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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on artificial operations and competing rules, and a summary of the problems from the film "Some Artificial Operations." (MK)

THE ARITHMETIC PROJECT COURSE FOR TEACHERS

TOPICS: Artificial Operations. Competing Rules.

FILM: Some Artificial Operations, Grade 4

SUPPLEMENT: Well-Adjusted Trapezoids

NAME ....

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The course is available from:

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BOOK ELEVEN

I.

One way to study the laws of arithmetic is to work with some new kinds of operations with numbers.

Suppose we take a new operation $\odot$ ("circle-dot"). We define what $\odot$ does as follows:

$\square \odot \triangle$ is the number halfway between $\square$ and $\triangle$.

Some examples:

$5 \odot 8 = 6\frac{1}{2}$ (pronounced "Five circle-dot eight equals six and one-half".)

$5 \odot -3 = 1$

Another way of describing what circle-dot does:

$\square \odot \triangle$ by definition is $\dfrac{\square + \triangle}{2}$

which can be abbreviated

$$\square \odot \triangle \overset{\text{df}}{=} \frac{\square + \triangle}{2}$$

1. $8 \odot 10 =$ __________

2. $10 \odot 20 =$ __________

3. $5 \odot 10 =$ __________

Continue with: $\square \odot \triangle \overset{df}{=} \frac{\square + \triangle}{2}$

4. $18 \odot 18 =$ ____________

5. $0 \odot 82 =$ ____________

6. $1000\frac{1}{2} \odot \frac{1}{2} =$ ____________

7. $1000\frac{1}{2} \odot -\frac{1}{2} =$ ____________

8. $1000\frac{1}{2} \odot -1 =$ ____________

9. $\triangle \odot 10 = 100$

10. $\square \odot 10 = 101$

11. $\diamond \odot 50 = 10$

12. $\square \odot 1000 = 0$

13. $492 = \square \odot 1000$

14. $(2 \odot 10) \odot 20 =$ ____________

15. $2 \odot (10 \odot 20) =$ ____________

Notice that in problems 14 and 15 the answers were different although the same numbers occurred in the same order. An expression without parentheses or loops, such as $2 \odot 10 \odot 20$, is not meaningful. It does not make clear which of the two answers in problems 14 and 15 is meant. Circle-dot is not an associative operation.

A similar expression with three numbers and +'s <u>is</u> meaningful: $17 + 3 + 5 = 25$. Addition is associative. Whichever operation you perform first, the result is the same: $(17 + 3) + 5 = 17 + (3 + 5)$.

In problems 16 through 22, continuing with $\square \odot \triangle \stackrel{\text{df}}{=} \dfrac{\square + \triangle}{2}$, insert parentheses to obtain the largest possible number, and give the number. The first one has been done for you.

16. $(1 \odot 3) \odot 6$ = 4 1

17. $1 \odot 1 \odot 49$ = ______

18. $20 \odot 20 \odot 2$ = ______

19. $6000 \odot 0 \odot 20$ = ______

20. $50 \odot 100 \odot 12$ = ______

21. $\frac{1}{2} \odot 400 \odot 400$ = ______

22. $14 \odot 18 \odot 14$ = ______

23. In a student's words, what is a good strategy to follow in doing the preceding problems?

24. Solve $(\triangle \odot \triangle) \odot \triangle = 3052$

25. Solve $\triangle \odot (\triangle \odot \triangle) = 3052$

II.

Here are some more problems with the operations "star" and "check" that you saw in the film. (For convenience we shall use the same symbols, but the choice of symbols is otherwise arbitrary.) Although the definitions were not written in this form, you will recall that

$$\square * \triangle \stackrel{df}{=} \square + \square + \triangle$$

$$\square \checkmark \triangle \stackrel{df}{=} \max(\square, \triangle)$$

(pronounced "the maximum of box and wedge")

Examples: $5 * 18 = 28$ $\qquad$ $18 * 5 = 41$

$5 \checkmark 18 = 18$ $\qquad$ $18 \checkmark 5 = 18$

$5 \checkmark 5 = 5$

1. $15\frac{2}{9} \checkmark 15 =$ __________

2. $-12 \checkmark 0 =$ __________

3. $-13 \checkmark -13\frac{1}{5} =$ __________

4. $8 * 5 =$ __________

5. $33 * 33 =$ __________

6. $\diamond * -10 = 0$

7. $\diamond * -10 = -10$

8. $\diamond * -10 = -11$

Continue with: $\square * \triangle \overset{\text{df}}{=} \square + \square + \triangle$

