information then needs careful review by mathematics teachers and other educators concerning the role that mathematics is expected to play in students' total educational preparation and how best to achieve this role in the mathematics program.

After these activities have been completed, the key to success is balance in the way that instruction is organized. Skills and concepts, applications, and problem solving need equal emphasis, sufficient instructional time, and concurrent development. No aspect of instruction should be sacrificed or subordinate to another. To assume otherwise is to risk the chance that some students will never have exposure to higher level outcomes for having failed to achieve other goals. Furthermore, the learning setting must present instructional activities that model real world situations currently relevant to students. Within these situations, the combination of abilities required to apply mathematics will naturally come into play. Planning and indepth thinking become a necessity. Motivation and reason for skill and knowledge mastery are self-evident. And transfer becomes less nebulous as mathematics bridges the gap between classroom and real-life activities.

Recommendations

For general mathematics programs to better meet the needs of the student audiences they serve, changes are needed in philosophy, goals, content, methodologies, and evaluation. The following recommendations address some of the problems identified in this paper.

1. General mathematics programs need a comprehensive reexamination that considers: (a) their roles in the total school and mathematics curricula; (b) community expectations concerning student competencies; (c) student learning characteristics and modalities; (d) instructional techniques and methodologies; and (3) breadth and depth of offerings as part of the total mathematics and school curricula.

2. Program experiences should be designed as an integrated sequence of instruction from grade seven through high school that assures a balance in program goals, student outcomes, and learning activities. Running throughout each course should be a continuing regard for the problem solving skills necessary to apply mathematics to relevant and realistic student situations.

3. Every effort should be made to preserve the integrity of general mathematics offerings as desirable and parallel alternatives to traditionally "academic-type" courses, differing primarily in goals only. If courses that are primarily remedial in character

are required to meet student needs, they should be developed as separate entities of the total mathematics curriculum rather than part of the general mathematics program.

4. Regular assessment of program activities should occur at several levels. Each course should be monitored against its particular goals. The total program mertis review in terms of success in achieving exit competencies (not restricted to minimums). And longitudinal studies are desirable to reflect changes in course and program effectiveness that signify the need for curriculum revisions.

5. Proficiency required with computations, written and mental, warrants extensive review in light of accessibility of hand calculators, changeover to metric measurement, and ways that mathematics is used in contemporary society.

6. Teachers need to work together more frequently in assessing, restructuring, and carrying on the general mathematics curriculum. Cooperation can improve articulation of effort, provide necessary manpower for study and changes, and enhance the overall image of the program.

Summary

We have seen a flurry of activity in the past few years that has surfaced the kinds of concerns that are most frequently addressed within the general mathematics program: NIE Basic Skills Conference, National Assessment information, NCSM Position Paper, back to basics movements, and minimum competency concerns. But is important to note that the need to reorganize and strengthen the general mathematics program is historically a concern of long standing and not a new problem. Only, the seriousness of the need is, perhaps, made more accute by the demand for greater mathematics literacy as society becomes more complex and places more demands upon the individual for personal coping skills.

Reviews of General Mathematics Textbooks

Publishers of general mathematics texts have responded in varying degrees to concerns for more emphasis on life-related skills and applications. The following textbook reviews of some of the major general mathematics books currently available have been prepared to assist you in making textbook selections appropriate for your needs. Highlighted in these reviews are the emphases which particular texts give to problem solving, applications, and calculators.

Bolster, L. Carey and Woodburn, H. Douglas.

Mathematics in Life. Glenview, Illinois: Scott Foresman, 1977 432 pp.

This text is designed for a first course in general mathematics. Emphasis is on basic mathematical skills, with consumer and career applications that reinforce these skills. The book is divided into six units of three chapters each. The topics included in these units are:

1. Arithmetic of whole numbers.
2. Decimals and the metric system.
3. Fractions, mixed numbers, and probability.
4. Ratio, proportion, similarity, percent, and statistics.
5. Signed numbers, equations, and graphing.
6. Perimeter, area, volume, the Pythagorean rule, and trigonometry.

Each chapter consists of a pretest, several skill lessons, followed by a posttest, a couple of application lessons, and one or two career lessons. Each chapter also includes one puzzle type problem as a "break time." There seems to be a good balance between skills and applications. The chapter tests cover both aspects. At the end of each unit there is a page of calculator exercises. These optional exercises involve skills taught in the unit and can be done on a simple four-function calculator.

The book follows an open format, and many pages are illustrated with photographs. End-of-book materials include answers to odd-numbered exercises, tables, a careers chart, glossary, and index.

The teacher's edition has answers to exercises overprinted in a second color, along with 48 pages of notes and tests. It also states an objective for each lesson. Other supplementary materials include a separate solution key and duplicating masters for tests and record keeping.

The index contains numerous references to problem solving. However, the authors apparently consider "problem solving" to be a synonym for "applications." No explicit instruction in problem solving strategies is given.

Bolster, L. Carey, Woodburn, H. Douglas, and Gipson, Joella H.
Consumer and Career Mathematics. Glenview, Illinois: Scott Foresman, 1978. 454 pp.

This text is recommended for a final course in general mathematics (following Mathematics in Life), or for a general course in consumer mathematics. Emphasis is on the mathematics needed in consumer and career situations, including a review of basic skills. There are six units of three chapters each. The topics of these units are:

1. Mathematics skills, including whole numbers, decimals, fractions, equations, proportions, percent, measurement, and statistics.
2. Income, banking, and credit.
3. Transportation, including travel and owning a car.
4. Housing, including renting, buying, building, and maintaining a home.
5. Taxes, insurance, and investments.
6. Purchasing and budgeting.

The chapters of the first unit emphasize the mathematics itself, while the other chapters treat various applications of mathematics in realistic situations. Each of these later chapters includes a "skills tune-up" page that is pure arithmetic drill.

Every chapter has a page of (optional) calculator exercises. These exercises usually involve numbers that make hand calculations tedious, and all of them can be done on a four-function calculator. Each chapter also includes a review page and a chapter test. Another interesting feature of chapters 4 - 18 is the discussion of various careers in which math skills are needed.

Most pages have a two-column format, illustrated with many tables, diagrams, sketches, and photographs. The two-color printing highlights key ideas on each page. End-of-book materials include a skills file of additional drill exercises, tables for taxes and measurement, a careers chart, glossary, selected answers, and index.

