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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) four finite geometries: (2) miniature geometries: (3) a coordinate approach to the 25-point miniature geometry: and (4) 25-point geometry. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

PREFACE

Any mathematical system is characterized by its undefined elements, its unproved postulates, and its definitions and theorems. A finite geometry is a geometry in which the set of postulates and undefined terms and relations are such that the system has only a finite number of points and a finite number of lines. Of course the words "point" and "line" as then used take on a somewhat different meaning from the classical Euclidean concepts of point and line.

Many different finite geometries are possible. The characteristics of any finite geometry are determined solely by the particular set of postulates chosen. Such geometries illustrate the basic logical structure of a mathematical system. Consider for example a simple three-point geometry in which the undefined terms are *point* and *line*, and in which the synonymous phrases "point on a line" and "line on a point" constitute an undefined relation. We may agree upon the four postulates:

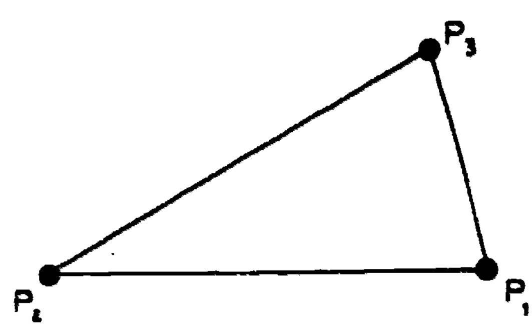
A_1. There exist exactly three points.

A_2. Not all points are on the same line.

A_3. On any two distinct points there is exactly one line.

A_4. On any two distinct lines there is at least one common point.

This trivial geometry may be represented by the following figure, and on the basis of the aforementioned postulates, A_1 — A_4, it is easy to prove these theorems:



T_1. Not all lines are on the same point.

T_2. On any two distinct lines there is not more than one point in common.

T_3. There exists three and only three distinct lines.

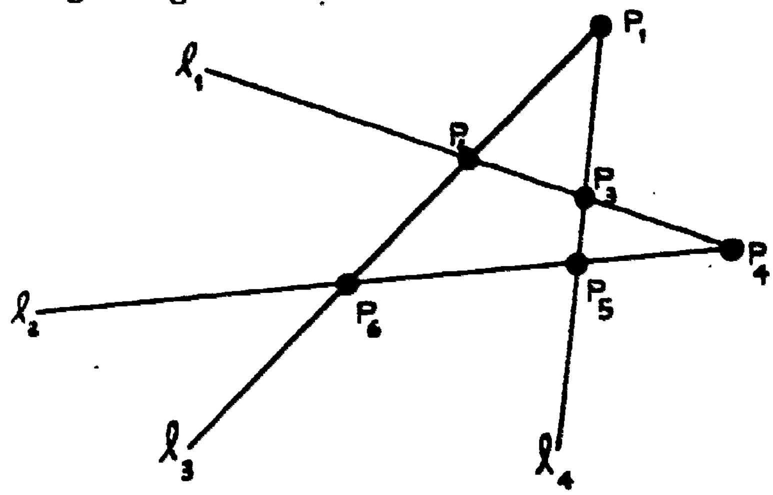
Interestingly enough, even though it is trivial, this geometry exhibits the property of *duality*, which means that if in any true statement we interchange the words "point" and "line," the new statement is also true.

Let us look briefly at another finite geometry, not quite as simple as the one above. This time we shall agree upon five postulates:

A_1. Each pair of lines is on at least one point.

A_2. Each pair of lines is on not more than one point.
A_3. Each point is on at least two lines.
A_4. Each point is on not more than two lines.
A_5. There exist exactly four lines.

This six-point geometry or mathematical model may be "interpreted" by the following configuration:



Among the theorems which can be derived from these five postulates we mention:

T_1. Exactly two lines pass through (lie on) each point.
T_2. Every line contains (lies on) exactly three points.
T_3. Not all lines pass through the same point.
T_4. Two distinct lines have exactly one point in common.

A system (S) consisting of the postulates A_1 —A_5, together with the theorems T_1 — T_4, among others, may be thought of as a mathematical model of some concrete situation. Instead of the above space configuration, the system (S) could just as readily be used to represent the officers of a corporation and its major departments. For example, the six "points" might represent the Chairman, President, Vice-president, Secretary, Treasurer, and Comptroller; and the four "lines" might represent the Manufacturing, Sales, Advertising and Finance Departments. All the statements made above about "points" and "lines" would then be consistent when the words "officer" and "department" are substituted for the words "point" and "line," respectively.

So an entirely new field has been opened up, and as you become more and more acquainted with these finite geometries you will see that much more than just "geometry" is involved—the theory of numbers, higher algebra, and the theory of groups—as the following essays will reveal.

—William L. Schaaf

CONTENTS

ACKNOWLEDGMENTS

The School Mathematics Study Group takes this opportunity to express its gratitude to the authors of these articles for their generosity in allowing their material to be reproduced in this manner. At the time the late Professor MacNeish published his paper he was Chairman of the Department of Mathematics of Brooklyn College, now an integral part of the City University of New York. Professor Burton W. Jones is associated with the University of Colorado at Boulder, Colorado. Professor Arthur F. Coxford, Jr. is associated with the University of Michigan at Ann Arbor, Michigan. Martha Heidlage is associated with Mount St. Scholastica College at Atchison, Kansas. Mr. H. Martyn Cundy, a Mathematical Master at Sherborne School (England), is well known in British mathematical education circles and is co-author with A. P. Rollett of a distinguished book entitled "*Mathematical Models*."

The School Mathematics Study Group at this time also wishes to express its appreciation to the several editors and publishers who have been kind enough to permit these articles to be published in this manner, namely:

THE MATHEMATICS TEACHER:

(1) Martha Heidlage, *A Coordinate Approach to the 25-Point Miniature Geometry*, vol. 58, pp. 109–113; February, 1965.

(2) Burton W. Jones, *Miniature Geometries*, vol. 52, pp. 66–71; February, 1959.

AMERICAN MATHEMATICAL MONTHLY:

(1) H. F. MacNeish, *Four Finite Geometries*, vol. 49, pp. 15–23; January, 1942.

MATHEMATICAL GAZETTE (England):

(1) H. Martyn Cundy, *25-Point Geometry*, vol. 36, pp. 158–166; September, 1952.

FOREWORD

These essays require somewhat greater mathematical maturity than most of the other pamphlets in this Series. Frankly, to understand and enjoy them, the reader should be at least somewhat familiar with modular arithmetic and congruences, and with the general nature of projective geometry.

The first essay, *"Four Finite Geometries,"* serves to set the stage by giving the reader somewhat of a perspective, both historical and mathematical. It should be noted that the author gives several examples of finite geometries, and that one need not read all of them.

The second essay, by Burton Jones, is based in large part on the concept of modular congruences and Galois fields, which are discussed in another article by the same author, entitled *"Miniature Number Systems"* (MATHEMATICS TEACHER, 51:226–231, April 1958). The reader unacquainted with these concepts or unfamiliar with the article referred to, may still understand a goodly part of the present essay.

In her paper on the 25-point geometry, Miss Heidlage uses the analytic approach, which, interestingly enough, reveals the intimate connection between finite geometries and modular arithmetic.

The concluding essay by H. M. Cundy uses a different approach, namely, that of geometric transformations. By so doing, he emphasizes some of the properties of configurations in a 25-point geometry, including not only conic sections, but also such matters as inverse points, the Simson line, the nine-point circle, and the projective plane. Here again (as in the preceding essay), little or no attention has been given to the question of the consistency and independence of the axioms.

The reader interested in pursuing the subject further may find the following references of interest:

Arthur Coxford, *"Geometric Diversions: A 25-Point Geometry."* The MATHEMATICS TEACHER 57:561–564; December, 1964.

W. L. Edge, *"31-Point Geometry."* MATHEMATICAL GAZETTE 39:113–121; May, 1955.

Martha Heidlage, *"A Study of Finite Geometry."* THE PENTAGON, vol. 23, pp. 18–27; Fall, 1963.

FOUR FINITE GEOMETRIES*

H. F. MacNeish

1. Introduction. A finite geometry is a geometry based on a set of postulates, undefined terms, and undefined relations which limits the set of all points and lines to a finite number. This is usually accomplished by a postulate limiting the number of points on a line. In the first three of the finite geometries considered in this paper there is the following postulate: "No line contains more than three points." In the Desargues finite geometry this is a theorem which follows from the postulates. The set of postulates should fulfill the three requirements of consistency, independence, and categoricalness.

Finite geometries were brought into prominence by the publication of the Veblen and Young *Projective Geometry* (Ginn and Co., Vol. I, 1910; Vol. II, 1918) and by Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (The Macmillan Co., 1911).

The simplest example is the finite geometry of 7 points and 7 lines given in Volume I, Chapter I of the *Projective Geometry* of Veblen and Young. This finite geometry was first considered by Fano in 1892 in 3 dimensions where there are 15 points and 35 lines, but in each plane there are 7 points and 7 lines.

The notion of a class of objects is fundamental in logic. The objects which make up a class are called the elements of the class. The notion of a class and the relation "belonging to a class" will be undefined. Given a set S with elements $A_1, A_2, A_3, \cdots$, let S have certain undefined sub-classes any one of which will be called an m-class, — or in particular, given a set of points $A_1, A_2, A_3, \cdots$, let certain sets of points be associated in an undefined way in sets called lines.

2. The seven point finite geometry. The postulates for the 7 point finite geometry may be stated as follows:

(1′). If A_1 and A_2 are distinct points (elements of S), there is at least one line (m-class) containing A_1 and A_2.

(2′). If A_1 and A_2 are distinct points (elements of S), there is not more than one line (m-class) containing A_1 and A_2.

(3′). Any two lines (m-classes) have at least one point (element of S)

* Presented at the organization meeting of the Metropolitan New York Section of the Mathematical Association of America at Queens College of the City of New York on April 19, 1941.

in common.

(4′). There exists at least one line (m-class).

(5′). Every line (m-class) contains at least three points (elements of S).

(6′). All the points (elements of S) do not belong to the same line (m-class).

(7′). No line (m-class) contains more than three points (elements of S).

Symbolic diagram where the vertical columns represent lines (m-classes):

$$
\begin{matrix}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\
A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_1 \\
A_4 & A_5 & A_6 & A_7 & A_1 & A_2 & A_3
\end{matrix}
$$

It is, however, unfortunate that it is necessary to assume that the three points A_1, A_3, and A_7, i.e., the diagonial points of the complete quadrangle, are collinear as indicated by the dotted line.

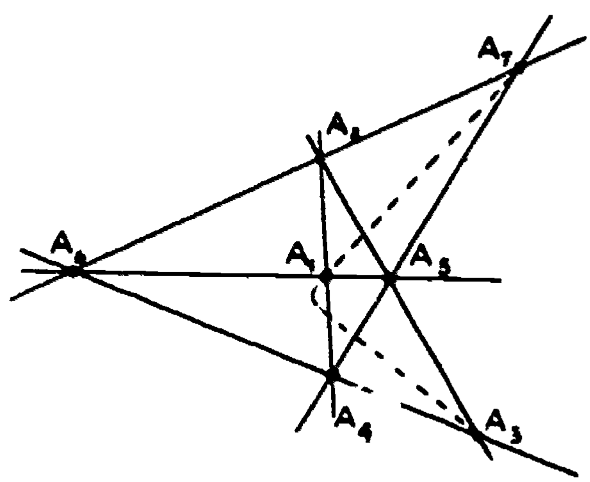


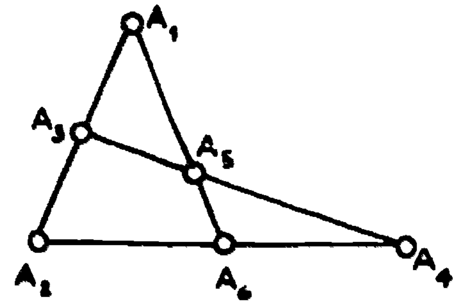
FIG. 1. Geometric diagram for 7 point finite geometry.

To prove that a postulate of a set of postulates is independent of the rest, it is sufficient to give an example which violates that postulate and fulfills all the rest. Since there are seven postulates in this set, it is necessary to give seven examples to complete the independence proof. This is not always easy to do, and in the case of Hilbert's postulates twenty-one examples would be necessary to complete the independence proof.

If the word "three" were changed to "two" in postulates 5 and 7, the entire geometry would consist of a single triangle which might well be considered as the simplest non-trivial finite geometry; so that seven independent postulates in this case define a geometry consisting of just one triangle.

