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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on number plane rules, a summary of the problems in the film "Jumping Rules in the Plane, Part II," and the supplement. (MK)

ED187600

THE ARITHMETIC PROJECT COURSE FOR TEACHERS

TOPIC: Number Plane Rules (continued).

FILM: Jumping Rules in the Plane, Part Two, Grade 6

SUPPLEMENT: More Work With Number Plane Rules

19

NAME: ....

This booklet is part of a course for teachers produced by The Arithmetic Project in association with Education Development Center. Principal financial support has come from the Carnegie Corporation of New York, the University of Illinois, and the National Science Foundation.

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BOOK NINETEEN

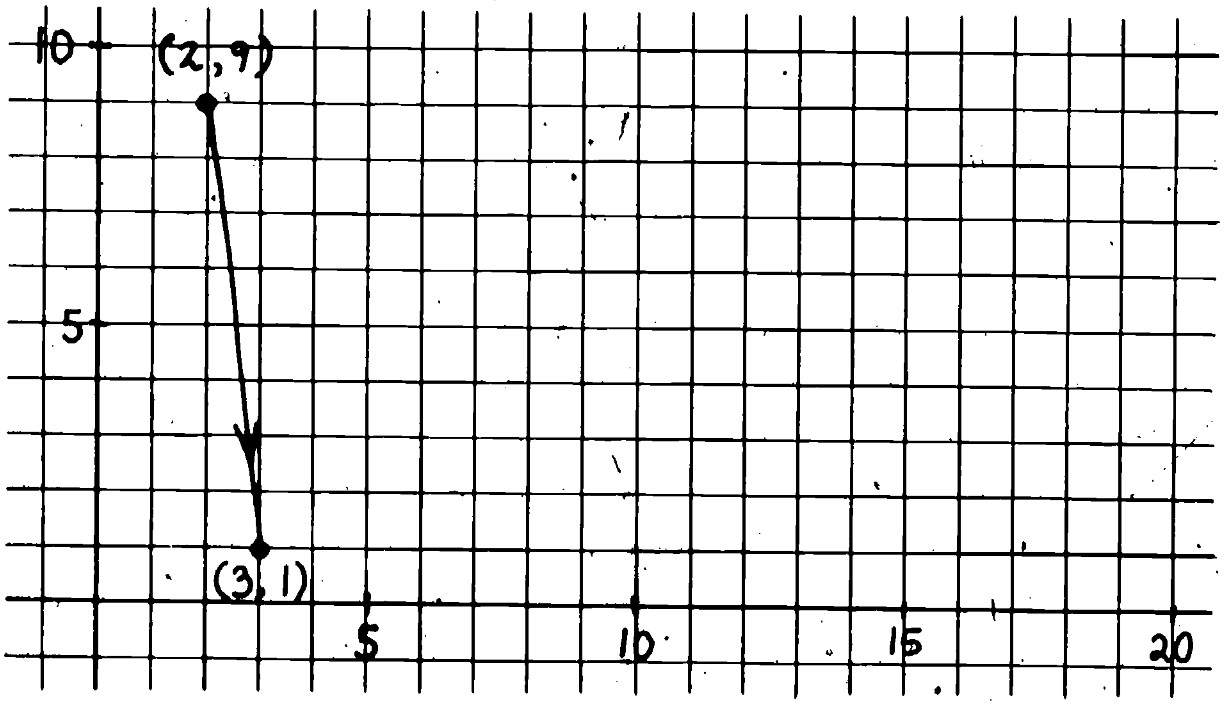
I.

Jumping rule: $(\square, \triangle) \longrightarrow (\square + 1, 10 - \triangle)$

1. Start at (2 , 9) and make one jump. Land: ____________
(The jump is shown below.)

2. Make a jump starting at (3 , 1). Land: ____________

3. Make five more consecutive jumps and draw them below.



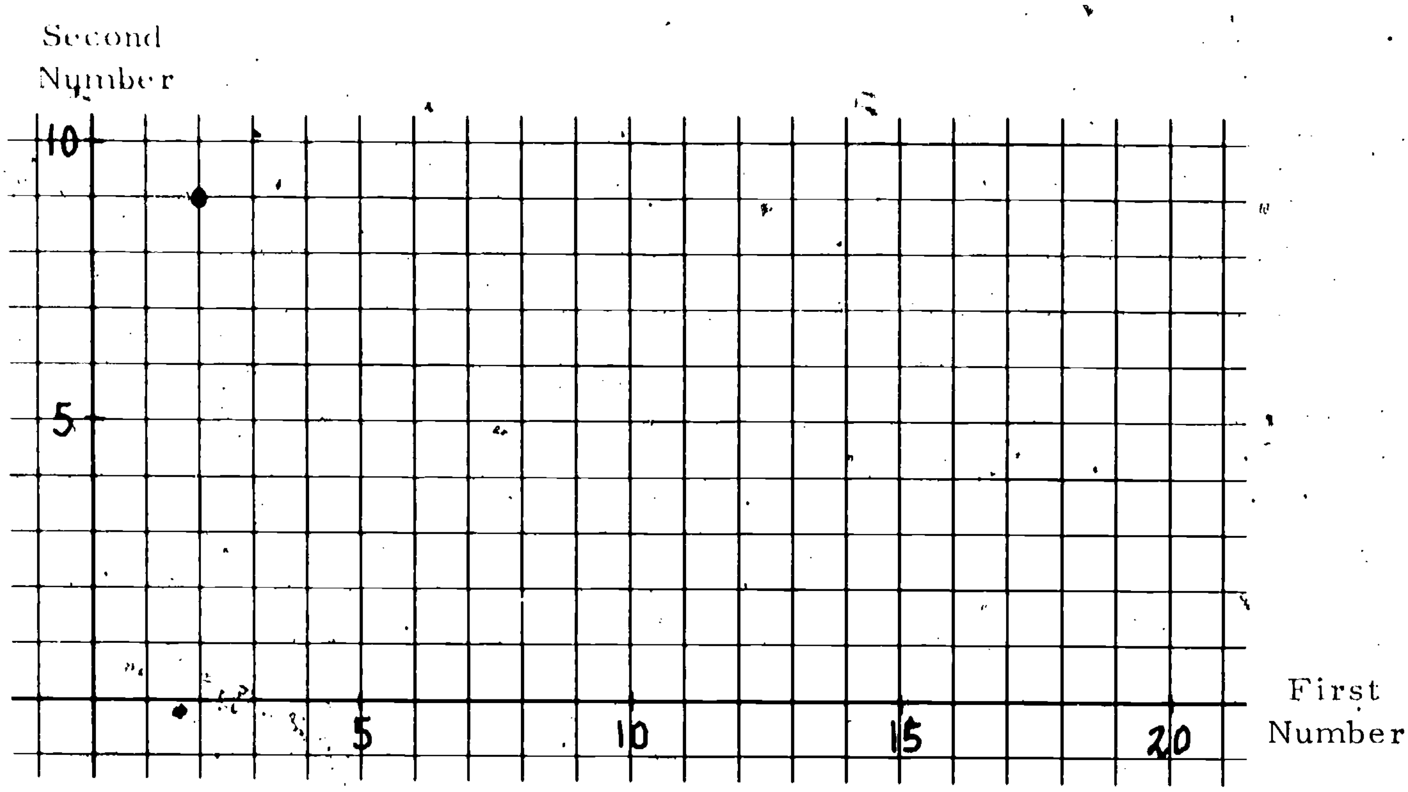
4. Name a point whose first number is larger than 50, on which you will land eventually, starting at (2 , 9). ____________

5. Starting at (2 , 9), will you ever hit (603 , 9) if you use the rule enough times? ____________

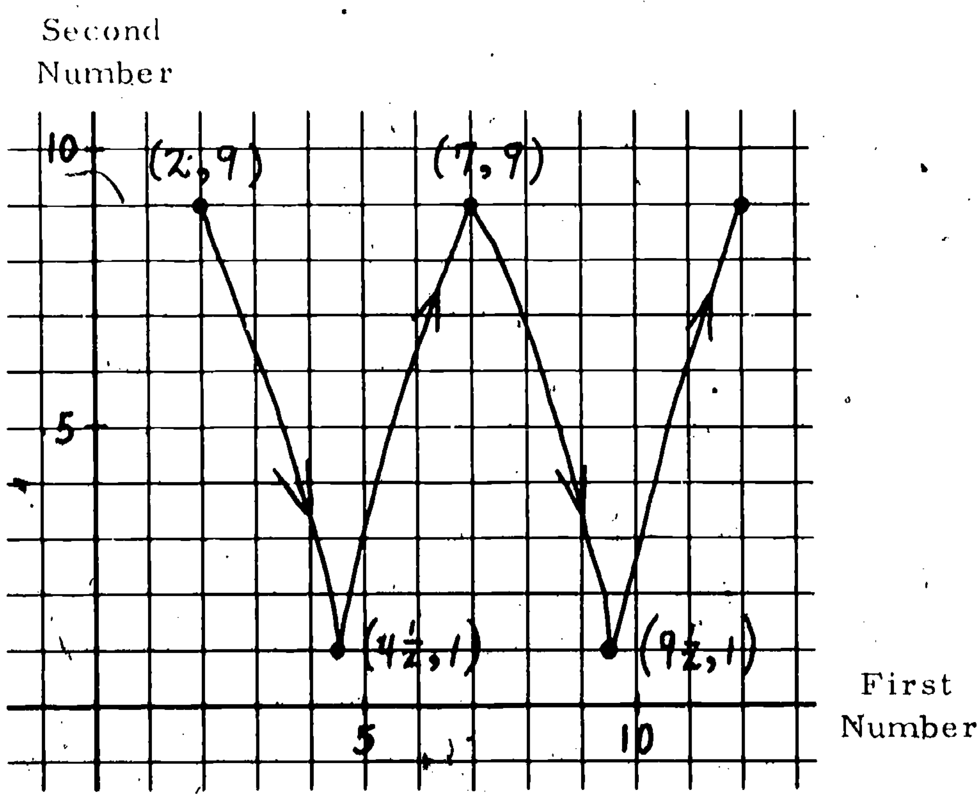
6. Where can you start so that after 21 consecutive jumps you land on (100 , 9)? ____________

New rule: $(\square, \triangle) \longrightarrow (\square + 5, 10 - \triangle)$

7. Again start at (2, 9). Make as many consecutive jumps as you can draw below.

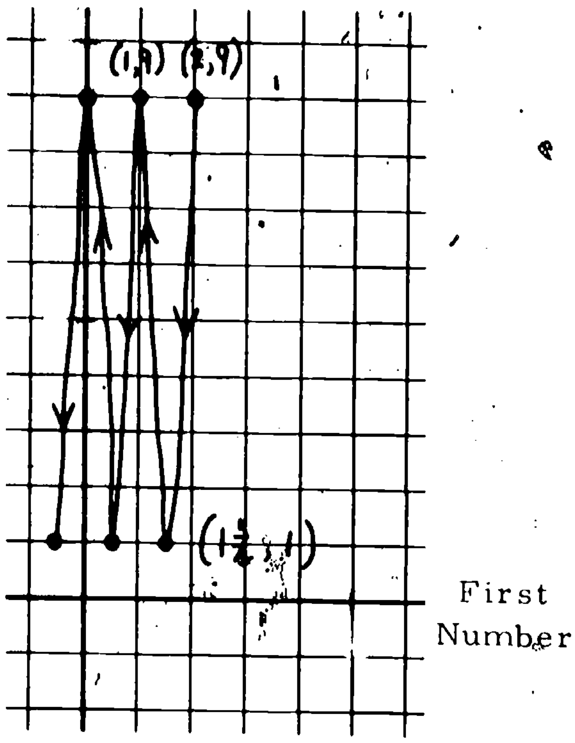


8. Write a rule which will give these jumps:



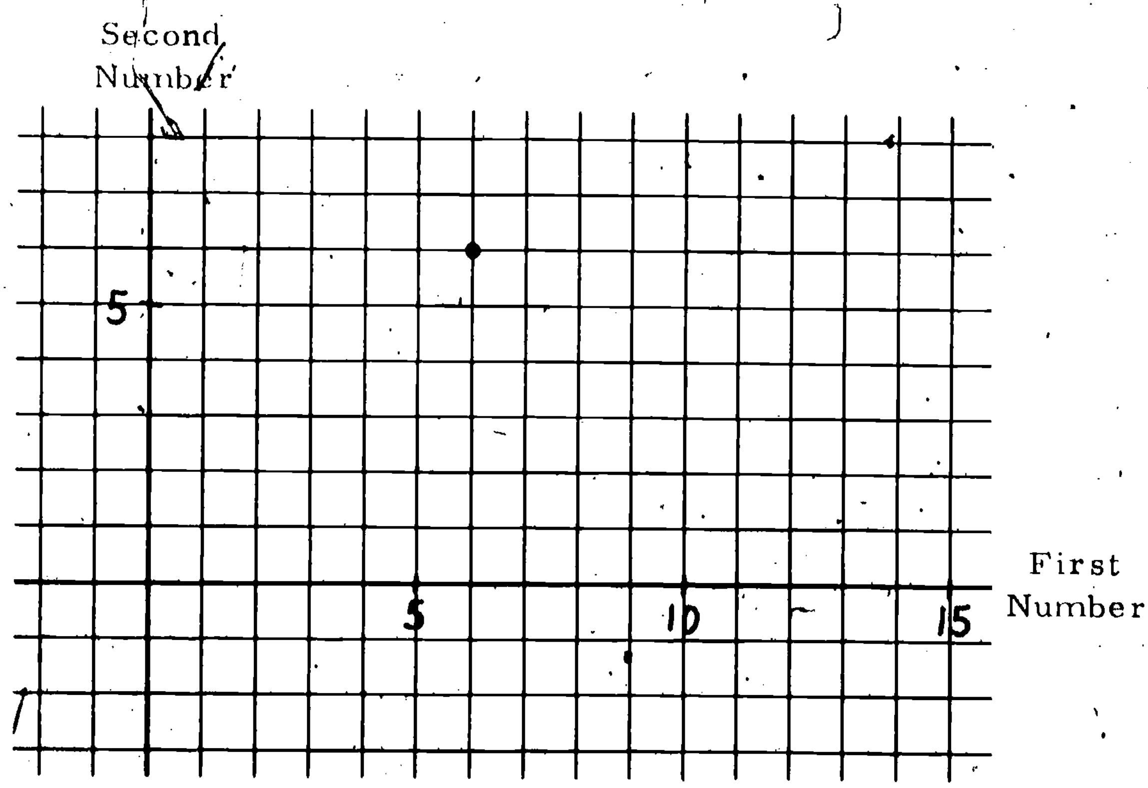
$(\square, \triangle) \longrightarrow$ ______________________

9. Write a rule which will give these jumps:



____

10. Using the original rule, $(\square, \triangle) \longrightarrow (\square + 1, 10 - \triangle)$, make and draw five consecutive jumps starting at (6, 6).

