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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on maneuvers on lattices, a summary of the problems in the film "A Periodic Lattice," and the supplement. (MK)

THE ARITHMETIC PROJECT COURSE FOR TEACHERS

TOPIC: Maneuvers on Lattices (continued).

FILM: A Periodic Lattice, Grade 5

SUPPLEMENT: More Suggestions for Lattices

9

NAME: ....

This booklet is part of a course for teachers produced by The Arithmetic Project in association with Education Development Center. Principal financial support has come from the Carnegie Corporation of New York, the University of Illinois, and the National Science Foundation.

The course is available from:

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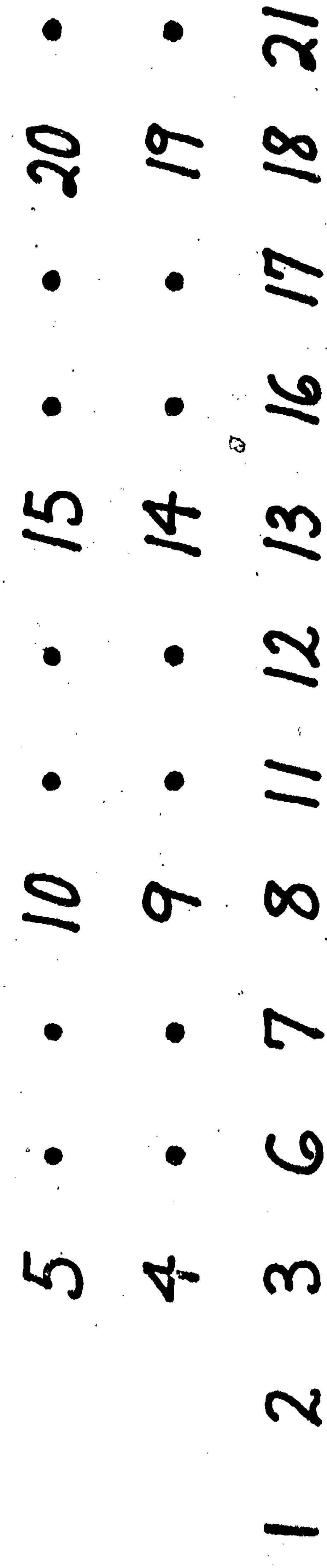
Tear out this page to use while viewing the film, "A Periodic Lattice."



BOOK NINE

I. A PERIODIC LATTICE

A lattice such as

●	3	●	6	●	9	●	12	●	15	●	18	. . .
1	2	4	5	7	8	10	11	13	14	16	17	

repeats the same pattern indefinitely to the right. The close three dots on the right indicate the continuation forever. The dots in

6	●	9	●	12
5	7	8	10	11

just tell you there are unnumbered "places" where you can move. If the instructions in a problem leave you at one of these unnumbered places, your answer is just "dot".

1. One and only one of the following examples is wrong. Find it.

   (a) $2 \rightarrow\rightarrow\rightarrow = 7$

   (b) $15 \rightarrow\rightarrow\rightarrow =$ ●

   (c) $8 \rightarrow\rightarrow\rightarrow\rightarrow = 14$

   (d) $27 \rightarrow\rightarrow\rightarrow\rightarrow = 33$

   (e) $27 \rightarrow\rightarrow\rightarrow\rightarrow\uparrow = 34$

   (f) $30 \leftarrow\leftarrow\downarrow\rightarrow\rightarrow\uparrow = 30$

   (We shall say that a symbol such as $13\downarrow$ has no meaning.)

Continue with the lattice:

●	3	●	6	●	9	●	12	●	15	●	18	. . .
1	2	4	5	7	8	10	11	13	14	16	17	

Give answers:

2. $4 \rightarrow\rightarrow = $ ______

3. $9 \rightarrow\rightarrow = $ ______

4. $17 \rightarrow\rightarrow = $ ______

5. $2 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = $ ______

6. $7 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = $ ______

7. $\boxed{} \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = 30$

8. $\boxed{} \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow = 30$

9. $19 \uparrow = $ ______

10. Give some numbers larger than 99 which are in the top row.

______ , ______ , ______ , ______

11. $105 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = $ ______

12. $104 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = $ ______

13. $104 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = $ ______

14. $105 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = $ ______

Continue with the lattice:

•	3	•	6	•	9	•	12	
1	2	4	5	7	8	10	11	. . .

Describe all the numbers that work:

15. $\square \rightarrow \; = \; \square + 1$

16. $\square \rightarrow \; = \; \square + 2$

17. $\square \rightarrow \; = \; \square + 3$

18. $\square \rightarrow \; = \; \bullet$

19. $\square \rightarrow \rightarrow \; = \; \square + 3$

20. $\square \searrow \nearrow \; = \; 54$

II. The remaining problems in this written lesson deal with the lattice shown below:

						2816	3328						
h						1408							
g						704							
f	32	96				352							
e	16	48	80			176	208						
d	8	24	40	56	72	88	104						
c	4	12	20	28	36	44	52				84		
b	2	6	10	14	18	22	26	30	34	38	42		
a	1	3	5	7	9	11	13	15	17	19	21	23	. . .

The dots indicate that the lattice continues indefinitely upward and to the right. Expressions such as 7 ↓ and 4 ← are undefined.

(A tear-out copy of this lattice appears at the end of this booklet.)