$\square \surd \triangle \overset{\text{df}}{=} \max(\square, \triangle)$

9. $\triangle * \triangle = -12$

10. $(2 \times \square) * 7 = 17$

11. $(2 \times \square) * (2 \times \square) = 30$

12. $\square * 2 = 12 \surd \square$

13. $\square * 2 = 26 \surd \square$

14. $\square * 26 = 26 \surd \square$

15. $\square * 4\frac{1}{2} = 4\frac{1}{2} \surd \square$

16. $\square * 4\frac{1}{2} = 3\frac{1}{2} \surd \square$

17. $4\frac{1}{2} * \square = 4\frac{1}{2} \surd \square$

☆18. $\square * 4 = (3 \times \square) \surd 10$

(Hint: Two numbers work.)

☆19. $\square * 2 = (3 \times \square) \surd 10$

☆20. Find a number for the wedge so that only one number will work in the box:

$\square * \triangle = (3 \times \square) \surd 10$

Comment?

Rules:

$$\square \xrightarrow{a} 4 \times \square + 1$$

$$\square \xrightarrow{b} \square \times \square$$

For the following starting places, which rule lands you on the larger number in one jump? (Write "a" or "b", as in the first example. Recall that $4 \times {}^{-}1 = {}^{-}4$, ${}^{-}1 \times {}^{-}1 = 1$, $4 \times {}^{-}3 = {}^{-}12$, and ${}^{-}3 \times {}^{-}3 = 9$. Don't compute specific landing points whenever you can be sure without computing.)

	Starting Place	Winner
1.	0	a
2.	1	
3.	$^{-}1$	
4.	2	
5.	$^{-}3$	
6.	4	
7.	$4\frac{1}{2}$	
8.	5	

Continue with the rules: $\square \xrightarrow{a} 4 \times \square + 1$

$\square \xrightarrow{b} \square \times \square$

	Starting Place	Winner
9.	8	
10.	100	
11.	10,081,314	
12.	-10	
13.	-400	
☆14.	$-\frac{1}{5}$	
☆15.	$-\frac{1}{4}$	

Epilogue

The artificial operations you have been working with were defined by using combinations of the ordinary arithmetic operations. Furthermore, they were defined for all numbers. We can define an artificial operation on just a few numbers by giving a table:

Operation ⊕ (rows: First Number; columns: Second Number)

⊕	1	4	10	11
1	11	10	10	10
4	10	11	4	4
10	10	1	11	4
11	10	4	4	11

According to the table:

$$4 \oplus 1 = 10$$

$$10 \oplus 11 = 4$$

$$11 \oplus 11 = 11$$

Don't look for any special pattern or system here—just follow the table. Is ⊕ an associative operation? With a little time you could check all possibilities. For now, try

$$(11 \oplus 4) \oplus 10 \stackrel{?}{=} 11 \oplus (4 \oplus 10)$$

$$4 \oplus 10 \stackrel{?}{=} 11 \oplus 4$$

$$4 = 4$$

So far it looks as though $\oplus$ might be associative. But now try

$$(1 \oplus 4) \oplus 10 \overset{?}{=} 1 \oplus (4 \oplus 10)$$

This one example shows that our operation is not associative. Is $\oplus$ commutative? If you try various pairs, they come out the same when the numbers are interchanged—with one exception:

$$4 \oplus 10 = 4$$
$$10 \oplus 4 = 1$$

But this makes it non-commutative.

The particular numbers used in the previous table had no significance in themselves. We can just as well say that we have four elements, a, b, c, and d and an operation $\oplus$ on them defined by the following table:

$\oplus$	a	b	c	d
a	d	c	c	c
b	c	d	b	b
c	c	a	d	b
d	c	b	b	d

We can begin studying this system. For example, you might notice that whenever you have $\square \oplus \square$ you get d for the answer. Also, there is only one way to get a for an answer: $c \oplus b = a$.

Again, notice that no matter what computation you do with $\oplus$ and a, b, c, and d, the table tells you that your answer will always be a, b, c, or d. If the table said that $a \oplus b = f$, for example, then you wouldn't know how to do other problems involving f unless the table were enlarged to include the element f.

To include an element f , a new table might look like this:

⊗	a	b	c	d	f
a	d	f	c	c	b
b	c	d	b	b	f
c	c	a	d	b	d
d	c	b	b	d	c
f	a	a	a	b	a

* * *

Using four elements such as a , b , c , d , and an operation ⊕ , it is possible to make up systems that are associative and commutative. Here is one such system that is commonly used in modern algebra:

⊕	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Can you show that ⊕ is commutative? Associative?

Another such four-element system, shown on the following page, is known as Cayley's four group. Is it associative? Commutative?

