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ABSTRACT

This is part three of a three-part teacher's guide to an SMSG textbook designed to be used as a one-semester course for twelfth-grade students. This guide contains a teacher's commentary and answers to materials that can be used to supplement the regular course found in the first ten chapters of the text. (MK)

ANALYTIC GEOMETRY

Teachers' Commentary

Part 3

(revised edition)

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Supplementary Chapters

Teacher's Commentary

PREFACE

The first ten chapters of this text constitutes a complete course. This supplement can be used as enrichment and extension in accordance with the interest of the students and teachers. The chapters can be used in any sequence.

There may be a little overlapping of material. However, where this occurs, the approach to the subject will usually be different. Several sections contain exercises and most solutions are presented in this commentary. Supplements A, B, and C are of general interest; the remaining sections supplement specific chapters.

Supplementary Chapters

ANALYTIC GEOMETRY

Teachers' Commentary

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   scale preserving, order reversing

2. $p' = -22$ $q' = 10$ $r' = 26$
   scale decreasing, order preserving

3. $p' = -1$ $q' = 1$ $r' = 2$
   scale increasing, order preserving

4. $p' = 15$ $q' = -9$ $r' = -21$
   scale decreasing, order reversing

5. $p' = \frac{17}{3}$ $q' = \frac{1}{3}$ $r' = -\frac{7}{3}$
   scale increasing, order reversing

6. $p' = 2$ $q' = 10$ $r' = 14$
   scale preserving, order preserving

7. Let P be the origin point, Q the unit point in the original system; i.e., $p = 0$ $q = 1$.

   (5) $p' = 3$ $q' = 2$
   (6) $p' = -2$ $q' = 2$
   (7) $p' = \frac{1}{4}$ $q' = \frac{1}{2}$
   (8) $p' = 0$ $q' = -3$
   (9) $p' = \frac{7}{3}$ $q' = \frac{5}{3}$
   (10) $p' = 7$ $q' = 8$

8. Let P be the origin of the new system, Q the new unit point; i.e., $p' = 0 \quad q' = 1$

(5) $p = 3 \qquad q = 2$

(6) $p = \frac{1}{2} \qquad q = \frac{3}{4}$

(7) $p = -1 \qquad q = 3$

(8) $p = 0 \qquad q = -\frac{1}{3}$

(9) $p = \frac{7}{2} \qquad q = 2$

(10) $p = -7 \qquad q = -6$

9. Suppose $a = 0$ in $x' = ax + b$.

Then for any point P with coordinate p, we would get $p' = 0 \cdot p + b$. So every point in the new system would have coordinate b, thus preserving neither measure nor order nor betweenness.

10. $x' = ax^3 + b$

Let p and p' be the intrinsic and the new coordinates of P, and similarly for q and q', etc. Let $d'(P,Q) = |p' - q'|$. Then

$$d'(P,Q) = |ap^3 + b - aq^3 - b| = |a|\;|p^3 - q^3|$$
$$= |a|\;|p - q|\;|p^2 + pq + q^2|.$$

Similarly

$$d'(R,S) = |a|\;|r - s|\;|r^2 + rs + s^2|.$$

Suppose $\overline{PQ} \cong \overline{RS}$. Then $|p - q| = |r - s|$. However, $d'(P,Q) = d'(R,S)$ only if $|p^2 + pq + q^2| = |r^2 + rs + s^2|$, which in general is false. For example, if $p = 1$, $q = 2$, $r = 3$ and $s = 4$, then $|p^2 + pq + q^2| = 7$ while $|r^2 + rs + s^2| = 37$. It is also true that we can have $d'(P,Q) = d'(R,S)$ although $\overline{PQ}$ and $\overline{RS}$ are not congruent. The example $p = 0$, $q = \sqrt[3]{7}$, $r = 1$ and $s = 2$ shows this. $p < q < r$ always implies $p^3 < q^3 < r^3$, so betweenness is preserved.

11. $x' = e^x$

$$d'(P,Q) = |e^p - e^q|$$
$$d'(R,S) = |e^r - e^s|$$

So $\overline{PQ} \cong \overline{RS}$ does not always imply $d'(P,Q) = d'(R,S)$

$p < q < r$ does always imply $e^p < e^q < e^r$, so betweenness is preserved.

12. $x' = \frac{1}{x}$ if $x \neq 0$

$x' = x$ if $x = 0$

If none of p, q, r, s is zero

$d'(P,Q) = \frac{1}{|pq|} |p - q|$

$d'(R,S) = \frac{1}{|rs|} |r - s|$

So $PQ \cong RS$ does not always imply $d'(P,Q) = d'(R,S)$. However, if $P = R = 0$ and $\overline{PQ} \cong \overline{RS}$, then $|q| = |s|$ and $d'(P,Q) = d'(R,S)$. Let $p < q < r$. Then betweenness is preserved only if $q = 0$ or $r < 0$ or $p > 0$.

13. $x' = \log_{10} x$

This cannot handle points on negative side of the origin since $\log_{10}$ is not defined for negative numbers or 0. Where it is defined

$d'(P,Q) = |\log_{10} \frac{p}{q}|$

$d'(R,S) = |\log_{10} \frac{r}{s}|$

So $\overline{PQ} \cong \overline{RS}$ does not always imply $d'(P,Q) = d'(R,S)$. Betweenness is preserved where $\log_{10}$ is defined.

The notion of a group will mean very little to the students unless they consider many examples. They should study carefully all those mentioned in the text and try to think of others. If they know something about complex numbers, they can be asked to prove that the three cube roots of 1 form a group under multiplication, as do the four fourth roots. These examples show that a group may be finite. If the students are asked for other finite groups, some of them may suggest the kind of arithmetic that suits clock faces. Finally, no complicated mathematical definition becomes clear to students until they have thought of examples that don't quite fit. What about the integers under multiplication, the non-negative integers under addition, and the rational numbers under multiplication?

Exercises S2-1b

1. Let f be the function defined by $f(x) = ax + b \quad a \neq 0.$
   Let g be the function defined by $g(x) = cx + d \quad c \neq 0.$

   We wish to prove $f(g)$ is a function defined by $(f(g))(x) = sx + t$ for real numbers $s \neq 0$, and t.

$$\begin{aligned}(f(g))(x) &= f(g(x)) \\ &= a(cx + d) + b \\ &= (ac)x + (ad + b)\end{aligned}$$

   Since $a \neq 0$ and $c \neq 0$ we know that $(ac) \neq 0$.
   Thus there do exist real numbers $s = ac \neq 0$, $t = ad + b$ such that $(f(g))(x) = sx + t$.

2. Consider f, g, h as three functions in our set:

$$f(x) = mx + n, \quad g(x) = px + q, \quad h(x) = rx + s \qquad m, p, r \neq 0$$

   We wish to show $(f(g))(h) = f(g(h))$
   We find that $f(g)$ is defined by $(f(g))(x) = (mp)x + (mq + n)$ and that $g(h)$ is defined by $(g(h))(x) = (pr)x + (ps + q)$

   Then for all x $\quad (f(g))(h)(x) = (mp)rx + (mp)s + mq + n$
   for all x $\quad f(g(h))(x) = m(pr)x + m(ps + q) + n$
   Hence for all x $\quad (f(g))(h)(x) = (f(g(h)))(x)$ which is the necessary and sufficient condition that the functions S, or for each x, $(f(g))(h) = (f(g(h)))$.

   Note: this is a special case of the theorem that if h maps set A into set B, g maps B into C, and, f maps C into D then $(f(g))(h) = f(g(h))$. The general proof follows: If $x \in A$, let $g = h(x) \in B$, $f = g(y) \in C$, and $f(z) \in D$. Let $k = f(g)$ mapping B into D, $\ell = g(h)$ mapping A into C. Then $(f(g))(h)(x) = k(h)(x) = k(h(x)) = k(y)$ but $k(y) = f(g)(y) = f(y)$. Also $f(g(h))(x) = (f(\ell))(x) = f(z)$ since $(x) = g(h)(x) = g(y) = z$. Therefore $(f(g))(h) = f(g(h))$.

3. Let f be defined by $f(x) = ax + b \quad a \neq 0$

g be defined by $g(x) = cx + d \quad c \neq 0$.

Then $(f(g))(x) = f(cx + d) = a(cx + d) + b = (ac)x + (ad + b)$

$(g(f))(x) = g(ax + b) = (ca)x + (cb + d)$

$f(g) = g(f)$ only if $ad + b = cb + d$.

To show that the commutative property does not hold, we need simply exhibit one case when it doesn't. Take $a = 1$, $c = 2$, $d = 1$, $b = 1$;

then $ad + b = 1 \cdot 1 + 1 = 2 \qquad cb + d = 2 \cdot 1 + 1 = 3$

$(f(g))(x) = 2x + 2 \qquad (g(f))(x) = 2x + 3 \qquad f(g) \neq g(f)$

4. To show that in any group the identity is unique.

Let e and e' be identity elements.

Then for all a, $a(e) = e(a) = a \quad (1)$

$a(e') = e'(a) = a \quad (2)$

So in particular $e'(e) = e(e') = e'$ from (1) letting $a = e'$

$e(e') = e'(e) = e$ from (2) letting $a = e$

Which gives us $e = e'$.

5. To show that in any groups G the inverse is unique. Let $a \in G$.

Suppose b and b' are both inverses; i.e.,

$a(b) = b(a) = e$

$a(b') = b'(a) = e$

Now consider $b(a(b')) = (b(a))(b')$ by associativity; but

$b(a) = e$ and $a(b') = e$ so

$b(e) = e(b')$;

but e is the identity element, so

$b = b'$.

6. To show that the inverse of the identity is the identity, let e be the identity, a its inverse.

Then $a(e) = e$ since a is the inverse of e,

but $a(e) = a$ since e is the identity, therefore $a = e$.

7. (a) $a^2x + ab + b$

(b) $apx + aq + b$

(c) $apx + bp + q$

(d) $p^2x + pq + q$

(e) $a^3x + a^2b + ab + b$

(f) $p^3x + p^2q + pq + q$

(g) $\frac{1}{p}x - \frac{q}{p}$

(h) $\frac{1}{ap}x - \frac{q + bp}{ap}$

(i) $\frac{1}{ap}x - \frac{b + aq}{ap}$

(j) $\frac{1}{ap}x - \frac{b + aq}{ap}$

(k) $\frac{p}{a}x - \frac{bp}{a} + q$

(l) $\frac{a}{p}x - \frac{aq}{p} + b$

* 8. Let f be defined by $f(x) = ax + b$ $a \neq 0$.

If $h(h) = f$ we must have $p \neq 0$ and q such that

$h(h(x)) = p^2x + pq + q = ax + b = f(x)$

Thus p and q must satisfy

$$p^2 = a \qquad pq + q = b$$

Case 1. $a < 0$

There is no real number whose square is negative so there is no function h such that $h(h) = f$.

Case 2. $a > 0$ and $a \neq 1$

Both $p = \sqrt{a}$ and $p = -\sqrt{a}$ satisfy $p^2 = a$. So we have, in general, two solutions to $h(h) = f$.

h_1 defined by $h_1(x) = \sqrt{a}\,x + \frac{b}{1 + \sqrt{a}}$

h_2 defined by $h_2(x) = -\sqrt{a}\,x + \frac{b}{1 - \sqrt{a}}$

h_1 is defined for all values of $a \neq 0$ and b. However, in the special case $a = 1$, h_2 is not defined because $1 - \sqrt{a} = 1 - 1 = 0$. So when $a = 1$ we get the unique solution $h(x) = x + \frac{b}{2}$.