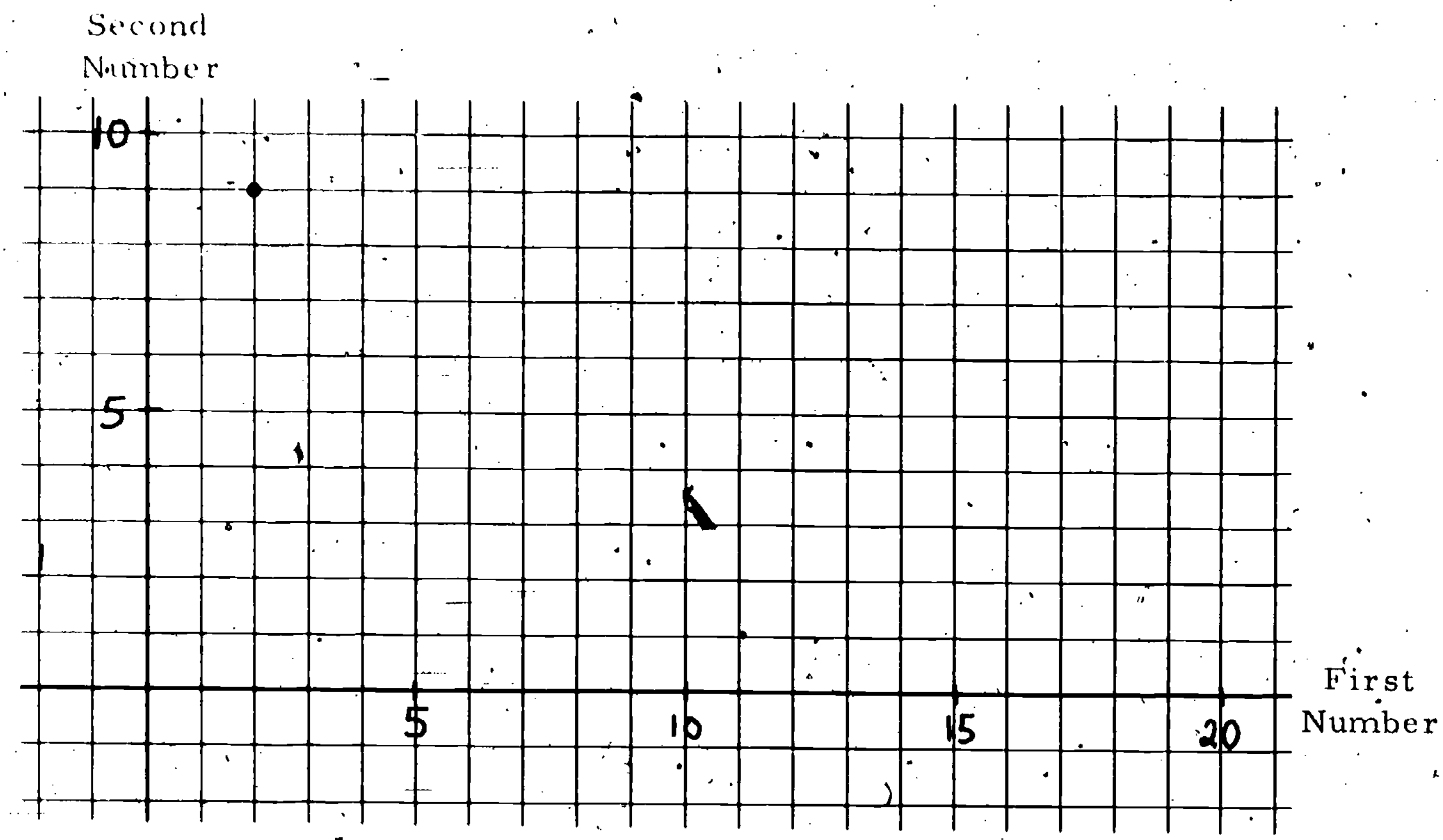


11. Using the same rule, make five consecutive jumps starting at (0, 5). Draw them on the graph paper above.

12. Using the rule

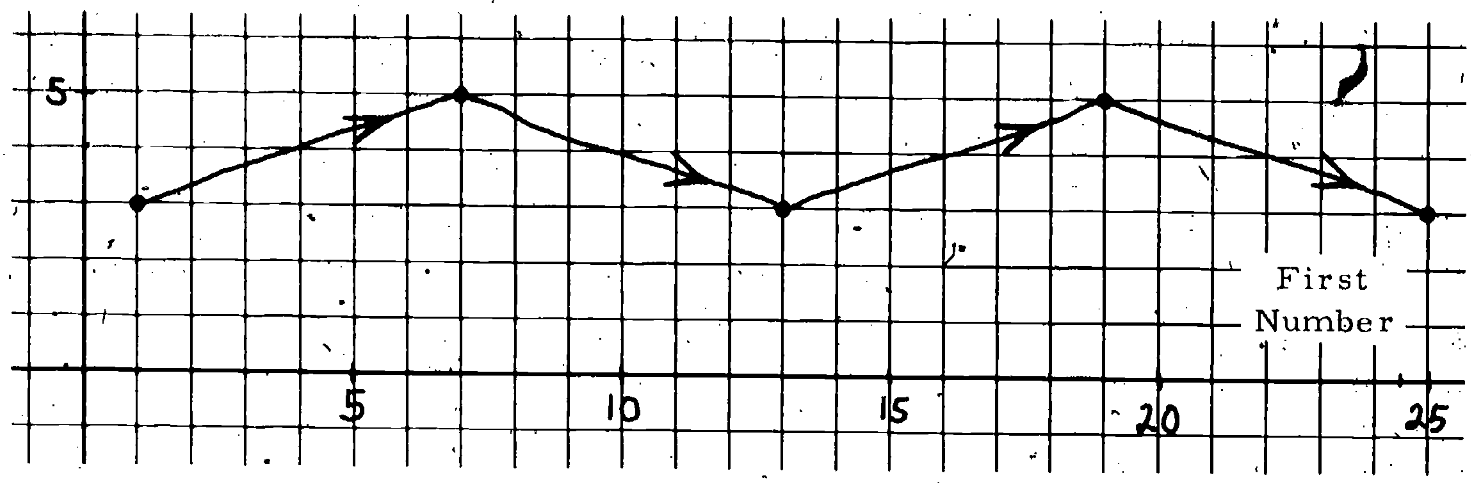
$$(\square , \triangle) \longrightarrow (\square + 1, 13 - \triangle),$$

make and draw five consecutive jumps starting at (2 , 9).



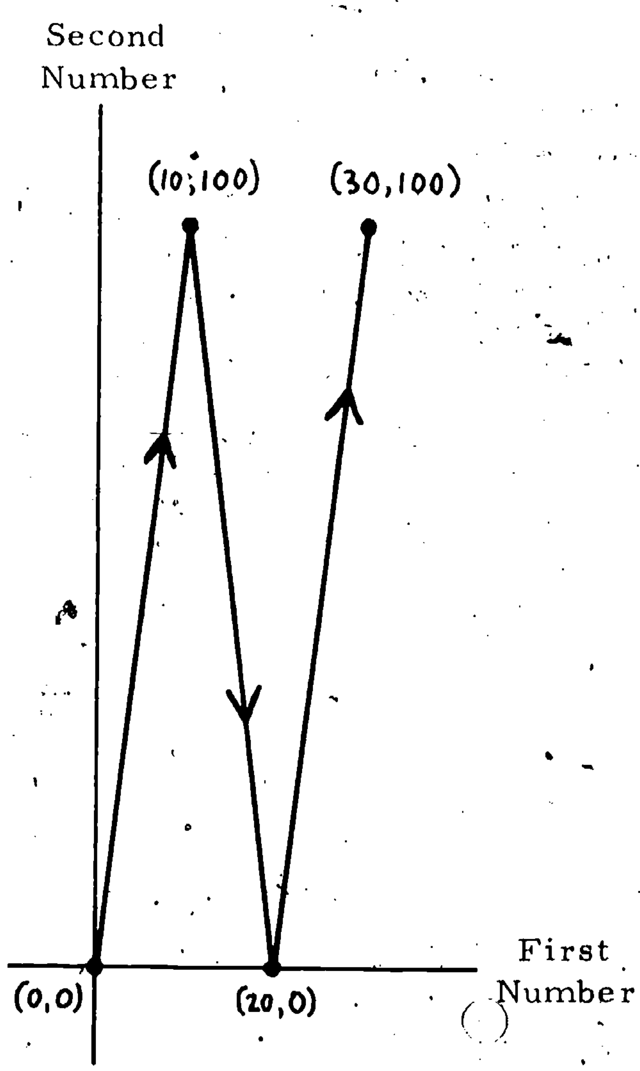
13. Where can you start so that your jump is horizontal; that is, so that it goes neither up nor down? ____________

14. Write a rule which will give these jumps:



$(\square , \triangle) \longrightarrow$ ______________________

15. Write a rule which will give these jumps:

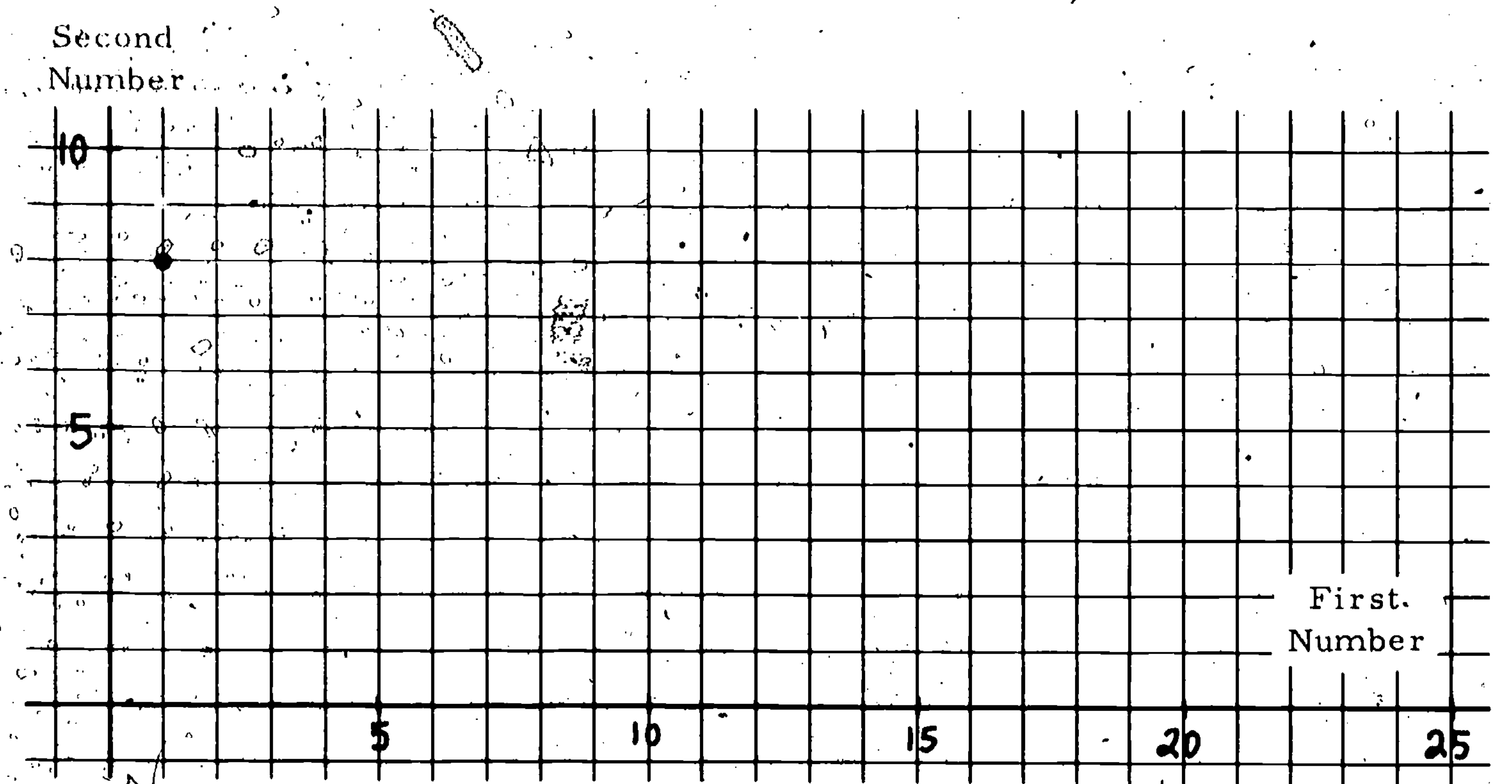


$(\square , \triangle) \longrightarrow$ ______________________

16. Instead of adding a constant amount to $\square$, this time we will add $\frac{\triangle}{2}$, so that the distance we move to the right will depend on how far up our starting point is.

The rule: $(\square, \triangle) \longrightarrow \left(\square + \frac{\triangle}{2},\ 10 - \triangle\right)$

(a) Start at (1, 8) and make five consecutive jumps. Show them below:



(b) Where can you start so that your jump goes neither up nor down?

____

(c) Where can you start so that your jump goes neither left nor right?

____

(d) Describe all the standstill points of this rule: ____________________

(e) If you want to say something about the rules we have been using so far, write it here:

II. Jumping rule:

$$(\boxed{}, \triangle) \xrightarrow{a} (10 - \triangle, 10 - \boxed{})$$

Do the following and, where indicated, show the jumps on the number plane on page 9.

1. Make a jump starting at (7 , 4).

$$(\boxed{7}, \triangle_{4}) \xrightarrow{a} (10 - \triangle_{4}, 10 - \boxed{7})$$

or $(7,4) \xrightarrow{a} (6,3)$

This jump has been plotted for you on page 9.

2. $(7,4) \xrightarrow{aa}$ = ____________

(Remember that $(7,4) \xrightarrow{aa}$ is the same as $((7,4) \xrightarrow{a}) \xrightarrow{a}$.)