Supplementary materials include a teacher's edition, consisting of the student edition with answers overprinted in red, along with stated objectives and warm-up exercises for each lesson. This teacher's edition also contains 40 pages of notes and a minimum competency test. A separate solution key and duplicating masters for consumer forms and tests are also available.

No explicit attention is paid to problem solving.

Bragg, Sadie; Couzins, Harry; Gilbert, Glenn; and McKillup, William, General Mathematics: Skills and Applications, Silver, Burdette, 1979, 472 p.

Computational skills and applications are equally emphasized in this general mathematics text. Sections in each chapter on calculator usage, problem solving skills, and "Something Different"(games, puzzles, etc.) are noteworthy features. Four case studies (e.g. Driving a Truck) are also included. Topics are:

1. Whole Numbers
2. Adding and Subtracting Decimals
3. Multiplying and Dividing Decimals
4. Fractions
5. Measurement
6. Ratio and Proportion
7. Percent
8. Geometry
9. Statistics and Probability

The format of each chapter is:

1. Pretest on skills
2. Skill development ("Sharpen Your Skills")
3. Practice ("Practice Your Skills")
4. Post test on skills
5. Problem solving skills
6. Applications (Section titles include: Sports, Food and Clothing, Wheels, Jobs and Careers, Environment and Ecology, Recreation, Travel, Money Matters, Earning a Living, Managing a Home)
7. Chapter test

The Applications portion constitutes 50 percent of each chapter. "Using Your Calculator" and "Something Different" are interspersed throughout. In addition to the customary drill exercises, there are magic squares, cross number puzzles, etc. to practice skills. The chapter tests, however, do not reflect the heavy emphasis on applications.

"Using Your Calculator" includes routines for certain calculations (e.g. adding two fractions), guessing numerical curiosities, problems involving large and small numbers, and the like. Fourteen problem solving skills (e.g. estimating the answer, draw a picture, use a table) are discussed. Problems with insufficient and extraneous data are included.

The book is visually stunning. Multiple colors are used to highlight important points (as many as seven colors are used on a single page); photographs are used extensively in the Applications sections as are charts, tables, maps etc. containing real data (e.g. EPA Mileage Chart for Subcompact Cars). No minority group of any kind could possibly be offended as stereotypes are conspicuously avoided.

The teacher's edition consists of the student text with answers overprinted in blue and margin notes on (1) teaching hints, (2) activity suggestions (e.g. bulletin board, field trip), (3) background information on application topics, and (4) information about the photographs. Diagnostic tests, competency tests, a suggested timetable, and information on other resources are also included.

Clarkson, David M. and Gamble, Andy. General Mathematics. New York: American Book Company, 1980. 370 p.

A reviewer will immediately be struck by the uses of color, graphics, and photographs in this text. The layout is impressive.

Fifty computational skills ("checkpoints") have been identified and are listed for emphasis in the 15 chapters, and students' abilities can be analyzed through the use of almost 50 pages of tests in the appendix. The skills, such as, "dividing by multiples of 10" are identified in the teacher's edition by chapter and page. Chapter titles follow:

1. Organizing Facts
2. Basic Operations
3. Measures and Rates
4. Lines and Angles
5. Ratio and Proportion
6. Graphs and Charts
7. Shapes and Constructions
8. Signed Numbers
9. Maps and Coordinates
10. Areas and Volumes
11. Chance and Prediction
12. Unusual Measures
13. Math and Ecology
14. Solving Equations
15. Indirect Measure

The text is unusual in many ways. The first section is about metric measure, which is used throughout the text. Each chapter includes several projects, experiments and games along with explanations and sample problems. Regretfully, all activities are highly structured so there is little opportunity for the student to do much independent thinking. The content of the problem settings are interesting, emphasizing topics such as data on nutrition and career descriptions and data. Although there are many impressive subjects for activities and interesting games, the students may be confused by the sequence of activities. Often it appears that the applications have dictated the sequence.

An Appendix section called "Calculator Workshop" has 20 calculator activities but most have no apparent educational goal. Except for another game in the body of the text, no real use of calculators appears.

Even if the book is not selected as a text, a teacher might find it useful as a reference.

(A second review of this book follows.)

Clarkson, David M. and Gamble, Andy. General Mathematics. New York: American, 1980. 370 pp.

This general mathematics text greatly limits content treatment to stress student involvement in applying mathematics. Each chapter begins with an interesting, but simple, application of mathematics involving a particular process. Students are challenged to solve similar but more difficult extensions of the same situation. One or more such lessons on different applications of the same process may follow. A related practice lesson requires students to use the ideas from previous applications in a novel situation without any model illustration. Intersperesed throughout chapters are also projects and experiments which require students to generate or collect their own data, organize and interpret it, and summarize their results. Interesting games are occasionally included which apply chapter concepts in competitive situations.

Although text lessons are predicated on prior student instruction with concepts and processes to be applied, provision is made for addressing missing prerequisitive skills. Each chapter is preceded by a checkpoint section which preassesses student concept and skill levels to be used in that chapter. An extensive appendix section contains drill lessons keyed to each idea of the checkpoint section.

The appendix also consists of a 10-page calculator workshop and a 4-page summary of career information in mathematics-applied fields. Occasional calculator and occupational reference pages also occur throughout the text.

Problem solving is not specifically stated as a goal for the book nor are any sections explicitly devoted to it. However, the underlying structure of the book is problem-oriented; therefore, the book is conducive to a problem solving approach.

Chapter topics are deceivingly suggestive of a more traditional content organization than the applications approach they encompass.

1. Organizing facts
2. Basic operations
3. Measures and rates
4. Lines and angles
5. Ratio and proportion
6. Graphs and charts
7. Shapes and constructions
8. Signed numbers
9. Maps and coordinates
10. Areas and volumes
11. Chance and prediction
12. Unusual measures
13. Math and ecology
14. Solving equations
15. Indirect measurement

A few early lessons introduce the calculator through flow charting; thereafter, calculator use is optional and referred to infrequently. Very few problem solving questions are presented and these as enrichment puzzles. No discussion of problem solving strategies is included.

Dilley, Clyde A. and Rucker, Walter E. Mathematics: Modern Concepts and Skills. Lexington, Massachusetts: D.C. Heath, 1974. 406 p.