Although Section S2-2 can be omitted without serious loss of continuity, there are a good many ideas in it which are important in other branches of mathematics. If you do not think there is time to cover it in class, perhaps the better students could study it and do some of the exercises.

In earlier courses, students have studied various number systems and learned to consider them as sets closed under certain operations but not under others. The fundamental operations of addition and [illegible]ation are commutative. In the set of linear transformations of a line onto itself we have an algebraic operation whose elements are not numbers but functions. The only operation--composition of functions--is not commutative. Nevertheless, the operation is associative. There is an element which plays the same role for composition as zero does for addition and one for multiplication. For each linear transformation there is a transformation which "undoes" the first, and thus acts like the reciprocal of a nonzero number when the operation is multiplication and like the negative of a number when the operation is addition.

It is the fact that so many different algebraic systems share these properties that led mathematicians to define a group. This concept was defined earlier, and the example treated here is one which is very important in advanced mathematics.

If the exercises on cardinal number are to be assigned, it will probably be necessary to prepare the way with a brief discussion in class. It can be pointed out that when we are asked whether two finite sets have the same number of members, we can count them. Now counting a set can be described as setting up a one-to-one correspondence between the set and part of a standard sequence of noises. If we do this for sets A and B and discover that we used the same part of the standard sequence of noises in both cases, we have set up a one-to-one correspondence between A and B. We could have done this without counting. Since we can't, in any ordinary sense, count the members of an infinite set, it is natural to define what we mean when we say that two such sets have the same number of members in terms of one-to-one correspondences. Although the students will probably be a bit disturbed by the fact that the set of positive integers and the set of odd positive integers have the same number of members, they will soon come to realize that no other definition seems reasonable.

The students should be asked to give detailed proofs, in class, for one or two cases of the theorem that an image is between two other images if and only if its pre-image is between the pre-images of the other two images. This will prepare them for the first exercise in the next set. Since we are dealing with a necessary and sufficient condition, two implications must be proved. The proof can be shortened, however, by noting that the inverse of a transformation of any of the four types is of the same type.

Exercises 3-6 of the following set justify that the linear transformation of a line onto itself forms a group under the operation of composition.

Exercises S2-2a

1. Let Q be between P and R; i.e., either $p < q < r$ or $p > q > r$ where p, q, r are coordinates of P, Q, R on line $\overleftrightarrow{PR}$. If T is a linear transformation, then there are numbers $a \neq 0$ and b such that the coordinate of $T(X) = ax + b$ where x is the coordinate of X.

$$T(P) \sim p' = ap + b \qquad T(Q) \sim q' = aq + b \qquad T(R) \sim r' = ar + b$$

If $p < q < r$ and $a > 0$ then $ap < aq < ar$ and $p' < q' < r'$
If $p < q < r$ and $a < 0$ then $ap > aq > ar$ and $p' > q' > r'$
If $p > q > r$ and $a > 0$ then $ap > aq > ar$ and $p' > q' > r'$
If $p > q > r$ and $a < 0$ then $ap < aq < ar$ and $p' < q' < r'$

Hence in all cases $T(Q)$ is between $T(P)$ and $T(R)$.

2. Let $\overline{PQ}$ and $\overline{RS}$ be congruent segments; i.e., $|p - q| = |r - s|$. Let T be a linear transformation, defined: $T(X) = X'$ has coordinate $x' = ax + b$.

$$T(P) \sim p' = ap + b \quad T(Q) \sim q' = aq + b, \quad |p'-q'| = |ap+b-aq-b| = |a|\ |p-q|$$
$$T(R) \sim r' = ar + b \quad T(S) \sim s' = as + b \quad |r'-s'| = |ar+b-as-b| = |a|\ |r-s|$$

But $\overline{PQ} \cong \overline{RS}$ implies $|p - q| = |r - s|$. So $|p' - q'| = |r' - s'|$ which means $\overline{P'Q'} \cong \overline{R'S'}$.

3. Let T_1, T_2 be arbitrary linear transformations of the line into itself defined by coordinate equations: $T_1(X) = X'$, $x' = ax + b$, $T_2(X) = X'$, $x' = cx + d$. We wish to know whether $T_1(T_2)$ is a linear transformation of the line.

$T_2(X)$ is a point Y with coordinate $cx + d$

T_1 is defined at Y; $T_1(y)$ is a point with coordinates $(ac)x + (ad + b)$.

But $ac \neq 0$ since $a \neq 0$ and $c \neq 0$. And $(ad + b)$ is a number. So $T_1(T_2)$ is defined for all points X by coordinate equation $x' = (ac)x + (ad + b)$. Thus it is a linear transformation of the line.

4. To show that composition of linear transformations is associative let T_1, T_2, T_3 be defined by coordinate equations $T_1(x) = ax + b$, $T_2(x) = cx + d$, $T_3(x) = ex + f$. Then $T_2(T_3)$ is the linear transformation taking x to $(ce)x + (cf + d)$ and $T_1(T_2)$ is the linear transformation taking x to $(ac)x + (ad + b)$. Let X_0 be an arbitrary point with coordinate x_0.

$T_3(X_0) = Y$ with coordinate $(ex_0 + f)$,

$(T_1(T_2))(Y) = Z$ with coordinate $(ac)(ex_0 + f) + (ad + b)$.

So $((T_1(T_2))T_3)(X_0) = Z$ with coordinate $(ace)x_0 + (acf + ad + b)$.

Now $(T_2(T_3))(X_0) = V$ with coordinate $v = (ce)x_0 + (cf + d)$,

$T_1(V) = Z'$ with coordinate $a((ce)x_0 + (cf + d)) + b$.

So $(T_1(T_2(T_3)))(X_0) = Z'$ with coordinate $(ace)x_0 + (acf + ad + b)$.

Therefore $Z = Z'$ since both have the same coordinate which means

$$T_1(T_2(T_3)) = (T_1(T_2))(T_3).$$

5. To show that the set of linear transformations of a line has an identity with respect to composition, consider line $\overleftrightarrow{OU}$ and the transposition I such that $I(X) = X$. I is given by the coordinate equation $I(x) = x = 1 \cdot x + 0$ so I is a member of the set of linear transformations. This I is an identity. By the definition of I we know

$$(I(T))(X) = I(T(X)) = T(X)$$

or $$(T(I))(X) = T(I(X)) = T(X)$$

so $$I(T) = T(I) = T.$$

Suppose I' were any other identity.

Then $I'(I) = I(I') = I$ since I' is an identity,

but $I(I') = I'(I) = I'$ since I is an identity.

Therefore $I' = I$, which means I is the unique identity.

6. To show that each element on the set S of linear transformations of the line has an inverse with respect to composition, let T be an arbitrary element of $S \cdot T(X)$ is the point Y such that $y = asc + b$, $b \neq 0$.

If there were an inverse T^{-1} to T we would have to have

$$T^{-1}(T) = T(T^{-1}) = I.$$

There would have to be numbers $c \neq 0$ and d, such that for all points S, with coordinate x,

$$c(ax + b) + d = a(cx + d) + b = 1x + 0.$$

This requires $\quad cax = acs = 1x \quad (1)$

$\quad cb + d = ad + b = 0 \quad (2)$

Since $a \neq 0$ we can choose $c = \frac{1}{a} \neq 0$ to satisfy (1) and then $d = -b$ along with $c = \frac{1}{a}$, $y - b$ will be the inverse of T, and is a linear transformation.

7. We exhibit one counter example to show that composition is not commutative. Consider

$T_1 : T_1(X) = Y, \; y = 2x + 0$ [":" is read "defined by"]

$T_2 : T_2(X) = Y, \; y = 1 \cdot x + 1$.

$T_1(T_2) : (T_1(T_2))(X) = 2(x + 1) + 0 = 2x + 2$

$T_2(T_1) : (T_2(T_1))(X) = 1(2x + 0) + 1 = 2x + 1$

Therefore $\quad T_2(T_1) \neq T_1(T_2)$.

Suppose we require $\quad T_1 : T_1(X) = Y, \; y = ax + b$ and

$T_2 : T_2(X) = Y, \; y = cx + d$

to be such that $\quad T_1(T_2) = T_2(T_1)$, i.e. $a(cx + d) + b = c(ax + b) + d, \forall x$.

So we must have $acx = cax$ and $ad + b = cb + d$.

The conditions are (1) $a = c = 1$ and b and d any real numbers.

(2) $a = c \neq 1$ and $b = d$ any real number.

(3) a, c any real numbers and $b = d = 0$.

8. Let $F : F(X) = Y$, $y = ax + b$ be a transformation.

Case (1) $a > 0$. $F = T(E)$ where $E : y = ax$ $T : y = x + b$

$\forall X$, $E(X)$ has coordinate ax, $T(E(X))$ has coordinate $ax + b$.

Case (2) $a < 0$. $F = T(E(R))$ where $R : y = -1x$ $E : y = |a|x$ $T : y = x + b$

$\forall X$, $R(X)$ has coordinate $-x$, $E(R(X))$ has coordinate

$|a|(-x) = ax$

$T(E(R(X)))$ has coordinate $ax + b$ hence $T(E(R)) = F$.

Exercises S2-2b

1. Let the points be R and S. We may assume $r < s$. The ratio of two non-zero numbers is positive if and only if both numbers have the same sign. $r < s$ means $r - s < 0$. Therefore $\frac{r' - s'}{r - s} > 0$ if and only if $r' - s' < 0$. But we have $r' - s' < 0$ if and only if $r' < s'$ which is the condition that the coordinate change be order preserving. Similarly, $\frac{r' - s'}{r - s} < 0$ if and only if $r' - s' > 0$ which is true if and only if the coordinate change is order reversing.

2. The coordinate change f determines an equation of the form $f(x) = x' = ax + b$. From $r' = ar + b$, $s' = as + b$, we find

$a = \frac{r' - s'}{r - s}$, $b = \frac{rs' - r's}{r - s}$

(a) f includes a contraction if and only if $0 < a < 1$ which is the condition $0 < \frac{r' - s'}{r - s} < 1$.

(b) f includes a contraction and reflection if and only if $-1 < a < 0$ which is the condition $-1 < \frac{r' - s'}{r - s} < 0$.

(c) f includes an expansion if and only if $a > 1$ which is the condition $\frac{r' - s'}{r - s} > 1$.

(d) f includes an expansion and reflection if and only if $a < -1$ which is $\frac{r' - s'}{r - s} < -1$.

3. The coordinate change f determines an equation of the form $f(x) = ax + b$. From $p' = ap + b$, $q' = aq + b$ we find $a = \frac{p' - q'}{p - q}$, $b = \frac{pq' - p'q}{p - q}$.

(a) f includes a translation if and only if $a = 1$ which is the condition $\frac{p' - q'}{p - q} = 1$.

(b) f includes a reflection if and only if $a = -1$ which is the condition $\frac{p' - q'}{p - q} = -1$.

4. We wish to show that the intrinsic coordinate systems are identical to the coordinate systems whose defining functions have the form $x' = x + b$ or $x' = -x + b$ with b, any real number.

Pick one intrinsic coordinate system, call its origin P_0 and refer to it as the P_0-system.

Consider any other intrinsic coordinate system (one having the same unit length) with origin P_1 and the same positive direction.