3. $(6,1) \xrightarrow{a}$ ____________ Plot starting and landing points, and draw the jump.

4. $(6,1) \xrightarrow{aaa}$ = ____________

5. $(6,6) \xrightarrow{a}$ ____________ Draw the jump as in problems 1 and 3.

6. $(6,6) \xrightarrow{aa}$ = ____________

7. $(4,6) \xrightarrow{a}$ ____________ Draw the jump.

Continue with the rule: $(\square, \triangle) \xrightarrow{a} (10 - \triangle, 10 - \square)$

8. $(4, 6) \xrightarrow{aaaaa}$ = ____________

9. $(4, 7) \xrightarrow{a}$ ____________ Draw the jump.

10. $(9, 1) \xrightarrow{a}$ = ____________

11. $(9, 8) \xrightarrow{a}$ ____________ Draw the jump.

12. $(9, 8) \xrightarrow{aaaaaaa}$ = ____________

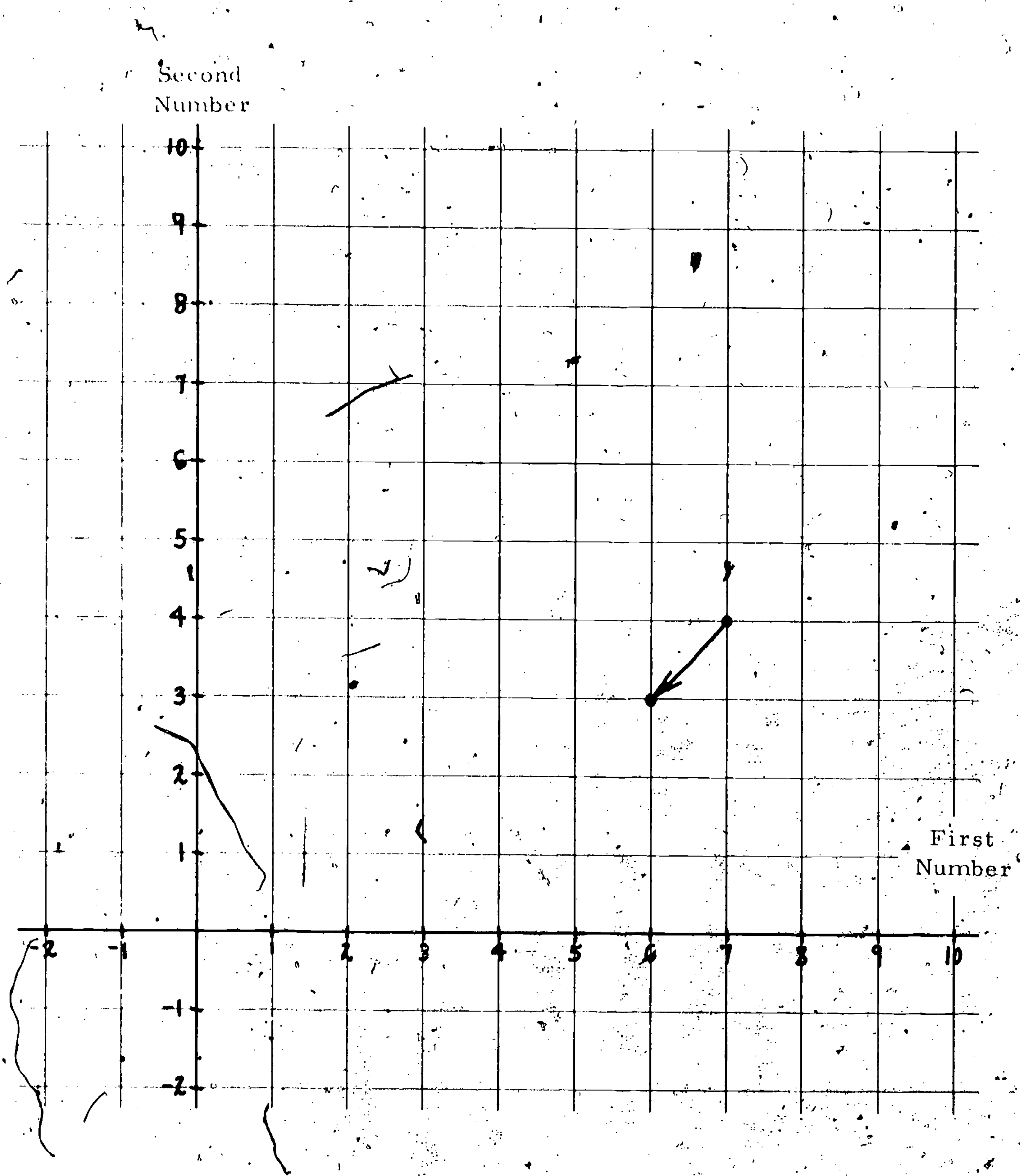
13. List 5 standstill points:

(,) (,) (11 ,) (,) (, 0)

Plot these and all other standstill points on the next page.

Continue with the rule: $(\square, \triangle) \xrightarrow{a} (10 - \triangle, 10 + \square)$

Picture of jumps on preceding two pages:



☆14. Describe a geometric method to find the landing point, given any starting point.

III. Jumping rules:

$$(\square, \triangle) \xrightarrow{b} (\triangle, \square)$$

$$(\square, \triangle) \xrightarrow{c} (-\triangle, -\square)$$

1. Start at (5, 4). Do the following in order:

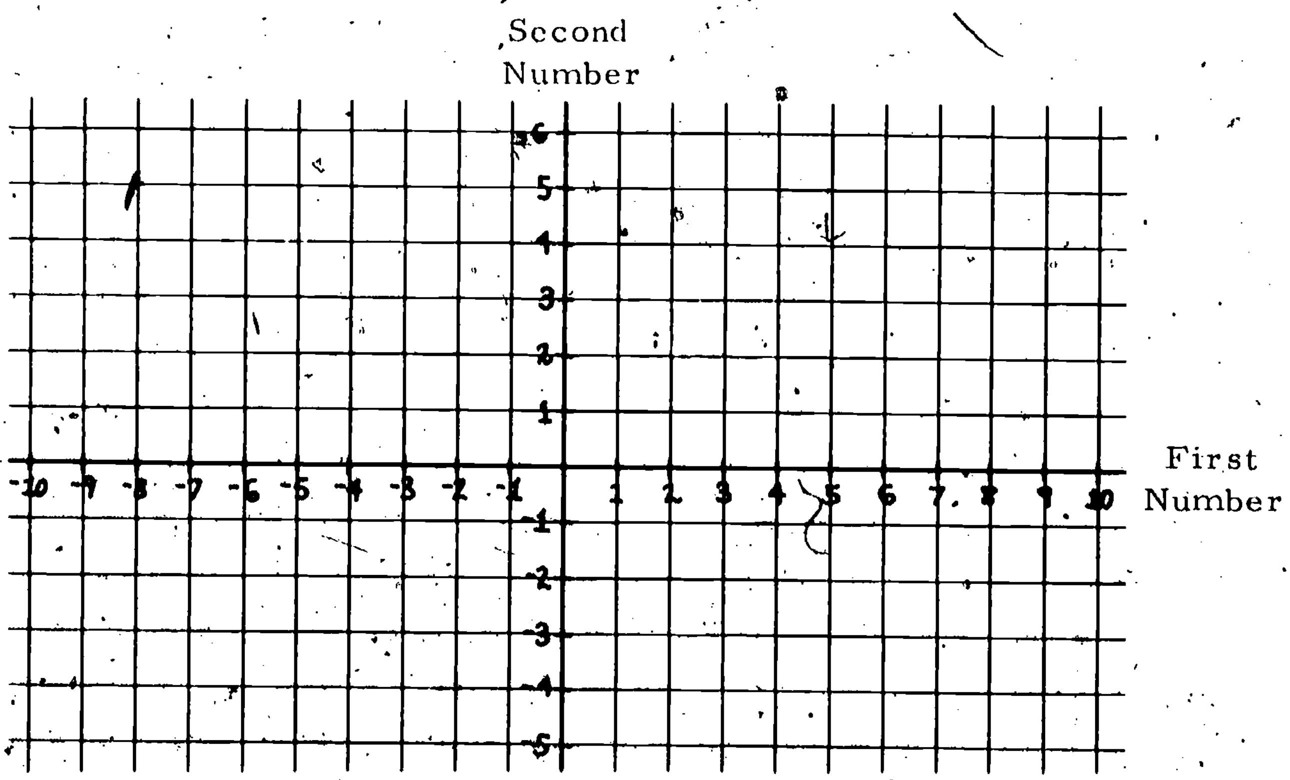
Jump with rule b ; land: (4,5).

Jump with rule c ; land: (-5,-4).

Jump with rule b ; land: ______.

Jump with rule c ; land: ______.

Plot (5, 4) and your various landing points below, and connect successive points with straight lines. (Keep an eye out for how one could describe geometrically the nature of the jumps given by each of these two rules.)

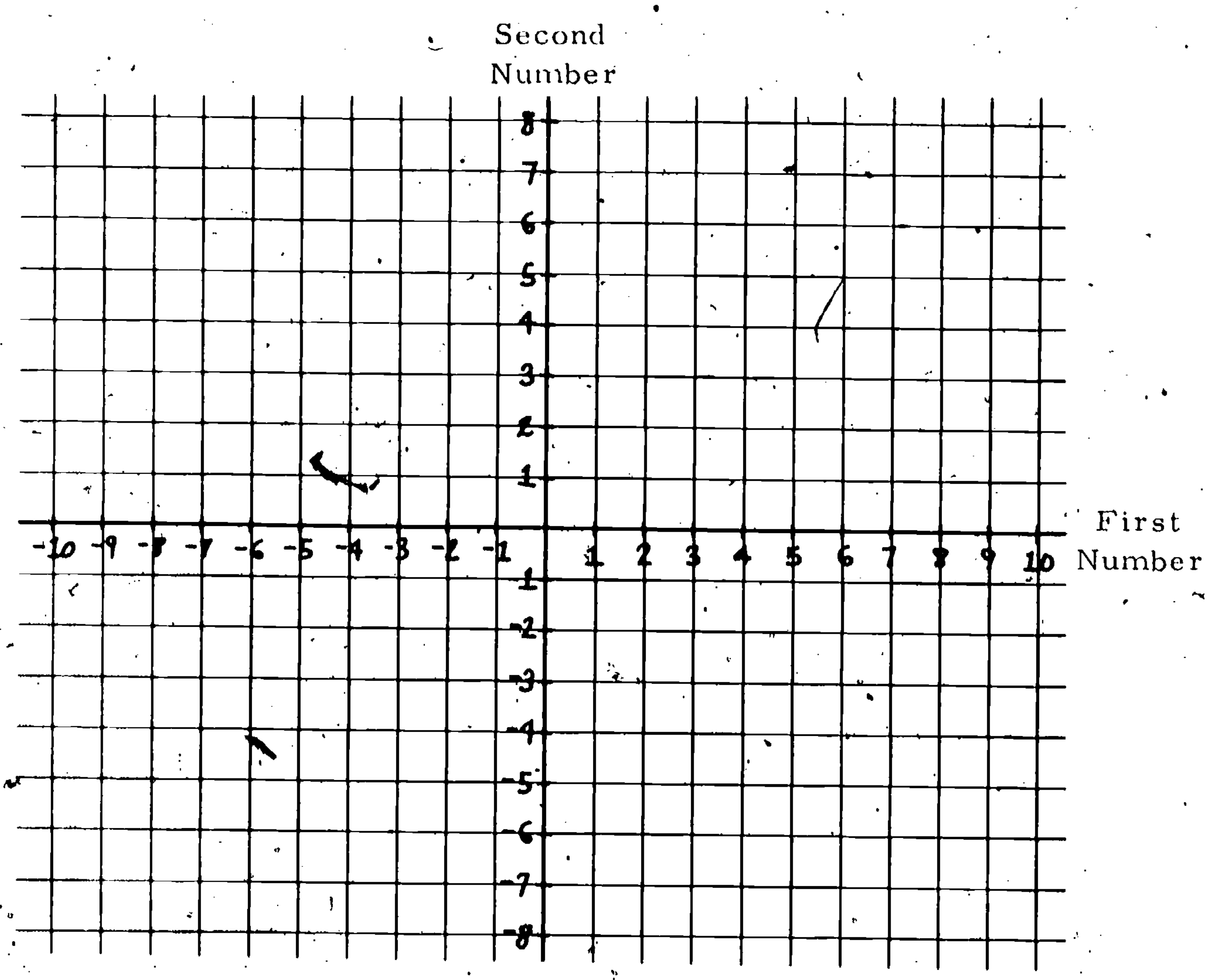


Continue with the rules: $(\square, \triangle) \xrightarrow{b} (\triangle, \square)$