1. 9 ↑ = ________

2. 40 ↑ = ________

3. 13 → = ________

4. 23 → = ________

5. 23 ↑ = ________

6. 23 ↑↑ = ________

7. 501 → = ________

8. 501 ↑ = ________

9. 501 ↑↑↑ = ________

10. $\boxed{}$ ↓↓ = 5

11. $\boxed{}$ ↓↓ = 15

12. $\boxed{}$ ↓↓↓ = 15

13. $\boxed{}$ ↑ = 1,000

14. $\boxed{}$ ↑↑ = 1,000

15. If one traveled straight down from 200, one would eventually reach an odd number. Which odd number is it? ____________

16. 10 is in row b, 16 is in row e, 104 is in row d, etc. In what row is 200? ____________

17. In what row is 120? ____________

18. In what row is 2,720 ? ____________

Above what odd number is 2,720 ? ____________

19. $20 \rightarrow \ = 20 + \square$

20. $44 \rightarrow \ = 44 + \square$

21. Describe all the numbers that work in the equation

$\square \rightarrow \ = \square + 32$

22. Write an equation for which the numbers that work are all the numbers in row b (and only the numbers in row b).

23. Teacher: "How can you tell what to add when you move one space to the right in some given row?"

Student:

24. Exactly one of the numbers in row d works in the equation

$\boxed{} \rightarrow \; = \; \boxed{} \; + \; \boxed{} \uparrow$

What is it? ____________

25. Lots of other numbers (not in row d) also work in

$\boxed{} \rightarrow \; = \; \boxed{} \; + \; \boxed{} \uparrow$

What are they?

26. Describe the smallest rectangular array in this lattice which includes the whole numbers from 1 to 100.

What is the number in the upper right hand corner of that array? ________

☆27. Are there any positive whole numbers which do not appear on this lattice? ____________ Explain:

☆28. Do any numbers appear more than once on the lattice? ____________ Explain:

12

Epilogue

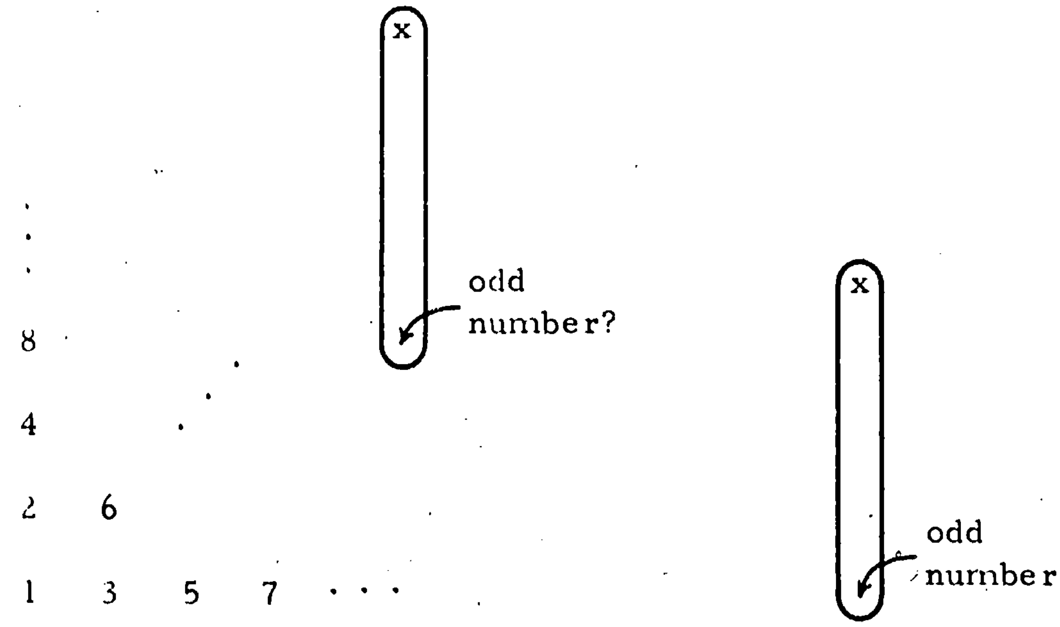
WARNING. DO NOT READ THIS EPILOGUE UNTIL YOU HAVE TRIED PAGE 8.

The answer to questions ☆27 and ☆28 on page 8 is no: each positive integer appears once and only once in the lattice. There are many ways to explain this; probably your lines of reasoning were not precisely the same as what appears below.

☆27. Suppose that some positive whole number does not appear. It can't be odd, for all the odd numbers occur in the bottom row. So it must be even. Write it as the product of some number of 2's times some odd number; that is, as $2 \times 2 \times \cdots \times 2 \times$ some odd number. Find that odd number in the bottom row and go up as many spaces as there are factors of 2: the number that appears in that position in the lattice is the number which was supposedly missing.

☆28. Suppose now that some number appears at least twice. Choose any such number; call it x. Further, choose two particular occurrences of x. These two positions can't be in the same row because in each row the arrow $\rightarrow$ adds some positive number. As we move to the right the numbers get bigger, but no number is bigger than itself. So one occurrence of x must be in a row lower than the other.

Consider the column of numbers under the lower occurrence of x, and notice that the last of these numbers, the bottom number, must be odd, since it's in the bottom row. Those same numbers must appear under the other occurrence of x, because the arrow down, $\downarrow$, always divides by 2. But this means that there must be an odd number which is not in the bottom row. (See diagram next page.)



We know this can't be true; every number above an odd number is even. So we could not have had two occurrences of the same number to begin with.

Summary of Problems in the Film "A Periodic Lattice"

5th Grade, Coolidge School, Watertown, Massachusetts
Teacher: Phyllis R. Klein

		5	●	●	10	●	●	15	●	●	20	
		4	●	●	9	●	●	14	●	●	19	
1	2	3	6	7	8	11	12	13	16	17	18	21

What goes over here next to 20? (A dot)

What goes next to this dot?