	P_0	P_1
P_0-system	0	P_1
new system	P'_0	0

$x < y$ if and only if X is left of Y.
$x' < y'$ if and only if X is left of Y

So $\vec{d}(P_0, P_1) = 0 - P'_0 = P_1 - 0$ since unit of measure is the same.

Solving $P_0' = a \cdot 0 + b$ and $0 = a \cdot P_1 + b$ we get $x' = x + (-P_1)$.

So this (intrinsic) coordinate system has defining function of the form $x' = x + b$ relative to the P_0-system. Conversely for any equation $x' = x + b$ we can find the intrinsic coordinate system whose origin has P_0 coordinate $(-b)$ and the P_0 positive direction.

Similarly we establish an identity between coordinate systems with positive sense opposite to that of the P_0-system and systems with defining functions $x' = -x + b$.

Notice

	P_0	P_1	
P_0-system	0	P_1	$x < y$ if and only if X is left of Y.
new system	P'_0	0	$x' < y'$ if and only if X is right of Y

$\vec{d}(P_0, P_1)$ is $p_1 - 0$ in P_0-system, but $p'_0 - 0$ in system with opposite positive sense.

5.

pre-images	-3	-2	$-\frac{3}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3
images	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	-1	-2	-4	0	4	2	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$

6. (a) Domain of $F(G(H))$ = domain of $H = \{w: w \text{ is real}\}$

range of $F(G(H)) = \{z: 0 < z < 1\}$

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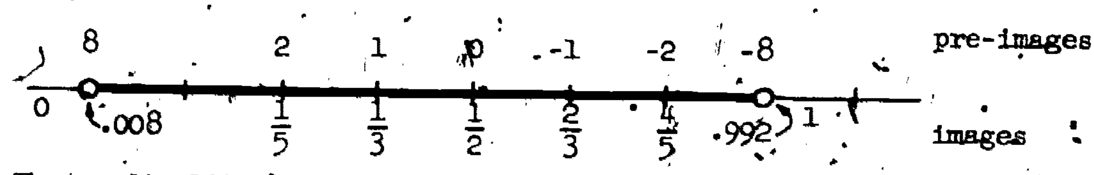
Transformation $F(G(H))$ is into the line, not onto.
It is one-to-one.

(b)



(c) The cardinality of the interior of a segment is the same as the cardinality of the line.

7. (a) Domain $D(E(F)) = \{w : w \text{ is real}\}$
Range $D(E(F)) = \{z : 0 < z < 1\}$
$D(E(F))$ maps the reals into but not onto the reals.
~~It~~ is one-to-one.

(b) The cardinality of R is infinite.

8. Let the coordinate change be given by $x' = ax + b$.

Then $\dfrac{p' - q'}{r' - s'} = \dfrac{(ap + b) - (aq + b)}{(ar + b) - (as + b)} = \dfrac{a(p - q)}{a(r - s)} \quad \dfrac{(b - b)}{(b - b)} \quad \dfrac{p - q}{r - s}$.

The operations are justified since $r \neq s$ and $a \neq c$ so that $r - s \neq 0$ and $\frac{a}{a} = 1$.

9. $x = \frac{11}{2}$

This may be obtained from the change of coordinate formula, or, using Problem 8, from ratios of directed distances (letting $A = P$, $B = R = Q$, $C = S$).

10. $x' = x\left(\dfrac{b' - a'}{b - a}\right) + \left(\dfrac{a'b - ab'}{b - a}\right)$

11. Let f be a linear transformation of the line into itself such that for two distinct points X and Y, $f(X) = X$ and $f(Y) = Y$. We wish to show that for all points Z, $f(Z) = Z$.

$f(X) = X$ and $f(Y) = Y$ yield coordinate equations

$$x = ax + b \quad \text{and} \quad y = ay + b$$

which implies $a = 1$ and $b = 0$. So for any point Z with coordinate Z, $f(Z)$ has coordinate

$$z' = 1 \cdot z + 0 = Z.$$

So f keeps all points fixed.

Teachers' Commentary

Chapter 2

Supplement D

(Supplement to Chapters 2,3,8)

POINTS, LINES, AND PLANES

In this chapter the student will face many problems arising from the relative positions of points, lines, and planes in space. Among these are the measurements of angles and distances, matters of parallelism and perpendicularity, and questions of incidence and separation.

Various schemes and devices are suggested as being appropriate in certain cases, but in the last analysis we believe that a student should not be told too much. He has many tools; therefore, he should be encouraged to find his own solution for any given situation.

Here is where a student begins to need some facility with determinants. There is help in Appendix A.

If the equation of a line is written in the form $ax + by + c = 0$, then the equations

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$
$$ax_3 + by_3 + c = 0$$

may be considered a system of 3 linear homogeneous equations in the 3 unknowns a , b , c . Equation (3) in the student's text is the necessary and sufficient condition that there are non-trivial solutions of the system.

Exercises D-2

1. (a) collinear (b) $k = 46.5$ (c) $|bc - ad|$ (d) collinear

2. ac; -ac; ac; -ac; yes; no. The direction of traverse of the triangle affects the sign (positive for counter-clockwise, negative for clockwise); the vertex at which one starts does not.

3. Consider the triangle with vertices $P_i = (x_i, y_i)$, $i = 1, 2, 3$. We know that the area is

$$K = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$ i.e. absolute value of determinant

$$= \frac{1}{2}\left|x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\left|x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2\right|$$

$$= \frac{1}{2}\left|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right|$$

$$= \frac{1}{2}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right)$$

4. $18\frac{1}{2}$

5. (a) $\begin{vmatrix} -2 & 1 & 1 \\ 2 & -2 & 1 \\ 6 & -5 & 1 \end{vmatrix} = -2(3) - 2(6) + 6(3) = 0$

(b) $\vec{B} - \vec{A} = [4,-3]$ $\quad \vec{C} - \vec{A} = [8,-6]$

Hence $\vec{B} - \vec{A} = \frac{1}{2}(\vec{C} - \vec{A})$

But $\overleftrightarrow{AB}$ is parallel to the line of $\vec{B} - \vec{A}$, and

$\overleftrightarrow{AC}$ is parallel to the line of $\vec{C} - \vec{A}$ which is the line of

$\vec{B} - \vec{A}$.

So $\overleftrightarrow{AB}$ coincides with $\overleftrightarrow{AC}$.

(c) $d(A,B) = 5$, $d(B,C) = 5$, $d(A,C) = 10$,

By the triangle inequality, this implies B lies on $\overleftrightarrow{AC}$.

If lines L_1, L_2, L_3 meet in a point (x_1, y_1), then

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1 &= 0 \\ a_2x_1 + b_2y_1 + c_2 &= 0 \\ a_3x_1 + b_3y_1 + c_3 &= 0 . \end{aligned}$$

This system of three linear equations in the two unknowns (x_1, y_1) has a common solution only if the determinant of the coefficients is zero; this condition is Equation (3) in the student's text.

It might be worthwhile to place considerable emphasis on the idea of <u>families</u>. This concept will appear later in connection with curves in the plane and in space.

<u>Exercises D-3</u>

1. (a) No (b) Yes, $(\frac{1}{2}, \frac{5}{2})$ (c) No, (the lines are parallel).

2. (a) 4

   (b) $k^3 + 4k - 16 = (k - 2)(k^2 + 2k + 8) = 0$; real value, $k = 2$.

3. General form, $3x - 2y + 5 + n(x + 4y - 1) = 0$.

   (a) $21x - 28y + 43 = 0$

   (b) $14x + 21y + 6 = 0$

   (c) $4x + 9y = 0$

   (d) $5x - 22y + 19 = 0$

   (e) $x - 3y + 3 = 0$

4. $9x - 3y + 8 = 0$

5. This exercise may be done in a variety of ways. If students use the methods in this section, some of the following may be useful in checking their work.

   (a) Centroid, $(\frac{a + c}{3}, \frac{b}{3})$

   (b) Orthocenter, $(0, -\frac{ac}{b})$

   (c) Circumcenter, $(\frac{a + c}{2}, \frac{b^2 + ac}{2b})$

(d). Evaluate determinant in (3) of text by factoring out $\frac{a+c}{60}$ from C_1, $\frac{1}{6b}$ from C_2, multiplying elements of R_2 by $-\frac{3}{2}$ and adding to elements of R_3:

$$\begin{vmatrix} 0 & \frac{-ac}{b} & 1 \\ \frac{a+c}{3} & \frac{b}{3} & 1 \\ \frac{a+c}{2} & \frac{b^2+ac}{2b} & 1 \end{vmatrix} = \frac{a+c}{6b \cdot 6b}\begin{vmatrix} 0 & -6ac & 1 \\ 2 & 2b^2 & 1 \\ 3 & 3b^2+3ac & 1 \end{vmatrix}$$

$$= \frac{a+c}{36b^2}\begin{vmatrix} 0 & -6ac & 1 \\ 2 & 2b^2 & 1 \\ 0 & 3ac & -\frac{1}{2} \end{vmatrix}$$

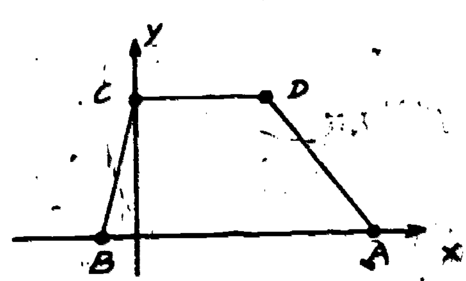
$$= \frac{a+c}{36b^2}(-2)(3ac - 3ac)$$

$$= 0$$

(e) Yes, because by appropriate choice of coordinates any triangle can have vertices with the coordinates given for A, B, C.

6. Consider trapezoid ABCD and choose coordinate system so that $A = (a,0)$, $B = (b,0)$, $C = (0,c)$, $D = (d,c)$. The diagonals are $cx + ay - ac = 0$, $cx + (b-d)y - bc = 0$. Joining midpoints of bases is the line $2cx + (a+b-d)y - (a+b)c = 0$

$$\begin{vmatrix} c & a & -ac \\ c & b-d & -bc \\ 2c & a+b-d & -(a+b)c \end{vmatrix} \doteq 0$$



The subject matter of this course can be grouped and developed in various ways. Although we have used some of the contents of this section in earlier sections, we now consider, in a more systematic way, the general topic of intersections and parallelisms.

We make extensive use of determinants, with which we assume some reasonable familiarity. An appendix presents a brief treatment of the topic, which was considered too algebraic to be part of the text. Matrices also, would have facilitated our development, particularly the concept of the rank of a matrix, and an augmented matrix; but these ideas were considered to be too far afield from our central theme, and so do not appear, even in an appendix. Teachers and interested students are referred to the SMSG text on Matrix Algebra, or to any of the recent elementary texts on matrices. We recommend strongly that students be encouraged to gain some competence in those aspects of matrix algebra which apply to the present content, and perhaps prepare oral or written reports on these applications.

Authors, as well as students and teachers, are not pleased with pages that seem overloaded with letters and subscripts. However, in three dimensions, equations of lines and planes do require many symbols. We chose to use fewer letters with different subscripts, rather than many different letters, because we felt that, with a bit of effort, the patterns of relationships could be more easily seen. Students should be encouraged to see these patterns, and to try to extend them to corresponding situations in higher dimensions, where subscripts become more significantly necessary. We have avoided here, and generally throughout the text, the use of Σ notation. If students have the proper background and ability, they might be encouraged to state, as far as possible, the results of this section that could be generalized to n dimensions, using whatever symbolism they think most appropriate.