The authors provide a well balanced text for general mathematics or pre-algebra. There is study of mathematical topics and of everyday applications of mathematics. There is an abundant number of drill exercises but the strongest aspect of the text is the large number of problems--not exercises in a given problem type but a variety of word problems.

Chapter titles follow:

1. Numeration Systems
2. Number Theory
3. Estimation
4. Algebra
5. Ratio and Proportion
6. Measuring Lengths and Angles
7. Geometric Figures
8. Area
9. Three Dimensional Figures
10. Algebra with Real Numbers
11. Probability
12. Statistics
13. Consumer Mathematics
14. Indirect Measurement

The standard topics of general mathematics, such as taxes, interest, and computational skill, are covered. More impressive is the inclusion of algebra, geometry (5 chapters!), probability, and statistics--all topics often denied to general mathematics students, yet of practical value.

As the authors claim,

> "Active involvement is stressed throughout this textbook. Each concept is developed intuitively by relating it to an appropriate model. Also, projects are included so that students can explore mathematical topics on their own."

Extensive use is made of artwork, color, graphics, and cartoon characters. "Thought clouds" are employed to explain the thinking behind calculations. There is no mention or use of calculators. As a matter of fact, problems appear to have been developed to control carefully the calculations involved.

Supplementary worksheets and tests are available as a book of duplicating masters (ditto sheets). Most teachers would find these 78 masters useful; however, the number of copies from each master is limited. This reviewer would rather have a set of the supplementary materials printed on paper so that thermal masters could be made as needed.

Lewis, Harry. Mathematics for Daily Living. Florence, Kentucky: McCormick Mathers, 1980. 534 p.

The text might be described simply as a financial survival kit. Each chapter is an analysis of a major component of personal finance. Chapter topics are the following:

1. Automobile Ownership
2. Commercial Transportation
3. Purchasing Consumer Goods
4. Personal Income
5. Income Tax
6. Banking
7. Banking Services
8. Small Loans
9. Investments in Stocks and Bonds
10. Insurance
11. Retirement Income
12. The Cost of Housing

Chapters consist of from 2 to 7 units which provide a "cook book" description of how to do the calculations for a subtopic. For example, the first chapter includes units on the loan, insurance, depreciation, operating costs, reading a road map, determining average speed, and travel costs. The chapter "Small Loans" includes units on the small loan agency, the pawnshop loan, the credit union loan, and the credit card loan. Discussions within units provide a brief overview of the topic and an explanation of how to solve a particular type of problem. A great many exercises follow in which the same format as the sample problem is used. Also, a supplementary workbook is 188 pages of exercises keyed to those in the text.

The exercises are very practical and realistic in the sense of using reasonable data or providing answers to questions useful to a consumer. Much of the data is taken from real life situations but the result of inflation may require very frequent revisions to maintain this characteristic. Problems, as opposed to exercises, are nonexistent. Also, no mention is made of using calculators even though arithmetic exercises, such as those with daily interest rates carried out to the seventh decimal place, would seem to justify their use.

The text appears to be most appropriate for the terminal student in the senior year, who will soon have to become concerned with personal finance.

Price, Jack; Brown, Olene; Charles, Michael; and Clifford, Miriam Lien. Mathematics for Everyday Life. Columbus, Ohio: Merrill, 1978. 415 p.

The emphasis in this text is on applications of mathematics in daily life activities. Computational and other related subskills are presented as key skill sublessons within the overall framework of applications-oriented lessons. Chapter topics are:

1. Tables and schedules
2. Reading and constructing graphs
3. Working with data
4. Reading measures
5. Useful calculations
6. Buying a car
7. Buying a house
8. Comparative shopping
9. Student supply store
10. Insurance
11. What's the angle?
12. Constructions
13. Polygons
14. Surface area and volume
15. Energy
16. Energy and conservation
17. Science and the environment
18. Health and nutrition
19. Mathematics in sports

The test is characterized by colorful visual presentations that use pictures of real-life materials (e.g. forms, tool readings) in lessons. The self-contained nature of individual lessons provides optimum flexibility to the teacher in selecting and using application topics. Career corner pages in each chapter provide short glimpses of mathematics applications in real jobs. Interwoven throughout many lessons are informative ideas and insights about consumer decision-making.

Each chapter is accompanied by pre-test and student self-test. The teacher's edition also includes diagnostic skill, chapter, and semester tests. Each lesson in the teacher's edition has student objective, answers, and suggestions overprinted in red. Daily assignment guides, unit introductions, chapter overviews, and lesson discussions make up a separate teacher's guide section.

Price, Jack; Brown, Olene; Charles, Michael; and Clifford, Miriam Lien. Mathematics for Today's Consumer with Career Applications. Columbus, Ohio: Merrill, 1979. 416 p.

This text is a special edition created by combining units from two other Merrill general mathematics books, Mathematics for the Real World and Mathematics for Everyday Life. Topics chosen from these books deal with consumer decisions. The result is a text devoted to the mathematics involved in consumer education. Chapter titles are:

1. Earning money
2. Checking account
3. Savings account
4. Credit cards
5. Income tax
6. Buying a car
7. Buying a house
8. Comparative shopping
9. Student supply store
10. Insurance
11. Tables and schedules
12. Reading and constructing graphs
13. Working with data
14. Reading measures
15. Useful calculations
16. Length and distance
17. Other common measures
18. Ratio and proportion
19. Area
20. Volume

Chapter 6-15 of this book are identical to chapters 1-10 of Mathematics for Everyday Life. Additional chapters from the other Merrill general mathematics book provide an older emphasis to the daily life applications of this text. It is probably better suited as a consumer mathematics text than a general mathematics book.

Therefore, the format, features, and supplementary sections of this book are similar to those of the preceding Merrill book. However, the few lessons using calculators are delayed until chapter 13, and problem solving is not directly addressed in either discussions or enrichment problems.

Price, Jack et al. Mathematics for the Real World. Columbus, Ohio: Charles E. Merrill, 1978. 416 p.