And what goes next to that dot continuing along the same line? (25)

This time, instead of building the chart upwards in your heads, you're going to build it so that it goes to the right.

4 → =

9 → → =

9 → → → =

8 → → → =

13 ↑ =

13 ↑ ↑ =

13 ↑ ↑ → =

● → = (Dot; 15; "It all depends what dot you are talking about.")

I'm talking about this one (the dot to the left of 9). (9)

Now I'm talking about this one (the dot to the left of 19). (19)

Now this one? (●)

And this one? (25)

A question like this one is pretty meaningless unless I give you some more information.

• → → → ("Depends on which dot you are talking about.")

("It's always dot.")

(Dot.)

Do I have any dots on the lower line? ("It will always be a dot when you have dot and three arrows.")

Let's call the numbers up here top numbers, and the ones below that middle numbers and those bottom numbers. I will give you a number, and you tell me if it's top, middle or bottom. 7? 9? 19?

Where is 27? (Top, bottom and middle all given.)

27 is on the bottom, and now where is 28? Why?

("I have a method that is up there on the board. On the top row, the numbers are going by 5, 10, 15, 20, ... and the middle row it goes 4, 9, 14, 19, and for the bottom row, it's three numbers and then you skip two and then three more numbers and you skip two...")

("I notice that on the bottom line that 7 is there and 17 is there so 27 must be there, too.")

Where is 147? (Bottom)

Where is 150?

Where is 1,346?

What about 13,468? (Bottom)

Why? ("8 is on the bottom")

("Sometimes you can tell where the numbers are by the last number"

13,468 ← = (13,467)

13,468 → =

Why is it 13,471?

17 →

3 →

16 →

What happened here? $(3 \to\ = 6)$

("When the number is on the bottom of the row that goes up, and you have an arrow going across, you add three, but that doesn't work out on the other numbers.")

$8 \to\to\to =$ (13)

$8 \to\to\to \quad \to\to\to =$

$8 \to\to\to \quad \to\to\to \quad \to\to\to =$

Who knows how many arrows I'm going to use this time?

$8 \to\to\to \quad \to\to\to \quad \to\to\to \quad \to\to\to =$

This is getting a little bit too long for me. If I wanted to write 30 arrows, you would have the answer before I could finish writing the arrows.

("You could write 8 and put parentheses to show how many arrows you want.")

I think I'll do exactly that. I'll write 8 with an arrow facing in the direction I want and I still want 12, with the parentheses.

$8 \xrightarrow{(12)} =$ (28)

$8 \xrightarrow{(15)} =$ (33)

$9 \xrightarrow{(3)} =$

$15 \xrightarrow{(3)} =$

$12 \xrightarrow{(6)} =$

$4 \xrightarrow{(6)} =$

Why?

("I knew there were two numbers between 14 and 4 and that was two and I added the four dots too.")

("First I added: 9 and 5 is 14, for each three arrows we add five; for 6 arrows I added 10 to each.")

("For $4 \xrightarrow{(12)}$, you can divide 12 by 3 four times and 4 times 5 is 20, so if you start at 4, you get 24.")

("If you put 9 arrows, I'm going to have to triple it.")

Here are 9 arrows: $7 \xrightarrow{(9)} =$ (22)

$6 \xrightarrow{(9)}$

$6 \xrightarrow{(12)}$ (Wrong answer of 19)

What row is 19 in?

What row is 6 in?

If I went straight across, could I go from 6 to 19?

Somebody explain the answer of 26. ("When you have 12, you add 10 to 6 and it's 16, and then you add another 10 and it's 26.")

What made you add 10? ("Every three arrows is 5 numbers.")

$6 \xrightarrow{(15)}$ Is the answer going to be more or less than 26?

How much more?

$6 \xrightarrow{(18)} =$ (36)

$6 \xrightarrow{(24)} =$

$6 \xrightarrow{(30)} =$

$6 \xrightarrow{(\ )} = 66$ (36)

$6 \xrightarrow{(\ )} = 86$

Who can explain that one? ("$6 \xrightarrow{(36)} = 66$ and when you put three more arrows it equals five more numbers, and so if you put six more arrows, that equals 10 more numbers; if you put 20 more numbers on, that equals 12 arrows, and 12 arrows added onto 36 is 48.")

$10 \xrightarrow{(90)} =$ (Wrong answer of 150 is given.)

You did $0 \xrightarrow{(90)}$; how can you fix your answer?

Can you tell how you did $10 \xrightarrow{(90)} = 160$ ("Each 30 arrows make 50 numbers and there are three 30's in 90, so that is 150 and add 10 is 160.")

$10 \xrightarrow{(91)} =$ (Dot)

$21 \xrightarrow{(300)} =$ and people who get that one work on this one:

$21 \xrightarrow{(301)} =$

Who wants to explain that?

("Three goes into 300 one hundred times; 100 times 5 is 500 and 21 more is 521. The second one is just one more. So I looked at one on the chart to make sure if it went over it would be just one number more instead of three numbers more, and it was; so I just said 521 is the first one, and 522 is the second one.")

Now this is 23, how will the answers change?

$23 \xrightarrow{(300)} =$

$23 \xrightarrow{(301)} =$

(Answers of 525 given.)

Where is 525, which row?

Where is 23?

Why is it 526?

Supplement

"More Suggestions for Lattices"

By Edward Esty

This supplement describes some more lattices and more things to do with them in the classroom. The questions are not sequenced in small steps; you will have to invent additional questions to fill in the gaps. Some of the problems are hard, but none requires excessive computation.