Solutions to Exercises D-4

1. (a) parallel
   (b) skew
   (c) skew
   (d) skew
   (e) skew
   (f) skew

2. (a) $\begin{cases} x = 1 + 3t \\ y = 2 - 4t \\ z = 3 - 2t \end{cases}$ (c) $\begin{cases} x = 1 + 3t \\ y = 2 - 2t \\ z = 3 - 8t \end{cases}$

(b) $\begin{cases} x = 1 + 6t \\ y = 2 + 2t \\ z = 3 + 4t \end{cases}$ (d) $\begin{cases} x = 1 + 3t \\ y = 2 + 4t \\ z = 3 - 6t \end{cases}$

3. (a) $M_1 : 4x + 18y - 3z - 34 = 0$

$M_2 : 4x + 18y - 3z - 69 = 0$

(b) $M_1 : 14x + 24y + 9z + 69 = 0$

$M_2 : 14x + 24y + 9z - 35 = 0$

4. (a) $4x + 18y - 3z - 34 = 0$

(b) $14x + 24y + 9z - 35 = 0$ Note $L_1 \parallel L_2$

5. (a) $2x - 8y + 7z = 0$

(b) $11x + 9y + 12z = 0$

(c) $22x + y + 8z = 0$

(d) $3y + 2z = 0$

6. (a) L_1 goes over L_4 (c) L_2 goes under L_4

(b) L_2 goes over L_3 (d) L_3 goes under L_4

7. If L_A goes over L_B and L_B goes over L_C, then it is <u>sometimes</u> true that L_A goes over L_C.

8. It is <u>false</u> that if L_A and L_B are distinct, then L_A goes over L_B or L_B goes over L_A. Consider the lines $L_A : x = 1$, $L_B : x = 2$. It is never the case that P_1 on L_A and P_2 on L_B have the same x-coordinate, hence, one criterion is never met.

9. (a) $[1,0,2] + t[5,11,7] = [x,y,z]$

(b) $[0,-11,-17] + t[1,7,7] = [x,y,z]$

(c) $[1,-1,0] + t[5,8,1] = [x,y,z]$

(d) $[3,2,4] + t[7,1,5] = [x,y,z]$

(e) $[1,-3,1] + t[5,2,4] = [x,y,z]$

(f) $[-5,-1,-6] + t[8,2,7] = [x,y,z]$

10. (a) $[\frac{11}{6}, \frac{11}{6}, \frac{19}{6}]$

(b) $[\frac{-2}{3}, \frac{-11}{3}, -\frac{1}{3}]$

(c) $(\frac{14}{9}, -\frac{1}{9}, \frac{1}{9})$

(d) $(\frac{58}{3}, \frac{13}{3}, \frac{47}{3})$

11. (a) $3x - 2y + z = 0$

(b) $2x + y - 3z = 0$

(c) $x + 3y - 2z = 0$

(d) $-2x + y + 2z = 0$

12. (a) $[\frac{7}{3}, \frac{14}{9}, \frac{10}{9}]$

(b) $[\frac{27}{11}, \frac{-53}{11}, \frac{15}{11}]$

(c) $[\frac{-23}{13}, \frac{50}{13}, \frac{57}{13}]$

(d) $[\frac{11}{2}, -4, 6]$

13. $L_1 \begin{cases} x = a_1 + \ell_1 t \\ y = b_1 + m_1 t \end{cases}$ $\qquad$ $L_2 \begin{cases} x = a_2 + \ell_2 t \\ y = b_2 + m_2 t \end{cases}$

L_1 and L_2 are coincident if and only if

$$\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} = 0$$

and there exists an s_0 such that

$$\begin{vmatrix} a_1 - a_2 & \ell_2 s_0 \\ b_1 - b_2 & m_2 s_0 \end{vmatrix} = 0 .$$

Note: This is equivalent to the existence of a t_0 such that

$$\begin{vmatrix} a_2 - a_1 & \ell_1 t_0 \\ b_2 - b_1 & m_1 t_0 \end{vmatrix} = 0 .$$

L_1 and L_2 are parallel if and only if

$$\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} = 0$$

and there is no s_0 such that

$$\begin{vmatrix} a_1 - a_2 & \ell_2 s_0 \\ b_1 - b_2 & m_2 s_0 \end{vmatrix} = 0 .$$

L_1 and L_2 intersect in a unique point if and only if

$$\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} \neq 0$$

It is traditional to talk about <u>the</u> angle between two lines, but present standards of precision require that we take account of the fact that at least four angles are formed when two lines intersect. These angles can be distinguished in a diagram by various methods, but all of these methods must induce a sense along each of the lines. We indicate explicitly in the text that such a sensing must underly any method of distinguishing these angles analytically.

It is convenient to carry through the development in the text using the parametric forms of equations for lines. We leave to an exercise (Problem 16) at the end of this section the development of some of these ideas, using the usual general forms of the equations of these lines, in 2-space. Students should be encouraged here, as in other places in the text, to use the corrdinate system and method of representation that seems most natural, and to be prepared to show the equivalence of the results obtained in different ways.

It is not expected that any class complete all the exercises at the end of this section. We have supplied sufficient exercises to give some variety in assignments, testing, etc.

<u>Solutions to Exercises D-5</u>

1. (a) $\sim 172^\circ$ $\quad \cos\theta = \frac{-7\sqrt{2}}{10} \sim 0.9898$

   (b) $\sim 75^\circ$ $\quad \cos\theta = \frac{3\sqrt{130}}{130} \sim 0.263$

   (c) $\sim 83^\circ$ $\quad \cos\theta = \frac{-\sqrt{65}}{65} \sim -0.124$

2. (a) $\begin{cases} x = 3 + 3t \\ y = 5 + t \end{cases}$ or $y - 5 = \frac{1}{3}x - \frac{1}{3}$

(b) $\begin{cases} x = 3 + 2t \\ y = 5 + t \end{cases}$ or $y - 5 = \frac{1}{2}x - \frac{3}{2}$

(c) $\begin{cases} x = 3 - 2t \\ y = 5 + 3t \end{cases}$ or $y = \frac{-3}{2}x + \frac{19}{2}$.

3. Lines $L_1 : y + 3x - 11 = 0$ direction pairs $\overline{L}_1 = [-1,3]$

$L_2 : y + 2x - 5 = 0$ $\overline{L}_2 = [-1,2]$

Bisectors $B_1 : (3 - 2\sqrt{2})x + (1 - \sqrt{2})y - 11 + 5\sqrt{2} = 0$ $\overline{B}_1 = [1-\sqrt{2}, -3+2\sqrt{2}]$

$B_2 : (3 + 2\sqrt{2})x + (1 + \sqrt{2})y - 11 - 5\sqrt{2} = 0$ $\overline{B}_2 = [-1-\sqrt{2}, 3+2\sqrt{2}]$

Let θ be one angle determined by L_1 and B_2

ϕ be one angle determined by L_2 and B_2

Since $\overline{L}_1$, $\overline{L}_2$ and $\overline{B}_2$ are in the same quadrant we can be sure that $\cos\theta = \cos\phi$ implies that $\angle\theta \cong \angle\phi$.

$$\cos\theta = \frac{\overline{B}_2 \cdot \overline{L}_1}{|\overline{B}_2||\overline{L}_1|} = \frac{10 + 7\sqrt{2}}{\left(\sqrt{20 + 14\sqrt{2}}\right)\sqrt{10}}$$

$$\cos\phi = \frac{\overline{B}_2 \cdot \overline{L}_2}{|\overline{B}_2||\overline{L}_2|} = \frac{7 + 5\sqrt{2}}{\left(\sqrt{20 + 14\sqrt{2}}\right)\sqrt{5}} = \frac{10 + 7\sqrt{2}}{\left(\sqrt{20 + 14\sqrt{2}}\right)\sqrt{10}}$$

This can also be checked by noticing that $\cos\theta$ is the cosine of half the angle between $\overline{L}_1$ and $\overline{L}_2$.

4. (a) $P_1 = [\frac{11}{4}, 3]$ $P_2 = [\frac{-40}{11}, \frac{43}{11}]$ $P_3 = [6,-7]$

(b) Alt. from $P_1 = [\frac{11}{4}, 3] + t[3,1]$ line through $P_1 \perp L_1$

Alt. from $P_2 = [\frac{-40}{11}, \frac{43}{11}] + t[2,1]$ line through $P_2 \perp L_2$

Alt. from $P_3 = [6,-7] + t[-2,3]$ line through $P_3 \perp L_3$

5. The lines are parallel. Therefore, $\theta = 0^\circ$.

6. (a) $\arccos \frac{2}{\sqrt{154}} \approx \arccos 0.161 \approx 80.5^\circ$ and 99.5°

(b) $\arccos \left(\frac{-11}{14}\right) \approx 180^\circ - \arccos (0.786) \approx 141.7^\circ$ and 38.3°

(c) $\arccos \left(\frac{-8}{\sqrt{.54}}\right) \approx 180^\circ - \arccos (0.654) \approx 130^\circ$ and 50°

7. (a) $[x,y,z] = [1,2,3] + t[a\ ,\ 3a - 2c\ ,\ c]$
(b) $[x,y,z] = [1,2,3] + t[a\ ,\ a + 3c\ ,\ c]$
(c) $[x,y,z] = [1,2,3] + t[a\ ,\ 3c - 2a\ ,\ c]$

for any a and c not both zero.

8. (a) $N_1 : [x,y,z] = t[0,3,1]$

(b) $N_2 : [x,y,z] = t[1,1,1]$

(c) $N_3 : [x,y,z] = t[5,11,2]$

9. (a) $-3x + y + 2z - 10 = 0$
(b) $x - y + 3z - 19 = 0$
(c) $2x + y - 3z + 10 = 0$

10. (a) $5x + 11y + 2z - 51 = 0$
(b) $x + y + z - 9 = 0$
(c) $5x + 11y + 2z - 53 = 0$
(d) $3y + z - 14 = 0$
(e) $x + y + z - 7 = 0$
(f) $3y + z - 10 = 0$

11. (a) 86° and 94°

(b) 69° and 111°

(c) 60° and 120°

12. (a) $7x - y + 11z - 55 = 0$
(b) $x + 3y + 0z - 11 = 0$
(c) $3x - 12y + 7z + 2 = 0$
(d) $-8x + 7y + 5z - 62 = 0$
(e) $x + 7y + 2z - 35 = 0$
(f) $3x + 0y - z - 7 = 0$
(g) $2x - y + z - 4 = 0$
(h) $x + 13y + 5z - 47 = 0$
(i) $3x - 3y + z - 1 = 0$

13. (a) $5x - 7y - 11z = 0$
    (b) $11x - 7y + z = 0$
    (c) $x + y - z = 0$

14. (a) 21° (d) 29.2° (g) 45.6°
    (b) 25.3° (e) 53.6° (h) 4°
    (c) 4° (f) 40.4° (i) 21°

15.

	with x-axis	y-axis	z-axis
(a)	32.3°	53.2°	15.5°
(b)	53.2°	15.5°	32.3°
(c)	15.5°	32.3°	53.2°

16. $$\text{Cos} = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\ \sqrt{a_2^2 + b_2^2}}$$

