DOCUMENT RESUME

ED 175 681 SE 028 669

AUTHOR	Clark, Ronald J., Ed.
TITLE	Supplementary and Enrichment Series: Radioactive Decay. SP-23.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	65
NOTE	17p.; For related documents, see SE 028 648-675; Contains occasional light and broken type
EDRS PRICE	MF01/PC01 Plus Postage.
DESCRIPTORS	Curriculum; *Enrichment; *Instruction; *Mathematical Applications; Mathematics Education; *Physics; *Radiation; Secondary Education; *Secondary School Mathematics; Supplementary Reading Materials
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include the law of decay, relative rate of change, and a general solution. (MP)

ED 177 084

SCHOOL MATHEMATICS STUDY GROUP

SP-23

SUPPLEMENTARY and ENRICHMENT SERIES

RADIOACTIVE DECAY

Edited by Ronald J. Clark

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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RADIOACTIVE DECAY

Introduction

As you probably know, the source of energy for the first atomic bomb which so dramatically inaugurated the present nuclear age in 1945 was atomic fission. The controlled fission of elements (or, in more common language, controlled "splitting") that produces bombs was a development of an earlier discovery that certain elements, the building blocks of nature, are unstable and disintegrate. Such elements are called radioactive. As they decompose, they release energy and, at the same time, there is an overall loss of mass. The equivalence between energy and mass is the foundation for much of Einstein's magnificent work. We are moving too fast, however; let us start at the beginning.

1. The law of decay.

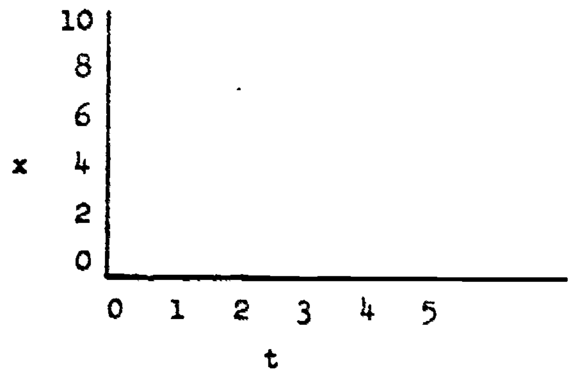
Suppose we have a rock containing uranium which is one of the unstable elements. If we measure the amount of uranium, in grams, in our sample each day, we find that the amount is decreasing. The uranium is disintegrating! If we were able to wait long enough, which means about 2,000 years or more in the case of uranium, all would have decomposed and turned into lead, an element that is not radioactive. By careful experimentation it has been discovered that the complete reaction proceeds in such a well-organized manner that it can be described mathematically.

Since uranium decays very slowly, let us imagine a radioactive element of our own, which we shall call quiddium, that obeys the same basic laws of decay as uranium but for which the arithmetic will be easier. Let us start with 10 grams of quiddium on the initial day.

The following table shows the amount of quiddium that remains in the sample when measured on the five succeeding days.

Day	Amount of Quiddium
0	10.0000
1	9.0000
2	8.1000
3	7.2900
4	6.5610
5	5.9049

We can make a graph showing the relation between t, the number of days, and x, the amount of quiddium



This is left for the reader to do as an exercise. The graph, if carefully drawn, will be very revealing and suggest certain conclusions. It is important, however, to investigate further.

The change in the amount of quiddium from one day to the next can be readily computed. For example, the amount of quiddium changes in the first day from 10 grams to 9 grams so the change is

$$\text{new value} - \text{old value} = 9 - 10 = -1$$

If we call x_0 the amount of quiddium at the start of the experiment and x_1 the amount of quiddium after one day, we can write

$$x_1 - x_0 = \Delta x_1$$

where Δx_1 stands for the change in amount during the first day. Likewise

$$x_2 - x_1 = \Delta x_2 .$$

We can arrange our information in a table.

Amount of Quiddium	Change in amount
$x_0 = 10.0000$	$\Delta x_1 = -1.0000$
$x_1 = 9.0000$	$\Delta x_2 = -.9000$
$x_2 = 8.1000$	$\Delta x_3 = -.8100$
$x_3 = 7.2900$	$\Delta x_4 = -.7290$
$x_4 = 6.5610$	$\Delta x_5 = -.6561$
$x_5 = 5.9049$	

Notice that the change in each case is <u>negative</u> since the amount of quiddium decreases.

