DOCUMENT RESUME

ED 175 682 SE 028 670

AUTHOR Bridgess, M. Philbrick, Ed.
TITLE Supplementary and Enrichment Series: Absolute Value. SP-24.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group
SPONS AGENCY Natio Science Foundation, Washington, D.C.
PUB DATE 66
NOTE 43p.; For related documents, see SE 028 648-675; Contains occasional light and broken type

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Algebra; Curriculum; *Enrichment; *Graphs; *Instruction; Mathematics Education; *Number Concepts; Secondary Education; *Secondary School Mathematics; Supplementary Reading Materials
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include absolute value, addition and multiplication in terms of absolute value, graphs of absolute value in the Cartesian plane, absolute value and quadratic expressions, and absolute value, complex numbers, and vectors. (MP)

SCHOOL MATHEMATICS STUDY GROUP

SP-24

SUPPLEMENTARY and ENRICHMENT SERIES

ABSOLUTE VALUE

Edited by M. Philbrick Bridgess

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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FOREWORD

This pamphlet is essentially a reprint of selected portions from the texts titled "First Course in Algebra" and "Intermediate Algebra" published by the School Mathematics Study Group.

The concept of absolute value is one of the most useful ideas of mathematics. We shall find an application of absolute value when we define addition and multiplication of real numbers. It is used to define the distance between points, and to express the correct result to the simplified form of a radical. Absolute value is used in open sentences in one and two variables. It gives us interesting functions and relations to graph. In later mathematics courses, such as the calculus and linear programming, the idea of absolute value is indispensable.

CONTENTS

ABSOLUTE VALUE

Chapter 1

INTRODUCTION

1-1. Definition of Absolute Value

You are familiar with binary operations (addition, for example) in the set of real numbers. There are, also, unary operations which are operations performed on a single number. For example, the "opposite" (or negative) of -2 is 2, of n is $-n$. Another example of a unary operation is "to take the reciprocal of a non-zero number". For any real number n, $n \neq 0$, $\frac{1}{n}$ is the reciprocal of n. Thus, the reciprocal of 2 is $\frac{1}{2}$, of $\frac{1}{3}$ is 3, and of $-\pi$ is $\frac{1}{-\pi}$. This pamphlet is concerned with another operation on a single real number, an operation which has many uses in mathematics. We shall find that it describes conveniently certain mathematical situations and that its use simplifies certain other operations. When we find a pattern occurring over and over again it is desirable to name it and to describe it carefully. We shall define this operation, called absolute value, and then subsequently restate the definition in three different forms. These restatements could be considered to be theorems as they can be shown to be equivalent to the original definition. In a given situation one statement can be applied better or more directly than another. We define absolute value as follows:

DEFINITION: The absolute value of a non-zero number is the greater of that number and its opposite. The absolute value of zero is 0.

Thus, the absolute value of 4 is 4, since 4 is greater than its opposite, -4. The absolute value of $-\frac{3}{2}$ is $\frac{3}{2}$. (Which is greater, $\frac{3}{2}$ or $-\frac{3}{2}$?). If n is any real number, the absolute value of n is non-negative. We are careful to say "non-negative" rather than "positive". Remember the absolute value of 0 is 0, and 0 is non-negative, not positive.

As usual, we find it convenient to agree on a symbol to indicate the operation. We write

$$|n|$$

to mean the absolute value of n. For example, $|5| = 5$, $|-\frac{2}{3}| = \frac{2}{3}$,

$|-\sqrt{2}| = \sqrt{2}$ and $|0| = 0$. Note that each of these is non-negative. In symbolic form our definition could be written:

$$|n| = \text{maximum}\,(n,-n).$$

Let us graph -4, 0, and 4 on the number line.

By definition the distance between any two points on the number line is a non-negative real number. The distance between 4 and 0 is 4,* but also the distance between -4 and 0 is 4. In fact, for any real number n, the distance between n and 0 is what we have just called the absolute value of n. This is true whether n is positive, negative, or zero.

We can restate the definition of absolute value.

> The distance between a real number and 0 on the real number line is the absolute value of the number.

Problem Set 1-1a

1. Find the absolute values of the following numbers:

   (a) -7

   (b) $-(-3)$

   (c) $(6 - 4 + 5)$

   (d) 14×0

   (e) $-(14 + 0)$

   (f) $-(-(-3))$

2. Which word, negative or non-negative, would correctly fill each blank?

   (a) $\frac{4}{3}$ is a __________ number.

   (b) $|\frac{4}{3}|$ is a __________ number.

   (c) If x is a non-negative number, then $|x|$ is a __________ number.

   (d) $-\frac{2}{5}$ is a __________ number.

   (e) $|-\frac{2}{5}|$ is a __________ number.

   (f) If x is a negative number, then $|x|$ is a __________ number.

   (g) For any real number x, $|x|$ is a __________ number.

3. If x is a negative number which is greater, x or $|x|$?

---

* Actually you cannot find the distance between numbers. You find the distance between points. This is an accepted abbreviation.

4\. (a) Let $S = \{-1,-2,1,2\}$. Let A be the set of absolute values of the elements of S. Write the set A. Is A a subset of S?

(b) Is the set $\{-1,-2,1,2\}$ closed under the operation of taking absolute values of its elements? (A set is closed under an operation if the number resulting from the operation on the element of the set is itself an element of the set).

5\. Is the set of all real numbers closed under the operation of taking absolute values?

We note that for a non-negative number, the greater of the number and its opposite is the number itself. That is:

For every real number x which is 0 or <u>positive</u>,

$$|x| = x.$$

What can be said of a negative number and its absolute value? Write common numerals for the following pairs.

$\|-5\| =$	$\|-3.1\| =$
$-(-5) =$	$-(-3.1) =$
$\|-\frac{1}{2}\| =$	$\|-467\| =$
$-(-\frac{1}{2}) =$	$-(-467) =$

(What kind of numbers are -5, $-\frac{1}{2}$, -3.1, -467?) You found that

$\|-5\| = -(-5)$	$\|-3.1\| = -(-3.1)$
$\|-\frac{1}{2}\| = -(-\frac{1}{2})$	$\|-467\| = -(-467)$

Is it now clear that we can say, "The absolute value of a <u>negative</u> number is the <u>opposite</u> of the negative number"? That is:

For every <u>negative</u> real number x,

$$|x| = -x$$

A third statement of the definition of absolute value is, therefore;

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The fourth statement of the definition that we might use, $|x| = \sqrt{x^2}$, will be considered later.

<u>Problem Set 1-1b</u>

1. Which of the following sentences are true?

(a) $|-7| < 3$

(b) $|-2| \leq |-3|$

(c) $|4| < |1|$

(d) $2 \nless |-3|$

(e) $|-5| \nless |2|$

(f) $-3 < 17$

(g) $-2 < |-3|$

(h) $|\sqrt{16}| > |-4|$

(i) $|-2|^2 = 4$

2. Write each as a common numeral

(a) $|2| + |3|$

(b) $|-2| + |3|$

(c) $-(|2| + |3|)$

(d) $-(|-2| + |3|)$

(e) $|-7| - (7 - 5)$

(f) $7 - |-3|$

(g) $|-5| \times 2$

(h) $-(|-5| - 2)$

(i) $|-3| - |2|$

(j) $|-2| - |-3|$

(k) $-(|-3| - 2)$

(l) $-(|-2| + |-3|)$

(m) $3 - |3 - 2|$

(n) $-(|-7| - 6)$

(o) $|-5| \times |-2|$

(p) $-(|-2| \times 5)$

(q) $-(|-5| \times |-2|)$

3. What is the truth set (solution set) of each open sentence?*

(a) $|x| = 1$

(b) $|x| = 3$

(c) $|x| + 1 = 4$

(d) $5 - |x| = 2$

(e) $5 + |x| = 2$

(f) $7 - 2|x| = 12 - 5$

4. Which of the following open sentences are true for all real numbers x?

(a) $|x| \geq 0$

(b) $x \leq |x|$

(c) $-x < |x|$

(d) $-|x| \leq x$

(Hint: Give x a positive value; a zero value; a negative value. Now come to a decision.)

5. Show that if x is a negative real number, then x is the opposite of the absolute value of x; that is, if $x < 0$, then $x = -|x|$. (Hint: What is the opposite of the opposite of a number?)

---

* The truth set (solution set) of an open sentence is the set of numbers in the domain of the variable for which the sentence is true.

The absolute value of a variable can occur in an open sentence such as $|x| < 2$. We notice that $-\frac{3}{2}$, $-\frac{1}{5}$, 0, 1, $\frac{5}{3}$, for example, are all in the truth set of $|x| < 2$, while -8, -4, -2.1, 2, 7, $\frac{25}{2}$ for example, are not. The truth set of $|x| < 2$ consists of all the real numbers between -2 and 2. The graph of the truth set of $|x| < 2$ on a Number Line is:

We use open circles at 2 and -2 to show that the points which have 2 and -2 as coordinates do <u>not</u> belong to the graph of the truth set of $|x| < 2$.

