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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include definitions, parallel and perpendicular lines, distance formula, midpoint formula, linear equations, and circles. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-11

SUPPLEMENTARY and ENRICHMENT SERIES

PLANE COORDINATE GEOMETRY

Teachers' Commentary

Edited by Thomas J. Hill

5H56

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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PLANE COORDINATE GEOMETRY

Commentary for Teachers

Sections 2 and 3 cover material that is familiar to most students, and classes should move on as quickly as possible. If students already know the terms "absicca" and "ordinate", there is no reason to object to their use of these words. The terms are superfluous, however, and need not be introduced by you.

Problem Set 1

1. "Cartesian" is used to honor the discoverer, Descartes.
2. (0,0).
3. -3.
4. The origin, or (0,0).
5. (2,1) and (2,0).
6. (a) IV. (b) II. (c) I. (d) III.
7. One of the coordinates must be 0.
8. D, B, C, A.
9. C, A, D, B.
10. (a) II. (b) I. (c) IV. (d) III. (e) IV. (f) I. (g) II. (h) III.

*12. (a) y-axis, x-axis, z-axis.
(b) xz-plane, yz-plane, xy-plane.
(c) 4, 2, 3.

When we define the slope of a line segment to be the quotient of the differences between pairs of coordinates, there is no need to introduce the notion of directed distance, but it is absolutely necessary to put the coordinates of the two points (x_1,y_1) and (x_2,y_2) in the proper position in the formula. That is, $m = \frac{y_2 - y_1}{x_2 - x_1}$ cannot be used as $m = \frac{y_2 - y_1}{x_1 - x_2}$ although $m = \frac{y_1 - y_2}{x_1 - x_2}$

is also correct. Notice that in finding the slope of $\overline{AB}$ it doesn't matter which point is labeled P_1 and which one is labeled P_2.

It is important to note that RP_2 and P_1R are positive numbers and we have to prefix the minus sign to the fraction $\frac{RP_2}{P_1R}$ if the slope is negative. However, the formula defining the slope of a segment will give the slope m as positive or negative <u>without</u> prefixing any minus sign.

For the Case (1) if $m > 0$, then $m = \frac{RP_2}{P_1R}$, $RP_2 = y_2 - y_1$ and $P_1R = x_2 - x_1$. For the Case (2) if $m < 0$, then $m = -\frac{RP_2}{P_1R}$, $RP_2 = y_2 - y_1$, and $P_1R = x_1 - x_2$. Therefore, Case (2) becomes $m = -\frac{y_2 - y_1}{x_1 - x_2}$ which is equivalent to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Problem Set 2

1. (a) 7 (b) -1 (c) y_1.

2. (a) 6 (b) -3 (c) x_1.

3. (a) 2 (b) 2 (c) 3

   (d) The two points in each part have the same y-coordinate.

   (e) If two points in a plane have the same y-coordinate, then the distance between them is the absolute value of the difference of their x-coordinates.

   (f) No.

4. (a) 3 (b) 2 (c) 4

   (d) $|y_1 - y_2|$ or $|y_2 - y_1|$.

   (e) The two points in each part have the same x-coordinate.

   (f) If two points in a plane have the same x-coordinate, the distance between them is the absolute value of the difference of their y-coordinates.

5. (2,3); (-1,-5); (3,-1).

6. PA = 2, QA = 2.
PB = 5, QB = 3.
PC = 7, QC = 3.

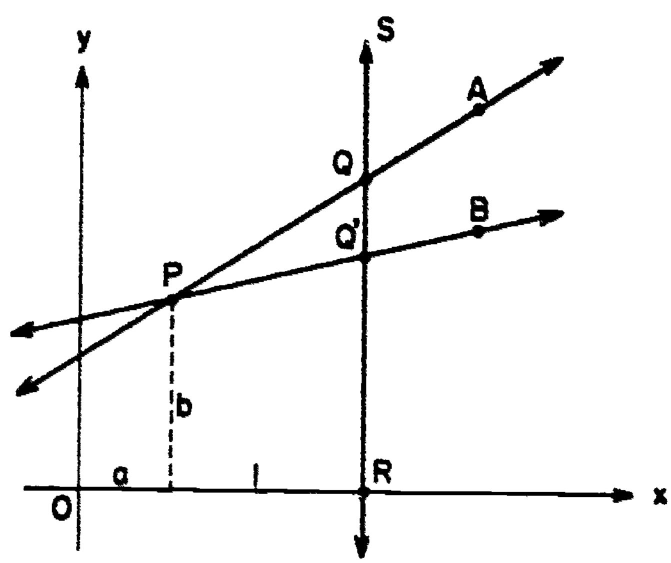
7. -1, $\frac{3}{5}$, $\frac{3}{7}$.

8. $\frac{1}{15}$.

9. (a) $\frac{1}{3}$ (e) $-\frac{15}{8}$
(b) -3 (f) $\frac{8}{15}$
(c) $\frac{7}{4}$ (g) -1
(d) $\frac{3}{4}$ (h) -3

10. (a) 6 (b) 4.5

*11. First assume that $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ have the same slope m. Let $P = (a,b)$, $R = (a + 1,0)$. Let $\overleftrightarrow{RS}$ be perpendicular to the x-axis. Neither $\overleftrightarrow{PA}$ nor $\overleftrightarrow{PB}$ is perpendicular to the x-axis; hence, neither $\overleftrightarrow{PA}$ nor $\overleftrightarrow{PB}$ is parallel to $\overleftrightarrow{RS}$. Let $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ intersect $\overleftrightarrow{RS}$ in Q, Q', respectively. Let $Q = (a + 1,c)$, $Q' = (a + 1,c')$. Then $\frac{c-b}{1} = m = \frac{c'-b}{1}$. Whence, $c = c'$ and, hence, $Q = Q'$. Hence, $\overleftrightarrow{PA} = \overleftrightarrow{PB}$. The converse has already been proved (Theorem 1). Hence, if $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ have different slopes, then P, A, B cannot be collinear.



12. (a) Yes (b) No

13. (a) -1 (b) $-\frac{3}{2}$ (c) $\frac{a - b}{2b}$

14. Slope of $\overleftrightarrow{AB}$ is $\frac{96}{96} = 1$. Slope of $\overleftrightarrow{BC}$ is $\frac{100}{100} = 1$. Point B is common. Therefore, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ coincide.

15. Slope of $\overleftrightarrow{AB}$ is $\frac{96}{96} = 1$; slope of $\overleftrightarrow{CD}$ is $\frac{1}{1} = 1$. We are tempted to say that $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$, but we must make sure that they are actually two different lines. We test by finding the slope of $\overleftrightarrow{AC}$, which is $\frac{101}{101} = 1$. Hence, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ must coincide so that C is on $\overleftrightarrow{AB}$ and the lines can't be parallel. It follows that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ coincide.

16. Draw the segment which joins (4,3) and the origin; any other segment through the origin lying on the line determined by this segment will also suffice.

The information concerning slopes of parallel and perpendicular lines constitutes a very important principle for the solving of geometric problems analytically. For instance, if a student were asked to show that two non-vertical lines were parallel, he would have to show that their slopes were equal; to show that a pair of oblique lines were perpendicular would require that he establish the slopes to be negative reciprocals of each other. Note that to show two segments parallel, it is not sufficient to show they have the same slope; it is necessary to show also that the segments are not collinear (see Problems 11 and 15 of Problem Set 2).

To show why $\triangle PQR \cong \triangle Q'PR'$ we first show that $\angle Q'PR'$ is complementary to $\angle QPR$. This follows from $m\angle Q'PR' + m\angle Q'PQ + m\angle QPR = 180$ and $m\angle Q'PQ = 90$. Therefore, $\angle Q'PR' \cong \angle PQR$ and $\angle PQ'R' \cong \angle QPR$. Since $PQ = PQ'$, the triangles are congruent by A.S.A.

In the converse we use S.A.S. to show $\triangle PQR \cong \triangle Q'PR'$. By construction, $R'P = RQ$ and $\angle R$ and $\angle R'$ are right angles. We get $R'Q' = PR$ as follows: $m = \frac{RQ}{PR}$ and $m = -\frac{R'Q'}{R'P}$. Then $m' = -\frac{1}{m}$ becomes $-\frac{R'Q'}{R'P} = -\frac{PR}{RQ}$, and since the denominators are equal we have $R'Q' = PR$.

Finally, we get $\angle Q'PQ$ a right angle by using the fact that $\angle Q'PR$ is an exterior angle of $\triangle PQ'R'$ and that $\angle QPR \cong \angle PQ'R'$.

Note that Theorem 2, and some theorems which follow, are stated after the proof rather than before. In this way, the full theorem seems to be a result of the discussion pertinent to the topic being considered.

Problem Set 3

1. Slope $\overline{AB} = \frac{3}{2}$; slope $\overline{CD} = \frac{3}{2}$; hence, $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ or $\overleftrightarrow{AB} = \overleftrightarrow{CD}$.
   Slope $\overline{AC} = \frac{-4}{5}$; hence, A, B, C are not collinear. (See Problems 11 and 15 of Problem Set 4.) Hence, $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$, so that $\overline{AB} \| \overline{CD}$.
   Similarly, prove $\overline{BC} \| \overline{AD}$.

2. Slope of $\overline{AB} = -\frac{2}{3}$; slope of $\overline{CD} = -\frac{2}{3}$.
   Slope of $\overline{BC} = -3$; slope of $\overline{DA} = -3$.
   Therefore, opposite sides are parallel and the quadrilateral is a parallelogram.

3. $L_1 \perp L_3$ and $L_2 \perp L_4$, by Theorem 3.

4. The second is a parallelogram, as can be shown from the slopes of $\overline{PQ}$, $\overline{RS}$, $\overline{QR}$, and $\overline{PS}$, which are, respectively, $\frac{2}{3}$, $\frac{2}{3}$, $-\frac{1}{5}$, $-\frac{1}{5}$. The first is not a parallelogram since the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$ are, respectively, 4, $\frac{1}{2}$, 5 and $\frac{3}{8}$.

5. (a) Slope of $\overline{AB} = -\frac{2}{7}$.
   Slope of $\overline{BC} = \frac{2}{9}$.
   Slope of $\overline{AC} = 0$.

   (b) Slope of altitude to $\overline{AB} = \frac{7}{2}$.
   Slope of altitude to $\overline{BC} = -\frac{9}{2}$.
   The altitude to $\overline{AC}$ has no slope; it is a vertical segment.

6. Both $\overline{AB}$ and $\overline{CD}$ have the same slope, -1; $\overline{AC}$ has slope 0. Therefore, $\overline{AB} \| \overline{CD}$, $\overline{AD}$ and $\overline{BC}$ have different slopes. Therefore, the figure is a trapezoid. Diagonal $\overline{AC}$ is horizontal since its slope is 0. Diagonal $\overline{BD}$ is vertical. A vertical and a horizontal line are perpendicular.

7. The slope in each case is the same, $-\frac{1}{3}$; the slope of line joining $(3n,0)$ to $(6n,0)$ is 0. Hence, the given lines are parallel.

8. The slope of the first line is $\frac{b}{a}$. The slope of the second is $-\frac{a}{b}$. Since the negative reciprocal of $\frac{b}{a}$ is $-\frac{a}{b}$, the lines are perpendicular.

*9. Application of the slope formula shows that the slope of $\overline{XY}$ is $\frac{b}{a}$, and that of $\overline{XZ}$ is $-\frac{a}{b}$. By Theorem 3, $\overline{XY} \perp \overline{XZ}$. Hence, $\angle X$ is a right angle.

10. $\angle PQR$ will be a right angle if $\overline{PQ} \perp \overline{QR}$. $\overline{PQ}$ will be perpendicular to $\overline{QR}$ if their slopes are negative reciprocals; that is, if

$$\frac{-6 - 2}{5 - 1} = -\frac{b - 5}{b + 6}$$

from which $b = -17$.

11. Slope $\overline{PQ} = \frac{-1}{a - 3}$; slope $\overline{RS} = \frac{-1}{b - 4}$; slope $\overline{QS} = 0$.
If $\overleftrightarrow{PQ}$ were the same as $\overleftrightarrow{RS}$, these three slopes would have to be equal; but neither of the first two can be zero for any value of a or b.

If $\overline{PQ} \parallel \overline{RS}$ then $\frac{-1}{a - 3} = \frac{-1}{b - 4}$, whence, $a = b - 1$.

Notice that it would be impossible for us to develop the distance formula without the Pythagorean Theorem, which in turn rests upon the theory of areas, parallels, and congruence.

It might be instructive with a good class to have them derive the distance formula with P_1 and P_2 in various positions in the plane. In working with the distance formula, it does not matter in which order we take P_1 and P_2 in as much as we will be squaring the difference between coordinates. The distance formula holds even when the segment $\overline{P_1P_2}$ is horizontal or vertical.

Problem Set 4

1. a and b. $AB = 1$, $AC = 3$, $AD = 4.5$, $BC = 4$, $BD = 3.5$, $CD = 7.5$.

2. (a) $|x_2 - x_1|$ or $\sqrt{(x_2 - x_1)^2}$.
   (b) $|y_2 - y_1|$ or $\sqrt{(y_2 - y_1)^2}$.

3. (a) 5 (e) 17
(b) 5 (f) $\sqrt{2}$
(c) 13 (g) 89
(d) 25 (h) $5\sqrt{5}$

4. (a) $(y_2 - y_1)^2 + (x_1 - x_2)^2$.

(b) $x^2 + y^2 = 25$.

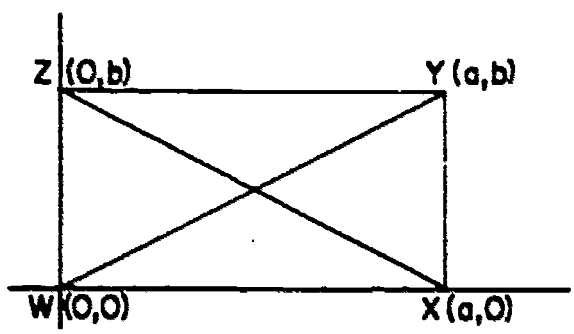
5. By the distance formula $RS = 5$, $RT = \sqrt{2}$ and $ST = 5$. Since $ST = RS$ the triangle is isosceles.

6. $\triangle DEF$ will be a right triangle with $\angle D$ a right angle only if $DE^2 + DF^2 = EF^2$. This is the case since $DE^2 = 5$, $DF^2 = 45$ and $EF^2 = 50$.

7. $AB = \sqrt{8} = 2\sqrt{2}$. $BC = \sqrt{72} = 6\sqrt{2}$. $AC = \sqrt{128} = 8\sqrt{2}$. Hence, $AB + BC = AC$, and therefore, from the Triangle Inequality, A, B, C are collinear. It now follows from the definition of "between" that B is between A and C.

8. (a) 7 (b) 5

9. (a) (a,b)
(b)



$WY = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$.

$XZ = \sqrt{(0 - a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$.

Hence, $WY = XZ$.

*10. (a) Let $A = (2,0,0)$, $B = (2,3,0)$. From the meaning of the x, y, and z-coordinates, $OA = 2$, $AB = 3$, and $BP = 6$. By the Pythagorean Theorem applied to $\triangle OAB$, $OB^2 = 13$, then applied to $\triangle OBP$, $OP^2 = 49$ and $OP = 7$. ($\overline{OP}$ may also be considered a diagonal of a rectangular block.)

10. (b) Generalizing the procedure in part (a), the distance is $\sqrt{x^2 + y^2 + z^2}$.

(c) $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

The midpoint formula will prove to be very useful in the work which follows. This will be true, for example, when we are speaking of the medians of a triangle. If we know the coordinates of the vertices of a triangle, and apply the definition of a median, we can find the coordinates of the point in which the median intersects the opposite side. This will give us the coordinates of its endpoints and enable us to find the length and slope of the median.

The proof of the midpoint formula is easily modified to hold for horizontal and vertical segments.

Problem Set 5

1. (a) (0,6)
   (b) (-2.5,0)
   (c) (2,0)
   (d) (0,0)
   (e) (0,0)

2. (a) (8,12)
   (b) (-5.5,-1.5)
   (c) $(\frac{5}{12}, \frac{13}{80})$
   (d) (1.58,1.11)
   (e) $(\frac{a + b}{2}, \frac{c}{2})$
   (f) $(\frac{s}{2}, \frac{r}{2})$

3. (a) (4,2)

   (b) $-9 = \frac{13 + x}{2}$, $30 = \frac{19 + y}{2}$,

   $x = -31$. $y = 41$.

   The other endpoint is at (-31,41).

4. $\overline{AC} \cong \overline{BD}$ since both have lengths $\sqrt{68}$ by the distance formula. $\overline{AC} \perp \overline{BD}$ since the slope of $\overline{AC}$ is 4 and the slope of $\overline{BD}$ is $-\frac{1}{4}$. These are negative reciprocals. $\overline{AC}$ and $\overline{BD}$ bisect each other since, using the midpoint formula, each has the midpoint (3,5).

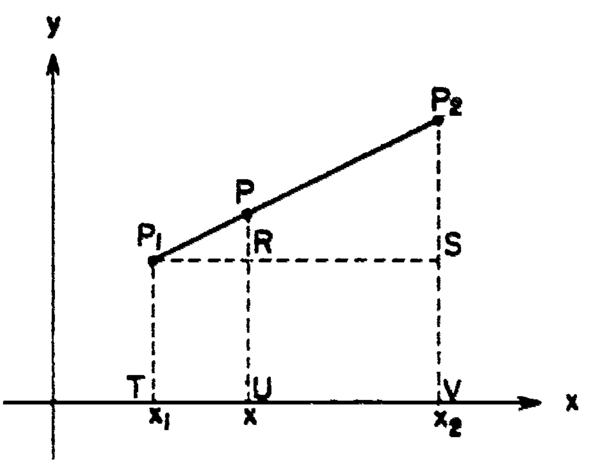
5. The midpoint X of $\overline{AB}$ is $(3,2)$.
The midpoint Y of $\overline{BC}$ is $(-1,3)$.
The midpoint Z of $\overline{CA}$ is $(1,0)$.