The text is organized into 4 major units with 5 or 6 chapters in each, as follows:

Money Management
1. Earning Money
2. Checking Account
3. Savings Account
4. Credit Cards
5. Income Tax

Measurement and Estimation
6. Length and Distance
7. Other Common Measurements
8. Ratio and Proportion
9. Area
10. Volume

Statistics and Probability
11. Gathering and Displaying Data
12. Describing Data
13. Experimental Probability
14. Sample Spaces
15. Using Statistics and Probability

Patterns in Mathematics
16. Patterns in Mathematics
17. Operations
18. Number Theory
19. Solving Open Sentences
20. Problem Solving
21. Sequence and Series

The preface to teacher's guide indicates that the text "...provides a comprehensive course in general mathematics. The approach is based upon educational research of the senior author along with research findings in the general math (sic) area." This reviewer doubts the validity of these statements. The text was found to be entirely skill oriented and completely devoid of any problem solving experiences--even in the 12 page chapter entitled "Problem Solving." Some of the sections of this chapter are: Formulas, Problems involving Multiplication and Division, Ratio, and Percent. The self test for this chapter has 30 "problems." Nine are formula "plugging" (e.g. d = rt given. Problem 3. r = 55, t = 3½; "evaluate" the formula.) Another 8 of 30 problems are "Solve the Proportion" (e.g. 14/b = 7/8). Another 8 are percent exercises (e.g. "30% of what number is 10?") The remaining 5 problems have words (e.g. 12. Find the cost of 3 items which cost 27¢ each.) The longest "problem" follows.

> "30. A supercharger will increase the horsepower on Tom's car by 20%. If the current horsepower is 150, how much can it be increased by installing a supercharger?" (p. 372)

Overall the text contains little explanation of mathematics nor of the real world. It contains hundreds and hundreds of exercises which make a reviewer grateful he is not a student assigned this text!

Shulte, Albert P. and Peterson, Robert E. Preparing to Use Algebra (2nd edition). River Forest, Illinois: Laidlaw, 1978. 493 p.

This text is designed for (1) students not ready for a first course in algebra ("contains all the basic mathematical content required for ... success in algebra") or (2) students who may not pursue a college-preparatory program ("emphasis on computational skills and practical uses of mathematics is most beneficial to these students"). Topics included are:

1. Simple sentences
2. Solving open sentences
3. Ordered pairs and graphing
4. Number theory
5. Operations and their properties
6. Operations with fractions
7. Operations with decimals
8. Integers
9. Sentences with integers
10. Problem solving
11. Ratio and proportion
12. Indirect measurement
13. Rational numbers
14. Using algebra
15. Approximation and measurement
16. Square root (optional)

Each chapter contains a vocabulary and chapter review, a chapter test, and activities, with special topics interspersed. An application of mathematics in a career (flying, photography, space travel, typesetting, air conditioning, power lines, plumbing, police work, etc.) is included at the end of each chapter (1-2 pages).

The problem-solving chapter focuses on writing equations and making tables. A brief introduction to each chapter sometimes cites problem solving, presumably as motivation. Thus, the introduction to chapter 1 indicates that: "Before solving problems, a good problem-solver needs a set of mathematical tools. In this chapter you will learn when to use some of those tools ..." (p. 1) Chapter 7 begins: "This will enable you to use decimals when solving some of the problems you meet every day" (p. 190). No other explicit attention is given to problem solving.

The format is open, with clear illustrations. The general procedure involves a given instance (example), statement of a rule, more examples, and practice exercises. The exercises are coded for oral, written or optional extension, and optional lessons are noted ("for flexibility"). The teacher's edition consists of the student textbook with answers overprinted in red, and a separate section with performance objectives, comments, and some suggestions on what to point out to students.

No reference is found on the use of calculators (or computers).

0.12

Wells, David W.; Shulte, Albert P.; and Choate, Stuart A. Mathematics for Daily Use. River Forest, Illinois: Laidlaw, 1980. 544 p.

The primary emphasis of this general mathematics textbook is on computational skills and practical applications. Topics included are:

1. Using tables and graphs
2. Making graphs
3. Equations and formulas
4. Ratio and proportion
5. Measurement
6. Plane figures
7. Scale drawings
8. Consumer skills
9. Solids
10. Chance and estimates

A "computing skills refresher", containing exercises on (1) whole numbers and decimals and (2) fractions and mixed numbers is included.

Special topics, a "career corner", and a "consumer corner" are interspersed. Each chapter includes a test, review work, and a preview test for the next chapter. Chapters begin with a "picture preview" of applications of mathematics to be developed in the chapter; other illustrations and photos are also included, in an attempt to provide a "highly visual", "readable" text with "practical content and a variety of applications of mathematics to everyday problems".

The lessons generally take the form of (1) illustration plus questions followed by (2) examples and exercises. Instance-rule-example-exercise is thus the pattern followed. The teacher's edition consists of an annotated student textbook plus a manual with background information, teaching aids, tests and answers, suggestions for organizing the course, and brief comments on each section.

Teaching of problem-solving strategies is not apparent, nor any allusions to the use of calculators (or computers).

P1.1

1980 Ohio Regional Conferences on Mathematics Education

Sharing Ideas with Colleagues

Dominoes

The 1980 Ohio Regional Conferences on Mathematics Education are supported by a grant from the National Science Foundation as a project in the program for "Dissemination of Information in Science Education." The purpose of such programs is to provide information to educational leaders and decision makers. The conferences themselves are not in-service activities for teachers. Rather, the intent is to reach superintendents, principals, supervisors, teacher leaders such as department heads, and college mathematics educators. Our goal is that this cadre of able people will reach the many teachers serving in Ohio's schools. The plan is that participants in the conferences will conduct some sort of in-service program for teachers in their own or nearby school systems. The Conference Development Team has attempted to assemble a packet of materials and to develop work sessions during the conferences that participants can use when they, in turn, provide information to their colleagues.

The purpose of this paper is to suggest some vehicles that participants can use to disseminate the ideas and resources of the 1980 conferences to their colleagues. In this way the project is designed to have a domino effect--and to reach a great many classroom teachers through a multiplier effect. The six members of the Conference Development Team come from different areas of Ohio. Through the conferences they will reach about 300 educational leaders in Ohio. If each participant transmits some of the conference ideas to 10 teachers, more than three thousand teachers will be reached. Some conference participants have the position and ability to reach one hundred or two hundred teachers--that is the challenge!

Using the Packet

The packet materials have been assembled as an aid for participants when they plan and conduct in-service activities. You may find one or more papers which you can use directly as they are written. There may be others that you want to modify for the special situation in your school. The major point is that the packets were assembled to be used. Participants should feel free to use, modify, or adapt the packet materials.