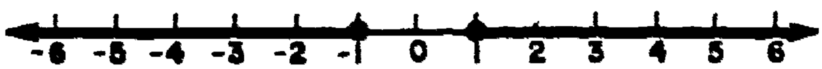
<u>Problem Set 1-1c</u>

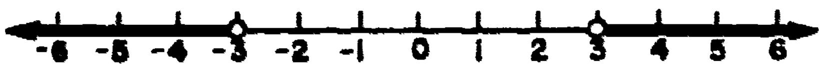
1. Graph the truth sets of the following sentences on Number Lines:

   (a) $|x| < 4$
   (b) $|x| > 2$
   (c) $|x| \leq 3$
   (d) $|x| \geq 4$
   (e) $|x| < -2$
   (f) $|x| > 0$
   (g) $|x| > -1$

2. Using the absolute value symbol, write an open sentence for each of the following graphs:

   (a)


   (b)


   (c)


Let us look again at the graph of $|x| < 2$.

The graph indicates that the truth set consists of those numbers which are coordinates of points <u>between</u> -2 <u>and</u> 2. Thus, the open sentence "$|x| < 2$" has the same truth set as the compound open sentence

"$x > -2$ <u>and</u> $x < 2$".

We say that they are equivalent open sentences since they have the same truth set. Of course, we can also write $-2 < x < 2$ to state the same thing.

What can we say about the open sentence for the graph below?

This is the graph of "$|x| > 2$". It is the same graph as that of the compound open sentence

$$"x < -2 \ \underline{\text{or}} \ x > 2".$$

The open sentence $2 < x < -2$, which means that x is greater than 2 <u>and</u> less than -2, is clearly an impossibility. It has the empty set, ϕ, as the truth set.

<u>Problem Set 1-1d</u>

1. Fill in the blank of each of the following with the correct word or symbol.

   (a) The graph of "$|x| < 5$" is the same as the graph of "$x > -5$ _____ $x < 5$".

   (b) The graph of "$|x| > 6$" is the same as the graph of "$x < -6$ _____ $x > 6$".

   (c) The truth set of "$x > -\frac{5}{2}$ and $x < \frac{5}{2}$" is the same as the truth set of __________.

2. $|x| \leq 8$ has the same truth set as which of the following:

   (A) $x \leq -8$ or $x \geq 8$
   (B) $x \leq -8$ and $x \geq 8$
   (C) $x \geq -8$ and $x \leq 8$
   (D) $x \geq -8$ or $x \leq 8$
   (E) $-8 \leq x \leq 8$

3. Which of the following open sentences suggests the English sentence "The temperature stayed within 5 degrees of zero today"?

   (A) $x < 5$
   (B) $x < 5$ and $x > -5$
   (C) $x < 5$ or $x > -5$
   (D) $|x| < 5$
   (E) $|x| > 5$

1-2. <u>Subtraction, Distance, and Absolute Value</u>

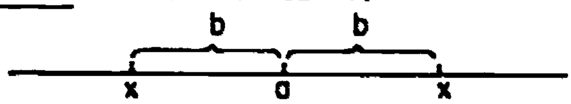
We have seen in Section 1-1 that the <u>distance between</u> a number and 0 on the real number line is the <u>absolute value</u> of that number. Let us see how we can relate "subtraction, distance, and absolute value". The distance between 13 and 9 is the same as the distance between 9 and 13, that is, a distance of 4. (Remember: We are actually referring to the distance between the points whose coordinates are 9 and 13). The distance between two numbers a and b on the number line is always non-negative. We remember that the absolute value of a number is non-negative. $|13 - 9| = |9 - 13| = 4$. Thus, the distance between a and b may be indicated by either

$$|a - b| \quad \text{or} \quad |b - a|$$

For any real number x, the distance between 4 and x can be written either $|4 - x|$ or $|x - 4|$. The sentence "$|x - 4| = 1$", therefore, can be interpreted as "the distance between x and 4 is 1". Its graph would be:

The distance between 4 and each of the points indicated by heavy dots on the number line is 1. Thus, the truth set of the sentence $|x - 4| = 1$ is $\{3,5\}$.

We can state this in general terms: The sentence $|x - a| = b$ indicates that the distance between x and a is b.



This is in agreement with our earlier discussion. If we take $a = 0$, then $|x - 0| = |x|$ is the distance between x and 0.

Problem Set 1-2a

1. Write each of the following using absolute value symbols:

   (a) The distance between 3 and 7
   (b) The distance between 6 and -2
   (c) The distance between -8 and -5
   (d) The distance between t and 6
   (e) The distance between y and 0
   (f) The distance between m and -4

2. Give the graph on the Number Line and truth set of each of the following:

   (a) $|x - 5| = 3$
   (b) $|y - 2| = 5$
   (c) $|4 - b| = 6$
   (d) $|a + 3| = 2$
   (e) $|s - 2| = -3$
   (f) $|m + 5| - 2 = 0$
   (g) $|2x - 3| = 1$
   (h) $|3d + 4| = 6$

$|x - 4| < 1$ can be interpreted as "the distance between x and 4 is less than 1". Its graph woul be

We can see that this is also the graph of "$x > 3$ <u>and</u> $x < 5$", usually written $3 < x < 5$.

The graph of $|x - 4| < 1$ is called an "open interval". It is a line-segment minus its endpoints. The graph of $|x - 4| \leq 1$, which would contain its endpoints, is called a "closed interval".

The graph of $|x - 4| > 1$ would be:

What compound open sentence (See Problem Set 1-1d, Ex. 1(b)) describes this graph?

<u>Problem Set 1-2b</u>

1. Give the graph and the truth set of each of the following:

   (a) $|x - 2| < 2$
   (b) $|y - 4| > 8$
   (c) $|r + 7| < 5$
   (d) $|b + 3| > 4$
   (e) $|c - 2| \geq 6$
   (f) $|d + 5| \leq 3$
   (g) $|x| + 3 > 0$ (Be careful!)
   (h) $|x - 4| + 5 < 0$

2. Which of the following sentences is <u>not</u> equivalent to the other three?

   (A) $|x| < 3$
   (B) $-3 < x$ and $x < 3$
   (C) $-3 < x$ or $x < 3$
   (D) $-3 < x < 3$

3. Are the graphs of "$|x| \geq 3$", "$|x| \nless 3$", and "$x \leq -3$ or $x \geq 3$" the same or different? If different in what way are they different?

4. Describe the truth sets of each of the following?

   (a) $|x| = x$
   (b) $-|m| < 2$
   (c) $|y| < -2$
   (d) $|u| = -u$
   (e) $|v| \geq 0$

5. Given the open sentence "$|x| < 3$". Draw the graph of its truth set if the domain of x is the set of:

   (a) real numbers
   (b) integers
   (c) non-negative real numbers
   (d) negative integers

6. Use the number line to show that the set of all x, $|x - a| < b$ may also be described as the set of all x such that $a - b < x < a + b$.

7. Show that if $|x - 2| < \frac{1}{4}$, then $|(5x + 3) - 13| < \frac{5}{4}$.

8. If x is a number in the interval defined by $|x + 2| < 5$, what is the largest interval inside of which x^2 may be found?

Chapter 2

ADDITION AND MULTIPLICATION IN TERMS OF ABSOLUTE VALUE

2-1. Definition of Addition

In this chapter we are going to state the processes of addition and multiplication in terms of absolute value and exhibit a few theorems. First of all, we want a rule for the sum of two numbers. It will be necessary to consider the various possibilities. If a and b are non-negative then we know the meaning of $a + b$ from our experience with arithmetic, namely, $a + b = |a| + |b|$. We shall base our rules for the other possibilities on this knowledge and the fact that $|a|$ and $|b|$ are non-negative.

What is a general rule for the addition of two negative numbers? We know that $(-4) + (-6) = -10$, that $(-2) + (-7) = -9$. In each case, the result is a negative number. We can find the sum by first adding 4 and 6 and taking the opposite of that sum. Now 4 is the absolute value of -4; likewise, $6 = |-6|$. Thus, $(-4) + (-6) = -(|-4| + |-6|)$. Similarly, $(-2) + (-7) = -(|-2| + |-7|)$.

A general rule for the sum of two <u>negative</u> numbers is:

> The sum of two negative numbers is negative. The absolute value of the sum is the sum of the absolute values of the numbers.