$$(\square, \triangle) \xrightarrow{c} (-\triangle, -\square)$$

2. Repeat the preceding for the starting point (8, −7) :

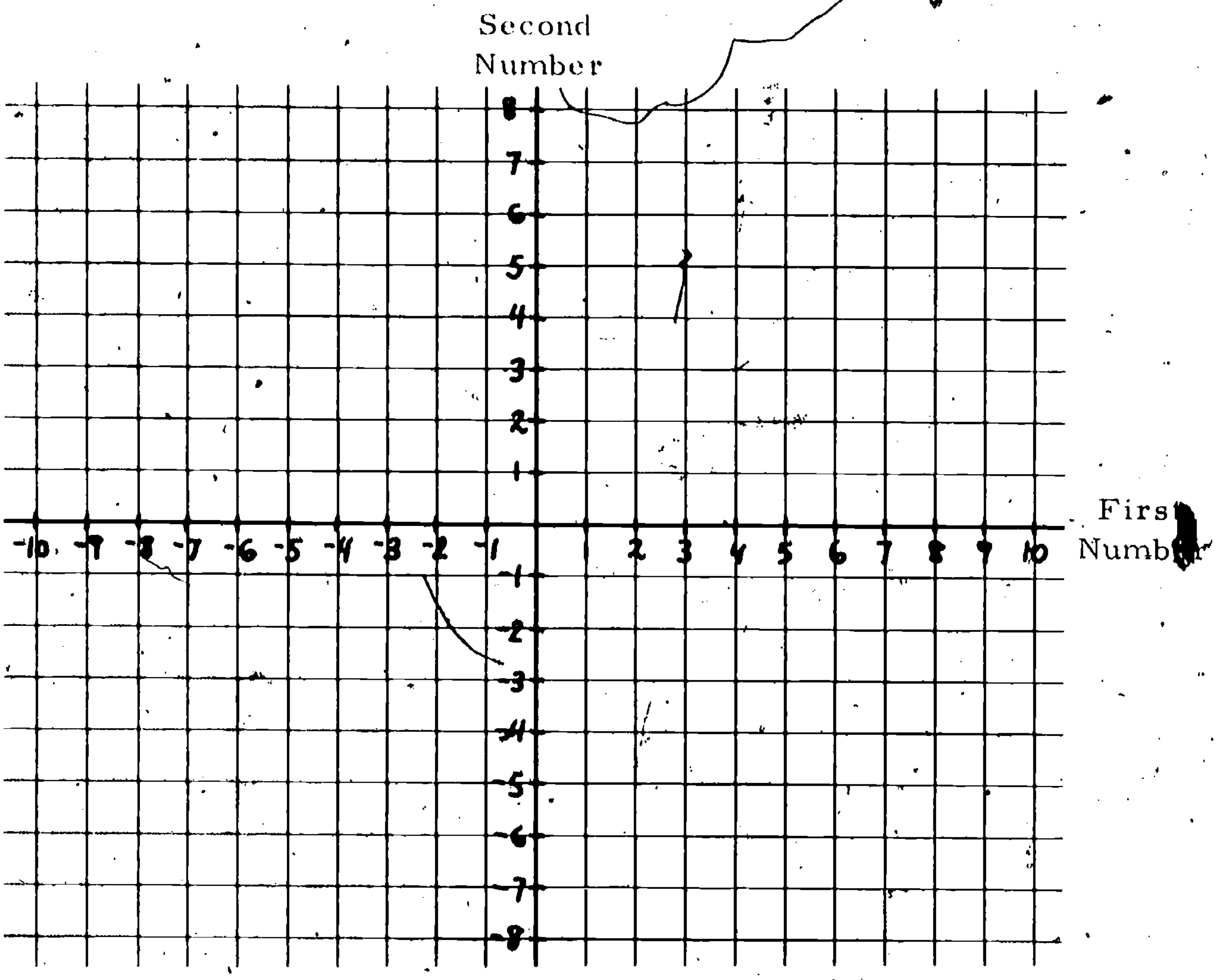
$(8, -7) \xrightarrow{b}$ ________ $\xrightarrow{c}$ ________ $\xrightarrow{b}$ ________ $\xrightarrow{c}$ ________ .



Continue with the rules: $(\square, \triangle) \xrightarrow{b} (\triangle, \square)$

$(\square, \triangle) \xrightarrow{c} (-\triangle, -\square)$

3. Using the rules in the order b , c , b , c , where could you start so that if you connected the points you would get a square? ____________



4. What happens if you start at (5 , 5) and use the rules in the order b , c , b , c ?

Continue with the rules: $(\square, \triangle) \xrightarrow{b} (\triangle, \square)$

$(\square, \triangle) \xrightarrow{c} (-\triangle, -\square)$

5. What happens if you start at $(-4, 4)$?

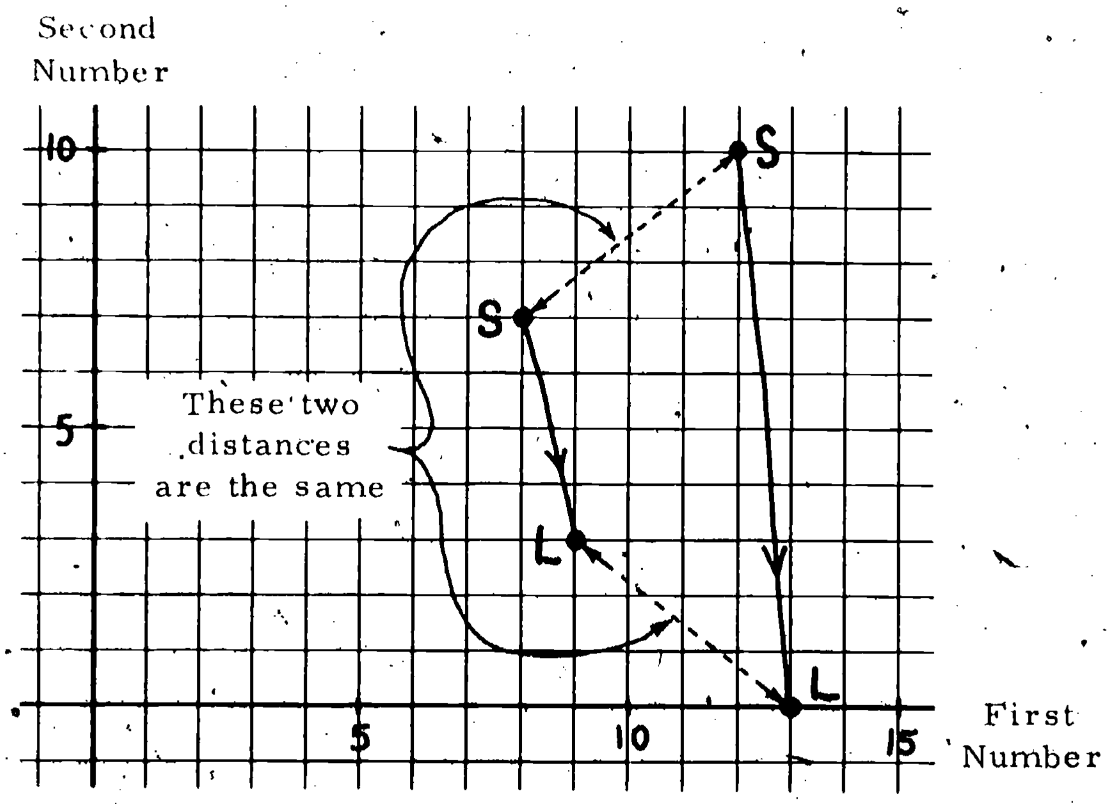
☆6. Where can you start so that when you connect the landing points you have a rectangle that is 3 times as long as it is wide? ____________

☆7. If you want to say something about these rules, write it here:

Epilogue

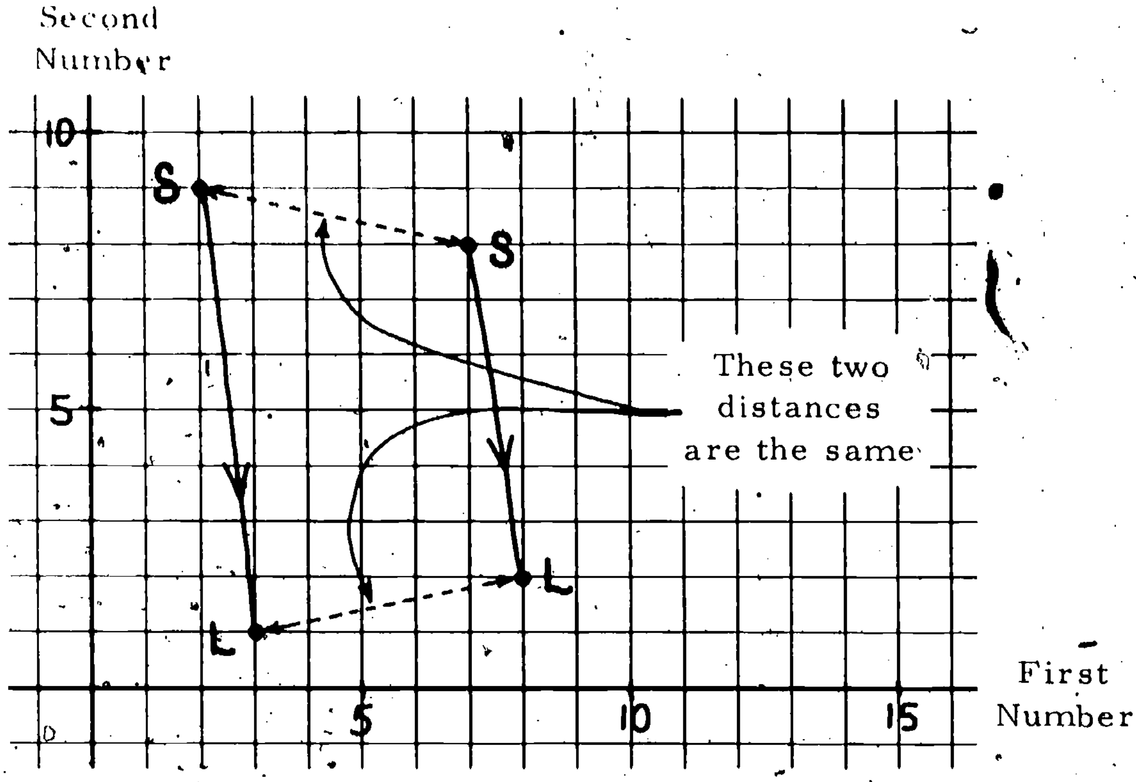
Each of the rules in this written lesson, with the exception of $(\Box, \triangle) \longrightarrow (\Box + \frac{\triangle}{2}, 10 - \triangle)$, is an example of an important class of number plane jumping rules—the isometries. An isometry, or rigid motion, is any jumping rule which preserves distances: the distance between any two starting points is the same as the distance between the two landing points given by the rule.

Consider, for example, the two starting points (8 , 7) and (12 , 10). (We picked these points for our first example because the distance between them is 5 units.) Using the rule $(\Box, \triangle) \longrightarrow (\Box + 1, 10 - \triangle)$, we obtain the landing points (9 , 3) and (13 , 0), respectively. It is not hard to calculate or see from a picture of these jumps that the distance between the starting points, (8 , 7) and (12 , 10), is the same as the distance between the landing points, (9 , 3) and (13 , 0).



Even in cases where the distance between the two points is very messy to compute, you can see from a picture that the distance between the starting points is the same as the distance between the landing points.

Another example: The rule $(\square, \triangle) \longrightarrow (\square + 1, 10 - \triangle)$ sends $(2, 9)$ to $(3, 1)$ and $(7, 8)$ to $(8, 2)$. As the picture below shows, the distance between the starting points is the same as the distance between the landing points.



An important fact about isometries is that the composite of any two isometries is also an isometry. To see this, suppose that rules A and B are both isometries. If we pick any two starting points and put them through rule A, the distance between the landing points will be the same as it was between the starting points. If we now apply rule B, again the distance remains unchanged. Thus, the composite rule AB, which has the same effect as first doing rule A and then rule B, preserves distances, and so it too is an isometry.

Consider the first rule in the written lesson:

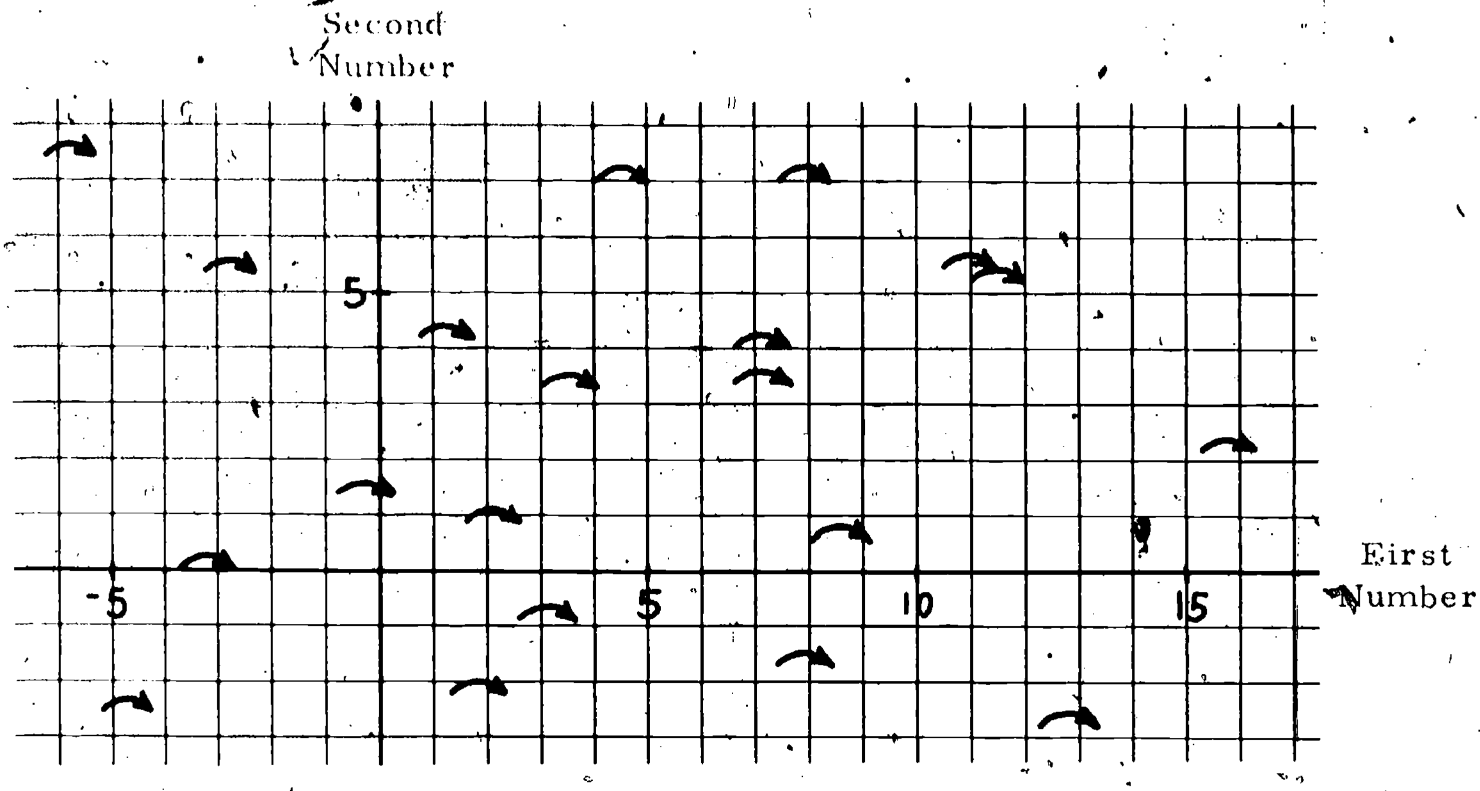
$$(\square, \triangle) \longrightarrow (\square + 1, 10 - \triangle)$$