By the distance formula $CX = \sqrt{37}$, $AY = \sqrt{52}$ or $2\sqrt{13}$, and $BZ = 5$.

6. By formula, the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are $W(0,1)$, $X(-1,6)$, $Y(4,6)$ and $Z(5,1)$, respectively. $\overline{WX}$ has length $\sqrt{26}$ and slope -5. $\overline{YZ}$ also has length $\sqrt{26}$ and slope -5. $\overline{XY}$ has slope 0, hence $\overleftrightarrow{WX} \neq \overleftrightarrow{YZ}$, so that, $\overline{WX} \parallel \overline{YZ}$. With the same two sides parallel and congruent, the figure is a parallelogram.

7. By the midpoint formula the other endpoint of one median is $(\frac{a}{2},\frac{3a}{2})$, and the other end of another median is $(\frac{-a}{2},\frac{3a}{2})$. By the slope formula, the slopes of these medians are 1 and -1. Since 1 is the negative reciprocal of -1, the medians are perpendicular.

8. From the similarity between $\triangle P_1PR$ and $\triangle P_1P_2S$, $P_1R = \frac{1}{3}P_1S$. Since $TU = P_1R$ and $TV = P_1S$, $TU = \frac{1}{3}TV$. In terms of coordinates $x - x_1 = \frac{1}{3}(x_2 - x_1)$, or $x = \frac{1}{3}(x_2 - x_1) + x_1$.



This can also be written $x = \dfrac{x_2 + 2x_1}{3}$. By a similar argument with $\overline{P_1P_2}$ projected into the y-axis

$$y = \frac{y_2 + 2y_1}{3}.$$

Therefore, the coordinates of P are $\left(\dfrac{x_2 + 2x_1}{3}, \dfrac{y_2 + 2y_1}{3}\right)$.

*9. (a) Replacing $\frac{1}{3}$ by $\frac{r}{r+s}$ in the solution of the previous problem, if $x_2 > x_1$, we get

$$x = \frac{r}{r+s}(x_2 - x_1) + x_1,$$

from which, $x = \dfrac{r(x_2 - x_1) + x_1(r + s)}{r + s}$,

1r

*9. (continued)

$$\text{or} \qquad x = \frac{rx_2 + sx_1}{r + s}.$$

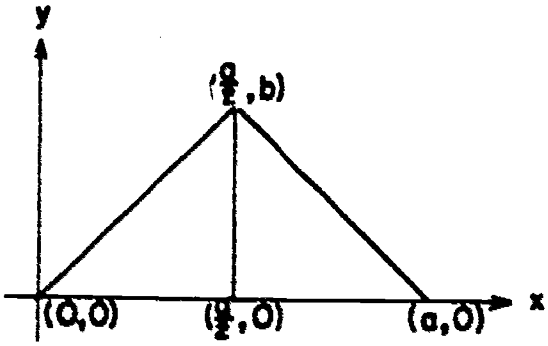
If $x_2 < x_1$, a similar argument leads to the same result. By a similar argument, with $\overline{P_1P_2}$ projected into the y-axis,

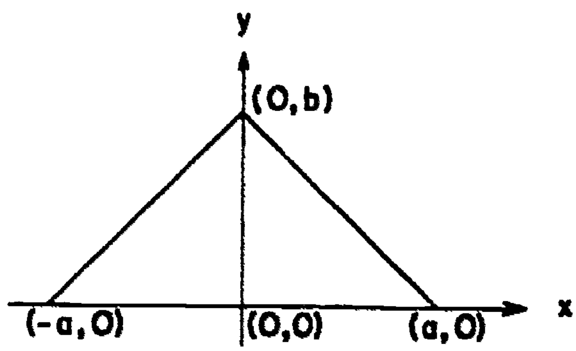
$$y = \frac{ry_2 + sy_1}{r + s}.$$

(b) $x = \dfrac{3 \cdot 25 + 5 \cdot 5}{3 + 5} = 12.5;$

$y = \dfrac{3 \cdot 36 + 5 \cdot 11}{8} = 20.375.$

Although we may place our axes in any manner we desire in relation to a figure, there are advantages to be had by a clever choice. For instance, if we are given an isosceles triangle, we may place the axes wherever we wish, then use the properties of an isosceles triangle to determine the coordinates of the vertices. Suppose we place it like this:

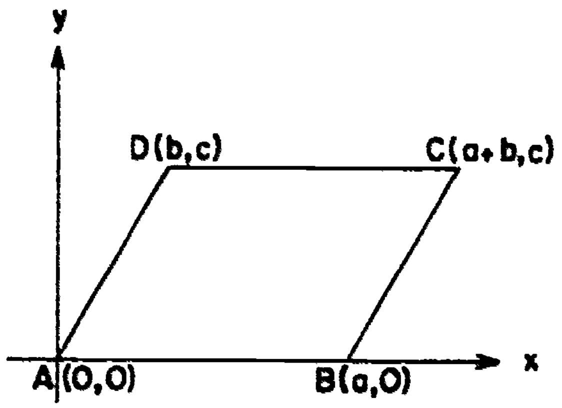


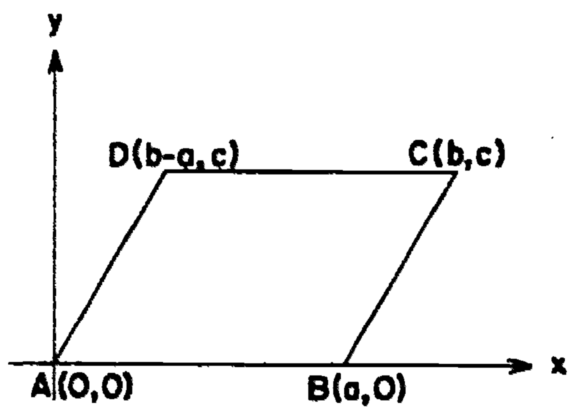


The student should be permitted to draw upon his knowledge of synthetic geometry and make use of the fact that the altitude to the base of an isosceles triangle bisects the base. Hence, the x-coordinate of the vertex should be half the x-coordinate of the endpoint of the base that is not at the origin. On the other hand the y-coordinate of the vertex is not determined by the coordinates of the other vertices and is an arbitrary positive number. Suppose we place the axes like this with the vertex on the y-axis:

Then, since the altitude bisects the base, the lengths of the segments into which it divides the base are equal, and therefore the endpoints of the base may be indicated by $(a,0)$ and $(-a,0)$.

There also are limits to what we can choose for coordinates. For parallelograms we find that three vertices may be labeled arbitrarily, but the coordinates of the fourth vertex are determined by those of the other three. Naturally there is more than one way in which we may label a parallelogram. Below in the figure on the left the coordinates of points A, B, and D were assigned first. Then the coordinates of C were determined in terms of the coordinates of the other three points. In the figure on the right A, B and C were chosen first. Notice how the coordinates of D are given in terms of the other coordinates.





One word of CAUTION. The above discussion is based upon the fact that such things as isosceles triangles or parallelograms are given in the problem. If the problem is to prove that a quadrilateral is a parallelogram or that a triangle is isosceles, then we cannot assume such properties to be true, and must establish, as part of the exercise, sufficient properties to characterize the figure.

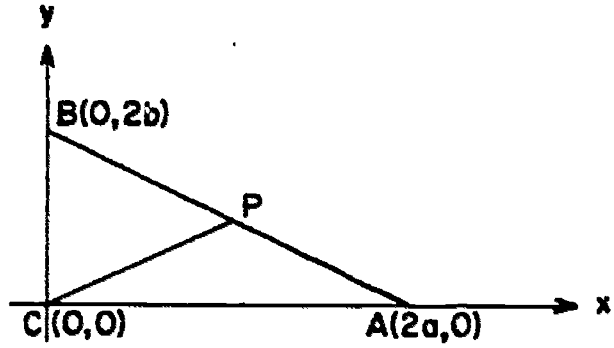
Problem Set 6

1. $DB = \sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}.$

$AC = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$

Therefore, $DB = AC$.

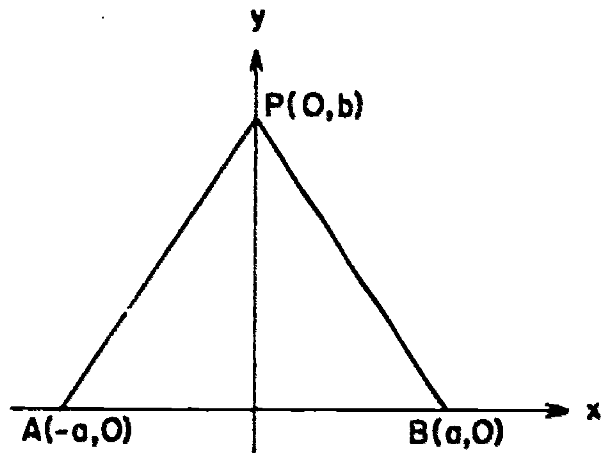
2. Locate the axes along the legs of the triangle as shown. By definition of midpoint PA = PB. Therefore, P = (a,b). It must be shown that PA = PC (or that PB = PC). By the distance formula



$$PA = \sqrt{(2a - a)^2 + (0 - b)^2} = \sqrt{a^2 + b^2} \text{ and}$$

$$PC = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

3. Let the x-axis contain the segment and the y-axis contain its midpoint. Then the y-axis is the perpendicular bisector of the segment. Let P(0,b) be any point of the y-axis, and A(-a,0) and B(a,0) be the endpoints of the segment. Then:

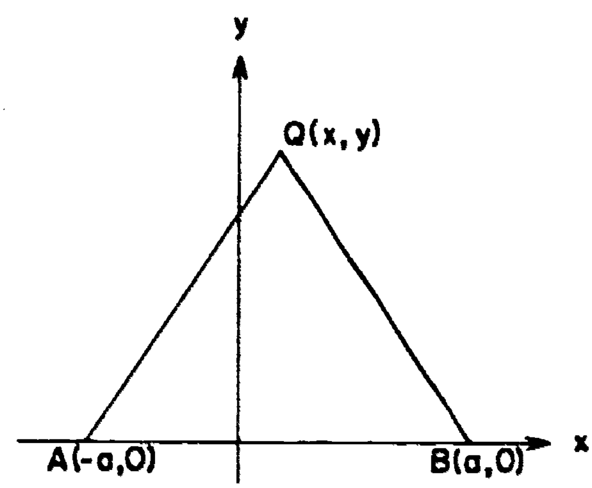


$$PA = \sqrt{(0 + a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

$$PB = \sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}.$$

Hence, PA = PB.

4. Place the axes so that the segment will have endpoints A(-a,0) and B(a,0), and the y-axis will be its perpendicular bisector. Let Q(x,y) be any point equidistant from A and B. From the distance formula



$QA^2 = (x + a)^2 + y^2$ and $QB^2 = (x - a)^2 + y^2$.

16

Since $QA = QB$, $QA^2 = QB^2$ or

$$(x + a)^2 + y^2 = (x - a)^2 + y^2.$$

Simplifying, $4ax = 0$.

$$x = 0, \quad \text{since} \quad a \neq 0.$$

Hence, Q must lie on the y-axis, which is the perpendicular bisector of $\overline{AB}$.

5. The midpoint of $\overline{AC} = \left(\frac{a + b}{2}, \frac{c + 0}{2}\right) = \left(\frac{a + b}{2}, \frac{c}{2}\right)$.

   The midpoint of $\overline{BD} = \left(\frac{a + b}{2}, \frac{0 + c}{2}\right) = \left(\frac{a + b}{2}, \frac{c}{2}\right)$.

   Since the diagonals have the same midpoints, they bisect each other.

6. $R = \left(\frac{d}{2}, \frac{c}{2}\right)$, $S = \left(\frac{b + a}{2}, \frac{c}{2}\right)$.

   Since R and S have the same y-coordinates, $\overleftrightarrow{RS} \parallel \overleftrightarrow{AB}$.
   Since $\overline{RS}$ is horizontal,

$$RS = \frac{b + a}{2} - \frac{d}{2} = \frac{b + a - d}{2}.$$

$$DC = d - b \quad \text{and} \quad AB = a.$$

   Therefore, $\frac{1}{2}(AB - DC) = \frac{a - (d - b)}{2} = \frac{b + a - d}{2}$.
   Hence, $RS = \frac{1}{2}(AB - DC)$, which was to be proved.

7. $R = (2a, 0)$, $S = (2a + 2d, 2e)$.
   $T = (2b + 2d, 2c + 2e)$, $W = (2b, 2c)$.
   Midpoint of $\overline{WS} = (a + d + b,\ e + c)$.
   Midpoint of $\overline{TR} = (a + b + d, c + e)$.
   Therefore, $\overline{WS}$ and $\overline{TR}$ bisect each other.

8. Area $\triangle ABC$ = area (XYBA) + area (YZCB) - area (XZCA).
   Area $\triangle ABC = \frac{1}{2}(s + r)(b - a) + \frac{1}{2}(t + s)(c - b) - \frac{1}{2}(r + t)(c - a)$.
   Multiplying out and combining terms,
   area $\triangle ABC = \frac{1}{2}(rb - sa + sc - tb + ta - rc)$, or
   area $\triangle ABC = \frac{a(t - s) + b(r - t) + c(s - r)}{2}$

9. $ZY^2 = (b - a)^2 + c^2$. $\quad XZ^2 = b^2 + c^2$.

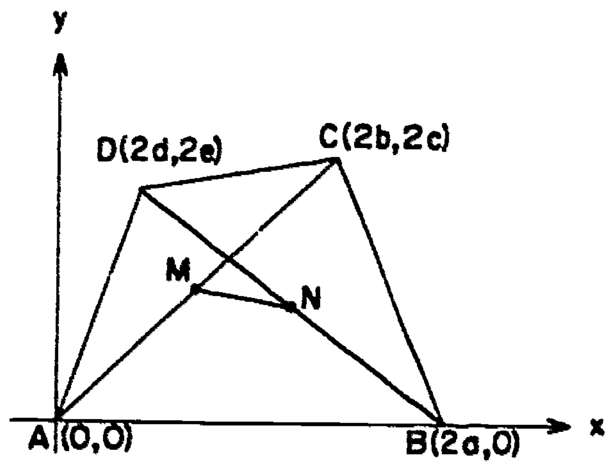
$XY = a$. $\quad XR = b$.

Since $(b - a)^2 + c^2 = (b^2 + c^2) + a^2 - 2ab$,

therefore, $ZY^2 = XZ^2 + XY^2 - 2XY \cdot XR$.

Observe that this proof remains valid if R lies between X and Y.

10. Select a coordinate system as indicated.



$M = (b,c)$, $\quad N = (a+d,e)$.

$AB^2 = 4a^2$.

$BC^2 = 4(a-b)^2 + 4c^2$.

$CD^2 = 4(b-d)^2 + 4(c-e)^2$.

$DA^2 = 4d^2 + 4e^2$.

$AC^2 = 4b^2 + 4c^2$.

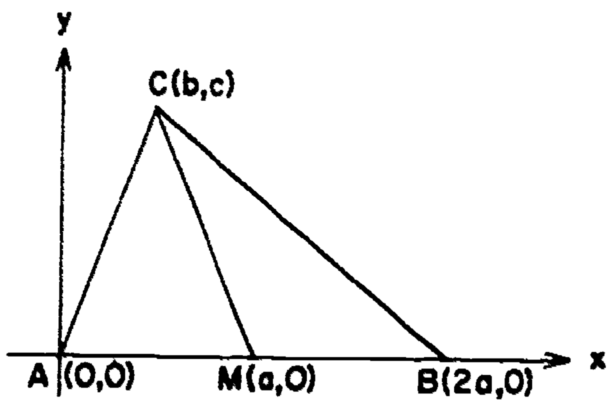
$BD^2 = 4(a-d)^2 + 4e^2$.

$MN^2 = (a+d-b)^2 + (e-c)^2$.

From these expressions the given equation can be verified. Note that

$$(a + d - b)^2 = a^2 + d^2 + b^2 + 2ad - 2ab - 2bd.$$

11. Place the axes and label the vertices as shown.



$AC^2 = b^2 + c^2$.

$BC^2 = (2a - b)^2 + c^2$.

$\frac{AB^2}{2} = 2a^2$.

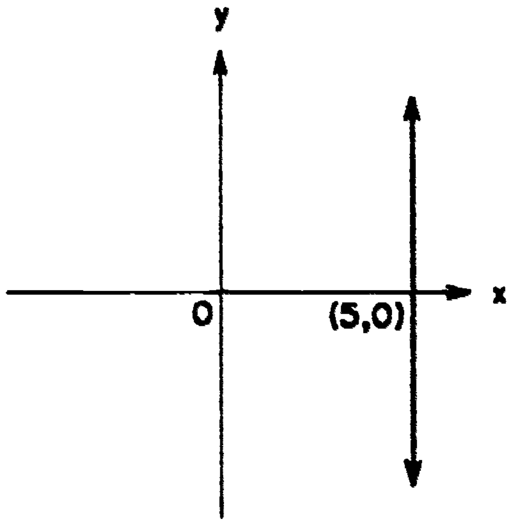
$MC^2 = (a - b)^2 + c^2$.

