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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exer ses, and sample test questions. Topics covered include directed lin sgments, applications to geometry, vectors and scalars, components, inner product, applications of vectors in physics, and vectors as a formal mathematical system. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-7

SUPPLEMENTARY and ENRICHMENT SERIES

THE SYSTEM OF VECTORS

Teachers' Commentary

Edited by Karl Kelman

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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Commentary and Answers

THE SYSTEM OF VECTORS

Introduction.

Vectors have both a geometric and algebraic aspect. The first part of the text is primarily geometric. The algebra of directed line segments is considered to be a pleasant device for solving geometric problems. In Sections 3 and 4 the algebra of vectors is worked out more carefully. Section 5 is about applications of vectors to physics. While this kind of discussion helped form the whole subject originally, it no longer is the central topic in vector studies. Section 6 is concerned with the system of vectors as a whole. Instead of examining individual vectors the student is exposed here to statements about all vectors.

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1. Directed Line Segments.

The main ideas of this section are equivalence of directed line segments, addition of directed line segments, and multiplication of directed line segments by real numbers. The student is required to translate statements of geometric relation into algebraic language.

Exercises 1. Answers.

1. $\overrightarrow{AA}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BB}$.

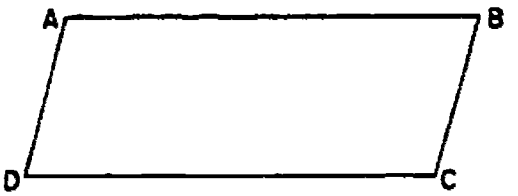
2. $\overrightarrow{AA}$, $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CC}$, $\overrightarrow{CB}$, $\overrightarrow{CA}$.

This is true whether the points are collinear or not.

3. $\overrightarrow{AA} \doteq \overrightarrow{BB} \doteq \overrightarrow{CC} \doteq \overrightarrow{DD}$

$\overrightarrow{AB} \doteq \overrightarrow{DC}$, $\overrightarrow{BA} \doteq \overrightarrow{CD}$

$\overrightarrow{AD} \doteq \overrightarrow{BC}$, $\overrightarrow{DA} \doteq \overrightarrow{CB}$



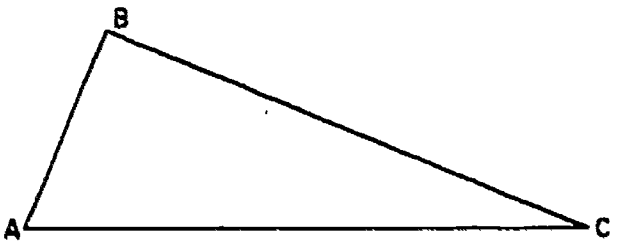
$\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BD}$, $\overrightarrow{DB}$ are also included in the list of directed line segments. From plane geometry the diagonals of a parallelogram have equal measure. This might lead one to say $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are equivalent. One needs to turn again to the Definition 1a for <u>equivalent</u> directed line segments. The same consideration can be invoked to convince one that $\overrightarrow{AC}$ and $\overrightarrow{CA}$ are not equivalent.

4. (a) $\overrightarrow{AC}$

(b) $\overrightarrow{AC}$

(c) $\overrightarrow{AC}$

(d) $\overrightarrow{BA}$



(e) $\overrightarrow{AA}$, for $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA}$.

(f) $\overrightarrow{BB}$, for $\overrightarrow{BA} + (\overrightarrow{AC} + \overrightarrow{CB}) = \overrightarrow{BA} + \overrightarrow{AB} = \overrightarrow{BB}$.

(g) $\overrightarrow{CB} + \overrightarrow{CA}$. Consider what must be added to $\overrightarrow{AC} + \overrightarrow{CB}$.

$\underline{\qquad\qquad} + \overrightarrow{AC} = \overrightarrow{CB}$

$\underline{\overrightarrow{CB} + \overrightarrow{CA}} + \overrightarrow{AC} = \overrightarrow{CB}$.

5. (a) A ——— X ——— B $\quad \overrightarrow{AX} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BX} = \frac{1}{2}\overrightarrow{BA}$,

$r = \frac{1}{2}$, $s = \frac{1}{2}$.

(b) A ——— B ——— X $\quad \overrightarrow{AX} = 2\overrightarrow{AB}$, $\overrightarrow{BX} = -1\overrightarrow{BA}$, $r = 2$, $s = -1$.

(c) B ——— A ——— X $\quad \overrightarrow{AX} = -1\overrightarrow{AB}$, $\overrightarrow{BX} = 2\overrightarrow{BA}$, $r = -1$, $s = 2$.

(d) A ——— X ——— B $\quad \overrightarrow{AX} = \frac{2}{3}\overrightarrow{AB}$, $\overrightarrow{BX} = \frac{1}{3}\overrightarrow{BA}$,

$r = \frac{2}{3}$, $s = \frac{1}{3}$.

(e) A ——— B ——— X $\quad \overrightarrow{AX} = \frac{3}{2}\overrightarrow{AB}$, $\overrightarrow{BX} = -\frac{1}{2}\overrightarrow{BA}$,

$r = \frac{3}{2}$, $s = -\frac{1}{2}$.

(f) B ——— A ——— X $\overrightarrow{AX} = -\frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BX} = \frac{3}{2}\overrightarrow{BA}$,

$r = -\frac{1}{2}$, $s = \frac{3}{2}$.

6. (a) $\frac{1}{2}$ (e) $\frac{1}{2}$

(b) 2 (f) $\frac{1}{2}$

(c) $\overrightarrow{CX}$ (g) 2

(d) $\overrightarrow{CA}$

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2. Applications to Geometry.

This section makes two main points. The first is that vectors can be manipulated according to some of the usual rules of algebra. The second is that certain problems of elementary geometry can be solved by such manipulations.

Each of the examples is worked out as an isolated problem. No hint is given about a general approach to all of them. There is such a general approach which the teacher may want to discuss. Each problem can be solved by

(1) Choosing two directed line segments on non-parallel lines.

(2) Expressing each of the other directed line segments in terms of the ones originally selected.

Exercises 2. Answers.

1. (a) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}$, by Definition 1b and equivalent directed line segments.

(b) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB}$.

(c) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \overrightarrow{AB} - \overrightarrow{BC}$ additive inverse.

(d) $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = -\overrightarrow{AD} + \overrightarrow{AB}$.

(e) $\overrightarrow{DB} = \overrightarrow{CB} + \overrightarrow{AB} = -\overrightarrow{BC} - \overrightarrow{BA}$.

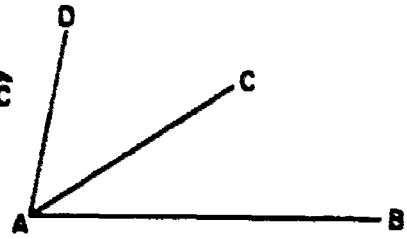
2. (a) The ray $\overrightarrow{AB}$.
   (b) The segment $\overline{AB}$.
   (c) The ray opposite to the ray $\overrightarrow{BA}$.
   (d) The segment whose midpoint is A and which has B as an endpoint.

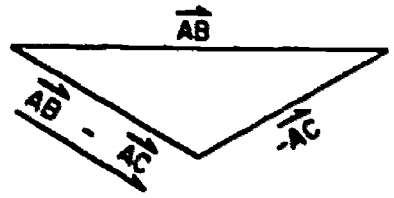
3. Hint: Note the development from the case where either r or s is zero and the other varies to the case where both are variable.

