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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include planar graphs, chains, and trees. (MP)

SCHOOL MATHEMATICS STUDY GROUP

SP-29

SUPPLEMENTARY and ENRICHMENT SERIES

The Mathematics of Trees and Other Graphs

By Marian M. Osborne

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

To My Father

who encouraged
my interest in mathematics

TABLE OF CONTENTS

THE MATHEMATICS OF TREES AND OTHER GRAPHS

1. Introduction

Once upon a time there was a rich mathematician who liked fresh air. He liked fresh air so much that he built himself a summer home in the mountains. Figure 1 is a sketch of the floor plan. He decided that the best way to be

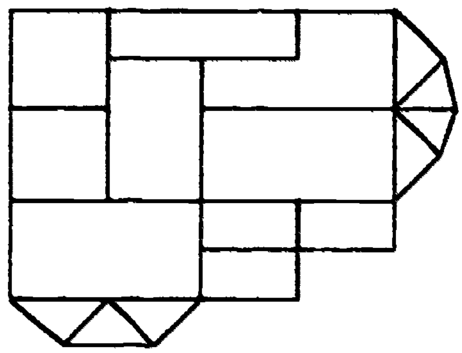


Figure 1

sure that he would have fresh air at all times, no matter which way the wind was blowing, was to put a door in each wall of each room; and that is exactly what he did.

This mathematician had a maid who kept house for him. Each year he would send her up to the summer home two days early with instructions to open every door and air the house. But the maid was lazy. She did not like to spend the morning opening doors and the evening closing them. She decided that the rooms would get aired well enough if each room had either an outside door open or doors open which were part of a series of open doors by which one could get outside. The only trouble with this idea was that it took too much time to decide which doors to open so as to open the least total number of doors. Could you have helped her?

This booklet will enable you to solve the maid's problem. In solving this problem you will encounter some new mathematical ideas which are part of an important branch of the subject known as graph theory. To get the most value out of your reading you should follow these general instructions:

1. Have paper and pencil with you each time you sit down to work.

2. Keep your work organized; avoid using scratch paper.

3. The numbered problems are discussed in the answer section, which begins on page 18. Unless you are really lost, you may learn more if you wait to look in the answer section until you have done several problems.

4. A number in square brackets, [], refers to the item with that number in "Books to Use," page 25.

It is hoped that the reader has already been introduced to sets, Venn diagrams, ordered pairs, and the induction principle.

Enjoy yourself.

2. Basic Definitions

Copy the set of dots in Figure 2.

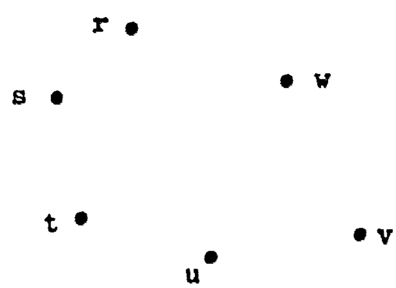


Figure 2

Pick a dot and draw a line from that dot to any other dot. Starting with the same dot or with a different dot, do the same thing seven more times.

Copy the set of dots again and draw a different set of eight lines.

You have now drawn representations of two graphs. The elements r, s, t, u, v, and w, represented by the six dots, are called vertices. The lines represent what are called edges. Graphs, vertices, and edges are abstractions. A representation of an abstraction is a visible object designed to aid the student in thinking about something which, by nature, is not visible. Representations of vertices, edges, and graphs can be three-dimensional, made using balls for vertices and strings for edges; or they can be two-dimensional, made using pencil or chalk dots for vertices and pencil or chalk lines for edges.[1]

Every graph can be described by a function, f. The notation used is as follows:

---

[1]Similarly, the number three is an abstraction usually represented by the numeral "3." The numeral "3" can be seen; but the number it represents is never seen, only talked about. Likewise, no one has ever seen a mathematical triangle. The only things one ever sees is a representation of a triangle, that representation usually being made with ink, chalk, wood, etc.

1. If x and y are any two vertices and n is the number of edges between them, then $f(x,y) = n$, read "f of xy equals n."

2. If $f(x,y) = n$, then $f(y,x) = n$. Therefore, in writing the set of equations for a graph, if $f(x,y)$ is included, then $f(y,x)$ need not be.

3. If s is a vertex and if there is no edge between s and any other vertex, then $f(s,x) = 0$, where x is any vertex. In this case s is said to be an <u>isolated</u> <u>vertex</u>.

Study Figure 3 and the set of equations given with it.

$f(s,t) = 1$

$f(s,u) = 2$

$f(t,u) = 0$

$f(v,x) = 0,\ x \in \{s,t,u\}$

$f(x,x) = 0,\ x \in \{s,t,u,v\}$

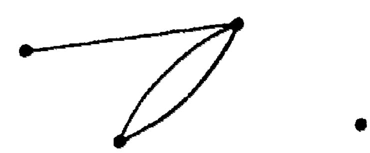


Figure 3

With respect to the above example, we have the following three sets:

1. The <u>set</u> <u>of</u> <u>vertices</u>, X, which is $\{s,t,u,v\}$.

2. The <u>domain</u> of f, which is the set of all pairs of elements in the set of vertices.

3. The <u>range</u> of f, which is $\{0,1,2\}$.

Note that the range of the function of a graph is always a subset of the non-negative integers.

<u>Problem 1</u>. Write the set of equations that goes with each of your two graphs. (Two possible cases are worked out on page 18.)

Having approached graphs inductively, we now state the formal definition.[2]

<u>Definition 1</u>. A <u>graph</u> is a non-empty set, X, of elements called vertices and a function, f:

1. whose domain is the set of all pairs of vertices,

2. whose range is a subset of the non-negative integers, and

---

[2]All of the definitions are important, but the numbered ones (there are six of them) should be learned especially thoroughly. As a start, it is suggested that you write them out and also memorize them. Note that because the definitions in the glossary are all given in the formal "if and only if" form, they are sometimes worded a little differently than in the text.

3. which is such that, if x and $y \in X$, then $f(x,y) = f(y,x)$.[3]

The following definitions describe four special types of graphs.[4] As you read the definitions, think about the relationships between the different types.

A graph is a <u>graph with multiple edges</u> if it has at least one pair of vertices connected by more than one edge. The graph which goes with Figure 3 is such a graph because s and u are connected by two edges. Is either of your graphs a graph with multiple edges?

It is possible for a function to have $\{0\}$ as its range. The set of edges for such a graph is the empty or null set. Thus, if a graph has no edges, it is called a <u>null graph</u>.

<u>Problem 2</u>. Write the set of equations for the null graph with the three vertices x, y, and z.

A graph is a <u>proper graph</u> if it has no isolated vertices. The graph which goes with Figure 3 is not a proper graph because v is an isolated vertex. The graph which goes with the mathematician's house is a proper graph.

A graph is a <u>universal graph</u> if each pair of vertices is connected by one and only one edge.

<u>Problem 3</u>. Finish the universal graphs for the sets of vertices in Figure 4. Does each graph have the same set of equations?