Since

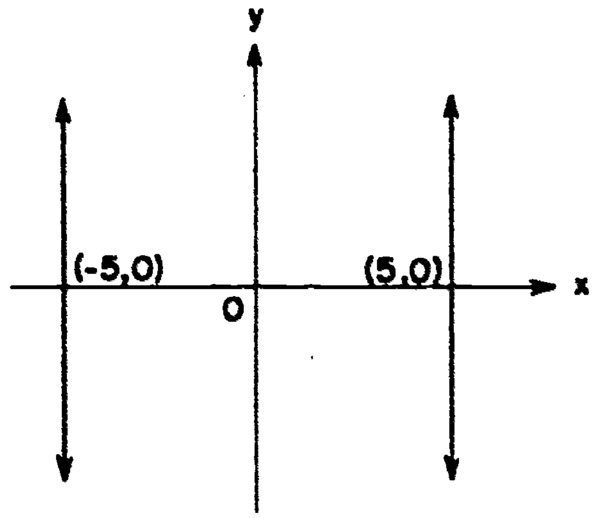
$$(b^2+c^2)+(4a^2 - 4ab + b^2 + c^2) = 2a^2 + 2(a^2 - 2ab + b^2 + c^2),$$
$$= 2a^2 + 2[(a - b)^2 + c^2].$$

Therefore, $AC^2 + BC^2 = \frac{AB^2}{2} + 2MC^2$.

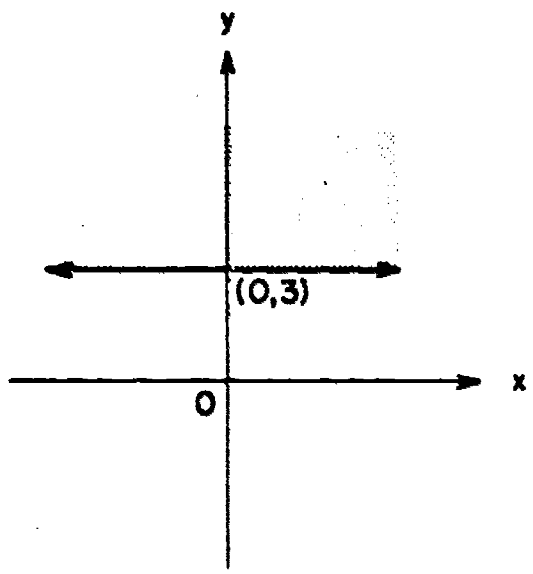
Problem Set 7



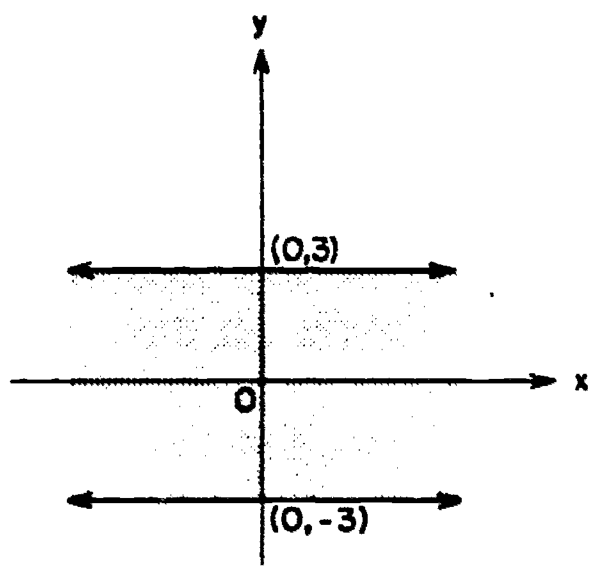
1a. The vertical line through (5,0).



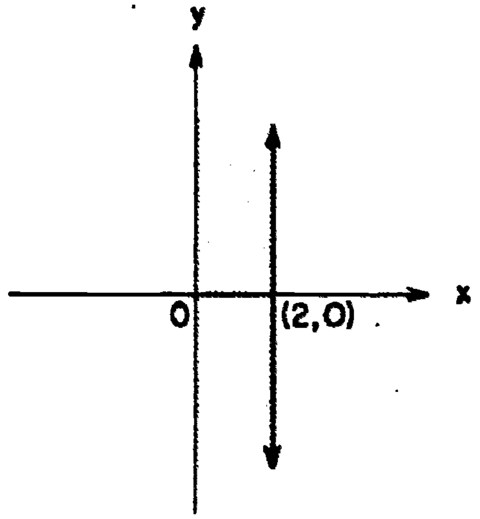
1b. The two vertical lines through (5,0) and (-5,0).

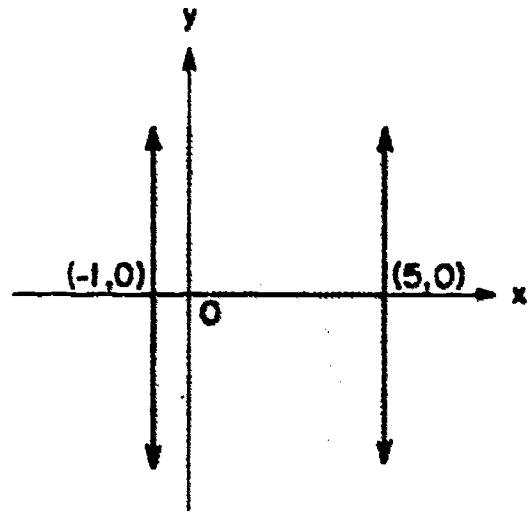


2a. The half-plane above the horizontal line through (0,3).



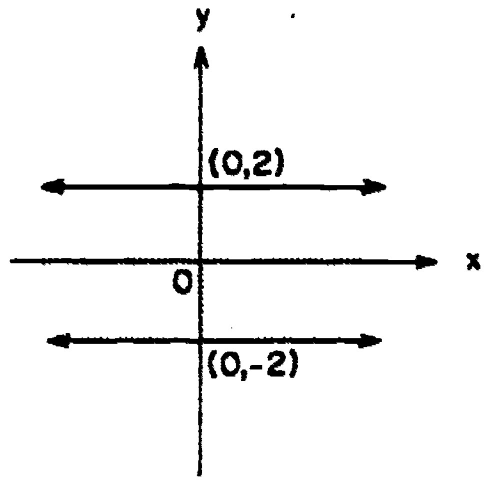
2b. All points between the lines $y = 3$ and $y = -3$.

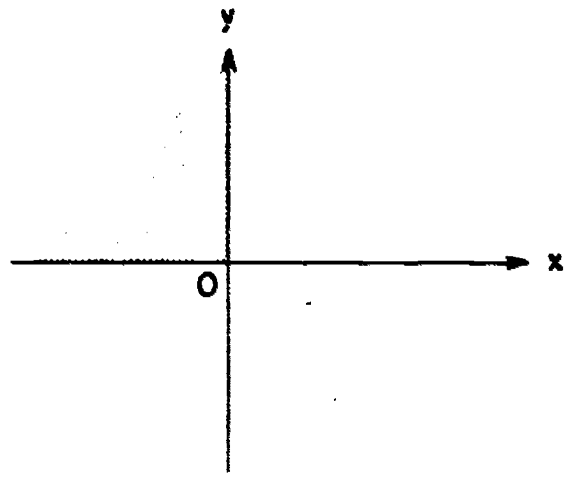




3. All points between the y-axis and the line $x = 2$.

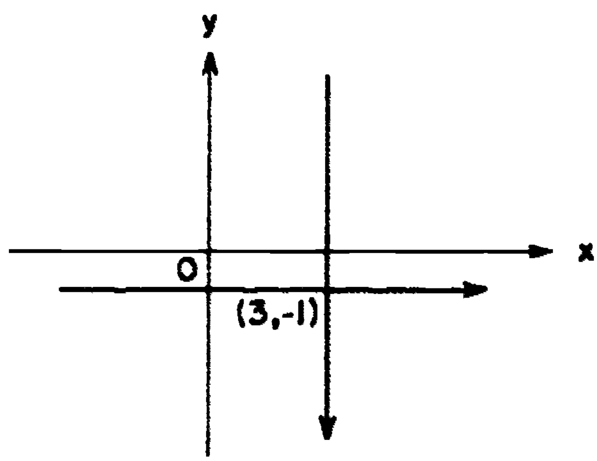
4. All points within or on the boundary of the indicated strip.



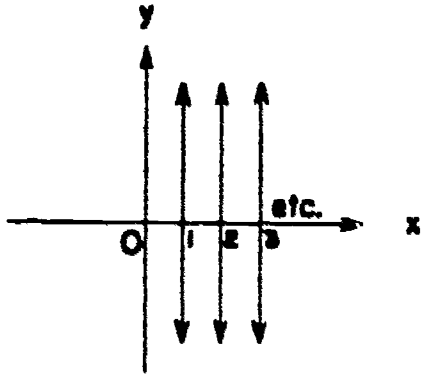


5. All points within, or on the lower boundary of, the indicated strip.

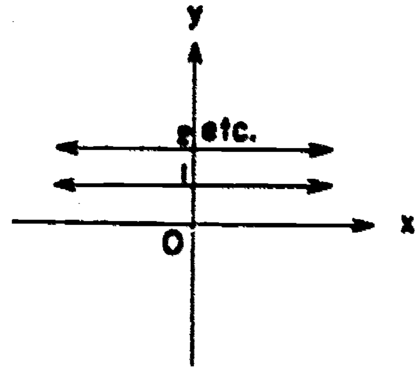
6. All points within the second quadrant.



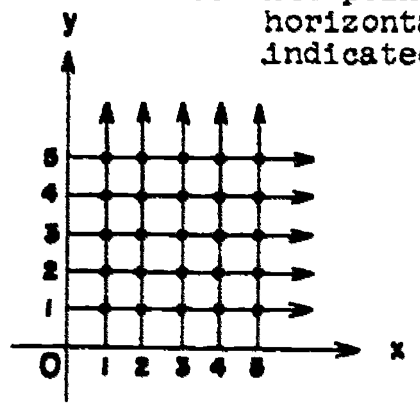
7. All points within indicated angle.



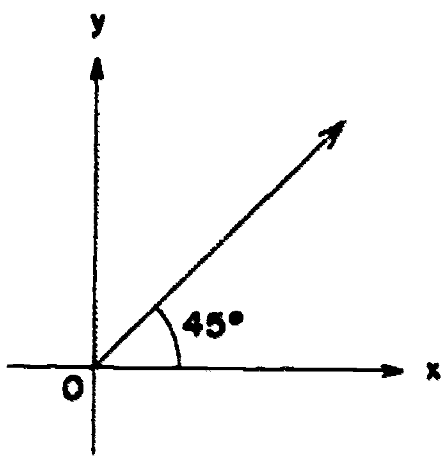
8a. All points on the vertical lines indicated.



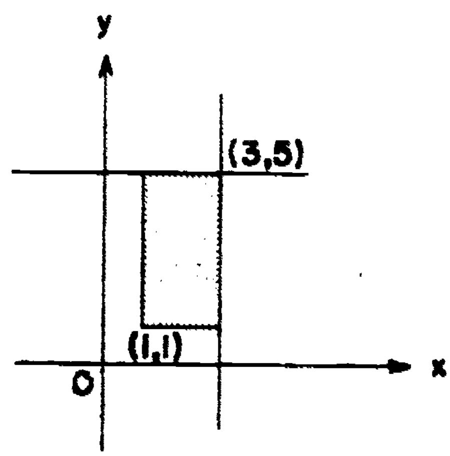
8b. All points on the horizontal lines indicated.



8c. The intersection of the solutions for (8a) and (8b). i.e., all points in the first quadrant with integral coordinates.

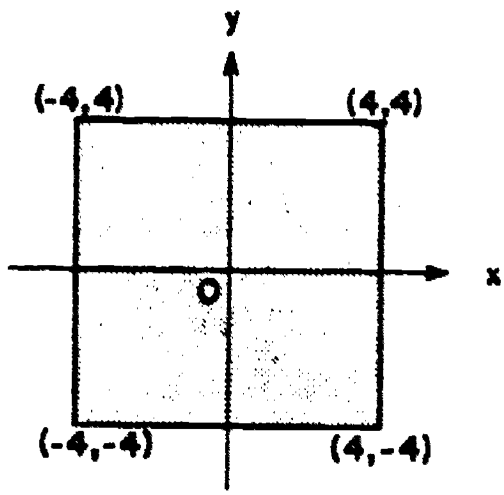


9. The intersection of the three half-planes formed by the three given conditions. i.e., all points within the angle formed by the positive part of the y-axis and the ray from the origin as shown.



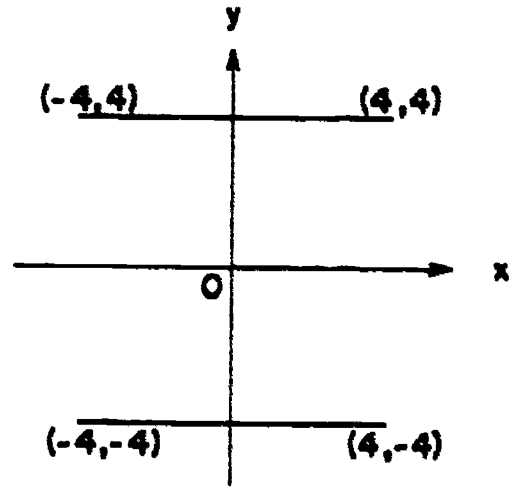
10. All points within or on the boundary of the indicated rectangle.

*11.



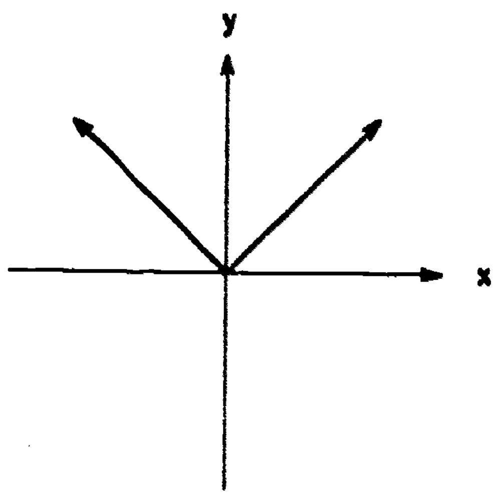
All points in the interior of the square with vertices (4,4), (-4,4), (-4,-4) (4,-4).

*12.



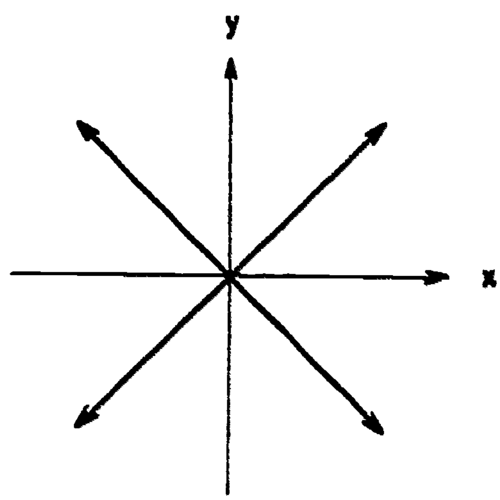
All points except the endpoints on the two segments joining (-4,4) and (4,4), and (-4,-4) and (4,-4).

*13.



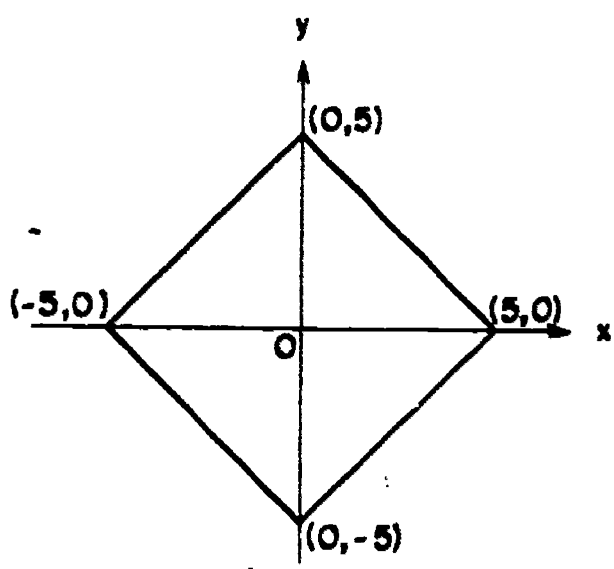
The rays bisecting the angles formed by the x and y-axes in first and second quadrants.

*14.



Lines bisecting the angles formed by the x and y-axes.

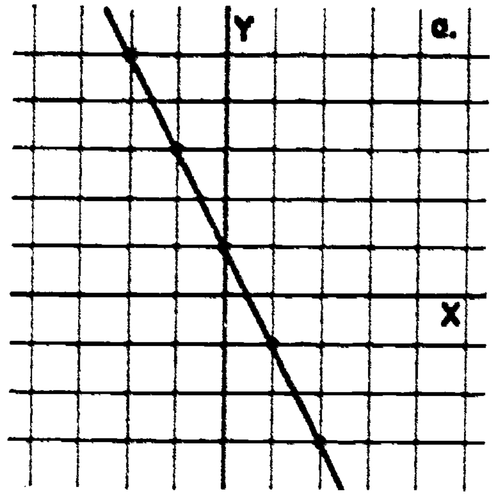
*15. The square with vertices (5,0), (0,5), (-5,0) and (0,-5).