Section I shows seven lattices different from those appearing in the written lessons and in the booklet Maneuvers on Lattices. Section II consists of more problems on the lattice of page 4 of this booklet. Section III introduces a different kind of arrow which can be used alone or in conjunction with the ordinary arrows. Finally, in Section IV we describe an activity with lattices which can be pursued with children of many ages.

Each section is independent of the others; it is not intended that the reader should complete this paper at one sitting. Answers start on page 34.

I.

A. "Serpentine" Lattices

These are a variety of periodic lattices. Example:

3	4	9	10	15	16	21	22	27	28		
2	5	8	11	14	17	20	23	26	29		...
1	6	7	12	13	18	19	24	25	30	31	

1. 14 → = __________

2. 14 → → = __________

3. 14 → → → → = __________

4. 14 $\xrightarrow{\text{(one hundred)}}$ = __________

5. Describe all the numbers that work in $\square \rightarrow \; = \; \square + 1$.

6. Describe all the numbers that work in $\square \rightarrow \; = \; \square + 3$.

7. $3 \xrightarrow{\text{(three)}} \; =$ ____________

8. $4 \xrightarrow{\text{(four)}} \; =$ ____________

9. $5 \xrightarrow{\text{(five)}} \; =$ ____________

10. $6 \xrightarrow{\text{(six)}} \; =$ ____________

11. $7 \xrightarrow{\text{(seven)}} \; =$ ____________

12. Either find a number for $\square$ so that $\square \xrightarrow{(\square)}$ is an odd number, or explain why it can't be done.

Here's another serpentine lattice:

4	5	12	13	
3	6	11	14	...
2	7	10	15	
1	8	9	16	17

13. Describe or draw two other serpentine lattices in which $8 \rightarrow \; = \; 9$. Do not include the trivial serpentine lattice

1 2 3 4 5 6 ...

in which there are no zigzags. (For now we will say that a serpentine lattice must have 1 in the lower left-hand corner, and, except for the trivial serpentine lattice, it must have 2 directly above 1.)

14. Including the trivial one, how many serpentine lattices are there in which $100 \rightarrow \; = \; 101$? (Hint: There are fewer than 50 of them.)

15. Find some numbers for $\square$ so that there are exactly two serpentine lattices (including the trivial one) in which $\square \rightarrow \; = \; \square + 1$.

16. In the serpentine lattice in which $3 \rightarrow \; = \; 100$, what is $1{,}000{,}000 \rightarrow \rightarrow$?

B. <u>The Tree Lattice</u>

1
2 3
4
5 6 7 8
9
10 11 12 13 14 15
16
17 18 19 20 21 22 23 24
25
⋮

(Probably this lattice should be used only with children who have had some experience with multiplication, and who will therefore recognize the "trunk" numbers — 1, 4, 9, 16, ... — as perfect squares: $1 \times 1 = 1$, $2 \times 2 = 4$, $3 \times 3 = 9$, etc.)

We shall say that $5\downarrow = 11$, $12\nearrow = 9$ and $9\downarrow = 16$.

1. $81\downarrow =$ ______________
2. $81\swarrow =$ ______________
3. $1{,}000{,}000\searrow =$ ______________
4. $1{,}000{,}001\downarrow =$ ______________
5. $10^{\text{(one hundred)}}\downarrow =$ ______________

C. A Spiral Lattice

Another lattice with interesting properties is one which spirals out forever and which therefore has no edges. (Compare it with the quilt lattice, page 12 of the booklet "Maneuvers on Lattices".)

				⋮			
	17	16	15	14	13		
	18	5	4	3	12		
. . .	19	6	1	2	11		. . .
	20	7	8	9	10	⋮	
	21	22	23	24	25	26	
			⋮				

Observe that the perfect squares appear in two diagonals, the odd squares starting with 1 and going down and to the right, and the even squares starting with 4 and going up and to the left.

1. $49 \searrow$ = ________
2. $144 \nwarrow$ = ________
3. $144 \uparrow$ = ________
4. $144 \rightarrow$ = ________
5. $144 \downarrow$ = ________
6. $144 \leftarrow$ = ________
7. $148 \uparrow$ = ________
8. $148 \downarrow$ = ________
9. Given any number $\square$ on this lattice, what is a procedure for finding $\square\uparrow$?
10. $1 \downarrow^{\text{(one hundred)}}$ = ________ (Hint: This is the same as $1 \searrow^{\text{(one hundred)}} \leftarrow^{\text{(one hundred)}}$.)

It is worthwhile to compare the spiral lattice with the tree lattice. Pairs of branches are turned into L-shaped arrays.

D. An Offset Lattice

31		32							
	26		27		28		29		30
21		22		23		24		25	
	16		17		18		19		20
11		12		13		14		15	
	6		7		8		9		10
1		2		3		4		5	

Notice that $22 \uparrow = 32$, as in a conventional 10-fold lattice, but $22 \nearrow = 27$.

1. $22 \nearrow \nwarrow \downarrow =$ __________
2. $18 \rightarrow \uparrow \swarrow =$ __________
3. $18 \rightarrow \uparrow \swarrow \swarrow =$ __________
4. $17 \rightarrow \rightarrow \uparrow \swarrow \searrow \uparrow \leftarrow \searrow \searrow \leftarrow \overset{\text{(one hundred)}}{\uparrow} \nearrow =$ __________
5. Construct a new lattice so that every equation on the offset lattice can be transformed into an equation on the new lattice by rotating each arrow one-eighth turn counterclockwise. For example, since $8 \nearrow = 14$ on the offset lattice, we want $8 \uparrow = 14$ on the new lattice; since $7 \nwarrow \nwarrow \downarrow \rightarrow = 7$ on the offset lattice, we want $7 \leftarrow \leftarrow \searrow \nearrow = 7$ on the new lattice.