What to Do

What can you as a participant do after the conference? This section is a listing of some of the activities conference participants might consider in order to disseminate ideas they find useful from the conferences.

P1.2

In-service

A great many applicants for the conferences indicated that their school system has scheduled in-service meetings for teachers and that they would have an opportunity to make presentations at these meetings. We would suggest that you don't try to do too much if you are preparing an in-service session. In particular, please don't try to condense two days of conference meetings and 300 pages of packet into a one hour presentation. We suggest that you carefully select a few elements of the conference or packets that you found interesting and useful. You might want to reproduce some papers or parts of papers to distribute before, during or after your in-service session.

Newsletter/circular

Even if you don't have an opportunity in the near future to conduct an in-service meeting *per se*, you might be able to review packet materials and notes from the conference to compile a small packet of materials for your colleagues. With perhaps a single cover page explaining what is attached and how it might be used, you could prepare copies of materials to disseminate. If your school system has a newsletter, you could make arrangements to submit small pieces over a period of time--perhaps "Problem Solving Ideas" or a "Calculator Corner" section could be established.

Resource Center

Supervisors and department heads might establish a resource center for teachers. Such a center might be a file drawer where all teachers can contribute ideas, problems, or papers to be used by everyone. A department meeting to discuss the resource center might be a vehicle to help establish it and improve its utilization.

Faculty Meetings

Perhaps the mathematics faculty of a school could meet together to consider what can be done to improve problem solving abilities in all or selected courses. We know one school where the principal has developed the schedule so that all mathematics teachers have the same planning period. These teachers meet almost every school day to work on improving the program and individual courses.

Discussion Meetings

If faculty meetings are not feasible, some department heads may consider a special discussion meeting--a very informal session for faculty to consider a selected topic.

Demonstrations

Those conference participants who have classroom responsibilities will, hopefully, want to try some activities out in their own classes. After a few try outs they might effectively disseminate some of the things they have learned by inviting other teachers to visit their classrooms during a problem solving or calculator class. We believe that seeing students in classroom settings using

 P1.3

new ideas is the best stimulus for making any modifications in a program. By opening your room to observation, you may be able to open a lot of doors for other teachers to try things.

Alternative techniques to demonstrations in your class are (1) demonstrations in other teachers' classes, (2) videotaping parts of your classes for other teachers to view or (3) describing your experiences in the classroom with other teachers.

Action Research

Some teachers like to try new techniques, curriculum modules, or instruction strategies. Often they can develop a mini-research study to ascertain the effect of the special treatment compared to a control group. Such evidence can be very useful to justify a more comprehensive change.

And...

The suggestions in this paper are only a part of what can be done. Follow-up of the 1978-79 Ohio Regional Conferences (Problem Solving and Use of Calculators in Elementary Schools) proved that participants can devise ingenious ways to share ideas and disseminate conference materials with their colleagues.

P2.1

Conducting Problem Solving Workshops

One purpose of this conference is to help prepare you to provide inservice instruction on teaching problem solving to colleagues in your district. To that end we have prepared outlines of several appropriate workshops. Two of the workshops are designed for junior high school teachers (an 8-hour workshop and a 2-hour workshop) and two are designed for high school teachers (also 8-hour and 2-hour workshops).

The only difference between the junior high and high school workshops is in the selection of problems to be used. In all cases the problems were selected to illustrate a wide variety of problem solving strategies. The problem lists are, of course, merely suggestions. You should feel free to substitute your own favorites.

The differences between the short and long workshops are in the numbers of problems to be solved and in the inclusion in the long workshops of time for participants to try out some of the ideas they have learned by leading other participants in problem solving sessions. These sessions should then be discussed in terms of the teaching skills introduced during the workshop.

Your participants should, of course, be aware that the workshop will not make them experts in teaching the problem solving processes. At best it can heighten their awareness of problem solving strategies, introduce them to teaching behaviors that will increase the probability that their students will learn to solve problems, and encourage them to broaden their teaching goals to include problem solving.

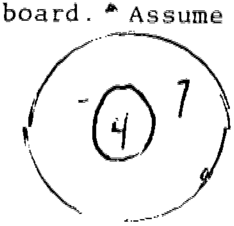
P2.2

Outline of an 8-hour Workshop on Teaching Problem Solving

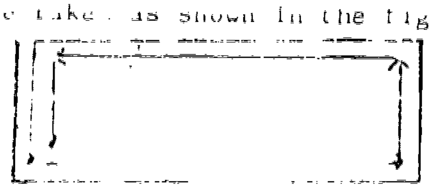
For Junior High School Teachers

I. Introduction (15 minutes)
   A. What is a problem?
      1. Blockage.
      2. Acceptance by students.
   B. Why teach problem solving skills?
      1. NCSM and NCTM lists of basic skills.
      2. National assessment and results.
   C. Schedule of workshop and purpose of each activity.

II. Solving problems (2 hr. 45 min.)
   A. Handshake problem - Suppose that 10 people meet for the first time and that each shakes hands with each other exactly once. How many handshakes will there be?
   (Some appropriate strategies that can be used are: do an experiment, draw a diagram, make a list, do simpler but similar problems looking for a pattern, logical reasoning)
   B. Dartboard problem - What is the greatest number that can not be scored on this dartboard. Assume that you have infinitely many darts.



   (Strategies: make a table and look for a pattern, deductive reasoning)
   C. Louie the Gambler - Louie the Gambler took his week's paycheck to the casino. He paid the $2 entrance fee, lost ½ of his remaining money and then tipped the hatcheck girl $1 on his way out. This bad luck persisted; the same sequence of events happened each day for the next 4 days. At the end of the fifth day he had $5 after he left the casino. What was the amount of his weekly paycheck?
   (Strategy: working backward)
   D. Using the digits 1, 4, 6, 7, and 9 form a 3 digit integer and a 2 digit integer whose product is as large as possible.
   (Strategies: systematic trial and error, exhausting all possibilities)
   E. Raking hay problem [illegible] field [illegible] long and 200 [illegible] How many circuits of the field must the farmer make to have raked [illegible] of the hay? He rakes as shown in the figure



[illegible] point of view (Note [illegible] to [illegible] the problem is to consider the area that has been raked and subtract that from the original area. Using another point of view is to consider the areas of the unraked rectangles after each full round of the rake.)

