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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quanitiy. Topics covered include order on the number line, properties of order, solution of inequalities, and graphs of open sentences in two variables. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-12

SUPPLEMENTARY and ENRICHMENT SERIES

INEQUALITIES

Edited by Philbrick Bridgess

SMSG

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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FOREWORD

This pamphlet on Inequalities is an exploration of the order properties of real numbers under the operations of addition and multiplication. It begins with order on the Number Line and proceeds to simple and compound open sentences in one variable involving inequalities and the graphs of their truth sets. The comparison and transitive properties of order are developed in some detail, after which there is considerable material on the addition and multiplication properties of order. Several proofs are included in the text, and additional ones are to be found in the exercises for students. A chapter is devoted to polynomial inequalities and the solution sets of rational inequalities. The pamphlet concludes with a chapter on the graphing of inequalities in two variables.

CONTENTS

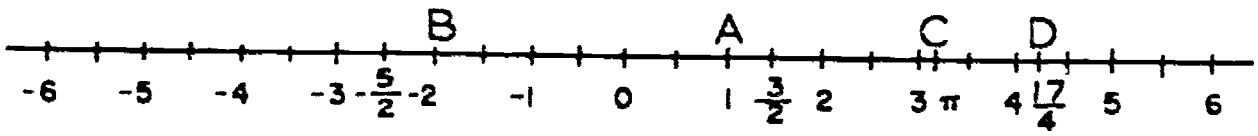
INEQUALITIES

Chapter 1

ORDER ON THE NUMBER LINE

1-1. The Number Line

We call a line by means of which we associate points with numbers, as we do below, a number line. The number associated with a point is called the coordinate of the point.



The coordinate of point A is 1, of B is -2 (negative two), of C is π, of D is $\frac{17}{4}$. It is assumed, also, that this line is extended indefinitely in both directions. The line is a set of points and the coordinates are a set of numbers. We say that there is a one-to-one (1-1) correspondence between the set of points and the set of numbers, because to each point there corresponds one and only one number and with each number there is associated one and only one point.

Just as a point is either to the right or to the left of a different point, so every number is either larger or smaller than a different number. The point D is to the right of the point A and the point B is to the left of the point A. $\frac{17}{4}$ is larger than 1; -2 is smaller than 1. The point whose coordinate is the larger number is farther to the right; the point whose coordinate is the smaller number is farther to the left.

Problem Set 1-1

1. Draw a number line and locate the points whose coordinates are given. Fill in the blanks below using R (right) or L (left).

   a. The point with coordinate 5 is to the _____ of the point with coordinate 2.

   b. The point with coordinate 3 is to the _____ of the point with coordinate 7.

   c. The point with coordinate -2 is to the _____ of the point with coordinate 4.

d. The point with coordinate -6 is to the _____ of the point with coordinate -3.

e. The point with coordinate 2 is to the _____ of the point with coordinate -5.

f. The point with coordinate 0 is to the _____ of the point with coordinate -6.

2. Draw a number line and locate:

a. 0 and the next 3 points to the left whose coordinates are integers.

b. -5 and the next 4 points to the right whose coordinates are integers.

c. $\frac{1}{2}$ and the next 6 points to the left whose coordinates follow the pattern $\frac{1}{2}, \frac{3}{8}, \frac{1}{4}$, etc.

d. the points between -4 and 3 whose coordinates are integers.

1-2. Sentences Involving Inequalities

Let us consider two numbers, like 7 and 4. We know that they are not equal and we can write this: $7 \neq 4$. The symbol "$\neq$" is translated, "is not equal to" or "does not equal". Thus, $3 \neq 8$ means "3 does not equal 8". This is a sentence. It is a true sentence.

When we make assertions about numbers we write sentences, such as $5 \neq 3$. Remember that a numerical sentence is either true or false, but not both. $5 \neq 3$ is a true sentence; $8 + 4 \neq \frac{24}{2}$ is a false sentence. The symbols of relationship, $=$, $\neq$, and others, are equivalent to the verbs in a sentence.

Another symbol which we shall use is $>$, which is read, "is greater than". We can write $7 > 4$; when we read it we say, "7 is greater than 4". Likewise, the symbol $<$ means "is less than". $4 < 7$ is read, "4 is less than 7". Note that the point of the arrow is always aimed at the numeral for the smaller of the two numbers.

Just as $\neq$ stands for the words "is not" or "is not equal to", $\not>$ means "is not greater than". What does $\not<$ mean?

Any sentence using the symbol, $=$, is called an equation. If we use the symbols $\neq$, $>$, $<$, $\not>$, $\not<$, we call the sentence an inequality.

Problem Set 1-2

1. Which of the following numerical sentences are true?

(a) $3 < -1$

(b) $2 < (-\frac{7}{2})$

(c) $-6 < 0$

(d) $0 > -4$

(e) $6 > -7$

(f) $1 \nless 1$

(g) $-1 \ngtr -2$

(h) $-3 \ngtr 5$

(i) $4 + 3 < 3 + 4$

(j) $5(2+3) > 5(2) + 3$

(k) $\frac{1}{2} + \frac{1}{3} \nless 1$

(l) $5 + 0 \nless 5$

(m) $2 > 2 \times 0$

(n) $0.5 + 1.1 = 0.7 + 0.9$

(o) $5.2 - 3.9 < 4.6$

(p) $2 + 1.3 > 3.3$

(q) $2 + 1.3 \ngtr 3.3$

(r) $4 + (3+2) < (4+3) + 2$

(s) $\frac{2}{3}(8+4) < \frac{2}{3}(8) + \frac{2}{3}(4)$

(t) $5 + (\frac{2}{5}+\frac{3}{5}) \neq (4 - 1)2$

2. Consider the following pairs of real numbers and tell which of the sentences <u>below</u> each pair are true and which are false. For example, for the pair:

$$-2 \text{ and } \frac{3 \times 4}{2},$$

$$-2 < \frac{3 \times 4}{2} \text{ is true}$$

$$-2 = \frac{3 \times 4}{2} \text{ is false}$$

$$-2 > \frac{3 \times 4}{2} \text{ is false}$$

(a) -3.14 and -3

$-3.14 < -3$

$-3.14 = -3$

$-3.14 > -3$

(b) 2 and -2

$2 < -2$

$2 = -2$

$2 > -2$

(c) $\frac{5 + 3}{2}$ and 2×2

$\frac{5 + 3}{2} < 2 \times 2$

$\frac{5 + 3}{2} = 2 \times 2$

$\frac{5 + 3}{2} > 2 \times 2$

(d) $-.001$ and $-(\frac{1}{1000})$

$-.001 < -(\frac{1}{1000})$

$-.001 = -(\frac{1}{1000})$

$-.001 > -(\frac{1}{1000})$

(e) What did you discover in each of (a), (b), (c) and (d)?

3. In the blanks below use one of $=, >, <$ to make a true sentence, if possible, in each case.

(a) $\frac{3}{5}$ ___ $-\frac{6}{10}$

(b) $\frac{3}{5}$ ___ $\frac{3}{6}$

(c) $\frac{9}{12}$ ___ $\frac{8}{12}$

(d) $-\frac{173}{29}$ ___ $-\frac{183}{29}$

(e) $-\frac{3}{5}$ ___ $\frac{3}{6}$

(f) $-\frac{3}{5}$ ___ $-\frac{3}{6}$

1-3. Open Sentences

We have some symbols, $=, \neq, >, <$ (our verb forms) and we have a set of numbers which are available for use (called the domain). Let us consider the sentence $x + 2 > 4$. We cannot state that it is true or false, anymore than we can say, "It is taller than the Washington Monument." (What is taller than the Washington Monument?" The White House? No. The Empire State Building? Yes.

In the sentence $x + 2 > 4$, if the domain is the set of integers, certain replacements for x will make the sentence true, others will make it false. If x is 1, the sentence is false; if x is 5, the sentence is true. The sentence acts as a sorter to sort the set of integers into two subsets, those which will make the sentence true, and those which will make the sentence false. Those which will make the sentence true are called the truth set or solution set of the open sentence. In general, the truth set will not include all the numbers in the domain, but it could.

What is the truth set of $x + 2 > 4$ if the domain is $\{-1,0,1,2,3,4\}$?

If $x + 2 > 4$ and

x	is	-1,	then	$-1 + 2 > 4$	False
x	is	0,	then	$0 + 2 > 4$	False
x	is	1,	then	$1 + 2 > 4$	False
x	is	2,	then	$2 + 2 > 4$	False
x	is	3,	then	$3 + 2 > 4$	True
x	is	4,	then	$4 + 2 > 4$	True

Therefore, the truth set is $\{3,4\}$.

Problem Set 1-3a

Substitute members of the given domain (the replacement set) in the open sentence. Tell whether the resulting sentences are true, and give the truth set for each.

1.	$x + 1 \neq 5$	Domain is $\{0,2,4,6\}$
2.	$a < 3$	Domain is $\{1,2,3,4\}$
3.	$w \nless 3$	Domain is $\{1,2,3,4\}$
4.	$2c + 3 = 1$	Domain is $\{-2,-1,0,1\}$
5.	$4y < 15$	Domain is $\{-10,-5,0,5\}$
6.	$(3+x) + 5 = 3 + (x+5)$	Domain is $\{-2,-1,0,1\}$
7.	$3(a+2) \ngtr 3a + 2$	Domain is $\{8,9,10,11\}$

How did you write the answer to Exercise 7? No element (member) of the domain $\{8,9,10,11\}$ will make the sentence true. In that case we say that the truth set (solution set) is the <u>empty set</u>. There are two symbols for the empty set, $\{\ \}$ or $\emptyset$. The latter is used more often. Note that we do <u>not</u> put curly braces around the symbol $\emptyset$.

If our domain is very large, for example, the set of all integers or the set of all numbers which are coordinates of points on the number line, then we cannot list the elements of the truth set, element by element. We must describe the truth set in words.

Consider the open sentence, $x \neq 4$, and the domain, the set of all integers. The truth set is the set of all integers except 4.

Problem Set 1-3b

Give the truth set in each of the open sentences (1-10). The domain is the set of all integers.

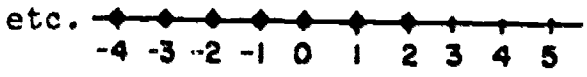
1. $5x > 20$
2. $3y \neq 18$
3. $m \nless 100$
4. $4(x+3) = 4x + 12$
5. $5 + y = 4$
6. $z + 3 - z - 5$
7. $9d < 8$
8. $\frac{2}{3}x \ngtr 7$
9. $(1000)(.06)y = (.06)(1000)y$
10. $\frac{t}{3} \neq \frac{3}{t}$

11. If there are so many elements in the truth set that they cannot be counted, we call it an infinite set; if they can be counted, we call it a finite set. Describe the truth sets in 1-10 as infinite sets or finite sets.

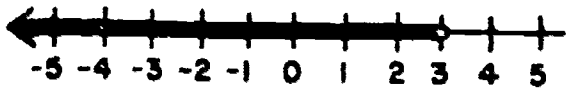
1-4. Graphs of Truth Sets

The graph of the truth set of an open sentence containing one variable (for example: $2x < 6$) is the set of all points on the number line whose coordinates are the permissible values of the variable which make the open sentence true. In our example the coordinates of the points would be those numbers in the domain which are less than 3.

In the example, $2x < 6$, if the domain of the variable is the set of integers, then our graph would be

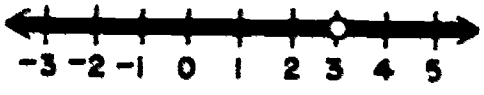


If, however, the domain is the set of real numbers, our graph would be



Note that we use an open circle to indicate that 3 is not in the truth set of the open sentence $2x < 6$. $2(3) < 6$ or $6 < 6$ is a false sentence. 2.998 is in the truth set as $2(2.998) < 6$ or $5.996 < 6$. How close to 3 can we get?

The graph of the truth set of the open sentence $x \neq 3$ (domain the set of real numbers) would be



The graph indicates that every member except 3 is a possible replacement value for x in the open sentence $x \neq 3$.

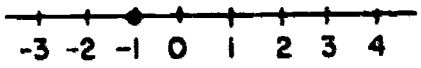
How do we construct the graph of the truth set of an open sentence when the truth set is the null set (empty set)? We can either draw a number line and put no marks on it, we can state "No graph", or at the end of the unmarked number line we can put a "$\emptyset$".

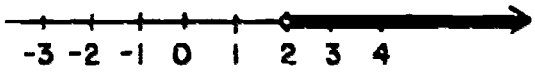
Problem Set 1-4

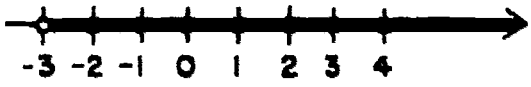
1. Draw the graphs of the truth sets of the following open sentences. Use the set of real numbers for the domain.

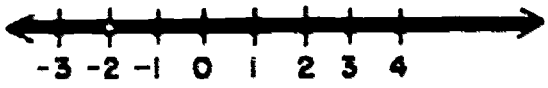
   (a) $a > 4$
   (b) $m < -2$
   (c) $y \neq 0$
   (d) $2x > 1$
   (e) $x + 2 > 1$

2. Below are some graphs. For each graph find an open sentence whose truth set is the set whose graph is given.

   (a)
   

   (b)
   

   (c)
   

   (d)
   

3. If the domain of the variable of each open sentence below is the set $\{0,1,2,3,4,5\}$, draw the graph of the truth set of each.

   (a) $x > 1$
   (b) $y + 2 < 6$
   (c) $5 - m \neq 1$
   (d) $2x + 6 = 2(x + 3)$
   (e) $y + 3 = y - 2$

1-5. Compound Inequali ies: Or

We can combine "<" with "=" and ">" with "=" to obtain two more symbols of comparison. The two symbols obtained are $\leq$ (we say, "less than or equal to") and $\geq$ (we say, "greater than or equal to"). It doesn't seem to make much sense to write $7 \geq 7$ (read, "7 is greater than or equal to 7). Later on, however, we shall be stating something like this: "I am thinking of a number greater than or equal to 3. Tell me number for which this statement is true." Do you see that if we use the number line, our graph of the answer would be as follows:

The graph tells us that the point with coordinate 3 and the coordinate of every point to the right of three is a possible solution to the problem. Just because we have labeled only the integers, don't think that integers 3 or larger are the only numbers which will make the statement true. There are also numbers like $\frac{11}{3}$, 4, 97, π, $\frac{35}{8}$, etc.

What shall we say about $7 \geq 7$? Is it true or is it false? The sentence consists of two parts:

(1) 7 is greater than 7 (obviously false)

(2) 7 is equal to 7 (obviously true)

The two parts are connected by the word or. Mathematicians have agreed that compound sentences of this kind with connective or will be true if either or both parts are true. If both parts are false, the compound statement is false.