The independence of the postulates for this finite geometry is shown by the following examples, whose numbers correspond to the number of the postulate which does not hold in the example.

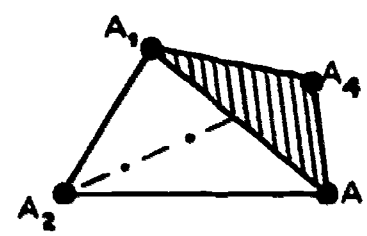
(1′). A complete quadrilateral.



$A_1A_1A_2A_3$
$A_2A_5A_6A_4$
$A_3A_6A_4A_5$

(2′). A tetrahedron $A_1A_2A_3A_4$, where the faces represent lines.

$A_1A_1A_1A_2$
$A_2A_2A_3A_3$
$A_3A_4A_4A_4$



(3′).

$A_1A_1A_1A_1A_2A_2A_2A_3A_3A_3A_4A_7$
$A_2A_4A_5A_6A_5A_4A_6A_4A_5A_6A_5A_8$
$A_3A_7A_9A_8A_8A_9A_7A_8A_7A_9A_6A_9$

(This is the Young finite geometry of 9 points and 12 lines, section 3.)

(4′). A single point (the remaining postulates are fulfilled vacuously).

(5′). A triangle with A_1, A_2, A_3 as vertices.

(6′). A single line containing 3 points A_1, A_2, A_3.

(7′). Projective geometry.

This seven point finite geometry has been generalized to give a finite geometry of thirteen points and thirteen lines if four is substituted for three in postulates 5 and 7; and to n^2+n+1 points and lines, if $n+1$ is substituted for three in postulates 5 and 7.

Finite geometries of this type have been treated extensively from the standpoint of algebra and finite groups.*

The question now arises as to what theorems there are in this seven point finite geometry. In the first place, the duals of the postulates may be proved as theorems and the geometry will then have duality. Postulates 1 and 3 are duals.

THEOREM 1. (Dual of Postulate 2). *Two distinct lines have only one point in common.*

THEOREM 2. (Dual of Postulate 4). *There exists at least one point.*

THEOREM 3. (Dual of Postulate 5). *At least three lines pass through every point.*

THEOREM 4. (Dual of Postulate 6). *All lines do not pass through the same point.*

* Veblen and Bussey, Finite projective geometries, Transactions of the American Mathematical Society, vol. 7, 1906, pp. 241–259.

THEOREM 5. (Dual of Postulate 7). *Not more than three lines pass through every point.*

These theorems are all easy to prove, and they show that the geometry has duality. Two other theorems suggest themselves.

THEOREM 6. *The geometry contains precisely seven points.*

THEOREM 7. (Dual of Theorem 6). *The geometry contains precisely seven lines.*

The entire body of theorems of this finite geometry of seven points and seven lines consists primarily of these seven theorems. The finite geometry has the characteristics of a projective geometry and might be considered as the simplest type of a projective geometry.

3. A FINITE GEOMETRY OF NINE POINTS AND TWELVE LINES. If in the postulates of section 2, the word "three" is changed to "four" in postulates 5 and 7, the postulates are satisfied by a finite geometry of 13 points and 13 lines. But, if in this geometry one line of four points is omitted, we obtain a geometry of 9 points and 12 lines, which is equivalent to projecting one line to infinity and converting the projective geometry of 13 points and 13 lines without parallel lines into a euclidean geometry of 9 points and 12 lines with parallel lines.

This is in some ways an advantage and a simplification, because in general there is a preference — for historical reasons — for a euclidean geometry in which the parallel postulate is true. This geometry has been used as an example of a complete logical system by Cohen and Nagel in their book, *An Introduction to Logic and Scientific Method* (Harcourt, Brace and Co., 1934).

It is remarkable that the 9 inflection points of a general plane cubic, as far as collinearity properties are concerned, fulfill all of the postulates of this finite geometry.

The following eight postulates define this finite geometry:

(1). If A_1 and A_2 are distinct points (elements of S), there exists one line (m-class) containing A_1 and A_2.

(2). If A_1 and A_2 are distinct points (elements of S), there exists not more than one line (m-class) containing A_1 and A_2.

(3). Given a line a (m-class a) not containing a point A (given element A of S), there exists one line (m-class) containing A and not containing any point of a (element of S belonging to m-class a).

(4). Given a line a (m-class a) not containing a point A (given element of S), there exists not more than one line (m-class) containing A and not containing any point of a (element of S belonging to m-class a).

(5). Every line (m-class) contains at least three points (elements of S).

(6). Not all points (elements of S) are contained by the same line (m-class).

(7). There exists at least one line (m-class).

(8.) No line (m-class) contains more than three points (elements of S).

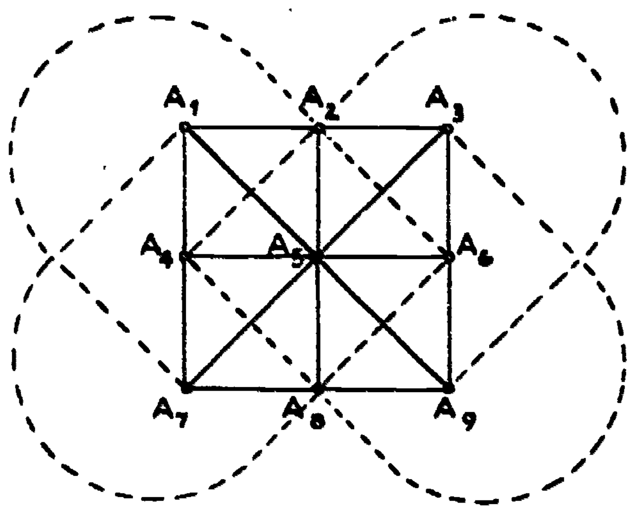


FIG. 2. Geometric diagram for the Young 9 point finite geometry.

Symbolic diagram where the vertical columns represent m-classes (lines):

$$\begin{matrix} A_1 A_1 A_1 A_1 A_2 A_2 A_2 A_3 A_3 A_3 A_4 A_7 \\ A_2 A_4 A_5 A_6 A_5 A_4 A_6 A_4 A_5 A_6 A_5 A_8 \\ A_3 A_7 A_9 A_8 A_8 A_9 A_7 A_8 A_7 A_9 A_6 A_9 \end{matrix}$$

The independence of the postulates is shown by the following examples,* where parentheses represent lines or m-classes.

(1'). Two lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

(2'). Six points (elements) A_1, A_2, A_3; A_4, A_5, A_6 taken three at a time to form twenty lines (m-classes).

$$\begin{matrix} (A_1A_2A_3), & (A_1A_2A_4), & (A_1A_2A_5), & (A_1A_2A_6), & (A_1A_3A_4), \\ (A_1A_3A_5), & (A_1A_3A_6), & (A_1A_4A_5), & (A_1A_4A_6), & (A_1A_5A_6), \\ (A_2A_3A_4), & (A_2A_3A_5), & (A_2A_3A_6), & (A_2A_4A_5), & (A_2A_4A_6), \\ (A_2A_5A_6), & (A_3A_4A_5), & (A_3A_4A_6), & (A_3A_5A_4), & (A_4A_5A_6). \end{matrix}$$

(This is a complete 6-point in space, where the planes represent lines.)

(3'). $(A_1A_2A_4)$, $(A_2A_3A_5)$, $(A_3A_4A_6)$, $(A_4A_5A_7)$, $(A_5A_6A_1)$, $(A_6A_7A_2)$, $(A_7A_1A_3)$.

* See article by A. Barshop, Brooklyn College Mathematics Mirror, Issue no. VII, 1939, p. 14.

(This is the seven point finite geometry of section 2).

(4′). $(A_1 A_4 A_5)$, $(A_2 A_5 A_6)$, $(A_3 A_6 A_7)$, $(A_4 A_7 A_8)$, $(A_5 A_8 A_9)$,
$(A_6 A_9 A_{10})$, $(A_7 A_{10} A_{11})$, $(A_8 A_{11} A_{12})$, $(A_9 A_{12} A_{13})$, $(A_{10} A_{13} A_{14})$,
$(A_{11} A_{14} A_{15})$, $(A_{12} A_{15} A_1)$, $(A_{13} A_1 A_2)$, $(A_{14} A_2 A_3)$, $(A_{15} A_3 A_4)$,
$(A_1 A_3 A_9)$, $(A_2 A_4 A_{10})$, $(A_3 A_5 A_{11})$, $(A_4 A_6 A_{12})$, $(A_5 A_7 A_{13})$,
$(A_6 A_8 A_{14})$, $(A_7 A_9 A_{15})$, $(A_1 A_8 A_{10})$, $(A_2 A_9 A_{11})$, $(A_3 A_{10} A_{12})$,
$(A_4 A_{11} A_{13})$, $(A_5 A_{12} A_{14})$, $(A_6 A_{13} A_{15})$, $(A_1 A_7 A_{14})$, $(A_2 A_8 A_{15})$,
$(A_1 A_6 A_{11})$, $(A_2 A_7 A_{12})$, $(A_3 A_8 A_{13})$, $(A_4 A_9 A_{14})$, $(A_5 A_{10} A_{15})$.

(5′). A complete quadrilateral.

$$(A_1A_2),\ (A_1A_3),\ (A_1A_4),\ (A_2A_3),\ (A_2A_4),\ (A_3A_4).$$

(6′). A single line of three points $(A_1A_2A_3)$.
(7′). A single point A_1—no lines.
(8′). Plane euclidean geometry.

This finite geometry is euclidean in the sense that through any point not on a line there is one and only one line parallel to that line. The geometry does not have the property of duality because any two distinct points determine one line, but any two distinct lines do not determine a point since they may be parallel.

Several theorems suggest themselves, such as the following:

THEOREM 1. *There exist exactly nine points.*

THEOREM 2. *There exist exactly twelve lines.*

THEOREM 3. *Every line has precisely two lines parallel to it.*

THEOREM 4. *Two lines parallel to a third line are parallel to each other.*

THEOREM 5. *The six points on two parallel lines determine a hexagon such that the intersection points of opposite sides are collinear.* (Pappus-Pascal theorem).

4. THE PAPPUS FINITE GEOMETRY. The postulates of the Pappus finite geometry may be stated as follows:

(1). There exists at least one line (m-class).
(2). Not all points (elements of S) belong to the same line (m-class).
(3). Not more than one line (m-class) contains any two points (elements of S).
(4). Every line (m-class) contains at least 3 points (elements of S).
(5). No line (m-class) contains more than 3 points (elements of S).
(6). Given a line (m-class) and a point (element of S) not on it, there

exists a line (m-class) containing the given point (element of S) which has no point (element of S) in common with the first line (m-class).

(7). Given a line (m-class) and a point (element of S) not on it, there exists not more than one line (m-class) containing the given point (element of S) which has no point (element of S) in common with the first line (m-class).

(8). Given a point (element of S) and a line(m-class) not containing it, there exists a point (element of S) contained in the given line (m-class) which is not on any line (m-class) with the first point (element of S).

(9). Given a point (element of S) and a line (m-class) not containing it, there exists not more than one point (element of S) contained in the given line (m-class) which is not on any line (m-class) with the first point (element of S).

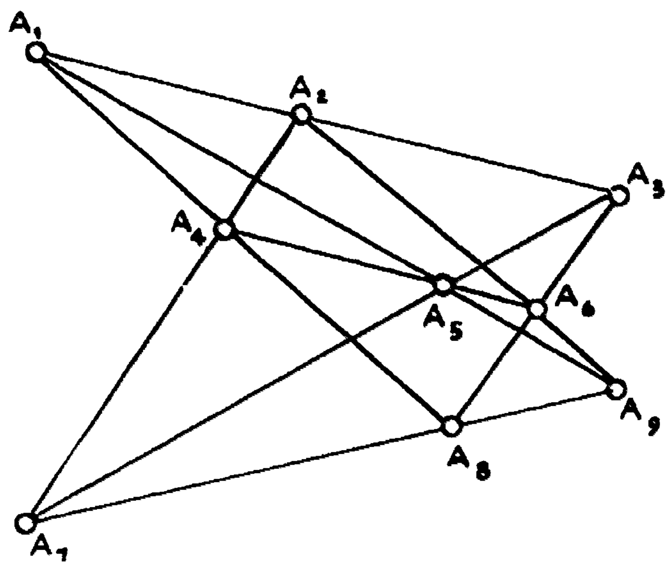
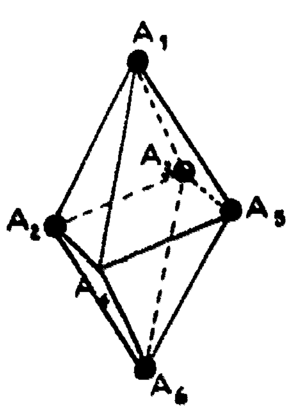


FIG. 3. Geometric diagram for the Pappus finite geometry.