16. Sample number pairs,

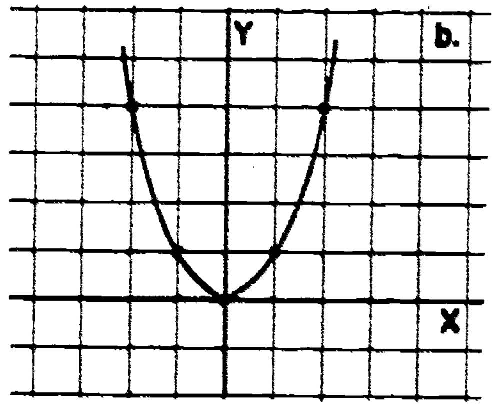
(a)

x	-2	-1	0	1	2
y	5	3	1	-1	-3



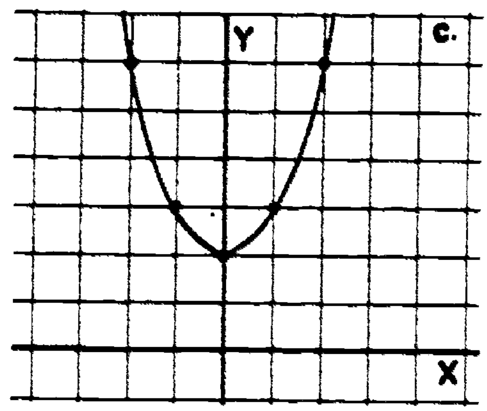
(b)

x	-2	-1	0	1	2
y	4	1	0	1	4



(c)

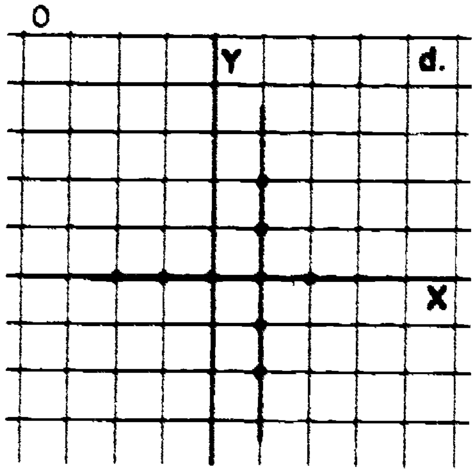
x	-2	-1	0	1	2
y	6	3	2	3	6



(d)

x	-2	-1	0	1	2
y	0	0	0	0, -1, -2, 1, 2...	0

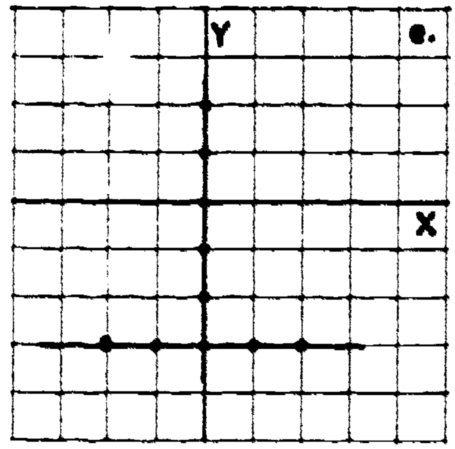
$\{(x,y): \ x - 1 = 0 \ \text{or} \ y = 0\}$



(e)

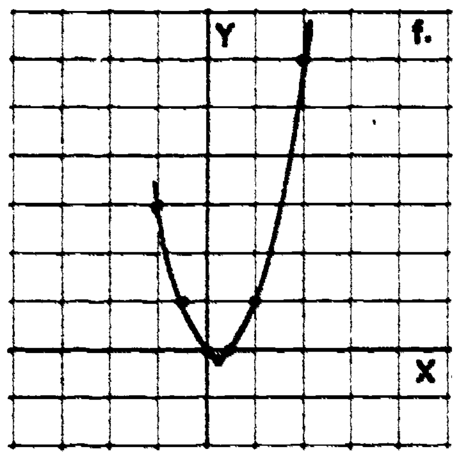
x	-2	-1	0	1	2
y	-3	-3	-3, -2, -1, 0, 1, 2, ...	-3	-3

$\{(x,y) : x = 0 \text{ or } y = -3\}$



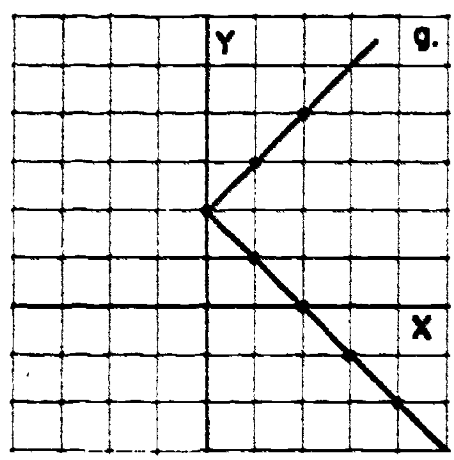
(f)

x	-1	-.5	0	.25	.5	1	2
y	3	1	0	-.125	0	1	6



(g)

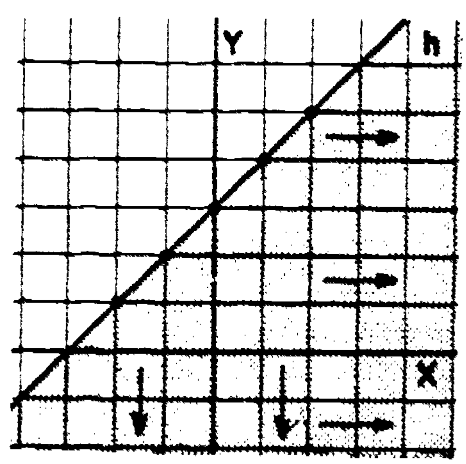
x	4	3	2	1	0	1	2
y	-2	-1	0	1	2	3	4



(h) Some number pairs for $y = x + 3$ are:

x	-2	-1	0	1	2
y	1	2	3	4	5

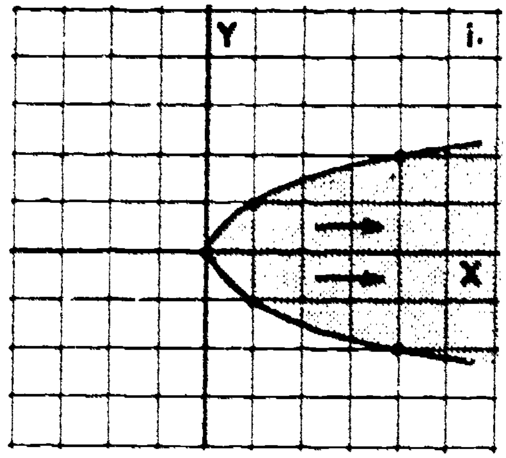
The graph is shown by the line <u>and</u> the region.



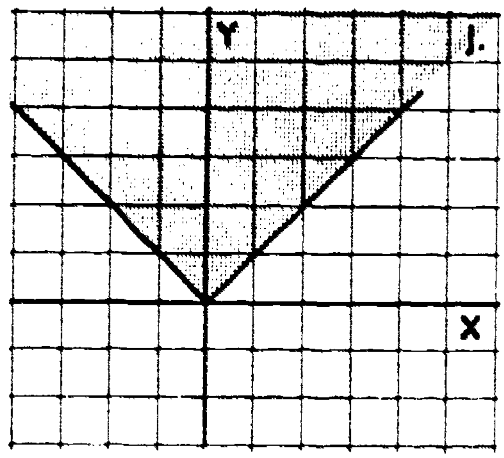
(i) Some number pairs for $x = y^2$ are,

x	4	1	0	1	4
y	-2	-1	0	1	2

The graph is $\{(x,y): x > y^2\}$, the shaded region.

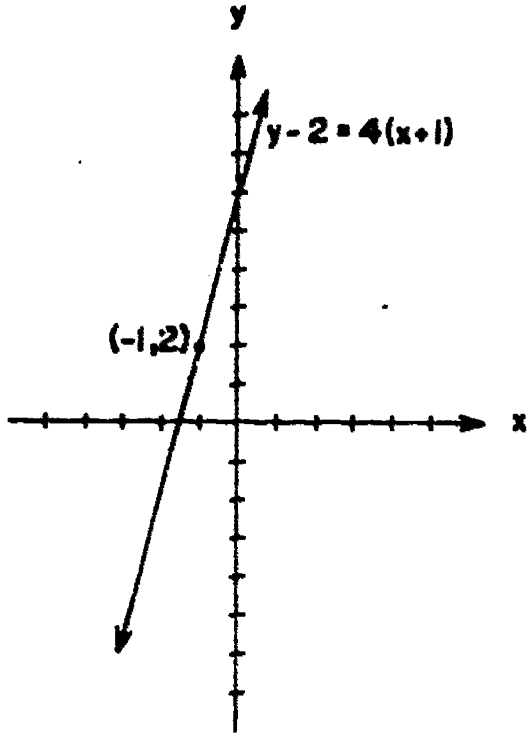


(j) The shaded region is the graph of $\{(x,y): y > |x|\}$.

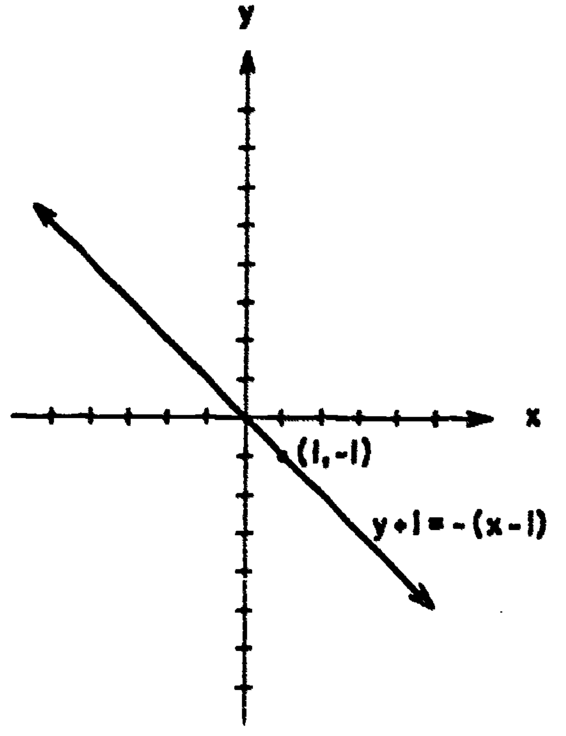


Problem Set 8

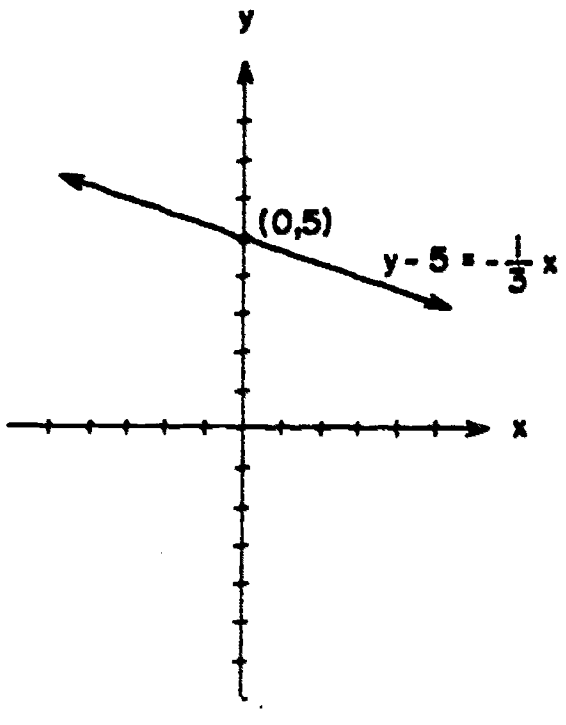
1.



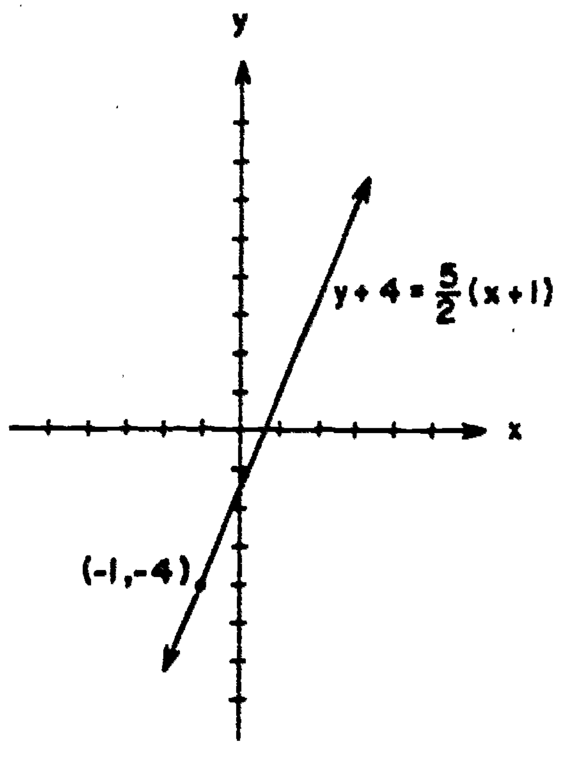
2.



3.

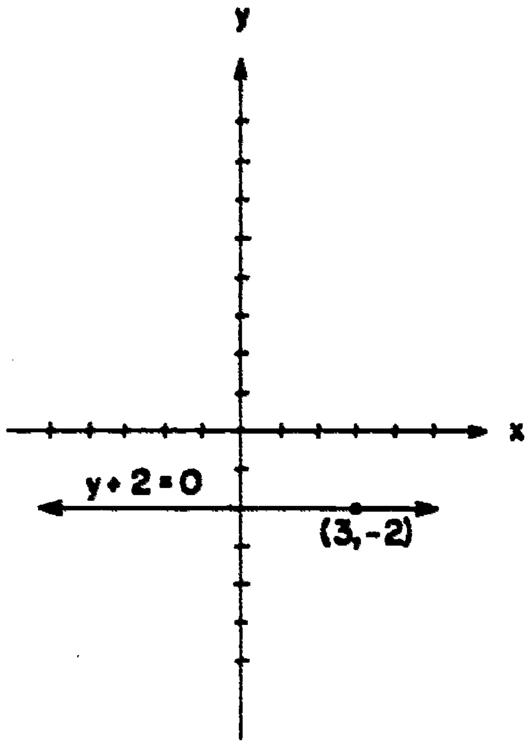


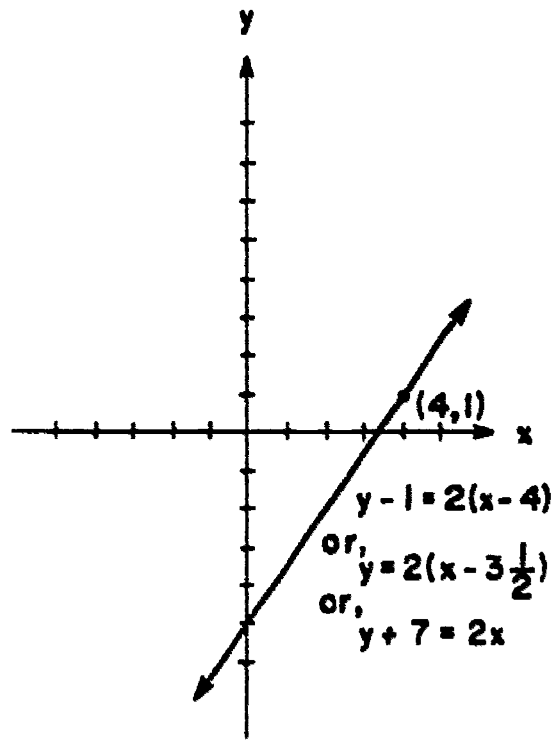
4.



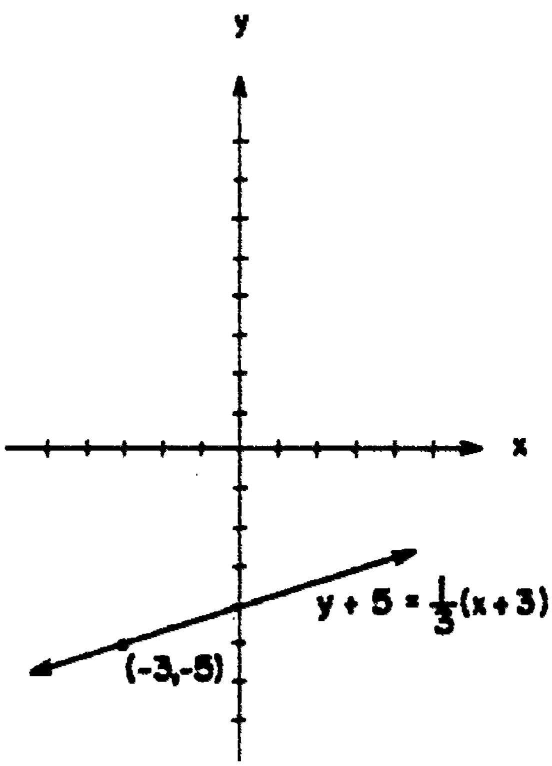
2C

5., 6., 7., and 8.

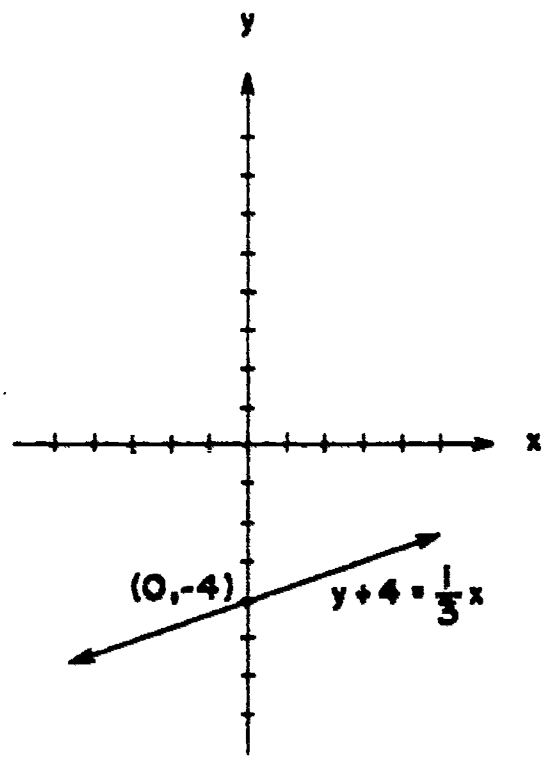




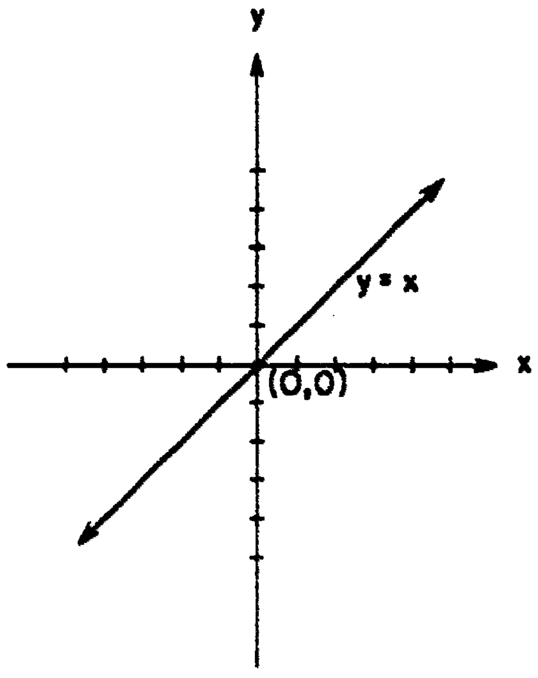
9.