Of more significance than the amount of change is the relative change, measured by the ratio

$$\frac{\text{change in amount}}{\text{original value}}$$

This gives us the following table:

No. of days t	Amt. of quiddium x	Change Δx	Relative change $\Delta x/x$
0	10.	-	-
1	9.0000	-1.0000	-.1
2	8.1000	- .9000	-.1
3	7.2900	- .8100	-.1
4	6.5610	- .7290	-.1
5	5.9049	- .6561	-.1

Since the time interval in each instance is one day, the relative rate of change as indicated by the table is approximately -.1 and is a constant.

Although quiddium was chosen so that the arithmetic would be easy, and our data was assumed to be without experimental error, contrary to what happens in the real world, our example illustrates the general law of radioactive decay which applies to all radioactive substances: the relative rate of change in the amount of a given sample is a negative constant.

It is only in theory that we assume that the rate of radioactive decay is constant. Experiment reveals it is approximately a constant as does our table. For the purposes of prediction, we assumed that the rate is constant. This (theoretically) makes our life easier. Actually the usefulness of our predictions about quiddium depends on how closely the actual rate of decay approximates the constant we use.

We may mention the reason for this law is that any particular atom of the radioactive element has a certain definite probability of disintegrating in a given interval of time. In our present example, we have assumed, really, that the probability of a quiddium atom disintegrating during a day is 0.1. Thus if we have a very large number of quiddium atoms, it is almost certain that $\frac{1}{10}$ of them will disintegrate during any one day.

Exercises 1

1-1. Using the tables on page 2, predict the amounts x_6, x_7, and x_8. Round off your results to four decimal places.

1-2. Make a graph showing the relation between t and x. (see page 2.)

1-3. Suppose that the measurements were made every half day, but that the same relative rate of change was observed. Make a table showing the values of x for $t = 0, .5, 1, 1.5, 2, \ldots, 5$.

1-4. Make a graph showing the relation in problem 3.

1-5. Here is a table of the amounts of centium in a sample measured at various times.

t in days	x in grams	change in x
0	100.0000	
1	99.0000	
2	98.0100	
3	97.0299	
4	96.0596	
5	95.0990	

The amounts were measured to the nearest .0001 of a gram. Calculate the change in x during each day. What is the relative rate of change in x per day? Is it constant? Predict the values of x_6 and x_7.

2. Relative rate of change.

If x_0 is the amount of quiddium at the start and x_1 is the amount a day later, then the change in x, labeled Δx for the interval is found by subtraction

$$x_1 - x_0 = \Delta x .$$

Since this pattern applies equally well to all intervals, we can say

$$x_n - x_{n-1} = \Delta x_n$$

where x_n is the amount of quiddium present at the end of the nth day. The relative change in x is the ratio of the change to the original value. That is

$$\text{relative change in x} = \frac{\text{change in x}}{\text{original value of x}} .$$

In symbols, we have for the nth day

$$\text{relative change in } x = \frac{\Delta x_n}{x_{n-1}} = \frac{x_n - x_{n-1}}{x_{n-1}}$$

The relative rate of change is the relative change per unit time. Since we have been using the day as our unit of time (that is the length of time between measurements was 1 day), the relative rate of change is

$$\frac{\text{relative change in } x}{1} .$$

As you recall, we assumed that the relative rate of change is a constant for each radioactive element. For quiddium, this constant is approximately $-.1$.

Let us look again at our pattern for determining the relative change. We have

$$\frac{x_1 - x_0}{x_0} = -.1$$

or

$$x_1 - x_0 = -.1x_0 .$$

Then

$$x_1 = -.1x_0 + 1x_0 ;$$

$$x_1 = .9x_0 ;$$

and

$$\frac{x_1}{x_0} = .9 .$$

If we calculate each of the ratios,

$$\frac{x_1}{x_0}, \frac{x_2}{x_1}, \frac{x_4}{x_3}, \frac{x_n}{x_{n-1}}$$

we learn that each is approximately equal to $.9$.

A sequence of numbers such that the ratio of any two consecutive numbers in the sequence is a constant is said to be a geometric progression. The constant is called the common ratio of the progression. As we have seen the amounts of quiddium on successive days form a geometric progression whose common ratio is $.9$.

Using this fact, we can form a series of equations:

(1) $$\frac{x_1}{x_0} = .9$$

(2) $$\frac{x_2}{x_1} = .9$$

(3) $$\frac{x_3}{x_2} = .9$$

$$\vdots$$

$$\frac{x_n}{x_{n-1}} = .9$$

If we substitute in equation (2) the value of x_1 from equation (1), we have

(4) $$\frac{x_2}{.9x_0} = .9$$

In a like manner, we can substitute the value of x_2 from equation (4) into equation (3); we have

$$\frac{x_3}{(.9)^2 x_0} = .9 .$$

If we repeat this pattern once more, we have

$$\frac{x_4}{(.9)^3 x_0} = .9$$

or $$x_4 = (.9)^4 x_0 .$$

By extrapolation of our reasoning, we have

$$x_n = (.9)^n x_0$$

which is a formula expressing the amount of quiddium that remains after n days in terms of the initial amount.