   (a) The line $\overleftrightarrow{AC}$.
   (b) The line $\overleftrightarrow{AB}$.
   (c) Any point on $\overleftrightarrow{AC}$ or $\overleftrightarrow{BX}$ or between $\overleftrightarrow{AC}$ and $\overleftrightarrow{BX}$ where $\overleftrightarrow{BX} \parallel \overleftrightarrow{AC}$.
   (d) Any point on $\overleftrightarrow{AB}$ or $\overleftrightarrow{CY}$ or between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CY}$ where $\overleftrightarrow{CY} \parallel \overleftrightarrow{AB}$.
   (e) Any point inside the parallelogram ABCD where D is the intersection of $\overleftrightarrow{BX}$ and $\overleftrightarrow{CY}$ or on its perimeter.
   (f) Any point on $\overleftrightarrow{BX}$ (line through B $\parallel \overleftrightarrow{AC}$) .
   (g) Any point on $\overleftrightarrow{CY}$ (line through C $\parallel \overleftrightarrow{AB}$) .
   *(h) Any point on $\overleftrightarrow{BC}$.
   *(i) Any point on $\overleftrightarrow{BC'}$ where C' is on $\overleftrightarrow{CA}$ and A is the midpoint of segment $\overline{C'C}$.
   *(j) Any point on $\overleftrightarrow{PQ}$ where P is located on $\overleftrightarrow{AB}$ so that $\overrightarrow{AP} = 2\overrightarrow{AB}$ and Q is located on $\overrightarrow{AC}$ so that $\overrightarrow{AQ} = 3\overrightarrow{AC}$.
   *(k) Any point on $\overleftrightarrow{EF}$ where $\overrightarrow{AE} = \frac{4}{3}\overrightarrow{AB}$ and $\overrightarrow{AF} = \frac{8}{7}\overrightarrow{AC}$ (E on $\overrightarrow{AB}$ and F on $\overrightarrow{AC}$) .
   *(l) Any point on $\overleftrightarrow{GH}$ where $\overrightarrow{AG} = -\frac{c}{a}\overrightarrow{AB}$ and $\overrightarrow{AH} = -\frac{c}{b}\overrightarrow{AC}$.

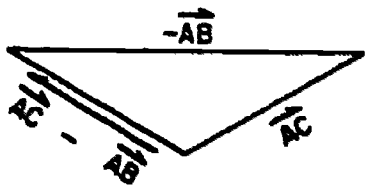
4. (a)

$\overrightarrow{AB} - \overrightarrow{AC}$

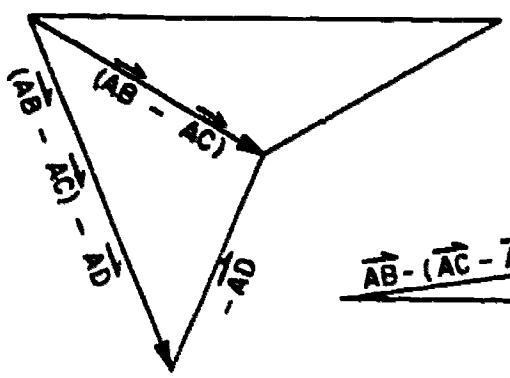


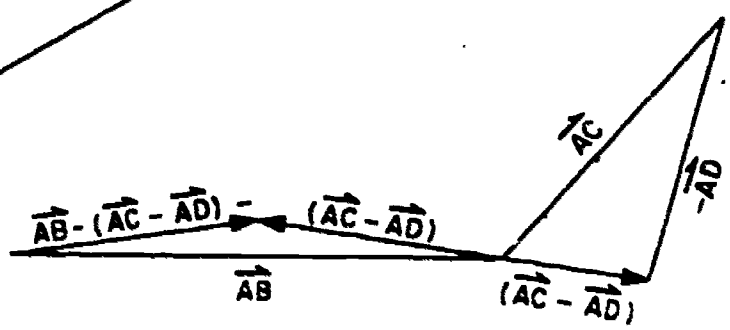


$\overrightarrow{AC} - \overrightarrow{AB}$



(b)





*5. Let $\overrightarrow{AH} = r\overrightarrow{AD}$ and $\overrightarrow{AE} = s\overrightarrow{AB}$. Then $\overrightarrow{HD} = (1 - r)\overrightarrow{AD}$ and $\overrightarrow{EB} = (1 - s)\overrightarrow{DC}$. Note that the opposite sides of a parallelogram are equal.

Let $\vec{V}_1 = \overrightarrow{AH} = \ell\overrightarrow{HF}$ and $\vec{V}_2 = \overrightarrow{AE} + m\overrightarrow{EG}$. We must show that these are values for ℓ and m for which $\vec{V}_1 = \vec{V}_2$ and that, for these values, either $\vec{V}_1$ or $\vec{V}_2$ is equal some constant times $\overrightarrow{AC}$. $\vec{V}_1 = \vec{V}_2$ implies

(1) $\overrightarrow{AH} + \ell(\overrightarrow{HD} + \overrightarrow{DF}) = \overrightarrow{AE} + m(\overrightarrow{EO} + \overrightarrow{OG})$.
Substituting for $\overrightarrow{AH}$, $\overrightarrow{AE}$, etc. in terms of $\overrightarrow{AD}$ and $\overrightarrow{DC}$ and collecting on $\overrightarrow{AD}$ and $\overrightarrow{DC}$ we obtain

(2) $(r + \ell - \ell r)\overrightarrow{AD} + \ell s\overrightarrow{DC} = (s + m - ms)\overrightarrow{DC} + mr\overrightarrow{AD}$.

This equality (2) is satisfied if

(i) $r + \ell - \ell r = mr$ and (ii) $\ell s = s + m - ms$.

Solving (i) and (ii) for ℓ and m in terms of r and s we obtain

(3) $\ell = \dfrac{r}{r + s - 1}$; $m = \dfrac{s}{r + s - 1}$.

For these values of ℓ and m , $\vec{V}_1 = \vec{V}_2$. Moreover

(4) $\vec{V}_1 = \overrightarrow{AH} + \ell\overrightarrow{HF} = r\overrightarrow{AD} + \dfrac{r}{r + s - 1}[(\ell - r)\overrightarrow{AD} + s\overrightarrow{DC}]$

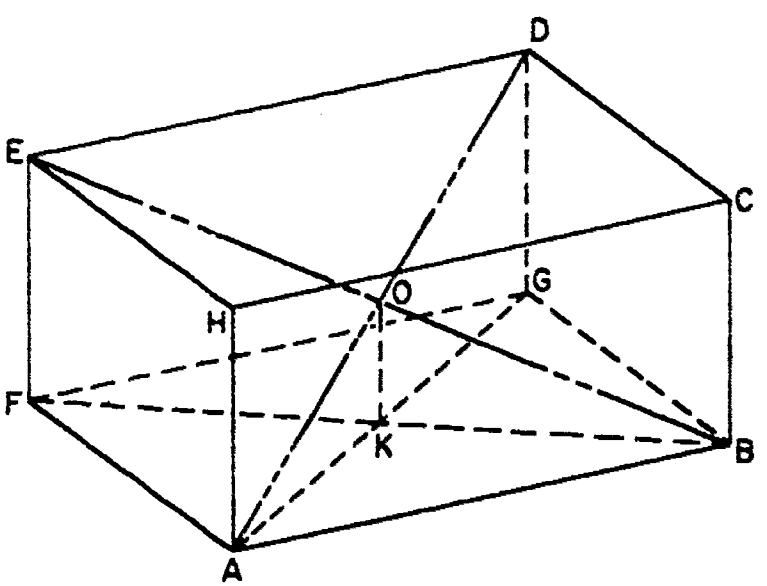
$= \dfrac{rs}{r + s - 1}(\overrightarrow{AD} + \overrightarrow{DC}) = \dfrac{rs}{r + s - 1} \cdot \overrightarrow{AC}$.