Figure 4

---

[3] This statement of the definition is not a standard one and is, as a matter of fact, quite unlike any the author has read. Its uniqueness is the result of the following two facts. First, the study of graphs is such a new area that no one definitional statement has really become standard. Second, most authors only define graphs in an informal and intuitive manner, whereas I have chosen to give formal definitions for all of the basic concepts.

[4] A graph with the function f is a <u>graph with a loop</u> if and only if there exists a vertex x such that $f(x,x) \neq 0$. Since none of the theorems or problems will involve graphs with loops, hereinafter the term "graph" will be understood to mean "graph without a loop."

Problem 4. If a set consists of n vertices, how many edges are in its universal graph?

Problem 5. To help you think about the relationships between the different types of graphs mentioned so far, draw a Venn diagram. Let the universal set, U, be the set of all graphs with more than one vertex. Let M stand for the set of graphs with multiple edges, N for the null graphs, P for the proper graphs, and V for the universal graphs. (Hint: To get started on this type of problem, of which there will be more, you might ask yourself these three questions about each pair of sets, A and B:

1. Is $A \cap B = \emptyset$?
2. Is $A \cap B = A$; that is, is $A \subseteq B$?
3. Is $B \cap A = B$; that is, is $B \subseteq A$?

In this case, the sets A and B would be chosen from U, M, N, P, and V as described above.)

Sometimes when you are given a representation of a graph, you can count the edges by putting a number beside each one and after a while you will come to a last edge. All of the examples given thus far have this property. A set such that it is possible to count the members and come to a last one is called a finite set, and thus a graph with a finite number of edges is called a finite graph.

It is possible to think of the real number line (Figure 5) as a graph. The vertices of such a graph would be in correspondence with the integers. The "real number line graph" is an infinite graph: there is no last edge no matter how the edges are numbered.

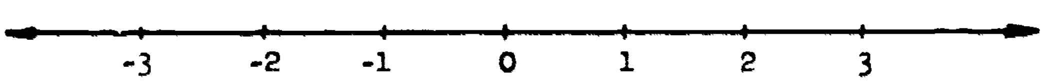


Figure 5

A mathematician sometimes has a choice when he makes a definition. For example, a finite graph could have been defined as one with a finite number of vertices. However, then it would have been more difficult to discuss how many edges such a graph could have, unless we restricted ourselves to graphs without multiple edges. With the definition that was given, it is fairly easy to prove the following theorem, Theorem 1, about the number of vertices. However, this theorem also includes a restriction, that of no isolated vertices. Graphs with isolated vertices are used less often in problems than are graphs with multiple edges. Thus, this is the preferable restriction from a practical standpoint.

<u>Theorem 1</u>. A finite proper graph has a finite number of vertices.

Proof: Let d be the number of edges. Assume that there are more than 2d vertices. Since each edge can "use up" at most two vertices, some of the vertices must be isolated. Thus, this assumption leads to a contradiction. Therefore, since 2d is a finite number and there are less than or at most 2d vertices, then the number of vertices is finite.

<u>Planar Graphs</u>

<u>Problem 6</u>. The function described below is to be used with each of Figures 6-10. For all but one of the figures, it is possible to draw (or finish) a representation of the graph in such a way that none of the lines intersect.

a. Find the one for which it is not possible by finding solutions for each of the others.

b. For the remaining one, find two solutions on a doughnut-shaped surface. It is recommended that a strip of paper, taped together as shown in Figure 11, be used for the doughnut-shaped surface.

Function: For each pair listed the value of the function is 1. The value of every other pair is 0.

(s,v)	(w,z)	(s,w)
(t,v)	(x,z)	(t,x)
(u,v)	(y,z)	(u,y)

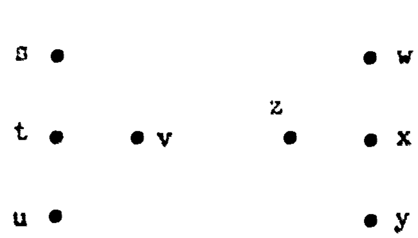


Figure 6

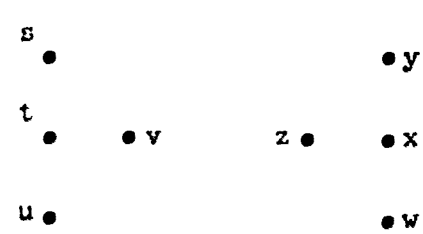


Figure 7

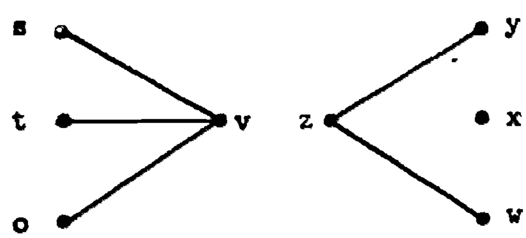


Figure 8

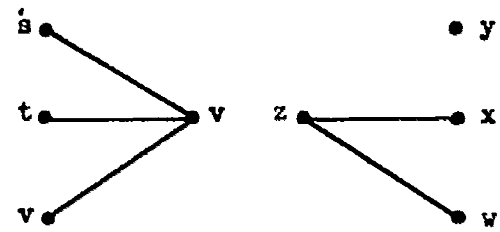


Figure 9

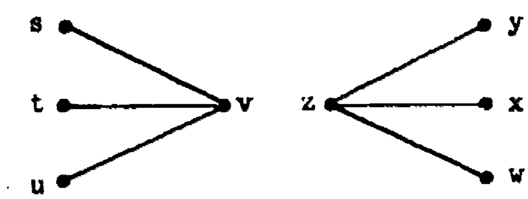


Figure 10

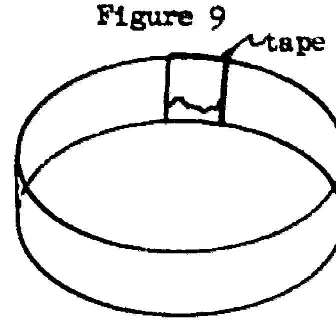


Figure 11

Any solution other than the doughnut solution shows that the graph which consists of eight vertices and the given function is a planar graph. A graph is a planar graph if it is possible to represent it on a plane in such a way that the vertices are all distinct points and no two edges meet except at vertices.

The point of the above problem is that whether intersections are necessary or not is determined by the type of surface on which the graph is represented. In three-dimensional space, every graph has a representation which has no intersecting lines.

Problem 7. Can you draw a planar representation of the universal graph which has three vertices? four vertices? five vertices?

Chains

In order to solve the maid's problem, we need to consider ordered lists of edges. Consider the two edges a and b in Figure 12 and the ordered pair of edges (a,b).

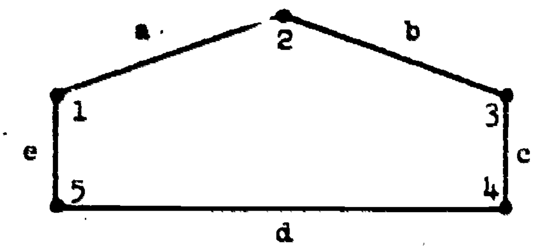


Figure 12

In this ordered pair, two edges have the vertex 2 in common. The ordered pair (b,a) has the same vertex in common but it is different because the edges are listed in a different order. Intuitively, the difference between (a,b) and (b,a) is like the difference between "walking" from 1 to 3 and "walking" from 3 to 1.