Exercises 2

2-1. (a) For centium, as described in Exercise 1, find the values of

$$\frac{x_1}{x_0}, \frac{x_2}{x_1}, \frac{x_3}{x_2}, \frac{x_4}{x_3}$$

(b) Find a formula for the amount of x_n of centium in the sample after n days.

2-2. In Exercise 1-3, find a formula for the amount of quiddium in the sample after n half-days; and also after t days. (How many half-days are there in t days?)

3. A general solution.

If we have a sample of radioactive material and measure the amounts every h days, that is we measure the amounts at the times

$$t = 0 , h , 2h , 3h , \ldots , nh , \ldots ,$$

we find the amounts

$$x = x_0 , x_1 , x_2 , x_3 , \ldots , x_n , \ldots .$$

We say that the relative change for the first h day period is

$$\frac{x_1 - x_0}{x_0} .$$

For the second h day period, it is

$$\frac{x_2 - x_1}{x_1} .$$

In general the relative change for the nth h day period is

$$\frac{x_n - x_{n-1}}{x_{n-1}} .$$

If we assume that the relative change is constant, (a convenient theory which is approximately borne out by the experimental evidence) we have

$$\frac{x_1 - x_0}{x_0} = \frac{x_2 - x_1}{x_1} = \ldots = \frac{x_n - x_{n-1}}{x_{n-1}}$$

The relative rate of change per day, however, is somewhat more difficult since we must consider the time interval of h days. We have

$$\text{relative rate of change} = \frac{\text{relative change}}{\text{time interval}}$$

$$= \frac{x_1 - x_0}{x_0} \Big/ h$$

$$= \frac{x_2 - x_1}{x_1} \Big/ h \ ..$$

As we have seen, in building our theory we assume the relative rate of change is a negative constant. If k is a positive number, then we can call the constant $-k$. Thus we can write

$$\frac{x_1 - x_0}{h\,x_0} = -k. \tag{6}$$

Also

$$\frac{x_2 - x_1}{h\,x_1} = -k. \tag{7}$$

Generally we can write

$$\frac{x_n - x_{n-1}}{h\,(x_{n-1})} = -k. \tag{8}$$

By a series of substitutions similar to those used in the previous section, we can obtain an expression for x_n in terms of x_0.

First we transform equation (6) and obtain

$$x_1 = (1 - kh)x_0 .$$

This we substitute into equation (7) obtaining

$$\frac{x_2 - (1 - kh)x_0}{h\,x_0(1 - kh)} = -k .$$

When we solve this last equation for x_2, we have

$$x_2 = (1 - kh)^2 x_0 .$$

Reasoning in this way we see that the amount, x_n, that will be left at the end of n time intervals is determined by the formula:

$$x_n = (1 - kh)^n x_0 .$$

This formula describes mathematically one of the very important processes out of which was developed the fascinating and powerful concept of controlled atomic energy that holds so much promise for all peoples.

Exercises 3

3-1. Express x_3 in terms of x_2.

3-2. Express x_3 in terms of x_0.

3-3. Express x_4 in terms of x_0.

3-4. Express x_n in terms of x_0.

3-5. Let $x(t)$ denote the amount at the time t. Give a formula of the form

$$x(t) = C^t x_0 ,$$

where C is a certain constant, expressing $x(t)$ in terms of t. (Notice that at the end of the nth time interval, $t = :$. Solve this equation for n in terms of t.)

3-6. Take $k = .1$. Compute C in problem 3-5 for $h = 1$, $.5$, $.1$, $.01$, $.001$:.

h	c
1	
.5	
.1	
.01	
.001	

What do you notice about the values of C?

3-7. The time T at which half of the radioactive substance has disintegrated is called the half-life. In other words, the half-life is the solution T of the equation

$$C^T x_0 = \frac{1}{2} x_0$$

Obtain a formula for T in terms of k and h.

3-8. It is sometimes easier to measure T than k. Find a formula expressing k in terms of T and h. For $T = 10$, compute k for these values of h:

h	k
1	
.5	
.1	
.01	
.001	

What do you notice about the value of k?

Note: Problems 3-6, 3-7, 3-8 are for students who have studied logarithms.