Problem Set 1-5a

List the numbers 1 through 18 on your paper. Then indicate by writing T or F whether each corresponding statement is true or false.

1.	$3 \geq -5$	7.	$0 \leq 8$	13.	$(.01)(3)(2) \neq (.01)(3) + 2$
2.	$-4 \neq 4$	8.	$-7 \geq -5$	14.	$\frac{1}{5} + \frac{1}{4} \not> (\frac{1}{5})(\frac{1}{4})$
3.	$0 \not> -3$	9.	$-2 \leq 4$	15.	$.008(.2+.3) \leq (.008)(.2) + (.008)(.3)$
4.	$2 \not\leq -2$	10.	$0 \not< 0$	16.	$1{,}000{,}000 + 1 \geq 1{,}000{,}000(1)$
5.	$-3 \geq 1$	11.	$-\frac{4}{3} \neq -\frac{3}{4}$	17.	$(.02)(85) \not< (85)(.02)$
6.	$-3 \geq -3$	12.	$5+2 \geq 2+5$	18.	$5 - 5 \neq \frac{0}{5}$

1

Problem Set 1-5b

1. Find a replacement for each indicated number which will (a) make the statement true, (b) make the statement false. Use the set of integers for your domain.

(a) $x - 3 \geq 4$	(f) $x + x \nless 2x$
(b) $2y \ngtr 17$	(g) $\frac{2y + 3}{2} \neq y + 3$
(c) $m + 2 \neq 2 + m$	(h) $3b \leq b + 3$
(d) $\frac{w}{5} \leq \frac{2}{3}$	(i) $h + 2 < 9$ or $5h > 9$
(e) $c + 5 \geq c + 8$	(j) $x > 0$ or $x < 0$

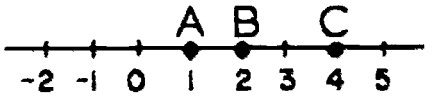
2. For each of the following sets, write an open sentence involving the variable x which has the given set as its truth set.
   (a) A is the set of all real numbers not equal to 3.
   (b) B is the set of all real numbers less than or equal to -2.
   (c) C is the set of all real numbers not less than $\frac{-5}{2}$.

3. Choose any positive real number p; choose any negative real number n. Which, if any, of the following sentences are true?

$$n < p, \quad p < n, \quad n \leq p, \quad n \neq p$$

1-6. Compound Inequalities: And

If we have three numbers, 1, 2, and 4, we can say: "1 is less than 2 and 2 is less than 4". We could write this: $1 < 2$ and $2 < 4$. However, it can be written more simply as: $1 < 2 < 4$. If we locate the points corresponding to these numbers, what do we observe?



Point B is between point A and point C which agrees with the fact that 2 is between 1 and 4. Could we also write: $4 > 2 > 1$? Yes, and we would say: "4 is greater than 2 which in turn is greater than 1" or "2 is between 4 and 1".

Problem Set 1-6a

1. List the letters a through f on your paper; then write T or F to indicate the truth or falsity of each sentence. Use the number line to help you decide.

   (a) $0 < \frac{5}{2} < \frac{5}{3}$ (d) $-6 < -7 < 3$

   (b) $-4 > -5 > -6$ (e) $5 > 0 > -2$

   (c) $-2 < -1 < 5$ (f) $3 \geq -1 > -3$

2. Using only integers, find, if possible, a replacement for the variable which will make each of the following sentences true.

   (a) $2 > x > -1$ (d) $6 < y < 0$

   (b) $-5 < n < 3$ (e) $4 > w > -5$

   (c) $-6 > b > -8$ (f) $-3 < d < -2$

Let us now look very carefully at some of the symbols which we have been using. In the case of $8 \geq 3$ (read "8 is greater than 3 <u>or</u> equal to 3) we have the connective <u>or</u>. The whole sentence is true when either part or both parts are true; false when both parts are false. However, in the case of $4 > 2 > 1$ (read "4 is greater than 2 <u>and</u> 2 is greater than 1"), both parts must be true if the whole sentence is to be true. If either part is false, the whole sentence is false. Look at the following:

1.	$4 > 3$ <u>or</u> $2 > 6$	True	One part true
2.	$5 < -2$ <u>or</u> $6 + 2 \neq 8$	False	Both parts false
3.	$3 > 1$ <u>and</u> $2 < 0$	False	One part false
4.	$-2 > -5$ <u>and</u> $8 < 10$	True	Both parts true

Problem Set 1-6b

1. Which of the following sentences are true?

   (a) $4 = 5 - 1$ and $5 = 3 + 2$

   (b) $5 = \frac{11}{2} - \frac{1}{2}$ and $6 < \frac{2}{3} \times 9$

   (c) $3 > 3 + 2$ and $4 + 7 < 11$

   (d) $3 + 2 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \neq 5$

   (e) $3.2 + 9.4 \neq 12.6$ and $\frac{7}{8} < \frac{11}{12}$

   (f) $3.25 + .3 \neq 6.25$ and $-2 \leq -2$

2. True or False?

(a) $3 = 5$ or $7 > 9$
(b) $-5 < 0$ and $-2 \neq -3$
(c) $-7 \geq 3$ or $2 \leq 9$
(d) $-1 \leq -1$ and $-6 \neq -8$
(e) $3 \neq -3$ and $-5 \geq -1$
(f) $8 \not> 3$ or $2 > 3$
(g) $-3 \neq -3$ or $4 \not< 7$
(h) $-5 < -4$ and $-2 \leq -1$
(i) $8 \not> -2$ or $0 < -3$
(j) $\frac{1}{2} > \frac{1}{3}$ and $-\frac{1}{3} > -\frac{1}{2}$

3. True or False?

(a) $3 = 5 - 1$ or $5 = 3 + 2$
(b) $7 = \frac{11}{2} + \frac{3}{2}$ or $2 = \frac{11}{2} - \frac{3}{2}$
(c) $4 > 3 + 2$ or $6 < 3 + 1$
(d) $2 + 3 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \neq 6$
(e) $6.5 + 2.3 \neq 8.8$ or $-\frac{3}{5} < -\frac{7}{15}$
(f) $5 + 4 < 9$ or $\frac{3}{4} < \frac{9}{12}$

4. A domain is given for each open sentence. Give the truth set.

(a) $4 > x$ and $x > -1$ $\{1,2,3,4,5\}$
(b) $3 \neq a$ or $a < 2$ $\{0,2,4\}$
(c) $-2 \leq c$ and $c < -4$ $\{-1,-2,-3,-4,-5\}$
(d) $n \not> 0$ and $-5 < n$ $\{0,-1,-2,-3,-4,-5\}$
(e) $y < 3$ and $2 < y$ $\{0,2,4,6\}$
(f) $-3 > x$ or $x > -2$ $\{-5,-3,-1,1\}$

5. In which of A, B, C, D, E does the sentence have the same truth set as the sentence "$x \leq 5$"?

(A) $x > 5$ or $x = 5$
(B) $x < 5$ and $x = 5$
(C) $x \not> 5$
(D) $x \not< 5$
(E) $x \neq 5$

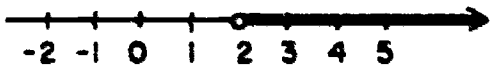
1-7. Graphs of Truth Sets of Compound Open Sentences

Our problems in graphing have so far involved only simple sentences. Graphs of compound open sentences require special handling. Let us consider the open sentence "$x \geq 2$" which we shall write as

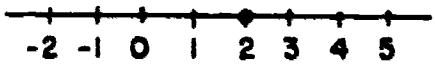
$$x > 2 \quad \text{or} \quad x = 2.$$

The clauses of this sentence and the corresponding graphs of their truth sets are (our domain being the set of all real numbers):

$x > 2$

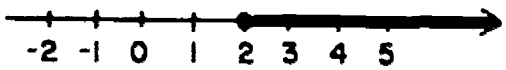


$x = 2$



If a number belongs to the truth set of the open sentence "$x > 2$" or to the truth set of the open sentence "$x = 2$", it is a number belonging to the truth set of the compound open sentence "$x > 2$ or $x = 2$". Therefore, every number greater than or equal to 2 belongs to the truth set. On the other hand, any number less than 2 makes both clauses of the compound sentence false and so fails to belong to the truth set. The graph of the truth set is then

$x > 2$ or $x = 2$



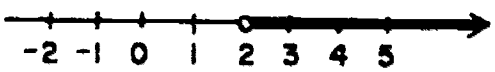
The graph of the truth set of a compound open sentence with connective <u>or</u> consists of the set of all points which belong to either one of the graphs of the two clauses of the compound sentence.

Now, let us consider the problem of finding the graph of an open sentence such as

$$x > 2 \text{ and } x < 4.$$

Again, we begin with the two clauses and the graphs of their truth sets:

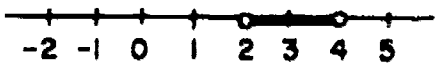
$x > 2$



$x < 4$

Then it follows (using an argument similar to that above) that the graph of the truth set of the compound open sentence is

$x > 2$ and $x < 4$



We see that the graph of the truth set of a compound open sentence with connective <u>and</u> consists of all those points which are common to the graphs of the truth sets of the two clauses of the compound sentence. Remember that we can write "$x > 2$ and $x < 4$" as "$2 < x < 4$". We observe that our graph consists of those points whose coordinates are between 2 and 4.

In our discussion we have referred to the <u>graph of the truth set of an open sentence</u>. In the future, let us shorten this phrase to the <u>graph of a sentence</u>. It will be a simpler description, and no confusion will result if we recall what is really meant by the description.

For the same reasons we shall find it convenient to speak of the point 3, or the point $\frac{1}{2}$, when we mean the point with coordinate 3 or the point with coordinate $\frac{1}{2}$. Points and numbers are distinct entities to be sure, but they correspond exactly on the number line.

Problem Set 1-7

Construct the graphs of the following open sentences:

1. $x = 2$ or $x = 3$
2. $x = 2$ and $x = 3$
3. $x > 5$ or $x = 5$
4. $x > -3$ and $x < \frac{7}{2}$
5. $x > -1$ and $x = -1$
6. $x + 1 = 4$ and $x + 2 = 5$
7. $x + 1 = 4$ or $x + 3 = 5$
8. $x < -2$ or $x = 2$
9. $x < -1$ or $x > 3$
10. $x \neq 3$ and $x \neq 4$
11. $x > -2$ or $x < 3$
12. $x < -2$ and $x > 3$

Chapter 2

PROPERTIES OF ORDER

2-1. Comparison Property

There are certain simple but highly important facts about the order of the real numbers on the real number line. If we choose any two different real numbers, we are sure that the first is less than the second or the second is less than the first, but not both. Stated in the language of algebra, this property of order for real numbers becomes the comparison or trichotomy property:

> If a is a real number and b is a real number, then exactly one of the following is true:
>
> $$a < b,\ a = b,\ b < a.$$

Problem Set 2-1

1. For each of the following pairs of numbers verify that the comparison property is true by determining which one of the three possibilities actually holds between the numbers:

   (a) -2 and -1.6

   (b) 0 and -2

   (c) $\frac{2\times3\times4}{5}$ and $-(\frac{2\times3\times4}{5})$

   (d) -16 and $-\frac{32}{2}$

   (e) 12 and $(5 + 2)(\frac{1}{7} \times \frac{36}{3})$

   (f) -2 and 2

2. Make up true sentences, using $<$, involving the following pairs:

   (a) $2,\ -3$

   (b) $\frac{4}{2},\ -\frac{5}{2}$

   (c) $-\frac{4}{5},\ -\frac{6}{5}$

   (d) $\frac{4}{5},\ \frac{11}{10}$

   (e) $-\frac{4}{50},\ \frac{11}{100}$

   (f) $\frac{103}{13},\ \frac{205}{26}$

   (g) $\frac{13}{15},\ \frac{2}{3}$

   (h) $\frac{12}{119},\ -(\frac{25}{238})$

   (i) $-\sqrt{2},\ -1.5$

   (j) $\sqrt{2} + \pi,\ 1.5 + 3$

3. The comparison property stated in the text is a statement involving "<". Try to formulate the corresponding property involving ">", and test it with the pairs of numbers in Problem 1.

4. Try to state a comparison property involving "$\geq$".

2-2. Transitive Property of Order

Which is less than the other, $\frac{4}{5}$ or $\frac{5}{6}$? You can find out by applying the multiplication property of 1 to each number to get $\frac{4}{5} = \frac{4}{5} \times \frac{6}{6} = \frac{24}{30}$ and $\frac{5}{6} = \frac{5}{6} \times \frac{5}{5} = \frac{25}{30}$. Then $\frac{4}{5} < \frac{5}{6}$ because $\frac{24}{30}$ is to the left of $\frac{25}{30}$ on the number line.

You should now be able to compare any two rational numbers. How would you decide which is the lesser, $\frac{337}{113}$ or $\frac{167}{55}$? (Describe the process; do not actually carry it out.) Perhaps you noticed, in comparing $\frac{337}{113}$ and $\frac{167}{55}$, that $\frac{337}{113} < 3$ (i.e., $\frac{337}{113} < \frac{339}{113}$) and $3 < \frac{167}{55}$ (i.e., $\frac{165}{55} < \frac{167}{55}$). Could you now decide about the order of $\frac{337}{113}$ and $\frac{167}{55}$ without writing them as fractions with the same denominator? How could you find out similarly which is lesser, $\frac{40}{27}$ or $\frac{\pi}{2}$? Or suppose that x and y are real numbers and that $x < -1$ and $-1 < y$. Again, using the number line, what can you say about the order of x and y?

The property of order used in these last three examples we call the <u>transitive property</u>:

> If a, b, c are real numbers
> and if $a < b$ and $b < c$,
> then $a < c$.

Problem Set 2-2

1. In each of the following groups of three real numbers, determine their order:

   For example, given $\frac{3}{4}$, $\frac{3}{2}$, $-\frac{4}{5}$. Since $-\frac{4}{5} < \frac{3}{4}$ and $\frac{3}{4} < \frac{3}{2}$, then $-\frac{4}{5} < \frac{3}{2}$.

   (a) $-\frac{1}{5}$, $\frac{3}{2}$, and 12.

   (b) π, $-\pi$, and $\sqrt{2}$.

   (c) 1.7, 0, and -1.7.

1. (continued)

(d) $-(\frac{27}{15})$, $-(\frac{3}{15})$, and $-(\frac{2}{15})$

(e) $\frac{12 \times (\frac{1}{2} + \frac{1}{3})}{3}$, $-\frac{5}{3}$, and $\frac{6}{3}$

(f) $\frac{3 \times (27 + 6)}{9}$, $\frac{(2 \times 3) + (7 \times 9)}{6}$, and $\frac{(99 \times 3) \times \frac{1}{3}}{2}$

(g) 3^2, 4^2, $(3 + 4)^2$

(h) $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{4}$

(i) $1 + \frac{1}{2}$, $1 + (\frac{1}{2})^2$, $(1 + \frac{1}{2})^2$

2. State a transitive property for ">", and illustrate this property in two of the exercises in Problem 1.

3. Art and Bob are seated on opposite ends of a see-saw (teeter-totter), and Art's end of the see-saw comes slowly to the ground. Cal gets on and Art gets off, after which Bob's end of the see-saw comes to the ground. Who is heavier, Art or Cal?