An equivalent statement of this fact is:

> If $a < 0$ and $b < 0$, then
> $$a + b = -(|a| + |b|)$$

Let us go on to consider the sum of two real numbers when one number is non-negative and the other is negative. The phrase "one number is non-negative" implies that one of the numbers may be 0. This possibility presents no special difficulties.

$$
\begin{aligned}
0 + (-7) &= -7 \\
0 + b &= b, \quad \text{for any negative number } b. \\
\left(-\frac{8}{3}\right) + 0 &= -\frac{8}{3} \\
a + 0 &= a, \quad \text{for any negative number } a.
\end{aligned}
$$

Now let us concentrate our attention on those cases where one of a or b is negative and the other is positive.

$6 + (-4) = 2$	$6 > 0,\ (-4) < 0$	$\|6\| > \|-4\|$
$4 + (-6) = -2$	$4 > 0,\ (-6) < 0$	$\|-6\| > \|4\|$
$7 + (-7) = 0$	$7 > 0,\ (-7) < 0$	$\|7\| = \|-7\|$

How do we get the "2" and how do we decide whether it is positive or negative? Let us consider the sum of 7 and -3. $7 + (-3) = 4$. We know that $7 > 0$, and that $(-3) < 0$, also that $|7| > |-3|$. The sum of 7 and (-3) has the form $a + b$, where $a > 0$ and $b < 0$ and $|a| \geq |b|$. 4 is $|7| - |-3|$. Thus, we can write:

$$\text{If } a > 0 \text{ and } b < 0, \text{ and if } |a| \geq |b|, \text{ then}$$
$$a + b = |a| - |b|.$$

Similarly, $5 + (-11) = -6$; $5 > 0$, and $(-11) < 0$; $|-11| > |5|$. Again, $5 + (-11)$ has the form $a + b$, where $a > 0$ and $b < 0$, but in this case $|b| > |a|$. -6 is $-(|-11| - |5|)$. Hence, we can write:

$$\text{If } a > 0 \text{ and } b < 0, \text{ and if } |b| > |a|, \text{ then}$$
$$a + b = -(|b| - |a|).$$

By now we have a pattern for completing the interpretation of $a + b$ in terms of absolute value.

$$\text{If } a < 0 \text{ and } b \geq 0, \text{ and if } |b| \geq |a|, \text{ then}$$
$$a + b = |b| - |a|$$
$$\text{If } a < 0 \text{ and } b \geq 0, \text{ and if } |a| > |b|, \text{ then}$$
$$a + b = -(|a| - |b|).$$

You are not expected to memorize these results. Given two real numbers, you should be able to add them without referring to the rules given here. On the other hand, we hope that you understand them and, therefore, will be able to use these results later in the pamphlet.

Problem Set 2-1

1. Fill in the blanks.

   (a) The absolute value of the sum of a negative and a non-negative number is equal to ____________ of the absolute values of the numbers.

   (b) It is non-negative if the non-negative number has the ____________ absolute value.

   (c) It is negative if the negative number has the greater ____________.

2. (a) Is the set all real numbers closed under the operation of addition?

   (b) Is the set of negative real numbers closed under the operation of addition?

3. (a) Does $|2| + |3| = |2 + 3|$?

   (b) Does $|-2| + |-3| = |(-2) + (-3)|$?

   (c) Does $|-2| + |3| = |(-2) + 3|$?

   (d) Does $|2| + |-3| = |2 + (-3)|$?

   (e) In (c) and (d) compare the left member to the right member of the expression?

   (f) Write a general statement for $|a| + |b|$ in terms of $|a + b|$. This is called "The Triangle Inequality."

4. Repeat Exercise 3 using numerical values for $|a| - |b|$ and $|a - b|$ and then make a general statement comparing $|a| - |b|$ with $|a - b|$.

5. (a) Show that $|c - d| = |d - c|$.

   (b) Show that $|c - d| \geq ||c| - |d||$ using numerical values for c and d.

   (c) Statement (b) says that the distance between c and d is at least as great as the distance between $|c|$ and $|d|$. Use the number line to visualize why this must be true.

6. Given that $|x| + |y| \geq |x + y|$.

   Prove: If a and b are real numbers, then $|a - b| \geq |a| - |b|$

   Let $x = a - b$, and $y = b$

   Then $|a - b| + |b| \geq |(a - b) + b|$

   Or $|a - b| + |b| \geq |a|$ Why?

   $|a - b| + |b| + (-|b|) \geq |a| + (-|b|)$ Why?

   Hence, $|a - b| \geq |a| - |b|$

   Supply the reasons.

2-2. Definition of Multiplication

As in the case of addition, the product of two real numbers can be stated in terms of absolute value. It is of primary importance here, as in the definition of addition of real numbers, that we maintain the "structure" of the number system. We want to insure that certain properties of numbers under multiplication (commutative, associative, distributive) which hold for non-negative real numbers still hold for all real numbers. On this basis, we define:

$$(-3)(0) = 0$$
$$(-3)(2) = -6$$
$$(3)(-2) = -6$$
$$(-3)(-2) = 6$$

How shall we interpret these results and state them in terms of absolute value?

It follows at once from the definition of absolute value that $(2)(3) = |2||3|$. Thus, if we formally state

$$ab = |a||b|$$

for positive numbers a and b, we shall not conflict with what we already know about the product of two positive numbers.

Now consider $(-2)(-3)$. Note that $|-2||-3| = (2)(3) = 6$. Hence, $(-2)(-3) = |-2||-3|$. It would appear that for any pair, a and b, of negative real numbers it is true that:

$$ab = |a||b|$$

Let us try the same approach on $(-2)(3)$. We know, $(-2)(3) = -6$. We know that $|-2||3| = 6$. Thus, $(-2)(3) = -(|-2||3|)$. From this example it appears that for a pair of real numbers, a and b, of which one is positive and the other is negative, it is true that

$$ab = -(|a||b|)$$

Finally, consider the product $(0)b$, where b is any real number. Our statement of multiplication must insure that $(0)b = 0$. Since b is non-negative, and since $|0| = 0$, we can state that:

$$|0||b| = 0$$

and

$$-(|0||b|) = 0 .$$

Here we conclude that for a pair of real numbers, a and b, of which at least one is zero, both

$$ab = |a||b|$$

and

$$ab = -(|a||b|)$$

should be valid.

This can be summarized as follows:

> Let a and b be any two real numbers. In case a and b are both negative, or both non-negative
>
> $$ab = |a||b|$$
>
> In case one of the numbers is negative and the other is non-negative
>
> $$ab = -(|a||b|)$$

Problem Set 2-2

1. Use $|7||-3|$ and $|(7)(-3)|$, $|-8||-6|$, and $|(-8)(-6)|$ to decide what is the relationship of $|a||b|$ to $|ab|$.

2. Demonstrate if $n \neq 0$, then $|\frac{1}{n}| = \frac{1}{|n|}$.

3. Demonstrate if $n \neq 0$, then $|\frac{m}{n}| = \frac{|m|}{|n|}$.

4. Suppose that we know only that $a \cdot 1 = a$ for $a \geq 0$. Fill in the missing reasons in the proof that $a \cdot 1 = a$, if a is negative.

Since a is negative and 1 is non-negative

$a \cdot 1 = -(\|a\|\|1\|)$	?
$= -(\|a\| \cdot 1)$	$\|1\| = 1$
$= -\|a\|$	Multiplication property of 1 for non-negative real numbers.

But, since a is negative

$\|a\| = -a$	?
Hence, $-\|a\| = -(-a)$	
$= a$	?

Which completes the proof.

5. Prove that $|a| = |-a|$ (Hint: Consider the possible cases)

6. Prove that for any number a, $-|a| \leq a \leq |a|$

7. (a) When does $|x - 2| = x - 2$?
   (b) When does $|x - 2| = 2 - x$?

8. Solve the equations:
   (a) $|x + 3| = x$ (Hint: If $x + 3 > 0$, then $|x + 3| = x + 3$. If $x + 3 < 0$, then $|x + 3| = -(x + 3)$)
   (b) $|x - 2| = 2x + 5$
   (c) $|3x + 4| = |5 - 2x|$
   (d) $|x - 2|^2 - 4|x - 2| + 3 = 0$ (Hint: Factor in terms of $|x - 2|$)
   (e) $|x - 3|^3 = 1$
   (f) $|x| = 2$ and $|x - 5| = 3$
   (g) $|x + 3| = 2$ or $|x - 1| = 2$

9. If we assume that $ab = ba$ for all non-negative numbers a and b, we can prove it for all real numbers. Fill in the missing reasons or statements

Case I Suppose that a and b are both negative

$ab = \|a\|\|b\|$	?
$= \|b\|\|a\|$	Commutative property of multiplication for non-negative real numbers
$= ba$	Definition of multiplication

Case II Suppose that one of the numbers a and b is non-negative and that the other is negative.