Symbolic diagram where the vertical columns represent lines (m-classes):

$$\begin{matrix} A_1 & A_1 & A_1 & A_2 & A_2 & A_3 & A_3 & A_4 & A_7 \\ A_2 & A_4 & A_5 & A_4 & A_6 & A_5 & A_6 & A_5 & A_8 \\ A_3 & A_8 & A_9 & A_7 & A_9 & A_7 & A_8 & A_6 & A_9. \end{matrix}$$



The independence of the postulates is shown by the following examples:

(1′). A single point A_1.

(2′). A single line containing points A_1, A_2, A_3.

(3′). The faces of an octahedron, where the faces represent lines:

$$(A_1A_2A_3),\ (A_1A_2A_4),\ (A_1A_4A_5),\ (A_1A_3A_5),$$
$$(A_6A_2A_3),\ (A_6A_2A_4),\ (A_6A_3A_4),\ (A_6A_3A_5).$$

(4′). A simple quadrilateral (A_1A_2), (A_2A_3), (A_3A_4), (A_4A_1).

(5′). $(A_1A_5A_9A_{13})$, $(A_1A_6A_{10}A_{14})$, $(A_1A_7A_{11}A_{15})$, $(A_1A_8A_{12}A_{16})$,
$(A_2A_5A_{10}A_{16})$, $(A_2A_6A_9A_{15})$, $(A_2A_7A_{12}A_{14})$, $(A_2A_8A_{11}A_{13})$,
$(A_3A_5A_{11}A_{14})$, $(A_3A_6A_{12}A_{13})$, $(A_3A_7A_9A_{16})$, $(A_3A_8A_{10}A_{15})$,
$(A_4A_5A_{12}A_{15})$, $(A_4A_6A_{11}A_{16})$, $(A_4A_7A_{10}A_{13})$, $(A_4A_8A_9A_{14})$.

(6′). $(A_1A_2A_3)$, $(A_1A_5A_6)$, $(A_2A_4A_6)$, $(A_3A_4A_5)$.

(7′). $(A_1A_5A_9)$, $(A_1A_6A_{10})$, $(A_1A_7A_{11})$, $(A_1A_8A_{12})$,
$(A_2A_5A_{10})$, $(A_2A_6A_9)$, $(A_2A_7A_{12})$, $(A_2A_8A_{11})$,
$(A_3A_5A_{11})$, $(A_3A_6A_{12})$, $(A_3A_7A_9)$, $(A_3A_8A_{10})$,
$(A_4A_5A_{12})$, $(A_4A_6A_{11})$, $(A_4A_7A_{10})$, $(A_4A_8A_9)$.

(8′). The finite geometry of section 3.

(9′). Two non-intersecting straight lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

The Pappus finite geometry is treated in the book *Fundamentals of Mathematics* by Moses Richardson (Macmillan, 1941).

The duals of the postulates can be proved, showing that the geometry has duality. The geometry has the euclidean property of parallelism of lines. It also has dual property of parallelism of points.

DEFINITION. *Two points which are not connected by any line will be called parallel points.*

The most important theorems are the following:

THEOREM 1. *If the six points of two (parallel) lines are connected to form a hexagon, the opposite sides intersect in three collinear points.* (Pappus-Pascal theorem).

THEOREM 2. *There are precisely nine points in the geometry.*

THEOREM 3. (Dual of Theorem 2). *There are precisely nine lines in the geometry.*

The Pappus geometry contains no artificial lines and is associated with one of the simplest non-trivial configurations in geometry. The Master's thesis of John E. Darraugh (Brooklyn College, 1940) lists thirty-five theorems for the Pappus finite geometry.

5. THE DESARGUES FINITE GEOMETRY. The postulates of this geometry may be stated as follows:

(1). There exists a point (element of S).

(2). Two distinct points (elements of S) are contained by at most one line (m-class).

DEFINITION. *Line p is called a polar line of point P if no point of p is connected to P by a line. Point P is called a pole of line p if no line* *through P contains a point of p.*

(3). For every line p (m-class), there is at most one pole P.

(4). There are at least three distinct points (elements of S) on a line (m-class).

(5). For every point (element of S), there exists a polar line (m-class).

(6). If a line p (m-class) does not contain a given point Q (element of S), the polar line q of point Q has a point in common with line p.

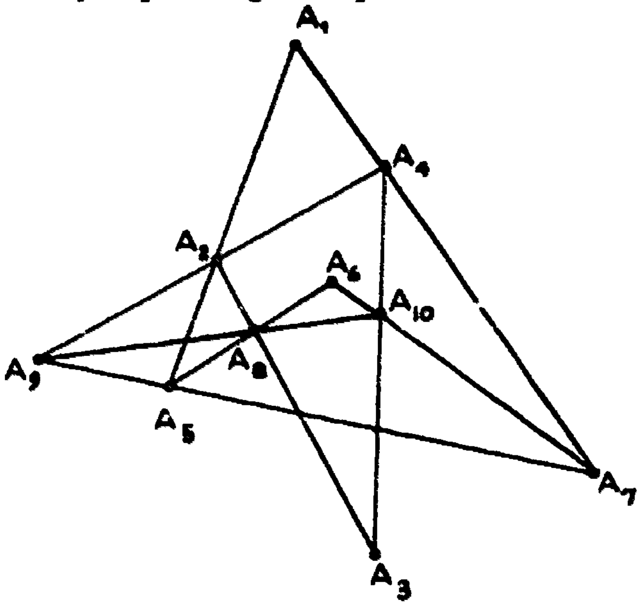


FIG. 4. Geometric diagram for the Desargues finite geometry.

Symbolic diagram where the vertical columns represent lines (m-classes):

$$\begin{matrix} A_1 & A_1 & A_1 & A_2 & A_2 & A_3 & A_5 & A_5 & A_6 & A_8 \\ A_2 & A_3 & A_4 & A_3 & A_4 & A_4 & A_6 & A_7 & A_7 & A_9 \\ A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_8 & A_9 & A_{10} & A_{10}. \end{matrix}$$

The independence of the postulates is shown by the following examples:

(1′). The null set — a geometry without points or lines.

(2′). $(A_1A_2A_3)$, $(A_1A_2A_7)$, $(A_1A_3A_8)$, $(A_2A_5A_7)$, $(A_3A_6A_8)$, $(A_4A_5A_6)$, $(A_4A_5A_7)$, $(A_4A_6A_8)$.

(3′) Two lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

(4′). A simple hexagon $A_1A_2A_3A_4A_5A_6$.

(A_1A_2), (A_2A_3), (A_3A_4), (A_4A_5), (A_5A_6), (A_6A_1).

(5′). A single point A_1.

$$(6'). \quad (A_1A_2A_6),\ (A_1A_3A_5),\ (A_1A_{10}A_{12}),\ (A_2A_3A_4),$$
$$(A_2A_{10}A_{11}),\ (A_3A_{11}A_{12}),\ (A_4A_5A_9),\ (A_4A_6A_8),$$
$$(A_5A_6A_7),\ (A_7A_8A_{12}),\ (A_7A_9A_{11}),\ (A_8A_9A_{10}).$$

The Desargues finite geometry has only six postulates, it has duality and polarity, it is non-euclidean in that a line may have as many as three lines parallel to it through a given point, and it is associated with a real configuration.

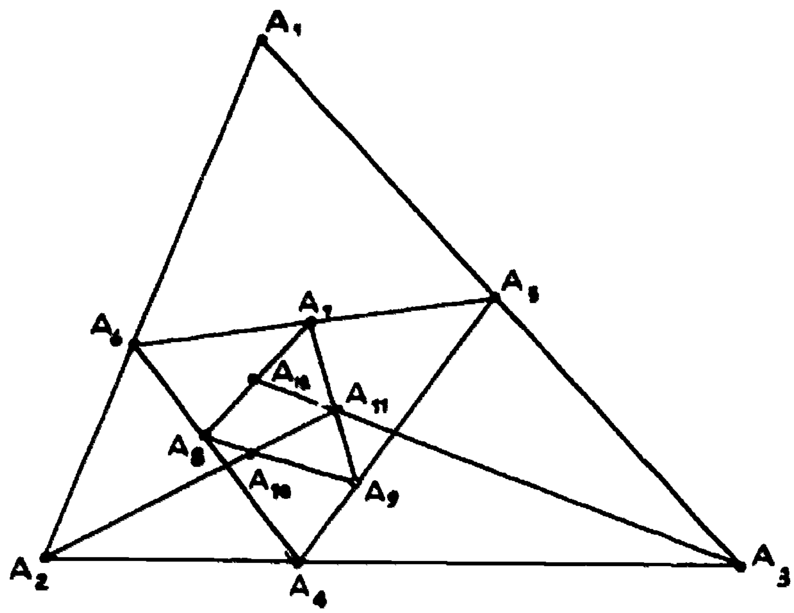


FIG. 5. Four triangles each inscribed in and circumscribed about another triangle of the set.

This finite geometry is treated in Part III, Chapter 1 of *Fundamental Mathematics* by Duncan Harkin (Prentice-Hall, 1941).

John E. Darraugh, in his Master's thesis (Brooklyn College, 1940) gives fifty-two theorems for this geometry. Among the most important theorems are the duals of the postulates, and also the following:

THEOREM 1. *If A lies on the polar line of B, then B lies on the polar line of A.*

THEOREM 2. *If b and c are both parallel to a, then b and c intersect in a point.*

THEOREM 3. *There exist precisely ten points.*

THEOREM 4. *There exist precisely ten lines.*

THEOREM 5. *If two triangles are perspective from a point, their corresponding sides intersect in collinear points.* (The Desargues theorem).

MINIATURE GEOMETRIES

Burton W. Jones

In a previous article[1] miniature number systems were developed—that is, number systems which have most of the usual properties of numbers with which we are familiar but which contain only a finite number of numbers. Similarly, there are geometries that are "miniature" in the sense that they contain only a finite number of points. Here, however, we are in the beginning forced to make a number of choices: Is it to be plane geometry, solid geometry, or a geometry of many dimensions? Is it to be Euclidean, projective, or any one of the other geometries we know about?

Our first arbitrary choice is to confine ourselves to plane geometry. That being the case, the following requirements are quite natural:

1. The geometry consists of a set of undefined elements called *points*.
2. It contains certain subsets (smaller sets) of points called *lines*.
3. If L is a line and p is a point in the set of points which comprise L, we call p "a point of L" or write "p lies on L." The same idea is expressed by writing that "L contains p" or "L passes through p."
4. Any two points "determine a line"; that is, given any two points p and q, there is exactly one line L passing through these two points.

Notice that in the first three statements we set up a terminology. This is especially true of the third statement. We could merely write "L contains p" or "p is contained in L," but if we are to get help from geometrical visualization (and this is our chief guide in this development) it is useful to preserve as much of the geometrical terminology as possible. Property 4, on the other hand, is the first really restrictive requirement.

It is natural to impose two other requirements, the first to make it a finite geometry and the second to keep it from being too trivial:

5. The geometry contains only a finite number of points.
6. The geometry contains four points, no three of which lie on a line.

Now we are faced with our second choice. If the geometry were to be Euclidean[2] we would specify:

E7. If L is a line and p a point not on L, there is exactly one line L_p through p which has no points in common with L. L_p is called "parallel to L."

[1] Burton W. Jones, "Miniature Number Systems," THE MATHEMATICS TEACHER, LI (April 1958), 226–231. This will hereafter be referred to as "Article I," and its contents will be presupposed in much of this article.

[2] R. H. Bruck, "Recent Advances in the Foundations of Euclidean Plane Geometry," *The American Mathematical Monthly*, 62 (August-September, II, 1955), 2–17.

On the other hand, if the geometry were to be projective we would specify:

P7. Every pair of lines has a common point.