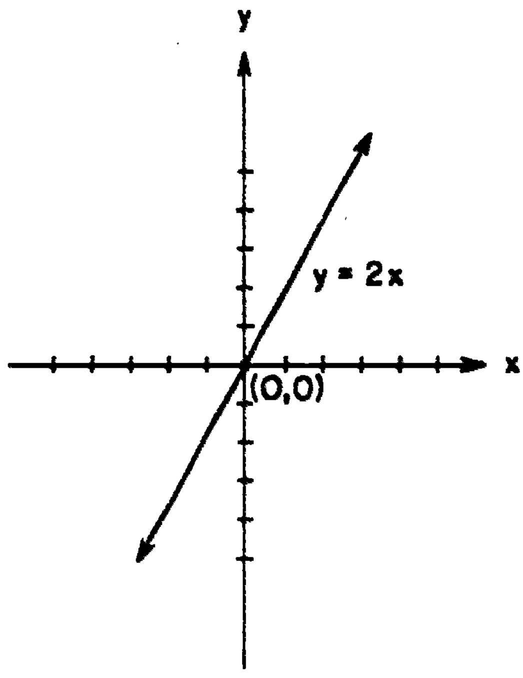
10.



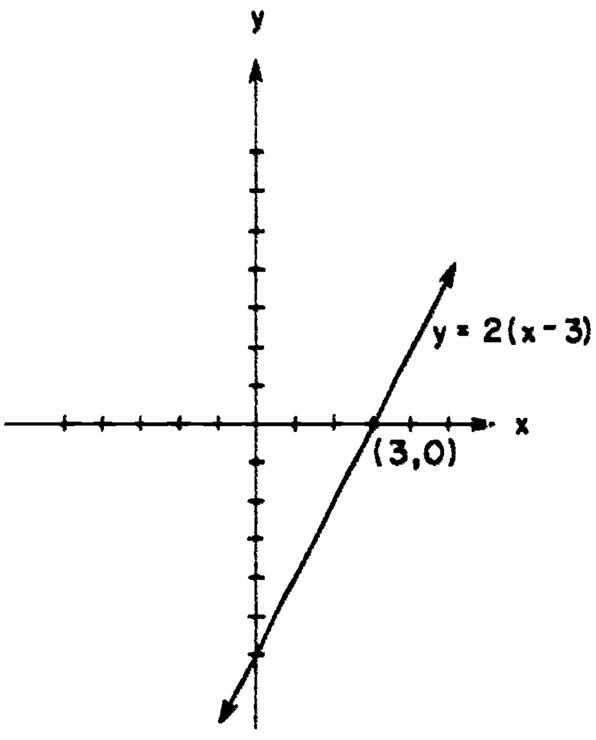
11.



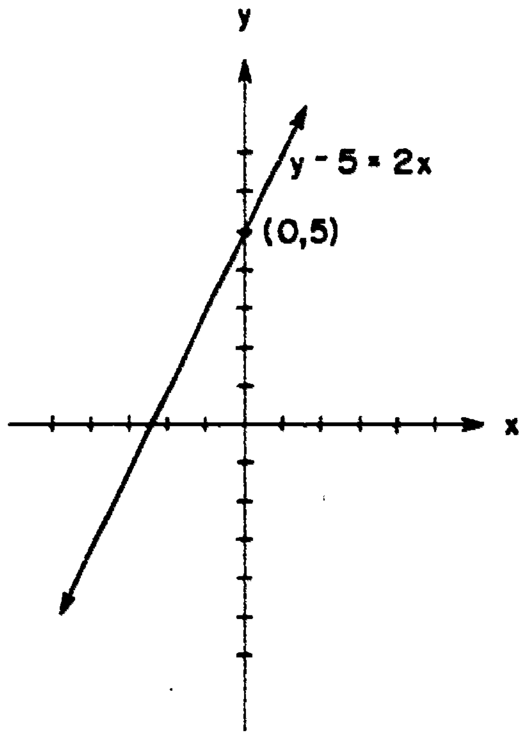
12.



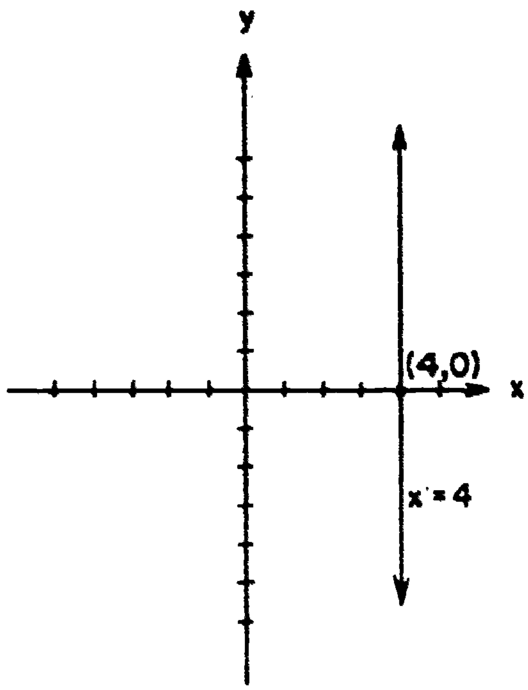
13.



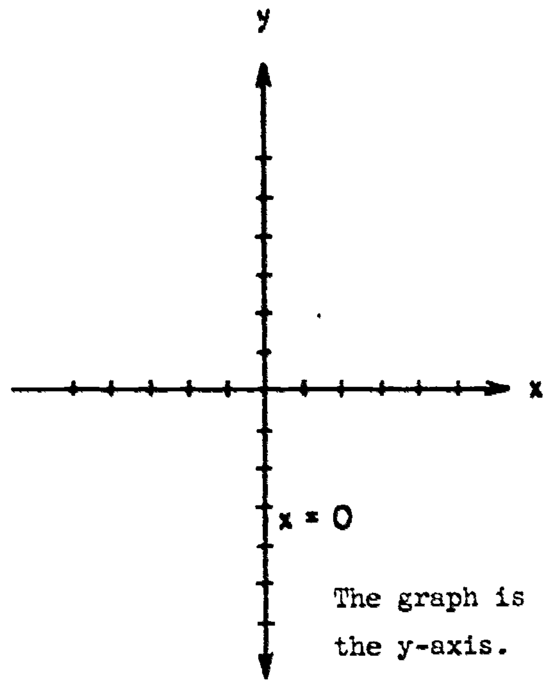
14.



15.

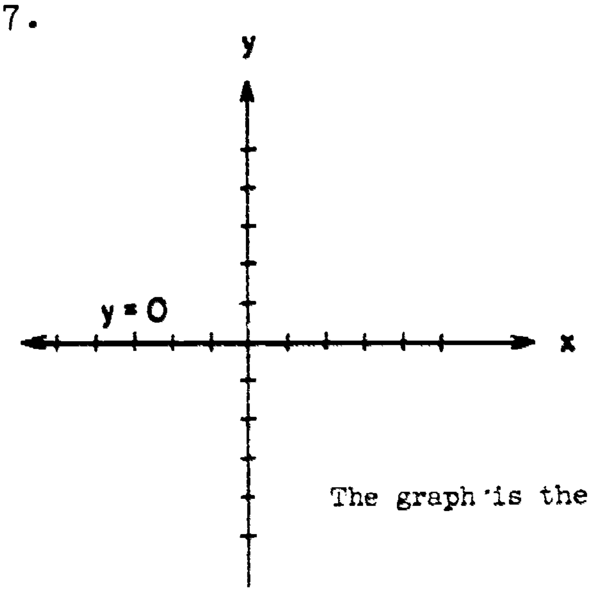


16.



The graph is the y-axis.

17.



The graph is the x-axis.

18. (a) The yz-plane.

(b) The xy-plane.

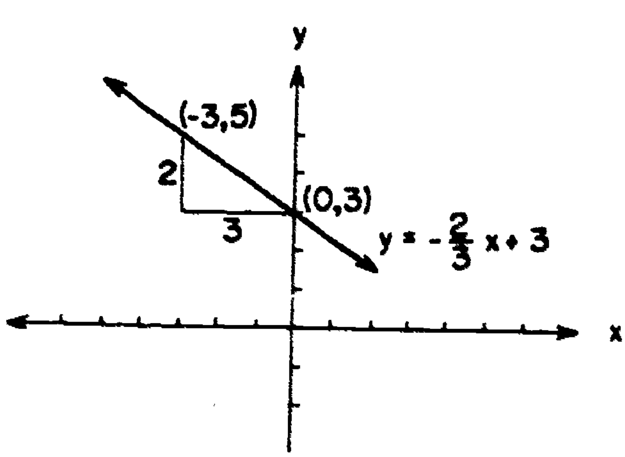
(c) A plane parallel to the yz-plane, intersecting the x-axis at $x = 1$.

(d) A plane parallel to the xz-plane, intersecting the y-axis at $y = 2$.

The material in Section 11 may have been previously covered in a first-year algebra course. If this is the case, do not spend any more time than is necessary on this section.

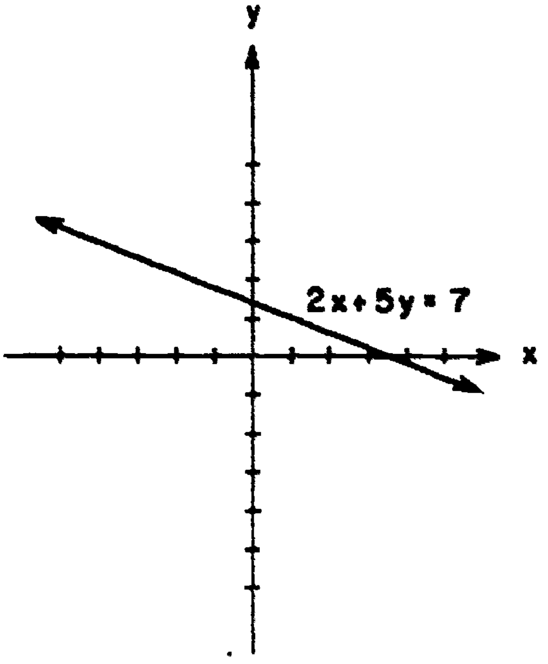
You will note that in the discussion on the first page of this section, it is necessary for us to find an additional point in order to plot the graph of the equation. We may do this in two ways. The first would be to assign to x a value, substitute this value in the given equation and compute the corresponding value of y (or we could assign a value to y and compute x). The second method depends upon the discussion here in the text. For we know how a line with a positive or negative slope will lie, and we also know that if a line has a positive slope then $m = -\frac{RP_2}{P_1R}$ and if its slope is negative, $m = -\frac{RP_2}{P_1R}$. Then, given one point on the graph and the slope, we can find a second point by counting the units in the legs of the right triangle. Consider the example used by the text, $y = 3x - 4$. We see immediately that the y-intercept is -4 and that the slope is 3. Since the slope is positive, the graph will rise to the right. Hence, we can find a second point by starting at $(0,-4)$ and counting 1 unit to the right and three units up to the point $(1,-1)$. We can check to see that we are correct by applying the slope formula to these coordinates.

Let us consider one more case; namely, when the slope of the given line is negative. Draw the graph of the equation $y = -\frac{2}{3}x + 3$. We see that the point $(0,3)$ lies on the graph, and to locate a second point by this method, we must realize that we will be working with a slope of $-\frac{2}{3}$. The graph, then, will rise to the left and we can locate a second point by counting 3 units to the left from $(0,3)$ and 2 units up, as in the figure below.

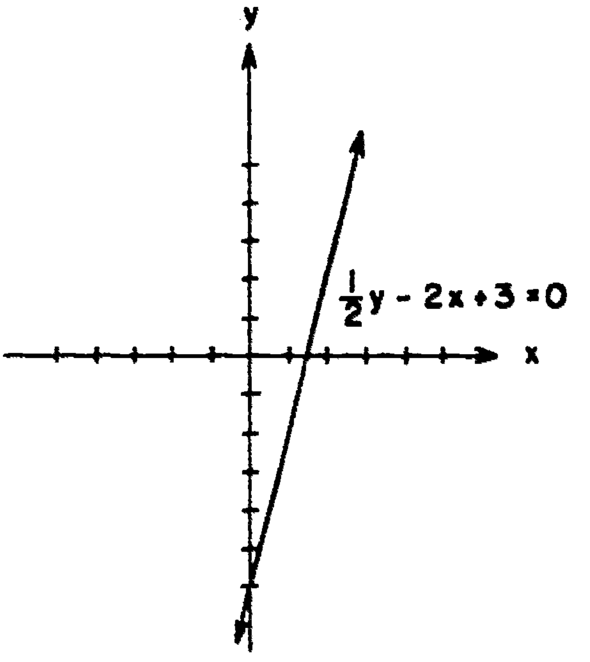


Problem Set 9

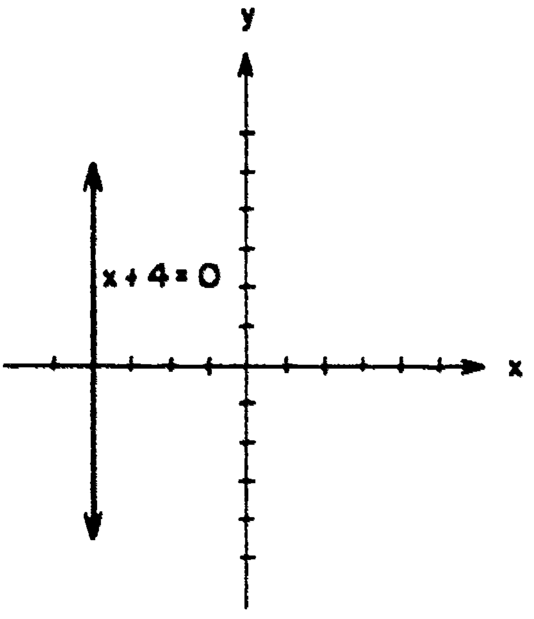
1.



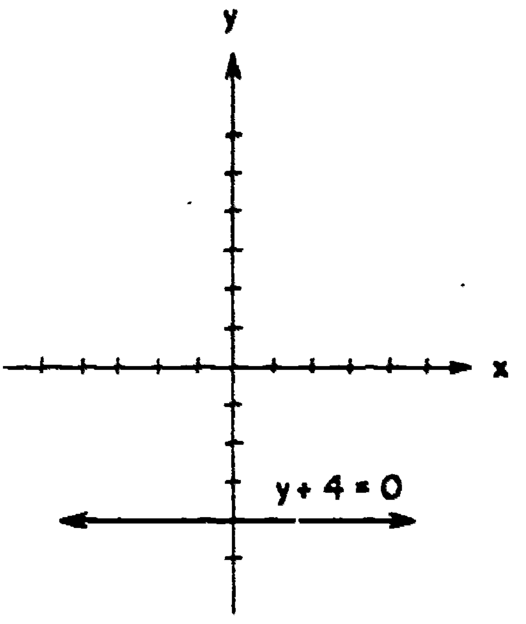
2.



3.



4.



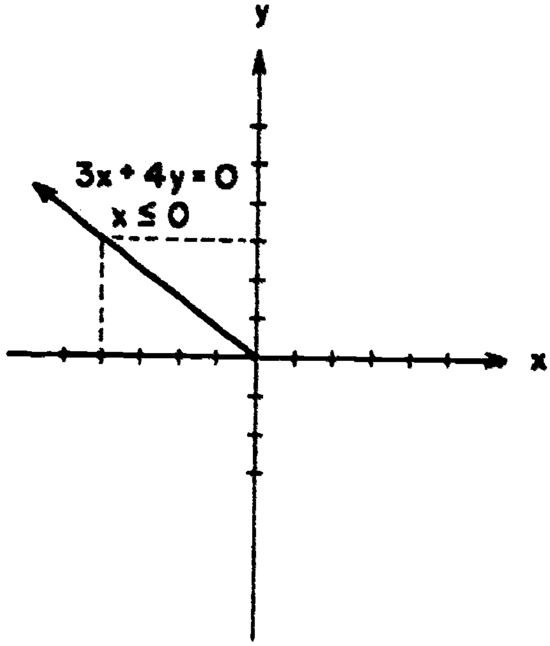
5. The graph is the whole xy-plane.