$ab = -(\|?\|\|?\|)$	Definition of multiplication
$= -(\|b\|\|a\|)$	?
$= ba$	?

10. (a) Show that $|mx + ma| = |m||x + a|$

(b) Use this to find the truth set of $|3x - 12| < d$, where d is a positive number.

2-3. Absolute Value and Simplification of Radicals

When we write $\sqrt{b}$, we mean the positive square root of b, or more simply "the square root of b". If $b = 0$, then $\sqrt{0} = 0$. Thus we can state that the $\sqrt{b} \geq 0$, provided that b is non-negative.

What is $\sqrt{x^2}$? Let us consider a few examples. If x is 3, then $\sqrt{x^2} = \sqrt{3^2} = \sqrt{9} = 3$, or $\sqrt{x^2} = x$. If x is -3, however, then $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$, or the $\sqrt{x^2} = -x$. If x is 0, then $\sqrt{x^2} = \sqrt{0^2} = 0 = x$. We can summarize this as follows:

$$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is exactly what we stated on page 3 for $|x|$, and thus we conclude that $\sqrt{x^2} = |x|$. $\sqrt{x^2}$ is a non-negative number for any value of x, positive, negative, or zero.

If the numbers with which we are working are coordinates of points on the Number Line, then we call them real numbers. They include such numbers as 5, π, $-\frac{2}{9}$, 3, and -5, but they do not include numbers like $\sqrt{-5}$, which is called an imaginary or complex number. You will meet complex numbers in Chapter 5.

How shall we write $\sqrt{x^3}$ in simpler form? If x is 3, then $\sqrt{x^3} = \sqrt{3^3} = \sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$ or $x\sqrt{x}$. What happens if x is -3? $\sqrt{x^3} = \sqrt{(-3)^3} = \sqrt{-27}$. Now $\sqrt{-27}$ is an imaginary number. If the numbers under consideration are real numbers only, then $\sqrt{-27}$ cannot be simplified. Thus we would say that $\sqrt{x^3} = x\sqrt{x}$ for $x \geq 0$. Note that we do not use the absolute value symbol here because it is not needed.

When working with radicals, we must be critical of the domain of the variable (the values of the variable which we can use) in a particular problem. We shall assume in the exercises that follow that we are always working with a real number.

Here is another example. Simplify $(2\sqrt{a})(3\sqrt{a})$ and indicate any restriction of the variable. If $a \geq 0$, then $(2\sqrt{a})(3\sqrt{a}) = 6\sqrt{a^2}$. If $a < 0$, then $2\sqrt{a}$ is not a real number, nor is $3\sqrt{a}$. Hence a must be non-negative. In this case $6\sqrt{a^2} = 6a$. We do not write $6|a|$, since a cannot be negative.

Let us try one more example. Simplify $\sqrt{\frac{b}{c}}$ and indicate the domain of the variables when they are restricted. Here we have two possibilities; either $b \geq 0$ and $c > 0$, or $b < 0$ and $c < 0$.

$$\text{If } b \geq 0,\ c > 0,\ \sqrt{\frac{b}{c}} = \sqrt{\frac{b}{c} \cdot \frac{c}{c}} = \sqrt{\frac{bc}{c^2}} = \frac{1}{c}\sqrt{bc}$$

$$\text{If } b < 0,\ c < 0,\ \sqrt{\frac{b}{c}} = \sqrt{\frac{b}{c} \cdot \frac{c}{c}} = \sqrt{\frac{bc}{c^2}} = \frac{1}{|c|}\sqrt{bc}$$

If you try $b = 2$ and $c = 3$, or $b = -2$ and $c = -3$ you will see the need for the absolute value symbol in the second case, but not in the first.

The possible values of the variable in the original expression determine whether or not we use the symbol "$|\ |$" in the answer.

Problem Set 2-3

Simplify. Indicate the domain of the variable when it is restricted.

1. (a) $\sqrt{24x^2}$ (b) $\sqrt{24x^3}$ (c) $\sqrt{24x^5}$

2. (a) $\sqrt{32a^4}$ (b) $\sqrt[3]{32a^4}$ (c) $\sqrt[4]{32a^4}$

3. (a) $\sqrt{x^4 + x^2}$ (b) $\sqrt{(x^4)(x^2)}$ (c) $\sqrt{x^4} + \sqrt{x^2}$

4. (a) $(2\sqrt{3x})(5\sqrt{6x})$ (b) $(3\sqrt{x^2y})(\sqrt{ay^2})$

5. (a) $\sqrt{\frac{x^2}{9}}$ (b) $\sqrt{\frac{4y}{9y^3}}$ (c) $\frac{\sqrt{xy}}{\sqrt{x^3}}$

6. (a) $\frac{\sqrt{3a^2}}{\sqrt{25x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$ (b) $\sqrt{\frac{2a}{45}} \cdot \sqrt{\frac{a^5}{2}}$

7. (a) $\sqrt{4(a + b)^2}$ (b) $\sqrt{\frac{(x - 2)^2}{9}}$ (c) $\sqrt{c^2 - 6c + 9}$

8. $2\sqrt{12a^2} - \frac{3|a|}{\sqrt{3}} - \frac{1}{4}\sqrt{48a^2}$

9. $\sqrt{\frac{4m^2}{q}} + \sqrt{98m^2q^3}$

10. Determine the square root of $(2x - 1)^2$ if

(a) $x < \frac{1}{2}$ (b) $x > \frac{1}{2}$ (c) $x = \frac{1}{2}$

Chapter 3

GRAPHS OF ABSOLUTE VALUE IN THE CARTESIAN PLANE

3-1. Graphs of Open Sentences with One Variable

In Section 1-1 we examined the graphs on the Number Line of open sentences involving absolute value and one variable. Now let us consider the graphs of these open sentences in the Cartesian plane. If we are given the open sentence $x = 3$, and are asked to find its graph in the Cartesian plane, we are looking for those ordered pairs which belong to the set $\{(x,y) : x = 3\}$. We can see that some members of this set are $(3,-10)$, $(3,0)$, and $(3,5)$. Thus the graph of $x = 3$ in the Cartesian plane is the set of points on a line parallel to the y-axis and 3 units to the right of it.

By definition the equation $|x| = 3$ is equivalent to the compound open sentence "$x = 3$ or $-x = 3$". This is equivalent to "$x = 3$ or $x = -3$". Since equivalent sentences have the same truth sets, they also have the same graphs. Hence the graph of $|x| = 3$ consists of two lines, one line is parallel to the y-axis and three units to the right of it, the other is a line parallel to the y-axis and three units to the left of it. In other words it is the union of the graph of $x = 3$ and the graph of $x = -3$, as shown in Figure 1.

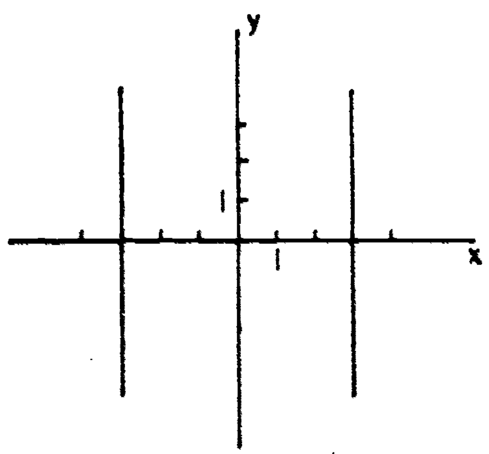


Figure 1

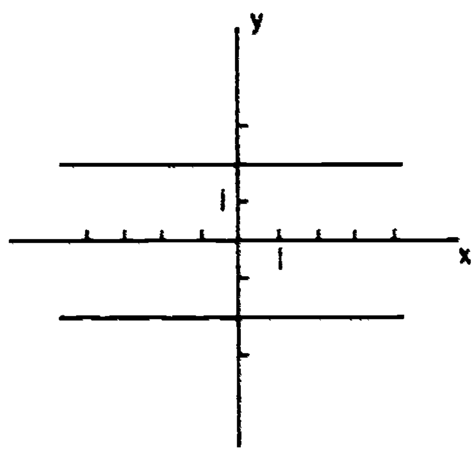


Figure 2

The graph of $|y| = 2$ is the union of the graph of $y = 2$ and the graph of $y = -2$. This graph is shown in Figure 2.

On the Number Line, "$|x - 3| = 2$" is interpreted as: "those numbers x such that the distance between x and 3 is 2".

Extending this notion to the number plane, we can say that $|x - 3| = 2$ describes the set of points whose x coordinates vary from 3 by exactly 2, i.e. those pairs with $x = 5$ or $x = 1$.

The graph in Figure 3 shows that the distance between the line $x = 1$ and the line $x = 3$ is 2; and the distance between the line $x = 5$ and the line $x = 3$ is 2.