In this article we make the second choice since it is simpler in a number of respects; if we are to manufacture a geometry we might as well make it as simple as possible. Notice that Property 4 with Property P7 affirms that *any pair of distinct lines has exactly one common point.*

A finite geometry with seven points

Before considering the general theory, let us look at one particular finite geometry: one with seven points and seven lines. (This is actually the "smallest" geometry.) Number the points 0, 1, 2, 3, 4, 5, 6 and the seven lines may be taken as the following seven sets of three points each: L_0: 0, 1, 3; L_1: 1, 2, 4; L_2: 2, 3, 5; L_3: 3, 4, 6; L_4: 4, 5, 0; L_5: 5, 6, 1; L_6: 6, 0, 2. Inspection shows that each pair of lines has exactly one common point and that each pair of points determines a line. Also each point has three lines through it, and each line contains three points. This can be seen from the sets of points above or from the diagram.

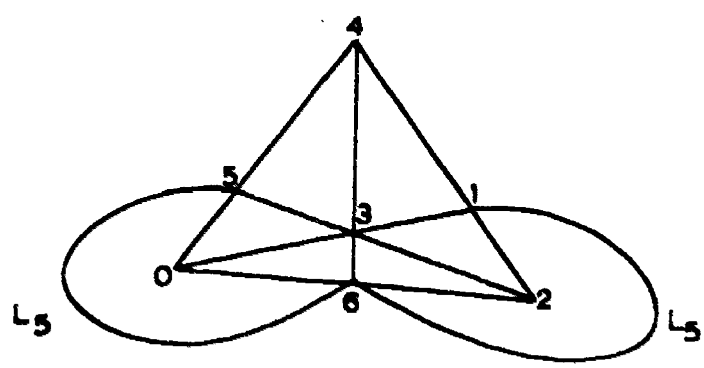


FIGURE 1

Notice that one of the lines, L_5, is not straight, but recall that the lines are sets of points and the only purpose of drawing the lines is to indicate the sets of points which comprise them.

Counting lines and points

More generally, suppose one line contains $m+1$ points. First we shall show that every line contains exactly $m+1$ points. Let L_0 be the given line and L any other line. From Property 6 there will be some point p not on either line. Let p_0 be the point common to L_0 and L and $p_1, p_2, \cdots, p_m$ the other points on L_0. Then the m lines determined by p and the points on L_0, except p_0, will intersect L in m distinct points. Thus L has at least as many points as L_0. By reversing the argument we can show that L_0 has at least as many points as L and hence both lines have the same number of points.

Next we can show that $m+1$ is the number of lines through a point. Given a point p and some line L not through p, it follows that L has $m+1$ points and that each of these with p determines a line through p. But any line through p intersects L in one of these points. Hence there are just as many lines through p as there are points on L, that is, there are $m+1$ lines through p.

Third, we can count the number of points in the geometry. Given a point p, every point of the geometry will lie on a line through p. Each such line contains m points besides p and there are $m+1$ such lines. Hence the total number of points is $m(m+1)+1=m^2+m+1$.

In a similar fashion it can be shown that there are m^2+m+1 lines in the geometry. Notice that 7 is the least number of lines in a finite geometry satisfying the properties listed above since that is the value of m^2+m+1 when $m=2$ and if $m=1$ there would be only three points in the geometry denying Property 6.

Duality

Perhaps the most important consequence of our choice of Property P7 is the *principle of duality:* any true statement about lines and points is also true when "line" and "point" are interchanged and the corresponding change in connective used. For instance, "any line has the same number of points as any other line" implies, by duality, " any point lies on the same number of lines as any other point." This principle seems like something too general to establish until we realize that by virtue of the fact that we assume o. ly Properties 1 to 6 and P7, we need merely verify duality for these seven properties. This can easily be done.

Suppose we consider "another" geometry defined by interchanging "point" and "line" in the Properties 1 through 6 and P7, that is, a dual geometry. We now show that this will have all the properties of the given geometry. In this dual geometry, a point would be identified with the set of lines through it. Property 3 is its own dual, and Property P7 is the dual of Property 4 if included in P7 is the italicized statement at the end of the first section. If the geometry contains only a finite number of lines, it must contain only a finite number of pairs of lines, and hence a finite number of points.

It remains to consider the dual of Property 6, namely 6d: the geometry contains four lines, no three of which have a common point. To show that this implies Property 6, let P_{ab} be the point determined by lines a and b, P_{bc} the point determined by lines b and c, and similarly define P_{cd} and P_{da}. Suppose P_{ab}, P_{bc} and P_{cd} were collinear. The first two of these lie on the line b and the second two on the line c; moreover they *determine* the lines b and c, respectively. But b and c are distinct lines and hence P_{ab}, P_{bc} and P_{cd} are not collinear. In similar fashion any three of the four designated points can be shown to be noncollinear. Thus we have shown that a geometry which satisfies the dual properties, satisfies the given ones. In similar fashion one may show that a geometry which satisfies the given properties, satisfies the dual ones. Hence the two geometries are the same, and the principle of duality is established.

CONSTRUCTION OF FINITE GEOMETRIES FOR $m=p^n$*

First, let us illustrate the construction for $p=3$, $n=1$, $m=3$. Associate with each point an ordered triple (a, b, c) where a, b, c are numbers of GF(3) (see Article I), not all are zero, and the triples (a, b, c) and (ka, kb, kc) are associated with the same point. For example, $(1, 1, 0)$ and $(-1, -1, 0)$ represent the same point. The number of triples excluding $(0, 0, 0)$ is $3^3-1=26$ and the number of proportionality factors different from zero is $3-1=2$. Hence the number of points in this geometry is $26/2=13$. Notice that $13=m^2+m+1$ for $m=3$. Then the following triples will represent the points of this geometry:

P_0: $(0, 0, 1)$; P_1: $(0, 1, 0)$; P_2: $(1, 0, 0)$; P_3: $(0, 1, -1)$; P_4: $(1, -1, 0)$; P_5: $(-1, 1, -1)$; P_6: $(1, 1, 1)$; P_7: $(1, -1, -1)$; P_8: $(-1, 0, -1)$; P_9: $(0, 1, 1)$; P_{10}: $(1, 1, 0)$; P_{11}: $(1, 1, -1)$; P_{12}: $(1, 0, -1)$.

Now we must determine the lines. Let the point P be associated with the ordered triple $t=(a, b, c)$ and the point P' with $t'=(a', b', c')$. Then the line determined by P and P' will be defined to be all those points associated with the triples $rt+r't'$, that is

$$(ra + r'a', \quad rb + r'b', \quad rc + r'c')$$

where r and r' range over the numbers of GF(3) excluding $r=0=r'$. Thus, if $P=P_0$, $P'=P_1$, $r=0$, $r'=1$ yields the point P_1 above; $r=1$, $r'=0$ yields P_0; $r'=1=-r$ yields P_3, and $r=1=r'$ yields P_9. (Notice that $r=-1$, $r'=1$, being proportional to 1 and -1, yield no new point.) We call L_0 the line composed of these four points. It can be seen that the quadruples of subscripts of points associated with the lines of this geometry will be:

L_0: (0, 1, 3, 9); L_1: (1, 2, 4, 10); L_2: (2, 3, 5, 11); L_3: (3, 4, 6, 12); L_4: (4, 5, 7, 0); L_5: (5, 6, 8, 1); L_6: (6, 7, 9, 2); L_7: (7, 8, 10, 3); L_8: (8, 9, 11, 4); L_9: (9, 10, 12, 5); L_{10}: (10, 11, 0, 6); L_{11}: (11, 12, 1, 7); L_{12}: (12, 0, 2, 8).

* In order for the remainder of this essay to be meaningful, the reader is urged to stop at this point and refer to an earlier article by the same author on "Miniature Number Systems," *The Mathematics Teacher* 51:226–231; April, 1958.

It can be shown without much trouble that this geometry satisfies the required properties of a finite geometry with $m=3$.

Similarly if $m=2$, there will be seven points determined by the triples: P_0: (0, 0, 1); P_1: (0, 1, 0); P_2: (1, 0, 0); P_3: (0, 1, 1); P_4: (1, 1, 0); P_5: (1, 1, 1); P_6: (1, 0, 1).

The lines will be those given above, and the numbers of the triples will be in GF(2).

In general, then for $m=p^n$ we consider the triples (a, b, c) where a, b, c range over the numbers in GF(m), not all are zero, and two triples in which a, b, c are proportional are associated with the same point. Then there will be m^3-1 triples not all zero and $m-1$ proportionality factors different from zero; hence $(m^3-1)/(m-1)=m^2+m+1$ points. The lines are defined as sets of points, as in the above example. This system can be shown to have all the required properties of a finite geometry.

Determination of lines by means of Galois Fields

The reader may have noticed a curious pattern in the two examples above for $m=2$ and $m=3$ in the formation of the successive lines. For instance, for $m=2$, the line L_0 is the set of points P_0, P_1, P_3 and the line L_1 is P_1, P_2, P_4. The subscripts of the points of the latter are those of the former increased by 1. The subscripts of the points in the line L_2 are 2, 3, 5, those for L_3 are 3, 4, 6 and for L_4 are 4, 5, 0 where 6+1 is replaced by 0, addition being (mod 7). The same pattern holds in the geometry with $m=3$.

Why does this pattern hold? To answer this for $m=2$, observe in a table given at the end of Article I, that x is a generator of GF(8), that is, each power of x will be of the form ax_2+bx+c (mod 2, x_3+x+1). Furthermore, every number of GF(8) will be a power of x. Thus x^i will be expressible in the form $a_ix^2+b_ix+c_i$ where a_i, b_i, c_i are in GF(2). Hence we can associate x^i with the point whose triple is (a_i, b_i, c_i). Thus $x^0=0\cdot x^2+1$ is associated with P_0: (0, 0, 1), $x=0\cdot x^2+1\cdot x+0$ is associated with P_1: (0, 1, 0), $\cdots$, $x^4=x^2+x+0$ with P_4: (1, 1, 0), etc. Since x is also a generator of GF(27), a similar development holds for $m=3$.

Now suppose that x is a generator of GF(m^3) and consider the following three triples:

$$t_i=(a_i, b_i, c_i);\ t_j=(a_j, b_j, c_j);\ t_k=(a_k, b_k, c_k),$$

where the a's, b's and c's are numbers in GF(m). Suppose the points represented by these triples lie on a line, that is:

$$a_k=ra_i+sa_j,\ \ b_k=rb_i+sb_j,\ \ c_k=rc_i+sc_j,$$

where r and s are numbers of GF(m) and not both are zero. Then $x^i=a_ix^2+b_ix+c_i$ with similar expressions for k and j imply $x^k=rx^i+sx^j$

2!

and thus, for every integer u, $x^{k+u} = rx^{i+u} + sx^{j+u}$. Hence $a_{k+u} = ra_{i+u} + sa_{j+u}$ and similarly for b and c. Thus $t_{k+u} = rt_{i+u} + st_{j+u}$ and the points P_{k+u}, P_{i+u}, and P_{j+u} are collinear.

Thus we have shown why the lines are formed from the initial one by the process described. There is one further question which should be answered: Since GF (m^3) contains m^3 numbers, there will be m^3 different powers of x but only m^2+m+1 different points; what is the explanation of this apparent discrepancy? The answer to this question is left to the interested reader.

In general, then to form a co-ordinate system with co-ordinates in GF(m), we may find a generator of GF(m^3) (not necessarily x as above) whose powers are all quadratic polynomials with coefficients in GF(m). We use the coefficients of the polynomials corresponding to successive powers of the generator, up to the power m^2+m+1, as the co-ordinates of the successive points of our geometry. This elegant device is due to James Singer.[3]

PERFECT DIFFERENCE SETS

Another approach to the problem of designating the points constituting the various lines of a finite geometry is by way of arithmetical properties of the set of subscripts of the points of line L_0. Consider the numbers 0, 1, 3 in the number system (mod 7). The six differences are: $1-0=1$, $0-1=6$, $3-1=2$, $1-3=5$, $3-0=3$, $0-3=4$ (mod 7). These are just the numbers 1, 2, 3, 4, 5, 6. In general, a set of numbers $0, 1, 2, \cdots, m$ whose differences (mod m^2+m+1) are $1, 2, 3, \cdots, m^2+m$ in some order is called a *perfect difference set*. Whenever such a set can be found, it can be used as the subscripts for a set of points on a line which may be called L_0. (For instance, for $m=3$, a perfect difference set is 0, 1, 3, 9.) The other lines then are gotten by increasing the subscripts by 1, by 2, etc. It is not too difficult to prove that such a difference set leads to a finite geometry. In fact, the first three properties of a finite geometry and the fifth already hold. Hence it remains only to show properties 4, 6, and P7. Here we shall only give an indication of how to show Property 4, leaving the rest to the reader.