2

E. An Offset Periodic Lattice

a				1		17		18		19		20		21						
b			2		3		14		15		16		22		23					
c		4		5		6		12		13		24		25		26				...
d	7		8		9		10		11		27		28		29		30		31	

1. $1 \xrightarrow{\text{(one hundred three)}} =$ __________

2. In which row does 3,721,386 appear? __________ (Do not do any computation.)

F. A Semi-Deleted Lattice

Another kind of lattice is one in which numbers are left out in some orderly way. For example:

			⋮		
38	39	41	42		
31	32	33	34	36	37
23	24	26	27	28	29
16	17	18	19	21	22
8	9	11	12	13	14
1	2	3	4	6	7

1. Find all numbers which work in $\square \uparrow\uparrow = \square + 15$.

2. Find all the numbers which work in $\square \rightarrow = \square + 2$.

3. $108 \uparrow =$ __________

4. $3{,}427 \uparrow =$ __________

5. $200 \overset{\text{(two hundred)}}{\uparrow} =$ __________ (Careful!)

6. $201 \overset{\text{(two hundred one)}}{\uparrow} =$ __________

G. A Lattice With Ordered Pairs

			⋮			
		(0, 2)				
	(−1, 1)	(0, 1)	(1, 1)	(2, 1)	(3, 1)	
. . . .	(−1, 0)	(0, 0)	(1, 0)	(2, 0)	(3, 0)	. . . .
	(−1, −1)	(0, −1)	(1, −1)	(2, −1)	(3, −1)	
		(0, −2)			(3, −2)	
			⋮		(3, −3)	

1. (2 , 1) ↑ = ____________

2. (−8 , −6) ↓↓↓ ← = ____________

3. (3 , 1) $\overset{\text{(one hundred)}}{\nearrow}$ = ____________

4. Use the fewest arrows you can in (5 , 1) $\boxed{\qquad\qquad}$ = (1 , 5) .

5. Do the same for (5 , 1) $\boxed{\qquad\qquad}$ = (12 , 3) .

6. How many arrows? (347 , 26) $\overset{\boxed{\qquad}}{\nwarrow}$ = (26 , 347) .

This lattice can be used as an introduction to locating points in the plane. Many of your early questions will probably be of the "What goes here?" type.

II.

We return to the lattice on page 4 of this booklet.

e	16								
d	8	24	40						
c	4	12	20	28	36	44			
b	2	6	10	14	18	22	26	...	
a	1	3	5	7	9	11	13	15	...

A. 1. You know that the numbers which work in $\square\rightarrow \;=\; \square + \square\uparrow$ are those in the first column. Describe all the numbers that work in $\square\rightarrow \;=\; \square + \square\leftarrow\leftarrow\leftarrow\uparrow$.

2. Write an equation such that the numbers that work are 101 and the numbers in the column above 101 .

3. Describe all the numbers that work in $\square \times \square \;=\; \square\uparrow \times \square\downarrow$. (Notice that 7 does not work in the equation above because so far no meaning has been assigned to $7\downarrow$.)

4. Write an equation so that the numbers that work are all the numbers above row d .

5. $$\frac{5 \xrightarrow{\text{(five)}}}{5} = \square$$

6. $$\frac{7 \xrightarrow{\text{(seven)}}}{7} = \square$$

7. $$\frac{1 \xrightarrow{\text{(one)}}}{1} = \square$$

8. $101 \xrightarrow{\text{(one hundred one)}} =$ __________ (3 is a likely wrong answer.)

9. $\dfrac{2 \xrightarrow{\text{(two)}}}{2} = \square$

10. $\dfrac{6 \xrightarrow{\text{(six)}}}{6} = \square$

11. Without doing any computation, predict what $\dfrac{12 \xrightarrow{\text{(twelve)}}}{12}$ is.

Prediction: ____________

12. Now do $\dfrac{4 \xrightarrow{\text{(four)}}}{4} = \square$ by looking at the lattice. Was your prediction right? ________

13. $\dfrac{40 \xrightarrow{\text{(forty)}}}{40} = \square$

14. $96 \xrightarrow{\text{(ninety-six)}} = \square$

B. Here is a jumping rule which we call rule a .

lattice arrow

$$\square \xrightarrow{\quad a \quad} \frac{\square \rightarrow \; - \; \square}{2}$$

arrow for jumping rule

If we start at 28 we have

$$28 \xrightarrow{\quad a \quad} \frac{28 \rightarrow \; - \; 28}{2}$$

or $\dfrac{36 - 28}{2}$

or 4

Continue with the rule: $\square \xrightarrow{a} \dfrac{\square\!\rightarrow \; - \; \square}{2}$

1. Start at 26. Use rule a. Land? ____________

2. Give three starting numbers larger than 100 for which 2 is the landing number (using rule a). ____________ , ____________ , ____________

3. 13,216 can be written as 32×413. Start at 13,216. Use rule a. Land? ____________

4. Start at 13,216. Make 13,216 jumps with rule a. Land? ____________

5. Start at 13,216 $\xrightarrow{\text{(thirteen thousand two hundred sixteen)}}$. Make 13,216 jumps with rule a. Land? ____________

Here is another jumping rule:

$$\square \xrightarrow{b} \frac{\square}{\left(\dfrac{\square\!\rightarrow \; - \; \square}{2}\right)}$$

6. Make some jumps with this rule so that you know what it does. If problems 1 through 5 on this page were to be answered using rule b, one of the problems would be impossible. Which problem? ____________ Of the remaining four problems, which would involve the most difficult computation? ____________

7. Start at 3,728,400,000. Make a jump with rule a and then a jump with rule b. What is your final landing number? ____________

Comment?