Since $\vec{V}_1$ equals a constant times $\overrightarrow{AC}$ the intersection of $\overrightarrow{HF}$ and $\overrightarrow{EG}$ lies on $\overrightarrow{AC}$. Q.E.D.

Question: What happens when $r + s = 1$?

6. (a) $\overrightarrow{OB} = \overrightarrow{OQ} + \overrightarrow{OP}$.
(b) $\overrightarrow{OC} = \overrightarrow{OQ} - \overrightarrow{OP}$.
(c) $\overrightarrow{OD} = -\overrightarrow{OQ} - \overrightarrow{OP}$.
(d) $\overrightarrow{OA} = -\overrightarrow{OQ} + \overrightarrow{OP}$.
(e) $\overrightarrow{DB} = 2\overrightarrow{OQ} + 2\overrightarrow{OP}$.
(f) $\overrightarrow{AC} = 2\overrightarrow{OQ} - 2\overrightarrow{OP}$.
(g) $\overrightarrow{CA} = -2\overrightarrow{OQ} + 2\overrightarrow{OP}$.
(h) $\overrightarrow{BD} = -2\overrightarrow{OQ} - 2\overrightarrow{OP}$.

7.



Let O be the midpoint of $\overline{AD}$, P the midpoint of $\overline{BE}$, Q the midpoint of $\overline{HG}$.

$$\overrightarrow{AO} = \tfrac{1}{2}\overrightarrow{AD} = \tfrac{1}{2}(\overrightarrow{AG} + \overrightarrow{GD}) = \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{BG} + \overrightarrow{GD}) = \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{AF} + \overrightarrow{AH})$$

$$\overrightarrow{AP} = \overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BE} = \overrightarrow{AB} + \tfrac{1}{2}(\overrightarrow{BF} + \overrightarrow{FE}) = \overrightarrow{AB} + \tfrac{1}{2}(\overrightarrow{BA} + \overrightarrow{AF} + \overrightarrow{FE})$$

$$= \overrightarrow{AB} - \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{AF} + \tfrac{1}{2}\overrightarrow{AH} = \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{AF} + \overrightarrow{AH}) .$$

$$\overrightarrow{AQ} = \overrightarrow{AH} + \tfrac{1}{2}\overrightarrow{HG} = \overrightarrow{AH} + \tfrac{1}{2}(\overrightarrow{HA} + \overrightarrow{AG}) = \overrightarrow{AH} + \tfrac{1}{2}(\overrightarrow{HA} + \overrightarrow{AB} + \overrightarrow{BF})$$

$$= \overrightarrow{AH} - \tfrac{1}{2}\overrightarrow{AH} + \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{AF} = \tfrac{1}{2}(\overrightarrow{AH} + \overrightarrow{AB} + \overrightarrow{AF})$$

$\therefore$ $\overrightarrow{AO} = \overrightarrow{AP} = \overrightarrow{AQ}$ and points O, P, and Q coincide.

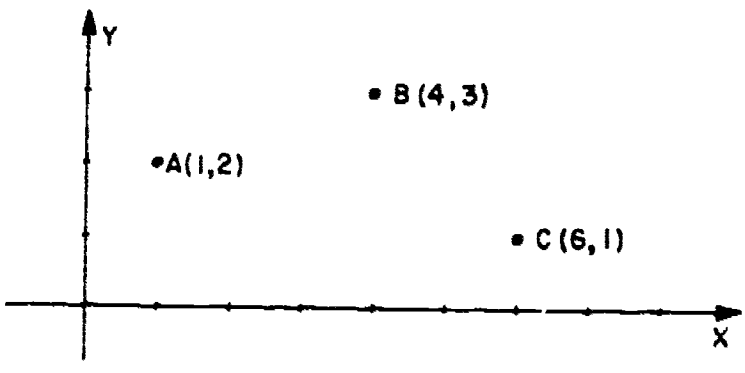
---

3. Vectors and Scalars; Components.

The main topic of this section is the algebra of vectors that are given in the component form [p,q] . The transition from coordinates (of points) to components (of vectors) is a little subtle. Once the change-over is made, the algebraic properties of vectors are easily established.

Exercises 3. Answers.

1.



(a) Let (a,b) be X .
Then $\overrightarrow{AB}$ is [(4 - 1) , (3 - 2)] .
Then $\overrightarrow{CX}$ is [(a - 6) , (b - 1)] .
Since $\overrightarrow{AB} \doteq \overrightarrow{CX}$,

$$a - 6 = 4 - 1 , \quad b - 1 = 3 - 2$$
$$a = 9 \qquad b = 2 .$$

The coordinates of X are (9,2) .

(b) $a - 1 = 4 - 6$, $b - 2 = 3 - 1$

$a = -1$ $\qquad$ $b = 4$

$X(-1,4)$

(c) $1 - a = 4 - 6$, $2 - b = 3 - 1$

$a = 3$ $\qquad$ $b = 0$

$X(3,0)$

(d) $1 - a = 6 - 4$, $2 - b = 1 - 3$

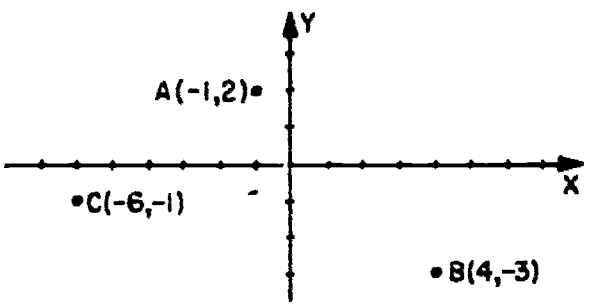
$a = -1$ $\qquad$ $b = 4$

$X(-1,4)$

2. (a) $4 - (-1) = a - (-6)$, $-3 - 2 = b - (-1)$

$a = -1$ $\qquad$ $b = -6$

$X(-1,-6)$



(b) $a - (-1) = 4 - (-6)$, $b - 2 = -3 - (-1)$

$a = 9$ $\qquad$ $b = 0$

$X(9,0)$

(c) $-1 - a = 4 - (-6)$, $2 - b = -3 - (-1)$

$a = -11$ $\qquad$ $b = 4$

$X(-11,4)$

(d) $X(9,0)$.

3. (a) $[3,2] + [4,1] = [(3 + 4) + (2 + 1)] = [7,3]$, by Theorem 3b.

(b) $[1,-1]$.

(c) $4[5,6] = [4 \cdot 5 , 4 \cdot 6] = [20,24]$, by Theorem 3c.

(d) $[-20,-24]$, by Theorem 3c.

(e) $[-5,-6]$, by Corollary of Theorem 3c.

(f) $[-5,-6]$.