Let m^2+m+1 be denoted by s and suppose

$$a_0, a_1, \cdots, a_m$$

is a perfect difference set (mod s). The points P with these subscripts will constitute the line L_0, and the line L_u will consist of the points P whose subscripts are

[3] James Singer, "A Theorem in Finite Projective Geometry and Some Applications to Number Theory," *Transactions American Mathematical Society*, 43 (1938), 377-385.

$$a_0+u, a_1+u, \cdots, a_m+u \pmod{s}$$

where u takes the values 0, 1, 2, $\cdots$, m. Let P_i and P_j be two points. We wish to show that they lie on one and only one line L, that is, there is only one set of three numbers: a_k, a_r, u such that $i=a_k+u$ and $j=a_r+u$. Since the a's form a perfect difference set, there will be exactly one pair such that $a_k-a_r=i-j \pmod{s}$. Then there is exactly one u in the range indicated such that $i=a_k+u \pmod{s}$. For example, if $m=3$ and the difference set is 0, 1, 3, 9 with $s=13$, suppose we find the line determined by P_5 and P_9. Then $9-0=5-9 \pmod{13}$ shows that $a_k=9$, $a_r=0$ and $5=9+u \pmod{13}$ shows that $u=9$. Hence the points P_5 and P_9 lie on L_9.

Block designs

Recent interest in finite geometries has arisen from their connection with designs used in experiments. Using as an example the finite geometry with seven lines and seven points, suppose one wanted to test seven different varieties of seeds by planting. One might have seven different plots of ground. Let these plots correspond to the lines of the geometry and the varieties of seed correspond to the points. A seed then would be planted in a given plot of ground if the corresponding point occurred on the corresponding line. Then there would be a desirable symmetry of treatment since each seed would be planted in three different plots of ground, each plot of ground would contain three different varieties of seed, and each seed would be competing with each other seed in a plot of ground exactly once. Each finite geometry yields a design with analogous properties.

Desargues' Theorem

One of the most interesting theorems of projective geometry is Desargues' Theorem. We shall show that this theorem holds in every finite geometry which has co-ordinatization as described above. Recall that this theorem may be stated as follows: If a, b, c and a', b', c' are two sets of three points each such that the lines aa', bb', cc' are concurrent, then the points determined by the following pairs of lines are collinear: ab, $a'b'$; ac, $a'c'$; bc, $b'c'$. To prove this let A, B, C and A', B', C' be triples corresponding to the given points. Then, since aa', bb', cc' are concurrent, numbers r_1, r_2, r_3, s_1, s_2, s_3 can be chosen so that

$$r_1A+s_1A'=r_2B+s_2B'=r_3C+s_3C'.$$

These equations may also be written in the following form:

$$\begin{aligned} r_2B-r_1A&=s_1A'-s_2B', \\ r_1A-r_3C&=-s_1A'+s_3C', \\ r_2B-r_3C&=-s_2B'+s_3C'. \end{aligned}$$

The left side of the first equation is a triple associated with a point on the line ab and the right-hand side is a triple associated with a point on the line $a'b'$. Since these triples are equal, each represents a point which is on each of the lines and hence their point of intersection. Similarly the triple on each side of the second equation represents the point of intersection of ac and $a'c'$, while the triples in the third equation represent the point of intersection of bc and $b'c'$. But $r_2B-r_1A+r_1A-r_3C=r_2B-r_3C$ shows the third point lies on the line determined by the first two, that is, the points of intersection of the following pairs of lines are collinear: $ab, a'b'$; $ac, a'c'$; $bc, b'c'$. This is what we wished to prove. The converse of this result may be proved in a similar manner or may be seen to follow from the principle of duality.

THE EXISTENCE OF FINITE GEOMETRIES

Since finite fields with p^n elements exist for every prime p and every natural number, one may, using the methods above, show that a finite geometry exists having $m+1$ points on a line whenever m is a power of a prime. Furthermore, if the geometry is constructed using the co-ordinatization above, Desargues' Theorem holds. There are, however, finite geometries with $m+1$ points on a line and m a power of a prime, for which Desargues' Theorem does not hold.

On the other hand, if m is 1 or 2 more than a multiple of 4 and if it contains to an odd power some prime factor which is 1 less than a multiple of 4, there is no finite geometry with $m+1$ points on a line. Thus there is no finite geometry for which $m=6$ since m contains to the first power the prime factor 3. But it is not known whether there is a finite geometry for $m=10$. In fact, the only finite geometries known have m a power of a prime. There is thus a wide gap between the values of m for which finite geometries are known to exist and those values for which it has been proved that no finite geometries exist. The existence of this gap in our knowledge is the reason for much of the interest in this subject today.

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A COORDINATE APPROACH TO THE 25-POINT MINIATURE GEOMETRY[1]

Martha Heidlage

A RATHER NOVEL WAY to study the familiar concepts of Euclidean geometry is to limit the number of points and lines in this geometry to a finite number and then examine the resulting system. Such a geometry, based on a set of postulates, undefined terms and relations, and restricted to 25 distinct points is called the 25-point miniature geometry. It can be characterized by the 25 letters *A* through *Y*, arranged in three distinct 5×5 arrays (Fig. 1), where the latter two of these three basic arrays were generated by a 60° rotation of the 25 letters on a lattice. In this geometry, the following definitions are given:

Definition 1. *Point*—any letter in one of the arrays.

Definition 2. *Line*—any row or column of five distinct letters in the three basic arrays (e.g., *ABCDE*).

Definition 3. *Parallel lines*—any two rows or columns having no "points" in common (e.g., $ABCDE \| FGHIJ$).

Definition 4. *Perpendicular lines*—any row and any column having one "point" in common (e.g., $ABCDE \perp AFKPU$).

Furthermore, some of the basic axioms[2] of this system are:

Axiom 1. Every line contains 5 and only 5 points.
Axiom 2. Not all points belong to the same line.
Axiom 3. Every point lies on 6 and only 6 lines.
Axiom 4. There are 30 and only 30 lines.

(1)					(2)					(3)				
A	B	C	D	E	A	I	L	T	W	A	H	O	Q	X
F	G	H	I	J	S	V	E	H	K	N	P	W	E	G
K	$\underline{L}$	M	$\underline{N}$	O	G	O	R	U	D	V	D	$\underline{F}$	M	$\underline{T}$
P	Q	R	S	T	Y	C	F	N	Q	J	L	S	U	C
U	V	W	X	Y	M	$\underline{P}$	X	B	$\underline{J}$	R	Y	B	I	K

FIGURE 1

THREE BASIC ARRAYS

[1] Support for study leading to this article came from the National Science Foundation. Miss Heidlage's faculty sponsor at Mount St. Scholastica College is Sister Helen Sullivan, O.S.B.

[2] It should be pointed out that nothing is said here about the consistency or independence of these axioms, or whether this set of axioms necessarily characterizes this geometry. (Editor)

During the remainder of this discussion,
this system shall be referred to as the miniature geometry.

Following Descartes' example, it is possible to coordinate this geometry to obtain the 25-point "mini-co geometry"; that is, a miniature coordinate geometry which is, in a sense, isomorphically related to the 25-point miniature noncoordinate geometry considered at the beginning of this article. Regard the first array (Fig. 2) of the 25-point miniature geometry as the key array, and associate an ordered pair of number from the set of residue classes (mod 5) with each of the 25 distinct letters. The coordinates, for example, of the point A are $x=0$, $y=4$. In the mini-co geometry, as in the miniature geometry, one can formulate the following basic definitions.

Definition 1. *Point*—an ordered pair of numbers from the set of residue classes (mod 5).

Definition 2. *Line*—set of five distinct points satisfying a single linear equation in two unknowns.

Thus, corresponding to each of the 30 lines of the miniature geometry, there is an algebraic expression in the mini-co geometry. For example, the equation of a line through the points L (1, 2) and W (2, 0) of the line $AILTW$ (Fig. 1) can be expressed in the familiar determinant form:

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

Expanding this determinant, one obtains

4	A	B	C	D	E
3	F	G	H	I	J
2	K	L	M	N	O
1	P	Q	R	S	T
0	U	V	W	X	Y
	0	1	2	3	4

FIGURE 2

ARRAY WITH COORDINATES

$y \equiv -2x+4$ or $y \equiv 3x+4$ (mod 5). Using the same method, the equation of the line through the points G (1, 3) and R (2, 1) of the line $GORUD$ (Fig. 1) is given by $y \equiv 3x$ (mod 5). It is possible to verify, by mere substitution, that the points of the two lines $AILTW$ and $GORUD$ do actually satisfy these two derived equations. As one might expect, the following familiar definition is stated in the mini-co system.

Definition 3. *Parallel lines*—line whose equations differ only in their constant terms.

Thus, the two equations just derived, $y \equiv 3x+4 \pmod 5$ and $y \equiv 3x \pmod 5$, corresponding to the rows *AILTW* and *GORUD* respectively, represent the equations of two parallel lines.

Since each of the three basic arrays (Fig. 1) consists of five parallel row lines and five parallel column lines, it follows that the algebraic expressions of these parallel lines will fall into one of six sets of equations, where the equations of each set differ only in their constant terms. In Figure 3, these equations of the mini-co geometry and their equivalent forms from the miniature geometry are arranged in six sets, such that the sets L_1, L_3, and L_5 represent the parallel columns of the arrays, while the sets L_2, L_4, and L_6 represent the parallel rows of the arrays.

In the mini-co geometry, perpendicularity follows from the notion of slope, where the slope of a line through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the familiar formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of the lines in set L_1 is actually undefined, since one obtains an expression of the form

$$m = \frac{y_2 - y_1}{0},$$

while the slope of the lines in L_2 is 0, in L_3 is 4, in L_4 is 3, in L_5 is 1, and in L_6 is 2.

L_1 (m_1 undefined)					
$x=0$	A	F	K	P	U
$x=1$	B	G	L	Q	V
$x=2$	C	H	M	R	W
$x=3$	D	I	N	S	X
$x=4$	E	J	O	T	Y

L_4 ($m_4=3$)					
$y=3x+4$	A	I	L	T	W
$y=3x+2$	S	V	E	H	K
$y=3x$	G	O	R	U	D
$y=3x+3$	Y	C	F	N	Q
$y=3x+1$	M	P	X	B	J

L_2 ($m_2=0$)						L_5 ($m_5=1$)					
$y=4$	A	B	C	D	E	$y=x+4$	A	N	V	J	R
$y=3$	F	G	H	I	J	$y=x+1$	H	P	D	L	Y
$y=2$	K	L	M	N	O	$y=x+3$	O	W	F	S	B
$y=1$	P	Q	R	S	T	$y=x$	Q	E	M	U	I
$y=0$	U	V	W	X	Y	$y=x+2$	X	G	T	C	K

L_3 ($m_3=4$)						L_6 ($m_6=2$)					
$y=4x+4$	A	S	G	Y	M	$y=2x+4$	A	H	O	Q	X
$y=4x+1$	I	V	O	C	P	$y=2x+1$	N	P	W	E	G
$y=4x+3$	L	E	R	F	X	$y=2x+3$	V	D	F	M	T
$y=4x$	T	H	U	N	B	$y=2x$	J	L	S	U	C
$y=4x+2$	W	K	D	Q	J	$y=2x+2$	R	Y	B	I	K

FIGURE 3

EQUATIONS (mod 5)

In general:

Definition 4. *Perpendicular lines*—two lines such that the slope m_1 of one line is twice the slope m_2 of the other line in the mod 5 system.

For example, the lines of L_3 are perpendicular to the lines of L_4, since $2 \cdot 4=8=3$ (mod 5), and the lines of L_5 are perpendicular to the lines of L_6, since $2 \cdot 1 \equiv 2$ (mod 5). By convention, consider the lines of L_1 to be perpendicular to the lines of L_2 in the mini-co geometry.