Exercises S7-6

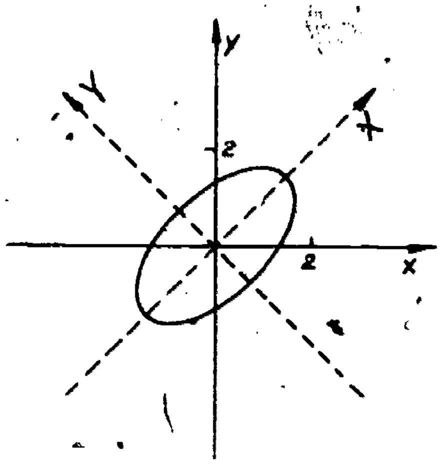
1. (a) 27° (d) 36°

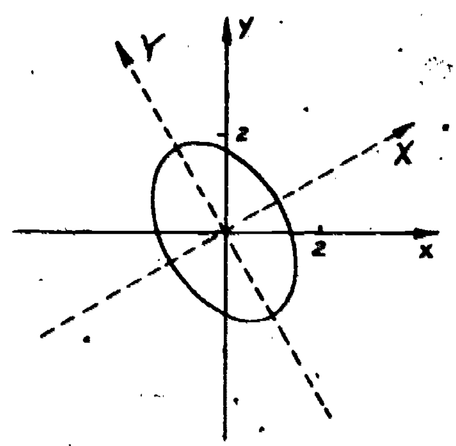
   (b) 60° (e) 30°

   (c) 22.5° (f) 63°

2. (a) $X^2 + 4Y^2 = 4$
   rotation through 45°
   ellipse

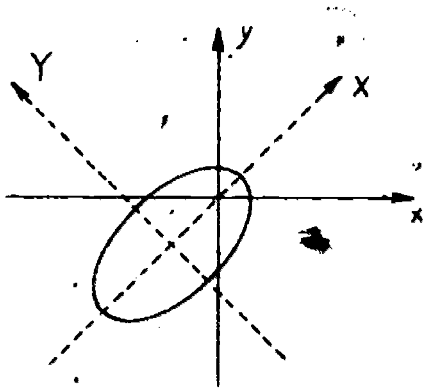
   (c) $2X^2 + Y^2 = 4$
   rotation through 30°
   ellipse

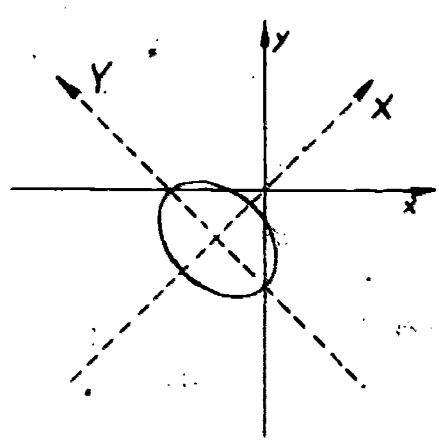




   (b) $X^2 + 4Y^2 = 4$
   rotate 45°
   translate $X = x + \sqrt{2}$
   ellipse

   (d) $2X^2 + Y^2 = 1$
   rotate $\theta = 45$
   translate $X = x + \sqrt{2}$
   ellipse





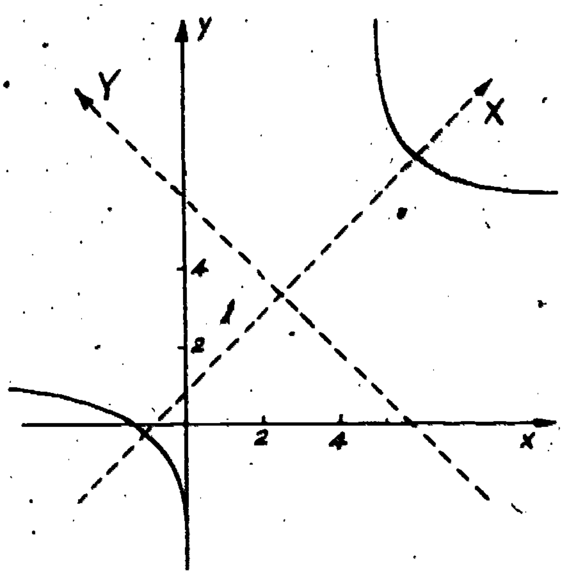
(e) $4X^2 - 8Y^2 = 99$

rotate 45°

translate $X = x - 3\sqrt{2}$,

$Y = y - \frac{\sqrt{2}}{4}$

hyperbola



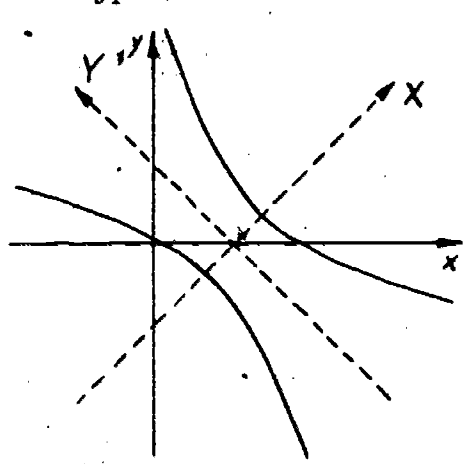
(f) $4X^2 - Y^2 = 4$

rotate $\arccos \frac{4}{5}$

translate $X = x - \frac{8}{5}$,

$Y = y + \frac{6}{5}$

hyperbola

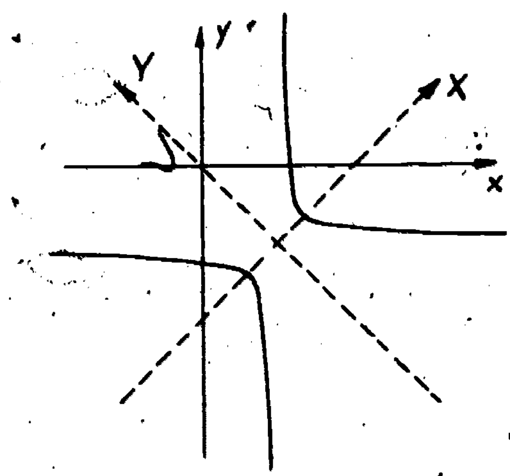


(g) $X^2 - Y^2 = 1$

rotate 45°

translate $X = x$, $Y = y + 2\sqrt{2}$

hyperbola

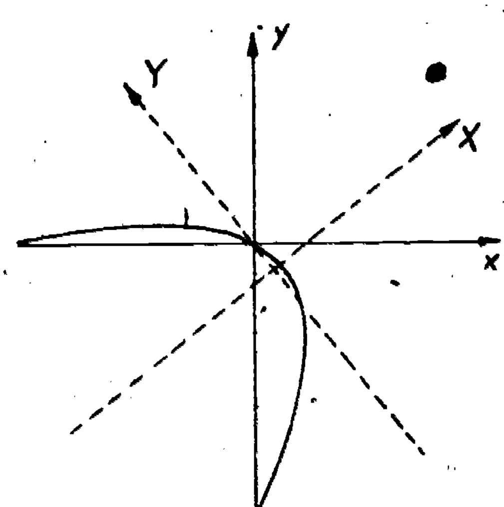


(h) $y^2 = -6x$

rotate $\arccos \frac{4}{5}$

translate $X = x - \frac{1}{6}$, $Y = y + 1$

parabola



Exercises S7-7a

1. Given that $x' = x + h$
   and $y' = y + k$
   and $4x^2 + y^2 - 8x + 4y + 4 = 0$

   Find h and k such that the first-degree terms will be eliminated.

   $$4x^2 + y^2 - 8x + 4y + 4 = 0 \qquad (1)$$
   $$x = x' - h$$
   $$y = y' - k$$

   Substituting in (1) and grouping terms, we find that the transformed equation is

   $$4x'^2 + y'^2 + (-8h - 8)x' + (-2k + 4)y' + (4h^2 + k^2 + 8h - 4k + 4) = 0$$

   Solving simultaneously

   $$-8h - 8 = 0 \qquad h = -1$$
   $$-2k + 4 = 0 \qquad k = 2.$$

   The transformed equation becomes

   $$4x'^2 + y'^2 = 4$$
   $$F' = -4$$

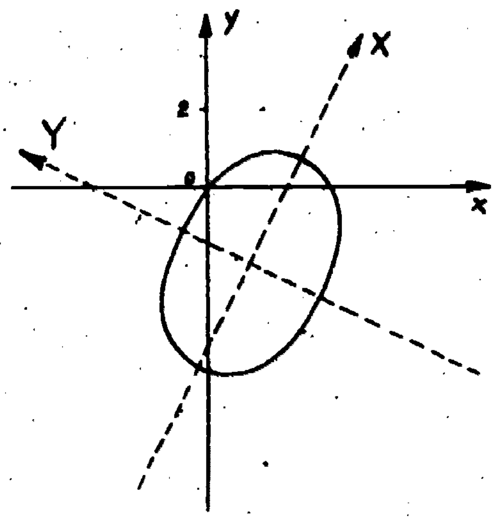
2\. (a) $8x^2 - 4xy + 5y^2 - 24x + 24y = 0$

Translate to center $(1,-2)$

$$8x'^2 - 4x'y' + 5y'^2 - 36 = 0$$

Rotate through arctan 2

$$4X^2 - 9Y^2 = 36$$



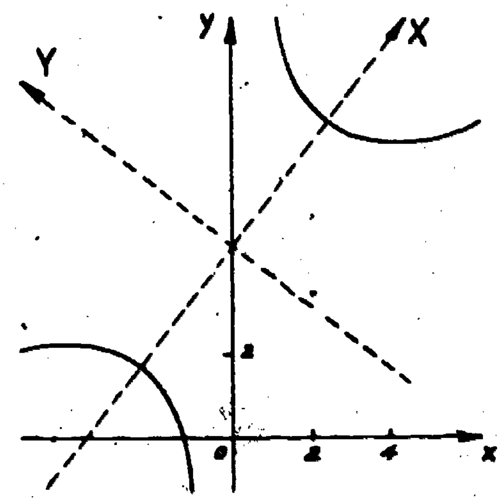
(c) $7x^2 - 24xy + 120x + 144 = 0$

Translate to center $(0,5)$

$$7x'^2 - 24x'y' + 144 = 0$$

Rotate through arctan $\frac{4}{3}$

$$9X^2 - 16Y^2 = 144$$



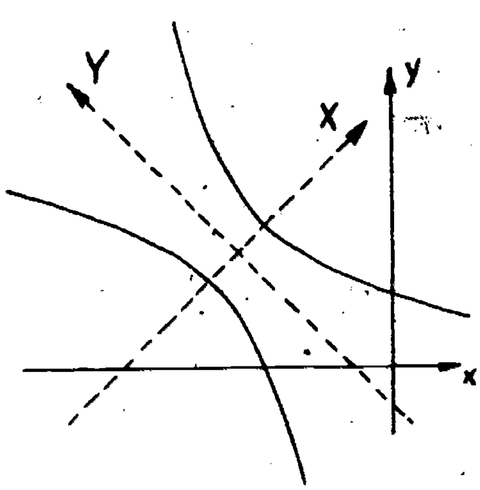
(b) $3x^2 + 10xy + 3y^2 - 6x + 22y - 53 = 0$