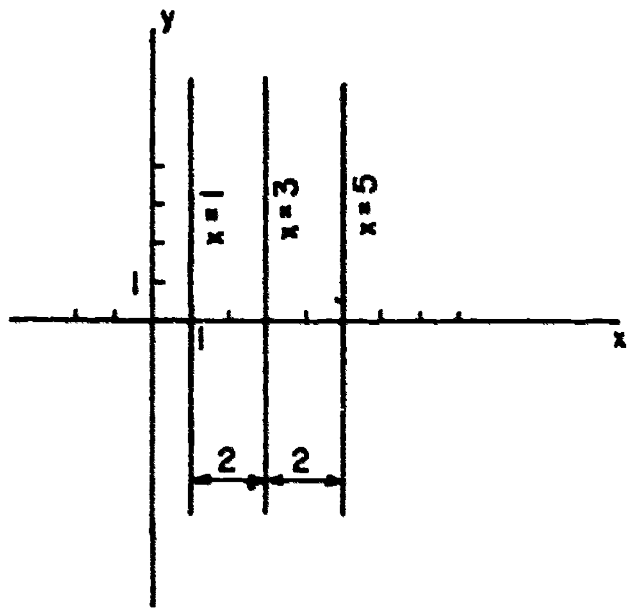


Figure 3

Problem Set 3-1a

1. (a) The graph of $\{(x,y) : |x| = k\}$ is a pair of vertical lines if k has what value?
   (b) The graph of $\{(x,y) : |x| = k\}$ is a single line if k has what value?

2. What is true of the graph of $\{(x,y) : |y| = k\}$ if:

   (a) $k > 0$ (b) $k = 0$ (c) $k < 0$

3. Draw the graph of the following open sentences in the Cartesian plane using a different set of axes for each sentence.

   (a) $|x| = 5$ (e) $|y + 2| = 3$
   (b) $|x - 2| = 3$ (f) $|x| - 3 = 1$
   (c) $|x + 2| = 4$ (g) $|y| + 4 = 3$
   (d) $|y - 3| = 2$ (h) $|x| = x$

Now let us consider the open sentence $|x| > 3$. What would be its graph? Recall that this sentence is equivalent to the sentence "$x > 3$ or $x < -3$". The graph will consist of all points whose x coordinates will be greater than 3 or less than -3. The graph is shown in Figure 4. Notice that the lines $x = 3$ and $x = -3$ are drawn with dashes to show that they are not included.

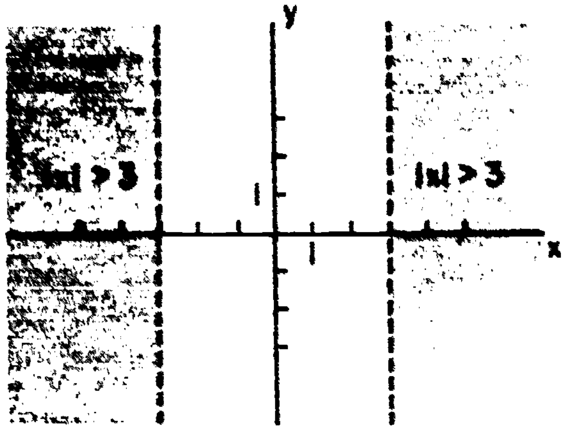


Figure 4

Problem Set 3-1b

1. Draw the graph of the following open sentences in the Cartesian plane using a different set of axes for each sentence.

(a) $|x| < 2$
(b) $|x| \geq 1$
(c) $|y| < 3$
(d) $|x - 2| \leq 3$
(e) $|y + 2| > 4$
(f) $|x + 1| > 0$

2. Same instructions as Exercise 1:

(a) $|x| > x$
(b) $|x| < x$

3-2. Graphs of Open Sentences with Two Variables

Let us consider the open sentence "$y = |x|$". If x is positive or negative, what is true of the absolute value of x? What, then, must be true of y for every value of x except 0? What is the value of y for $x = 0$?

x	-3	-2	-1	0	1	2	3
$\|x\|$	3	2	1	0	1	2	3

From Figure 5 we notice something new to us: the graph of the simple sentence "$y = |x|$" turns out to be the two sides of a right angle. Is it possible to have a simple equation whose graph would be two lines which do not form a right angle? Suggest one.

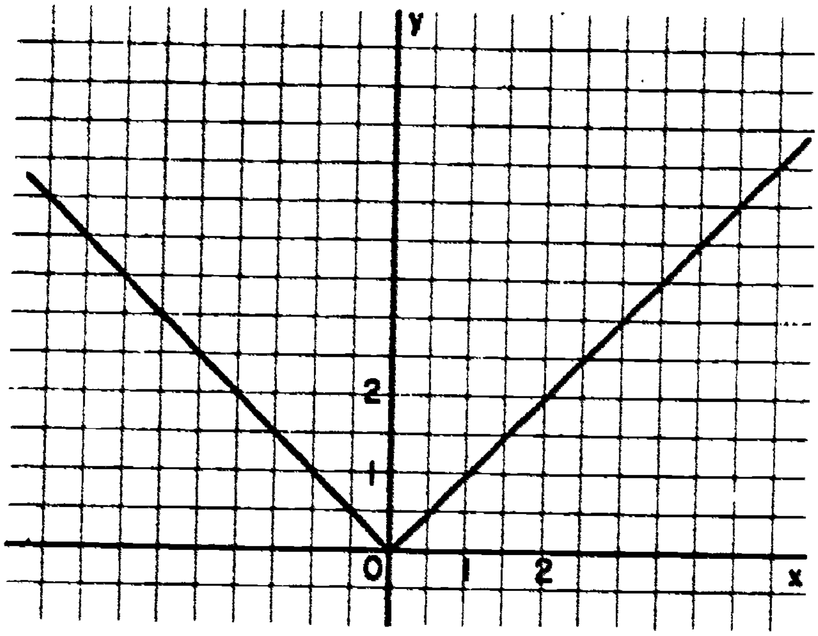


Figure 5

Problem Set 3-2a

1. Draw the graph of each of the following with reference to a separate set of axes.

   (a) $y = 2|x|$

   (b) $y = \frac{1}{2}|x|$

   (c) $y = -|x|$

   (d) $y = -2|x|$

   (e) $x = -|y|$

   (f) $x = |-2y|$

2. Draw the graph of each of the following with references to a separate set of axes.

   (a) $y = |x| + 3$

   (b) $y = |x| - 7$

   (c) $y = 2|x| + 1$

   (d) $x = |y| + 3$

   (e) $x = 2|y| - 1$

   (f) $y = -|x| - 1$

3. Draw the graph of each of the following with reference to a separate set of axes.

   (a) $y = |x - 2|$
   (b) $y = |x + 3|$
   (c) $y = 2|x + 3|$
   (d) $y = |x + 3| - 5$
   (e) $y = \frac{1}{2}|x - 1| + 3$

4. How would you get each of the graphs in Problems 1(c), (e), 2(a), (b), (d), (f), 3(a), (b), (d) from the graph of either $y = |x|$ or $x = |y|$ by revolving or sliding the graph? Examples: The graph of "$y = |x - 2|$" can be obtained by sliding the graph of "$y = |x|$" to the right 2 units. The graph of "$x = -|y|$" can be obtained by revolving the graph of "$x = |y|$" about the y-axis. The graph of "$y = |x| - 7$" can be obtained by sliding the graph of "$y = |x|$" down 7 units.

5. Plot the graph of:

   (a) $y = |x| + x$
   (b) $y = \frac{|x|}{x}$

6. Plot the following open sentences using a different set of axes for each sentence:

   (a) $y > |x|$
   (b) $y \leq |x - 2|$
   (c) $y < -|x| + 2$
   (d) $y \geq \frac{1}{2}|x - 3| - 5$

So far we have concerned ourselves with the graphs of open sentences in two variables in which there was only one absolute value symbol and only one variable whose absolute value was being considered. Let us consider a few open sentences of a somewhat more complicated nature. For example, what is the graph of $|x - y| = 0$, or the graph of $|x - y| = 2$?

By our definition of absolute value we know that $|x - y| = x - y$ if $x - y \geq 0$, and $|x - y| = -(x - y)$ if $x - y < 0$. Since $x - y = 0$ and $-(x - y) = 0$ are equivalent, the graph of $|x - y| = 0$ is the same as the graph of $y = x$. Now, what about $|x - y| = 2$? Again, if $x - y \geq 0$, then $|x - y| = 2$ is equivalent to $x - y = 2$, if $x - y < 0$, then $|x - y| = 2$ is equivalent to $-(x - y) = 2$ or $x - y = -2$. If we make a table of values we find:

x	-4	-2	0	2	4
y	-2,-6	0,-4	2,-2	4,0	[illegible],2

These points are on the graphs of one or the other of the two parallel lines; $x - y = 2$ $(y = x - 2)$ or $x - y = -2$ $(y = x + 2)$. Can we tell which? If $x = -4$ and $y = -6$, then $x - y \geq 0$; so $(-4,-6)$ is on the graph of $x - y = 2$. On the other hand, if $x = -4$ and $y = -2$, then $x - y < 0$; so $(-4,-2)$ is on the graph of $x - y = -2$. See Figure 6.