6. The graph is the empty set; i.e., there are no points whose coordinates satisfy the equation.

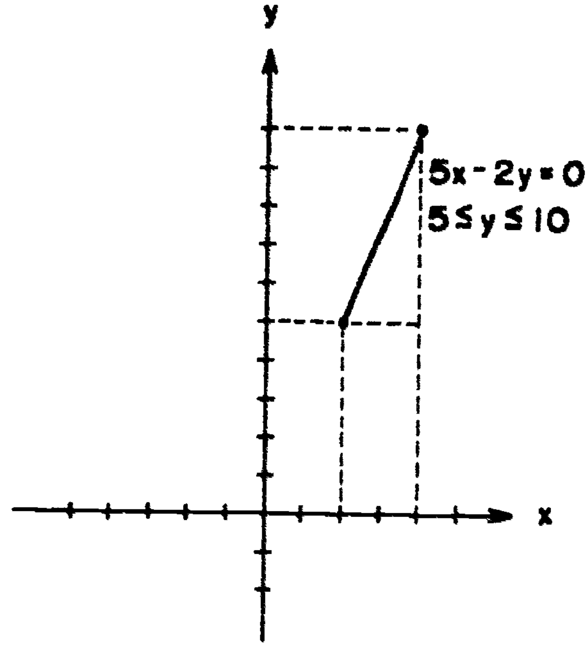
7. The graph contains a single point, the origin (0,0).

8. The graph is the empty set.

9.



10.



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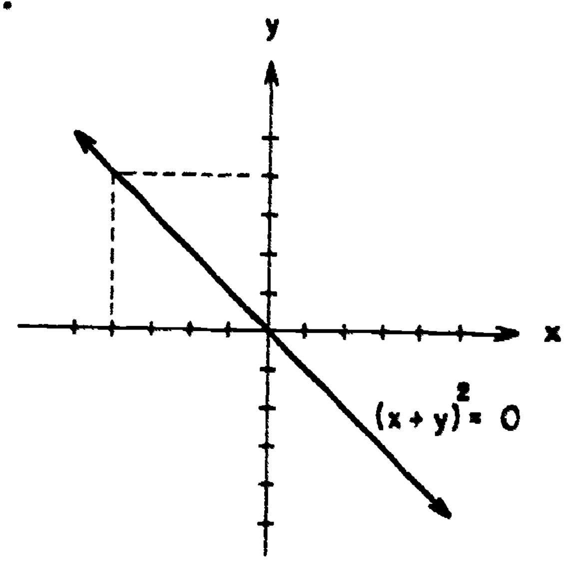

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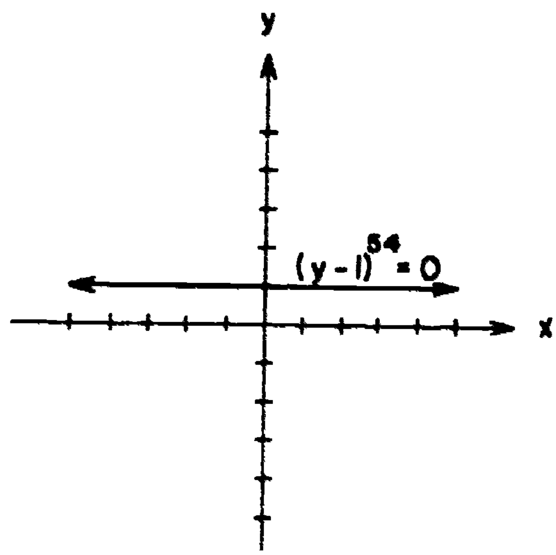


13. $3x - y - 1 = 0$. $A = 3$, $B = -1$, $C = -1$.

14. $x + y - 1 = 0$. $A = 1$, $B = 1$, $C = -1$.

15. $2x - y - 4 = 0$. $A = 2$, $B = -1$, $C = -4$.

16. $y = 0$. $A = 0$, $B = 1$, $C = 0$.

17. $x = 0$. $A = 1$, $B = 0$, $C = 0$.

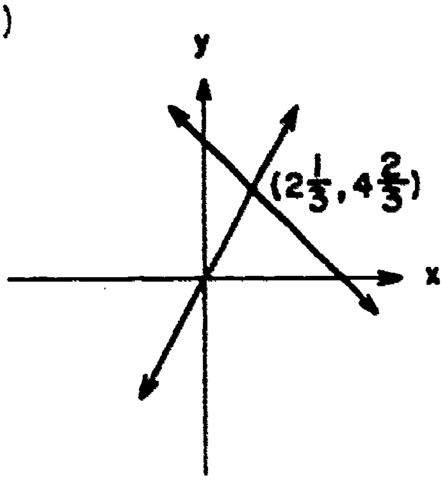
18. $y + 3 = 0$. $A = 0$, $B = 1$, $C = 3$.

19. $x + 5 = 0$. $A = 1$, $B = 0$, $C = 5$.

20. $x - 5y = 0$. $A = 1$, $B = -5$, $C = 0$.

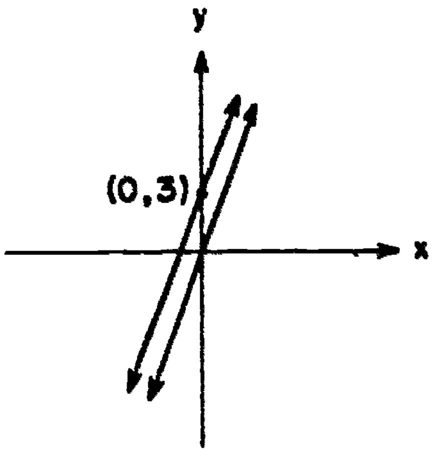
Problem Set 10

1. (a)



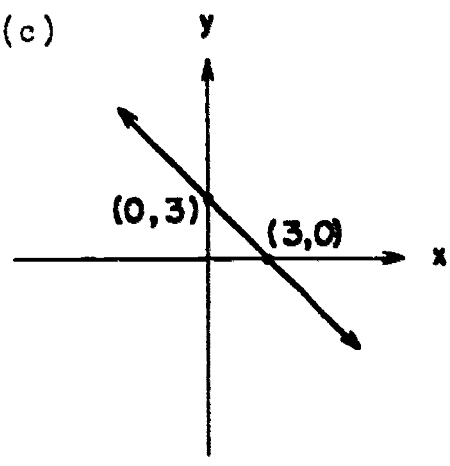
$x = 2\frac{1}{3}$; $y = 4\frac{2}{3}$.

(b)



The empty set.

(c)



The equations are equivalent. Any pair of numbers whose sum is 3 is a common solution.

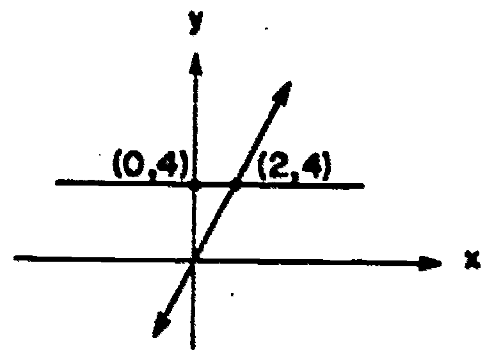
2. (a) (1) and (4), (3) and (4).

   (b) (1) and (2), (2) and (3), (2) and (4).

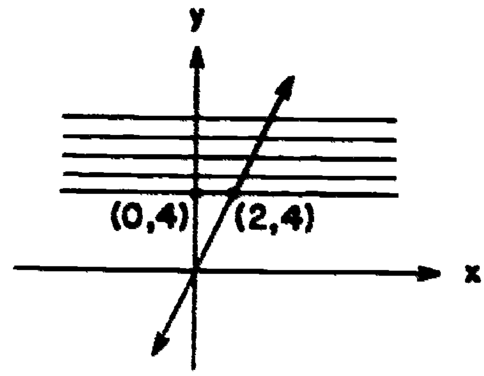
   (c) (1) and (3).

3. 4000 miles.

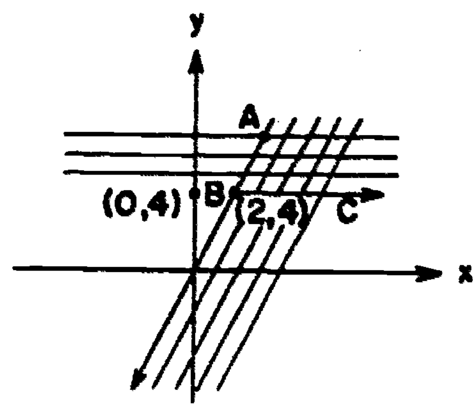
4. (a) The intersection is point $(2,4)$.



(b) The intersection is the ray shown with endpoint $(2,4)$.



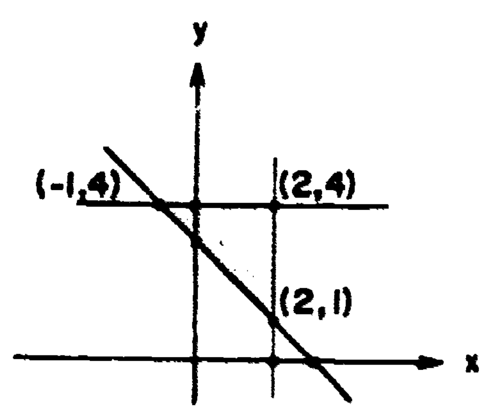
(c) The intersection is the interior of $\angle ABC$.



(d) The conditions are $y < 2x$ and $y < 4$.

5. (a) The intersection is the interior of the triangle with vertices $(2,1)$, $(2,4)$, and $(-1,4)$.

(b) $x + y < 3$,
$x > 0$,
$y > 0$.



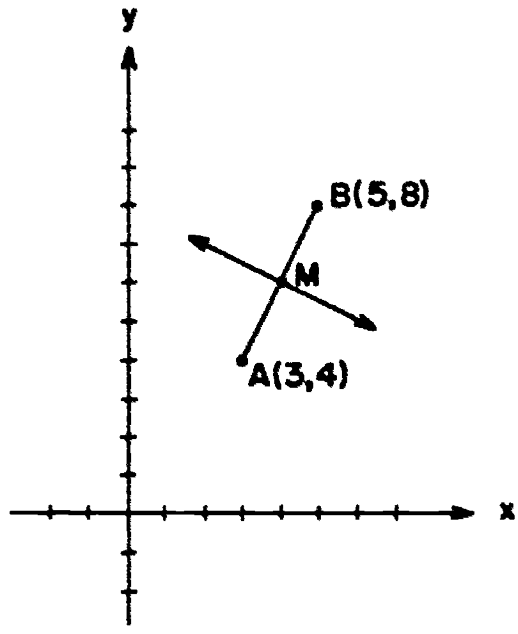
6. The midpoint M has coordinates $\left(\frac{3+5}{2}, \frac{4+8}{2}\right) = (4,6)$.

The slope of $\overline{AB}$ is $\frac{8-4}{5-3} = 2$, so the slope of L is $-\frac{1}{2}$ and its equation is

$y - 6 = -\frac{1}{2}(x - 4)$,

$y - 6 = -\frac{1}{2}x + 2$,

$x + 2y = 16$.



Alternate solution: L is the set of points P(x,y) for which PA = PB. This gives

$$\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-5)^2 + (y-8)^2}$$

which reduces to $x + 2y = 16$.

7. In the preceding problem, we found the equation

$$L: \quad x + 2y = 16.$$

Similarly, for M and N we find

$$M: \quad 3x - y = -3,$$

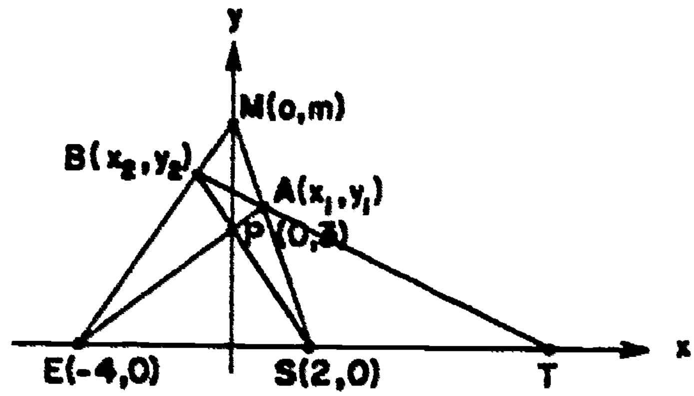
$$N: \quad 2x - 3y = -19.$$

The intersection G of L and M is obtained by solving their equations:

$$G = \left(\frac{10}{7}, \frac{51}{7}\right).$$

Substituting in the third equation, we find that G lies on N also.

*8. Take a coordinate system in which Queen's Road is the x-axis and King's Road is the y-axis.



The coordinates of the elm, spruce, and pine are as indicated. The maple is gone, but its assumed position is labeled (0,m). The slope of $\overleftrightarrow{EP}$ is $\frac{3}{4}$, so its equation (in slope-intercept form) is

$$\overleftrightarrow{EP}: \quad y = \frac{3}{4}x + 3.$$

The slope of $\overleftrightarrow{SM}$ is $-\frac{m}{2}$, so its equation (in point-slope form) is

$$\overleftrightarrow{SM}: \quad y = -\frac{m}{2}(x - 2).$$

Solving these two equations simultaneously, we find the coordinates of A:

$$A: \begin{cases} x_1 = \dfrac{4(m - 3)}{2m + 3}, \\ y_1 = \dfrac{9m}{2m + 3}. \end{cases}$$

Similarly, we get the equations

$$\overleftrightarrow{SP}: \quad y = -\frac{3}{2}x + 3,$$

$$\overleftrightarrow{EM}: \quad y = \frac{m}{4}(x + 4),$$

and the point of intersection is

$$B: \begin{cases} x_2 = -\dfrac{4(m - 3)}{m + 6}, \\ y_2 = \dfrac{9m}{m + 6}. \end{cases}$$

The line $\overleftrightarrow{AB}$ has the equation,

$$\overleftrightarrow{AB}: \quad y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

The intersection T of $\overleftrightarrow{AB}$ and the x-axis is found by letting $y = 0$ and solving for x:

$$x = x_1 - y_1\left(\frac{x_2 - x_1}{y_2 - y_1}\right),$$

$$x = \frac{x_1y_2 - x_2y_1}{y_2 - y_1}.$$

Now

$$x_1y_2 - x_2y_1 = \frac{4(m - 3)}{2m + 3} \cdot \frac{9m}{m + 6} + \frac{4(m - 3)}{m + 6} \cdot \frac{9m}{2m + 3} \cdot$$

$$= \frac{72m(m - 3)}{(m + 6)(2m + 3)},$$

$$y_2 - y_1 = \frac{9m}{m + 6} - \frac{9m}{2m + 3},$$

$$= \frac{9m(m - 3)}{(m + 6)(2m + 3)}.$$

Dividing, we get $x = 8$. Therefore, the treasure was buried 8 miles east of the crossing.

Suppose now that the pine were also missing. Assume coordinates $(0,p)$, for P and carry through the calculation in terms of both m and p. The algebra is a little more complicated, but if it is done correctly, both m and p drop out in the final result, which is again $x = 8$.

*9. The y-axis is a line through C, perpendicular to the base $\overline{AB}$, i.e., it contains the altitude from C. If $\overleftrightarrow{AM}$, where m is its slope, contains the altitude from A, it has the equation

$$y = m(x + 4),$$

Since $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$, $m = -\dfrac{1}{\text{slope } \overleftrightarrow{BC}}$.

But slope $\overleftrightarrow{BC} = -\frac{8}{7}$, so $m = \frac{7}{8}$, and the equation of $\overleftrightarrow{AM}$ is

$$y = \tfrac{7}{8}(x + 4).$$

To find the y-intercept, let $x = 0$:

$$y = \tfrac{7}{8} \cdot 4 = \tfrac{7}{2}.$$

Now do the same for $\overleftrightarrow{BN}$, which contains the altitude from B. Slope $\overleftrightarrow{AC} = \frac{8}{4} = 2$, so the slope of $\overleftrightarrow{BN}$ is $-\frac{1}{2}$, and its equation is

$$y = -\frac{1}{2}(x - 7).$$

Letting $x = 0$, we get the y-intercept

$$y = -\frac{1}{2}(-7) = \frac{7}{2}.$$

Therefore, $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ meet at the point $(0,\frac{7}{2})$ on the line containing the altitude from C.

For the general triangle,

slope $\overleftrightarrow{BC} = -\frac{c}{b}$,

slope $\overleftrightarrow{AM} = \frac{b}{c}$, so

$\overleftrightarrow{AM}$: $y = \frac{b}{c}(x - a)$, and

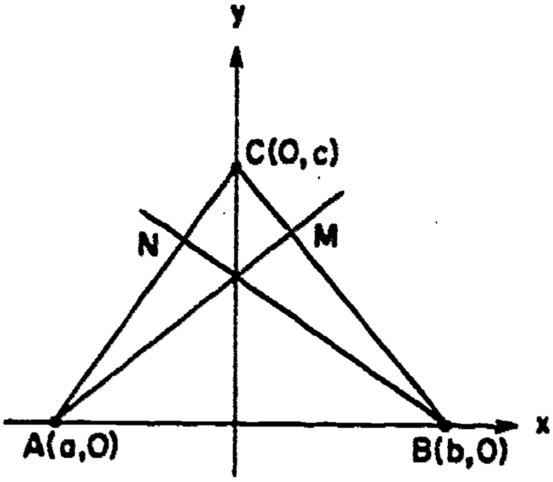
the y-intercept is $-\frac{ba}{c}$.

Similarly, slope $\overleftrightarrow{AC} = -\frac{c}{a}$,

slope $\overleftrightarrow{BN} = \frac{a}{c}$, so

$\overleftrightarrow{BN}$: $y = \frac{a}{c}(x - b)$, and

the y-intercept is $-\frac{ab}{c}$.



Therefore, the three altitudes meet at the point $(0,-\frac{ab}{c})$. Note that this proof does not depend on the signs of a, b, and c, but only on the fact that A, B, lie on the x-axis and C on the y-axis.