III.

A. We introduce a new kind of arrow—the superarrow. Here's how superarrows work, using this lattice:

		⋮		
20	21	22		
15	16	17	18	19
10	11	12	13	14
5	6	7	8	9
0	1	2	3	4

(Assume that nothing has been said about edges; that $9 \rightarrow$, for instance, is undefined.)

$11 \Downarrow \; = \; 1$

$16 \Rightarrow \; = \; 19$

$19 \Rightarrow \; = \; 19$

$3 \Nwarrow \; = \; 15$

$8 \Uparrow$ does not exist, because there is no upper edge.

$15 \Swarrow \; = \; 15$

$10 \leftarrow \Downarrow$ is not defined (because $10 \leftarrow$ isn't).

$10 \Leftarrow \Downarrow \; = \; 0$

• Now do the problems beginning on the next page.

1. $13 \Leftarrow \;=\; \square$

2. $13 \Swarrow \Nwarrow \;=\; \square$

20	21	22		
15	16	17	18	19
10	11	12	13	14
5	6	7	8	9
0	1	2	3	4

Find all numbers that work in problems 3 - 8 :

3. $\square \Rightarrow \;=\; 14$

4. $\square \Rightarrow \Nearrow \;=\; 14$

5. $\square \Rightarrow \;=\; 2{,}338$

6. $26{,}942 \Downarrow \Nwarrow \;=\; \square$

7. $26{,}942 \Nwarrow \Downarrow \;=\; \square$

8. $\square \Rightarrow \Rightarrow \;=\; \square \Rightarrow$.

Compare this equation with

$$\left|\,|\square|\,\right| = |\square| \quad \text{and} \quad \left\lfloor \lfloor \square \rfloor \right\rfloor = \lfloor \square \rfloor$$

9. Use only superarrows to make this true:

$8 \; \boxed{} \;=\; 19$

(It <u>is</u> possible to do this problem.)

10. Using only superarrows, can you get from 8 to every number on an edge? Explain.

11. Same problem, starting at 12.

B. One reason for introducing superarrows is that they can be used to give general descriptions of whole classes of lattices.

1. Still on the original five-fold lattice, find all numbers that work in

$\square \;=\; \square \Leftarrow \;+\; \square \Downarrow$

Does 3 work? Does 10 work? Does 0 work?

31

Now the lattice is

$\vdots$

21	22					
14	15	16	17	18	19	20
7	8	9	10	11	12	13
0	1	2	3	4	5	6

2. Same problem: $\square = \square\Leftarrow + \square\Downarrow$

Now the lattice is

$\vdots$

21				
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

3. Find all the numbers that work in $\square = \square\Leftarrow + \square\Downarrow$. Where is the 1 coming from?

4. Using only boxes and superarrows (and + or − , etc.—no numbers), fix up the equation in problem 3 so that all numbers work.

$\square = \square\Leftarrow + \square\Downarrow$ ______________________________
Write something here

Your equation should work even for this lattice:

$\vdots$

18,842		
17,921	18,228	18,535
17,000	17,307	17,614

(There's no number below 17,000 on this lattice.)

5. Build a single lattice for which equations (a), (b), (c) and (d) are always true. $\uparrow$ should add a constant amount; $\rightarrow$ should also add a constant amount (except for numbers on the right-hand edge, where $\rightarrow$ is undefined).

(a) $\Box \Rightarrow \; - \; \Box \Leftarrow \; = \; 10$

(b) $\Box \Downarrow \Leftarrow \; = \; 3$

(c) $\Box \Downarrow \Leftarrow \rightarrow \rightarrow \; = \; 7$

(d) $\Box \Downarrow \Rightarrow \nwarrow \; = \; 58$

Here is a multiplicative lattice:

625				
125	250	500		
25	50	100	200	. . .
5	10	20	40	

6. $250 \Downarrow \times 250 \Leftarrow \; = \; 250 \times \Box$

7. Write an equation similar to the one you wrote for problem 4 that will be true for all multiplicative lattices.

8. Write a superarrow equation for the lattice of Section II.

C. Another reason for introducing superarrows is that since they always take us to edges, we can use them to describe extensions of lattices at edges. Example: consider a ten-fold lattice:

$\vdots$

30	31								
20	21	22	23	24	25	26	27	28	29
10	11	12	13	14	15	16	17	18	19
0	1	2	3	4	5	6	7	8	9

$9 \rightarrow$ is not yet defined. If we define it by saying that $\Box \rightarrow$ shall always be $\Box + 1$, we will have defined the "reversed typewriter"[†] extension, but only for lattices (like the one above) in which $\Box \rightarrow$ is already $\Box + 1$ for some number in the boxes. E.g., the description is inadequate for the lattices in section 6, pages 9 and 10 of Maneuvers on Lattices.