It is also possible to have ordered triples such as (a,b,c) or even (b,e,b). <u>Sequence</u> is the general term used to describe a set of ordered items without specifying the number of items being ordered.

<u>Problem 8</u>. Using Figure 12, give three sequences of edges.

<u>Definition 2</u>. A non-empty sequence of edges $(u_1, u_2, \ldots)$ is a <u>chain</u> if each u_k has one vertex in common with the preceding edge u_{k-1}, and the other vertex in common with the succeeding edge u_{k+1}.

In Figure 13, (a, b, c, e, a, d, h) is a chain.

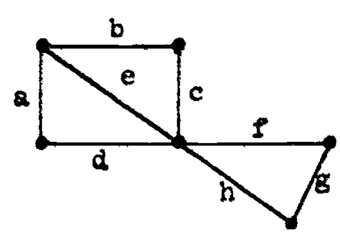


Figure 13

<u>Problem 9</u>. With respect to Figure 13, which of the following are chains?

1. (a, b, c, d, a, e)
2. (a, d, f, g, h, c)
3. (e, c, h, g)
4. (b, c, a, d)

There are several special types of chains. As you read the following definitions, think about the relationships between the different types.

A chain is a <u>simple chain</u> if no edge is used more than once. The chain (f, g, h, c) is simple, but the chain (f, g, h, f) is not.

A chain is a <u>finite chain</u> if it has a last edge. An <u>infinite chain</u> has no last edge.

<u>Problem 10</u>. The sequence of edges $(v_1, v_2, v_3, \ldots)$ of the non-negative real number line (Figure 14) is an example of an ______ ______ chain. (two words)

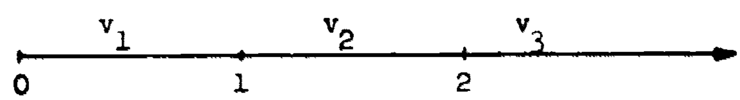


Figure 14

<u>Problem 11</u>. Draw a Venn diagram which shows the relationships among the set of chains, C, the set of simple chains, SC, and the set of finite chains, FC. Let the universal set, U, be the set of non-empty sequences of edges.

<u>Definition 3</u>. A finite chain $(u_1, u_2, \ldots, u_n)$ is a <u>cycle</u> if the first vertex of u_1 is the same as the last vertex of u_n.

Intuitively, one can think of a cycle as beginning and ending at the same vertex; but, technically, the cycle itself contains no vertices, only edges.

In Figure 15, is (a, f, l, h, b) a cycle? Why not?

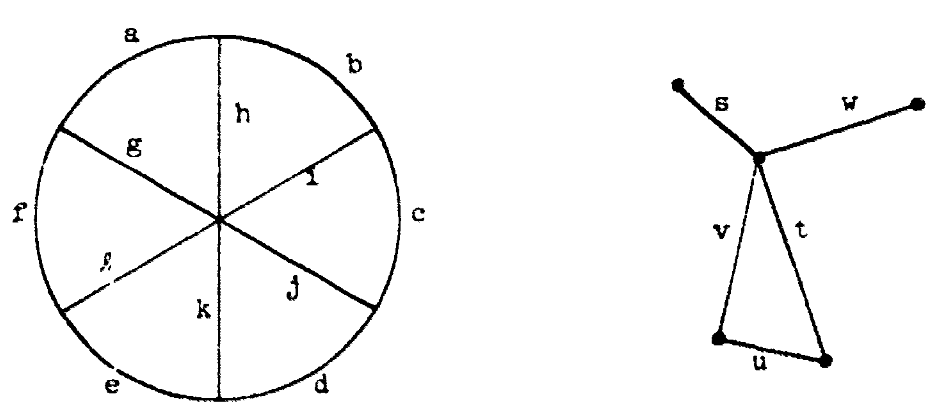


Figure 15

Note that all graphs except for null graphs contain such trivial cycles as (a, b, b, a) and (a, a).

Just as there are different types of chains, so there are also different types of cycles. A cycle is a <u>simple cycle</u> if no edge is used more than once. Does the "real number line graph" (page 5) contain any simple cycles?

<u>Definition 4</u>. A cycle is an <u>elementary cycle</u> if:

1. The vertices used to define the edges are used by the edges in the given cycle exactly twice, and
2. no edge is used more than once.[5]

In Figure 15, (a, b, c, d, e, f) is an elementary cycle, but (a, g, j, d, k, h) is not.

<u>Problem 12</u>. Draw a Venn diagram which shows the relationship between simple cycles, SO, and elementary cycles, EO. Let the universal set be the set of cycles, O.

<u>Problem 13</u>. Using Figure 15, identify each of the following chains, choosing one of a - e.

1. (s, w)	a. finite chain
2. (j, i, c)	b. finite simple chain
3. (s, t, u, v, w)	c. cycle
4. (a, f, l, h, a, g)	d. simple cycle
5. (a, g, l, e, d, c, i, h)	e. elementary cycle
6. (a, g, i, b, a, g, h)	

<u>Problem 14</u>. Draw a Venn diagram which shows the relationships among the following sets:

1. non-empty sequences of edges, U (the universal set)
2. chains, C
3. simple chains, SC
4. finite chains, FC
5. cycles, O
6. simple cycles, SO
7. elementary cycles, EO

You may find it easiest to use rectangles to represent the sets. Be sure to have a large enough piece of paper. (Hint: Draw two other diagrams first: one with everything but No. 3, and the other with just Nos. 3, 5, and 6.)

[5]Without this second restriction, cycles such as (a, a) would be elementary cycles. By not stating this second restriction, some authors have, in effect, contradicted themselves in later definitions.

Trees

If you need to review the definitions in the first section, "Basic Definitions," do so now.

Consider the set of all towns with railroad stations as a set of vertices, and let the railroad tracks represent the edges of a graph. For this graph, there exist pairs of vertices, such as those represented by London and New York, for which there is no chain, no sequence of tracks, which connects the two. The telephone system, on the other hand, represents a different type of graph; anyone with a telephone can talk with anyone else with one (although special arrangements have to be made for persons on the same party line). The "telephone system graph" is an example of a <u>connected graph</u>.

<u>Definition 5</u>. A graph is a <u>connected graph</u> if, for every pair of distinct vertices, there is a chain going from one to the other.

<u>Problem 15</u>. Draw a Venn diagram which shows the relationships among the set of connected graphs, C; the set of proper graphs, P; and the set of null graphs, N. Let the universal set, U, be the set of all graphs with more than one vertex.

The next problem, the last Venn diagram problem, is given to help you understand the definition which follows it.

<u>Problem 16</u>. Draw a Venn diagram which shows the relationship between the set of graphs which contain simple cycles, GSO, and the set of graphs which contain elementary cycles, GEO. (This is a tricky problem!)