III. Teaching problem solving skills (4 hrs. 30 min.)

A. Objectives
   1. Selecting objectives that involve the use of problem solving skills.
   2. Assigning priorities to objectives.

B. Reinforcing students.
   1. Reinforcing according to objectives.
   2. Reinforcing approximations of final behaviors.

C. Testing problem solving skills.
   1. Testing the ability to use selected strategies.
   2. Testing the ability to choose appropriate strategies.
   3. Formal and informal evaluation.

D. Providing help - asking questions.
   1. When to help.
   2. Giving as little help as possible.
   3. Giving help without guiding toward specific strategies or solutions.
   4. Wait time after asking questions.

E. Holding review sessions after problems are solved.
   1. How was the problem solved?
   2. What made the solver think of his solution or method of solution?
   3. What are the advantages and disadvantages of the method?
   4. Is there a different method or different solution?

F. Practice (micro teaching) sessions for workshop participants.

IV. Classroom organization. (30 minutes)

A. Teaching a unit on problem solving and then applying the skills regularly.

B. Using one day a week for problem solving.

C. Skill centers.

D. Using a problem solving approach as a basic teaching strategy.

Outline of a 2-hour Workshop on Teaching Problem Solving

For Junior High School Teachers

I. Introduction (15 minutes)
  A. What is a problem?
    1. Blockage.
    2. Acceptance by students.
  B. Why teach problem solving skills?
    1. NCSM and NCTM lists of basic skills.
    2. National assessment results.
  C. Schedule of workshop and purpose of each activity.

II. Solving problems (1 hr.)
  A. Handshake problem - Suppose that ten people meet for the first time and that each shakes hands with each other exactly once. How many handshakes will there be? (Some strategies that can be used are: do an experiment, draw a diagram, make a list, do simpler but similar problems looking for a pattern, deductive reasoning.)
  B. Using the digits 1, 4, 6, 7, and 9 form a 3-digit integer and a 2-digit integer whose product is as large as possible. (Strategies: systematic trial and error, exhausting all possibilities.)

III. Teaching problem solving strategies (30 minutes)
  A. Objectives.
    1. Selecting objectives that involve the use of problem solving skills.
    2. Assigning priorities to objectives.
  B. Reinforcing students.
    1. Reinforcing according to objectives.
    2. Reinforcing approximations to final behaviors.
  C. Testing problem solving abilities.
    1. Testing the ability to use selected strategies.
    2. Testing the ability to select appropriate strategies.
    3. Formal and informal evaluation.
  D. Providing help - asking questions.
    1. When to help.
    2. Giving as little help as possible.
    3. Giving help without guiding students toward specific strategies or solutions.
    4. Wait time after asking questions.
  E. Holding review sessions after problems are solved.
    1. How was the problem solved?
    2. What made the solver think of the [illegible]
    3. What are the advantages or disadvantages of the method?
    4. Is there a different method or different solution?

[illegible] organization. (30 minutes)
  A. Teaching a unit on problem solving and then applying the skills regularly.
  B. Using one day a week for problem solving.
  C. Skill centers.
  D. Using a problem solving approach as a [illegible] teaching strategy.

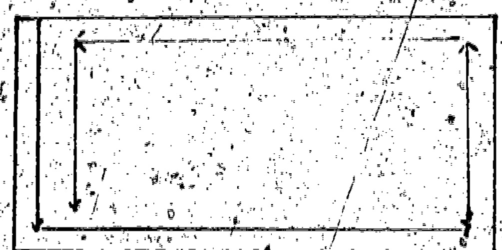
Outline of a 8-hour Workshop on Teaching Problem Solving

P2.5

For High School Teachers

I. Introduction (15 minutes)
  A. What is a problem?
    1. Blockage.
    2. Acceptance by students.
  B. Why teach rpoblem solving skills?
    1. NCSM and NCTM lists of basic skills.
    2. National assessment results.
  C. Schedule of workshop and purpose of each activity.

II. Solving problems (2 hrs. 45 min.)
  A. Painted cube problem - Imagine a large wooden cube that is painted black and then cut into 1000 congruent small cubes. How many of the small cubes have 0 faces painted? One face painted? Two faces painted? Three faces? etc. (Strategies that can be used include these: experiment, solving simpler but similar problems looking for a pattern, make a drawing, systematic counting.)
  B. What is the remainder when 5,999,999 is divided by 7? (Stragegy: Simpler but similar problems.)
  C. Raking hay problem - Given a hay field 600 m long and 200 m wide, how many circuits of the field must the farmer make to have raked one half of the hay? He rakes as shown in the figure.



Write and solve an equation. (Strategies: Draw a picture, shift point of view. Note: One way to do the problem is to consider the area that has been raked and substract that from the original area. Using another point of view is to consider the areas of the unraked rectangles after each full round of the rake.)
  D. What is the maximum number of pieces into which an orange can be cut with 10 cuts? (Strategies: experiment, simpler but similar problems.)
  E. Calculate--
  $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$
  (Strategy: working backward and looking for a pattern.)

III. Teaching problem solving skills (4 hrs. 30 min.)
  A. Objectives
    1. Selecting objectives that involve the use of problem solving skills.
    2. Assigning priorities to objectives.
  B. Reinforcing students.
    1. Reinforcing according to objectives.
    2. Reinforcing approximations of final behaviors.

P2.6

C. Testing problem solving skills.
   1. Testing the ability to use selected strategies.
   2. Testing the ability to choose appropriate strategies.
   3. Formal and informal evaluation.

D. Providing help - asking questions.
   1. When to help.
   2. Giving as little help as possible.
   3. Giving help without guiding toward specific strategies or solutions.
   4. Wait time after asking questions.

E. Holding review sessions after problems are solved.
   1. How was the problem solved?
   2. What made the solver think of his solution or method of solution?
   3. What are the advantages and disadvantages of the method?
   4. Is there a different method or different solution?