If, on the other hand, we say that the equation $\Box \Rightarrow \rightarrow \;=\; \Box \Leftarrow \uparrow$ is always true, we will have described the reversed typewriter extension independently of the particular lattice. Notice that if we put 13 in the boxes we have $13 \Rightarrow \rightarrow \;=\; 13 \Leftarrow \uparrow$, or $19 \rightarrow \;=\; 10 \uparrow$. The equation tells us that $19 \rightarrow$ is $10 \uparrow$, or 20; it doesn't say anything about $13 \rightarrow$, which we already know to be 14. If we put a number in the $\Box$ which is already on the right-hand edge, the first superarrow has no effect. Since $\nearrow$ is equivalent to $\rightarrow \uparrow$, we have $\Box \Rightarrow \nearrow \;=\; \Box \Leftarrow \uparrow\uparrow$, which tells us that in this system $29 \nearrow$ equals 40.

It could be argued that the reversed typewriter notion can be described generally by saying that for all numbers in $\Box$ and $\triangle$, $\Box \rightarrow - \Box \;=\; \triangle \rightarrow - \triangle$. That is, $\rightarrow$ adds a constant amount (not necessarily 1). But the superarrow approach is more geometric.

Superarrows can be used to define or describe other edge extensions. The "marching men at a cliff—don't go!" idea is simply $\Box \Rightarrow \rightarrow \;=\; \Box \Rightarrow$. (Can this be done without using superarrows?)

[†] See Maneuvers on Lattices, page 11.

"Go directly to zero" is $\square \Rightarrow \rightarrow \; = \; 0$, but if zero is not in the lattice, we probably want to go directly to the lower left-hand corner: $\square \Rightarrow \rightarrow \; = \; \square \Leftarrow \Downarrow$.

1. How would systems IV, V, and VI, pages 11 and 12 of <u>Maneuvers</u>, be done? What about system IX?

IV. Consider this lattice:

20	21			
15	16	17	18	19
10	11	12	13	14
5	6	7	8	9
0	1	2	3	4
A	B	C	D	E

(Throughout this section, all lattices will have 0 in the lower left-hand corner.)

1. Pick some number in column D. Pick some number in column B. Add the two numbers. In what column is your answer? ____________

2. Pick another number in column D and another number in column B. In what column is their sum? ____________

A few more examples will convince you that no matter which numbers from B and D you pick, their sum will be in column E. We can express this by writing D + B = E.

In the following problems, fill in the frames with letters only.

3. C + D = $\boxed{}$

4. E + $\boxed{}$ = B

5. $\boxed{}$ + $\boxed{}$ + $\boxed{}$ = E

6. $\boxed{}$ + $\boxed{}$ + $\boxed{}$ + $\boxed{}$ = E

7. $\boxed{}$ + $\boxed{}$ + $\boxed{}$ + $\boxed{}$ + $\boxed{}$ = E

8. Pick two numbers, one from column C and the other from D. <u>Multiply</u> them. In what column is your answer? ________ Will this always work? ________

9. $D \times E = \square$

10. $D \times \square = D$

11. $C \times A = \square$

12. $\square \times A = E$

13. $A \times B \times C \times D \times E = \square$

14. $A + B + C + D + E = \square$

15. $A \times A + B \times B + C \times C + D \times D + E \times E = \square$

16. Explain why $\square \times \triangle = A$ is impossible unless you use A in at least one of the frames.

17. Since $C \times E = D$, it might make sense to write $D \div E = C$. But if we pick 13 from D and 7 from E and divide, we get $13 \div 7 = 1\frac{6}{7}$, and $1\frac{6}{7}$ isn't in any column. How should we interpret $D \div E$?

18. $B \div E = \square$

19. $B \div A = \square$

Try the same sort of thing with this lattice:

⋮					
18	19				
12	13	14	15	16	17
6	7	8	9	10	11
0	1	2	3	4	5
A	B	C	D	E	F

20. $E + D = \square$

21. $A + B + C + D + E + F = \square$ (Cf. problem 14)

22. Do not use A in either frame: $\square \times$ ⏢ $= A$

23. Use 12 and 2 as representatives from columns A and C :

$A \div C =$ ______

24. Use 18 and 2 as representatives from columns A and C :

$A \div C =$ ______

25. We have seen that division (except by column A) makes sense in the lattice with five columns but not in the lattice with six columns. What about this lattice?

			⋮			
14						
7	8	9	10	11	12	13
0	1	2	3	4	5	6
A	B	C	D	E	F	G

26. Describe all lattices in which division (except by column A) can be defined.

27. Describe all the lattices for which

$$A + B + C + D + \ldots + (\text{last column}) = A$$

(Hint: Try a lattice with 9 columns before you decide.)

Answers

I. A

1. 17
2. 20
3. 26
4. 314
5. Positive multiples of 3
7. 10
8. 16
9. 20
10. 24
11. 30

6. Numbers in the middle row; that is, numbers that are 2 more than some multiple of 3.

12. If $\square$ is even, $\square \xrightarrow{(\square)} = \square +$ multiple of 6, which is even.

If $\square$ is odd, $\square \xrightarrow{(\square)} = \square +$ multiple of 6 + (1 or 5), which is also even.

13.

2	3	6	7		...
1	4	5	8	9	

and

8	9		
7	10		
6	11		
5	12		...
4	13		
3	14		
2	15		
1	16	17	

14. 9 (the number of positive divisors of 100)

15. Any and all primes for $\square$

16. We have

51	52		
⋮	⋮		...
3	100	⋮	
2	101	104	
1	102	103	

, so $\rightarrow\rightarrow$ adds 102.

Thus $1{,}000{,}000 \rightarrow\rightarrow = 1{,}000{,}102$.