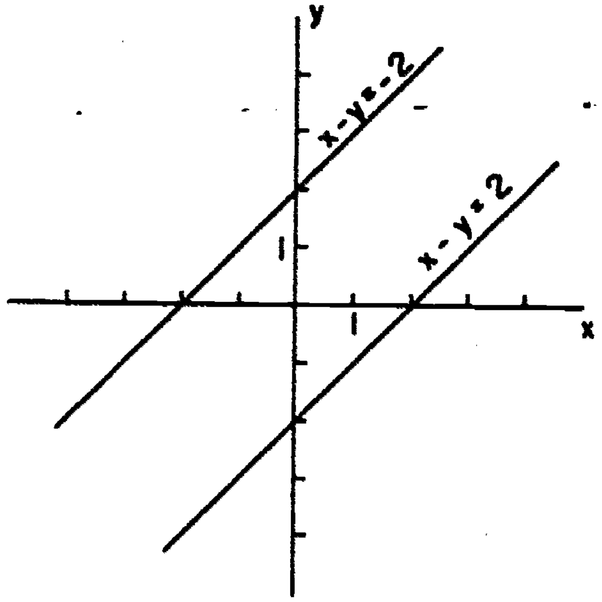


Figure 6

What does the graph of $|x| + |y| = 5$ look like? Let us make a table of values first. Suppose we start with the intercepts. Let $y = 0$ and find some possible values of x which will make the sentence true. Then let $x = 0$, and find some values of y. Now fill in some of the other possible values. Suppose $x = 6$, what can you say about possible values for y? If $x = 3$, then $|x| = 3$, and $|y| = 2$; what possible values may y have? Fill in the blanks in the table below.

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
$\|x\|$	5	3	3	1		0	0					5
$\|y\|$	0		2	4		5	5					0
y	0		-2	4		5	-5					0

We can write four open sentences which give the same graph, provided we limit the values of x :

$$x + y = 5, \text{ and } 0 \le x \le 5,$$
$$x - y = 5, \text{ and } 0 \le x \le 5,$$
$$-x + y = 5, \text{ and } -5 \le x \le 0,$$
$$-x - y = 5, \text{ and } -5 \le x \le 0.$$

Problem Set 3-2b

1. Draw the graph of $|x| + |y| = 5$. Why was it necessary to limit the values of x in the four sentences stated above?

2. Draw the graph of each of the following on a separate set of axes.

   (a) $|x| + |y| > 5$ (c) $|x| + |y| \le 5$

   (b) $|x| + |y| < 5$ (d) $|x| + |y| \nless 5$

3. Make a table of values for the open sentence

$$|x| - |y| = 3,$$

and draw the graph of the open sentence. Write four open sentences, as above the Problem Set, whose graphs form the same figure.

If more than one absolute value symbol is present the problem of breaking the original sentence into all possible cases may become quite complicated. The graphs of some of these sentences are very unusual. It is possible to have an open sentence whose graph is a regular polygon of $2n$ sides. For instance, an equation of a regular hexagon is:

$$|y| + \left|\frac{y - \sqrt{3}\,x}{2}\right| + \left|\frac{-y - \sqrt{3}\,x}{2}\right| = 2\sqrt{3}\ .$$

This hexagon has its center at the origin and one vertex at $(2,0)$.

Problem Set 3-2c

1. On separate set of axes plot the graph of:

   (a) $|x| = |y|$ (b) $|x| < |y|$

2. Plot the graphs of the following on a separate set of axes:

(a) $x - 1 + |x - y| = 0$

(b) $|x| + 2 = |x - y|$

(c) $|x| - 2 + |x + y - 2| = 0$

(d) $-x + |x| - |x - 2| + |y - 2| = 0$

Chapter 4

ABSOLUTE VALUE AND QUADRATIC EXPRESSIONS

4-1. Absolute Value and Quadratic Equations

There is a type of equation, not a quadratic equation, which involves quadratic equations in its solution. We recall that if $x \geq 0$, $|x| = x$, while if $x < 0$, then $|x| = -x$. In particular, since $x^2 \geq 0$ for all real numbers x, $|x^2| = x^2$. We recall, too, the useful theorem: For all real numbers a and b, $|ab| = |a||b|$. Hence, we can see that $|x^2| = |x||x| = |x|^2$. In solving equation involving absolute value, we often use the fact that $|x|^2 = x^2$.

If $|x| = x + 1$ is true for some x, then $|x|^2 = (x + 1)^2$ or $x^2 = x^2 + 2x + 1$ is true for the same x. We conclude that if $|x| = x + 1$ has a solution, it is $-\frac{1}{2}$. If x is $-\frac{1}{2}$, then the sentence $|x| = x + 1$ means $\left|-\frac{1}{2}\right| = (-\frac{1}{2}) + 1$. This last sentence is a true sentence which makes it certain that $\{-\frac{1}{2}\}$ is the truth set of the given equation.

We solved $|x| = x + 1$ by squaring both members. We could, however, solve the same equation in another way. Either $x \geq 0$ and $|x| = x$, so that $x = x + 1$, <u>or</u> $x < 0$, and $|x| = -x$, so that $-x = x + 1$. This compound open sentence has two main parts connected by <u>or</u>. Each main part, in turn, is a compound sentence using <u>and</u>. The truth set of $x \geq 0$ and $x = x + 1$ is ϕ, the empty set. The truth set of $x < 0$ and $-x = x + 1$ is $\{-\frac{1}{2}\}$. The truth set of the compound open sentence: "$x \geq 0$ and $x = x + 1$ <u>or</u> $x < 0$ and $-x = x + 1$" is $\{-\frac{1}{2}\}$. This latter method is not as simple as the first method; nevertheless it should be understood.

Problem Set 4-1

1. Solve each of the following equations:

(a) $|2x| = x + 1$

(b) $x - |x| = 1$

(c) $2x = |x| + 1$

(d) $|x - 3| = 4$

(e) $x = |2x| + 1$

(f) $x^2 - |x| - 2 = 0$

2. The distance between x and 3 is 2 more than x. Find the truth set.

4-2. Solution of Quadratic Inequalities

It is possible to solve quadratic inequalities by using compound open sentences involving inequalities and also by graphs. We shall make use of the fact that $\sqrt{x^2} = |x|$ to develop another method.

Let us consider first how we would apply this to an equation. If $(x - 1)^2 = 4$, then $\sqrt{(x - 1)^2} = \sqrt{4}$ and $|x - 1| = 2$. Then, if $x - 1 > 0$, $x - 1 = 2$; if $x - 1 < 0$, $-(x - 1) = 2$ or $x - 1 = -2$. Hence, $x = 3$ or $x = -1$. The truth set is $\{3,-1\}$.

If we are given the quadratic inequality

$$x^2 - 2x - 3 < 0$$

we can write it as

$$x^2 - 2x \quad < 3$$

(Since $a < b$ implies $a + c < b + c$) and adding 1 to both members completes the square

$$x^2 - 2x + 1 < 4$$

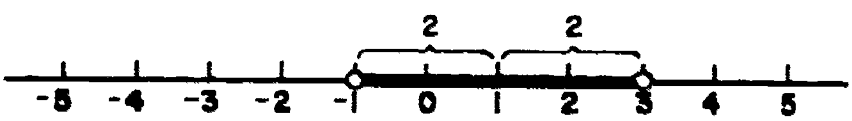
which is equivalent to

$$(x - 1)^2 < 4$$

and

$$|x - 1| < 2^*$$

We have treated the method of finding the truth set of this sentence in Section 1-2. The distance between x and 1 is less than 2.



Thus, $x > -1$ and $x < 3$, or $-1 < x < 3$.

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* The theorem that we use here states that if $a^2 < b^2$ and $b > 0$ then $|a| < b$.

This method has definite advantages in the case of quadratic inequalities which can be expressed in terms of perfect squares. It is not, however, a general method which can be applied to examples of the type: $(2x - 3)(x + 4)(x - 3) < 0$. Here we either make use of the graphical interpretation or use the method of determining positive and negative factors. These methods are described in the pamphlet on Inequalities.

Problem Set 4-2

Find the solution set and draw the graph on the number line.