4. Is there a transitive property for the relation "="? If so, give an example.

5. State a transitive property for "$\geq$", and give an example.

6. The set of numbers greater than 0 we have called the positive real numbers, and the set of numbers less than 0 the <u>negative real numbers</u>. Describe the

(a) non-positive real numbers,

(b) non-negative real numbers.

7. Find the order of each of the following pairs of numbers:

(a) $-\frac{15}{8}$ and $-\frac{25}{12}$

(b) $-\frac{17}{35}$ and $-\frac{7}{13}$

(c) $-\frac{145}{28}$ and $-\frac{104}{21}$

(d) $-\frac{192}{46}$ and $-\frac{173}{44}$

2-3. Order of Opposites

When we attach the dash "-" to a variable x we are performing on x the operation of "determining the opposite of x". Do not confuse this with the binary operation of subtraction, which is performed on two numbers; for example, $3 - x$ means "x subtracted from 3". What kind of number is $-x$ if x is a positive number? If x is a negative number? If x is 0?

We shall read "$-x$" as the "opposite of x". Thus, if x is is a number to the right of 0 (positive), then $-x$ is to the left (negative); if x is to the left of 0 (negative), then $-x$ is to the right (positive).

The ordering of numbers on the real number line specifies that $-\frac{1}{2}$ is less than 2. Is the opposite of $-\frac{1}{2}$ less than the opposite of 2? Make up other similar examples of pairs of numbers. After you have determined the ordering of a pair, then find the ordering of their opposites. You will see that there is a general property for opposites:

For real numbers a and b,
if $a < b$, then $-b < -a$.

Problem Set 2-3a

1. What kind of number is $-x$ if x is positive? if x is negative? if x is zero?

2. What kind of number is x if $-x$ is a positive number? if $-x$ is a negative number? if $-x$ is 0?

3. (a) Is every real number the opposite of some real number?

   (b) Is the set of all opposites of real numbers the same as the set of all real numbers?

   (c) Is the set of all negative numbers a subset of the set of all opposites of real numbers?

   (d) Is the set of all opposites of real numbers a subset of the set of all negative numbers?

   (e) Is every opposite of a number a negative number?

Problem Set 2-3b

1. Write true sentences for the following numbers and their opposites, using the relations "<" or ">".

   Example: For the numbers 2 and 7, $2 < 7$ and $-2 > -7$.

   (a) $\frac{2}{7}$, $-\frac{1}{6}$

   (b) $\sqrt{2}$, $-\pi$

   (c) π, $\frac{22}{7}$

   (d) $3(\frac{4}{3} + 2)$, $\frac{5}{4}(20 + 8)$

   (e) $-(\frac{8 + 6}{7})$, -2

   (f) $-\big((3 + 17)0\big)$, $-\big((5 + 0)\big)3$

2. Graph the truth sets of the following open sentences: (Hint: Use order property of opposites before graphing.)

   (a) $x > 3$

   (b) $x > -3$

   (c) $-x > 3$

   (d) $-x > -3$

3. Describe the truth set of each open sentence:

   (a) $-x \neq 3$

   (b) $-x \neq -3$

   (c) $x < 0$

   (d) $-x < 0$

   (e) $-x \geq 0$

   (f) $-x \leq 0$

4. For the following sets, give two open sentences each of whose truth sets is the given set (use opposites when convenient):

   (a) A is the set of all non-negative real numbers.

   (b) B is the set of all real numbers not equal to -2.

   (c) C is the set of all real numbers not greater than -3.

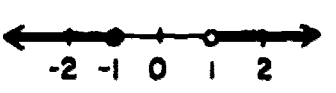
   (d) $\emptyset$.

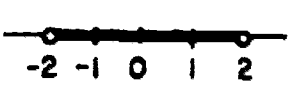
   (e) E is the set of all real numbers.

5. Write an open sentence for each of the following graphs:

   (a)
   

   (b)
   

   (c)
   

   (d)
   

6. For each of the following numbers write its opposite, and then choose the greater of the number and its opposite:

(a) -7.2	(f) -0.01
(b) 3	(g) $-(-2)$
(c) 0	(h) $(1 - \frac{1}{4})^2$
(d) $\sqrt{2}$	(i) $1 - (\frac{1}{4})^2$
(e) 17	(j) $-(\frac{1}{2} - \frac{1}{3})$

7. Let us write "$\succ$" for the phrase "is further from 0 than" on the real number line. Does "$\succ$" have the comparison property enjoyed by ">"; that is, if a and b are different real numbers, is it true that $a \succ b$ or $b \succ a$ but not both? Does "$\succ$" have a transitive property? For which subset of the set of real numbers do "$\succ$" and ">" have the same meaning?

8. Translate the following English sentences into open sentences, describing the variable used:
   (a) John's score is greater than negative 100. What is his score?
   (b) I know that I don't have any money, but I am no more than \$200 in debt. What is my financial condition?
   (c) Paul has paid \$10 of his bill, but still owes more than \$25. What was the amount of Paul's bill?

9. What is the order of "$-\frac{13}{42}$" and "$-\frac{15}{49}$"?
   (Hint: If you know the order of $\frac{13}{42}$ and $\frac{15}{49}$, what is the order of their opposites?)
   Now state a general rule for determining the order of two negative rational numbers.

2-4. Order Relation for Real Numbers

We speak of the relation "is less than" for real numbers as an order relation. It is a binary relation since it expresses a relation between two numbers. So far we have discussed three basic properties of the order relation for real numbers.

<u>Comparison property</u>: If a is a real number and b is a real number, then exactly <u>one</u> of the following is true:

$$a < b, \quad a = b, \quad b < a.$$

<u>Transitive property</u>: If a, b, and c are real numbers and if $a < b$ and $b < c$, then $a < c$.

<u>Order property of opposites</u>: If a and b are real numbers and if $a < b$, then $-b < -a$.

You may wonder at this point why we are so careful in giving these properties to avoid talking about "greater than". As a matter of fact, the relation "is greater than", for which we use the symbol ">" is also an order relation. Does this order relation have the properties given above? Since it does, we actually have two <u>different</u> (though very closely connected!) order relations for the real numbers, and we have chosen to concentrate our attention on "less than". We could have decided to concentrate on "greater than"; but if we are going to study an order relation and its properties, we must not confuse the issue by shifting from one order relation to another in the middle of the discussion.

Thus, we state the last property mentioned above in terms of ">", but in <u>applying</u> the property we feel free to say, "If $a < b$, then $-a > -b$".

In the next two sections we obtain some properties of the order relation "<" which involve the operations of addition and multiplication. Such properties are essential if we are to make much use of the order relation in algebra.

<u>Problem Set 2-4</u>

1. For each pair of numbers, determine their order.

   (a) $-\frac{3}{2}, -\frac{4}{3}$

   (b) $-(-7), -(7)$

   (c) $m, 1$ (Consider the comparison property.)

2. Continuing Problem 1(c), what can you say about the order of m and 1 if it is known that $m > 4$? What property of order did you use here?

3. Decide in each case whether the sentence is true.

(a) $-3 + (-2) < 2 + (-2)$ (c) $(-3) + 5 < 2 + 5$
(b) $(-3) + (0) < 2 + 0$ (d) $(-3) + a < 2 + a$

4. Decide in each case whether the sentence is true.

(a) $(-3)(5) < (2)(5)$ (c) $(-3)(-2) < (2)(-2)$
(b) $(-3)(0) < (2)(0)$ (d) $(-3)(a) < (2)(a)$ (What is the truth set of this sentence?)

5. A given set may be described in many ways. Describe in three ways the truth set of
(a) $3 < 3 + x$ (b) $3 + x < 3$

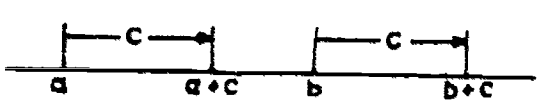
6. Determine the truth set of

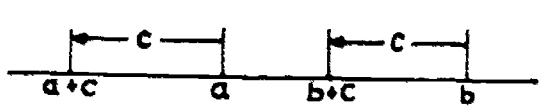
(a) $y < 3$ (c) $-y < -3$
(b) $-y < 3$ (d) $-(-y) < -3$

2-5. Addition Property of Order

What is the connection between order of numbers and addition of numbers? We shall find a basic property, and from it prove other properties which relate order and addition. As before, we concentrate on the order relation "<"; similar properties can be stated for the order relation ">".

It is helpful to view addition and order on the number line. We remember that adding a positive number means moving to the right; adding a negative number means moving to the left. Let us fix two points, a and b on the number line, with $a < b$. If we add the same number c to a and to b, we move to the right of a and of b if c is positive, to the left if c is negative. We could think of two men walking on the number line carrying a ladder between them. At the start the man at a is to the left of the man at b. If they walk c units in either direction, the fixed length of the ladder will insure that the man to the left will stay to the left. In their new positions the man at $a + c$ will still be to the left of the man at $b + c$. Thus,





Here we have found a fundamental property of order which we shall assume for all real numbers.

> <u>Addition Property of Order</u>. If a, b, c are real numbers and if $a < b$, then
>
> $$a + c < b + c.$$

Illustrate this property for $a = -3$ and $b = -\frac{1}{2}$, with c having, successively, the values -3, $\frac{1}{2}$, 0, -7. Here $-3 < -\frac{1}{2}$. Is "$(-3) + (-3) < (-\frac{1}{2}) + (-3)$" a true sentence? Continue with the other values of c. Phrase the addition property of order in words. Is there a corresponding property of equality?

Problem Set 2-5a

1. By applying the addition properties of order, determine which of the following sentences are true.

   (a) $(-\frac{6}{5}) + 4 < (-\frac{3}{4}) + 4$

   (b) $(-\frac{5}{3})(\frac{6}{5}) + (-5) > (-\frac{5}{2}) + (-5)$

   (c) $(-5.3) + (-2)(-\frac{4}{3}) < (-0.4) + \frac{8}{3}$

   (d) $(\frac{5}{2})(-\frac{3}{4}) + 2 \geq (-\frac{15}{8}) + 2$

2. Formulate an addition property of order for each of the relations "$\leq$", "$>$", "$\geq$".

3. An extension of the order property states that:

   If a, b, c, d are real numbers such that $a < b$ and $c < d$, then $a + c < b + d$.

   This can be proved in three steps. Give the reason for each step:

   If $a < b$, then $a + c < b + c$;
   if $c < d$, then $b + c < b + d$;
   hence,

   $$a + c < b + d.$$

4. Find the truth set of each of the following sentences.

   Example: If $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$ is true for some x,

   then $x < (-5) + \frac{3}{2} + \frac{3}{2}$ is true for the same x.

   $x < -2$ is true for the same x.

Thus, if x is a number which makes the original sentence true, then $x < -2$.

Let us reverse the steps.

If $x < -2$ is true for some x,

then $(-\frac{3}{2}) + x < (-\frac{3}{2}) + (-2)$ is true for the same x,

and $(-\frac{3}{2}) + x < (-\frac{3}{2}) + \big((-5)+3\big)$ is true for the same x,

also, $(-\frac{3}{2}) + x < (-5) + \big(3+(-\frac{3}{2})\big)$ is true for the same x.

Finally, $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$ is true for the same x.

Hence, the truth set is the set of all real numbers less than -2.

(a) $3 + x < (-4)$

(b) $x + (-2) > -3$

(c) $2x < (-5) + x$

(d) $3x > \frac{4}{3} + 2x$

(e) $(-\frac{2}{3}) + 2x \geq \frac{5}{3} + x$

(f) $(-x) + 4 < (-3) + \big(-(-3)\big)$

(g) $(-5) + (-x) < \frac{2}{3} - (-\frac{4}{3})$

(h) $(-2) + 2x < (-3) + 3x + 5$

(i) $(-\frac{3}{4}) + \frac{5}{4} \geq x + \frac{3}{2}$

5. Graph the truth sets of parts (a), (c), and (h) of Problem 4.

6. In Section 2-3 the following property of order was stated:

"If $a < b$, then $-b < -a$." Prove this property, using the addition property of order. (Hint: Add $\big((-a)+(-b)\big)$ to both members of the inequality $a < b$; then use the property of additive inverses.)

7. Show that the property:

"If $0 < y$, then $x < x + y$."

is a special case of the addition property of order. (Hint: In the statement of the addition property of order let $a = y$, $b = 0$, $c = x$.)

Many results about order can be proved as consequences of the addition property of order. Two of these are of special interest to us, because they give direct translations back and forth

between statements about order and statements about equality.

The first of these results will be a special case of the property. Let us consider a few numerical examples of the property with $a = 0$. If $a = 0$, then "$a < b$" becomes "$0 < b$"; that is, b is a positive number. Thus, we may write: If $0 < b$, then $c + 0 < c + b$.

Let $a = 0$, $b = 3$, and $c = 4$; then $4 + 0 < 4 + 3$.

Since $7 = 4 + 3$, then $4 < 7$.

Let $a = 0$, $b = 5$, and $c = -4$; then $(-4)+0 < (-4) + 5$.

Since $1 = (-4) + 5$, then $-4 < 1$.

These two examples can be thought of as saying:

Since $7 = 4 + 3$ and 3 is a positive number, then

$$4 < 7.$$

Since $1 = (-4) + 5$ and 5 is a positive number, then

$$-4 < 1.$$

This result we state as

<u>Theorem 2-5a</u>. If $z = x + y$ and y is a positive number, then $x < z$.

<u>Proof</u>: We may change the addition property of order to read:

If $a < b$, then $c + a < c + b$. (Why?)

Since the property is true for <u>all</u> real numbers a, b, c, we may let $a = 0$, $b = y$, $c = x$. Thus,

$$\text{if } 0 < y, \text{ then } x + 0 < x + y.$$

If $z = x + y$, then "$x + 0 < x + y$" means "$x < z$". Hence, we have proved that if $z = x + y$ and $0 < y$ (y is positive), then $x < z$.

Theorem 2-5a now gives us a translation from a statement about equality, such as

$$-4 = (-6) + 2,$$

to a statement about order, in this case,

$$-6 < -4.$$

Notice that adding 2, a positive number, to (-6) yields a number to the right of -6.

Change the sentence

$$4 = (-2) + 6$$

to a sentence involving order.