B. 1. 100

2. $81 + \sqrt{81} = 90$

3. $1{,}000{,}000 + \sqrt{1{,}000{,}000} + 1 = 1{,}001{,}001$

4. $1{,}000{,}001 + 2 \times \sqrt{1{,}002{,}001} = 1{,}002{,}003$

5. $$10 \overset{(100)}{\downarrow} = 10 \rightarrow \rightarrow \nearrow \overset{(100)}{\downarrow} \swarrow \leftarrow \leftarrow$$

$$= 9 \overset{(100)}{\downarrow} \swarrow \leftarrow \leftarrow$$

$$= (103)^2 \swarrow \leftarrow \leftarrow$$

$$= (103)^2 + 103 - 2 = 10{,}710$$

C. 1. 81
2. 196
3. 195
4. 143
5. 101
6. 145
7. 147
8. 149

9. Here's one procedure. Locate $\square$ between two successive odd squares, m and n ; $m \leq \square < n$.

If $\square = m$,
then $\square\uparrow = \square - 4 \times \sqrt{m} + 5$.

If $m < \square \leq m + \sqrt{m}$,
then $\square\uparrow = \square + 1$.

If $m + \sqrt{m} < \square \leq m + 2 \times \sqrt{m} + 2$,
then $\square\uparrow = \square + 4 \times \sqrt{m} + 7$.

If $m + 2 \times \sqrt{m} + 2 < \square \leq m + 3 \times \sqrt{m} + 3$,
then $\square\uparrow = \square - 1$.

If $m + 3 \times \sqrt{m} < \square < n$,
then $\square\uparrow = \square - 4 \times \sqrt{m} - 3$.

10. $(201)^2 - 100 = 40{,}301$

D. 1. 22
2. 24
3. 18
4. 1,024

5.

			40				
		39	35	30			
	38	34	29	25	20		
37	33	28	24	19	15	10	
36	32	27	23	18	14	9	5
31	26	22	17	13	8	4	
21	16	12	7	3			
	11	6	2				
		1					

E. 1. 419
2. Row c

F. 1. All numbers on the lattice
2. All numbers 1 less than a multiple of 5 which are not 1 less than a multiple of 30.
3. 116
4. 3,434
5. 200 is not on the lattice.
6. 1,708

G. 1. (2, 2)
2. (−9, −9)
3. (103, 101)
4. ↖ ↖ ↖ ↖
5. → → → → → ↗ ↗ in any order
6. 321 (which is 347 − 26)

II. A.

1. 7 and numbers in the column above 7
2. $\rightarrow = \square + \square \xleftarrow{(50)} \uparrow$
3. Everything in row b or above
4. $\times = \uparrow\uparrow\uparrow\uparrow \times \downarrow\downarrow\downarrow\downarrow$
5. 3
6. 3
7. 3
8. 303
9. 5
10. 5
11. 7 or ~~9~~
12. 9
13. 17
14. 65 × 96 = 6,240

B.

1. 2
2. 102, 106, 110, etc.
3. 32
4. 32
5. 32
6. (2) impossible, (5) hard
7. 1. Rules a and b commute.

III. A.

1.	10	5.	impossible
2.	5	6.	10
3.	10, 11, 12, 13, 14	7.	0
4.	10, 11, 12, 13, 14	8.	everything

9. 8 $\Downarrow \nwarrow \Rightarrow$ = 19 (other ways possible)

10. Yes. Trial and error shows you can reach edge numbers ≤ 24. For larger edge numbers use appropriate ↖ and ⇗ alternately.

11. No. Can reach only even edge numbers.

B. 1. Everything works.

2. Everything works.

3. Nothing works. 1 comes from lower left hand corner.

4. $\square = \square \Leftarrow + \square \Downarrow - \square \Leftarrow \Downarrow$.

5.

58	60				
47	49	51	53	55	57
36	38	40	42	44	46
25	27	29	31	33	35
14	16	18	20	22	24
3	5	7	9	11	13

6. 5

7. $\square = \square \Downarrow \times \square \Leftarrow \div \square \Downarrow \Leftarrow$

or $\square \Downarrow \times \square \Leftarrow = \square \Downarrow \Leftarrow \times \square$

8. $\square = \square \Downarrow \times \square \Leftarrow$

C. 1. System IV : $\square \Rightarrow \rightarrow = \square \Rightarrow \leftarrow$

$\square \Rightarrow \nearrow = \square \Rightarrow \swarrow$, etc

System V : $\square \Rightarrow \rightarrow = \square \Rightarrow$

$\square \Rightarrow \nearrow = \square \Rightarrow \uparrow\uparrow$, etc.

System VI : $\square \Rightarrow \rightarrow = \square \Leftarrow$, etc.

System IX : $\square \Rightarrow \searrow = \square \Rightarrow \uparrow$

$\square \Rightarrow \rightarrow = \square \Rightarrow \uparrow$

$\square \Rightarrow \nearrow = \square \Rightarrow \uparrow$, etc.

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IV.

1. E
2. E
3. A
4. C
5. D
6. B
7. Impossible
8. B , Yes.
9. C
10. B
11. A
12. Impossible
13. A
14. A
15. A
16. 5 is prime.
17. Pick two numbers d and e from columns D and E respectively, so that d ÷ e is a whole number.
18. E.
19. Impossible
20. B
21. D
22. C × D or D × E
23. A
24. D
25. Yes.
26. Those where $0 \uparrow$ is prime.
27. Those where $0 \uparrow$ is odd.

tear here

						2816	3328						
h						1408							
g						704							
f	32	96				352							
e	16	48	80			176	208						
d	8	24	40	56	72	88	104						
c	4	12	20	28	36	44	52				84		
b	2	6	10	14	18	22	26	30	34	38	42		
a	1	3	5	7	9	11	13	15	17	19	21	23	. . .