Translate to center $(-4,3)$

$$3x'^2 + 10x'y' + 3y'^2 - 8 = 0$$

Rotate through 45°

$$4X^2 - Y^2 = 4$$



(d) $4x^2 - 8xy + 4y^2 - 9\sqrt{2}x \quad 7\sqrt{2}y + 14 = 0$

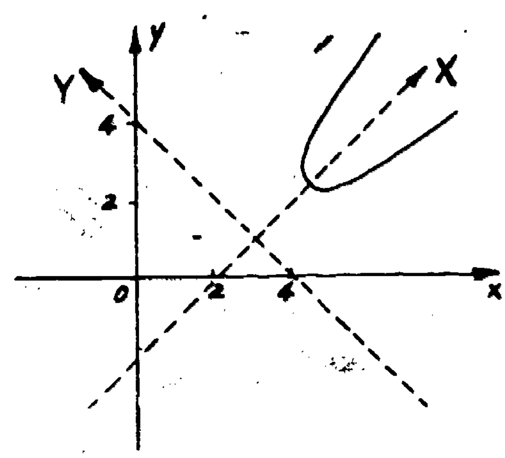
Translate to $(3,1)$

$$X = 2Y^2 + 2$$

Rotate through 45°

$$4y'^2 - 8y' - 2x' + 14 = 0$$

Parabola: $\delta = 0$



Exercises S7-7b

1. Center $(2,-5)$ Axes of symmetry $(y + 5) = \pm(x - 2)$

2. Center $(-\frac{11}{7}, -\frac{5}{7})$ Axes of symmetry $(y + \frac{5}{7}) = (\sqrt{17} - 4)(x + \frac{11}{7})$

$$(y + \frac{5}{7}) = -(\sqrt{17} + 4)(x + \frac{11}{7})$$

Exercises S7-8

1. (a) $0x^2 + 6xy + 0y^2 + 3x - 8y - 4 = 0$

$$\Delta = \begin{vmatrix} 0 & 6 & 3 \\ 6 & 0 & -8 \\ 3 & -8 & -8 \end{vmatrix} = -6(-24) - 6(24) = 0$$

Thus it is a degenerate conic: $(2y + 1)(3x - 4) = 0$

Lines: $2y + 1 = 0$, $3y - 4 = 0$

(b) $2x^2 + 8xy + 0y^2 - x + 4y - 1 = 0$

$$\Delta = \begin{vmatrix} 4 & 8 & -1 \\ 8 & 0 & 4 \\ -1 & 4 & -2 \end{vmatrix} = 4(-16) - 8(-12) - 32 = 0$$

Thus it is a degenerate conic: $(2x + 1)(x + 4y - 1) = 0$

Lines: $2x + 1 = 0$, $x + 4y - 1 = 0$

(c) $4x^2 - 5xy + 9y^2 - 1 = 0$

$$\Delta = \begin{vmatrix} 8 & -5 & 0 \\ -5 & 18 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 8(-36) + 5(10) = -288 + 50 \neq 0$$

Thus it is not a degenerate conic.

(d) $2x^2 - 1xy - 6y^2 = 0$

$$\Delta = \begin{vmatrix} 4 & -1 & 0 \\ -1 & -12 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

So it is a degenerate conic: $(2x + 3)(x - 2y) = 0$

Lines: $2x + 3 = 0$, $x - 2y = 0$

2. Consider $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
where $\Delta = 0$ and $\delta \neq 0$.

Case 1. Suppose the factors of the left member represent dependent linear equations. Then we could write the left member as
$(Mx + Ny + P)(kMx + kNy + kP) = 0$ where $k \neq 0$.
But then we get

$$kM^2x + 2kMNxy + kN^2y^2 + 2kMPx + 2kNPy + kP^2 = 0$$

$\delta = 4(kM^2)(kN^2) - (2kMN)^2 = 0$ which contradicts our hypothesis $\delta \neq 0$.

Case 2. Supposing the factors represent inconsistent equations, we get that

$$(Mx + Ny + P)(kMx + kNy + hP) = 0 \quad \text{for } k \neq 0 ,\ h \neq k .$$

But again this implies that $\delta = 0$ contrary to our hypothesis, $\delta \neq 0$.

3. Consider $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
where

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 2F \quad - E(2AE - BD) + D(BE - 2CD) = 0$$

and

$$\delta = \begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 0 .$$

Then $$-2AE^2 + BDE + BDE - 2CD^2 = 0$$

or $$-2AE^2 + BDE = 2CD^2 - BDE = 0 .$$

Expression (5) is $(B^2 - 4AC)x^2 + 2(BE - 2CD)x + E^2 - 4CF$.

$\delta = 4AC - B^2 = 0$ makes the coefficient of x^2 vanish.
It remains to show that the coefficient of x is 0.

From $\Delta = 0$ and $B^2 = 4AC$ we get

$$0 = -AE^2 + BDE - CD^2 .$$

Multiply by $-4A$ and use $B^2 = 4AC$ to get

$$0 = 4A^2E^2 - 4ABDE + 4ACD^2$$

$$0 = 4(AE)^2 - 4(AE)(BD) + 4(BD)^2$$

$$0 = (2AE - BD)^2 .$$

Hence $BD - 2AE = 0$ which completes the proof.

Exercises S7-10

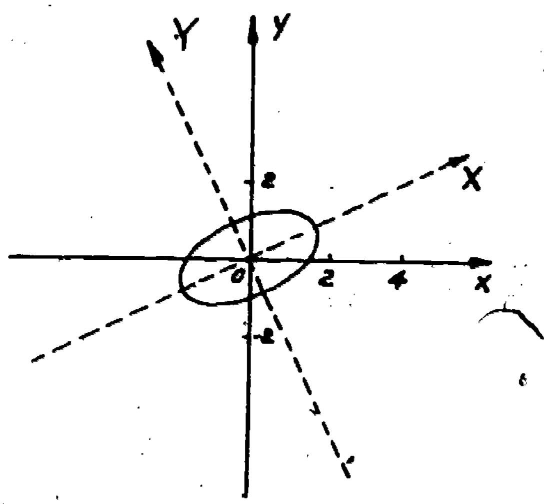
1. $8x^2 - 12xy + 17y^2 - 20 = 0$

   $\delta = 400 \qquad \Delta = -16000$

   Rotate through $\frac{1}{2}$ arctan $\frac{4}{3}$

   $$X^2 + 4Y^2 = 4$$

   ellipse



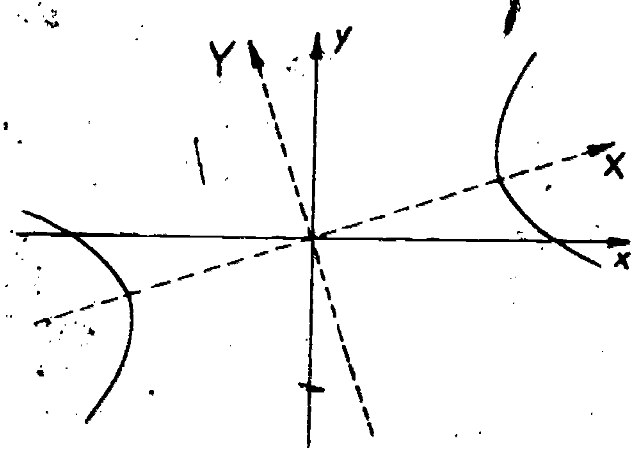
2. $3x^2 + 12xy - 13y^2 - 135 = 0$

   $\delta = -300 \quad \Delta = 81000$

   Rotate through $\frac{1}{2}$ arctan $\frac{3}{4}$

   $$X^2 - 3Y^2 = 27$$

   hyperbola



3. $5x^2 - 6xy + 5y^2 - 16x + 16y + 8 = 0$

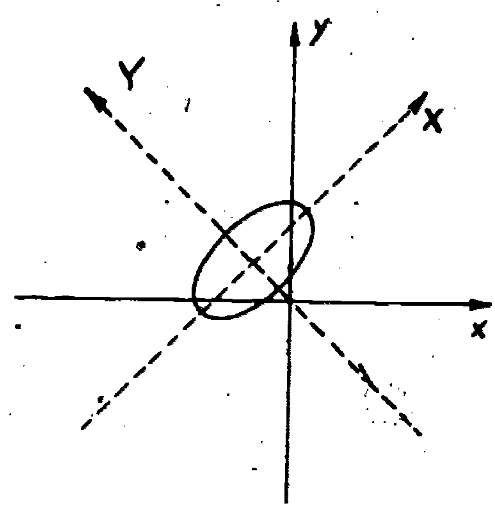
   $\delta = 64 \qquad \Delta = -1024$

   Translate $h = 1$, $k = -1$

   Then rotate through 45°

   $$X^2 + 4Y^2 = 4$$

   ellipse



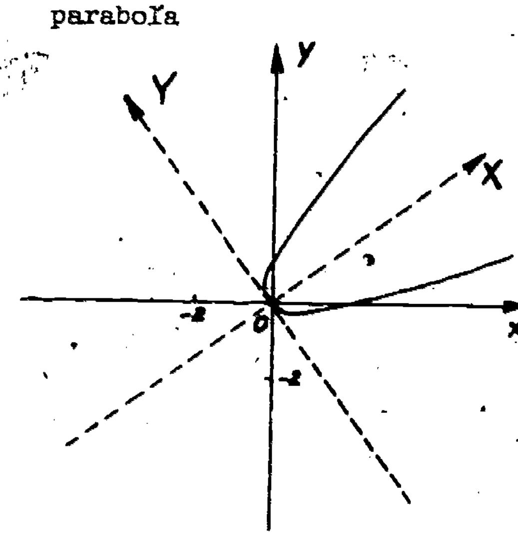
4. $9x^2 - 24xy + 16y^2 - 20x - 15y = 0$