(g) $3[4,1] + 2[-1,3] = [12,3] + [-2,6]$

$= [12 + (-2) , 3 + 6] = [10,9]$.

(h) $[14,-3]$.

4. (a) $x[3,-1] + y[3,1] = [5,6]$

$[3x,-x] + [3y,y] = [5,6]$

$[(3x + 3y) , (-x + y)] = [5,6]$

$\begin{cases} 3x + 3y = 5 \\ -x + y = 6 \end{cases}$

The solution set of the system is $\{(\frac{-13}{6},\frac{23}{6})\}$.

That is, $x = \frac{-13}{6}$ and $y = \frac{23}{6}$.

(b) The resulting system is,

$\begin{cases} 3x + 2y = 1 \\ 2x + 3y = 2 \end{cases}$ whose solution set is $\{(-\frac{1}{5},\frac{4}{5})\}$.

That is, $x = -\frac{1}{5}$ and $y = \frac{4}{5}$.

(c) $x = \frac{27}{13}$ and $y = \frac{8}{13}$.

(d) The solution set of the system

$\begin{cases} 3x + 6y = -3 \\ 2x + 4y = -2 \end{cases}$ is $\{(a,\frac{-a - 1}{2})\}$ for all real a .

For instance, one element of the solution set is $(3,\frac{-3 - 1}{2})$, or $(3,-2)$. Ask students to find other pairs of numbers which belong to the solution set. There will be an infinite number of such pairs.

5. (a) $[3,1] = a[1,0] + b[0,1]$

$[3,1] = [a,0] + [0,b]$

$[3,1] = [(a + 0) ,(0 + b)]$

$a + 0 = 3$ and $0 + b = 1$

$a = 3$ and $b = 1$

(b) $a = 1$ and $b = -3$.

(c) $\vec{1} = a[-3,1] + b[1,-3]$

$[1,0] = a[-3,1] + b[1,-3]$

$[1,0] = [-3a,a] + [b,-3b]$

$[1,0] = [(-3a + b) , (a - 3b)]$.

Hence a and b satisfy

$\begin{cases} -3a + b = 1 \\ a - 3b = 0 . \end{cases}$

We conclude that $a = -\frac{3}{8}$ and $b = -\frac{1}{8}$.

(d) $\vec{j} = a[-3,1] + b[1,-3]$

$[0,1] = a[-3,1] + b[1,-3]$.

Hence a and b satisfy

$$\begin{cases} -3a + b = 0 \\ a - 3b = 1 \end{cases}.$$

We conclude that $a = -\frac{1}{8}$ and $b = -\frac{3}{8}$.

6. $3\vec{i} - 2\vec{j} = a(3\vec{i} + 4\vec{j}) + b(4\vec{i} + 3\vec{j})$

$3\vec{i} + (-2)\vec{j} = (3a\vec{i} + 4a\vec{j}) + (4b\vec{i} + 3b\vec{j})$

$3\vec{i} + (-2)\vec{j} = (3a + 4b)\vec{i} + (4a + 3b)\vec{j}$.

Hence a and b satisfy

$$\begin{cases} 3a + 4b = 3 \\ 4a + 3b = -2 \end{cases}.$$

We conclude that $a = -\frac{17}{7}$ and $b = \frac{18}{7}$.

---

4. <u>Inner Product</u>.

The system of vectors before the inner product is introduced is not adequate to handle all of geometry. Only a few problems relating to angles and distance can be covered. The introduction of the inner product enriches vector algebra to the point that it is capable of being a completely adequate substitute for Euclidean Geometry.

The student is not likely to see these implications of the introduction of inner product. He should only be expected to compute them and to use them in the simple applications indicated.

Exercises 4. Answers.

1. Given $\vec{i} = [1,0]$ and $\vec{j} = [0,1]$.

   (a) $\vec{X} \cdot \vec{Y} = [1,0] \cdot [0,1] = 1 \cdot 0 + 0 \cdot 1 = 0$.
   (b) $[1,0] \cdot [1,0] = 1 \cdot 1 + 0 \cdot 0 = 1$.
   (c) $[0,1] \cdot [1,0] = 0$.
   (d) 1 .
   (e) 0 .
   (f) -7 .
   (g) -7 .
   (h) $ac + bd$.
   (i) $4a^2 + 4b^2$.
   (j) $sa^2 + sb^2$.

2. $\vec{X} \cdot \vec{Y} = |\vec{X}||\vec{Y}| \cos\theta$.

   (a) $2 \cdot 3 \cos\theta = 0$; therefore, $\theta = 90^\circ$.
   (b) 81.4° .
   (c) 109.5° .
   (d) 60° .
   (e) 131.8° .
   (f) 33.6° .
   (g) 0 .
   (h) 180° .

3. If $\vec{Y} \perp \vec{X}$ then $\vec{X} \cdot \vec{Y} = 0$.

   (a) $[3,4][a,4] = 0$
   $3a + 16 = 0$; $a = -\frac{16}{3}$.
   (b) $\frac{16}{3}$.
   (c) -3 .
   (d) 4 .

4. (a) $\sqrt{1^2 + 0^2}\sqrt{0^2 + 1^2} \cos\theta = 0$

   $\cos\theta = 0$; $\theta = 90^\circ$.
   (b) 0 .
   (c) 90° .
   (d) 0 .
   (e) 90° .
   (f) 107.6° .
   (g) 107.6° .

(h) $\cos\theta = \dfrac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$.

(i) $\sqrt{a^2 + b^2}\sqrt{16(a^2 + b^2)}\cos\theta = 4(a^2 + b^2)$

$\cos\theta = 1$

$\theta = 0^\circ$.

(j) 0° .

5. Note that $(c\vec{i} + d\vec{j}) \cdot (-d\vec{i} + c\vec{j}) = -cd + cd = 0$. Therefore since $c^2 + d^2 \neq 0$, $c\vec{i} + d\vec{j}$ is perpendicular to $-d\vec{i} + c\vec{j}$. A non-zero vector is perpendicular to one of these if and only if it is parallel to the other.

6. (a) Component of $\vec{Y}$ in the direction of $\vec{X}$ is $|\vec{Y}|\cos\theta$

$|\vec{Y}| = \sqrt{3^2 + 4^2} = 5$. To find θ we note two expressions of $\vec{X} \cdot \vec{Y}$.

(i) $\vec{X} \cdot \vec{Y} = |\vec{X}| \cdot |\vec{Y}| \cos\theta$ and

(ii) $\vec{X} \cdot \vec{Y} = x_1y_1 + x_2y_2$ where $\vec{X} = x_1i + x_2j$ and $\vec{Y} = y_1i + y_2j$.

From (i) and (ii) we have $x_1y_1 + x_2y_2 = |\vec{X}|\,|\vec{Y}| \cos\theta$

$1 \cdot 3 + 0 \cdot 4 = \sqrt{1^2 + 0^2} \cdot \sqrt{3^2 + 4^2} \cos\theta$

$3 = 5\cos\theta \longrightarrow \cos\theta = \frac{3}{5}$

Desired component $= 5 \cdot \frac{3}{5} = 3$.

(b) Using same plan as in Part (a) we obtain $4 = 5\cos\theta$.

$\therefore$ Component of $\vec{Y}$ in direction $\vec{X} = 4$.

(c) $3 \cdot 1 + 4 \cdot 0 = 5 \cdot 1 \cos\theta$. $\cos\theta = \frac{3}{5}$.