F. Practice (micro teaching) sessions for workshop participants.

IV. Classroom organization. (30 minutes)

A. Teaching a unit on problem solving and then applying the skills regularly.

B. Using one day a week for problem solving.

C. Skill centers.

D. Using a problem solving approach as a basic teaching strategy.

Outline of a 2-hour Workshop on Teaching Problem Solving

For High School Teachers

I. Introduction (15 minutes)
   A. What is a problem?
      1. Blockage.
      2. Acceptance by students.
   B. Why teach problem solving skills?
      1. NCSM and NCTM lists of basic skills.
      2. National assessment results.
   C. Schedule of workshop and purpose of each activity.

II. Solving problems (1 hr.)
   A. How many games are played in a single elimination tournament with 275 entrants? (Strategies that can be used include: paper-and-pencil experiment, systematic counting, simpler but similar problems.)
   B. Painted cube problem - Imagine that a large wooden cube is painted black and then cut into 100 congruent small cubes. How many of the small cubes have 0 faces painted? 1 face painted? 2 faces painted? etc. (Strategies: experiment, solving simpler but similar problems, make a drawing, systematic counting.)
   C. Calculate - $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$

      (Strategies: working backward and looking for a pattern.)

III. Teaching problem solving strategies (30 minutes)
   A. Objectives.
      1. Selecting objectives that involve the use of problem solving skills.
      2. Assigning priorities to objectives.
   B. Reinforcing students.
      1. Reinforcing according to objectives.
      2. Reinforcing approximations to final behaviors.
   C. Testing problem solving abilities.
      1. Testing the ability to use selected strategies.
      2. Testing the ability to select appropriate strategies.
      3. Formal and informal evaulation.
   D. Providing help - asking questions.
      1. When to help.
      2. Giving as little help as possible.
      3. Giving help without guiding students toward specific strategies or solutions.
      4. Wait time after asking questions.
   E. Holding review sessions after problems are solved.
      1. How was the problem solved?
      2. What made the solver think of the solution or the method of solution?
      3. What are the advantages or disadvantages of the method?
      4. Is there a different method or different solution?

IV. Classroom organization. (30 minutes)
   A. Teaching a unit on problem solving and then applying the skills regularly.
   B. Using one day a week for problem solving.
   C. Skill centers.
   D. Using a problem solving approach as a basic teaching strategy.

SECTION Q.01-Q.17 "MATHEMATICS EDUCATION PUBLICATIONS FROM THE ERIC CLEARINGHOUSE FOR SCIENCE, MATHEMATICS, AND ENVIRONMENTAL EDUCATION" DELETED DUE TO MARGINAL LEGIBILITY.

WORKSHOP EVALUATION

Ohio Regional Conferences on Mathematics Education

Evaluation of the 6 Conferences

The staff conducting the 6 regional conferences are interested in the participants' reactions to the conferences. Each participant will be asked to complete a questionnaire, (pink form) which includes a set of multiple choice questions and a section for open responses.

The goal of the conferences is to help the participants prepare to conduct workshops or other in-service activities with teachers in their own or nearby districts. We will contact you later to see if the conferences resulted in such workshops. (This is an evaluation of the effectiveness of the Ohio Regional Conferences.) At that time you may have other suggestions for the staff and we hope you will send them.

Evaluation of Follow-up Activities

If you present a workshop or other activity for teachers, you may want to use the evaluation form in this section (white form). Answer sheets are available from:

Len Pikaart
Ohio University
College of Education
McCracken Hall
Athens, Ohio 45701

The answer sheets will be provided free (as long as the supply lasts). We ask that you reproduce the questionnaire itself. Please request your best estimate of the number of answer sheets needed and return unused ones so they will be available for others. We will score the answer sheets and send you a complete analysis if you will permit us to include the data from your program in our summary data.

We emphasize that no individual workshop or activity will be identifiable in any report of analysis we conduct. We would like to examine the reactions to all the programs conducted as a result of the conferences to determine if we should use the same sort of conference format in the future--thus, we want to evaluate us--not you. Positive results would have great implications for the National Science Foundation (supporting this project), and perhaps for the Ohio State Department of Education. (Negative results would have implications too!)

We hope you will decide to use the evaluation form and to share the data with us.

TEACHER IN-SERVICE ACTIVITY

Evaluation Form

Select the single best answer for you to each question and mark all answers on the answer sheet. If the question is not appropriate, leave it blank.

1. Are you:
   A. Teacher, grades 5-9.
   B. School teacher, grades 9-12..
   C. School principal.
   D. School system administrator.
   E. Other.

2. This activity focused on:
   A. Problem solving.
   B. Using calculators.
   C. Both problem solving and calculators.
   D. Some other topic but referred to problem solving or to calculators.
   E. Some other topic.

3. The objectives and purposes:
   A. Were clearly outlined from the beginning.
   B. Became clear as the activities developed.
   C. Became somewhat clear as the activities progressed.
   D. Were referred to only indirectly.
   E. Were never made clear.

4. The agreement between the announced purposes of the activity and what was actually presented was:
   A. Superior.
   B. Above average.
   C. Average.
   D. Below Average.
   E. Poor.

5. How well was the activity organized?
   A. It was extremely well organized and integrated.
   B. It was adequately organized.
   C. It had less organization than would seem desirable.
   D. It had no apparent organization.
   E. It was too tightly organized; there was not enough flexibility to meet participant needs and desires.

6. How well did this activity contribute to your professional needs?
   A. Made a very important contribution.
   B. Was valuable, but not essential.
   C. Was moderately helpful.
   D. Made a minor contribution.
   E. Made no significant contribution.

7. How would you rate the usefulness of the materials on Problem Solving?
   A. Extremely valuable.
   B. Very useful.
   C. Useful.
   D. May be of use.
   E. Useless.

8. How would you rate the usefulness of the materials on <u>Calculators</u>?
   A. Extremely valuable.
   B. Very useful.
   C. Useful.
   D. May be of use.
   E. Useless.

9. How clearly were your responsibilities during this activity defined?
   A. I always knew what was expected of me.
   B. I usually knew what was expected of me.
   C. I usually had a general idea of what was expected of me.
   D. I was often in doubt about what was expected of me.
   E. I seldom knew what was expected of me.

10. Considering the size of the group, do you feel that the leaders were willing to give personal help?
   A. I felt <u>welcome</u> to seek personal help as often as I needed it.
   B. I felt free to seek personal help.
   C. I felt he or she would give personal help if asked.
   D. I felt hesitant to seek personal help.
   E. I felt that he or she was unsympathetic and uninterested in participant problems.