   $\delta = 0 \quad \Delta = -8750$

   Rotate through arccos $\frac{4}{5}$

   $$Y^2 = X$$

   parabola



5. $9x^2 - 24xy + 16y^2 + 60x - 80y + 100 = 0$

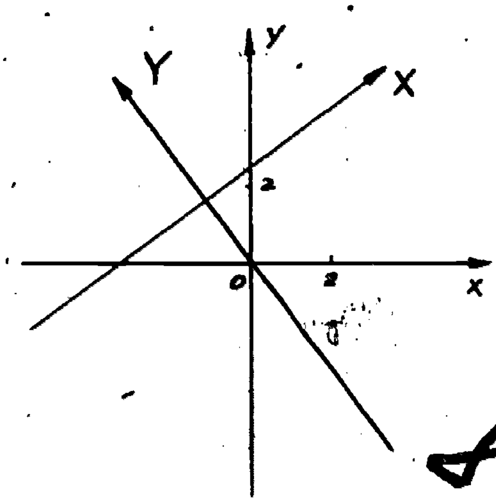
$\delta = 0 \quad \Delta = 0$

Rotate through $\arccos \frac{4}{5}$

Translate $Y = y - 2$, $X = x$

$Y = 0$

coincident lines



7. $5x^2 + 6xy + 5y^2 - 16x - 16y + 8 = 0$

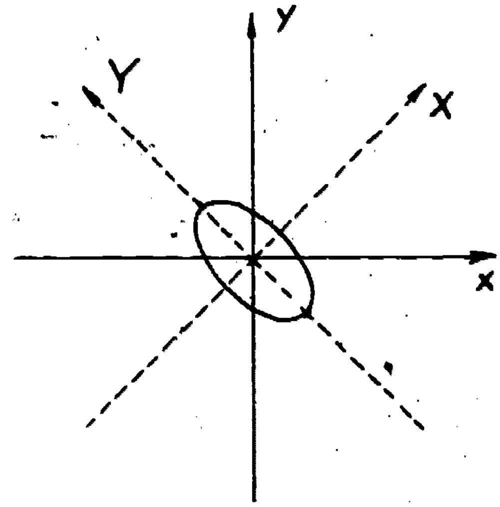
$\delta = 64 \quad \Delta = -1024$

Rotate through 45°

Translate $X = x - \sqrt{2}$; $Y = y$

$4X^2 + Y^2 = 4$

ellipse



6. $3x^2 + 10xy + 3y^2 + 16x + 16y + 24 = 0$

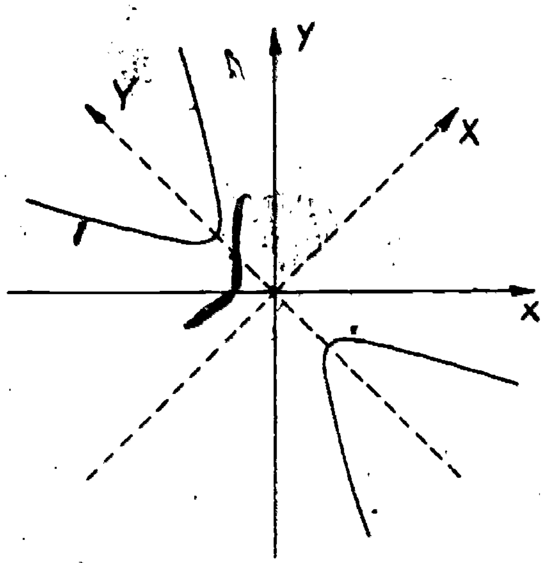
$\delta = -64 \quad \Delta = 512$

Rotate through 45°

Translate $Y = y$, $X = x + \sqrt{2}$

$Y^2 - 4X^2 = 4$

hyperbola



8. $27x^2 - 48xy + 13y^2 - 12x + 44y - 77 = 0$

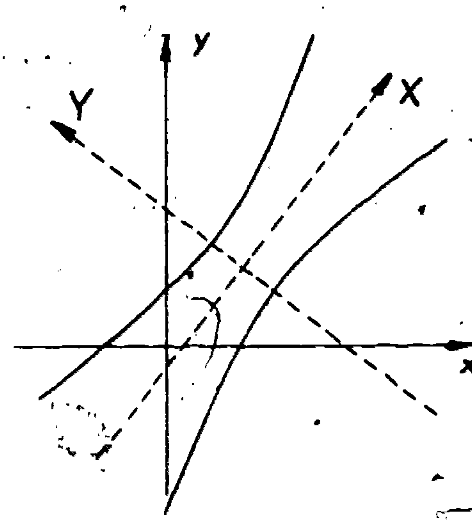
$\delta = -900 \quad \Delta = -196200$

Rotate through $\arccos \frac{3}{5}$

Translate $X = x - \frac{14}{5}$, $Y = y + \frac{2}{5}$

$9Y^2 - X^2 = 9$

hyperbola



9. $12x^2 - 7xy - 12y^2 - 41x + 38y + 22 = 0$

$\delta = -625 \quad \Delta = 0$

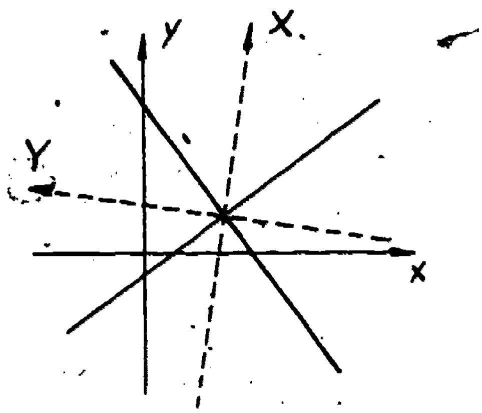
Rotate through arccos $\frac{1}{5\sqrt{2}}$

Translate $X = x - \frac{9}{5\sqrt{2}}$,

$Y = y + \frac{13}{5\sqrt{2}}$

$(X + Y)(X - Y) = 0$

Intersecting lines



10. $13x^2 + 48xy + 27y^2 + 44x + 12y - 77 = 0$

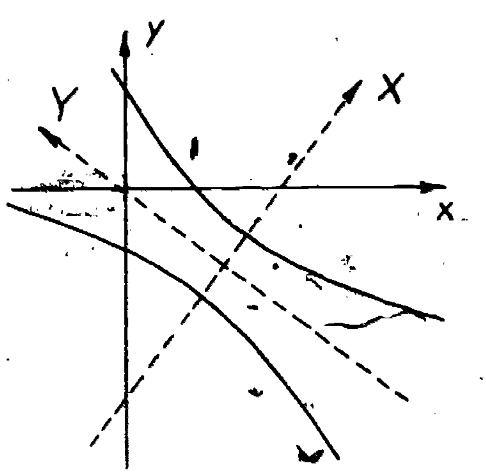
$\delta = -900 \quad \Delta = -196200$

Rotate arccos $\frac{3}{5}$

Translate $X = x + \frac{2}{5}$, $Y = y + \frac{14}{5}$

$9X^2 - Y^2 = 9$

hyperbola



11. $9x^2 - 24xy + 16y^2 + 90x - 120y + 200 = 0$

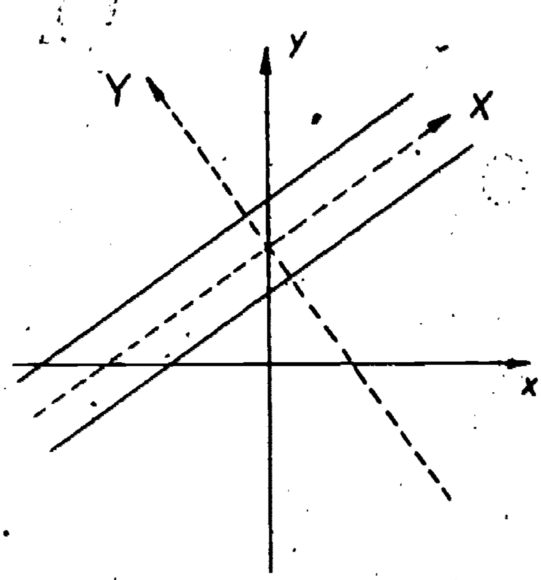
$\delta = 0 \quad \Delta = 0$

Rotate through arccos $\frac{4}{5}$.

Translate $X = x$, $Y = y - 3$

$(Y - 1)(Y + 1) = 0$

Parallel lines



12. $10xy + 4x - 15y - 6 = 0$

$\delta = -100 \quad \Delta = 0$

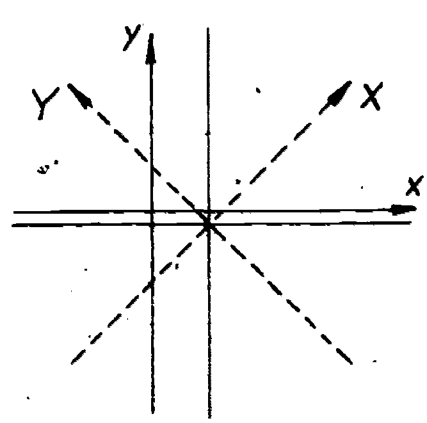
Rotate 45°

Translate $X = x - \frac{11\sqrt{2}}{20}$,

$Y = y + \frac{19\sqrt{2}}{20}$

$(X + Y)(X - Y) = 0$

intersecting lines



Teachers' Commentary

Chapter 4

Supplement to Chapter 10

GEOMETRIC TRANSFORMATIONS

In a sense, this chapter can be thought of as a review of the early chapters. It is essentially a summary of the various treatments of transformations, but now they are observed from a more sophisticated point of view. The concepts of mappings and groups constitute the background for the discussion.

The writers would be interested in knowing how the teachers feel about including this type of material and also, if it is included, whether it should come earlier in the presentation--perhaps even near the front of the book.

Exercises S10-2

1. The reflection about the $x = 1$ line is $(x,y) \longrightarrow (x',y') = (-x+2,y)$.
   The reflection about the $x = 4$ line is $(x',y') \longrightarrow (x'',y'') = (-x'+8,y')$.
   Taking $x = 1$ then $x = 4$ we get

   $$\boxed{x'' = x + 6,\ y'' = y}$$

   Taking $x = 4$ then $x = 1$ we get

   $$\boxed{x'' = -x' + 2 = -(-x + 8) + 2 = x - 6} \quad \boxed{y'' = y}$$

   So they don't commute.

2. Mapping of reflection about $x = h$

$$\boxed{(x,y) \longrightarrow (x',y') = (-x + 2h, y)}$$

Mapping of reflection about $y = k$

$$\boxed{(x,y) \longrightarrow (x',y') = (x, -y + 2k)}$$

3. Two successive reflections about horizontal lines:

$(x,y) \longrightarrow (x',y') = (x, -y + 2k)$, $(x',y') \longrightarrow (x'',y'') = (x', -g' + 2n)$

$x'' = x' = x$ $\quad \boxed{x'' = x}$

$y'' = -y' + 2n = \boxed{y + 2(n - k) = y''}$

Two successive reflections about vertical lines:

$(x,y) \longrightarrow (x',y') = (-x + 2h, y)$, $(x',y') \longrightarrow (x'', y'') = (-x' + 2m, y')$

$x'' = -x' + 2m = \boxed{x + 2(m - h) = x''}$

$y'' = y' = y \quad \boxed{y'' = y}$

4. $(x,y) \longrightarrow (x',y') = (-x + 2h)y$, $(x',y') \longrightarrow (x'',y'') = (x', -y' + 2k)$

$x'' = x' =$ $\quad \boxed{-x + 2h = x''}$

$y'' = -y' + 2k = \boxed{-y + 2k = y''}$

5. The mappings in (3) will commute only if $k = n$ and $h = m$.

The mappings in (4) will commute.

Exercises 10-3

1. Suppose they have the rotation

$$\phi'' = \phi + 2(\theta_2 - \theta_1)$$

$$r'' = r$$

Then rewrite

$$\phi'' = 2\theta_2 - (2\theta_1 - \phi)$$

$$r'' = r$$

Then let $r = r'$ and $2\theta_1 - \phi = \phi'$ and we have $\phi'' = 2\theta_2 - \phi'$, $r'' = r$.

Then we see that the rotation is the product of the line reflections

$(r,\phi) \longrightarrow (r',\phi') = (r, 2\theta_1 - \phi)$ and

$(r',\phi') \longrightarrow (r'',\phi'') = (r', 2\theta_2 - \phi')$

2. $R_L R_M$ where $R_m : (r,\phi) \longrightarrow (r',\phi') = (r, 2\theta_2 - \phi)$

$R_L : (r',\phi') \longrightarrow (r'',\phi'') = (r', 2\theta_1 - \phi')$

$\phi'' = 2\theta_1' - \phi' =$ $\boxed{\begin{array}{r} \phi + 2(\phi_1 - \phi_2) = \phi'' \\ r = r'' \end{array}}$

$r'' = r$

Exercises S10-4

1. $(x,y) \xrightarrow{J} (x',y') = (ax + by, cx + dy)$ where $ad - bc \neq 0$

Now solve for x and y in terms of x' and y'.