1. $x^2 - 4x + 3 < 0$
2. $x^2 + 4 \geq 5x$
3. $x^2 - x < -1$
4. $x^2 - x > -1$
5. $x^2 - 6x + 9 \leq 0$
6. $5x < 2 - 3x^2$
7. $6(-x^2 + 1) > 13x$
8. $2x - 1 > x - x^2$
9. $x > \frac{1}{x}$
10. $\frac{x + 3}{x - 5} > 0$
11. $|x| > x^2$
12. $|x| < x^2$
13. $|x| > x + 5$
14. $|2x - 1| > x^2$
15. Which of the following inequalities are equivalent to $|x| > 2$?

    (A) $\frac{x + 2}{x - 2} > 0$ (B) $x^2 - 4 > 0$
16. What is the truth set of $|x|(x - 2)(x + 4) < 0$?
17. Describe the truth set of $\frac{|x - 3|}{x - 2} > 0$.

4-3. Graphs of Quadratic Inequalities in Two Variables

Some very unusual graphs occur when we consider quadratic inequalities in two variables which involve absolute value. For example, draw the graph of $\{(x,y): |x| > y^2\}$. Since $|x| = x$ if $x \geq 0$, and $-x$ if $x < 0$, we actually have two cases here. Either $-x > y^2$ or $x > y^2$. At first glance we might say that it is not possible for y^2 to be less than $-x$. Don't forget that x can be replaced by a negative number. The graph is the shaded region in Figure 7.

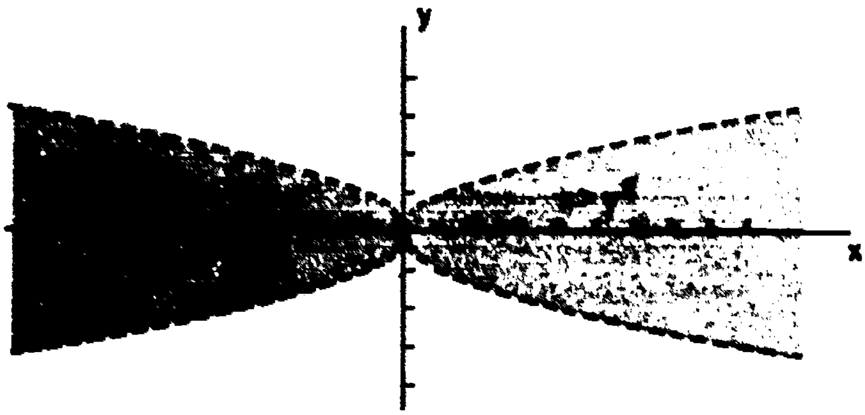

Figure 7

To do these graphs it is very important to remember the definition of absolute value and to consider both possible cases.

<u>Problem Set 4-3</u>

Draw the graphs of the following:

1. $|x^2 + x| < y$
2. $y = |4 - x^2|$ for $-3 \leq x \leq 3$.
3. $y = x^2 - 6|x| + 5$
4. $|x| + 2|y| < 5$

Chapter V

ABSOLUTE VALUE, COMPLEX NUMBERS AND VECTORS

5-1. Absolute Value of a Complex Number

If we are to have a solution for the quadratic equation $x^2 + 1 = 0$, then we must have a number which is not a real number, since the square of all real numbers is zero or positive. We do have such a number, the symbol for which is "i". We state that it has the property $i \cdot i = i^2 = -1$. It is a special element of the set of complex numbers and is called the imaginary unit. The set of complex numbers is comprised of numbers of the type $a + bi$, where a and b are real numbers: a is the real part of the complex number and b is the imaginary part. If b is 0, then $a + bi$ represents a real number, if a is 0, then $a + bi$ represents a pure imaginary. A full discussion of complex numbers can be found in another pamphlet of this series entitled "Complex Numbers".

Each member of C (the set of complex numbers) can be expressed in the form $a + bi$. It can be proved that this representation is unique; given any complex number z, there is only one pair of real numbers a, b such that $z = a + bi$. When a complex number is written in the form $a + bi$ it is said to be in standard form.

We recall that ordered pairs of real numbers formed the starting point of coordinate geometry. We are able to represent the complex numbers by points in the xy-plane. We agree to associate z with the point (a,b) if and only if $z = a + bi$. We set up a one-to-one correspondence between the elements of C and the points in the xy-plane.

It is customary to use the expression "Argand diagram" to describe the figure obtained when the point (a,b) of the xy-plane is used to represent the complex number $a + bi$. (The standard form of $4 - 3i$ is $4 + (-3)i$, of 0 is $0 + 0i$, of i^3 is $0 + (-1)i$). Figure 8 is an example of an Argand diagram showing three points $(0,2)$, $(4,-5)$, and $(-4,3)$ and the complex numbers that they represent. Note that points on the x-axis correspond to real numbers and points on the y-axis correspond to pure imaginary numbers. For the

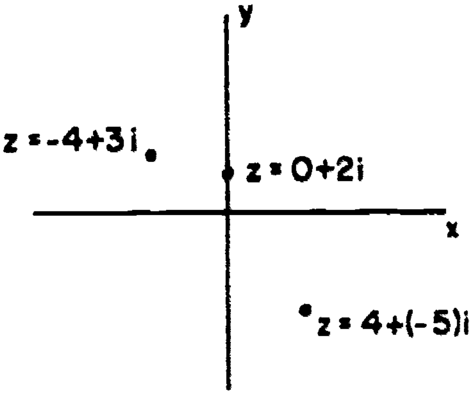

Figure 8

sake of brevity we shall often say "the point $z = x + iy$" instead of "the point (x,y) corresponding to the complex number $z = x + iy$."

The geometric representation of complex numbers by means of an Argand diagram serves a double purpose. It enables us to interpret statements about complex numbers geometrically and to express geometric statements in terms of complex numbers. The geometric representation suggests a definition of absolute value of a complex number. Recall that when real numbers are represented by points on a line the absolute value of a real number is equal to its distance from the origin. Accordingly, we define the absolute value $|z|$ of a complex number $z = a + bi$ to be the distance from the origin to the point (a,b). If we use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we may state the definition algebraically as follows:

DEFINITION. If $z = a + bi$, where a and b are real numbers, we write

$$|z| = \sqrt{a^2 + b^2}$$

and call $|z|$ the absolute value of z.

We can readily see this from the Argand diagram shown in Figure 9.

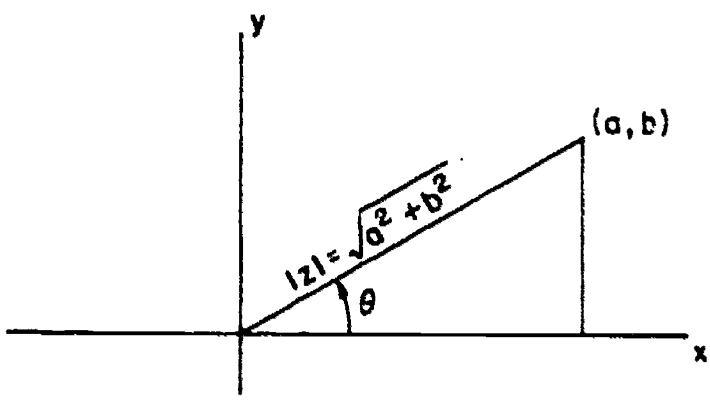

Figure 9

Example: Show that the distance between the points z_1 and z_2 is $|z_2 - z_1|$.

Solution: If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ where x_1, y_1, x_2, y_2 are real numbers, then from the definition of subtraction

$$z_2 - z_1 = (x_2 - x_1) + (y_2 - y_1)i$$

By the definition of absolute value

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and this is the distance between the points (x_1,y_1) and (x_2,y_2).

When z_1 and z_2 are real numbers we have the following relations involving absolute value and the algebraic operations:

(5-1a) $$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

(5-1b) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

(5-1c) $$|z_1 + z_2| \leq |z_1| + |z_2|$$

(5-1d) $$||z_1| - |z_2|| \leq |z_1 - z_2|$$

These relations continue to hold when z_1 and z_2 are complex numbers. Formulas (5-1a) and (5-1b) can be demonstrated by calculations. The algebraic proof of (5-1c) is quite difficult but we can give an easy geometric proof. Consider the triangle whose vertices are 0, z_1, $z_1 + z_2$. The lengths of its sides are $|z_1|$, $|z_2|$, $|z_1 + z_2|$. Why?