One way to look at this rule is as the composite of two isometries:

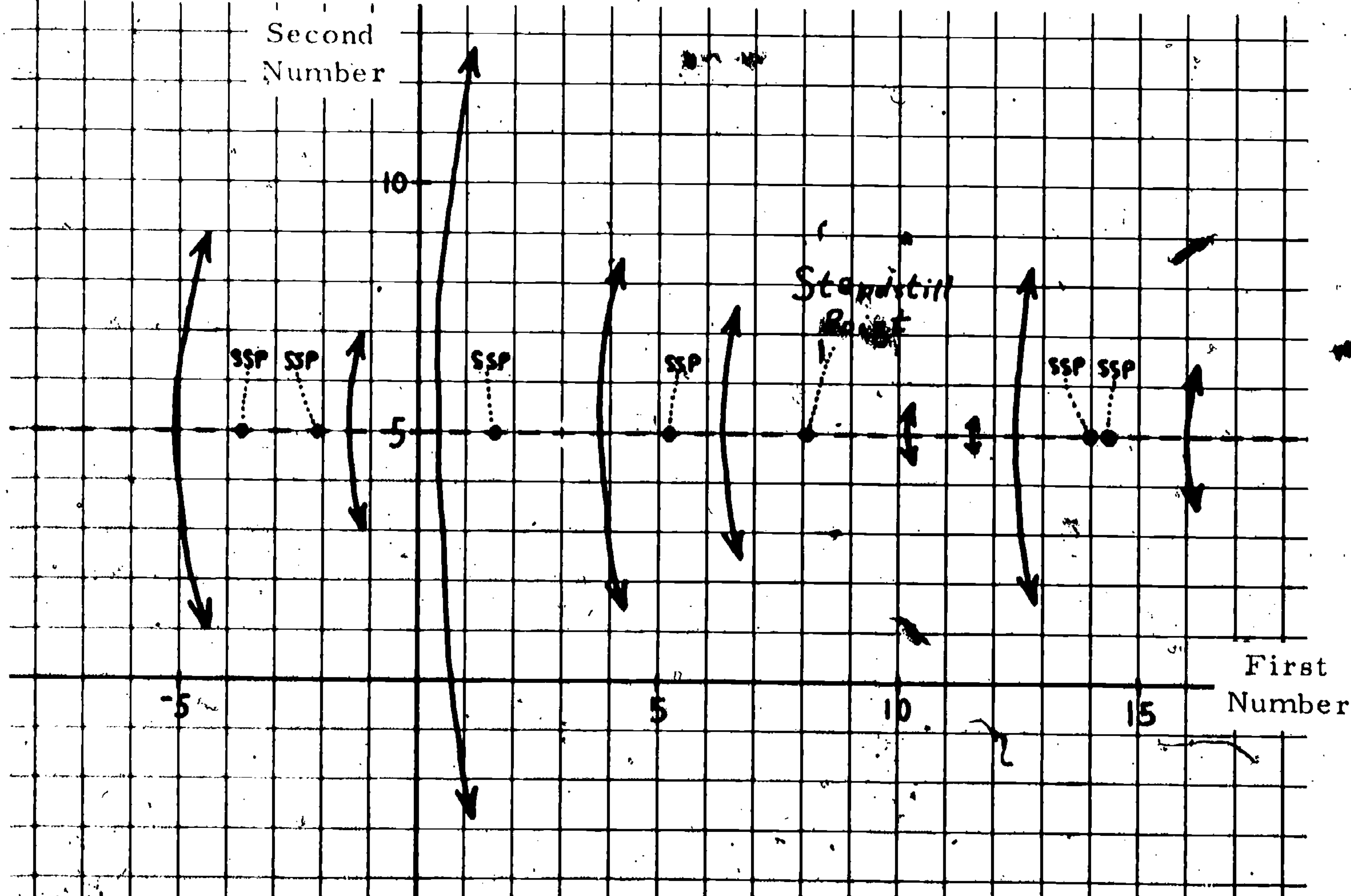
$$(\square, \triangle) \xrightarrow{A} (\square + 1, \triangle)$$

and $$(\square, \triangle) \xrightarrow{B} (\square, 10 - \triangle)$$

Rule A, $(\square, \triangle) \xrightarrow{A} (\square + 1, \triangle)$, slides every point in the plane one space to the right:



Rule B, $(\square, \triangle) \xrightarrow{B} (\square, 10 - \triangle)$, flips the whole plane about a horizontal line which is five spaces up from (0 , 0):



The kind of isometry which, like rule A, moves every point the same distance and direction might well be called a glide in the elementary classroom; technically, it is called a translation. The composite of any two translations is again a translation. Since a translation which moves any particular point moves all other points the same distance and direction, no translation except the simple one, $(\square, \triangle) \longrightarrow (\square, \triangle)$, has any standstill points: Every point is a standstill point of the rule $(\square, \triangle) \longrightarrow (\square, \triangle)$ because nothing is moving at all.

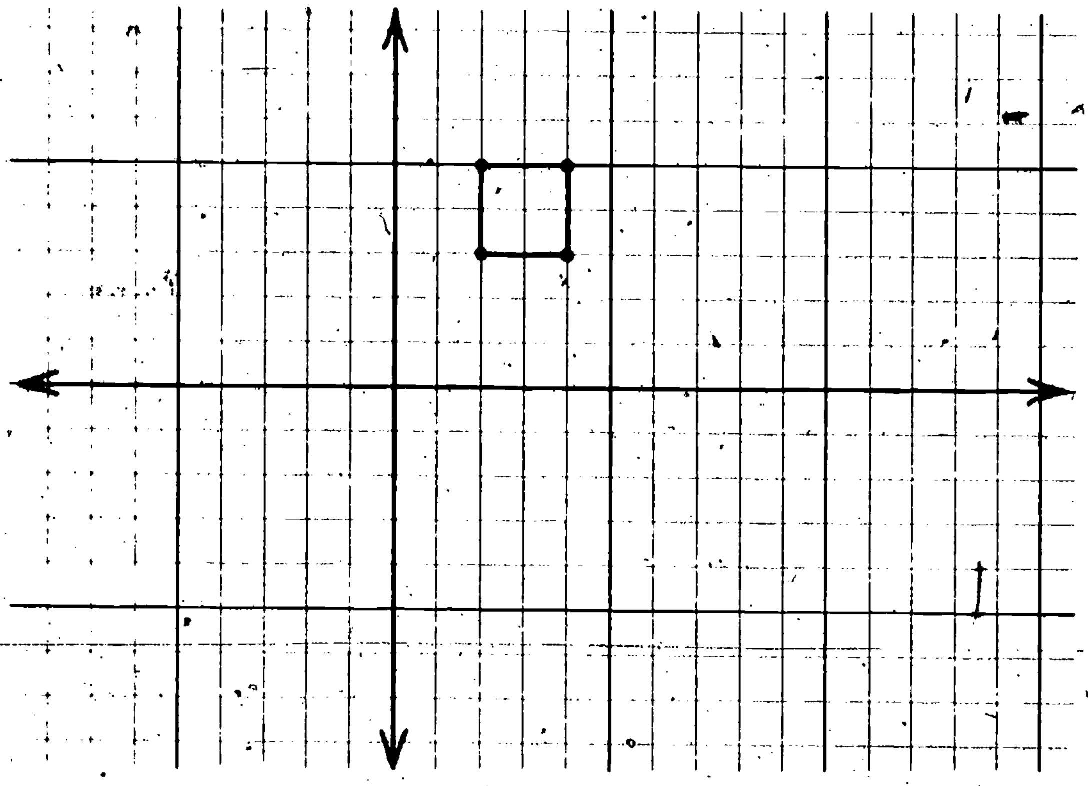
The kind of isometry which, like rule B, spins the whole plane about some straight line can be spoken of in the classroom as a flip; technically, it is called a reflection. The standstill points of a reflection are all the points on the line around which the plane is turned.

Rule a of page 7 is a reflection; this time the flip is made around the diagonal line through (10, 0) and (0, 10).

Rules b and c of page 10 are reflections too; rule b flips the plane around the line going through (0, 0) and (1, 1), and rule c flips about the line passing through (0, 0) and (1, −1). Interestingly enough, the composite of these two reflections, $(\square, \triangle) \xrightarrow{bc} (-\square, -\triangle)$, is not itself a reflection. Instead, it is a rotation through 180° (a half turn) about the origin, (0, 0). Notice that the only standstill point for this composite rotation is (0, 0).

Summary of Problems in the Film
"Jumping Rules in the Plane, Part II"

6th Grade, James Russell Lowell School, Watertown, Massachusetts
Teacher: Lee Osburn



Here is a new rule: $(\square , \triangle) \longrightarrow (\square \times 5 , \triangle)$

Will the area change or stay the same now?

Teacher changes the rule to $(\square , \triangle) \longrightarrow (\square \times 3 , \triangle)$.

Try (2 , 5)	(6 , 5)
Try one of the other points up there.	(2 , 3) goes to (6 , 3)
Maybe you are right and the area does stay the same.	
Try another point.	(4 , 3) goes to (12 , 3)
Another?	(4 , 5) goes to (12 , 5)
	"It has gone three times bigger."
	"The space between the two sets of dots is three of those boxes."

New problem: Write a rule so that when you start with that same basic figure that we have up there, your rule will make the area 15 times bigger and put the figure down here in the lower left hand quarter.

Following are some of the students' written answers and the teacher's comments:

$(\square , \triangle) \longrightarrow (\square , \triangle \times 15)$

That will make the area 15 times bigger, but will all of the picture be down here?

Other similar answers were:

$(\square , \triangle) \longrightarrow (15 \times (\square) , \triangle)$

$(\square , \triangle) \longrightarrow (\square \times 5 , \triangle \times 3)$

$(\square , \triangle) \longrightarrow (\square \times 3 , \triangle \times 5)$

$(\square , \triangle) \longrightarrow (\square \times 3 , \triangle \times 5) - 6$

If you want to subtract 6, where do you want to subtract it from? (From $\triangle \times 5$)

Then you have to put it in here: $(\square , \triangle) \longrightarrow (\square \times 3 , \triangle \times 5 - 6)$

$(\square , \triangle) \longrightarrow (\square - 20 , \triangle - 5)$

That's not going to make the area bigger.

$(\square , \triangle) \longrightarrow (\square \times 15 , \triangle \times 15)$

That's going to be more than 15 times bigger.

$(\square , \triangle) \longrightarrow (\square \times 15 , \triangle \div 15)$

That does funny things.

Let's see if we can get a rule that will make it 15 times bigger. Then we will worry about moving it.

$(\square , \triangle) \longrightarrow (\square \times 15 , \triangle)$

Can you do something to that rule so that it will come down over here?

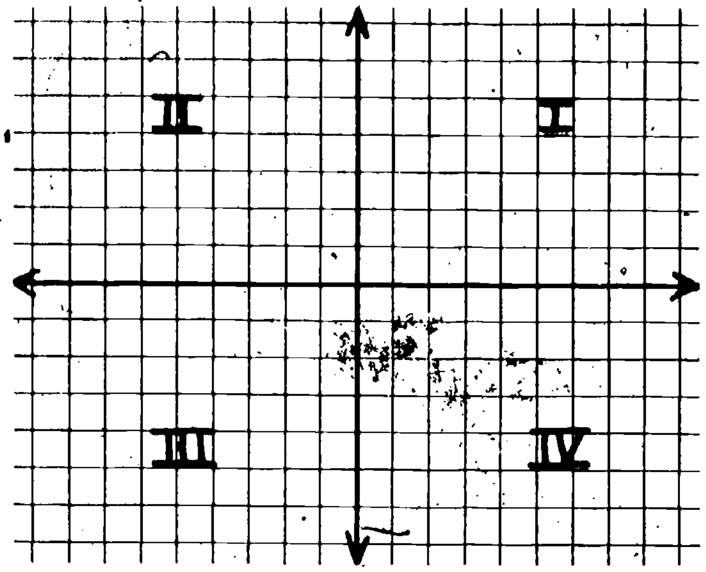
$(\square , \triangle) \longrightarrow (15 \times (-\square) , -\triangle)$

Let's try all four of these points on the corners. Where will (2 , 5) go with this rule?

$(-30 , -5)$

Where is that point? Is it going to be in I, II, III, or IV?

(In III.)



How about (4, 5)? "It's going to get you in III."

What's the name of the point? (−60, −5)

Is the area going to be 15 times the original?

How do you know? "... it would be 30 across, and 2 × 30 is 60 and 4 × 15 is 60."

(□, △) ⟶ (5 × □, 7 × △): How many times bigger will the area be?

(□, △) ⟶ (100 × □, 5 × △): How many times bigger now?

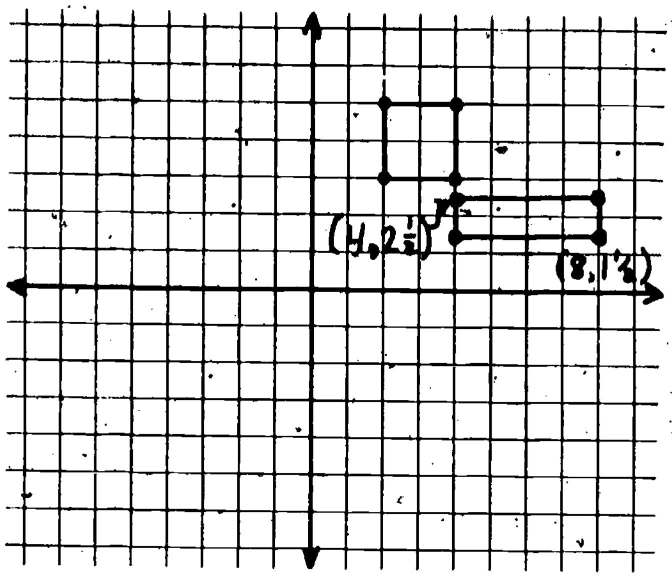
(□, △) ⟶ (70 × □, 6 × △): How many times bigger?

(□, △) ⟶ (2 × □, $\frac{1}{2}$ × △): How many times bigger?

Using (□, △) ⟶ (2 × □, $\frac{1}{2}$ × △), check a couple of points out. Where will (2, 5) go?

(4, $2\frac{1}{2}$)

Where will (4, 5) go? (2, 3)? (4, 3)?



What happened to the figure? "The shape changed."

It changed from a square to a rectangle.