$|\vec{Y}| \cos\theta = 1 \cdot \frac{3}{5} = \frac{3}{5}$.

(d) $3 \cdot 0 + 4 \cdot 1 = 5 \cdot 1 \cos\theta$. $\cos\theta = \frac{4}{5}$.

$|\vec{Y}| \cos\theta = \frac{4}{5}$.

(e) $3 \cdot 3 + 4 \cdot 4 = 5 \cdot 5 \cos\theta$. $\cos\theta = 1$.

component $= 5 \cdot 1 = 5$.

(f) $15 + 8 = 5 \cdot \sqrt{29} \cos\theta$. $\cos\theta = \dfrac{23}{5\sqrt{29}}$

$|\vec{Y}|\cos\theta = \sqrt{29} \cdot \dfrac{23}{5\sqrt{29}} = \dfrac{23}{5} = 4.6$

(g) $3a + 4b = 5 \cdot \sqrt{a^2 + b^2} \cdot \cos\theta$

$\dfrac{3a + 4b}{5\sqrt{a^2 + b^2}} = \cos\theta$

$|\vec{Y}| \cos\theta = \dfrac{3a + 4b}{5}$ = desired component.

(h) $pa + qb = \sqrt{p^2 + q^2} \cdot \sqrt{a^2 + b^2} \cos\theta$

Desired component $= \sqrt{a^2 + b^2} \cos\theta = \dfrac{pa + qb}{\sqrt{p^2 + q^2}}$.

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5. Applications of Vectors in Physics.

The main topic of this section is the use of vectors in solving certain problems of physics. The student does not have to know much in the way of physics to handle the material, but there are a few bits of information which are taken for granted in the problems (for instance, that the direction of a force transmitted by a cord must be along the line of the cord). Primarily the student should come to this work knowing about addition of vectors, scalar multiplication, and inner products. He should see how this knowledge can help him to learn substantial amounts of physics easily. For instance, forces in equilibrium can be discussed readily in vector language.

Two extreme points of view should be avoided.

(1) The student could get the impression that his knowledge of vectors makes him an expert physicist. This is not so. He needs to learn a little physics as well as vector algebra to solve these problems.

(2) The student could get the impression that in spite of his knowledge of vectors he is unable to solve the simple problems given here without a lot of supplementary study of physics. This is not so. He is given a few observations on forces, resultant of forces, forces in equilibrium, work, velocity. These should not be made to appear so formidable as to discourage him.

Exercises 5a. Answers.

1. $5\sqrt{2}$ lb.

2. $\vec{R} = (|\vec{R}| \cos 120^\circ , |\vec{R}| \sin 120^\circ)$

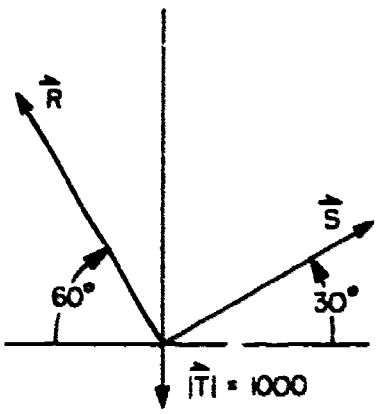
$= (-\frac{1}{2} |\vec{R}| , \frac{|\vec{R}|\sqrt{3}}{2})$.

$\vec{S} = (|\vec{S}| \cos 30^\circ , |\vec{S}| \sin 30^\circ)$

$= (\frac{|\vec{S}|\sqrt{3}}{2} , \frac{1}{2} |\vec{S}|)$.

$\vec{T} = (0, -1000)$.

$\vec{R} + \vec{S} + \vec{T} = \vec{0}$.



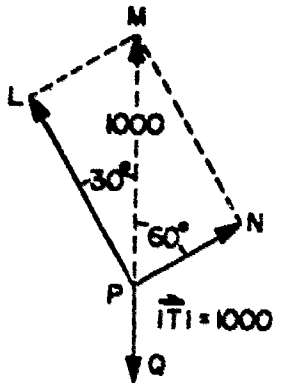
$(-\frac{1}{2} |\vec{R}| + \frac{|\vec{S}|\sqrt{3}}{2} , \frac{|\vec{R}|\sqrt{3}}{2} + \frac{1}{2} |\vec{S}| - 1000) = (0,0)$.

$|\vec{R}| = 500\sqrt{3} \approx 866$.

$|\vec{S}| = 500$.

The force of wire AC on C is approximately 866 pounds; the force of wire BC on C is 500 pounds; for equally strong wires, CW is more likely to break since the greatest force is on it and BC is least likely to break.

An alternate solution can be gained using "free" vectors, right triangles, and the resultant of $\vec{R}$ and $\vec{S}$ as shown in the sketch. Using the parallelogram law for the addition of the vectors, $\overrightarrow{PM}$ must be the hypotenuse of a $30^\circ - 60^\circ$ right triangle and have a length of 1000 units. Hence, $\overrightarrow{PN}$ which lies opposite the 30° angle has a length of 500 units. Similarly in right triangle LMP, $\overrightarrow{LP}$ lies opposite the 60° angle; it has a length of $500\sqrt{3}$ units.



3. Force in AC is $10000/\sqrt{3} \approx 5770$.

   Force in BC is $5000/\sqrt{3} \approx 2885$.

   Force in CW is 5000 pounds.

4. $\overrightarrow{OP} = (|\overrightarrow{OP}| \cos 23^\circ , |\overrightarrow{OP}| \sin 23^\circ)$

   $\overrightarrow{OQ} = (|\overrightarrow{OQ}| \cos 113^\circ , |\overrightarrow{OQ}| \sin 113^\circ)$ $\quad \overrightarrow{OW} = (0,-300)$

   $\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OW} = 0$; i.e., $(|\overrightarrow{OP}| \cos 23^\circ + |\overrightarrow{OQ}| \cos 113^\circ ,$

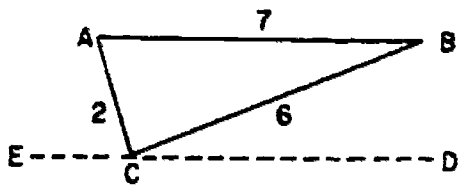
   $|\overrightarrow{OP}| \sin 23^\circ + |\overrightarrow{OQ}| \sin 113^\circ - 300) = (0,0)$

   Solving $|\overrightarrow{OP}| \approx 117$ and $|\overrightarrow{OQ}| \approx 276$.

5. From the Law of Cosines,

$$\cos B = \frac{7^2 + 6^2 - 2^2}{(2)(7)(6)}$$

$$= \frac{27}{28} \approx 0.971 .$$

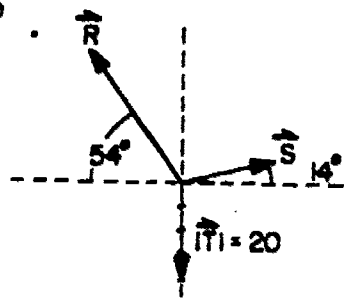


Angle B $\approx 14^\circ$ = angle BCD .

Also, $\cos C = \dfrac{2^2 + 6^2 - 7^2}{(2)(2)(6)} = -\dfrac{3}{8} \approx -0.3750$, and $C \approx 112^\circ$.

Angle ACE $\approx 180^\circ - (14^\circ + 112^\circ) = 54^\circ$.