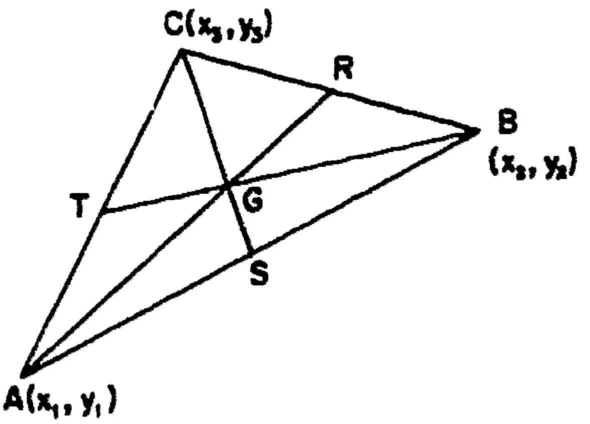
*10. Let $A = (x_1,y_1)$, $B = (x_2,y_2)$, $C = (x_3,y_3)$.

Then we have

$$R = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right),$$

$$S = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),$$

$$T = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right).$$



The slope of $\overleftrightarrow{AR}$ is

$$m_1 = \frac{\frac{y_2 + y_3}{2} - y_1}{\frac{x_2 + x_3}{2} - x_1} = \frac{y_2 + y_3 - 2y_1}{x_2 + x_3 - 2x_1} .$$

If $G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, then the slope of $\overleftrightarrow{AG}$ is

$$m_1' = \frac{\frac{y_1 + y_2 + y_3}{3} - y_1}{\frac{x_1 + x_2 + x_3}{3} - x_1} = m_1,$$

so G is on the median $\overleftrightarrow{AR}$. Similarly, the slope of $\overleftrightarrow{BT}$ is

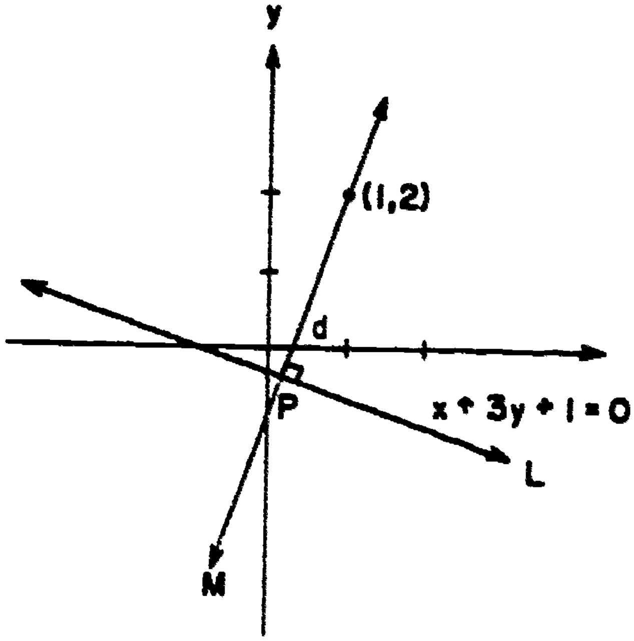
$$m_2 = \frac{\frac{y_1 + y_3}{2} - y_2}{\frac{x_1 + x_3}{2} - x_2} = \frac{y_1 + y_3 - 2y_2}{x_1 + x_3 - 2x_2} ,$$

and the slope of $\overleftrightarrow{BG}$ is

$$m_2' = \frac{\frac{y_1 + y_2 + y_3}{3} - y_2}{\frac{x_1 + x_2 + x_3}{3} - x_2} = m_2,$$

so G is on the median $\overleftrightarrow{BG}$. Similarly, we find that G is on the median $\overleftrightarrow{CS}$. Hence, the three medians intersect in the point G whose coordinates are the averages of the coordinates of the vertices.

*11.



The equation $x + 3y + 1 = 0$ is equivalent to $y = -\frac{1}{3}x - \frac{1}{3}$, which is in slope-intercept form. Therefore, the slope is $-\frac{1}{3}$. The line M through (1,2) perpendicular to L has slope 3, so an equation for it is

$$M: \quad y - 2 = 3(x - 1),$$
$$y = 3x - 1.$$

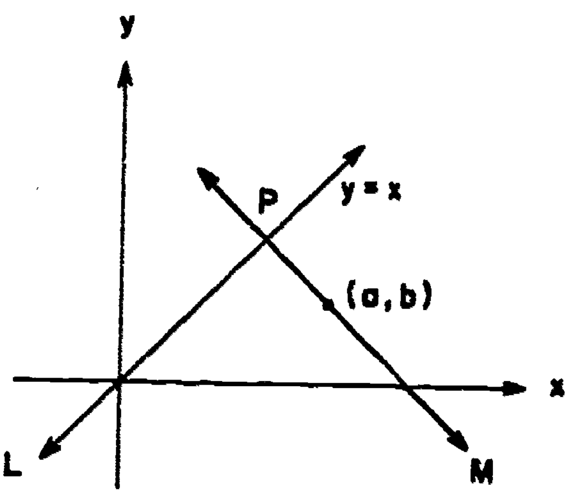
Solving the equations for M and L simultaneously to find their intersection P, we get

$$P = (\tfrac{1}{5}, -\tfrac{2}{5}).$$

Computing the distance d from (1,2) to P by the distance formula, we find $d = \frac{4}{5}\sqrt{10}$.

*12. The line L with equation $y = x$ has slope 1, so the line M through (a,b) perpendicular to L has slope −1. An equation for M is

$$M: \quad y - b = -(x - a),$$
$$x + y = a + b.$$



*12. (continued)

Solving for the point of intersection P, we get

$$P = \left(\frac{a+b}{2}, \frac{a+b}{2}\right).$$

The distance is obtained from

$$d^2 = \left(\frac{a+b}{2} - a\right)^2 + \left(\frac{a+b}{2} - b\right)^2 = \left(\frac{a-b}{2}\right)^2,$$

$$d = \frac{|a-b|}{\sqrt{2}}.$$

*13. From Problem 9 we have $H = (0, -\frac{ab}{c})$.

From Problem 10 we have $M = (\frac{a+b}{3}, \frac{c}{3})$.

To find D we get the perpendicular bisectors u, v of $\overline{AB}$ and $\overline{BC}$:

$$u: \quad x = \frac{a+b}{2},$$

$$v: \quad y - \frac{c}{2} = \frac{b}{c}\left(x - \frac{b}{2}\right).$$

Therefore, $$D = \left(\frac{a+b}{2}, \frac{c^2+ab}{2c}\right).$$

Now

$$HM^2 = \left(\frac{a+b}{3}\right)^2 + \left(\frac{c^2+3ab}{3c}\right)^2 = \frac{c^2(a+b)^2 + (c^2+3ab)^2}{(3c)^2},$$

$$HD^2 = \left(\frac{a+b}{2}\right)^2 + \left(\frac{c^2+3ab}{2c}\right)^2 = \frac{c^2(a+b)^2 + (c^2+3ab)^2}{(2c)^2},$$

$$MD^2 = \left(\frac{a+b}{6}\right)^2 + \left(\frac{c^2+3ab}{6c}\right)^2 = \frac{c^2(a+b)^2 + (c^2+3ab)^2}{(6c)^2}.$$

From these equations we get,

$$HM = 2MD, \quad HD = 3MD,$$

$$HM + MD = HD.$$

This shows that H, M, and D are collinear, that M is between H and D, and that M trisects $\overline{HD}$: $MD = \frac{1}{3}HD$.

Problem Set 11

1. In each case the result is 25. This becomes obvious if radii are drawn to the points on the circle.

2. (a) (1), (3), (4), (6).

   (b) (3), (4).

   (c) (1).

3. (a) Center $(0,0)$; $r = 3$.

   (b) $(0,0)$; $r = 10$.

   (c) $(1,0)$; $r = 4$.

   (d) $(0,0)$; $r = \sqrt{7}$.

   (e) $(0,0)$; $r = 2$.

   (f) $(4,3)$; $r = 6$.

   (g) $(-1,-5)$; $r = 7$.

   (h) $(1,0)$; $r = 5$.

   (i) $(1,0)$; $r = 5$.

   (j) $(-3,2)$; $r = 5$.

4. (a) Replacing x and y in the equation by the given coordinates satisfies the equation.

   (b) $x^2 - 10x + y^2 = 0,$

   $(x^2 - 10x + 25) + y^2 = 25,$

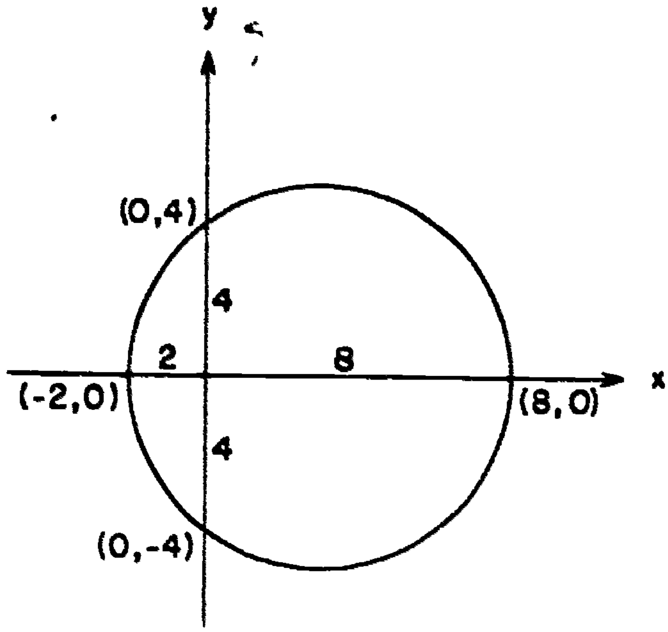
   $(x - 5)^2 + (y - 0)^2 = 5^2.$

   The center of the circle is $(5,0)$; the radius is 5.

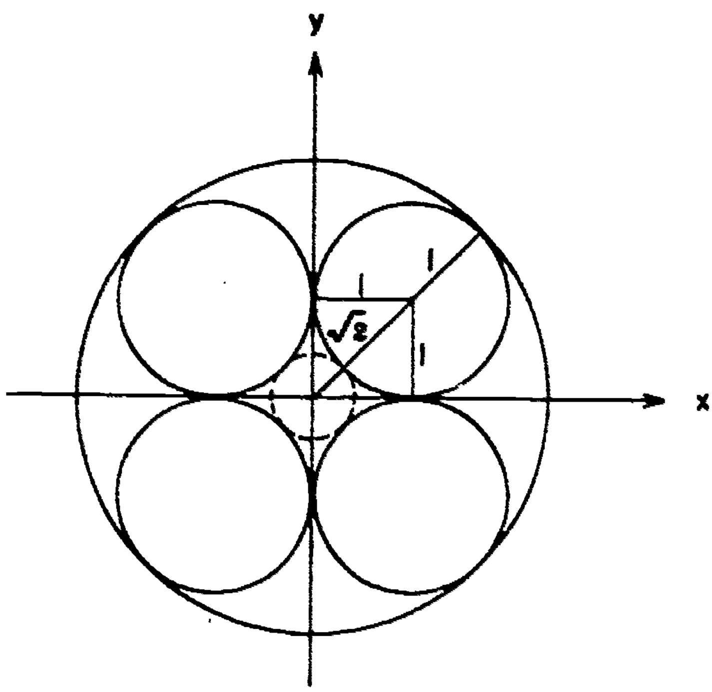
   (c) The ends of the diameter along the x-axis are $(0,0)$ and $(10,0)$. The slope of the segment joining $(0,0)$ and $(1,3)$ is 3. The slope of the segment joining $(10,0)$ and $(1,3)$ is $-\frac{1}{3}$. Since 3 and $-\frac{1}{3}$ are negative reciprocals, the lines are perpendicular and a right angle is formed.

5. (a) The x-axis intersects the circle where $y = 0$, that is where $(x - 3)^2 = 25$, or at points $(-2,0)$ and $(8,0)$. The y-axis intersects the circle where $x = 0$; that is, where $9 + y^2 = 25$, or at points $(0,4)$ and $(0,-4)$.

(b) $2 \cdot 8 = 4 \cdot 4 = 16.$



6.



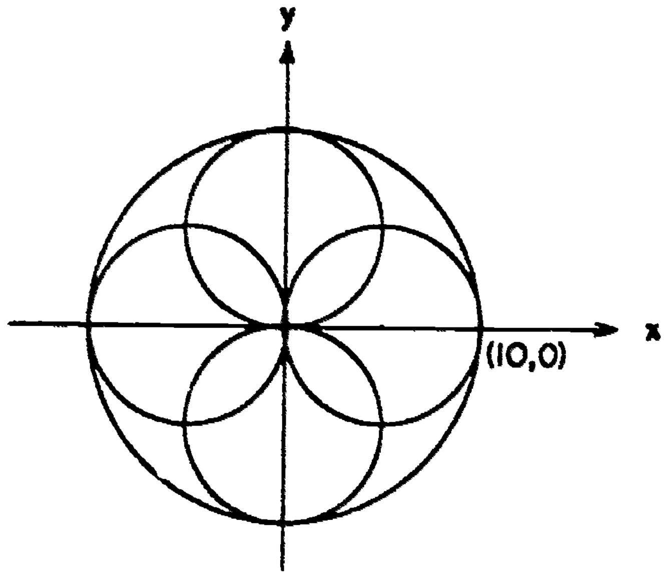
The radius of the larger circle is $1 + \sqrt{2}$. So the equation is

$$x^2 + y^2 = (1 + \sqrt{2})^2.$$

There would be another tangent circle of radius $\sqrt{2} - 1$ and the same center.

4

7.



The including circle is $x^2 + y^2 = 100$.

8. (a) $y = m(x + 7)$.

(b) $x^2 + \{m(x + 7)\}^2 = 16$,

$$(1 + m^2)x^2 + 14m^2x + (49m^2 - 16) = 0,$$

$$x = \frac{-14m^2 \pm \sqrt{(14m^2)^2 - 4(1 + m^2)(49m^2 - 16)}}{2(1 + m^2)}$$

$$= \frac{-14m^2 \pm \sqrt{4(16 - 33m^2)}}{2(1 + m^2)}$$

$$= \frac{-7m^2 \pm \sqrt{16 - 33m^2}}{1 + m^2}$$

$$y = m(x + 7) = \left(\frac{-7m^2 \pm \sqrt{16 - 33m^2} + 7 + 7m^2}{1 + m^2}\right)m$$

$$= \frac{m(7 \pm \sqrt{16 - 33m^2})}{1 + m^2}.$$

If $16 - 33m^2 > 0$, there are two points of intersection:

$$P_1 = \frac{-7m^2 + \sqrt{16 - 33m^2}}{1 + m^2}, \frac{m(7 + \sqrt{16 - 33m^2})}{1 + m^2},$$

$$P_2 = \frac{-7m^2 - \sqrt{16 - 33m^2}}{1 + m^2}, \frac{m(7 - \sqrt{16 - 33m^2})}{1 + m^2}.$$

8. (c) If $16 - 33m^2 = 0$, there is one point of intersection:

$$P = \left(\frac{-7m^2}{1 + m^2}, \frac{7m}{1 + m^2}\right)$$

and $$m^2 = \frac{16}{33}, \quad m = \pm\frac{4}{\sqrt{33}} .$$

This means that the two lines

$$y = \frac{4}{\sqrt{33}}(x + 7),$$

$$y = -\frac{4}{\sqrt{33}}(x + 7)$$

are tangent to the circle.

If $16 - 33m^2 < 0$, there is no point of intersection.

9. Put the given equation in standard form:

$$(x - 5)^2 + (y - 3)^2 = 2^2.$$

The given circle has center $(5,3)$, radius 2. Let the required circle have center (a,b) and radius r. Then $b = a = r$, since the circle touches the x- and y-axes, and the distance from center (a,b) to center $(5,3)$ is $r + 2$. Hence,

$$r + 2 = \sqrt{(r - 5)^2 + (r - 3)^2}$$

$$r^2 + 4r + 4 = 2r^2 - 16r + 34$$

$$r^2 - 20r + 30 = 0$$

$$r = \frac{20 \pm \sqrt{400 - 120}}{2}$$

$$r = 10 \pm \sqrt{70}.$$

Thus, there are two solutions:

$$(x - r)^2 + (y - r)^2 = r^2,$$

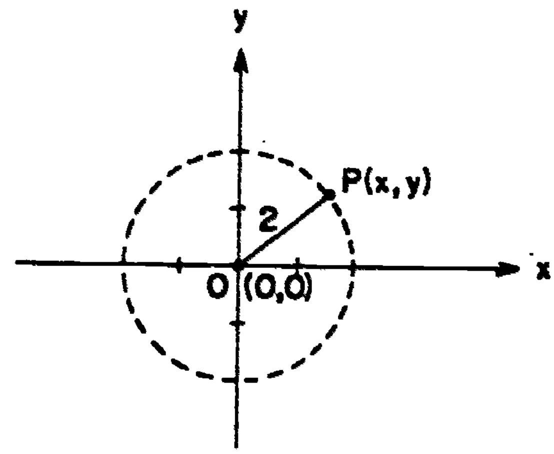
where $r = 10 + \sqrt{70}$ (approx. 18.37) and $r^2 = 337.3$ (approx.) or $r = 10 - \sqrt{70}$ (approx. 1.63) and $r^2 = 2.7$ (approx.).

Problem Set 12

1. $OP = \sqrt{x^2 + y^2}$

$2 = \sqrt{x^2 + y^2}$

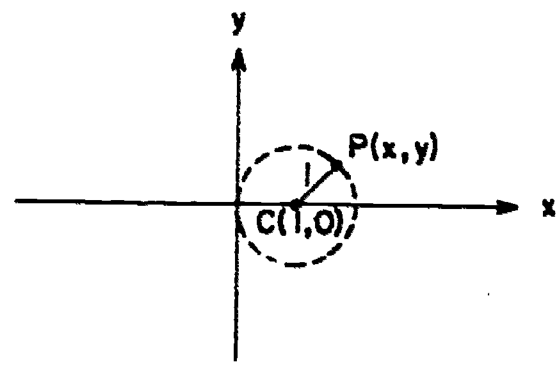
$4 = x^2 + y^2$, the equation.