Thus far, the definitions formulated have closely paralleled those found in ordinary Cartesian geometry The concept of distance, however, requires a slightly different definition. In the miniature geometry, distance between two points of an array is defined as the least number of steps separating the letters or points on the line which joins them. According to this interpretation, distance can be measured only horizontally or vertically in a row or column, but not diagonally, and yields only two nonzero units of length—1 or 2. To illustrate this, let $d_r[a, b]$ denote the row distance from a to b and let $d_c[a, c]$ denote the column distance from a to c. Since the points on the ten lines in Figure 4 are cyclically permutable, the following distances are obtained:

$$\begin{aligned} d_r[A, B] &= d_r[A, E] = d_c[A, F] \\ &= d_c[A, U] = 1 \\ d_r[A, C] &= d_r[A, D] = d_c[A, K] \\ &= d_c[A, P] = 2 \end{aligned}$$

In the mini-co geometry, distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula

$$d = \sqrt{(x_1 - x_2)^2 + 2(y_1 - y_2)^2} \pmod 5.$$

This particular expression was chosen since it always yields positive distances in the mini-co geometry and automatically distinguishes between

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

FIGURE 4

BASIC ARRAY

row distances d_r and column distances d_c. Thus, the nonzero lengths in the miniature geometry have equivalent expressions in the mini-co geometry, where the single expression $d[a, b]$ is used to denote either the row or the column distance from a to b as in the following:

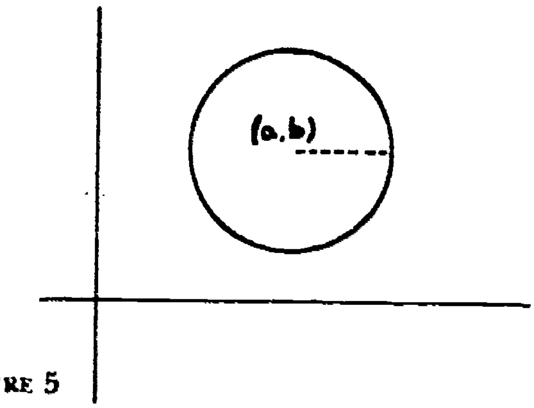
$$d[A, B] = d[A, E] \equiv 1 \pmod 5$$
$$d[A, C] = d[A, D] \equiv 2 \pmod 5$$
$$d[A, F] = d[A, U] \equiv \sqrt{2} \pmod 5$$
$$d[A, K] = d[A, P] \equiv 2\sqrt{2} \pmod 5$$

It is evident from the above scheme that the only nonzero row lengths in the mini-co geometry are 1 and 2, while the only nonzero column lengths are $\sqrt{2}$ and $2\sqrt{2}$ and distances can be measured horizontally, vertically, and diagonally by merely applying the distance formula.

With this machinery available, the study of conic sections falls into place, since a definition of any of the conic sections requires only the notions of point, line, and distance. Consider the four cases: circle, parabola, ellipse, and hyperbola. In both the miniature and mini-co geometries, a circle is defined as the locus of all points equidistant from a fixed point, called the center. For example, let M be the center and let the radius be determined by a row distance of length one. In the three basic arrays (Fig. 1), the six points, L and N of the first array, P and J of the second array, and F and T of the third array, are the only points on a circle of center M, with radius of row length one. This same circle of center M in the miniature geometry also has a quadratic expression in the mini-co geometry. In general, the equation of a circle (Fig. 5) with center (a, b) and radius r is given by:

$$(x + 4a)^2 + 2(y + 4b)^2 \equiv r^2 \pmod 5.$$

Note the coefficient 2 in the second term, this factor being a consequence of the mini-co definition of distance. Moreover, since the mini-co geometry allows only four possible distances, the radius r is restricted to four values. Thus, if each of the 25 points serving as the center of a circle can admit of only four possible values, there must be only 100 circles in the entire system!



Center: (a, b)
Radius: r

$$(x+4a)^2+2(y+4b)^2\equiv r^2 \pmod 5$$
$$r=1,\ 2,\ \sqrt{2},\ 2\sqrt{2}$$

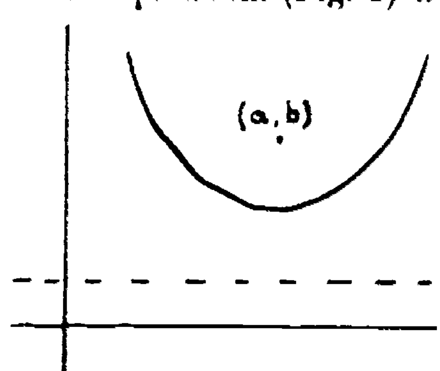
FIGURE 5

GENERAL EQUATION OF CIRCLE

The remaining three conics are defined as follows: Let F be a fixed point called the focus and let d be a fixed line, not through F, called the directrix. Then a conic is the locus of a point P which moves so that its distance from the focus F bears a constant ratio, or more commonly, eccentricity e to its distance from the directrix d. Again, since distances in the mini-co geometry are only four in number, this ratio e is also limited in the values it can assume. Thus, the conic just defined is

1 a parabola when $e=1$;
2 an ellipse when $e<1$ or $e=\frac{1}{2}\equiv 3 \pmod 5$;
3 a hyperbola when $e>1$ or $e\equiv 2 \pmod 5$.

The general equation of a parabola (Fig. 6) with focus (a, b) and row



Focus: (a, b)
Row directrix: $y=d$

$$(x+4a)^2\equiv 4y(b+4d)+2(d^2+4b^2) \pmod 5$$

FIGURE 6

GENERAL EQUATION OF PARABOLA

directrix $y=d$ is given by

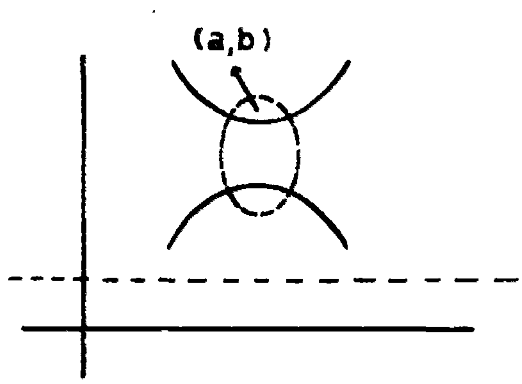
$$(x+4a)^2=4y(b+4d)+2(d^2+4b^2) \pmod 5.$$

Similarly, the general equation of an ellipse or a hyperbola (Fig. 7) with focus (a, b) and row directrix $y=d$ is given by

$$(x+4a)^2+2(y+4b)^2\equiv 2e^2(y+4d)^2 \pmod 5,$$

where $e=3$ for an ellipse and $e=2$ for a hyperbola. It is to be noted that the algebraic expressions of these conics have similar representations for column directrices. Furthermore, while the general definition of a parabola determines five distinct points for each such conic, the definition of an ellipse or a hyperbola directly determines only three distinct points. However, an observation of the symmetrical arrangement of these three points in the arrays conveniently reveals the existence of a fourth likely prospect. When this point is substituted in the equation for the ellipse or hyperbola, it is seen to satisfy the equation and thus is regarded as a point on the conic in the mini-co geometry.

It is interesting to note that there are certain pairs of ellipses or pairs of hyperbolas which are related in such a way that six of the eight points on the two conics also describe a circle! For example, consider the pair of ellipses E_1 and E_2 with the common focus M. Applying the definition of an ellipse to the three basic arrays, one obtains the following pair of ellipses: $E_1=\{J, N, T, K\}$ and $E_2=\{F, O, P, L\}$. Note, however, that the points J, N, T, P, L, and F are also points on the circle of center M and radius one which was defined earlier. Such ellipses or hyperbolas will be called symmetric conics such that six of the eight points of a pair of symmetric ellipses or symmetric hyperbolas also constitute a circle!



Focus: (a, b)
Row directrix: $y=d$

$$(x+4a)^2+2(y+4b)^2\equiv 2e^2(y+4d)^2 \pmod 5$$

where $e=3\rightarrow$ellipse
$e=2\rightarrow$hyperbola

FIGURE 7

GENERAL EQUATION OF ELLIPSE AND HYPERBOLA ROW DIRECTRIX

This 25-point miniature geometry and its coordinate counterpart illustrate the wide range of applicability of finite geometries to the basic Euclidean concepts. It thus seems reasonable to conclude that treatment of these familiar concepts should be considered initially in a finite system and secondarily in Euclid's "world of the infinite"!

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25-POINT GEOMETRY

H. Martyn Cundy

FINITE Galois arithmetics are well-known; finite geometries however, though more interesting to the amateur, have not really acquired professional status and do not appear to any great extent in standard works. The following example arose from a chance remark in *Mathematics for T. C. Mits*, by L. R. and H. G. Lieber; from this I deduce that a full theory has been worked out, but I have not seen it, and as far as I am concerned what follows is original, and I hope readers of the *Gazette* may find it new and stimulating.

1. *Basic structure.*

Consider the array of letters

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array}$$

and denote by p the operation of replacing this array by

$$\begin{array}{ccccc} A & I & L & T & W \\ S & V & E & H & K \\ G & O & R & U & D \\ Y & C & F & N & Q \\ M & P & X & B & J. \end{array}$$

The reader can discover for himself the rule by which the transposition is effected. If the operation is repeated we obtain a new array, but a further repetition merely gives the first array with the order of rows and columns altered; the first row reads *AEDCB* and the first column *AUPKF*. The three arrays formed by the operations 1, p, p^2 are here set out in a table:

$$\begin{array}{ccccccccccccccccc} A & B & C & D & E & & A & I & L & T & W & & A & H & O & Q & X \\ F & G & H & I & J & \rightarrow & S & V & E & H & K & \rightarrow & N & P & W & E & G \\ K & L & M & N & O & p & G & O & R & U & D & p & V & D & F & M & T \\ P & Q & R & S & T & & Y & C & F & N & Q & & J & L & S & U & C \\ U & V & W & X & Y & & M & P & X & B & J & & R & Y & B & I & K. \end{array}$$

We regard the letters as *points*; every row and column in any array as a *line*; two rows or two columns in the *same* array are lines having no point in common and will be called *parallel*; a row and column in the *same* array are *perpendicular*.

Distance.

The lines are regarded as closed and the points on them as cyclically permutable. The distance between two points is the shortest number of steps separating them on the line which joins them; row-wise and column-wise distances are regarded as incommensurable. Thus we write

$$AB=AE=LT=DF=1, \qquad AC=AD=LW=DM=2,$$
$$AF=AM=1', \qquad AK=AY=2',$$

column-wise distances being denoted by dashed numerals. Sense is not taken into account at present, and there are no axioms of order. Note that

$$2\times1=2, \quad 2\times2=1, \quad 2\times1'=2', \quad 2\times2'=1'.$$

Axioms.

It is easy to verify the following axioms:

(a) There is one and only one line joining any two points.

(b) Two lines meet in one point unless they are parallel.

(c) Through any point there is one and only one line parallel to a given line.

(d) Through any point there is one and only one line perpendicular to a given line.

The geometry is therefore *plane* and partially metrical. The concept of angle cannot be developed satisfactorily in any manner which satisfies the fundamental congruence axiom (*SAS*).

2. *The fundamental transformation-group.*

I have already defined the operator p. Let us regard the identity operator as including any *cyclic* permutation of rows or columns or both; that is, let us confine ourselves for the moment to transformations which keep one point, say A, fixed. Denote by i the operation of reversing the cyclic order in the rows. The operation of reversing the order of both rows and columns (if we like, of turning the plane through 180°) commutes with all other operations considered and is conveniently denoted by -1. These three operations preserve "distance" and "angle" and generate the group of *congruent* rotations about A. The group is of order 12, containing the elements ±1, $\pm i$, $\pm p$, $\pm p^2$, $\pm ip$, $\pm ip^2$, which are connected by the relations.

$$i^2=1,\ p^3=-1,\ pi=-ip^2,\ ip=-p^2i.$$

(This is the *dihedral group* of the regular hexagon D_6, i being a reflection in the y-axis, and p a rotation about the origin through $\pi/3$.) The relations are easily verified by direct operations on the arrays of letters.

The only further operations which preserve right-angles and parallels, but not distance, are the elements of the product of this group with the operation q, defined as the operation of doubling distances in rows and interchanging rows and columns. (Neither of these operations separately preserves parallels when combined with p.)

Thus, the first array becomes

$$
\begin{array}{ccccc}
A & B & C & D & E \\
F & G & H & I & J \\
K & L & M & N & O \\
P & Q & R & S & T \\
U & V & W & X & Y
\end{array}
\quad \xrightarrow{\;q\;} \quad
\begin{array}{ccccc}
A & F & K & P & U \\
C & H & M & R & W \\
E & J & O & T & Y \\
B & G & L & Q & U \\
D & I & N & S & X
\end{array}
$$

The reader can now verify that $q^4=-1$, $qi=-iq$, $pq=-qp^2$, $pq^2=q^2p$, $ip^2q=pqi$. The extended group is of order 48, and includes all the *similar rotations* about A, since q obviously leaves *ratios of distances* unaltered. (Remember that $2\times 2=1$ and that 1 and $1'$ are incommensurable.) These rotations carry B into any one of the 24 points other than A, combined with the "reflection"—i about the line AB. This last operation transforms the right-angled triangle ABG into ABV, and the 24 "rotations" transform these into 24 similar pairs of right-angled triangles with one vertex at A and any one of the other 24 points at the right-angled corner, including of course the identical pair themselves.