Forming a vector diagram in a coordinate system with a vector unit of 1 pound of force, we see that the components of the vectors are:



$\vec{R} = (|\vec{R}| \cos 54^\circ , |\vec{R}| \sin 54^\circ) \approx (-0.588 |\vec{R}| , 0.809 |\vec{R}|)$;

$\vec{S} = (|\vec{S}| \cos 14^\circ , |\vec{S}| \sin 14^\circ) \approx (0.970 |\vec{S}| , 0.242 |\vec{S}|)$;

$\vec{T} = (0, -20)$.

Since $\vec{R} + \vec{S} + \vec{T} = \vec{0}$, adding the left member vectors gives the equal vectors

$$(-0.588 |\vec{R}| + 0.970 |\vec{S}| , 0.809 |\vec{R}| + 0.242 |\vec{S}| - 20) = (0,0) .$$

Equating corresponding components, we have

$$-0.588 |\vec{R}| + 0.970 |\vec{S}| = 0 ,$$

and

$$0.809 |\vec{R}| + 0.242 |\vec{S}| = 20 .$$

Solving these equations simultaneously, we have

$$|\vec{R}| = \frac{0.970}{0.588} |\vec{S}| .$$

$$0.809 \left(\frac{0.970}{0.588}\right) |\vec{S}| + 0.242 |\vec{S}| = 20 .$$

$$(1.334 + 0.242) |\vec{S}| = 20 .$$

$$|\vec{S}| = \frac{20}{1.576} \approx 12.7 .$$

$$|\vec{R}| = \frac{0.970}{0.588} (12.7) \approx 21.0 .$$

The force on wire AC is approximately 21 pounds; on wire BC , approximately 12.7 pounds. Wire AC is the one which is most likely to break.

6. The force on wire BC at C is $500\sqrt{3} \approx 866$ pounds; on AC at C , it is 1000 pounds; and on CW at C , it is 500 pounds.

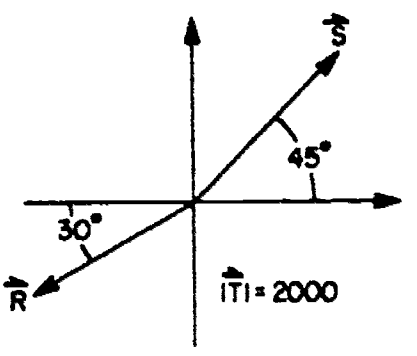
7. $\vec{R} = (|\vec{R}| \cos 210^\circ , |\vec{R}| \sin 210^\circ)$

$= (-\frac{\sqrt{3}}{2}|\vec{R}| , -\frac{1}{2} |\vec{R}|)$.

$\vec{S} = (|\vec{S}| \cos 45^\circ , |\vec{S}| \sin 45^\circ)$

$= (\frac{1}{\sqrt{2}} |\vec{S}| , \frac{1}{\sqrt{2}} |\vec{S}|)$.

$\vec{T} = (0, -2000)$.



Since $\vec{R} + \vec{S} + \vec{T} = \vec{0}$, addition gives two equal vectors; thus

$$\left(-\frac{\sqrt{3}}{2}\,|\vec{R}| + \frac{1}{\sqrt{2}}\,|\vec{S}|\ ,\ -\frac{1}{2}\,|\vec{R}| + \frac{1}{\sqrt{2}}\,|\vec{R}| - 2000\right) = (0,0)\ .$$

Equating corresponding components gives the following pair of simultaneous equations:

$$-\frac{\sqrt{3}}{2}\,|\vec{R}| + \frac{1}{\sqrt{2}}\,|\vec{S}| = 0\ ,$$

$$-\frac{1}{2}\,|\vec{R}| + \frac{1}{\sqrt{2}}\,|\vec{S}| - 2000 = 0\ .$$

In the first equation,

$$|\vec{R}| = \frac{2}{\sqrt{6}}\,|\vec{S}|\ .$$

Using this value in the second equation, we obtain

$$\left(-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}\right)|\vec{S}| = 2000\ .$$

Hence, $$|\vec{S}| = \frac{\sqrt{12}\,(2000)}{6 - 2} \approx 6750$$

$$|\vec{R}| = \frac{\sqrt{2}\,(4000)}{\sqrt{6} - \sqrt{2}} \approx 5465\ .$$

8. The vectors are placed in a coordinate system using 1,000 pounds of force as a convenient vector unit. The vector components are as follows:

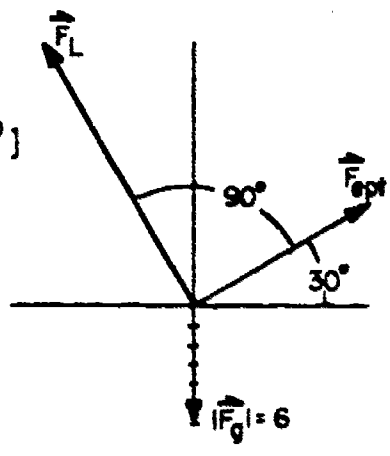
$$\overrightarrow{F_{ept}} = [\,|\overrightarrow{F_{ept}}|\ \cos 30^\circ\ ,\ |\overrightarrow{F_{ept}}|\ \sin 30^\circ\,]$$

$$\overrightarrow{F_{ept}} = \left[\frac{|\overrightarrow{F_{ept}}|\sqrt{3}}{2}\ ,\ \frac{|\overrightarrow{F_{ept}}|}{2}\right];$$

$$\vec{F_L} = [\,|\vec{F_L}|\ \cos 120^\circ\ ,\ |\vec{F_L}|\ \sin 120^\circ\,]$$

$$= \left[\frac{-|\vec{F_L}|}{2}\ ,\ \frac{|\vec{F_L}|\sqrt{3}}{2}\right];$$

$$\vec{F_g} = (0,-6)\ .$$



Since the airplane is moving in a straight line at constant speed,

$$\overrightarrow{F_{ept}} + \overrightarrow{F_L} + \overrightarrow{F_g} = 0 .$$

Adding the vectors in the left member we obtain

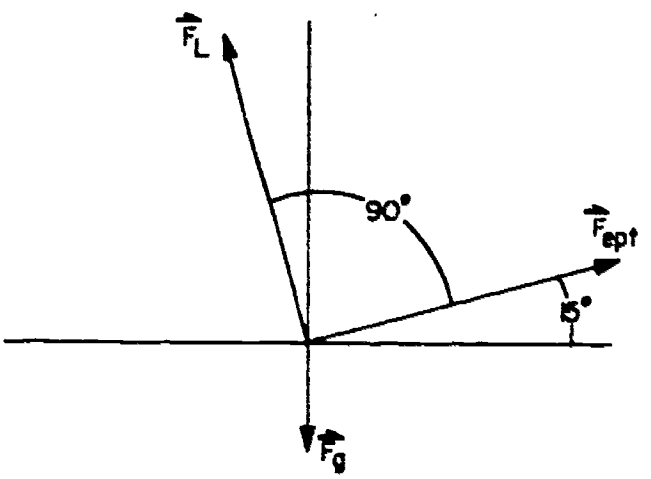
$$\left[\frac{|\overrightarrow{F_{ept}}|\sqrt{3}}{2} - \frac{|\overrightarrow{F_L}|}{2} + 0 , \frac{|\overrightarrow{F_{ept}}|}{2} + \frac{|\overrightarrow{F_L}|\sqrt{3}}{2} - 6 \right] = (0,0) .$$

Equating corresponding components and solving the equations simultaneously, we obtain

$$|\overrightarrow{F_{ept}}| = 3.000 , \text{ and } |\overrightarrow{F_L}| = 3\sqrt{3} \approx 5.196 .$$

Hence, the effective propeller thrust is 3000 pounds and the lift force is approximately 5196 pounds.