The second result of the addition property is a theorem which translates from order to equality, instead of from equality to order, as Theorem 2-5a does. You have seen that if y is positive and x is any number, then x is always less than $x + y$. If $x < z$, then does there exist a positive number y such that $z = x + y$? Consider, for example, the numbers 5 and 7 and note that $5 < 7$. What is the number y such that

$$7 = 5 + y?$$

How did you determine y? Did you find y to be positive? Consider the numbers -3 and -6, noting that $-6 < -3$. What is the truth set of

$$-3 = -6 + y?$$

Is y again positive?

$$4 < 9, \qquad 9 = 4 + (\ )$$

$$-3 < 5, \qquad 5 = (-3) + (\ )$$

$$-4 < -1, \qquad -1 = (-4) + (\ )$$

$$-6 < 0, \qquad 0 = (-6) + (\ )$$

What kind of number makes each of the above equations true? In each case you added a positive number to the smaller number to get the greater.

By this time you see that the theorem we have in mind is

<u>Theorem 2-5b</u>. If x and z are two real numbers such that $x < z$, then there is a positive real number y such that

$$z = x + y.$$

<u>Proof</u> There are really two things to be proved. First, we must <u>find</u> a value of y such that $z = x + y$; second, we must prove that if $x < z$, then the y that we found is <u>positive</u>.

It is not hard to find a value of y such that $z = x + y$. Your experience with solving equations probably suggests adding $(-x)$ to both members of "$z = x + y$" to obtain $y = z + (-x)$.

Let us try this value of y. Let

$$y = z + (-x).$$

Then

$$\begin{aligned} x + y &= x + \big(z + (-x)\big) && \text{(Why?)} \\ &= \big(x + (-x)\big) + z && \text{(Why?)} \\ &= 0 + z. \\ x + y &= z. \end{aligned}$$

Thus, we have found a y; namely, $z + (-x)$, such that $z = x + y$.

It remains to be shown that if $x < z$, then this y is positive. We know there is exactly one true sentence among these: y is negative, y is zero, y is positive. (Why?) If we can show that two of these possibilities are false, the third <u>must</u> be true. Try the first possibility: If it were true that y is <u>negative</u> and $z = x + y$, then the addition property of order would assert that $z < x$. (Let $a = y$, $b = 0$, $c = x$.) But this contradicts the fact that $x < z$; so it cannot be true that y is negative. Try the second possibility: If it were true that y is <u>zero</u> and $z = x + y$, then it would be true that $z = x$. This again contradicts the fact that $z < x$; so it cannot be true that y is zero. Hence, we are left with only one possibility, y is <u>positive</u>, which must be true. This completes the proof.

Theorem 2-5b allows us to translate from a sentence involving order to one involving equality. Thus,

$$-5 < -2$$

can be replaced by

$$-2 = (-5) + 3,$$

which gives the same "order information" about -5 and -2. That is, there is a positive number, 3, which when added to the lesser, -5, yields the greater, -2.

<u>Problem Set 2-5b</u>

1. For each pair of numbers, determine their order and find the positive number b which when added to the smaller gives the larger.

(a) -15 and -24

(b) $\frac{63}{4}$ and $-\frac{5}{4}$

(c) $\frac{6}{5}$ and $\frac{7}{10}$

(d) $-\frac{1}{2}$ and $\frac{1}{3}$

(e) -254 and -345

(f) $-\frac{33}{13}$ and $-\frac{98}{39}$

(g) 1.47 and -0.21

(h) $(-\frac{2}{3})(\frac{4}{5})$ and $(\frac{3}{2})(-\frac{5}{4})$

2. Show that the following is a true statement: If a and c are real numbers and if $c < a$, then there is a negative real number b such that $c = a + b$. (Hint: Follow the similar discussion for b positive.)

3. Which of the following sentences are true for all real values of the variables?

   (a) If $a + 1 = b$, then $a < b$.

   (b) If $a + (-1) = b$, then $a < b$.

   (c) If $(a + c) + 2 = (b + c)$, then $a + c < b + c$.

   (d) If $(a + c) + (-2) = (b + c)$, then $b + c < a + c$.

   (e) If $a < -2$, then there is a positive number d such that $-2 = a + d$.

   (f) If $-2 < a$, then there is a positive number d such that $a = (-2) + d$.

4. (a) Use $5 + 8 = 13$ to suggest two true sentences involving "<" relating pairs of the numbers 5, 8, 13.

   (b) Since $(-3) + 2 = (-1)$, how many true sentences involving "<" can you write using pairs of these three numbers?

   (c) If $5 < 7$, write two true sentences involving "=" relating the numbers 5, 7.

5. Show on the number line that if a and c are real numbers and if b is a negative number such that $c = a + b$, then $c < a$.

6. Which of the following sentences are true for all values of the variables?

   (a) If $b < 0$, then $3 + b < b$.
   (b) If $b < 0$, then $3 + b < 3$.
   (c) If $x < 2$, then $2x < 4$.

7. Translate the following into open sentences and find their truth sets.

   (a) The sum of a number and 5 is less than twice the number. What is the number?

(b) When Joe and Moe were planning to buy a sailboat, they asked a salesman about the cost of a new type of a boat that was being designed. The salesman replied, "It won't cost more than $380." If Joe and Moe had agreed that Joe was to contribute $130 more than Moe when the boat was purchased, how much would Moe have to pay?

(c) Three more than six times a number is greater than seven increased by five times the number. What is the number?

(d) A teacher says, "If I had twice as many students in my class as I do have, I would have at least 26 more than I now have." How many students does he have in his class?

(e) A student has test grades of 82 and 91. What must he score on a third test to have an average of 90 or higher?

(f) Bill is 5 years older than Norman, and the sum of their ages is less than 23. How old is Norman?

2-6. Multiplication Property of Order

In the preceding section we stated a basic property giving the order of $a + c$ and $b + c$ when $a < b$. Let us now ask about the order of the products ac and bc when $a < b$.

Consider the true sentence

$$5 < 8.$$

If each of these numbers is multiplied by 2, the products are involved in the true sentence

$$(5)(2) < (8)(2).$$

What is your conclusion about a multiplication property of order? Before making a decision, let us try more examples. Just as above, where we took the two numbers 5 and 8 in the true sentence "$5 < 8$" and inserted them in "$(\ )(2) < (\ )(2)$" to make a true sentence, do the same in the following.

1. $7 < 10$ and $(\ )(6) < (\ )(6)$
2. $-9 < 6$ and $(\ )(5) < (\ )(5)$
3. $2 < 3$ and $(\ )(-4) < (\ )(-4)$
4. $-7 < -2$ and $(\ )(2) < (\ )(2)$
5. $-1 < 8$ and $(\ )(-3) < (\ )(-3)$
6. $-5 < -4$ and $(\ )(-6) < (\ )(-6)$

We are concerned here with the order relation "<", observing the pattern when each of the numbers in the statement "$a < b$" is multiplied by the same number. Did you notice that it makes a difference whether we multiply by a positive number or a negative number?

The above experience suggests that if $a < b$, then

$ac < bc$, <u>provided</u> c <u>is a positive number</u>;
$bc < ac$, <u>provided</u> c <u>is a negative number</u>.

Thus, we have found another important set of properties of order.

How can you use these properties to tell quickly whether the following sentences are true?

Since $\frac{1}{4} < \frac{2}{7}$, then $\frac{5}{4} < \frac{10}{7}$.

Since $-\frac{5}{6} < -\frac{14}{17}$, then $\frac{14}{51} < \frac{5}{18}$.

Since $\frac{5}{3} < \frac{7}{4}$, then $-\frac{7}{16} < -\frac{5}{12}$.

These properties of order turn out to be consequences of the other properties of order, and we state them together as

<u>Theorem 2-6a</u>. <u>Multiplication Property of Order</u>. If a, b, and c are real numbers and if $a < b$, then

$ac < bc$, if c is positive,
$bc < ac$, if c is negative.

<u>Proof</u>: There are two cases. Let us consider the case of <u>positive</u> c. Here we must prove that if $a < b$, then $ac < bc$. You fill in the reason for each step of the proof.

1. There is a positive number d such that $b = a + d$.
2. Therefore, $bc = (a + d)c$.
3. $bc = ac + dc$.

4. The number dc is positive.
5. Hence, $ac < bc$.

The proof of the case for <u>negative</u> c is left to the pupil in the problems.

We could equally well have discussed the multiplication property of the order relation "is greater than" instead of "is less than".

When we are comparing numbers, the two statements "$a < b$" and "$b > a$" say the same thing about a and b. Thus, when we are concerned primarily with numbers rather than a particular order relation, it may be convenient to shift from one order relation to another and write such sentences as:

Since $3 < 5$, then $3(-2) > 5(-2)$.
Since $-2 > -5$, then $(-2)(8) > (-5)(8)$.
Since $3 > 2$, then $(3)(-7) < (2)(-7)$.

Verify that these sentences are true.

When we are focusing on the numbers involved instead of on an order relation, we can say that

$$\text{if } a < b, \text{ then } \begin{cases} ac < bc & \text{if } c \text{ is positive,} \\ ac > bc & \text{if } c \text{ is negative.} \end{cases}$$

State these properties of order in your own words.

In our study of inequalities we shall also need some results such as

<u>Theorem 2-6b</u>. If $x \neq 0$, then $x^2 > 0$.

<u>Proof</u>: If $x \neq 0$, then either x is negative or x is positive, but not both. If x is positive, then

$$\begin{aligned} x &> 0, \\ (x)(x) &> (0)(x), \qquad \text{(Why?)} \\ x^2 &> 0. \end{aligned}$$

If x is negative, then

$$\begin{aligned} x &< 0, \\ (x)(x) &> (0)(x), \qquad \text{(Why?)} \\ x^2 &> 0. \end{aligned}$$

In either case the result is the desired one.

Theorem 2-6b states that the square of a non-zero number is positive. What can be said about x^2 for any x?

The properties of order can be used to advantage in finding truth sets of inequalities. For example, let us find the truth set of

$$(-3x) + 2 < 5x + (-6).$$

By the addition property of order we may add $(-2) + (-5x)$ to both members of this inequality to obtain

$$\big((-3x) + 2\big) + \big((-2) + (-5x)\big) < \big(5x + (-6)\big) + \big((-2) + (-5x)\big),$$

which when simplified is

$$-8x < -8.$$

Since $\big((-2) + (-5x)\big)$ is a real number for every value of x, the new sentence has the same truth set as the original. (What must we add to the members of "$-8x < -8$" to obtain the original sentence; that is, to reverse the step?)

Then, by the multiplication property of order.

$$(-8)(-\tfrac{1}{8}) < (-8x)(-\tfrac{1}{8})$$

$$1 < x$$

Here we multiplied by a non-zero real number. Thus, this sentence is equivalent to the former sentence; that is, it has the same truth set. (What must we multiply the members of "$1 < x$" by to obtain the former sentence?) Obviously, the truth set of "$1 < x$" is the set of all numbers greater than 1, and this is the truth set of the original inequality.

Problem Set 2-6

1. Solve each of the following inequalities, using the form of the following example. (Recall that to "solve" a sentence is to find its truth set.)

   Example: $(-3x) + 4 < -5$.

   This sentence is equivalent to

   $-3x < (-5) + (-4)$, (add (-4) to both members)

   $-3x < -9$,

   which is equivalent to

   $(-\frac{1}{3})(-9) < (-\frac{1}{3})(-3x)$, (multiply both members by $(-\frac{1}{3})$

   $3 < x$.

   Thus, the truth set consists of all numbers greater than 3.

(a) $(-2x) + 3 < -5$

(b) $(-2) + (-4x) > -6$

(c) $(-4) + 7 < (-2x) + (-5)$

(d) $5 + (-2x) < 4x + (-3)$

(e) $(\frac{2}{3}) + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3x)$

(f) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$

(g) $2x < 3 + (-2)(-\frac{4}{3})$

(h) $(-2) + 5 + (-3x) < 4x + 7 + (-2x)$

(i) $-(2 + x) < 3 + (-7)$

2. Graph the truth sets of parts (a) and (b) of Problem 1.

3. Translate the following into open sentences and solve.
   (a) Sue has 16 more books than Sally. Together they have more than 28 books. How many books does Sally have?

   (b) If a certain variety of bulbs are planted, less than $\frac{5}{8}$ of them will grow into plants. If, however, the bulbs are given proper care, more than $\frac{3}{8}$ of them will grow. If a careful gardener has 15 plants, how many bulbs did he plant?

4. Prove that if $a < b$ and c is a negative number, then $bc < ac$. Hint: There is a negative number e such that $a = b + e$. Therefore, $ac = bc + ec$. What kind of number is ec? Hence, what is the order of ac and bc?

5. If c is a negative number, then $c < 0$. By taking opposites, $0 < (-c)$. Since $(-c)$ is a positive number, we may prove the theorem of Problem 4 by noting that if $a < b$, then $a(-c) < b(-c)$; i.e., $-(ac) < -(bc)$. Why does the conclusion then follow?

6. If $a < b$, and a and b are both positive real numbers, prove that $\frac{1}{b} < \frac{1}{a}$. Hint: Multiply the inequality $a < b$ by $(\frac{1}{a} \cdot \frac{1}{b})$. Demonstrate the theorem on the number line.

7. Does the relation $\frac{1}{b} < \frac{1}{a}$ hold if $a < b$ and both a and b are negative? Prove it or disprove it.

9. State the addition and multiplication properties of the order ">".

10. Prove: If $0 < a < b$, then $a^2 < b^2$. Hint: Use properties of order to obtain $a^2 < ab$ and $ab < b^2$.

Review Problems

1. For each pair of numbers, determine their order.

   (a) $-100, -99$

   (b) $0.2, -0.1$

   (c) $-(-3), -(-7)$

   (d) $\frac{6}{7}, \frac{5}{8}$

   (e) $3.4 - 4, \quad 3(4 - 4)$

   (f) $x^2 + 1, \quad 0$

2. If $p > 0$ and $n < 0$, determine which sentences are true and which are false.

   (a) If $5 > 3$, then $5n < 3n$.

   (b) If $a > 0$, then $ap < 0$.

   (c) If $3x > x$, then $3px > px$.

   (d) If $(\frac{1}{n})x > 1$, then $x > n$.

   (e) If $p > n$, then $\frac{1}{p} < \frac{1}{n}$.

   (f) If $\frac{1}{p} > \frac{1}{x}$ and $\frac{1}{x} > 0$, then $p < x$ and $x > 0$.

3. Which of the following pairs of sentences are equivalent?

   (a) $3a > 2, \quad (-3)a > (-2)$

   (b) $3x > 2 + x, \quad 2x > 2$

   (c) $3y + 5 = y + (-1), \quad 2y = (-6)$

   (d) $-x < 3, \quad x > (-3)$

   (e) $-p + 5 < p + (-1), \quad 6 > 2p$

   (f) $\frac{1}{m} < \frac{1}{2}$ and $m > 0, \quad m < 2$

4. Solve each of the following inequalities.

   (a) $-x > 5$

   (b) $(-1) + 2y < 3y$

   (c) $(-\frac{1}{2})z < 3$

   (d) $(-4) + (-x) > 3x + 8$

   (e) $b + b + 5 + 2b + 12 \leq 381$

   (f) $x(x + 1) < x$

5. Find the truth set of each of the following sentences.

(a) $\frac{1}{x} < \frac{1}{2}$ and $x > 0$

(b) $\frac{1}{x} = \frac{1}{2}$

(c) $\frac{1}{x} < \frac{1}{2}$ and $x < 0$

(d) $(\frac{1}{x})^2 > 0$ and $x \neq 0$

(e) $0 \leq 2x < 180$

(f) $x^2 + 1 = 0$

6. If the domain of the variable is the set of integers greater than -4 and less than 4, find the truth sets of the following sentences.