If now we consider the 25 cyclic changes of rows and columns which carry A into any other point (the "pure translation"), including the identity, and form the product, we obtain 1200 similarity transformations of the configuration into itself, of which 300 are congruent transformations. The four operations 1, q, q^2, q^3 can be considered as "magnifications without rotation." To sum up, the congruence group of rotations is generated by the elements -1, i, p; and the similarity group is the product of this group by the cyclic group on the additional generator q. The operator q changes 1 to $2'$, 2 to $1'$, $1'$ to 1, and $2'$ to 2. Further, the full congruence transformation group is transitive on all the 25 points.

3. *Triangles and parallelograms.*

I shall not attempt here to develop the geometry logically from the minimum of axioms. It will be more interesting, I think, to indicate some of the methods of proof and to outline the results that can be obtained. It will be found that almost every euclidean theorem expressible in this geometry is true in it. To prove any particular result, we have only to verify it for a few cases which we can show are transformed into all other possible cases by the operations of the congruence or similarity groups, C and S. We begin with triangles. There are $25 . 24 . 20/3! = 2000$ of these, formed by any three non-collinear points. They are of three types only. 1200 are scalene right-angled triangles, obtained from ABF by the operations of S. The sides of ABF are $11'2'$ and the others are similar to it, of four "sizes," found by magnifying ABF by $1, q, q^2, q^3$. A set of one of each size is ABF $(11'2')$, APB $(2'12')$, ADP $(22'1')$ and AUD $(1'21)$; (q removes B to the position of P; that is, AB becomes $AB' = AP = 2'$; operating on lengths, q is the cycle $(1'12'2)$). A further 200 are equilateral, for example, ABI, with four lengths of side. The remaining 600 are isoceles, similar to ABH $(112')$. The group S carries the representative triangles ABF, ABI, ABH into every other triangle; 6 operations of S carry ABI into itself, and 2 operations ABH, owing to their symmetry.

The important *midpoint theorem*, that the line joining the midpoints of the sides of a triangle is parallel to the base of the triangle and equal to half the base, can be verified by the reader in the three basic cases by the following figures; the theorem is invariant under S, therefore it is true for all cases. The fact that opposite sides of a parallelogram are

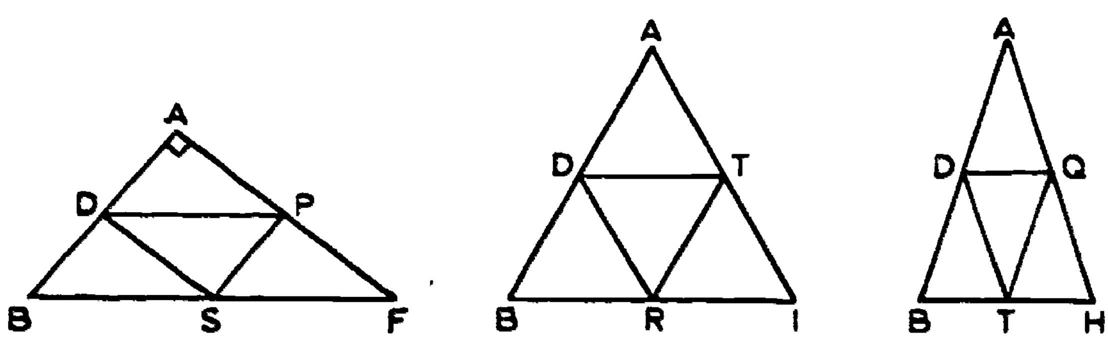

FIGURE 1

equal follows from the observation that if we regard the 25 points as forming a rectangular array in a fundamental cell of a euclidean point-lattice (or on the universal covering-surface of a torus), the operations p and q preserve euclidean parallels. Hence a parallelogram in the finite geometry is a euclidean parallelogram on the lattice and its opposite sides are equal in both geometries. By the same argument its diagonals bisect one another.

4. *Circles.*

There are six points distant 1 from A, namely B, E, I, W, H, X. These lie on a *circle*. There will be 100 circles, with 25 centres and 4 radii. Since in the triangles ABI and ABH, AR is perpendicular to BI and AT

to *BH*, the altitudes of all isosceles (and equilateral) triangles bisect the bases, so that the centre of a circle through three non-collinear points can be found uniquely by the usual construction. It follows from the midpoint theorem that the angle in a semi-circle is a right angle, for instance, *BIE*. There is only one line through any point on the circle which does not meet it again, and it is perpendicular to the radius; for example, for the circle *BEIWHX*, the *tangent* at *B* is *BGLQV*, and it is perpendicular to *AB*.

The Simson line.

For the triangle *BEI* and the point *H* on its circumcircle, the feet of the perpendiculars are the collinear points *QYC*. The general investigation is left to the reader.

The isosceles triangle.

The triangle *ABH* has midpoints *TQD*, centroid *L* which divides the medians in the ratio 2 : 1, circumcentre *I*, feet of altitudes *TOC*, orthocentre *W*, Euler line *AILTW*, on which $WL=2LI$. The "nine points centre" is *A*, and the "nine points circle" contains the points *TOCQDL*.

The equilateral triangle.

The triangle *BWH* has in addition an incentre *A*; its circumcircle is *BEIWHX*, also with centre *A*; the triangle *IEX* is congruent to *BWH*, and the triangles *LDQ*, *OCT* are congruent and equilateral.

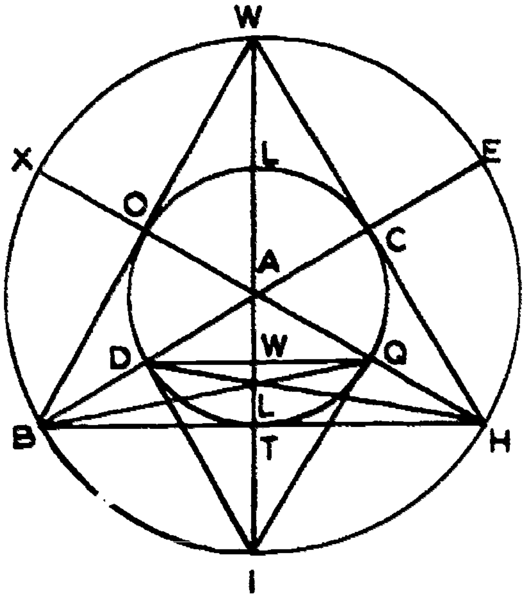

FIGURE 2

The right-angled triangle.

The properties are left to the reader.

5. *Polar properties and inversion.*

Consider the circle centre M and radius 1; it contains the six points $LNPJTF$. The remaining eighteen points fall into two classes: (i) points on tangents, $BCDGIKOQSVWX$, each on two tangents, and (ii) points which are midpoints of chords, $AEHRUY$. The polars of the points (i) can be identified as chords of contact of tangents; they are perpendicular to the lines joining the poles to the centre. For example, B lies on $BGLQV$, the tangent at L, and $OWFSB$, the tangent at F. Thus LF is the polar of B and is perpendicular to MB at X. For six of these points $BDKOVX$, the line joining the point to the centre meets the circle in two points, and the polar is the perpendicular to this line through the fifth point on it. For the remaining six, and the points in class (ii), the lines joining them to the centre do not meet the circle, but the polars can be obtained by the reciprocal property. For example, A lies on the polar of O ($AFKPU$), and of D ($ANVJR$). Hence the polar of A is OD, which is perpendicular to AM, and parallel to the chord TAL with midpoint A. The reciprocal property can be shown to hold throughout.

Inverse points.

For the above circle, inverse points are the pairs

$$BDKAEHRUY, \quad XVOGIWCQS.$$

Inspection shows that, if we now take sense into account, inversion in a circle of radius 1 is the transformation $1\to1$, $2\to-2$, $1'\to-2'$, $2'\to-1'$. Inversions in circles of other radii are obtained by transforming these relations by the operations q. Since two circles do not necessarily intersect, the standard euclidean procedure cannot be carried through, but it will be found that a straight line inverts into a circle through the centre; thus $SVEHK$ invert into $YDIWO$, which lie on the circle $YDIWOM$, centre S and radius $2'$. Also a circle not through M inverts into a circle; for example, $GDISPT$, centre R and radius 2, inverts into $AVEXPT$, centre W and radius 1. The centres are not inverse.

6. *Parabolas.*

The locus of a point which moves so that its distance from A is equal to its distance from $CHMRW$ has five points on it, namely, $BIXTO$. There are 600 such parabolas, each occurring twice if the operations of S are applied to one of them. We have then only to verify results for this particular case. The following familiar results are seen to be true (Fig.8):

$ABCDE$ is the axis, IT, XO are focal chords. The tangent at the vertex B is $BGLQV$. The tangent at I is $QEMUI$; the foot of the perpendicular from A to this is Q, on the tangent at B. The tangents at the ends of a focal chord IT meet at right angles at M on the directrix. AM is perpendicular to IT. The tangents at I, T, X are $QEMUI$, $VDFMT$, $LERFX$. They form a triangle MEF, whose orthocentre M lies on the directrix, and whose circumcircle $MWEFAJ$ passes through the focus. The chords IT, BX are parallel; their midpoints lie on a line LM parallel to the axis, which meets the curve at O and the directrix at M, where the tangents at I, T meet. The tangents at B, X meet at L, also on the line, and the tangent at O, $GORUD$, is parallel to the chords.

7. Projective geometry.

We now add to the rows in the first block an "infinity point" r_1, in which the "parallel lines" formed by the rows meet; and to the columns another "infinity point" c_1. We define r_2, c_2, r_3, c_3 similarly for the second and third blocks. If we consider r_1, r_2, r_3, c_1, c_2, c_3, to lie on a single line, we obtain a configuration of 31 points and 31 lines such that six lines pass

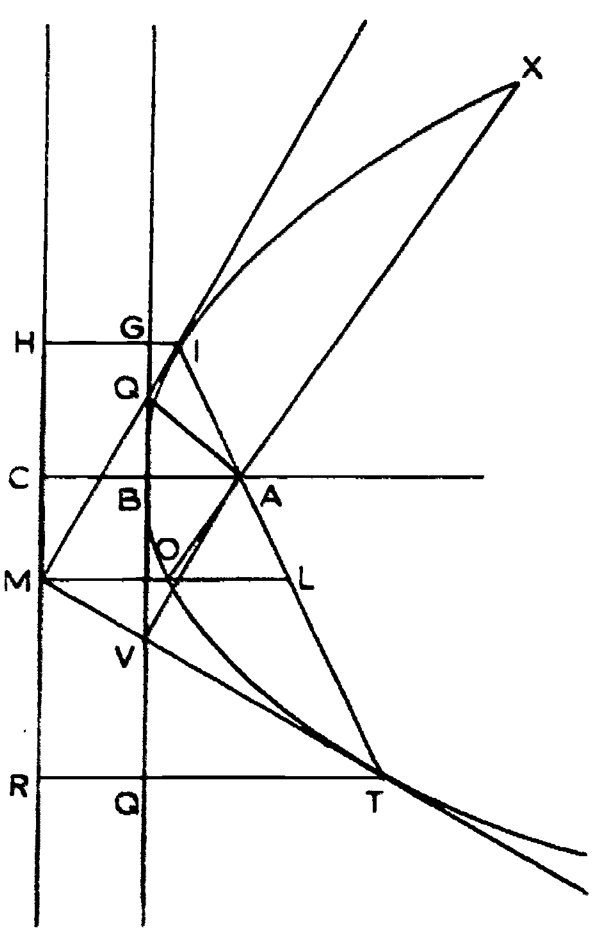

FIGURE 3

through every point, six points lie on every line; every two lines without exception meet in a unique point, and every two points are joined by a unique line. In addition, the complete quadrangle construction leads to a unique harmonic conjugate. Fig. 4 shows three constructions for the harmonic conjugate of W with respect to C and M. Note that R, W; H, c_1 are both harmonic pairs we respect to C, M and similarly for all other ranges.

The axioms of projective geometry are therefore satisfied, and the projective theory of the conics can be developed. In particular, Pascal's theorem is true and enables us quickly to obtain the six points which comprise a conic, given four or five of them. For example, through $AGQW$ we find, apart from line-pairs, only three conics (Fig. 5, p. 36).