ANSWERS

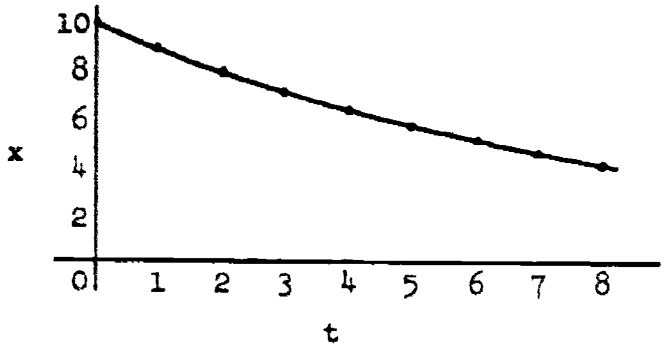
Exercises 1

1-1. $x_6 = 5.3144$

$x_7 = 4.7830$

$x_8 = 4.3047$

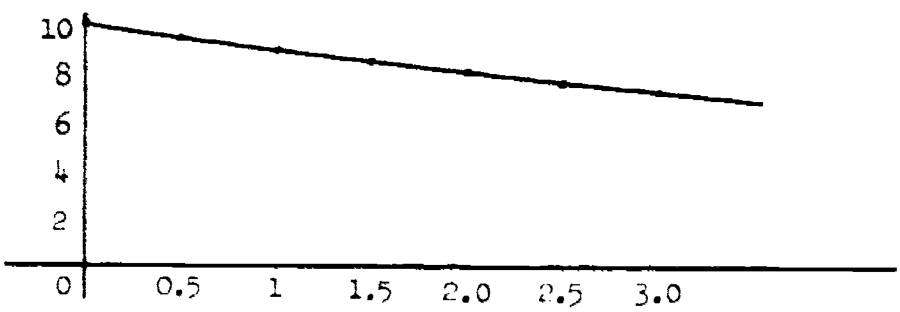
1-2.



1-3. Relative rate of change per $\frac{1}{2}$ day $-.05$

t	$\frac{1}{2}$	1	1.5	2.0	2.5	3	3.5	4.0	4.5	5
x	9.5000	9.0250	8.5738	8.1451	7.7378	7.3409	6.9739	6.6252	6.2939	5.9792

1-4.



1-5.

t	Amt. of Centium	Change in Amt.	Relative change
0	100.0000	---	----
1	99.0000	1.0000	-.01
2	98.0100	.9900	-.01
3	97.0299	.9801	-.01
4	96.0596	.9703	-.01
5	95.0990	.9606	-.01
6	94.1480	.9510	-.01
7	93.2065	.9415	-.01

Exercises 2

2-1. (a) $\frac{x_1}{x_0} = \frac{99}{100} = .99$

$\frac{x_2}{x_1} = \frac{98.01}{99} = .99$

$\frac{x_3}{x_2} = \frac{97.03}{98.01} = .99$

$\frac{x_4}{x_3} = \frac{96.06}{97.03} = .99$

(b) $x_n = (.99)^n x_0$

2-2. $x_n = (.95)^n x_0$ (n = number of $\frac{1}{2}$ days)

Exercises 3

3-1. $x_3 = (1 - kh)x_2$

3-2. $x_3 = (1 - kh)^3 x_0$

3-3. $x_4 = (1 - kh)^4 x_0$

3-4. $x_n = (1 - kh)^n x_0$

3-5. $x_t = (1 - kh)^{\frac{t}{h}} x_0$

$t = hn \rightarrow n = \frac{t}{h}$

If $c = (1 - kh)^{\frac{1}{h}}$

$x_t = c^t x_0$

3-6. If $C = (1 - kh)^{\frac{1}{h}}$, then

$h = 1 \rightarrow C = 0.90000$

$h = 0.5 \rightarrow C = 0.90250$

$h = 0.1 \rightarrow C = 0.90448$

$h = 0.01 \rightarrow C = 0.90574$

$h = 0.001 \rightarrow C = 0.91202$

As h approaches zero, C approaches 1.

3-7. $(1 - kh)^{\frac{t}{h}} x_0 = \frac{1}{2} x_0$

$$T = \frac{h \log \frac{1}{2}}{\log(1 - kh)} = \frac{-.3010\ h}{\log(1 - kh)}$$

3-8. $k = \dfrac{1 - (.5)^{\frac{h}{t}}}{h}$

Let $T = 10$

$h = 1 \rightarrow k = 0.06696$

$h = 0.5 \rightarrow k = 0.06812$

$h = 0.1 \rightarrow k = 0.06900$

$h = 0.01 \rightarrow k = 0.07000$

$h = 0.001 \rightarrow k = 0.07500$