In order to solve the maid's problem it is necessary to learn one more definition and two theorems.

<u>Definition 6</u>. A finite connected graph is a <u>tree</u>

1. if it has at least two vertices and
2. if it has no elementary cycles.

By the previous problem, problem 16, you also know that a tree has no simple cycles.

Problem 17. In Figure 1 , which drawings are not of trees? Justify your answer.

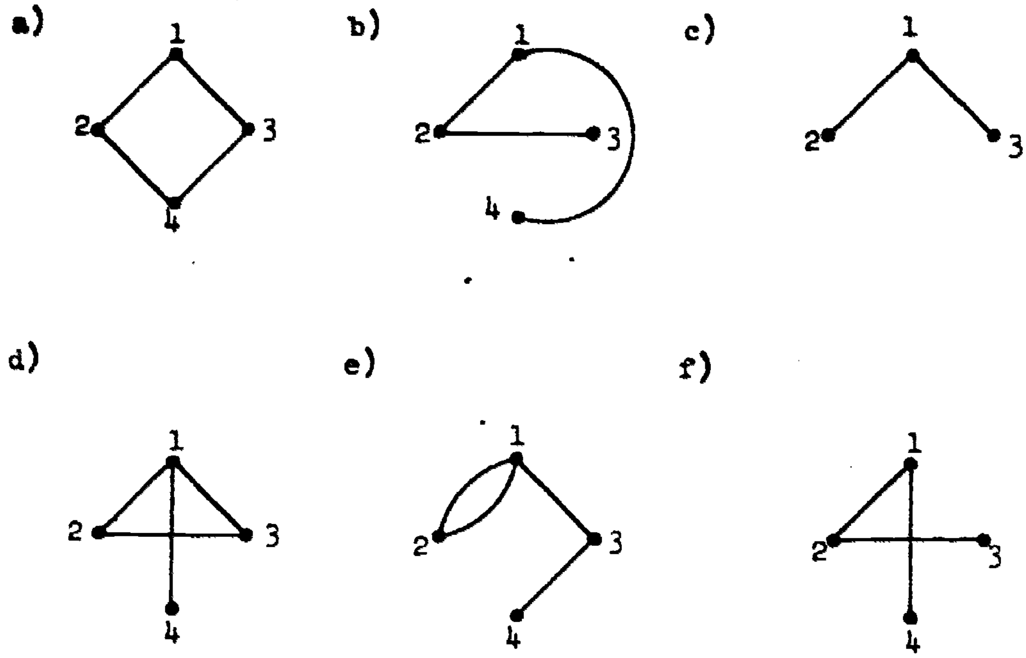


Figure 16

Problem 18. Using Figure 17, draw a representation for each of the trees which has exactly four vertices. Do not stop too soon!

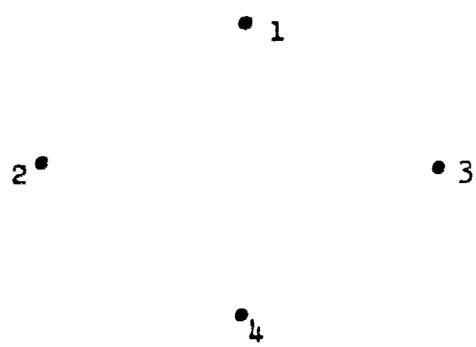


Figure 17

Problem 19. What are the possible numbers of edges a tree with four vertices can have? with three vertices? with five vertices?

The mathematician's house can be considered as representing a finite connected graph, and the opening of a door can be thought of as the removal of an edge. Do you see that as long as there are any elementary cycles in the house, the part of the house which is so enclosed cannot be aired? In order to know which doors to open, it is necessary to know more about finite connected graphs without elementary cycles, that is, trees. The following two theorems describe some of the characteristics of a tree.

Theorem 2. If a graph is a tree and if x and y are any two distinct vertices, then there is one and only one simple chain beginning at x and ending at y.

Proof: If there were not a chain, then the graph would not be connected. By definition, a tree is a connected graph. If there were more than one simple chain, then any two together would form a cycle which, by the proper elimination of extra edges, would yield at least one elementary cycle. (Figure 18 should suggest to you how this procedure works.) By definition, trees do not have elementary cycles. Therefore, there is only one simple chain between any two vertices.

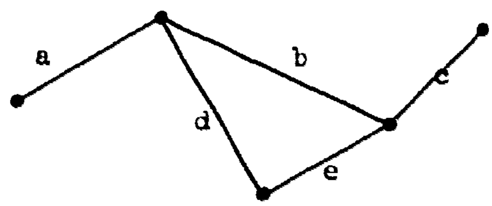


Figure 18

Theorem 3. If a tree has n vertices, then it has n - 1 edges.

Comment: A corollary of this theorem is that every tree has one less edge than it has vertices. Every tree has a finite number of edges by the definition of a finite graph. By theorem 1, page , we know that every tree has a finite number of vertices. Therefore, even when a finite number of vertices is not postulated for a particular tree, we will know after proving theorem 3 that that tree has one less edge than it has vertices.

Preliminary exercise: Draw a representation of a tree with, say, between five and ten edges. Add one edge to it in such a way that it still represents a tree. How many vertices did you add? Can you add an edge in another way so as to add a different number of vertices?

Proof: The proof of this theorem is by induction. Let S(n) be the statement that a tree with n vertices has n - 1 edges. A tree must have at least two vertices. Since, for a tree, two vertices are connected by one and only one simple chain, the simplest tree has only two vertices and one edge. Therefore, S(2) is true.

Suppose that S(k) is true, that is, that a tree with k vertices has k - 1 edges. One way to get a tree with k edges is to add an edge to a tree with $k - 1$ edges. As we add the edge, can we add no vertices? one vertex? more than one vertex? If no vertices are added, then the new edge will be a simple chain between the two vertices which are its endpoints. By theorem 2, there already was a simple chain between those vertices; so now there are two simple chains between them. Since that theorem says that there must be exactly one chain, such a graph would not be a tree. Therefore, it is necessary to add at least one vertex. If we add more than one vertex, then the graph is not connected; either the new edge is disconnected or else there are isolated vertices. Since trees must be connected, we cannot add more than one vertex. Because it is necessary to add one and only one vertex while adding the extra edge, a tree with k edges must have $k + 1$ vertices. Thus, $S(k)$ implies $S(k + 1)$.

Since $S(2)$ is true and $S(k)$ implies $S(k + 1)$, then $S(n)$ is true for all $n > 1$; and the theorem is proved by induction.

Restatement of the lazy maid's problem: You remember that she wanted to get rid of all air traps (elementary cycles) by opening as few doors (removing as few edges) as possible. Since a finite proper graph has a finite number of vertices (theorem 1, page 6), the problem, stated in general terms, is twofold:

1. For a finite connected graph with n vertices and m edges, $n \leq m$ what is the smallest number of edges that must be removed if there are to be no elementary cycles?
2. How can appropriate edges be chosen?