11. Would you recommend this conference to a good friend whose interests and background are similar to yours?
   A. Recommend highly.
   B. Generally recommend.
   C. Recommend with reservations.
   D. Definitely not.

12. How would you rate your understanding of the use of <u>problem solving</u> as a result of this conference?
   A. I learned a lot.
   B. My understanding improved.
   C. A few ideas were new to me.
   D. I learned very little.
   E. I learned almost nothing.

13. How would you rate your understanding of the use of <u>calculators</u> in schools as a result of this conference?
   A. I learned a lot.
   B. My understanding improved.
   C. A few ideas were new to me.
   D. I learned very little.
   E. I learned almost nothing.

14. How would you rate the activity in general?
   A. Outstanding and stimulating.
   B. Very good.
   C. Good.
   D. Adequate, but not stimulating.
   E. Poor and inadequate.

PLEASE TEAR OFF THIS PAGE AND TURN IT IN WITH YOUR ANSWER SHEET.

1. Best features of the activity were:

2. Worst aspects of the activity were:

3. I would suggest the following:

OHIO REGIONAL CONFERENCES ON MATHEMATICS EDUCATION

Conference Evaluation Form

Select the single best answer for you to each question and mark all answers on the answer sheet. If the question is not appropriate, leave it blank.

Participant Position

1. Are you:
   A. Superintendent.
   B. Supervisor.
   C. Principal.
   D. Mathematics Education (College Level).
   E. Teacher.

Conference Objectives and Purposes

2. How clear were the objectives or purposes of this conference? The objectives and purposes:
   A. Were clearly outlined from the beginning.
   B. Became clear as the conference developed.
   C. Became somewhat clear as the conference progressed.
   D. Were referred to only indirectly.
   E. Were never made clear.

3. The agreement between the announced purpose of the conference and what was actually presented was:
   A. Superior.
   B. Above average.
   C. Average.
   D. Below Average.
   E. Poor.

Organization

4. How well was the conference organized?
   A. The conference was extremely well organized and integrated.
   B. The conference was adequately organized.
   C. The conference had less organization than would seem desirable.
   D. The conference had no apparent organization.
   E. The conference was too tightly organized; there was not enough flexibility to meet participant needs and desires.

5. Concerning the mixture of participants, do you think:
   A. The mixture was about right.
   B. There should have been more superintendents.
   C. There should have been more supervisors.
   D. There should have been more classroom teachers.
   E. The groups should have met separately.

Conference Content

6. How well did this conference contribute to your professional needs?
   A. Made a very important contribution.
   B. Was valuable, but not essential.
   C. Was moderately helpful.
   D. Made a minor contribution.
   E. Made no significant contribution.

7. How would you rate the usefulness of the Resource Packet materials on Problem Solving?
   A. Extremely valuable.
   B. Very useful.
   C. Useful.
   D. May be of use.
   E. Useless.

8. How would you rate the Resource Packet materials on Calculators?
   A. Extremely Valuable.
   B. Very Useful.
   C. Useful.
   D. May be of use.
   E. Useless.

Participant Participation

9. How clearly were your responsibilities in this conference defined?
   A. I always knew what was expected of me.
   B. I usually knew what was expected of me.
   C. I usually had a general idea of what was expected of me.
   D. I was often in doubt about what was expected of me.
   E. I seldom knew what was expected of me.

Presenter-Participant Relationships

10. Do you feel that the presenters were willing to give personal help in this conference?
   A. I felt welcome to seek personal help as often as I needed it.
   B. I felt free to seek personal help.
   C. I felt he or she would give personal help if asked.
   D. I felt hesitant to seek personal help.
   E. I felt that he or she was unsympathetic and uninterested in participant problems.

11. Freedom of participation in conference meetings: questions and comments were:
   A. Almost always sought.
   B. Often sought.
   C. Usually allowed.
   D. Seldom allowed.
   E. Usually inhibited.

Conference Effectiveness

12. Did the conference help prepare you to lead in-service activities on problem solving and calculators?
   A. Definitely.
   B. It was a help on both.
   C. On one of the topics.
   D. It was little help.
   E. It was no help.

13. Would you recommend this conference to a good friend whose interests and background are similar to yours?
   A. Recommend highly.
   B. Generally recommend.
   C. Recommend with reservations.
   D. Definitely not.

14. How would you rate your understanding of Problem Solving as a result of this conference?
   A. I learned a lot.
   B. My understanding improved.
   C. A few ideas were new to me.
   D. I learned very little.
   E. I learned almost nothing.

15. How would you rate your understanding of the use of Calculators in schools as a result of this conference?
   A. I learned a lot.
   B. My understanding improved.
   C. A few ideas were new to me.
   D. I learned very little.
   E. I learned almost nothing.

16. How would you rate the presenters' sensitivity to what you consider to be the important problems in school mathematics?
    A. They were well aware of the important problems.
    B. They were aware of these problems.
    C. They had a general idea of the problems.
    D. They had a vague knowledge of some problems.
    E. They did not seem informed of significant problems.

17. How would you rate the presentations, in general?
    A. Outstanding and stimulating.
    B. Very good.
    C. Good.
    D. Adequate, but not stimulating.
    E. Poor and inadequate.

18. Would you like to attend conferences on other (like these) topics in this geographic area?
    A. Definitely.
    B. Yes, but in a bigger city.
    C. It would be a good idea.
    D. Probably not.
    E. Definitely not.

19. How would you rate the use of instructional media in this conference?
    A. The uses of media were almost always effective.
    B. The uses of media were usually effective.
    C. The uses of media were sometimes effective.
    D. The uses of media were seldom effective.
    E. The uses of media were never effective.

20. Do you believe that the conference helped establish (or improve) positive linkages between school system personnel and college mathematics educators?
    A. Definitely.
    B. Somewhat.
    C. Very little improvement.
    D. No improvement.
    E. The linkages should not be established.

PLEASE TEAR OFF THIS PAGE AND TURN IT IN WITH YOUR ANSWER SHEET.

1. Best features of the conference were:

2. Worst aspects of the conference were:

3. I would suggest the following:

4. Were there materials on display that you would like to see included in the Resource Packets? (Which ones?)

5. Do you feel you are prepared to lead in-service activities on problem solving and/or calculators for teachers? Can you tell us what activities you expect to organize? For how many teachers? When?