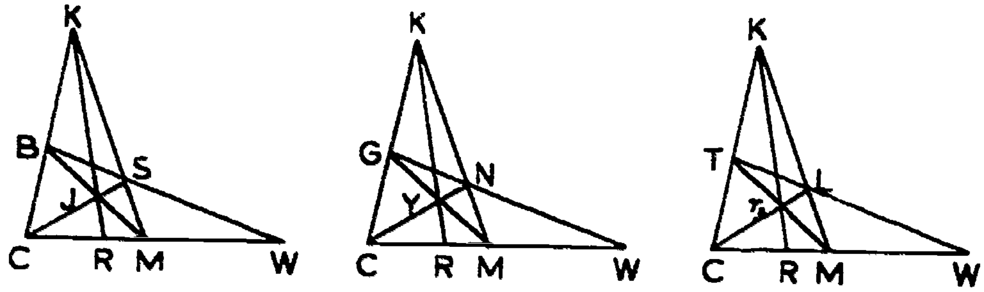

FIGURE 4

In this case, because AW, GQ have a common midpoint L, though not in general, these three conics have the same centre, L. The first two can be called *ellipses*; the first has two equal diameters and one different; the second has all its diameters different. The third is a *hyperbola*; the tangents at r_1, c_3 are $KLMNO$ and $HPDLY$, which meet at L, the centre.

In this case we begin with four points not on a parabola or circle. If we take four points on a parabola, say $BIXT$, we again obtain three conics, namely,

an ellipse $BIXTEV$, with centre L,
an ellipse $BIXTFQ$, with centre M,
the parabola $BIXTOr_1$.

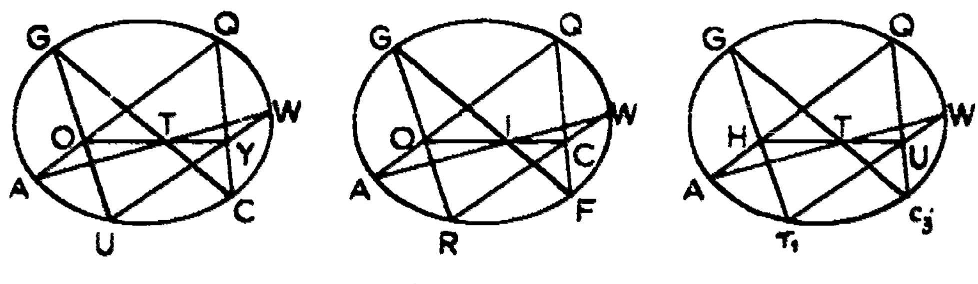

FIGURE 5

The centres (including those of the line-pairs) are the six points K, L, M, N, r_1, r_2, of which the first five lie on a line.

Similarly, if we begin with four concyclic points, for example, $HRGN$, we obtain

the circle $HRGNKQ$, with centre O,
the ellipse $HRGNSL$, with centre M,
the ellipse $HRGNXY$., with centre A.

The centres, and those of the line-pairs, are the six points O, M, A, W, J, U which, as is easily verified, lie on a conic. Note also that in each case the "new" six points, namely, $UCRFr_1c_2$, $EVFQOr_1$, $KQSLXY$, also lie on a conic. (The first is the hyperbola, centre L, asymptotes Lr_1, Lc_2; the second the parabola focus H and directrix $DINSX$; the third an ellipse centre R.) We conclude then that through any four points three conics can be drawn, excluding line-pairs, each containing two other points. These extra six points themselves lie on a conic. The centres of the three conics and the three line-pairs lie on a further conic. Of course, there is one and only one conic through five points, no three of which are collinear.

8. *The rectangular hyperbola.*

The conic through G touching Ar_1, Ac_1 at r_1, c_1 is $RNGYr_1c_1$. This is a rectangular hyperbola, centre A; RAN, GAY are diameters. The tangent at G is $XGTCK$; it meets the asymptotes at C, K and $CG=GK$. All the triangles formed by three points on the curve are right-handed, so that the orthocentric property has no significance. In fact, $NGRY$ is a parallelogram in which each diagonal is perpendicular to a pair of opposite sides (Fig. 6).

9. *Conclusion.*

I have said enough to indicate the very large scope and some of the fascination of this geometry. I have not investigated at all its many peculiar properties in which it differs from euclidean geometry, but it is amusing to see all the familiar results coming out. What more will you have? Why bother about a continuum when 25 points will do all the tricks? Or are there really only 25 points? In considering the parallelogram we had recourse to an infinite lattice. This approach suggests a euclidean model for the geometry. Suppose the length 1 is called k, and $1'=l$. Then if $l=k/\sqrt{3}$, the operation $-p$ is a plane rotation of the rectangular euclidean lattice, through $2\pi/3$, with a reduction of all distances, modulo $5k$ or $5l$. (It is then clear why the group generated by ± 1, p, i is the dihedral group D_6.) This is apparent from Fig. 7, in which $AB\equiv k$, $AF\equiv l$, $AI\equiv -k$ (mod. $5k$ or $5l$) and $\angle BAI=2\pi/3$.

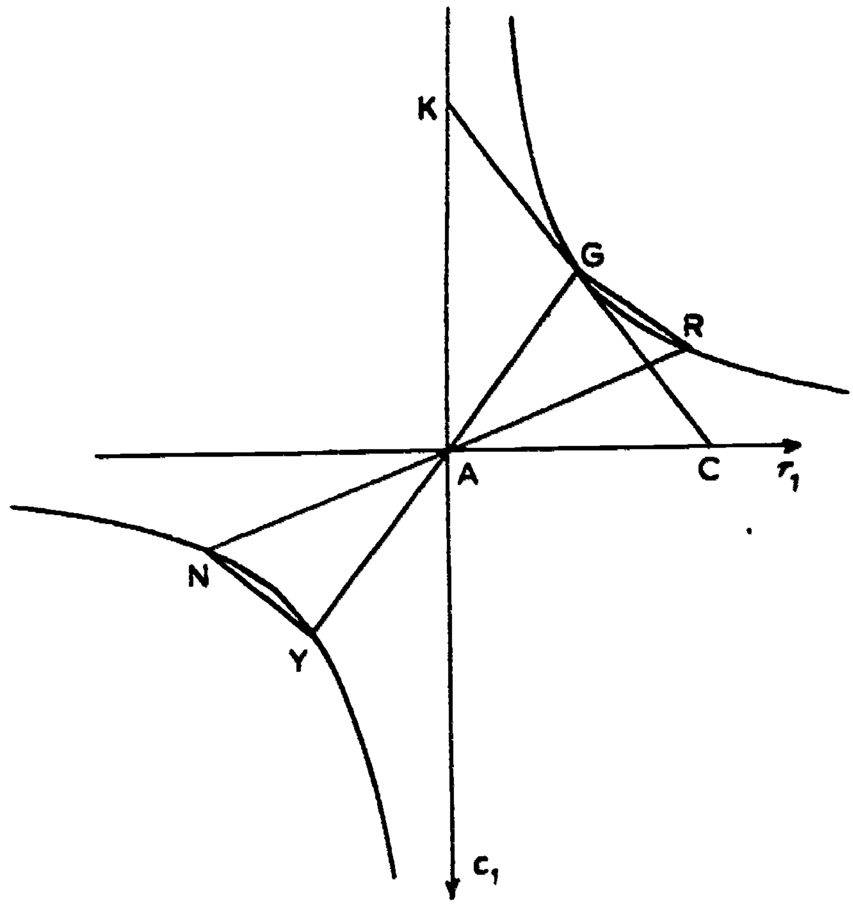

FIGURE 6

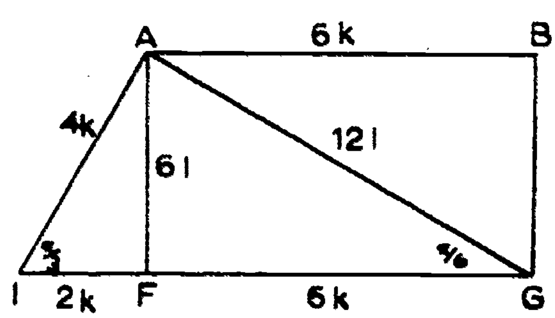

FIGURE 7

p carries *AI* to *AB*, and *AS* to *AF*.

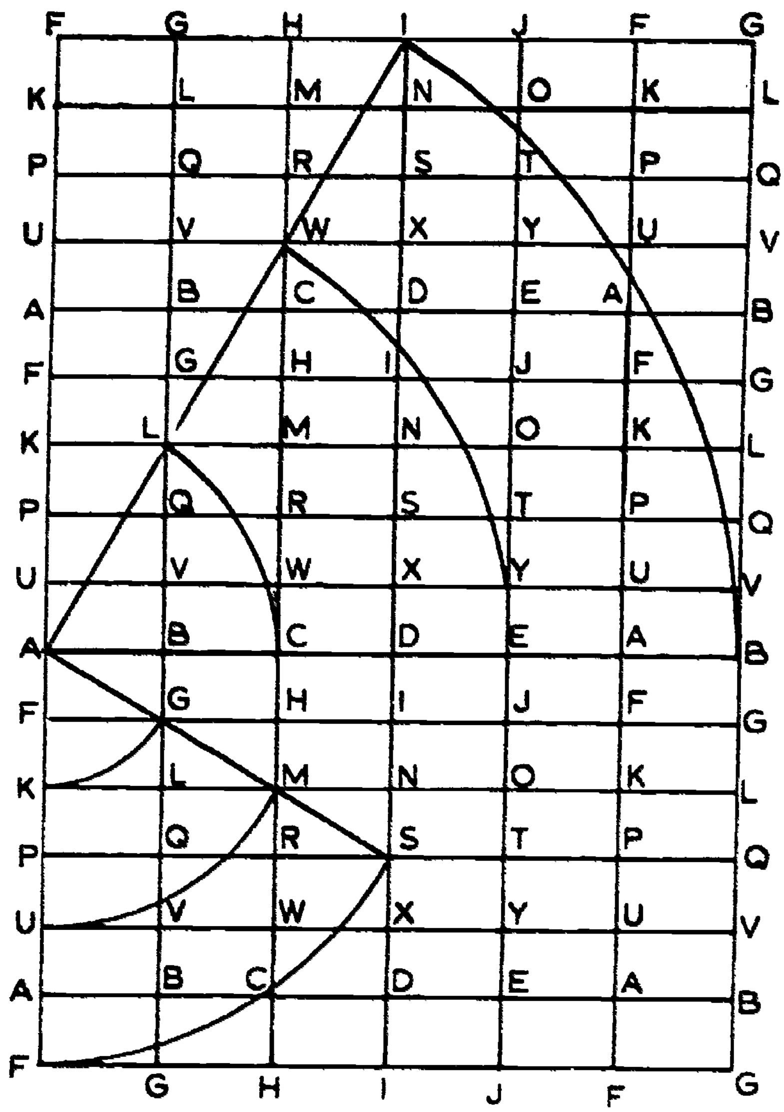
F G H I J F G
K L M N O K L
P Q R S T P Q
U V W X Y U V
A B C D E A B
F G H I J F G
K L M N O K L
P Q R S T P Q
U V W X Y U V
A B C D E A B
F G H I J F G
K L M N O K L
P Q R S T P Q
U V W X Y U V
A B C D E A B
F
G H I J F G

FIGURE 8

If this rotation is followed by a reflection in the centre A (the operation -1), reversing the sign of k, it is seen to be equivalent to p. It is easier to see the rotation on the lattice (Fig. 8); p is a rotation in the clockwise direction through $\pi/3$. The operation qi in this model is now seen to be a magnification by $2/\sqrt{3}$ coupled with a rotation through $\frac{1}{2}\pi$; AB is carried to AP, and AF (6 units) to AE. We expect euclidean results therefore to be true in this geometry in so far as they apply to points on the lattice. Finally, it becomes clear that what we have really been investigating is the geometry of a rectangular lattice of this type. We have for example proved that three non-degenerate conics pass through four points of this lattice, if we may select the cells appropriately in which points are to lie, that is, if we may replace any point by an equivalent point, and if we insist that the conic contains an additional lattice-point. Further every conic through five points of the lattice contains a sixth, possibly at infinity, provided equivalent points are suitably selected. If we remove this restriction, the result provides that every conic through five points of the lattice contains another *rational* point of the lattice; that is, any conic through five points with rational coordinates contains a sixth such point; but this is obvious anyway by Pascal's theorem.