Then $y = \dfrac{cx' - ay'}{bc - ad}$ and $x = \dfrac{dx' - by'}{ad - bc}$.

Now substitute these into the line $kx + \ell y + m = 0$ and we see that

$$kdx' - kby' + \ell cx' - \ell ay' + m = 0$$

or

$$(kd + \ell c)x' + (-kb - \ell a)y' + m = 0$$

which means that any transformation of the group in Theorem S10-3 will map a line into a line.

2. (a) $(x,y) \longrightarrow (2x,2y)$

$x' = 2x$, $y' = 2y$

$x'^2 + y'^2 = 4(x^2 + y^2)$ so the circle $x^2 + y^2 = 1$ maps into $x'^2 + y'^2 = 4$.

(b) $(x,y) \longrightarrow (2x,3y)$

$x' = 2x$, $y' = 3y$

$x^2 + y^2 = \frac{1}{4}x'^2 + \frac{1}{9}y'^2 = 1$ so the circle $x^2 + y^2 = 1$

maps into the ellipse $\frac{1}{4}x'^2 + \frac{1}{9}y'^2 = 1$

3. $(x,y) \longrightarrow (x',y') = (x + y, 2x + 2y)$

$x' = x + y$, $y' = 2x + 2y$

Consider the point a, $2a$ on $2x = y$, then $a = x + y$ and $2a = 2x + y$ so all points mapped into a point on $2x = y$ satisfy the equation $x + y - a = 0$. This is the equation of a line.

4. Show that the angle is preserved between two lines through the origin under $z \longrightarrow z' = kz$.

Let $z = r(\cos\theta + i\sin\theta)$, then let L_1 be $r(\cos\theta_1 + i\sin\theta_1)$ and L_2 be $r(\cos\theta_2 + i\sin\theta_2)$. Now the angle between L_2 and L_1 will simply be $|\theta_2 - \theta_1|$. Under the mapping $L_1 \longrightarrow L_1'$ where L_1' is $Kr(\cos\theta_1 + i\sin\theta_1)$ and $L_2 \longrightarrow L_2'$ where L_2' is $Kr(\cos\theta_2 + i\sin\theta_2)$. So we see the angle between L_1' and L_2' again equals $|\theta_2 - \theta_1|$. Therefore the angle is preserved.

5. Discuss $z - z' = \frac{1}{z}$

$$z = x + iy, \quad \frac{1}{z} = z' = \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2}$$

so $x' = \dfrac{x}{x^2 + y^2}$ and $y' = \dfrac{-y}{x^2 + y^2}$, in non-linear coordinates.

Then the circles $(x - \frac{1}{k})^2 + y^2 = \frac{1}{4k^2}$ are mapped onto $x' = k$ and the circles $x^2 + (y + \frac{1}{k})^2 = \frac{1}{4k^2}$ are mapped onto $y' = k$.

Also we have $x'^2 + y'^2 = \dfrac{x^2 + y^2}{(x^2 + y^2)^2} = \dfrac{1}{x^2 + y^2}$, hence the circles $x^2 + y^2 = r$ are mapped onto the circles $x'^2 + y'^2 = \frac{1}{r}$, in the z' plane.

6. (a) It is simplest to consider this problem in polar coordinates then the solution is $(r,\phi) \longrightarrow (r',\phi') = (\frac{1}{r},\phi')$ where the origin is defined to map onto the origin.

(b) A second form would be $(x,y) \longrightarrow (x',y') = \left(\dfrac{1}{x(1 + a^2)}, y\right)$ where $y = ax$ is the line involved. Again the origin would have to be defined as mapping onto the origin.

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Exercises S10-5a

1. $R_x \quad R_y$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. (a) Reflection about $y = x$

$x' = y = 0 \cdot x + 1 \cdot y$
$y' = x = 1 \cdot x + 0 \cdot y$

$$\boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(b) Reflection about $y = -x$

$x' = -y = 0 \cdot x + -1 \cdot y$
$y' = -x = -1 \cdot x + 0 \cdot y$

$$\boxed{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$$

3. Reflection in $y = x$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

rotation $\frac{\pi}{2}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

composition is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

4. $$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 , & -\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 \\ \cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1 , & \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

This mapping is the same as a mapping of a single rotation through $\theta_1 + \theta_2$ radians.

5. $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \left[\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \right] = k$

$$K = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1c_1 + b_2c_2 & b_1c_2 + b_2c_4 \\ b_3c_1 + c_3b_4 & b_3c_2 + b_4c_4 \end{pmatrix}$$

$$K = \begin{pmatrix} a_1b_1c_1 + a_1b_2c_3 + a_2b_3c_1 + a_2c_3b_4 & a_1b_1c_2 + a_1b_2c_4 + a_2b_3c_2 + a_2b_4c_4 \\ a_3b_1c_1 + a_3b_2c_3 + a_4b_3c_1 + a_4c_3b_4 & a_3b_1c_2 + a_3b_2c_4 + a_4b_3c_2 + a_4b_4c_4 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right] \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = K'$$

$$K' = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$K' = \begin{matrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & a_1b_1c_2 + a_2b_3c_2 + a_1b_2c_4 + a_2b_4c_4 \\ a_3b_1c_1 + a_4b_3c_1 + a_3b_2c_3 + a_4b_4c_3 & a_3b_1c_2 + a_4b_3c_2 + a_3b_2c_4 + a_4b_4c_4 \end{matrix}$$

and so we see that $K = K'$ and matrix multiplication is associative.

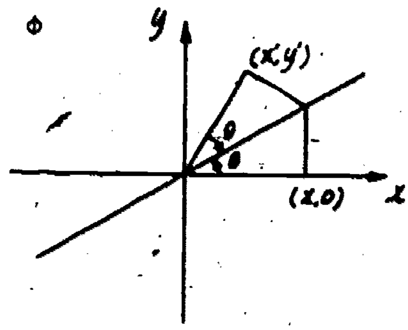
$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = L$$

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} b_1a_1 + b_2a_3 & b_2a_1 + b_4a_2 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{pmatrix} = L'$$

and so we see that $L \neq L'$ hence matrix multiplication doesn't commute.

6. In polar coordinates

$r' = r$ and $\phi' = 2\theta - \phi$



$x' = r\cos(2\theta - \phi) = r\cos\phi\cos 2\theta + r\sin\phi\sin 2\theta = x\cos 2\theta + y\sin 2\theta$

$y' = r\sin(2\theta - \phi) = r\sin 2\theta\cos\phi - r\cos 2\theta\sin\phi = x\sin 2\theta - y\cos 2\theta$

hence the matrix is:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

When $\theta = 0$, we get $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which was previously shown to be a reflection about the x-axis, when $\theta = \frac{\pi}{4}$ we get $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which was previously shown to be a reflection in $y = x$, when $\theta = \frac{\pi}{2}$ we get $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which is a reflection in the y-axis, when $\theta = \frac{3\pi}{4}$ we get $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which is a reflection in the $y = -x$-axis.

7. $$\begin{pmatrix} \cos 2\theta_2 & \sin 2\theta_2 \\ \sin 2\theta_2 & -\cos 2\theta_2 \end{pmatrix} \cdot \begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ \sin 2\theta_1 & -\cos 2\theta_1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos 2\theta_2 \cos 2\theta_1 + \sin 2\theta_2 \sin 2\theta_1 & \cos 2\theta_2 \sin 2\theta_1 - \cos 2\theta_1 \sin 2\theta_2 \\ \cos 2\theta_1 \sin 2\theta_2 - \cos 2\theta_2 \sin 2\theta_1 + \sin 2\theta_1 \sin 2\theta_2 + \cos 2\theta_1 \cos 2\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2(\theta_2 - \theta_1) & -\sin 2(\theta_2 - \theta_1) \\ \sin 2(\theta_2 - \theta_1) & \cos 2(\theta_2 - \theta_1) \end{pmatrix}$$

This is the matrix of a rotation where $\theta = 2(\theta_2 - \theta_1)$

Exercises S10-5b

1. $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ or $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

By Problem 7 (S10-5a) we saw that the product of two matrices of the form $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ is of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

By Problem 4 (S10-5a) we saw that the product of two matrices of the form $\begin{matrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha \end{matrix}$ is another matrix of the same form.

We see that the product $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & +\cos\alpha \end{pmatrix}$

is of the form $\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$.

Finally $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & +\sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$ is of the form

$\begin{pmatrix} \cos\alpha+\beta & \sin\alpha+\beta \\ \sin\alpha+\beta & -\cos\alpha+\beta \end{pmatrix}$.

Hence we see that the matrix multiplication is closed. From Problem 5 (S10-5a) we see that the multiplication obeys the associative law, and because $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ is included in this set and it is the identity matrix, that this set forms a group.

2. $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$

$$\begin{vmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_2b_1 + a_2b_3 & a_3b_2 + a_4b_4 \end{vmatrix} = (a_1b_1 + a_2b_3)(a_3b_2 + a_4b_4) - (a_3b_1 + a_4b_3)(a_1b_2 + a_2b_4)$$

$$= a_1b_1a_4b_4 + a_2a_3b_2b_3 - a_2a_3b_1b_4 - a_1a_4b_2b_3$$

$$= (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3)$$

$$= \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}.$$

3. The matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ isn't an isometry as the vector $(0,1) \longrightarrow (2,1)$ and hence distance isn't preserved, yet the det = 1

4. The matrix must be of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ or $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ by Theorem 10-5.

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1$$

Hence the det of the matrix that represents an isometry is 1 or -1.

5. If $\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = \pm 1$ then $a_1a_4 - a_2a_3 = \pm 1$; also, we have $a_1^2 = a_3^2 = 1$, $a_1^2 + a_2^2 = 1$, $a_3^2 + a_4^2 = 1$ and $a_2^2 + a_4^2 = 1$. Now, if the sum of two squares = 1, the numbers can be written as sin and cos of some angle θ. Hence we have $a_1 = \pm\sin\alpha$ or $\pm\sin\alpha$, $a_2 = \pm\cos\alpha$ or $\pm\sin\alpha$, $a_3 = \pm\sin\alpha$ or $\pm\cos\alpha$, $a_4 = \pm\cos\alpha$ or $\pm\sin\alpha$. Now, from these, we obviously get matrices that belong to S but we get other as well:

$a_3 = \pm\sin\alpha$ or $\pm\cos\alpha$, $a_4 = \pm\cos\alpha$ or $\pm\sin\alpha$. Now from these we obviously get matrices that belong to S but we get others as well:

$\begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, $\begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$, $\begin{pmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix}$

$\begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$, and $\begin{pmatrix} -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$. All of these cases can be reduced to members of S by letting $\alpha = -\beta$, $\alpha = \beta + \frac{\pi}{2}$ or $\alpha = \beta + \pi$. Hence, these conditions are enough to make the matrix belong to S.

Exercises S10-6

1. Answers given in text

2. Answers given in text

3. I-1 Reflection in x-y plane

I-2 Reflection in y-z plane

I-3 Reflection in x-z plane

I-4 Identity

I-5 Reflection in plane through x-axis with 45° to y-axis

I-6 Reflection in plane through y-axis with 45° angle to z-axis

I-7 Reflection in plane through z-axis with 45° angle to x-axis

I-8 Reflection in plane through x-axis with 135° angle to y-axis

I-9 Reflection in plane through y-axis with 135° angle to z-axis

I-10 Reflection in plane through z-axis with 135° angle to x-axis