Cayley's four group:

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

* * *

People are often curious about the symbols that appear in the written lessons. Although many of the symbols have been devised by the Project, others (such as absolute value) appear in standard mathematics texts. The symbols $\odot$, $*$, and $\surd$ do not necessarily have to mean the operations that are used in this written lesson. In the next lesson, the symbol $\surd$ will be used for a different artificial operation. You should notice, however, that although special binary symbols (such as $\surd$ or $\odot$) are used, the ideas they stand for are not necessarily new. Averaging, $\frac{\square + \triangle}{2}$, and maximum, $\max(\square, \triangle)$, are standard mathematical operations.

Summary of Problems in the Film "Some Artificial Operations"

4th Grade, Phillip's School, Watertown, Massachusetts
Teacher: Phyllis R. Klein

Here is a symbol that does something to numbers. You figure out what it does.

$$10 * 3 = 23$$

$$50 * 7\frac{1}{2} = 107\frac{1}{2}$$

$$22 * \frac{1}{8} = 44\frac{1}{8}$$

Tell me the answer if you know what star does.

$100 * \frac{2}{5} =$ $\quad(200\frac{2}{5})$

$\frac{1}{3} * 8 =$ $\quad(8\frac{2}{3})$

More problems:

$\square * 10 = 20$ $\quad(5)$

$\square * 17 = 18$ $\quad(\frac{1}{2})$

$\square * 17 = 17$ $\quad(0)$

Tell me what to write: "* means: Double the first number and add the second.

$5 * 1 = 11$"

Don't do any arithmetic. Where do you put the numbers to get the biggest answer?

$*$ 287 1422 (1422 first)

Think of what 10 * 3 is. Now do:

$$(10 * 3) * \frac{1}{2} = \qquad (46\frac{1}{2})$$

$$10 * (3 * \frac{1}{2}) =$$

Difference between top and bottom?

$$(50 * \frac{1}{3}) * 4 =$$

$$50 * (\frac{1}{3} * 4) =$$

Difference between top and bottom?

Predict what the difference is:

$$(4 * 5) * 100 = \qquad (126)$$

$$4 * (5 * 100) = \qquad (118)$$

Student: You double the first number and that tells you how far apart.

Problem: Give me 3 numbers (and we'll put stars and parentheses between them) so the difference between the top one and bottom one is 0.

Student: $(25 * \frac{1}{3}) * 14 =$

$25 * (\frac{1}{3} * 14) =$

How far apart? (50)

Student: $(0 * 0) * 0 =$

$0 * (0 * 0) =$

How far apart? (It works.)

Get the difference to be 0 but one of the numbers has to be 10.

Student: $(10 * 10) * 10 =$ (70)

(20 apart)

$10 * (10 * 10) =$ (50)

Student: $(0 * 10) * 0 =$ (20)

(0 apart)

$0 * (10 * 0) =$ (20)

Student: Put 0 for the first number.

Write what goes in the box to make this true:

(Kept on board: $(10 * 10) * 10 = 70$)

$(\square * \square) * \square = 21$ (3)

$(\square * \square) * \square = 56$ (8)

$(\square * \square) * \square = 14$

$(\square * \square) * \square = 22$

How are you getting your answers?

Student: If you look at $(\square * \square) * \square = 14$ there are 7 boxes and you think $7 \times 2 = 14$

Student: Divide the number by 7.

Let's do something different—a new operation, $\checkmark$. It's very simple.

Examples: $10 \checkmark 14 = 14$

$100\frac{1}{3} \checkmark 55 = 100\frac{1}{3}$

Guess: $88 \checkmark 0 =$ (88)

$1047 \checkmark 1048 =$ (1048)

$17 \checkmark 17 =$

$17 \checkmark -17 =$

$-17 \checkmark -18 =$

Silly question: $-18 \checkmark -17 =$

On side board:

$*$ means: Double first number and add second.

$5 * 1 = 11$

$\checkmark$ means: Take the larger number.

$2 \checkmark 5 = 5$

More problems:

	$\square \checkmark 100 = 100$	(Any number under 100)
	$\square \checkmark 5 = 16$	(16)
	$\square \checkmark 5 = 2$	(Impossible)

Give me some numbers to write on the board:

$$\left(\left(\left(\left(1 \checkmark 6\right) \checkmark 8\tfrac{1}{2}\right) \checkmark 8\frac{1}{1{,}000{,}000}\right) \checkmark 3{,}000\right) \checkmark 0 \quad =$$

Use: $\checkmark$ $*$ ( ) 51 0 100

Arrange them so you get the biggest answer.

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