Note:

1. We do not have fewer edges than vertices, so we are not starting with a tree. $(m > n - 1)$
2. A graph, by definition, must have at least one vertex $(n \geq 1)$. Since $m \geq n$, the type of graph we are considering here also has at least one edge and thus at least two vertices.

Solution: We begin by eliminating an edge belonging to an elementary cycle, say the edge a_k between the vertices x_j and x_k. The graph is still connected because the other part of the cycle forms a chain from x_k to x_j. If there is another elementary cycle, another edge can be eliminated in the same way. Since the graph is finite, there will come a time when there are no elementary cycles left. Then we shall have a finite connected graph with no elementary cycles and with at least two vertices, in other words, a tree. This tree has n vertices, the same number as for the original graph. It has $n - 1$ edges, by theorem 3. If r was the number of edges removed, then

$$n - 1 = m - r$$

or

$$r = m - n + 1.$$ [6]

Note that in answering the first half of the problem the methodology for the second was given.

Problem 20. What is the smallest number of doors that the maid has to open? (Hint: In order to count the edges, copy or trace figure 1.) Is there more than one set of doors which consists of this "least number" and which will air out the house?

Summary Quiz

Finish all the problems before checking the answers.

Problem 21. For each of a and b, fill in the blank in the following statement with as many of p - s as are true. The graph with this set of equations is a ____________________ .

a.	$f(x,y) = 1$	p. universal graph
	$f(y,z) = 1$	q. proper graph
	$f(x,z) = 0$	r. graph with multiple edges
b.	$f(x,y) = 1$	s. tree
	$f(y,z) = 1$	
	$f(x,z) = 1$	

Problem 22. Refer to figures 19 - 21.

a. Is (a, c, b) a chain?

b. Is (e, f, g, h) an elementary cycle?

c. Is G a representation of a tree?

In each case, justify your answer.

---

[6] Adapted from [4], p. 37.

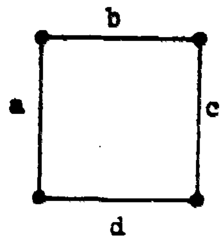


Figure 19

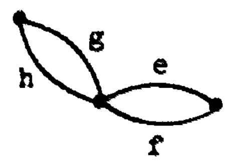


Figure 20

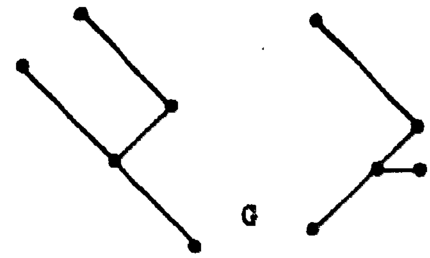


Figure 21

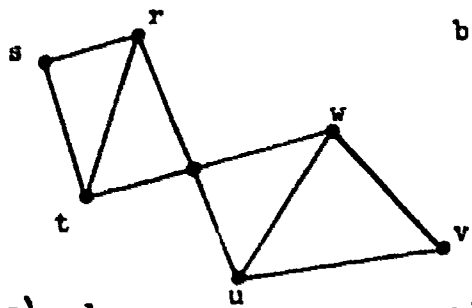
<u>Problem</u> <u>23</u>. With respect to the "real number line graph," for what pairs is the value of the function 1?

APPENDIXES

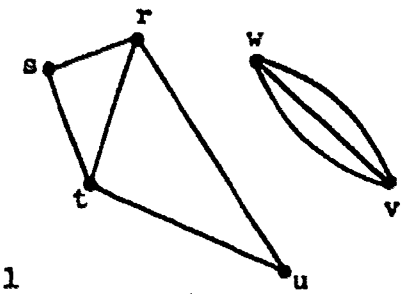
Answers

1. Two possible sets of equations and representations:

a)



b)



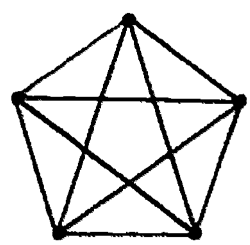
a) $f(r,s) = 1$
$f(r,t) = 1$
$f(r,u) = 1$
$f(s,t) = 1$
$f(t,w) = 1$
$f(u,w) = 1$
$f(v,w) = 1$
$f(u,v) = 1$

b) $g(r,s) = 1$
$g(r,t) = 1$
$g(r,u) = 1$
$g(s,t) = 1$
$g(t,u) = 1$
$g(v,w) = 3$

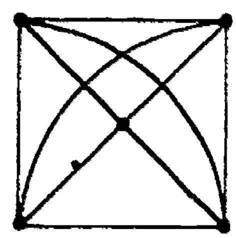
For all other pairs of vertices, the values of f and g are 0.

2. $f(x,y) = 0$ $f(y,z) = 0$ $f(x,z) = 0$

3. a)



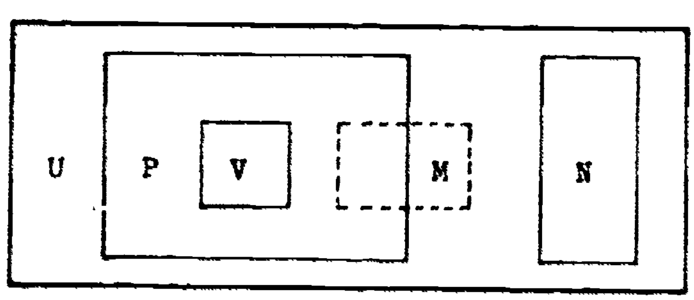
b)



Yes, each does have the same set of equations.

4. $\frac{1}{2}n(n - 1)$

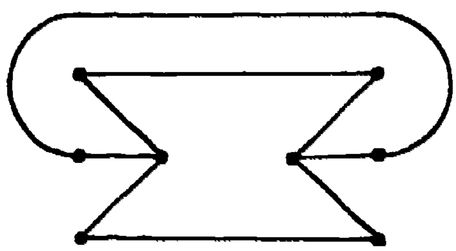
5.



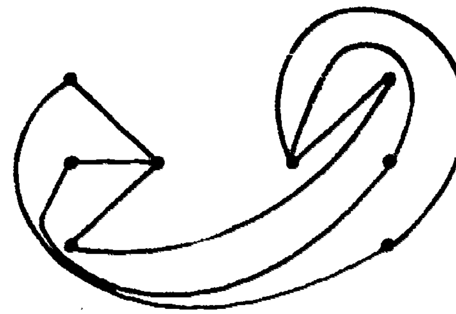
2

6. a) Figure 10 has no solution. There is a story, "The Story of the Persian Caliph and His Daughter's Boyfriends," which goes with that figure. See [1], page 8. Possible solutions for figures 6 - 9 are:

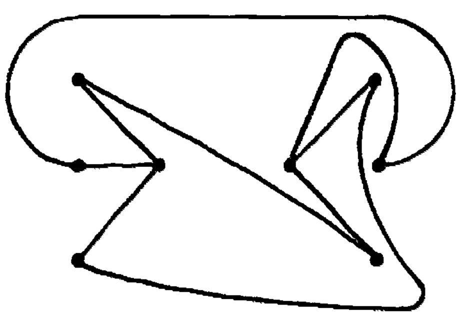
6)