2. $CP = \sqrt{(x - 1)^2 + y^2}$

$1 = \sqrt{(x - 1)^2 + y^2}$

$1 = (x - 1)^2 + y^2$



3. $CP = \sqrt{x^2 + (y - 2)^2}$

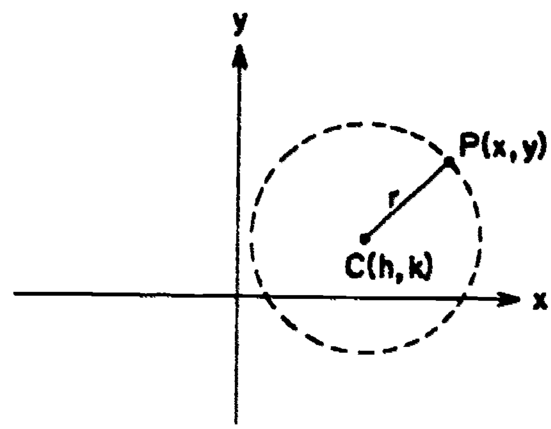
$3 = \sqrt{x^2 + (y - 2)^2}$

$9 = x^2 + (y - 2)^2$

4. $CP = \sqrt{(x - 2)^2 + (y - 3)^2}$

$5 = \sqrt{(x - 2)^2 + (y - 3)^2}$

$25 = (x - 2)^2 + (y - 3)^2$



5. $CP = \sqrt{(x + 1)^2 + (y - 3)^2} = k$

$(x + 1)^2 + (y - 3)^2 = k^2$

6. $CP = \sqrt{(x - h)^2 + (y - k)^2} = r$

$(x - h)^2 + (y - k)^2 = r^2$

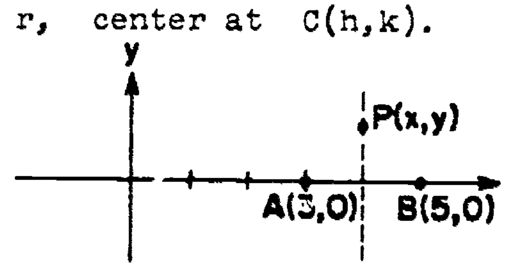
The set is a circle with radius r, center at $C(h,k)$.

7. $AP = BP$

$\sqrt{(x - 3)^2 + y^2} = \sqrt{(x - 5)^2 + y^2}$

$(x - 3)^2 + y^2 = (x - 5)^2 + y^2$

$x = 4.$



8. $\sqrt{(x + 2)^2 + (y + 5)^2} = \sqrt{(x - 3)^2 + (y - 2)^2}$

$5x + 7y = -8.$

*9. $P_1P = P_2P$

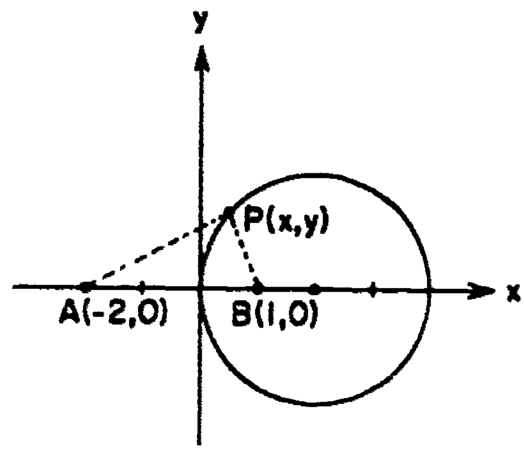
$\sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2}$

$2x(x_2 + x_1) + 2y(y_2 - y_1) + (x_1^2 + x_2^2 + y_1^2 + y_2^2) = 0$

10. $PA = 2(PB)$

$\sqrt{(x + 2)^2 + y^2} = 2\sqrt{(x - 1)^2 + y^2}$

$x^2 - 4x + y^2 = 0$



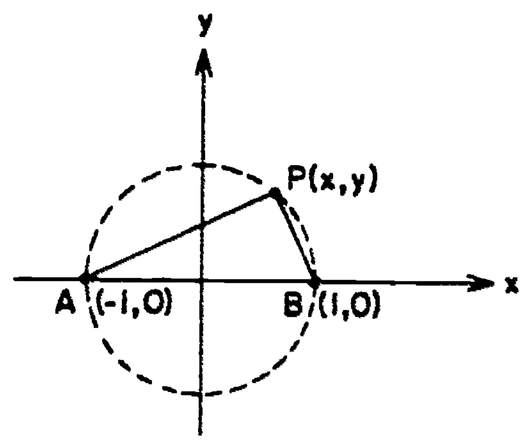
11. $m(\overline{PA}) \cdot m(\overline{PB}) = -1$, the condition for $\overline{PA} \perp \overline{PB}$

$m(\overline{PA}) = \dfrac{y}{x + 1}$; $m(\overline{PB}) = \dfrac{y}{x - 1}$

$\dfrac{y}{x + 1} \cdot \dfrac{y}{x - 1} = -1$

$x^2 + y^2 = 1,\ y \neq 0.$

This set consists of the set of all points except A and B on the circle with center (0,0) and the length of the radius equal to 1.

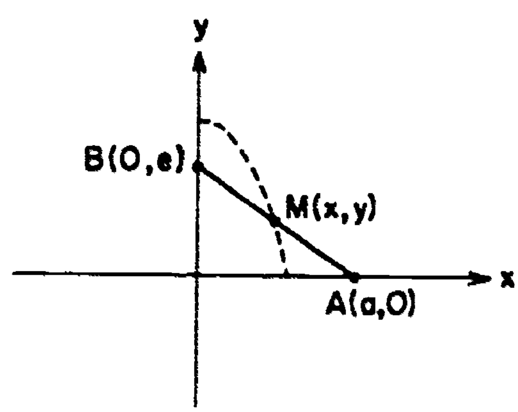


*12. Using the midpoint formula, $P(x,y) = P(\frac{a}{2},\frac{b}{2})$; hence, $a = 2x$, $b = 2y$. Using the distance formula,

$AB = \sqrt{a^2 + b^2} = 2$

$\sqrt{(2x)^2 + (2y)^2} = 2$, by substitution.

$x^2 + y^2 = 1.$

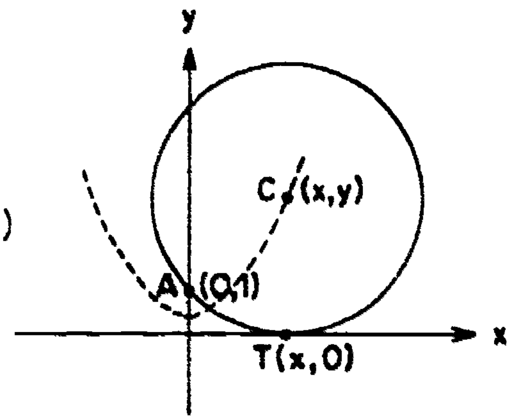


13. CT = CA

$$y^2 = \sqrt{x^2 + (y-1)^2}$$

$$y = \frac{1}{2}x^2 + \frac{1}{2}$$

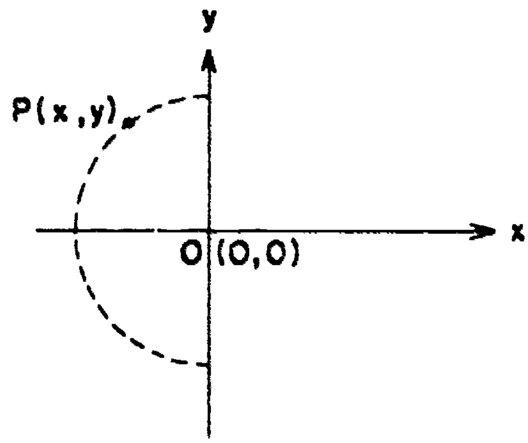
(The set of points is a parabola.)



14. This problem can be considered as that of finding the set of all points 0 distance from the origin. Hence,

$$PO = \sqrt{x^2 + y^2} = 0$$

$$x^2 + y^2 = 0$$



15. $PO = \sqrt{x^2 + y^2} = 2$

$$x^2 + y^2 = 4.$$

But, this includes all values for x and y. To exclude the "right" part of the circle, the description may be written as any one of the following:

(a) $\{(x,y) : x = -\sqrt{4 - y^2}$

(b) $\{(x,y)\ x < 0 : x^2 + y^2 = 4\}$

16. $PA = \sqrt{y^2} = |y|$

$OB = 3$

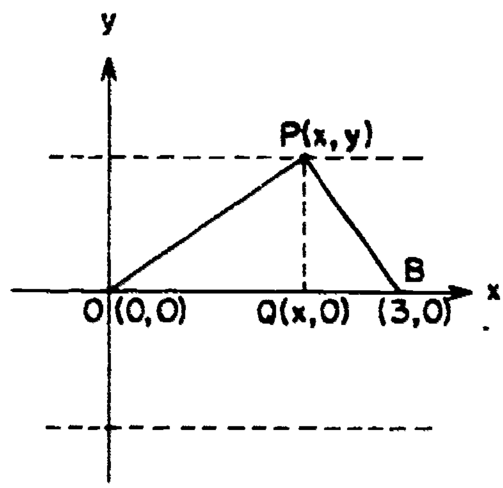
$\text{Area} = \frac{1}{2}(PA)(OB)$

$2 = \frac{1}{2}|y| \cdot 3$

$\frac{4}{3} = |y|$

$y = \pm\frac{4}{3}$, which means that the set of points (x,y) is the graph of two lines parallel to the x-axis. This may be described as,

$\{(x,y) : y = \frac{4}{3}\} \cup \{(x,y) : y = -\frac{4}{3}\}$.



18

Review Problems

1. $(5,0)$.

2. $(-1,5)$.

3. $\frac{b}{3a}$. The median is vertical and has no slope.

4. $\frac{a}{b}$.

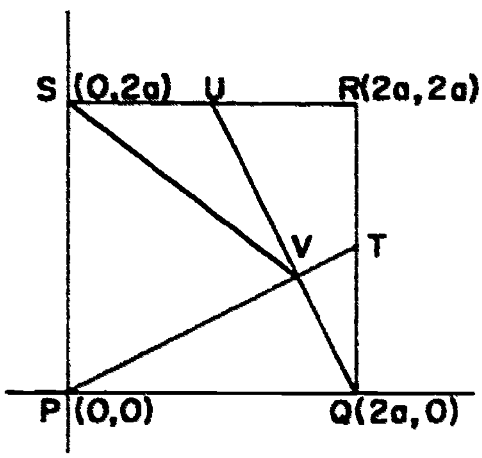
5. $2b$; $\sqrt{9a^2 + b^2}$; $\sqrt{9a^2 + b^2}$.

6. $-\frac{3}{2}$.

7. $5\sqrt{2}$; $6\sqrt{2}$.

8. $(\frac{1}{2}, 3\frac{1}{2})$; $(3,6)$; $(6,3)$; $(3\frac{1}{2}, \frac{1}{2})$; $(4,4)$; $(2\frac{1}{2}, 2\frac{1}{2})$.

9. Place the axes and assign coordinates as shown.



(a) $T = (2a,a)$, $U = (a,2a)$.

$PT = \sqrt{4a^2 + a^2} = a\sqrt{5}$.

$QU = \sqrt{a^2 + 4a^2} = a\sqrt{5}$.

Therefore, $PT = QU$.

(b) The slope of $\overline{PT} = \frac{a - 0}{2a - 0} = \frac{1}{2}$.

The slope of $\overline{QU} = \frac{0 - 2a}{2a - a} = -2$.

Since -2 is the negative reciprocal of $\frac{1}{2}$, the segments are perpendicular.

9. *(c) Using the point-slope form the equation of $\overleftrightarrow{PT}$ is:

$$y - 0 = \frac{1}{2}(x - 0)$$

or $$y = \frac{1}{2}x.$$

The equation of $\overleftrightarrow{QU}$ is:

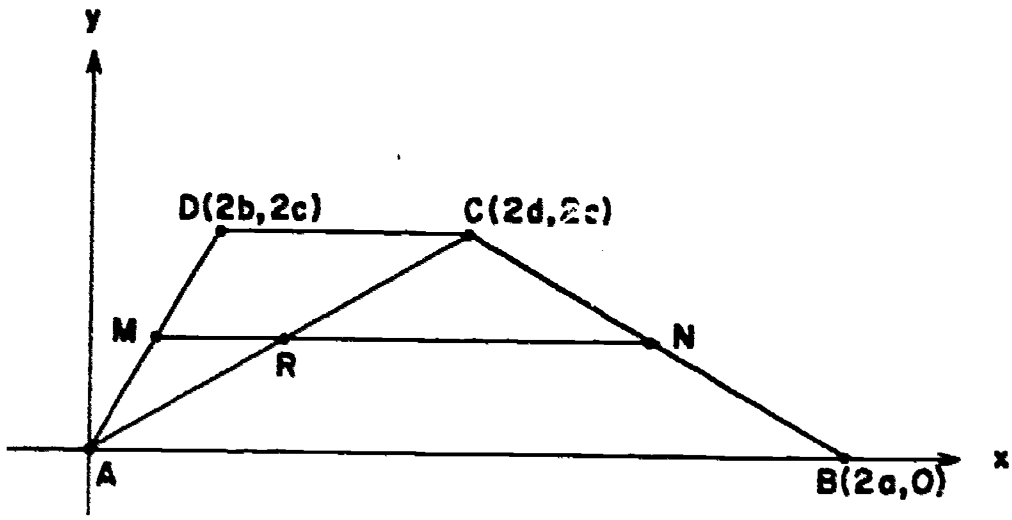
$$y - 0 = -2(x - 2a)$$

or $$y = -2x + 4a.$$

The coordinates of V, given by the common solution of the equations of $\overleftrightarrow{PT}$ and $\overleftrightarrow{QU}$ are $(\frac{8a}{5}, \frac{4a}{5})$. The distance VS is then

$$\sqrt{(\frac{8a}{5} - 0)^2 + (\frac{4a}{5} - 2a)^2} = \sqrt{\frac{100a^2}{25}} = 2a = \text{length of side.}$$

10.



Take coordinate system as shown. Then $M = (b, c)$; $N = (a + d, c)$.

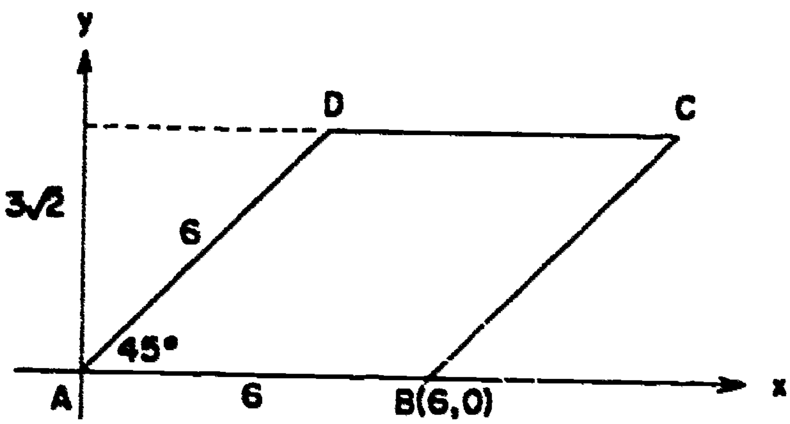
Equation of $\overleftrightarrow{MN}$ is: $y = c$.

Equation of diagonal $\overline{AC}$ is: $y = \frac{c}{d}x$.

Point R of intersection is (d, c), which is also the midpoint of $\overline{AC}$.

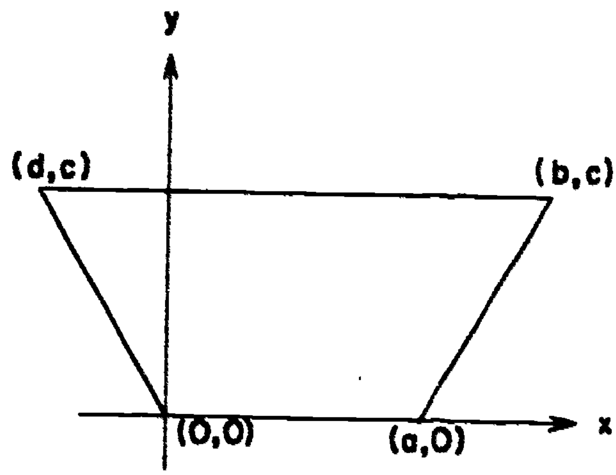
11. $x = 0$.

12.



Equation of $\overleftrightarrow{AB}$ is $y = 0$. Slope $\overline{BC} = 1$.
Equation of $\overleftrightarrow{BC}$ is $y = x - 6$.
Equation of $\overleftrightarrow{CD}$ is $y = 3\sqrt{2}$.

13. Lengths of parallel sides are: $|a|$, $|b - d|$.
Altitude is $|c|$.
Hence,
area $= \frac{1}{2}|c|(|a| + |b - d|)$.



14. $(2,1)$.

15. A circle with center at the origin and radius 2.

16. (a) $x^2 + y^2 = 49$.

(b) $x^2 + y^2 = k^2$.

(c) $(x - 1)^2 + (y - 2)^2 = 9$.

*17. Find first the intersection of the line $x + y = 2$ and the circle. Now $x = 2 - y$.

Therefore, $(2 - y)^2 + y^2 = 2$,

$$4 - 4y + y^2 + y^2 = 2,$$

$$(y - 1)^2 = 0,$$

so that $y = 1$ and $x = 1$.

Thus, the point $(1,1)$ is the only point of intersection, so that the line is tangent to the circle.