(a) $3x + 2x \geq 10$

(b) $x + (-1) < 3x + 1$

(c) $2x + 1 \not> 3x + (-9)$

(d) $2(x + (-3)) = 5$

(e) $3x + 5 < 2x + 3$

(f) $(\frac{1}{2}) + (-x) > (-\frac{1}{2}) + (-2x)$

7. If the domain of the variable is the set of real numbers, find the truth sets of

(a) $6 < 3x + 2 < 10$

(b) $-4 < 2y + 7 < 4$

(c) $-2 < \frac{2w + 3}{5} < 2$

(d) $-1 < 3 - x < 1$

(e) $2 < \frac{5 - 3y}{2} < 15$

(f) $2 < 3 - 2m \leq 3$

(g) $-3 \leq 4 - 2a < 3$

(h) $-\frac{1}{2} < 3p - \frac{1}{3} < \frac{1}{2}$

(i) $-1 \leq \frac{4 - 3x}{2} \leq 1$

(j) $-1 \leq \frac{4 - 5x}{-2} \leq 1$

8. The length of a rectangle is known to be greater than or equal to 6 units and less than 7 units. The width is known to be 4 units. Find the area of the rectangle.

9. The length of a rectangle is known to be greater than or equal to 6 units and less than 7 units. The width is known to be greater than or equal to 4 units and less than 5 units. Find the area of the rectangle.

10. The length of a rectangle is known to be greater than or equal to 6.15 inches and less than 6.25 inches. The width is known to be greater than or equal to 4.15 inches, and less than 4.25 inches. Find the area of the rectangle.

11. (a) A certain variety of corn plant yields 240 seeds per plant. Not all the seeds will grow into new plants when planted. Between $\frac{3}{4}$ and $\frac{5}{6}$ of the seeds will produce new plants. Each new plant will also yield 240 seeds. From a single corn plant whose seeds are harvested in 1965 how many seeds can be expected in 1966?

    (b) Suppose instead that a corn plant did not yield exactly 240 seeds, but between 230 and 250 seeds. Under this condition how many seeds can be expected in 1966 from the 240 seeds planted at the beginning of the season?

12. Write open sentences and find the solution to each of the questions which follow.

    (a) A square and an equilateral triangle have equal perimeters. A side of the triangle is 3.5 inches longer than a side of the square. What is the length of the side of the square?

    (b) A boat traveling downstream goes 10 miles per hour faster than the rate of the current. Its velocity downstream is not more than 25 miles per hour. What is the rate of the current?

    (c) Mary has typing to do which will take her at least 3 hours. If she starts at 1 P.M. and must finish by 6 P.M., how much time can she expect to spend on the job?

    (d) Jim receives $1.50 per hour for work which he does in his spare time, and is saving his money to buy a car. If the car will cost him at least $75, how many hours must he work?

13. A man needs 7 gallons of paint to paint his house. He bought three times as much grey paint at $6 a gallon as white paint at $7 a gallon. If his paint bill was less than $50, how many gallons of each color paint did he buy? (Assume the smallest size paint can available is the quart size.)

14. Prove: If $a > 0$, $b > 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$.
Hint: Use the comparison property in an indirect proof.

15. Prove: If $a > b$, then $a - b$ is positive.

16. If $(a - b)$ is a positive number, which of the statements $a < b$, $a > b$, is true? What if $(a - b)$ is a negative number? What if $(a - b)$ is zero?

17. If a, b, and c are real numbers and $a > b$, what can we say about the order of $a - c$ and $b - c$? Prove your statement.

Chapter 3

SOLUTION OF INEQUALITIES

3-1. Equivalent Inequalities

In Chapter 2 we solved certain inequalities by obtaining simpler equivalent inequalities. Recall that we often used the properties:

For real numbers a, b, c, $a < b$
if and only if $a + c < b + c$,

and

for c positive, $a < b$ if and only if $ac < bc$,
for c negative, $a < b$ if and only if $ac > bc$.

It turns out that the operations we may perform on an inequality to yield an equivalent inequality are somewhat like those for equations. The only difference is that when we multiply both members of an inequality, we must distinguish between multiplication by positive numbers and multiplication by negative numbers. For example, $x^2 + 1$ is always positive for every value of x; $-\frac{1}{x^2 + 2}$ is always negative for every value of x; but $x^2 - 1$ is negative for some values, positive for other values, and 0 for others. Hence, we shall not use $x^2 - 1$ as a multiplier.

To summarize, some operations which yield equivalent inequalities are:

(1) adding a real number to both members,

(2) multiplying both members by a <u>positive</u> number, in which case the order of the resulting products is unchanged,

(3) multiplying both members by a <u>negative</u> number, in which case the order of the resulting products is reversed.

<u>Example</u> 1. Solve $\frac{4}{5}y - 6 < \frac{2}{3}y + \frac{5}{6}$

We may first multiply both members by the positive real number 30 to obtain a sentence free of fractions:

$$24y - 180 < 20y + 25.$$

Now we add the real number $-20y + 180$ to both members:

$$4y < 205.$$

Finally, we multiply by the positive real number $\frac{1}{4}$:

$$y < \frac{205}{4}.$$

What is the truth set of the original inequality? Explain why all these sentences are equivalent.

<u>Example</u> 2. Solve $-\frac{1}{x^2 + 1} > -1$.

Since $-(x^2 + 1)$ is a negative real number for every value of x, we may multiply both members by $-(x^2 + 1)$ to obtain the equivalent sentence

$$1 < x^2 + 1.$$

By adding -1 to both members, we have the equivalent sentence

$$0 < x^2.$$

The truth set of this final sentence is the set of all non-zero real numbers. This is also the truth set of the original inequality.

Problem Set 3-1

1. Solve the following inequalities by changing to simpler equivalent inequalities.

   (a) $x + 12 < 39$

   (b) $\frac{5}{7}x < 36 - x$

   (c) $\sqrt{2} + 2x > 3\sqrt{2}$

   (d) $t\sqrt{3} < 3$

   (e) $8y - 3 > 3y + 7$

   (f) $\frac{t}{3} < 4 + \frac{t}{6} - 2$

   (g) $x^2 + 5 \geq 4$

   (h) $\frac{3}{x^2 + 4} < -2$

   (i) $-\frac{2}{x^2 + 2} \geq -1$

2. Solve the following sentences.

   (a) $1 < 4x + 1 < 2$ (This is equivalent to "$1 < 4x + 1$ and $4x + 1 < 2$".)

   (b) $4t - 4 < 0$ and $1 - 3t < 0$

   (c) $-1 < 2t < 1$

   (d) $6t + 3 < 0$ or $6t - 3 > 0$

   *(e) $|x - 1| < 2$

   *(f) $|2t| < 1$

   *(g) $|x + 2| < \frac{1}{2}$

   *(h) $|y + 2| > 1$

* Omit if pupil has not studied "absolute value".

3. Graph the truth sets of the sentences in Problems 2(a), (c), (e) and (h).

4. Solve $3y - x + 7 < 0$ for y; that is, obtain an equivalent sentence with y alone on the left side. What is the truth set for y when $x = 1$? Now solve $3y - x + 7 < 0$ for x. What is the truth set for x if $y = -2$?

5. If the area of a rectangle is 12 square inches and its length is less than 5 inches, what is its width?

6. Write an open sentence expressing the fact that a certain negative number is less than its reciprocal. Solve the sentence.

3-2. Polynomial Inequalities

Is $(-4)(3)(5)(-6)(-8)$ a positive number? A negative number? Did you need to perform the multiplication to answer this question?

When we multiply several non-zero numbers together, their product is positive if the number of negative factors is even, and their product is negative if the number of negative factors is odd.

This means that we can tell immediately whether a factored polynomial, such as

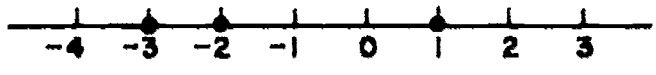
$$(x + 3)(x + 2)(x - 1),$$

is positive, negative, or 0 for any given x. How about this polynomial for $x = 2$? For $x = 0$? For $x = -1$? For $x = -\frac{5}{2}$? For $x = -4$? You need not compute the value of the polynomial; just check how many factors are negative.

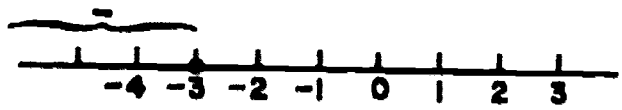
We can do better than choosing a few points at random. We can first find the set of values of x for which

$$(x + 3)(x + 2)(x - 1) \text{ is } 0$$
$$(\text{the truth set of } (x + 3)(x + 2)(x - 1) = 0).$$

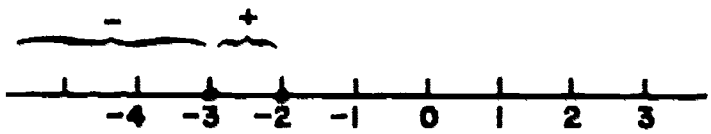
What is this set? Then we draw the graph of this set on the number line.



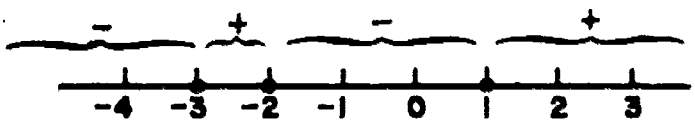
What can we say about each of the factors $(x + 3)$, $(x + 2)$, $(x - 1)$ for any x less than -3? Try $x = -4$. We find that all three factors are negative numbers, and therefore their product is negative. We indicate this on the number line as follows:



What about these factors when x is between -3 and -2? Try $x = -\frac{5}{2}$. The factor $(x + 3)$ is now positive, while the other two remain negative. We can think of $(x + 3)$ as "changing over" from negative to positive as x crosses -3. The product is now positive for x between -3 and -2. We indicate this by a mark "+" over the interval.



Probably you can see what is going to happen when x crosses -2 and finally 1. When x crosses -2, the next factor $(x + 2)$ changes from negative to positive, so that for any x between -2 and 1 there are two positive factors and one negative, the product now being negative. Finally, when x crosses 1, the last factor changes from negative to positive, so that for x greater than 1, all factors are positive.



Using this final diagram, we can read off the truth sets of certain related inequalities. For example, the truth set of the sentence

$$(x + 3)(x + 2)(x - 1) < 0$$

is graphed on page 43. This is the set of all numbers x for

which the product of the factors is negative; namely, the set of all x such that

$$x < -3 \quad \text{or} \quad -2 < x < 1.$$

What is the truth set of the sentence $(x + 3)(x + 2)(x - 1) > 0$? Draw its graph. Of the sentence $(x + 3)(x + 2)(x - 1) \geq 0$?

To find the truth set of

$$x^2 - 3 \leq 2x,$$

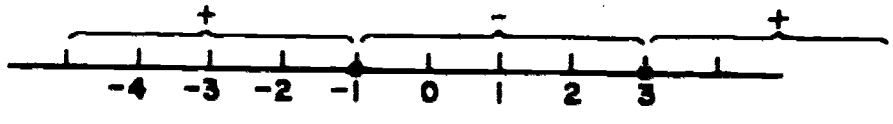
we first change to the equivalent inequality with 0 on the right side:

$$x^2 - 2x - 3 \leq 0.$$

Then we factor the left side into <u>first degree</u> polynomial factors:

$$(x + 1)(x - 3) \leq 0.$$

Proceeding as before, we get the diagram:



Thus, the truth set of the inequality $x^2 - 3 \leq 2x$ has the following graph (since the product of the factors $(x + 1)(x - 3)$ must be negative or zero).

This is the set of all x such that $-1 \leq x \leq 3$.

<u>Problem Set 3-2a</u>

1. Using the above discussion as a model, draw the graph and describe the truth sets of the following inequalities:

   (a) $(x - 1)(x + 2) > 0$

   (b) $y^2 < 1$

   (c) $t^2 + 5t \leq 6$

(d) $x^2 + 2 \geq 3x$

(e) $(s + 5)(s + 4)(s + 2)(s)(s - 3) < 0$

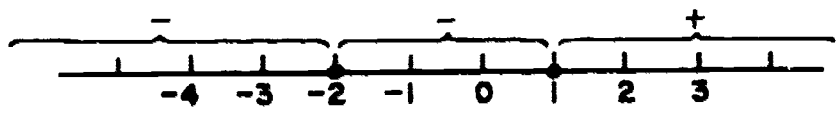
(f) $2 - x^2 < x$

2. What is the truth set of the sentence

$$(x + 2)(x - 1) > 0 \text{ and } x < 3?$$

3. Is there a single polynomial inequality equivalent to the sentence of Problem 2?

There is one danger point which we should explain. Suppose a factor is repeated in a polynomial one or more times, as in $(x + 2)^2(x - 1)$. When x crosses -2, there are <u>two</u> factors of the polynomial changing together from negative to positive. The number of negative factors drops from 3 to 1, and the product <u>remains</u> <u>negative</u> as x crosses -2. The diagram is then



What is the truth set of

$$(x + 2)^2(x - 1) > 0?$$

(i.e., for what values of x is the product of the factors positive?)

Notice that the truth set of

$$(x + 2)^2(x - 1) \geq 0$$

is the set of all x such that $x \geq 1$ <u>or</u> $x = -2$.

What happens if a factor occurs three times, as in $x(x - 1)^3$? What is the truth set of

$$x(x - 1)^3 < 0?$$

Of

$$x(x - 1)^3 \geq 0?$$

Sometimes we have a quadratic factor, such as $x^2 + 2$, which cannot be factored and which is <u>always</u> positive for all values of x. Does such a factor have any effect on the way the product

changes from positive to negative? What is the truth set of

$$(x^2 + 2)(x - 3) < 0?$$

Or

$$(x^2 + 2)(x - 3) \geq 0?$$

Problem Set 3-2b

Solve and graph:

1. $x^2 + 1 > 2x$
2. $x^2 + 1 < 0$
3. $(t^2 + 1)(t^2 - 1) \geq 0$
4. $4s - s^2 > 4$
5. $(x - 1)^2(x - 2)^2 > 0$
6. $(y^2 - 7y + 6) \leq 0$
7. $(x + 2)(x^2 + 3x + 2) < 0$
8. $3y + 12 \leq y^2 - 16$
9. $x^2 + 5x > 24$
*10. $|x|(x - 2)(x + 4) < 0$

3-3. Rational Inequalities

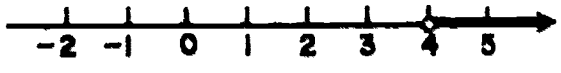
We have observed that we can find the truth set of polynomial inequalities by determining whether or not the product of the factors is positive or negative for a given value of the variable.