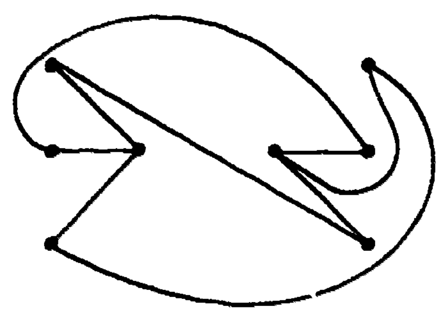
7)



8)

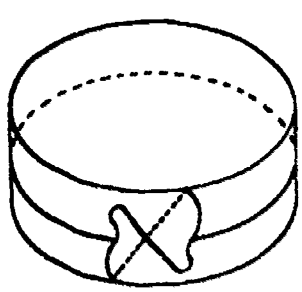


9)

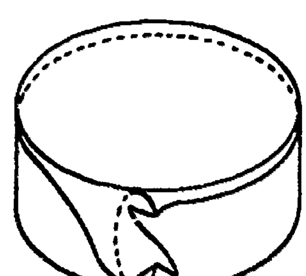


b) Two possible solutions are:

1)

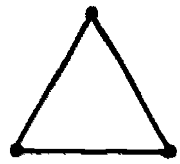


2)

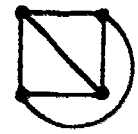


For further study on some of the differences between planar surfaces and doughnut-shaped surfaces, see [2] or [6].

7. a)



b)



c) The universal graph with five vertices is not planar. Neither is the graph which goes with the function whose value is 1 at each of the pairs listed and 0 elsewhere.

(u,x)	(v,x)	(w,x)
(u,y)	(v,y)	(w,y)
(u,z)	(v,z)	(w,z)

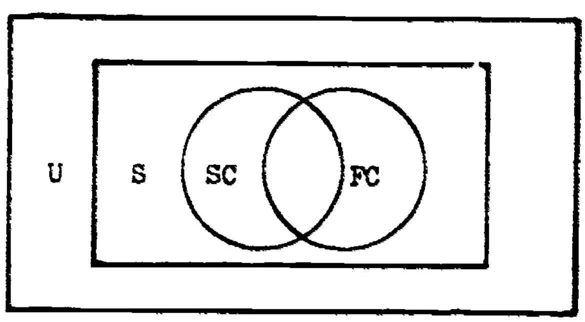
It has been proved that a graph is not planar if and only if part of if is, in a certain technical sense, "like" either of the two non-planar graphs mentioned above. See [4], p. 96; [6], p. 42; [10], p. 192; or [9], p. 211. There is a proof in [9], and [10] mentions another place to find a good one.

8. Three possible sequences are (a, b, d), (e, a, b), and (e, b, c, d, a, e).

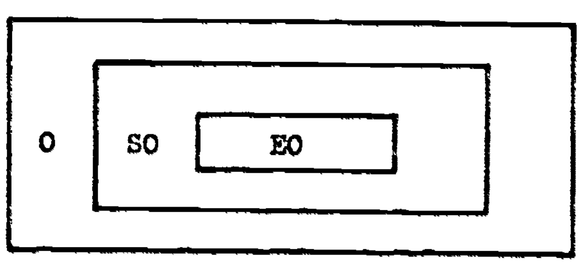
9. Number 1 and Number 2.

10. Infinite simple

11.

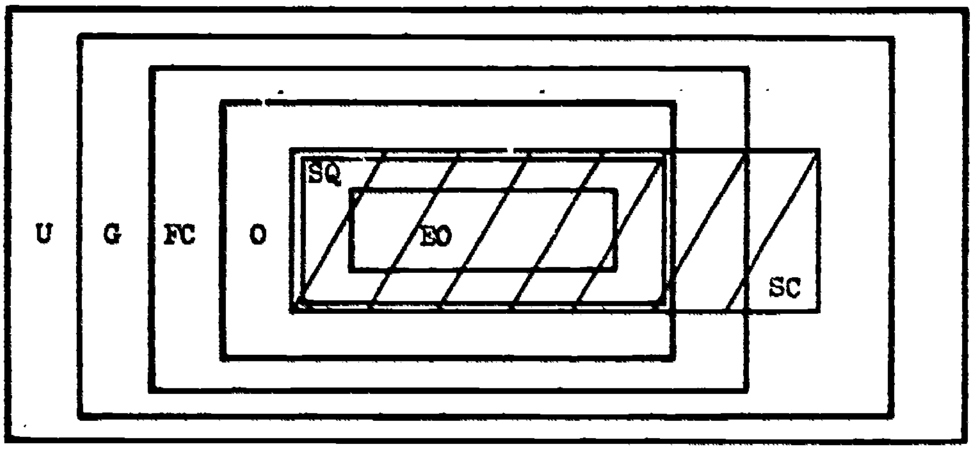


12.

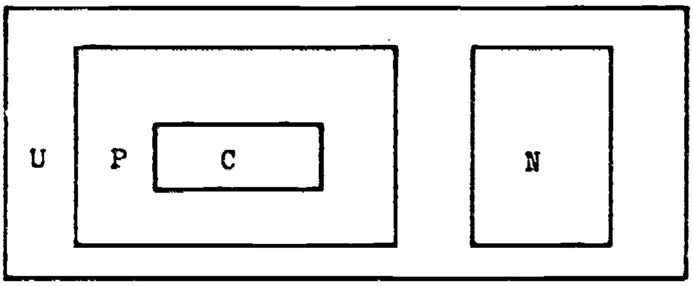


13. 1 - b, 2 - e, 3 - b, 4 - a, 5 - d, 6 - c.

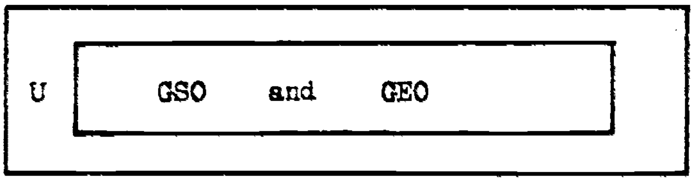
14.



15.



16.



Any elementary cycle is a simple cycle; therefore, GEO $\leq$ GSO. Any simple cycle which is not an elementary cycle can be broken up into elementary cycles; therefore, GSO $\leq$ GEO.

17. Refer to figure 22.
   a) This graph is not a tree because (a, b, c, d) is an elementary cycle.
   b) This graph is a tree.
   c) This graph is not a tree because it is not connected; 4 is an isolated point.

d) This graph is not a tree because (e, f, g) is an elementary cycle.

e) This graph is not a tree because (u, v) is an elementary cycle.

f) This graph is a tree.