Using that rule, draw a rectangle so that after you use that rule once, the rectangle will be a square.

Supplement

More Work With Number Plane Rules

by Edward Esty

I.

The written lesson Epilogue in this booklet introduces isometries, which are jumping rules that do not alter distances. Translations, reflections and rotations are examples of isometries. The purpose of this paper is to explore number plane isometries in some greater depth, and to develop ways of combining simple isometries by composition to form more complicated ones.

As was suggested in the written lessons, one of the best ways to see what a jumping rule does is to try making jumps with it. Trying things out is the way a child starts to learn about jumping rules (among other things). You also will learn from this process in the same way as your students. Most definitely, you should not wait until you "understand all about" number plane rules before beginning to play with them.

After doing the written lessons, you may have made up and tried to solve such questions as, "Here is a triangle. Can I make up a rule that will move it to a specific place, or turn it upside down?" or "Here is a rule that I made up. What will it do?" One can ask what one jump with a rule will do to figures located in various places, what will happen with successive jumps with the rule, and where its standstill points are. Whether or not you and your class were able to solve all the problems you thought of, this supplement will sum up some of the discoveries you made, and introduce you to ways to pursue other interesting problems.

Central to what follows is the fact that isometries, when combined by the operation of composition, enjoy the following four properties:

1. The composite of two isometries is again an isometry. (See page 15 in this booklet.)

2. Composition is associative. This is true regardless of whether the rules (functions) being composed are isometries or not. Let A, B, and C be three jumping rules, and compare (AB)C with A(BC). Each of these is

the rule which has the same effect as first doing rule A, then rule B, and finally rule C. So (AB)C is the same as A(BC). Saying anything more about this would only confuse things.

3. The rule $(\square, \triangle) \xrightarrow{i} (\square, \triangle)$ is an isometry, and has no effect. This means that no matter what rule b is, bi is the same as b, and ib is also the same as b. In this paper we will always use the letter i to name the rule $(\square, \triangle) \longrightarrow (\square, \triangle)$. Rule i can be viewed either as a translation that slides every point a distance of zero or else as a rotation through zero degrees.

4. If rule t is any isometry, then there is an isometry $\bar{t}$ (read "t bar") which undoes whatever rule t did; that is,

$$t\bar{t} = i$$

Rule $\bar{t}$ is called the <u>inverse</u> of rule t. (See problems 5, 6, and 7 on pages 4 and 5 of the written lesson for Book 16.) Furthermore, rule t is the inverse of rule $\bar{t}$, so we also have $\bar{t}t = i$. For example, if t is the

translation $(\square, \triangle) \longrightarrow (\square + 4\frac{1}{2}, \triangle - 100)$,

then $\bar{t}$ is $(\square, \triangle) \longrightarrow (\square - 4\frac{1}{2}, \triangle + 100)$.

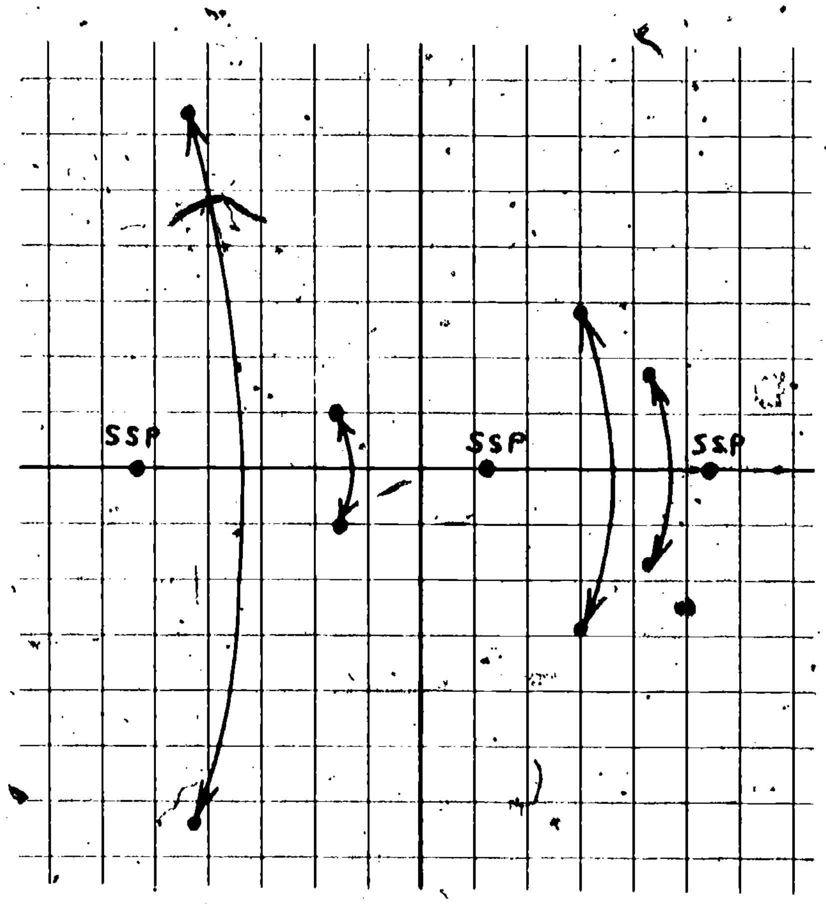
Translations are simple. Some problems:

1. Write the translation that sends (1, 2) to (10, 10).
   $(\square, \triangle) \xrightarrow{c}$ ______________________
2. Where does the rule you wrote for problem 1 send (0, 0)? __________
3. What point gets sent <u>to</u> (0, 0) using the same rule (rule c)? __________
4. Where does the inverse of rule c send (0, 0)? __________
5. Where is $(3\frac{1}{2}, 6)$ sent, using rule $c\bar{c}$? __________
6. Starting at (0, 0) make three consecutive jumps with rule c. What is your final landing point?
7. Starting at (0, 0) make one jump with rule ccc. Where do you land? __________
8. One jump with the translation d sends the point $(798, -14\frac{1}{2})$ 897 spaces straight down.
   Write the <u>inverse</u> of rule d.
   $(\square, \triangle) \xrightarrow{\bar{d}}$ ______________________

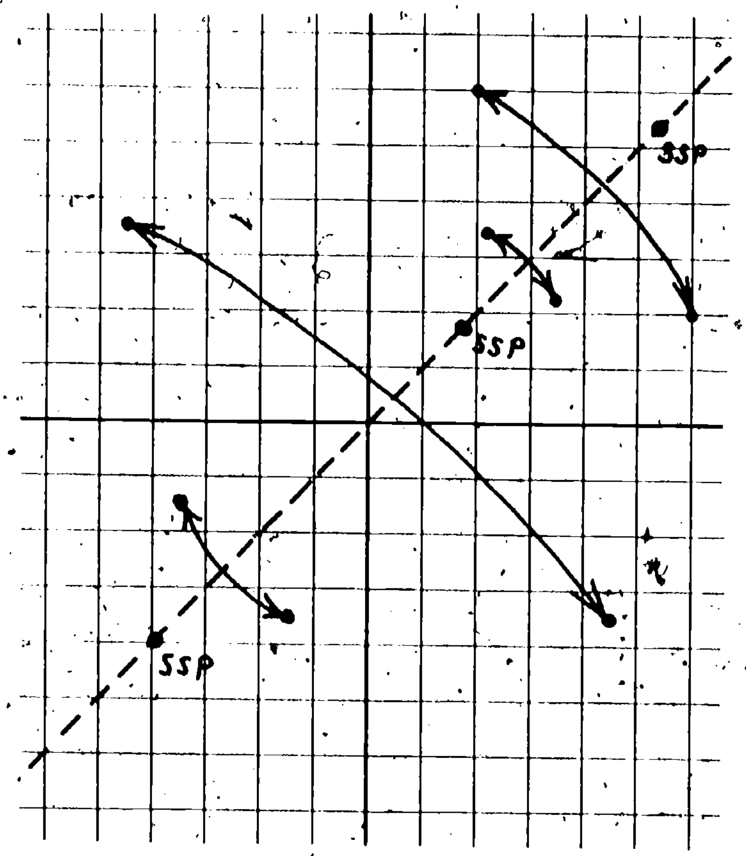
We turn now to reflections and rotations, considering for the moment two very simple reflections. The first is

$$(\square, \triangle) \xrightarrow{f} (\square, -\triangle)$$

which flips the plane around the first number line. Its effect on number pairs is to keep the first component the same but to replace the second component by its opposite. Notice that standstill points for f are exactly all points on the first number line. Also, f is its own inverse: ff = i. (Of course, this is true of all reflections—not just f.)



The second reflection we consider is $(\square, \triangle) \xrightarrow{g} (\triangle, \square)$ which is the flip about the diagonal line through (0, 0) and (1, 1).* Its effect on number pairs is simply to interchange the components. The fixed points for g are those points whose coördinates are equal. And again, $g = \bar{g}$ —that is, g is its own inverse.

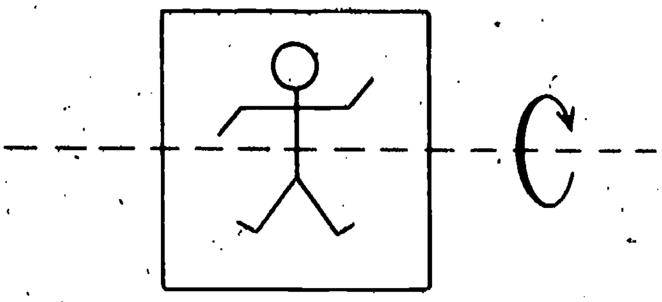


*We used the letter f because it would suggest "flip". The letter g is near f in the alphabet and thus seems appropriate to indicate another flipping rule. Because rules f and g are used so frequently, we rewrite them at the top of all appropriate pages.

$(\boxtimes, \triangle) \xrightarrow{f} (\square, -\triangle)$

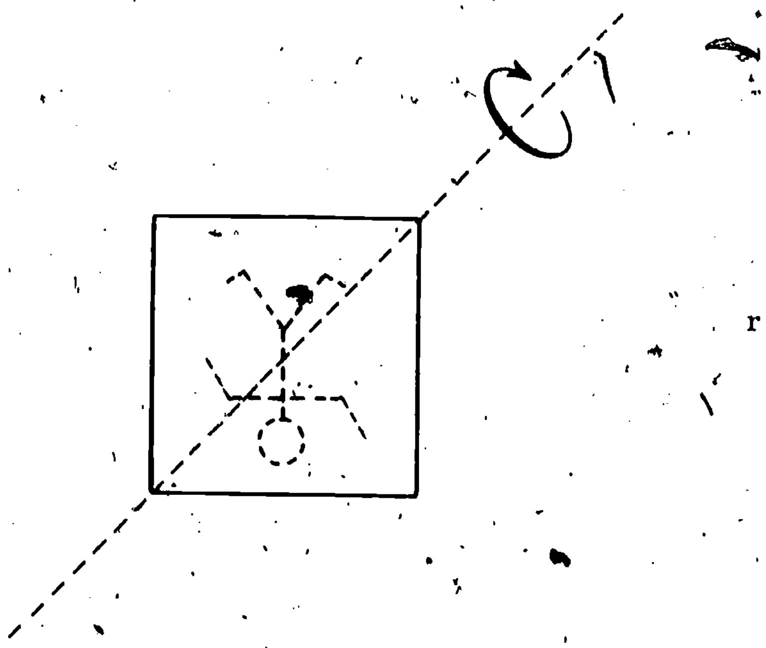
$(\square, \triangle) \xrightarrow{g} (\triangle, \square)$

What does the composite rule fg do? The reader is encouraged to cut out the square at the bottom of this page, give it a flip like this:



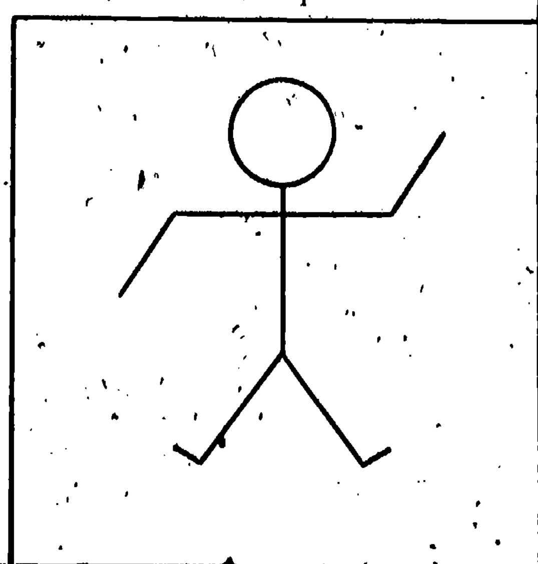
rule f

and then a flip like this:



rule g

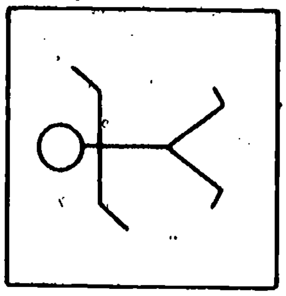
Cut out this square



$(\square, \triangle) \xrightarrow{f} (\square, -\triangle)$

$(\square, \triangle) \xrightarrow{g} (\triangle, \square)$

The final result looks like this:



The little man has been rotated 90° counterclockwise. (This is hard to show in pictures. It's much more convincing if you actually do it yourself.)

In terms of boxes and wedges the rule fg is $(\square, \triangle) \longrightarrow (-\triangle, \square)$, because

$$(\square, \triangle) \xrightarrow{f} (\square, -\triangle) \xrightarrow{g} (-\triangle, \square)$$

Take the opposite of second component.

Interchange components.