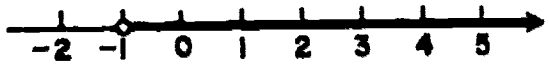
Let us consider how we might apply the fact that $ab = 0$ if $a = 0$ or $b = 0$ to the solution of $ab > 0$. When is $ab > 0$? $ab > 0$ if $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

For example, find the solution set of $x^2 - 3x - 4 > 0$:

$(x - 4)(x + 1) > 0$ if

$x - 4 > 0$ and $x + 1 > 0$, or $x - 4 < 0$ and $x + 1 < 0$.

The graph of $x - 4 > 0$ is 

The graph of $x + 1 > 0$ is 

The solution set of these two clauses with connective and is the set of all real numbers greater than 4.

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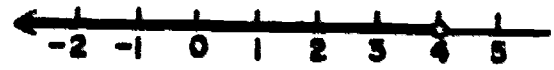
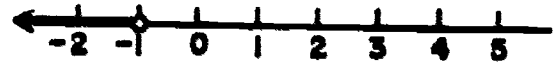
*Omit if pupil has not studied "absolute value".

The graph of $x - 4 < 0$ is 

The graph of $x + 1 < 0$ is 

The solution set of these two clauses with connective <u>and</u> is the set of all real numbers less than -1. Therefore, the solution set of the original sentence is the set of all real numbers greater than 4 or less than -1.

Problem Set 3-3a

Use the above method to find the solution sets of the following sentences:

1. $x^2 - 3x + 2 > 0$
2. $x^2 - 5x + 6 < 0$
3. $3x^2 - 2x - 8 < 0$
4. $2x^2 + 1 < 3x$
5. $2x^2 - 6 > 7x - 2$
6. $x^2 - x > x - 1$
7. $(x + 3)^2 > 4$
8. $-x^2 - 4x + 5 < 0$
9. $2x^2 + 4x + 5 < 0$
10. $x^2 - 2\sqrt{5}\,x - 4 < 0$

This method can be used to find the solution set of a rational inequality of the form $\frac{x + 2}{x - 3} > 0$.

If we multiply both members of the inequality by $x - 3$, we would not know if $x + 2 > 0$ or $x + 2 < 0$. It would depend upon the value of $x - 3$. Is $x - 3 > 0$, or is $x - 3 < 0$? If we multiply both members, however, by $(x - 3)^2$, we know that $(x - 3)(x + 2) > 0$. The square of any real number is what kind of number? We now have

$$x - 3 > 0 \text{ and } x + 2 > 0, \quad \text{or} \quad x - 3 < 0 \text{ and } x + 2 < 0$$
$$x > 3 \text{ and } x > -2, \quad \text{or} \quad x < 3 \text{ and } x < -2.$$

Therefore, the solution set is the set of real numbers greater than 3 or less than -2. Let us check 4 and -3 which are in the solution set.

$$\frac{4 + 2}{4 - 3} = \frac{6}{1}, \quad \frac{6}{1} > 0; \quad \frac{-3 + 2}{-3 - 3} = \frac{-1}{-6} = \frac{1}{6}, \quad \frac{1}{6} > 0.$$

Problem Set 3-3b

1. Find the solution sets of the following inequalities:

(a) $\frac{2x + 3}{x - 4} < 0$

(b) $\frac{5x - 2}{x + 3} > 0$

(c) $\frac{3x + 2}{4x + 7} > 0$

(d) $\frac{2x - 5}{2 - 5x} < 0$

(e) $\frac{(3x + 4)(x + 2)}{x - 1} > 0$

(f) $\frac{(x - 4)(2x - 3)}{5 - 3x} > 0$

(g) $\frac{(5x + 1)(x + 4)}{2x + 5} < 0$

(h) $\frac{(3x - 4)^2}{(7x + 2)^2} < 0$

2. Solve and graph:

$$\sqrt{1 + 2x} < x - 1$$

3. Graph the truth set of each of the following sentences.

(a) $(x - 3)(x - 1)(x + 1) > 0$

(b) $(x - 3)(x - 1)(x + 1) > 0$ and $x \geq 0$

(c) $(x - 3)(x - 1)(x + 1) > 0$ or $x \geq 0$

Chapter 4

GRAPHS OF OPEN SENTENCES IN TWO VARIABLES

4-1. Inequalities in Two Variables

With respect to a set of coordinate axes locate points for which the abscissas are

$$-2, -1, 0, 1, 2, 3$$

respectively, and for which each ordinate is equal to 3 times the abscissa. Do these points lie on a line?

Now locate points having these same abscissas, but for which each ordinate is greater than 3 times the abscissa. Do these new points lie on a line? Does each one lie above the corresponding point of the first set?

The points in the first set satisfy the sentence

$$y = 3x$$

while those in the second set satisfy the sentence

$$y > 3x.$$

The sentence "$y = 3x$" is the equation of a line, and the graph of "$y > 3x$" is the set of all points above this line, as shown by the shaded portion of Figure 2. Thus, the graph of a sentence such as "$y > 3x$" is the set of all points of the plane for which the sentence is true. If the verb is "is greater than or equal to", that is, "$\geq$", we make the boundary line solid, as in Figure 1, while the verb "is greater than" is indicated by using a dashed line for the boundary between the shaded and the unshaded regions as in Figure 2. In these two illustrations, the line is the graph of the sentence $y = 3x$. This graph, which is a line, separates the plane into two half-planes. The graph of $y < 3x$ is the half-plane such that every ordinate is less than three times the abscissa; it is the set of points below the line $y = 3x$. The graph of $y \leq 3x$ is the lower half-plane including the line $y = 3x$.

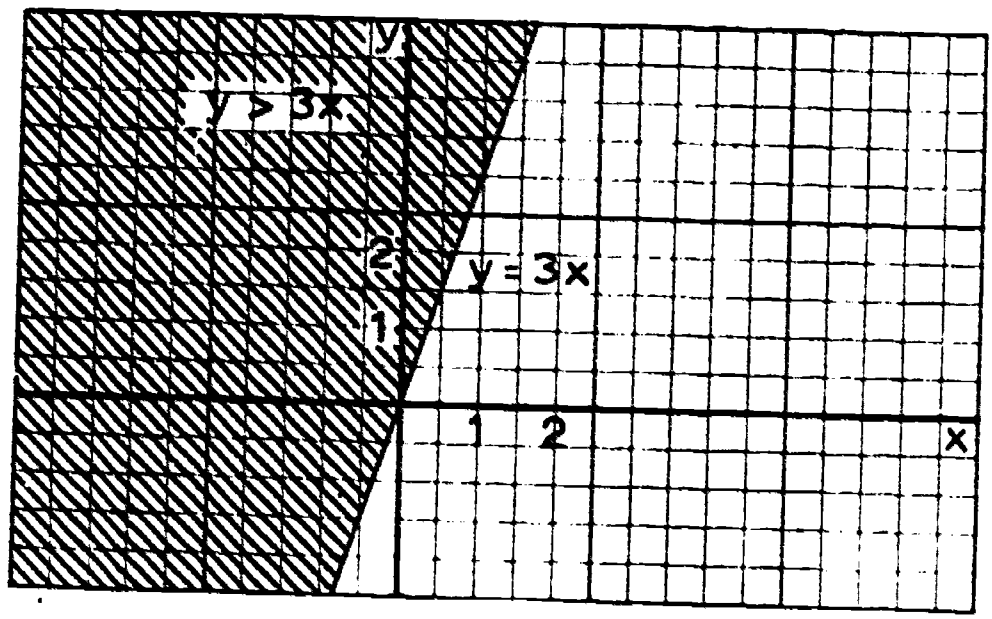


Figure 1. $y \geq 3x$

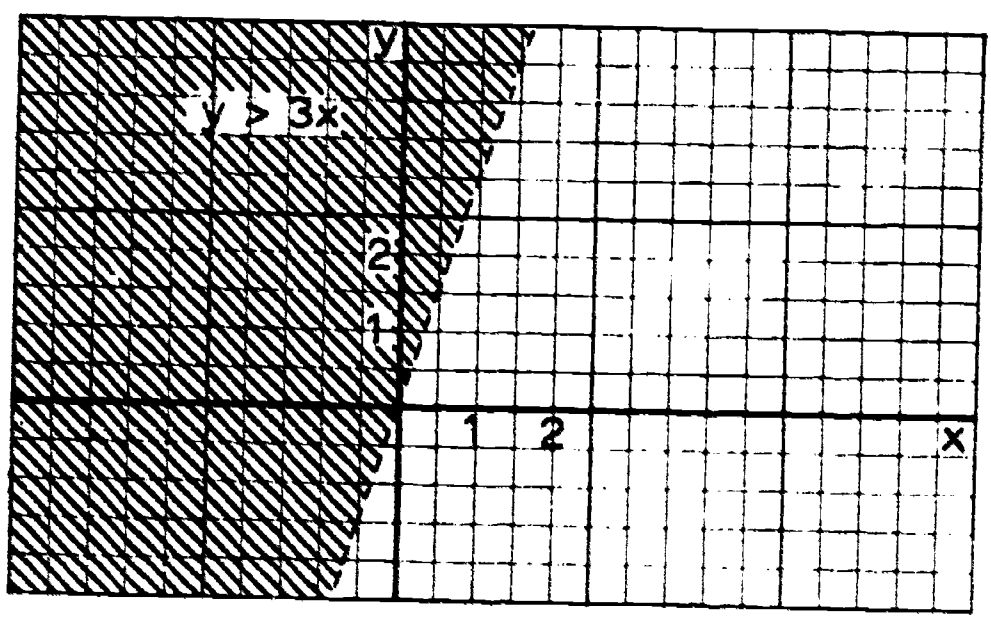


Figure 2. $y > 3x$

Problem Set 4-1

1. With reference to a set of coordinate axes indicate the set of points associated with the ordered pairs of numbers such that each has an ordinate two greater than the abscissa. What open sentence can you write for this set? Now draw the graph of the following open sentences.

   (a) $y > x + 2$ (b) $y \geq x + 2$

   Is it possible to draw both of these graphs with reference to the same coordinate axes?

2. Draw the graph of each of the following with reference to a different set of axes.

   (a) $y \leq 3x$ (d) $x < 1.5$

   (b) $y < \frac{x}{2} - 5$ (e) $-2 < x < 3$

   (c) $y > 3$ (f) $y > 2$ or $y < -1$

3. Draw the graph of each of the following with reference to a different set of axes.

   (a) $y > 2x + 4$ (c) $y \geq \frac{3x}{4} - 1$

   (b) $y < \frac{2x}{3} + 7$ (d) $y \leq 2x - 1$

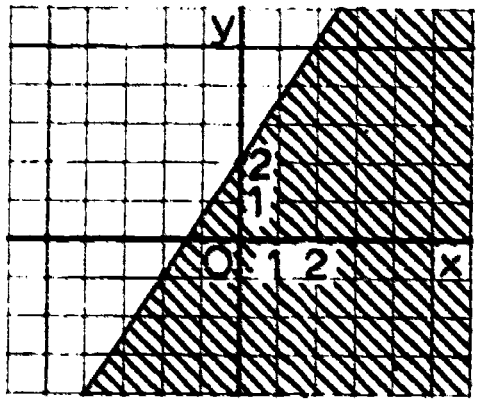
4. Draw the graph of each of the following with reference to a different set of axes.

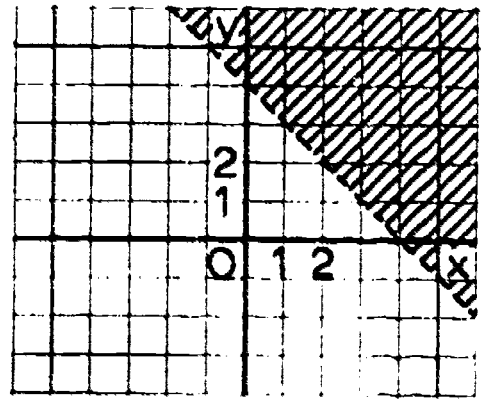
   (a) $2x + y > 3$ (c) $x - 2y \leq 4$

   (b) $x + 2y \geq 4$ (d) $2x - y \leq 3$

5. Write the open sentence for each graph in y-form.

   (a) (b)





4-2. Graphs of Open Sentences Involving Integers Only

In drawing graphs of open sentences, we must keep in mind that every point of a graph is associated with some pair of real numbers. Suppose we consider a sentence in which the value of the variables are restricted to integers so that the coordinates of points on the graph must be integers. What would such a graph look like?

First let us consider the coordinate axes. Would they still be straight lines? It seems that they are sets of points, such as (0,1), (0,2), (0,3), etc., since we are restricting ourselves to integers, and we might wish to distinguish the axes for such cases from the coordinate axes for all real numbers. However, a series of dots or short dashes would be apt to be confused with the graph itself; so we shall use a line segment for each axis, and remember that our coordinates are limited to integers.

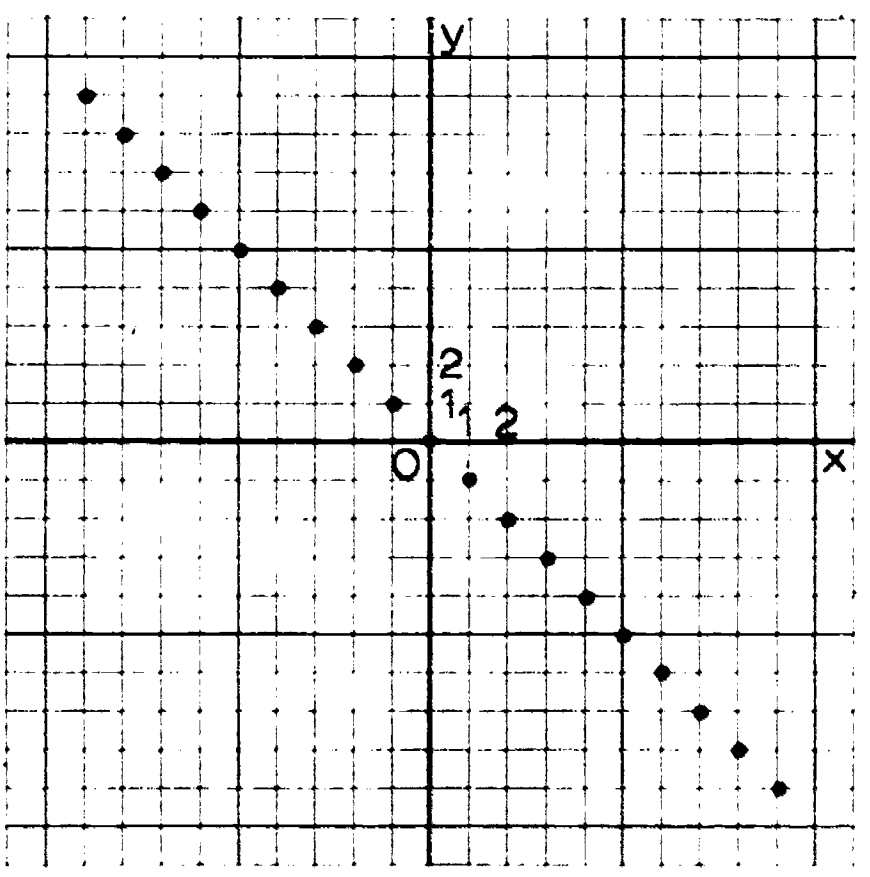


Figure 3.