Since the length of a side of a triangle is less than the sum of the lengths of the other two sides, we have,

$$|z_1 + z_2| < |z_1| + |z_2|.$$

When the parallelogram collapses into a straight line we have the equation

$$|z_1 + z_2| = |z_1| + |z_2|$$

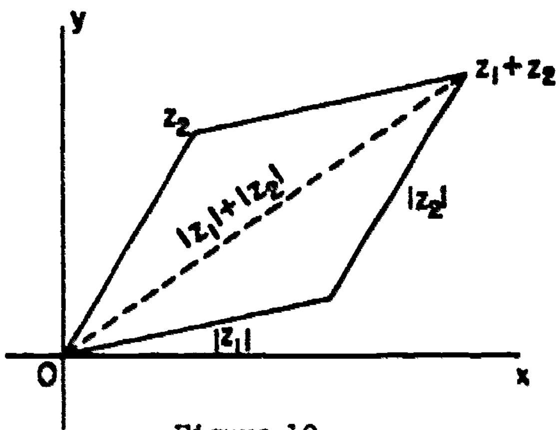

Figure 10

This will complete the proof of Formula (5-1c), which is often called the "triangle inequality". The discussion of (5-1d) is left as an exercise.

Problem Set 5-1

1. Find $|z|$ if:

   (a) $z = 3 - 4i$

   (b) $z = -2i$

   (c) $z = 1 + i^2$

   (d) $z = i^4 + i^7$

   (e) $z = \pi + \sqrt{2}\, i$

2. Show that, if $z \neq 0$, $\left|\frac{z}{|z|}\right| = 1$.

3. Find the set of points described by each of the following equations:

(a) $z = 1$ (b) $z = |z|$ (c) $z = \frac{z}{|z|}$

4. Give an algebraic proof of the equation

$$|z_1 z_2| = |z_1| \cdot |z_2|,$$

if z_1 and z_2 are complex numbers. (Hint: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$)

5. Give an algebraic proof of the equation

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

if z_1 and z_2 are complex numbers, and $z_2 \neq 0$. Use hint of Exercise 4.

6. Give a geometric proof of the inequality

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

7. Prove that the triangle with vertices 0, 1, and z is similar to the triangle with vertices 0, z, and z^2 by showing that the corresponding sides are proportional. (Hint: Note that the length of each side of the second triangle is equal to $|z|$ multiplied by the length of each side of the first triangle.) Use the results to describe a geometric construction for z^2.

5-2. Complex Conjugates

For further discussion of the applications of absolute value to complex numbers it is convenient to introduce the notion of the complex conjugate.

DEFINITION. If $z = a + bi$, in standard form (a and b real), we call $a + (-b)i$ the complex conjugate, or simply the conjugate of z and write

$$\bar{z} = \overline{a + bi} = a + (-b)i.$$

Since $a + (-b)i = a - bi$ we may also write

$$\bar{z} = \overline{a + bi} = a - bi.$$

Whenever we solve a quadratic equation which has a negative discriminant we encounter conjugates. For example, the solution set of the equation, $x^2 - 2x + 4 = 0$ is $1 + \sqrt{3}\,i,\ 1 - \sqrt{3}\,i$. Also the formula for the multiplicative inverse of $z = a + bi$ can be written

$$\frac{1}{z} = \frac{1}{a + bi} = \frac{a + (-b)i}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

or

(5-2a) $$\frac{1}{z} = \frac{\bar{z}}{|z|^2}\,.$$

This gives, in turn,

(5-2b) $$z \cdot \bar{z} = |z|^2$$

This last equation is important enough to deserve statement as a theorem and to be proved by a different method.

Theorem 5-2. $$z \cdot \bar{z} = |z|^2$$

Proof: If $z = a + bi$ in standard form, then

$$\begin{aligned} z \cdot \bar{z} &= (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 - b^2(-1) \\ &= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2 \end{aligned}$$

Now that we have proved Equation (5-2b) independently of Equation (5-2a) we can derive (5-2a) from (5-2b). In fact, it is convenient to use Theorem 5-2 directly in dividing complex numbers.

(5-2c) $$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2}\,.$$

Let us see how we might apply Theorem 5-2 to a different problem: Show that the circle of radius 1 with center at the origin is the set of all points z which satisfy the equation

$$z \cdot \bar{z} = 1$$

Solution: We start with the definition of this circle as the set of points whose distance from the origin is 1, and use the fact that the distance of the point $z = x + yi$ from the origin is $|z|$. Then z is on the circle if and only if

$$|z| = 1.$$

Squaring both sides of this equation and using Theorem 5-2 we get

$$z \cdot \bar{z} = |z|^2 = 1$$

Here is a second problem: Show that the segments which join the points $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2$ to the origin are perpendicular if and only if the product $z_1 \cdot \overline{z}_2$ is a pure imaginary.

Solution: We can express the geometric conditions in terms of z_1 and z_2. The segments joining z_1 and z_2 to the origin will be perpendicular if and only if the triangle with vertices 0, z_1, z_2 is a right triangle. By the Pythagorean Theorem this will be true if and only if

$$|z_1|^2 + |z_2|^2 = |z_1 - z_2|^2.$$

In our proof of this we shall need two theorems about conjugates.

(1) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$ (Conjugate of difference equals difference of conjugates)

(2) A complex number is a pure imaginary if and only if $z = -\overline{z}$
(For example: $0 + 5i = -(0 - 5i)$)

Proof: $|z_1|^2 + |z_2|^2 = |z_1 - z_2|^2$

By Theorem 5-2: $z_1\overline{z}_1 + z_2\overline{z}_2 = (z_1 - z_2)(\overline{z_1 - z_2})$

By (1) above $= (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)$

Multiplying $= z_1\overline{z}_1 - z_1\overline{z}_2 - z_2\overline{z}_1 + z_2\overline{z}_2$

Or $0 = -z_1\overline{z}_2 - z_2\overline{z}_1$

Hence $z_1\overline{z}_2 = -z_2\overline{z}_1$

But $\overline{z_1\overline{z}_2} = z_2\overline{z}_1$

Therefore $z_1\overline{z}_2 = -\overline{z_1\overline{z}_2}$

By (2) above, this equation can hold if and only if the product $z_1\overline{z}_2$ is a pure imaginary. Since each step above is reversible, this completes the proof.

Here is a third problem. Show that $|z_1 \cdot z_2| = |z_1||z_2|$.

Solution: Since $|z_1 \cdot z_2|$, $|z_1|$, and $|z_2|$ are positive it will suffice to prove

$$|z_1 \cdot z_2|^2 = |z_1|^2|z_2|^2. \qquad \text{Why?}$$

We have
$$|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1 \cdot z_2}) \qquad \text{(Theorem 5-2)}$$

A theorem on conjugates states that $\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$.

Thus, we have
$$= (z_1 \cdot z_2)(\overline{z}_1 \cdot \overline{z}_2)$$
$$= (z_1\overline{z}_1)(z_2\overline{z}_2)$$
$$= |z_1|^2|z_2|^2.$$

This completes the proof.

Problem Set 5-2

1. If $z = 2 - 3i$, evaluate

$$-z,\ \overline{z},\ |z|,\ |\overline{z}|,\ \frac{1}{z},\ |z|^2,\ |z^2|, \text{ and } \frac{4 + 5i}{z}.$$

2. Use formula (5-2c) to compute the following quotients.

(a) $\frac{2 + i}{1 + i}$ (e) $\frac{7 + 6i}{3 - 4i}$

(b) $\frac{i}{i + 3}$ (f) $\frac{3 + 2i}{4i}$

(c) $\frac{-i + 1}{2 + 3i}$ (g) $\frac{-5 + 6i}{-3 - 4i}$

(d) $\frac{-4 + 3i}{2 + 5i}$ (h) $\frac{3 - 6i}{2i}$

3. Sketch the set of points z which satisfy each of the following conditions.

(a) $|z - 2| = 3$ (c) $|z - 2i| < 4$

(b) $|z + 2| > 3$ (d) $|z - z_0| \leq 5$

4. Write an equation in x and y which is equivalent to the equation $|z - (2 + 3i)| = 5$.
Describe the set of points in an Argand diagram which satisfy the equation.

5. If $z = x + yi$ show that

$$x \leq |z| \text{ and } y \leq |z|$$

(Hint: Start with $y^2 \geq 0$ and remember that $|x| \geq x$)

6. Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

(Hint: Express each term of left member in the form $z \cdot \bar{z} = |z|^2$)

7. Use the relation $z \cdot \bar{z} = |z|^2$ to show that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

8. Give a geometrical interpretation for the following relations.

(a) $|z_1| < |z_2|$ (b) $|z| = 5$

9. Find all complex numbers z such that (Real part of z) = (Imaginary part of z), and $|z| = 1$.

10. Let $z_0 = x_0 + y_0 i$. Describe the set of points $z = x + yi$ which satisfy the inequality

$$\frac{|z - \bar{z}_0|}{|z - z_0|} < 1$$

11. If $z = x + yi$, show that

$$|x| + |y| \leq \sqrt{2}\,|z|$$