9.



$$\overrightarrow{F_L} = (|\overrightarrow{F_L}| \cos 105^\circ , |\overrightarrow{F_L}| \sin 105^\circ)$$

$$= (-0.26 |\overrightarrow{F_L}| , 0.97 |\overrightarrow{F_L}|) .$$

$$\overrightarrow{F_{ept}} = (|\overrightarrow{F_{ept}}| \cos 15^\circ , |\overrightarrow{F_{ept}}| \sin 15^\circ)$$

$$= (0.97 |\overrightarrow{F_{ept}}| , 0.26 |\overrightarrow{F_{ept}}|) .$$

$$\overrightarrow{F_g} = (0.-10,000) .$$

Since $\vec{F_L} + \vec{F_{ept}} + \vec{F_g} = \vec{0}$, we have the two simultaneous equations:

$$(1) \quad -0.26\ |\vec{F_L}| + 0.97\ |\vec{F_{ept}}| = 0\ ,$$

$$(2) \quad 0.97\ |\vec{F_L}| + 0.26\ |\vec{F_{ept}}| = 10{,}000\ .$$

Solving this system of equations, we get

$$|\vec{F_L}| \approx 3.7\ |\vec{F_{ept}}|\ ,$$

and $$|\vec{F_{ept}}| = 2{,}600 \text{ pounds.}$$

$$|\vec{F_L}| = 9{,}500 \text{ pounds.}$$

10. $\vec{F_L} = (|\vec{F_L}| \cos 100^\circ\ ,\ |\vec{F_L}| \sin 100^\circ) \approx (-0.174\ |\vec{F_L}|\ ,\ 0.985\ |\vec{F_L}|)$

$|\vec{F_d}| = (|\vec{F_d}| \cos 10^\circ\ ,\ |\vec{F_d}| \sin 10^\circ) \approx (0.985\ |\vec{F_d}|\ ,\ 0.174\ |\vec{F_d}|)$

$\vec{F_g} = (0,-500)$.

$-0.174\ |\vec{F_L}| + 0.985\ |\vec{F_d}| = 0$.

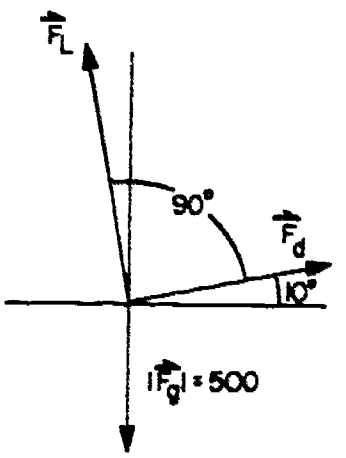
$0.985\ |\vec{F_L}| + 0.174\ |\vec{F_d}| - 500 = 0$.

$|\vec{F_L}| = \frac{0.985}{0.174}\ |\vec{F_d}|$.

$(0.985)\ (\frac{0.985}{0.174}\ |\vec{F_d}| + 0.174\ |\vec{F_d}| = 500$

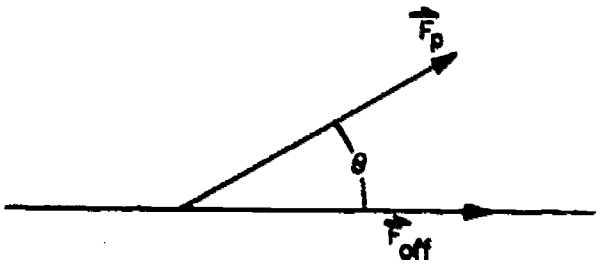
$|\vec{F_d}| = \frac{500}{5.56 + 0.174} \approx 87.3$.

$|\vec{F_L}| \approx \frac{0.985}{0.174}\ (87.3) \approx 494$.



Exercises 5b. Answers.

1.



$$\overrightarrow{F_{ebb}} = \vec{F_p} \cos \theta$$

$$W = d \cdot \overrightarrow{F_{ebb}}$$

(a) $\overrightarrow{F_{ebb}} = 10 \cos 10^\circ \approx 10 \times .985 = 9.85$ lb.

Work $= d \cdot \overrightarrow{F_{ebb}} \approx 10 \times 9.85 = 98.5$ lb.

(b) $W = 100 \cdot 10 \cdot \cos 20^\circ \approx 1000 \cdot .940 = 940$ ft. lb.

(c) $W = 8660$ ft. lb.

(d) $W = d \cdot 10 \cos 10^\circ$

$1000 \approx d \cdot 10 \cdot .985$

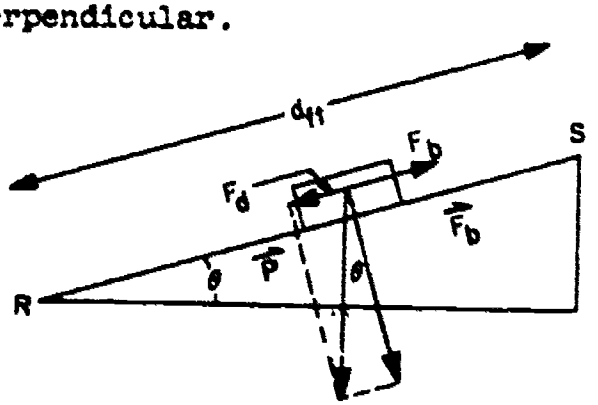
$d = \frac{1000}{9.85} = 101.5$ ft.

(e) $d \approx \frac{1000}{100 \cdot .940} = \frac{100}{94} = 10.6$ ft.

(f) $d = \frac{1000}{100 \cdot \cos 0^\circ} = 10$ ft.

(g) $d = \frac{1000}{100 \cos 89^\circ} = \frac{1000}{100 \cdot .0175} = \frac{1000}{1.75} \approx 571.4$ ft.

2. $\angle$ between P and $F_b = \theta$ since sides of $\angle$S are mutually perpendicular.



(a) $|\vec{F_1}| = |\vec{P}| \cos(\frac{\pi}{2} - \theta)$

$|\vec{F_d}| = |\vec{P}| \sin\theta \quad \vec{F_b} = -\vec{F_d}$

$W = \vec{F_b} \cdot \vec{d}$

$W = d\,\vec{P} \sin\theta$ (Note that this is equivalent to left-P in a vertical direction from R to S .)

$W = 10 \cdot 10 \cdot \sin 10^\circ$

$W \approx 100 \cdot .174 = 17.4$ ft. lb.

(b) $W = 342$ ft. lb.

(c) $W = 500$ ft. lb.

(d) $d = \dfrac{W}{P \sin\theta}$

$d = 575$ ft.

(e) $d = 292$ ft.

(f) $d = 571$ ft.

(g) $d = 10$ ft.

Exercises 5c. Answers.

1. ≈ 1.8 miles.

2. From Figure (a) we determine the angle which the path of the boat makes with the shore line ($\angle\alpha$) and the speed of the boat along OQ . Let length of OQ = d .