What is the open sentence associated with the graph in Figure 3? In determining this, we first note that the graph includes points with integral coordinates only, and second that each ordinate is the opposite of the corresponding abscissa. This may be stated as follows: "$y = -x$, where x and y are integers such that $-10 < x < 10$ and $-10 < y < 10$".

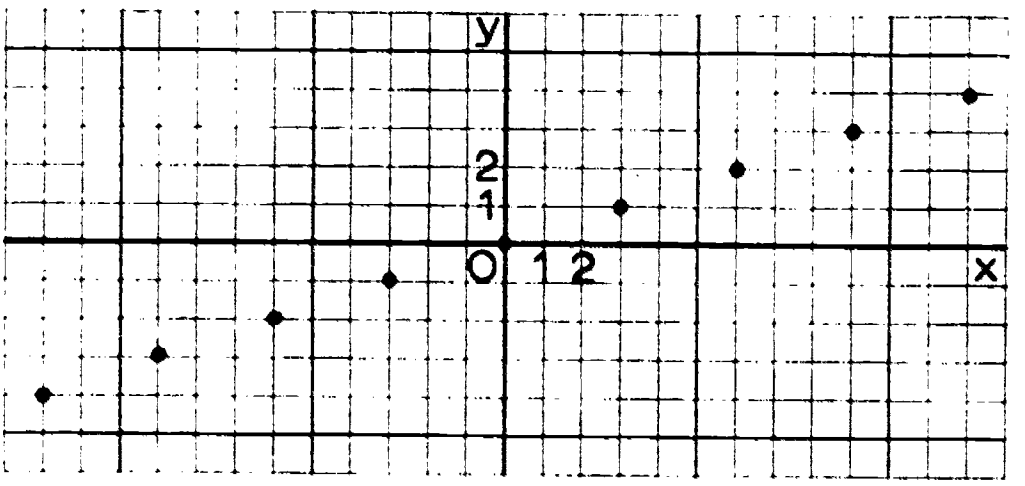


Figure 4.

In Figure 4 the points go on beyond the limit of this diagram. Note that there are no points for $x = 1$, $x = 2$, $x = 4$, $x = -1$, and others. What do you notice about the ordinate corresponding to each abscissa, if we assume that all the points lie on a straight line, as these points seem to indicate? We would write the open sentence: "$y = \frac{x}{3}$ where x and y are integers". Why can the abscissa not be 1 or 2?

Consider Figure 5. For this set of twelve points it seems there is no simple open sentence. Can you describe the limitations on the abscissas? What statement can you make about the ordirates?

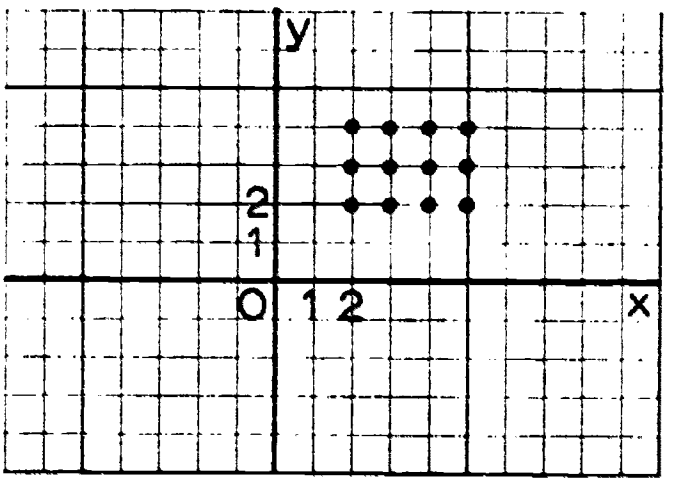


Figure 5.

These facts could be stated in a compound open sentence as follows: "$1 < x < 6$ and $1 < y < 5$, where x and y are integers".

Notice that here the connective for the compound sentence is <u>and</u>; note also that the points whose coordinates make the sentence true are only those which belong to the truth sets of <u>both</u> parts of the compound sentence.

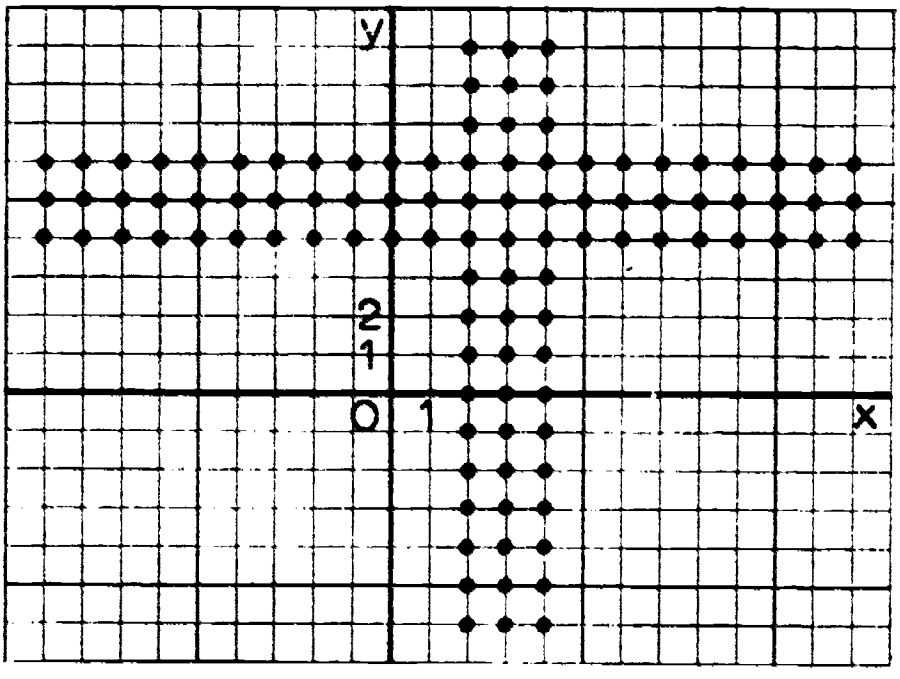


Figure 6.

In Figure 6 a different situation exists. Let us see what open sentence will describe this graph. The three horizontal rows of dots could be the graph of the sentence: "$3 < y < 7$, where x and y are integers". Then we write a sentence which describes the three vertical rows of dots: "$1 < x < 5$ where x and y are integers". The open sentence which describes the total set of points is "$1 < x < 5$ or $3 < y < 7$, where x and y are integers". Another way of stating this would be: "$2 \leq x \leq 4$ or $4 \leq y \leq 6$ where x and y are integers". Notice that the connective here is <u>or</u>, and that the graph includes all points which belong to the truth sets of either of the two parts of the compound sentences, or to both of them.

Problem Set 4-2

1. With reference to separate sets of coordinate axes, and for x and y integers, draw the graph of each of the following.

   (a) $y = \frac{x}{2}$, for $-6 < x < 6$

   (b) $y = 3x - 2$

   (c) $y = 2x + 4$

2. Draw the graphs of each of the following with reference to a separate set of coordinate axes.

   (a) $-3 < x < 2$ and $-2 < y < 1$, where x and y are integers.

   (b) $-3 < x < 2$ or $-2 < y < 1$, where x and y are integers.

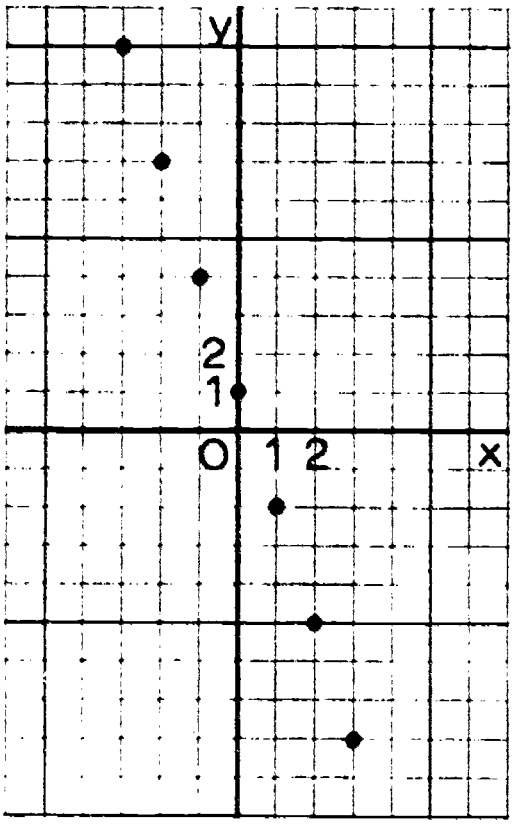
   (c) $5 \leq x \leq 6$ or $1 \leq y \leq 3$, where x and y are integers.

   (d) $5 \leq x \leq 6$ and $y = 0$, where x and y are integers.

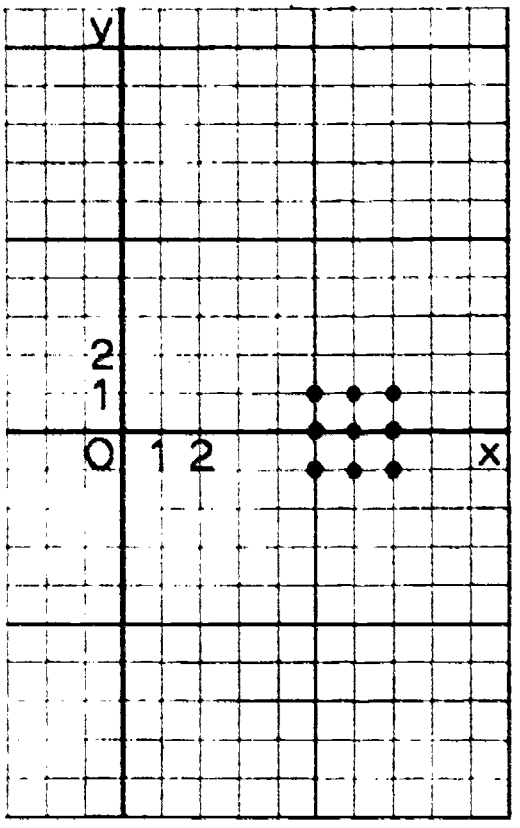
3. Write a compound open sentence whose truth set is $\{(-1,3)\}$.

4. Write open sentences whose truth sets are the following sets of points:

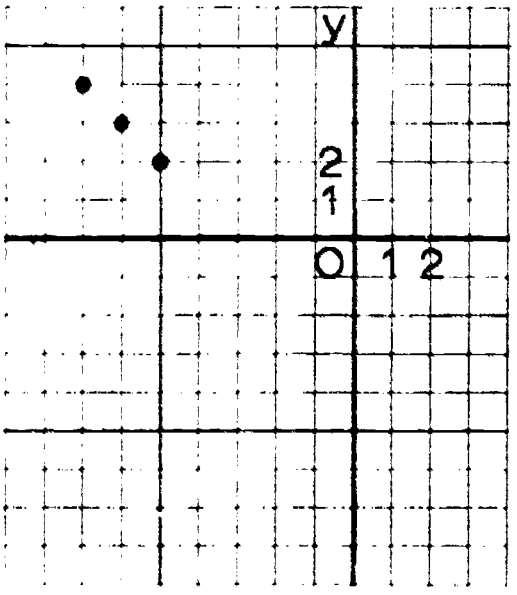
(a)



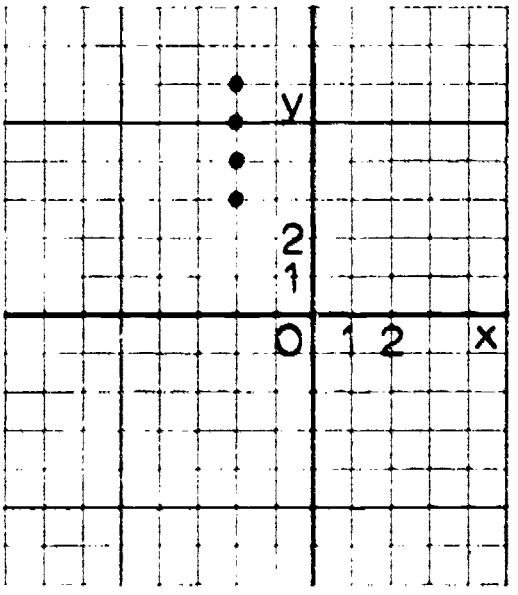
(b)



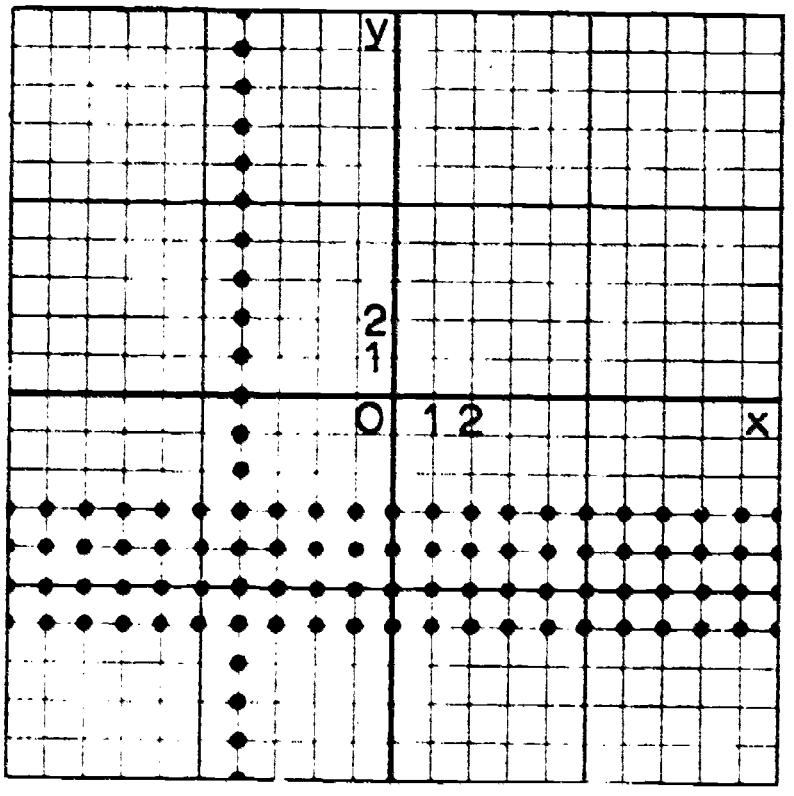
(c)