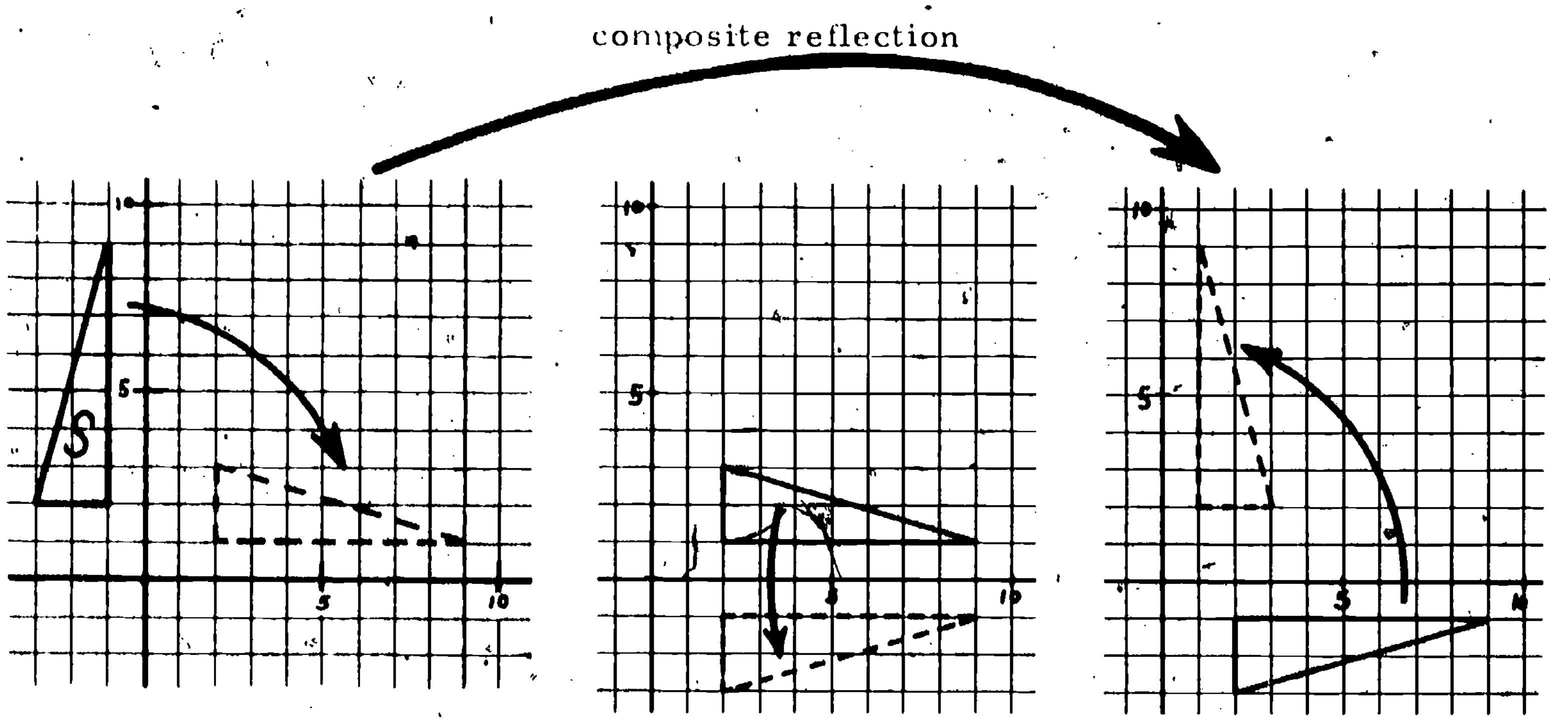
The origin, $(0, 0)$, is the sole fixed point of this rotation.

Problems:

1. If fg is a counterclockwise rotation of 90°, what is fgfg ? That is, what motion has the same effect as two successive 90° counterclockwise rotations ?
2. Rule fgfgfg is a rotation of ______ degrees counterclockwise, or, what amounts to the same thing, a rotation of ______ degrees clockwise.
3. What is a much simpler name for rule fgfgfgfg ? __________
4. The rule gffg must be i because we can cross off the ff in the middle (since ff = i) and then we are left with gg , which is also i . So, since (gf)(fg) = i, gf must be the inverse of fg. Describe what rule gf does.

$$(\square\,,\,\triangle) \xrightarrow{f} (\square\,,\,-\triangle)$$
$$(\square\,,\,\triangle) \xrightarrow{g} (\triangle\,,\,\square)$$

You can doubtless figure out how to get a reflection about the second number line. Instead of taking the opposite of the second component as we did in rule f, we must take the opposite of the first component. The rule is then $(\square\,,\,\triangle) \longrightarrow (-\square\,,\,\triangle)$. But let's see how we could have obtained this rule using only f's and g's. Since we want to flip around the second number line, we first do a rotation of 90° clockwise, using rule gf which sends the second number line into the first. Then we flip using rule f, and finally rotate 90° counterclockwise, using rule fg. The effects of these rules on a triangle, S, are shown below.



gf
rotate 90° clockwise

f
flip

fg
rotate back again

The composite rule is therefore gfffg, which we can simplify into gfg by crossing out ff. With boxes and wedges we have:

$$(\square\,,\,\triangle) \xrightarrow{g} (\triangle\,,\,\square) \xrightarrow{f} (\triangle\,,\,-\square) \xrightarrow{g} (-\square\,,\,\triangle)$$

switch components

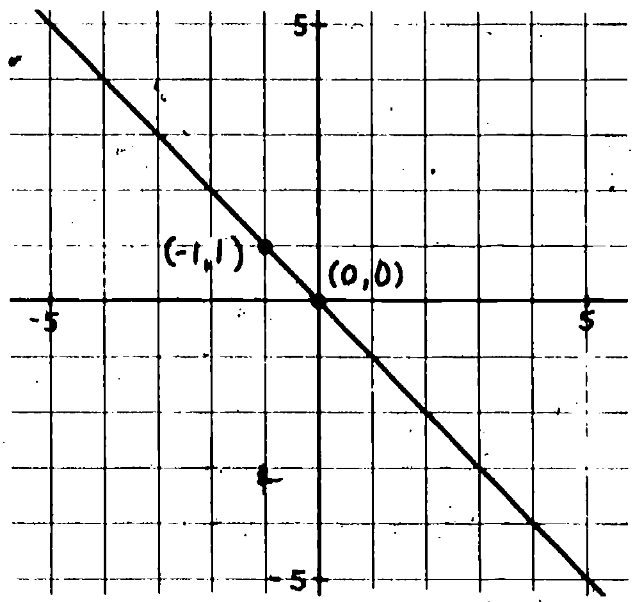
take opposite of second component

switch components again

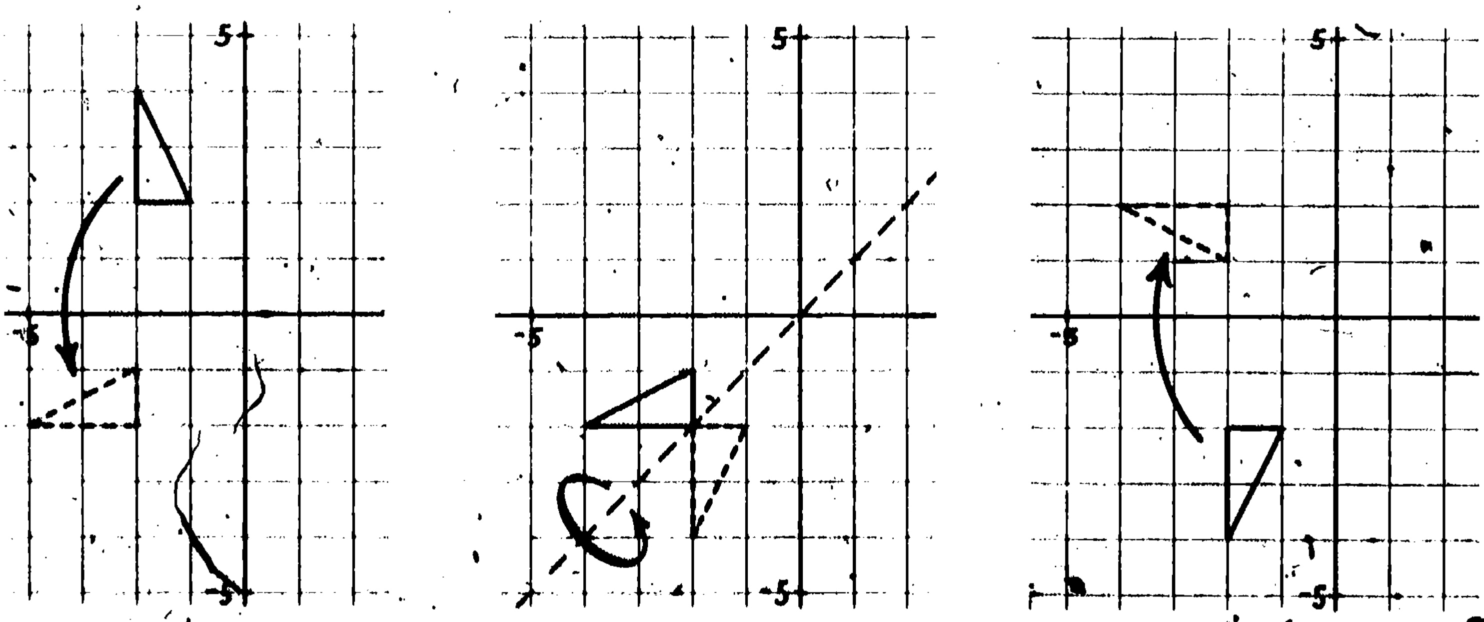
$$(\square\,,\triangle) \xrightarrow{f} (\square\,,-\triangle)$$
$$(\square\,,\triangle) \xrightarrow{g} (\triangle\,,\square)$$

The same sort of thing is happening with the boxes and wedges as is happening in the pictures; we move the box into the second slot, take its opposite, and then move it back. An analogous thing occurs in hospitals. The patient is moved from his room to the operating room, is operated on, and then moved back to his own room. Moving the surgical equipment to the patient's room and operating there would have the same effect, but of course the standard procedure is far easier.

Our last simple reflection will be one about the other diagonal—the line through $(-1\,,\,1)$ and $(0\,,\,0)$.



Rotating 90° counterclockwise will bring the diagonal line shown onto the diagonal through $(0\,,\,0)$ and $(1\,,\,1)$. Then we flip using rule g and finally rotate 90° clockwise. We illustrate these steps by showing their effect on another triangle.



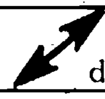
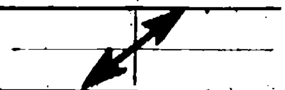
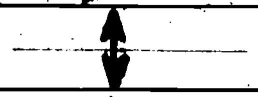
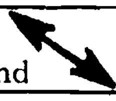
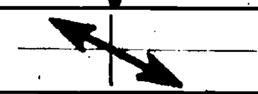
fg g gf

$$(\square, \triangle) \xrightarrow{f} (\square, -\triangle)$$
$$(\square, \triangle) \xrightarrow{g} (\triangle, \square)$$

This time the composite rule is fgggf, or, more simply, fgf.

$$(\square, \triangle) \xrightarrow{f} (\square, -\triangle) \xrightarrow{g} (-\triangle, \square) \xrightarrow{f} (-\triangle, -\square)$$

We summarize the reflections and rotations we have so far:

	Rule	Name	Effect	Standstill Point(s)
ROTATIONS	$(\square, \triangle) \rightarrow (-\triangle, \square)$	fg	90° counterclockwise or 270° clockwise	(0 , 0)
	$(\square, \triangle) \rightarrow (-\square, -\triangle)$	fgfg or gfgf	180° counterclockwise or 180° clockwise	(0 , 0)
	$(\square, \triangle) \rightarrow (\triangle, -\square)$	fgfgfg or gf	270° counterclockwise or 90° clockwise	(0 , 0)
	$(\square, \triangle) \rightarrow (\square, \triangle)$	fgfgfgfg or i	0° counterclockwise or clockwise	every point
REFLECTIONS	$(\square, \triangle) \rightarrow (\square, -\triangle)$	f	flip around first number line	
	$(\square, \triangle) \rightarrow (\triangle, \square)$	g	flip around  diagonal	
	$(\square, \triangle) \rightarrow (-\square, \triangle)$	gfg	flip around second number line	
	$(\square, \triangle) \rightarrow (-\triangle, -\square)$	fgf	flip around  diagonal	

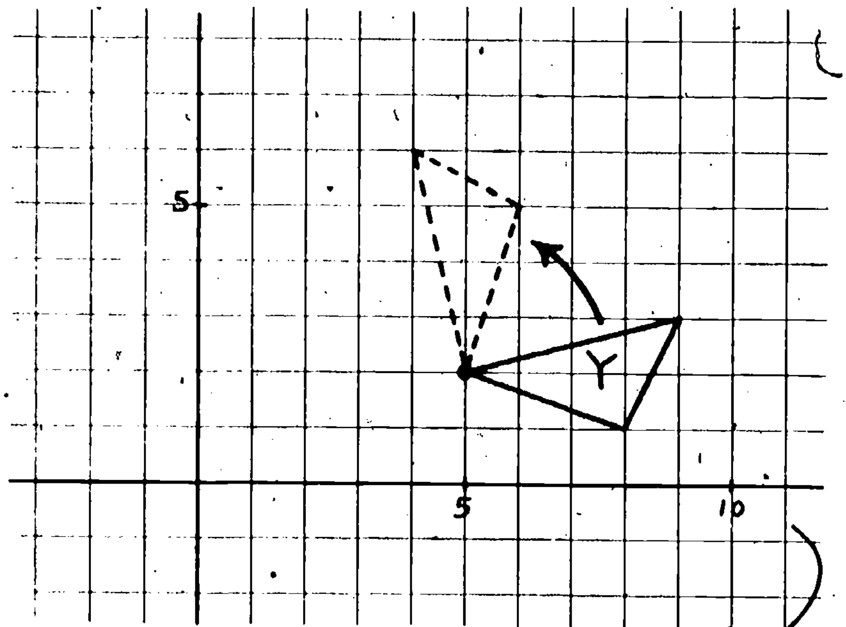
The reader may be wondering, "How can I remember all those rules? They all look so much alike." Indeed they do, and this is why nobody does memorize them. All you have to remember is the two reflections $(\square, \triangle) \xrightarrow{f} (\square, -\triangle)$ and $(\square, \triangle) \xrightarrow{g} (\triangle, \square)$, because all the others can be generated from these two. (If this is reminiscent of lists of rules discussed in an earlier supplement, it should be. Rules f and g generate a list of eight rules—those appearing in the chart above.)

$(\square, \triangle) \xrightarrow{f} (\square, -\triangle)$

$(\square, \triangle) \xrightarrow{g} (\triangle, \square)$

III.

So far, our rotations have been around the origin and our reflections about lines through the origin. Can we find rotations and reflections through other points and lines? Techniques similar to those we have already used are brought into play again; we illustrate by deriving the 90° counter-clockwise rotation around the point (5, 2), whose effect on triangle Y is shown below.