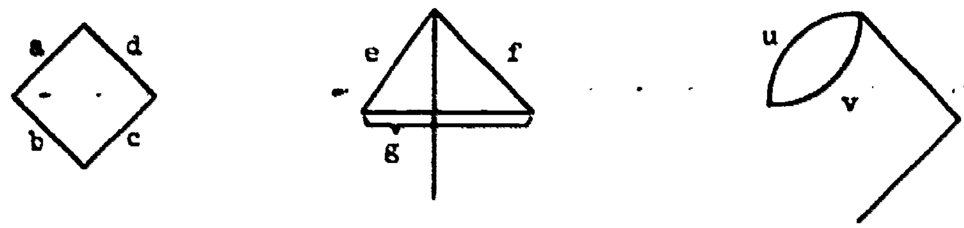
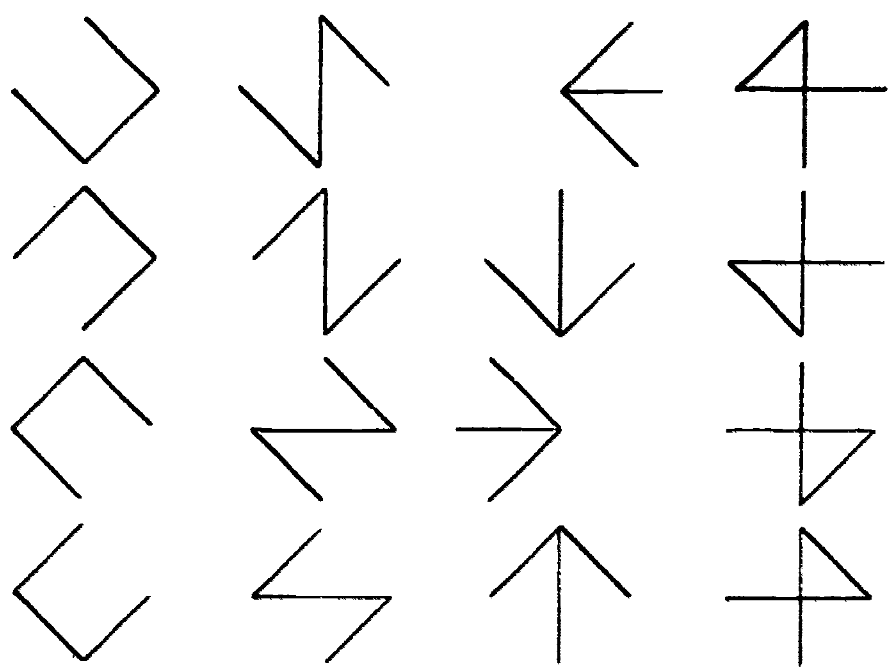


Figure 22

18.[7]



19. four vertices - three
    three vertices - two
    five vertices - four

20. 17 = 44 - 28 + 1.
    Yes, there is more than one set.
    Q: When is a house not a house?
    A: When it is a tree.

21. a - q, s; b - p, q.

---

[7]Adapted from [9], p. 160.

22. a) No. Edges a and c have no vertex in common.

b) No. The vertex between f and g is used more than once.

c) No. G is not connected.

23. $f(x,y) = 1$ if and only if $x = y + 1$.

Further Study

By describing some other problems related to graph theory and by providing guidance concerning the choice and use of other sources, it is hoped that the reader will be stimulated to study other aspects of graph theory and topology.

Problems to Consider

The following five problems illustrate some additional ideas which are part of graph theory. Answers to the starred problems appear after the fifth problem. The sources and page numbers are listed after each problem, and each answer is taken from the same source as the corresponding problem.

1. Road Construction. If a person were given a set consisting of n cities and were given the cost of constructing a road between any pair of these cities, how could he determine the cheapest way to construct a road network which would connect all n cities? [4-38].

*2. Round Tour of the World. Pretend that the world is shaped like a dodecahedron and that there is a town at each of the twenty vertices. Using figure 23, plan a world tour such that each town is visited once and only once, travel is done along the edges, and the last town is the same as the first. [4-28] and [9-107]. See [8] and [11] for discussions of Hamilton circuits, the generalization of the idea behind this problem.

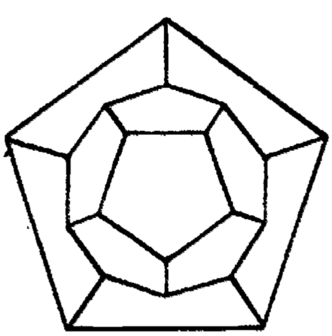


Figure 23

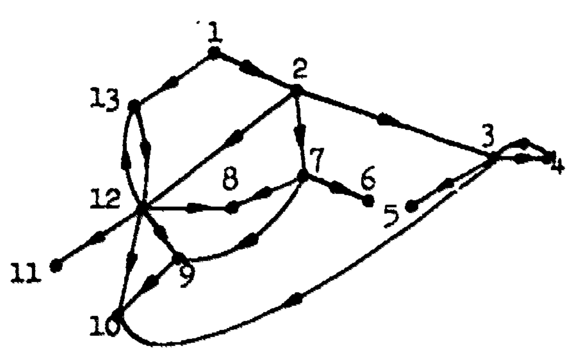


Figure 24

*3. Who is the Most Powerful? This is a directed graph problem. In figure 24, the vertices represent people; and the "arrows" represent influence. An arrow goes from x to y if the influence of x on y is of significance. Decide who the most powerful person is. [9-136]. (A teacher who has an especially troublesome class is sometimes advised to make a diagram like this one. The information for it is obtained from student questionnaires.)

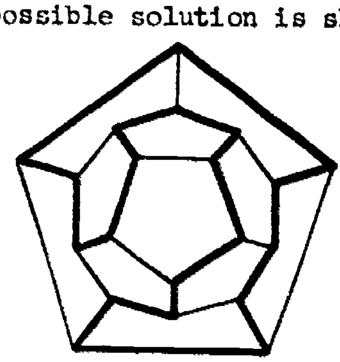
4. Jobs and Applicants. A firm has n vacant jobs of various types and has a group of n applicants such that each workman is qualified for one or more of the jobs. Under what conditions is it possible to assign each man to a position for which he is qualified? [4-43] and [9-92].

5. Optimum Personnel Assignment. There are n workers in a firm who are to be assigned to n different machines. The output of each worker with respect to each machine is known. How should assignments be made so as to achieve the maximum total output? [7-vii, 79] and [9-229].

Even though there are similarities between problems four and five, problem five is much more difficult.

Answers to Selected Problems

2. One possible solution is shown below.



3. One might think that because person number 12 has the most direct influence that he was the most powerful, but the people whom he dominates are not very influential. Person number 2 is the most powerful because the three people whom he influences are themselves very influential.

Books to Use

Graph theory is a branch of topology. Two types of works have been included in this list: all the basic works on graph theory and some additional books which are suitable for high school students and which are about other areas of topology. The first six books are listed in their order of difficulty. The first five were written for high school students. Those which contain both exercises and solutions are numbers 1, 3, 4, and 5. The last five books are more difficult. They are listed alphabetically.