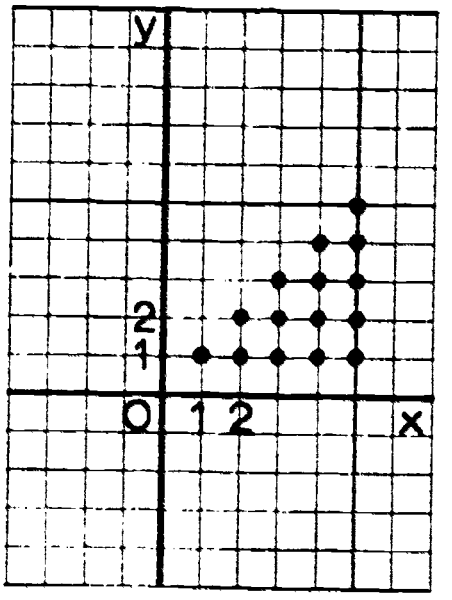
(d)



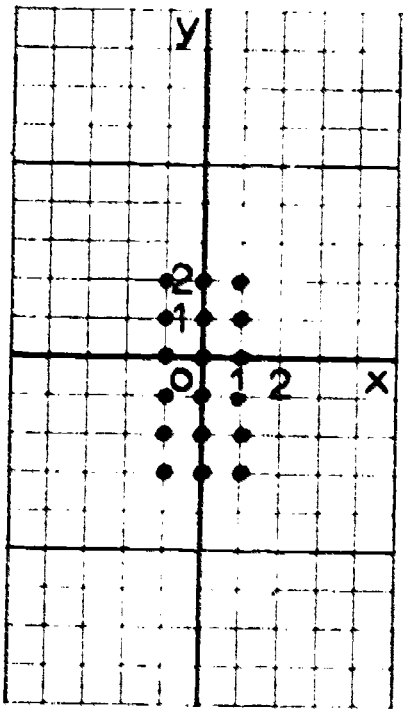
(e)



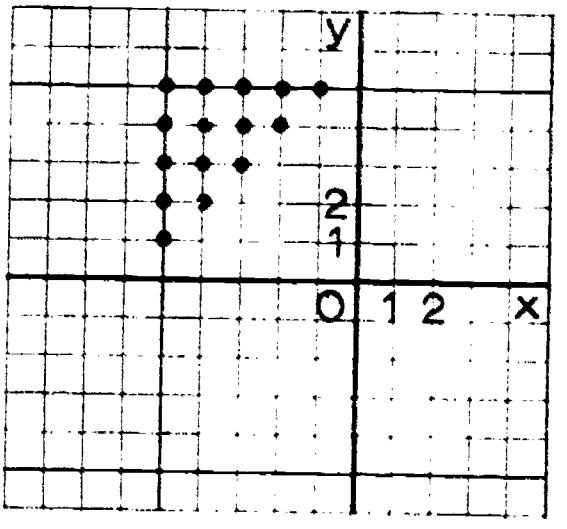
(g)



(f)



(h)



4-3. Systems of Inequalities

We define a system of equations as a compound open sentence in which two equations are joined by the connective "and". We also introduce a notation for this, "$\{$". Carrying the idea over to inequalities, let us consider systems like the following:

(a) $\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$ (b) $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$

(c) What would the graph of $x + 2y - 4 > 0$ be? You recall that we first draw the graph of

$$x + 2y - 4 = 0,$$

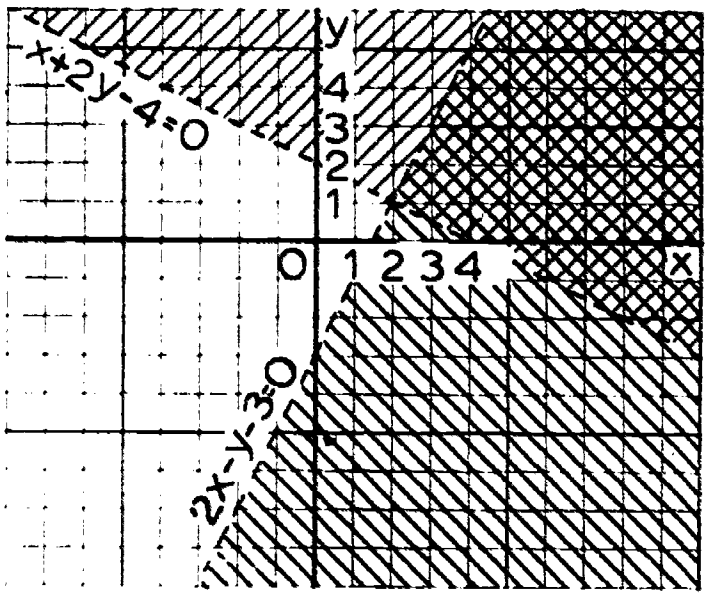


Figure 7.

using a dashed line along the boundary. Why? Then we shade the region above the line, since the graph of "$x + 2y - 4 > 0$", i.e., of "$y > -\frac{1}{2}x + 2$", consists of all those points whose ordinate is greater than "two more than $-\frac{1}{2}$ times the abscissa". In a similar way, we shade the region where "$y < 2x - 3$". This is the region below the line whose equation is "$2x-y-3=0$". Why is the line here also dashed? When would we use a solid line as boundary?

(c) (continued)

Since the truth set of a compound open sentence with the connective <u>and</u> is the set of elements common to the truth sets of the two clauses, it follows that the truth set of the system (a) is the region indicated by double shading in Figure 7.

(d) What would be the graph of a system in which we have one equation and one inequality, such as Example (b)? What is the graph of "3x - 2y - 5 = 0"?

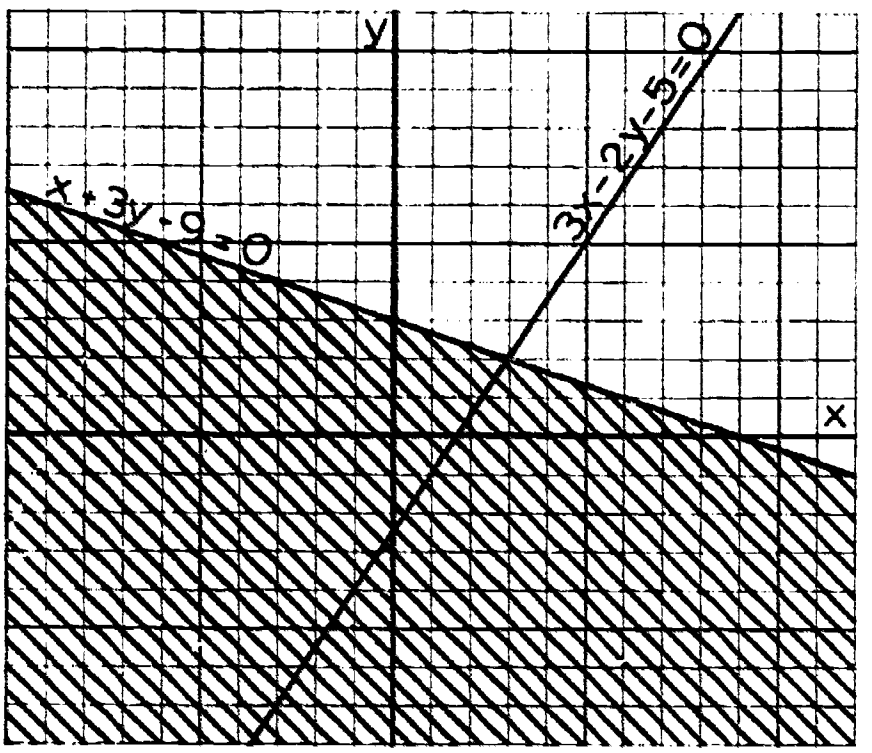


Figure 8.

Is the graph of "$x + 3y - 9 \leq 0$" the region above or below the line

$$x + 3y - 9 = 0?$$

Is the line included? Study Figure 8 carefully, and describe the graph of the system

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0. \end{cases}$$

Problem Set 4-3a

Draw graphs of the truth sets of the following systems:

1. $\begin{cases} 2x + y > 8 \\ 4x - 2y \leq 4 \end{cases}$

2. $\begin{cases} 6x + 3y < 0 \\ 4x - y < 6 \end{cases}$

3. $\begin{cases} 5x + 2y + 1 > 0 \\ 3x - y - 6 = 0 \end{cases}$

4. $\begin{cases} 4x + 2y = -1 \\ y - x \geq 4 \end{cases}$

5. $\begin{cases} 2x + y < 4 \\ 2x + y > 6 \end{cases}$

6. $\begin{cases} 2x + y > 4 \\ 2x + y < 6 \end{cases}$

7. $\begin{cases} 2x - y \leq 4 \\ 4x - 2y < 8 \end{cases}$

Let us consider the graph of the compound open sentence

$$x - y - 2 > 0 \quad \text{or} \quad x + y - 2 > 0.$$

First we draw the graphs of the clauses "$x - y - 2 > 0$" and "$x + y - 2 > 0$".

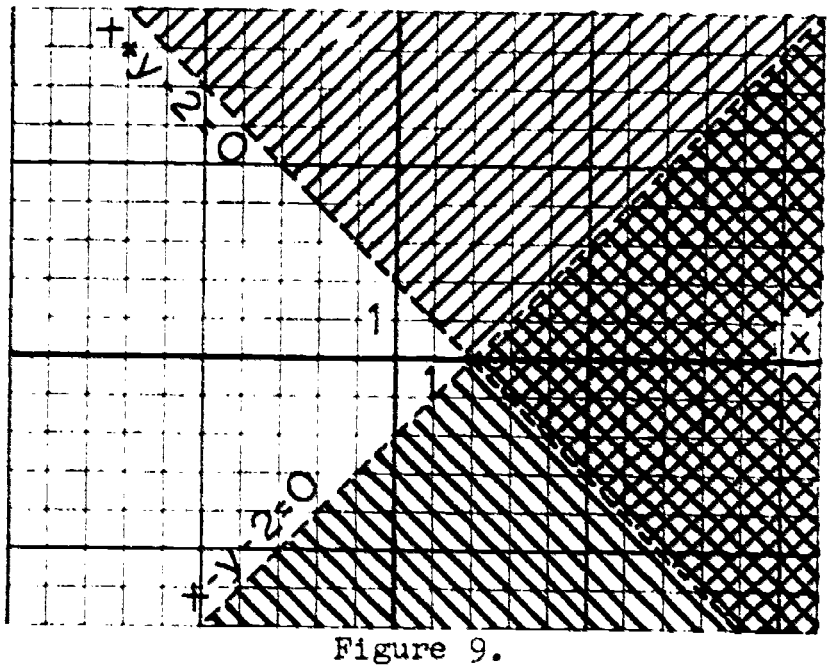


Figure 9.

Next we recall that the truth set of a compound open sentence with the connective <u>or</u> is the set of all elements in either of the truth sets of the clauses. Hence, the graph of the compound open sentence under consideration includes the entire shaded region in Figure 9.

Problem Set 4-3b

Draw the graphs of truth sets of the following sentences:

1. $2x + y + 3 > 0$ or $3x + y + 1 < 0$
2. $2x + y + 3 < 0$ or $3x - y + 1 < 0$
3. $2x + y + 3 \leq 0$ or $3x + y + 1 \geq 0$
4. $2x + y + 3 > 0$ and $3x - y + 1 < 0$

To complete the picture, let us consider the compound open sentence:

$$(x - y - 2)(x + y - 2) > 0.$$

Remember that "$ab > 0$" means that "the product of a and b is a positive number". What can be said of a and b if $ab > 0$? Thus, we have the two possibilities:

$$x - y - 2 > 0 \text{ and } x + y - 2 > 0,$$

<u>or</u>

$$x - y - 2 < 0 \text{ and } x + y - 2 < 0.$$

In Figure 3, the graph of "$x - y - 2 > 0$ and $x + y - 2 > 0$" is the region indicated by double shading, while the graph of "$x - y - 2 < 0$ and $x + y - 2 < 0$" is the unshaded region. So the graph of

$$(x - y - 2)(x + y - 2) > 0$$

consists of all the points in these two regions of the plane.

Which areas form the graph of the open sentence

$$(x - y - 2)(x + y - 2) < 0?$$

(If $ab < 0$, what can be said of a and b?)

To summarize, we list the following pairs of equivalent sentences (a and b are real numbers):

$ab = 0$: $a = 0$ or $b = 0$.
$ab > 0$: $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.
$ab < 0$: $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$.

Verify these equivalences by going back to the definition of the product of real numbers.

Problem Set 4-3c

1. Draw the graphs of the truth sets of the following open sentences.
   (a) $(2x - y - 2)(3x + y - 3) > 0$
   (b) $(x + 2y - 4)(2x - y - 3) < 0$
   (c) $(x + 2y - 6)(2x + 4y + 4) > 0$
   (d) $(x - y - 3)(3x - 3y - 9) < 0$

2. Draw the graphs of the truth sets of the following open sentences.
   (a) $x - 3y - 6 = 0$ and $3x + y + 2 = 0$
   (b) $(x - 3y - 6)(3x + y + 2) = 0$
   (c) $x - 3y - 6 > 0$ and $3x + y + 2 > 0$
   (d) $x - 3y - 6 < 0$ and $3x + y + 2 < 0$
   (e) $x - 3y - 6 > 0$ and $3x + y + 2 = 0$
   (f) $x - 3y - 6 < 0$ or $3x + y + 2 < 0$
   (g) $x - 3y - 6 = 0$ or $3x + y + 2 \geq 0$
   (h) $(x - 3y - 6)(3x + y + 2) > 0$
   (i) $(x - 3y - 6)(3x + y + 2) < 0$

3. Draw the graph of the truth set of each of these systems of inequalities. (The brace again indicates a compound sentence with connective <u>and</u>.)
   (a) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 3x + 4y \leq 12 \end{cases}$ (b) $\begin{cases} y \geq 2 \\ 4y \leq 3x + 8 \\ 4y + 5x \leq 40 \end{cases}$ (c) $\begin{cases} -4 < x < 4 \\ -3 < y < 3 \end{cases}$

4. A football team finds itself on its own 40 yard line, in posession of the ball, with five minutes left in the game. The score is 3 to 0 in favor of the opposing team. The quarterback knows the team should make 3 yards on each running play, but will use 30 seconds per play. He can make 20 yards on a successful pass play, which uses 15 seconds. However, he usually completes only one pass out of three. What combination of plays will assure a victory, or what should be the strategy of the quarterback?