We first perform the translation that sends (5, 2) to (0, 0); then we do the rotation fg; finally we slide back using the inverse of our original translation. The first translation, t, is

$$(\square, \triangle) \xrightarrow{t} (\square - 5, \triangle - 2)$$

Our rotation is

$$(\square, \triangle) \xrightarrow{fg} (-\triangle, \square)$$

and the inverse translation is

$$(\square, \triangle) \xrightarrow{\bar{t}} (\square + 5, \triangle + 2)$$

This gives us

$$(\square, \triangle) \xrightarrow{t} (\square - 5, \triangle - 2) \xrightarrow{fg} (-(\triangle - 2), \square - 5) \xrightarrow{\bar{t}}$$

$$(-(\triangle - 2) + 5, \square - 5 + 2)$$

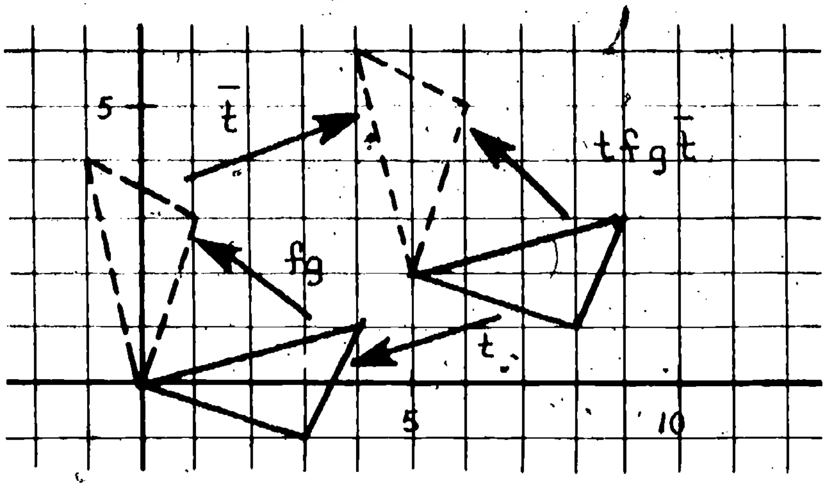
so $(\square, \triangle) \xrightarrow{tfg\bar{t}} (-(\triangle - 2) + 5, \square - 5 + 2)$

or, simplifying, $(\square, \triangle) \xrightarrow{tfg\bar{t}} (7 - \triangle, \square - 3)$

$$(\square\ ,\ \triangle) \xrightarrow{f} (\square\ ,\ -\triangle)$$

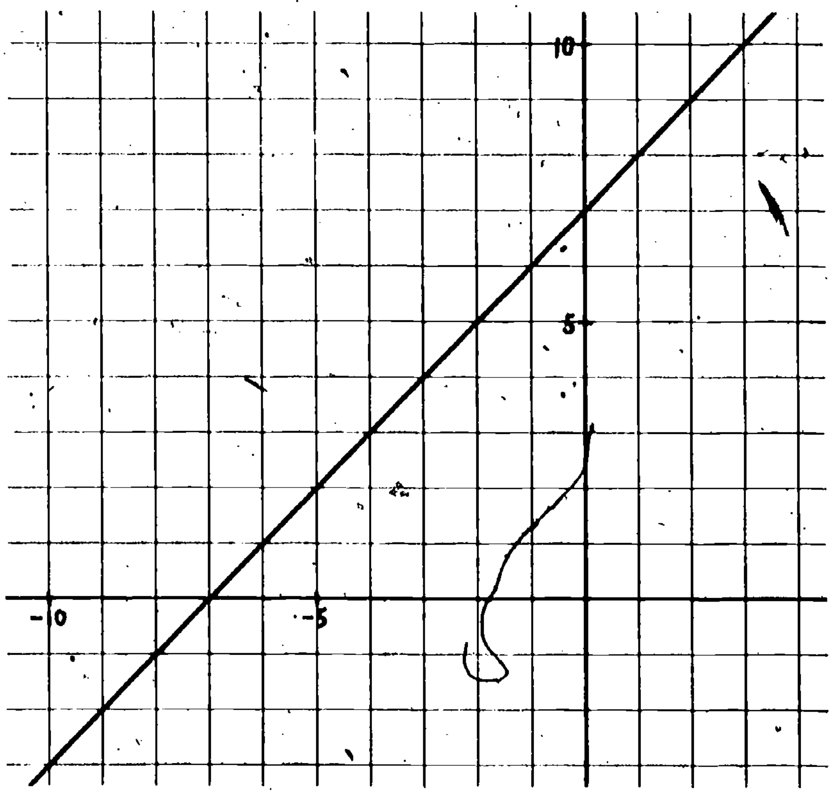
$$(\square\ ,\ \triangle) \xrightarrow{g} (\triangle\ ,\ \square)$$

The effects on triangle Y of the succession of rules t, fg, and $\bar{t}$ (and hence the effect of the composite rule $tfg\bar{t}$) are shown here:



The reader is encouraged to try this rule on a few points to see that it really performs as advertised.

One last example: What is the reflection about the line through (−7 , 0) and (0 , 7) ?



Pick any point on the line in question and find the translation which sends that point to (0 , 0). (Call the translation s .) Then do the flip $(\square\ ,\ \triangle) \xrightarrow{g} (\triangle\ ,\ \square)$. Follow that with the inverse of the translation. Suppose we pick (−1 , 6), which is on the line through (−7 , 0) and (0 , 7).

$$(\square\ ,\ \triangle) \xrightarrow{s} (\square + 1\ ,\ \triangle - 6) \xrightarrow{g} (\triangle - 6\ ,\ \square + 1) \xrightarrow{\bar{s}} (\triangle - 6\ -\ 1\ ,\ \square + 1 + 6)$$

So, $$(\square\ ,\ \triangle) \xrightarrow{sg\bar{s}} (\triangle\ -\ 7\ ,\ \square\ +\ 7)$$

$(\square, \triangle) \xrightarrow{g} (\triangle, \square)$

Would it have made any difference if we had picked a point other than $(-1, 6)$? Try $(-4, 3)$. The rules then look like this:

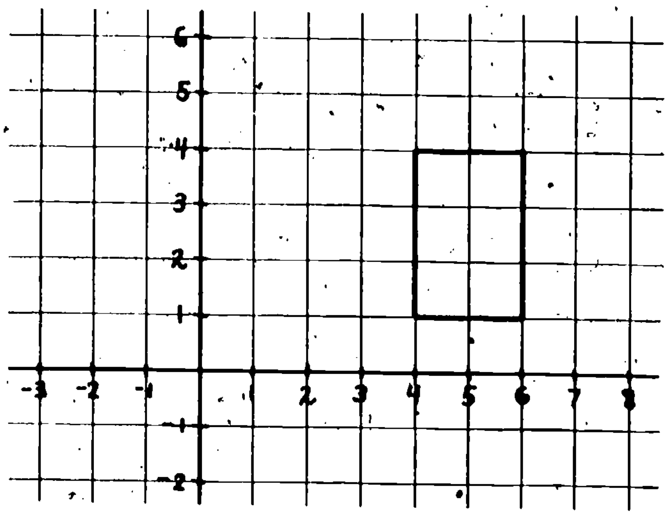
$(\square, \triangle) \xrightarrow{n} (\square + 4, \triangle - 3) \xrightarrow{g} (\triangle - 3, \square + 4) \xrightarrow{\bar{n}} (\triangle - 3 - 4, \square + 4 + 3)$

Again we have $(\square, \triangle) \longrightarrow (\triangle - 7, \square + 7)$

More Problems:

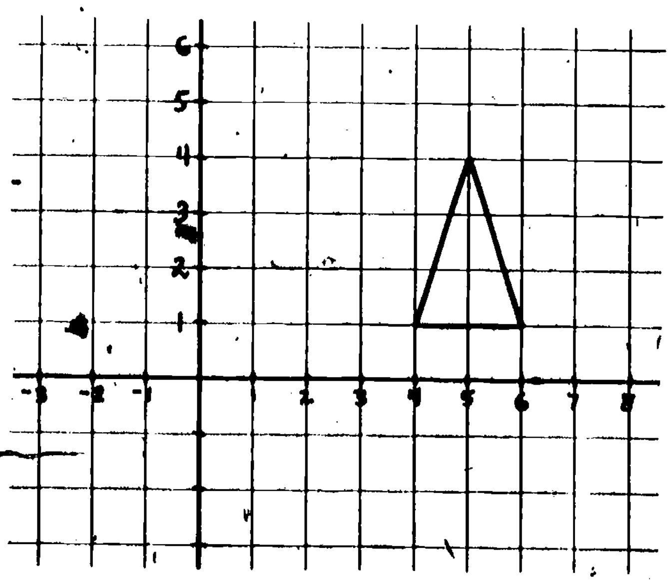
1. Using the rules s, g and n of pages 32 and 33, write the rule $sg\bar{n}$ using boxes and wedges. (This kind of isometry is like the rules of the first five pages of the written lesson in this booklet and is called a <u>glide reflection.</u>)

2. What are all the isometries that do not alter the position of the rectangle shown below?

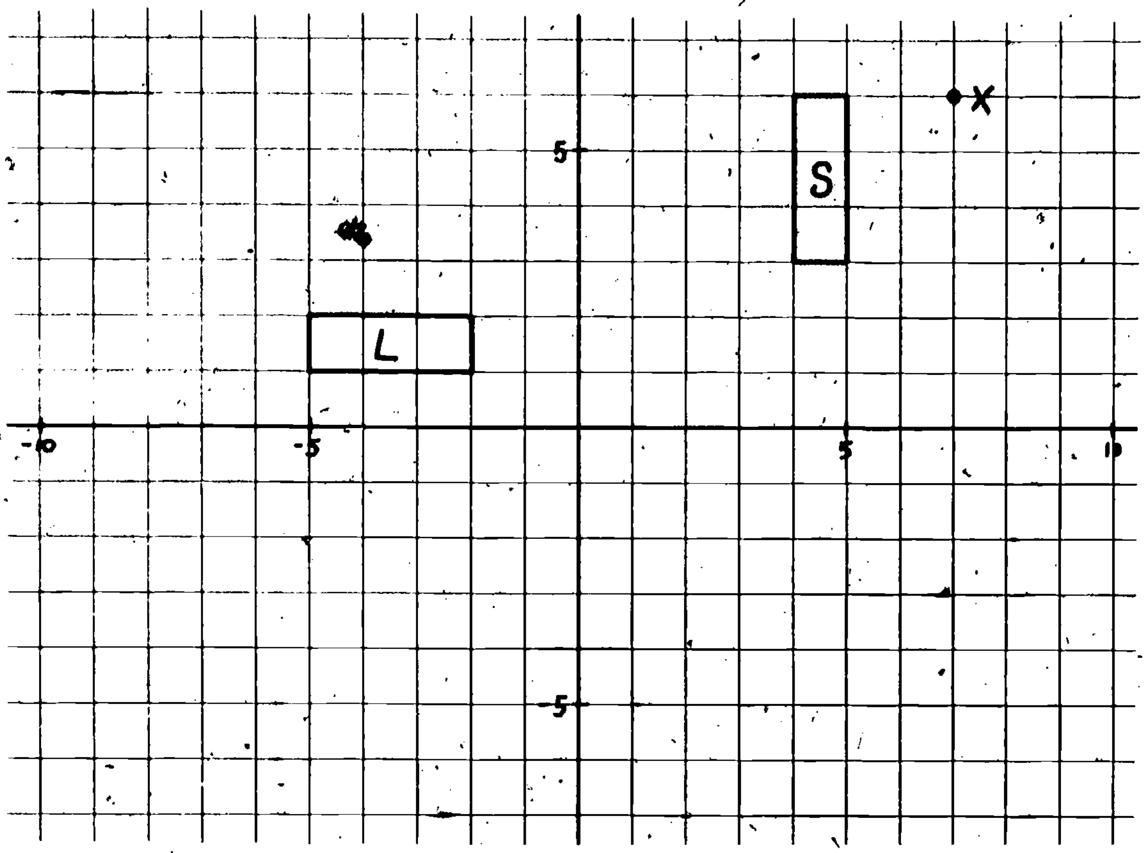


(Hint: There are four of them, two of which are reflections.)

3. What are all the isometries that do not alter the position of the triangle shown below?



4. There are four isometries which take the points of rectangle S into the points of rectangle L. Of these four, how many take the point X into the third (lower left-hand) quadrant?



5. Find a point Y such that exactly <u>one</u> of the four isometries of problem 4 takes Y into the third quadrant.

6. Replace the underlined word of problem 5 by <u>three</u>; by <u>four</u>; by <u>none</u>. Which are possible?

IV.

We have said nothing about rotations through angles other than 0°, 90°, 180°, or 270°, nor have we mentioned reflections about lines other than those that make angles of 0°, 45°, 90°, or 135° with the first number line. The reason for this is that even though other reflections and rotations are as simple conceptually as the ones we have considered, they are devilishly difficult to express in box-wedge jumping rule form. If the angles involved are simple, like 30° or 60°, then some of the numbers involved are irrational (like $\sqrt{3}$); if the numbers are relatively simple the angles are never easy to handle. This is just the way the ball bounces (reflects); but, for the determined reader we append two formulas:

1. The reflection about the line that goes through (0, 0) and (1, n) is:

$$(\square, \triangle) \longrightarrow \left(-\frac{n^2-1}{n^2+1} \times \square + \frac{2n}{n^2+1} \times \triangle \; , \; \frac{2n}{n^2+1} \times \square + \frac{n^2-1}{n^2+1} \times \triangle\right)$$

(The symbol "1" stands for the number <u>one</u>, and is not a lower case letter L.)

2. The rotation which keeps (0, 0) fixed and sends (1, 0) to (a, b) is given by

$$(\square, \triangle) \longrightarrow (a\square - b\triangle, \; b\square + a\triangle)$$

The rotation jumping rule looks pretty straightforward until you realize that the point (a, b) must be on the circle of radius 1 centered at (0, 0), because the distance between (0, 0) and (a, b) must be 1, just as it is between (0, 0) and (1, 0). Even a rotation as seemingly simple as 45° counterclockwise necessitates irrational numbers, for we must send (1, 0) to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ making the rule

$$(\square, \triangle) \longrightarrow \left(\frac{\sqrt{2}}{2}\square - \frac{\sqrt{2}}{2}\triangle, \; \frac{\sqrt{2}}{2}\square + \frac{\sqrt{2}}{2}\triangle\right)$$