There are two famous problems that are included in most of these books. The first is a proposed theorem about the number of colors needed to color a map. Because map coloring is easy to explain, this topic is often included in beginning books even though the proofs for the part of the theorem that have been proved are quite difficult. Map coloring is discussed in numbers 1 - 4, 6, 8, and 10. The other famous problem is the Koenigsberg bridge problem. It was the first problem in the first graph theory book, which was written by Euler a little over 200 years ago. This problem is discussed in numbers 1, 2, 4, 6, 8, 9, and 11.

Elementary Works

[1] Johnson, D. A., and Glenn, W. H., Topology, the Rubber-Sheet Geometry. Pasadena: Webster Publishing Company, 1966. 40pp. This booklet is a general introduction to topology and thus includes a number of different topics. One section of it is devoted to tricks and puzzles which involve topological concepts.

[2] Barr, Stephen, Experiments in Topology. New York: Thomas Y. Crowell Company, 1964. 210 pp. This book is written in a very readable style. To help the reader understand certain concepts, instructions are given on how to make paper models of a number of different topological surfaces. The book also includes a chapter on sets and Venn diagrams.

[3] Dynkin, E. B., and Uspenski, V. A., Multicolor Problems. Boston: D. C. Heath and Company, 1963. 66 pp. This booklet is very thorough. It includes a short appendix on coloring spheres.

[4] Ore, Oystein, Graphs and Their Uses. New York: Random House, Inc., 1963. 131 pp. As can be seen from the table on page 40, Ore's terminology is different from that used here. The beginning of the book is fairly informal, but the rest of it requires thorough work.

[5] Chinn, W. G., and Steenrod, N. E., First Concepts of Topology: The Geometry of Mappings of Segments, Curves, Circles, and Disks. New York: Random House, Inc., 1966. 160 pp. Both [4] and [5] are part of the same series: the New Mathematical Library, the Monograph Project of the School Mathematics Study Group.

[6] Arnold, B. H., Intuitive Concepts in Elementary Topology. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1962. 182 pp. This book includes an explanation of mathematical proofs by induction. It gives a proof for a special case of the Jordan curve theorem, a theorem which is referred to frequently in topology. The book contains problems but not solutions. It was written as a text for college sophomores. It is my impression that only the last two chapters require more than high school material. However, what makes the book a college text is that the student needs to have had experience in working with proofs.

Advanced Works

[7] Avondo-Bodino, G., Economic Applications of the Theory of Graphs. New York: Gordon and Breach, 1962. 108 pp. The subject matter of this book is the solution of four practical but difficult economic problems, one of which is problem five on page 24. Most of the book is on a very abstract level. The first two chapters are a little easier than [11], but the rest of the book is much more difficult.

[8] Ball, W. Rouse, Mathematical Recreations and Essays. New York: Macmillan, 1962. (Reprint of an 1892 edition.) 418 pp. The introductions to the Koenigsberg bridge problem, the fifteen girls problem, and especially the map coloring problem are very clear and understandable. However, after a topic has been introduced, the discussion becomes rather complex.

[9] Berge, C., The Theory of Graphs and its Applications. Translated by Alison Doig. New York: John Wiley and Sons, Inc., 1962. 247 pp. This book is very difficult. However, it contains a number of problems, some with solutions, which are fun to read. They are found on the pages listed below.

30	41	72	109	135-36	178	188
35	42	92	110	165	179	202
36	66	107	112	176-77	187	204

[10] Harary, Frank, "Combinatorial Problems in Graphical Enumeration," Applied Combinatorial Mathematics. Edited by Edwin F. Beckenbach. New York: John Wiley and Sons, Inc., 1964. This chapter discusses some of the difficult problems in graph theory which have been solved and a number which have not.

[11] Ore, Oystein, Theory of Graphs. American Mathematical Society Colloquium Publications, Volume XXXVIII. Providence, Rhode Island: American Mathematical Society, 1962. 270 pp. This book contains problems but not solutions.

Table of Terminologies

Graph theory is a fairly new area of study; and, as yet, the terminology is not standardized. To aid you in further study, the following chart lists the terms used by some of the different authors. This booklet uses Berge's terminology for non-directed graphs.

Berge non-directed	Berge directed	Avondo-Bodino non-directed	Avondo-Bodino directed	Ore
vertex or point	vertex or point	point	point	vertex
edge	arc	line	arc	edge
chain	path	chain	course	connected sequence
simple chain	simple path			path
	elementary path			arc or simple path
cycle	circuit	cycle	circuit	
simple cycle		simple cycle		cyclic path
elementary cycle	elementary circuit	elementary cycle		circuit
tree	arborescence	tree	arborescence	tree

Glossary

Chain.
A non-empty sequence of edges $(u_1, u_2, \ldots)$ is a chain if and only if each u_k has one vertex in common with the preceding edge u_{k-1}, and the other vertex in common with the succeeding edge u_{k+1}. (Page 8).

Connected graph.
A graph is a connected graph if and only if, for every pair of distinct vertices, there is a chain going from one to the other. (Page 11).

Cycle.
A finite chain $(u_1, u_2, \ldots, u_n)$ is a cycle if and only if the first vertex of u_1 is the same as the last vertex of u_n. (Page 9).

Elementary cycle.
A cycle is an elementary cycle if and only if:
1. the vertices used to define the edges are used by the edges in the given cycle exactly twice, and
2. no edge is used more than once. (Page 10).

Finite chain.
A chain is a finite chain if and only if it has a last edge. (Page 9).

Finite graph.
A graph is a finite graph if and only if the set consisting of its edges is a finite set. (Page 5).

Finite set.
A set is a finite set if and only if there exists a positive integer m such that the elements of the set can be put into a one-to-one correspondence with a proper subset of $\{1, 2, \ldots, m\}$. (Page 5).

Graph.
The set consisting of a set, X, and a function, f, is a graph if and only if:
1. X is a non-empty set of vertices, and
2. f is a function
   a. whose domain is the set of all pairs of vertices,
   b. whose range is a subset of the non-negative integers, and
   c. which is such that, if x and $y \in X$, then $f(x,y) = f(y,x)$. (Page 3).

Graph with multiple edges.
A graph is a graph with multiple edges if and only if it has at least one pair of vertices connected by more than one edge. (Page 4).

Infinite chain.
A chain is an infinite chain if and only if it is not a finite chain. (Page 9).

Infinite graph.

A graph is an infinite graph if and only if it is not a finite graph. (Page 5).

Null graph.

A graph is a null graph if and only if it has no edges. (Page 4).

Planar graph.

A graph is a planar graph if and only if it is possible to represent it on a plane in such a way that the vertices are all distinct points and no two edges meet except at vertices. (Page 7).

Proper graph.

A graph is a proper graph if and only if it has no isolated vertices. (Page 4).

Simple chain.

A chain is a simple chain if and only if no edge is used more than once. (Page 9).

Simple cycle.

A cycle is a simple cycle if and only if no edge is used more than once. (Page 10).

Tree.

A graph is a tree if and only if:

1. it is finite,
2. it is connected,
3. it has at least two vertices, and
4. it has no elementary cycles. (Page 11).

Universal graph.

A graph is a universal graph if and only if each pair of vertices is connected by one and only one edge. (Page 4.)