$d^2 = 1.3^2 + 0.5^2$

$d = \sqrt{1.94} = 1.39$ = distance traveled.

The boat covers the distance d in 25 minutes. Hence if s is the speed of the boat along OQ

$s \cdot \frac{5}{12} = 1.39$

$s = 3.34$.

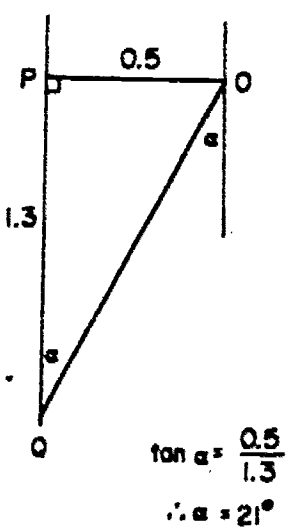


Figure (a)

Figure (b) is our force diagram. We have

$$\overrightarrow{OT} + \overrightarrow{TR} = \overrightarrow{OR}$$

$|\overrightarrow{OT}| = 4$, $|\overrightarrow{OR}| = 3.34$

and $\angle TOR = 21^\circ$.

By the Cosine Law

$$|\overrightarrow{TR}|^2 = 4^2 + (3.34)^2 - 2 \cdot 4 \cdot 3.34 \cos 21^\circ$$

$$|\overrightarrow{TR}| \approx 1.52 .$$

By the Sine Law applied to ΔRTO

$$\frac{3.34}{\sin \theta} = \frac{1.52}{\sin 21^\circ} . \quad \theta \approx 51^\circ$$

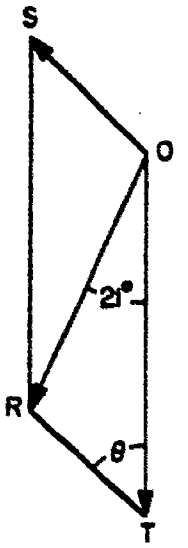
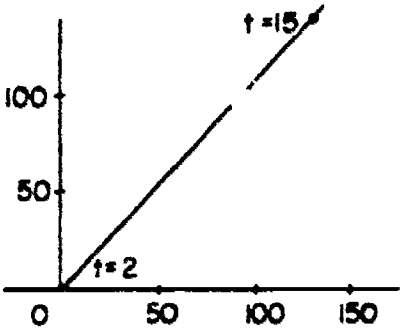


Figure (b)

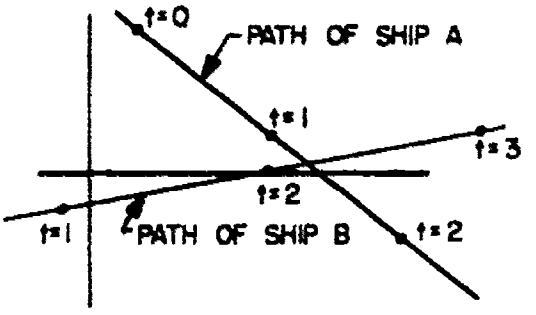
3. $\sqrt{37} \approx 6.08$ miles per hour.

4. Since the velocity is constant, in one second the body will reach the point (2,1.5) . Thus, the velocity vector is $2\vec{i} + 1.5\vec{j}$. The velocity of the body is 200 feet per second to the right, and 150 feet per second upward. Its speed is 250 feet per second.

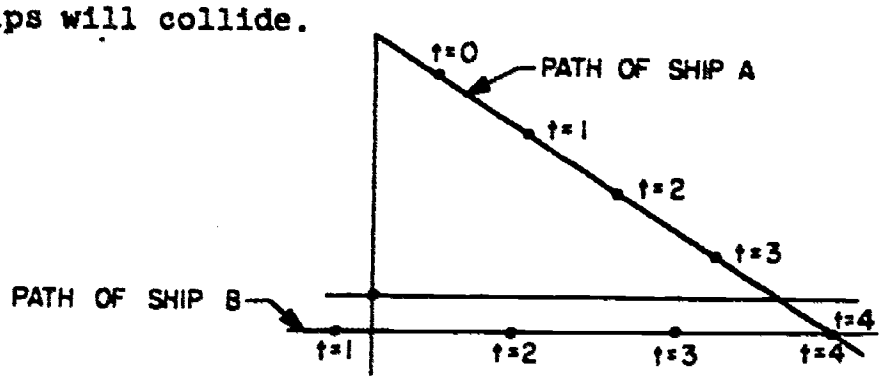
5. At $t = 15$ the body is at the point (130,131) . Thus, it has moved 1300 miles to the right, and 1310 miles upward.



6. Since ship B does not cross the wake of ship A until after $t = 2$, the ships will not collide.



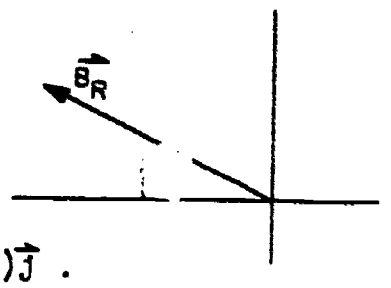
7. Since both ships are at the point $(14,-1)$ when $t = 4$, the ships will collide.



8. We compute the displacements that would result from one hour of travel. Thus

$$\vec{R_L} = -4\vec{j} ,$$

$$\vec{B_R} = -3(\cos\theta)\vec{i} + 3(\sin\theta)\vec{j} .$$

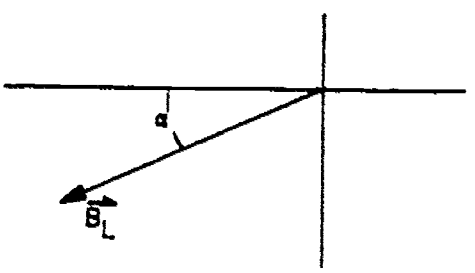


Consequently,

$$\vec{B_L} = \vec{R_L} + \vec{B_R}$$

$$= (-3\cos\theta)\vec{i} + (3\sin\theta - 4)\vec{j} .$$

The scalar components of $\vec{B_L}$ are both negative. This means that the boat will actually be carried downstream. The situation is illustrated by the diagram below.



In order to drift downstream as little as possible θ must be determined so that $\tan\alpha$ is minimum:

$$\tan\alpha = \frac{-3\sin\theta + 4}{3\cos\theta} .$$

This problem can be handled easily by using calculus. However, by making use of a table or graph we can obtain an approximate solution without using calculus. Thus, the smallest value of $\tan \alpha$ occurs for

$$\sin \theta = \frac{3}{4} ,$$

$$\theta \approx 49^\circ .$$

For this value of θ ,

$$\vec{B}_L = -3(.66)\vec{i} + (3 \cdot \tfrac{3}{4} - 4)\vec{j}$$

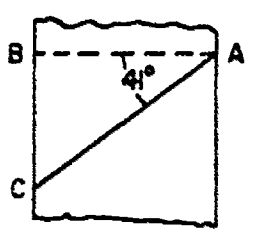
$$= -1.98\vec{i} - 1.75\vec{j} .$$

The corresponding value of α is given by

$$\tan \alpha = \frac{-3\,(\frac{3}{4}) + 4}{\frac{3\sqrt{7}}{4}} = \frac{7}{3\sqrt{7}} = 0.88 .$$

$$\approx 41^\circ .$$

Traveling in this direction, the boat will land at C .



Thus $\overline{AB} = \frac{1}{2}$ mile,

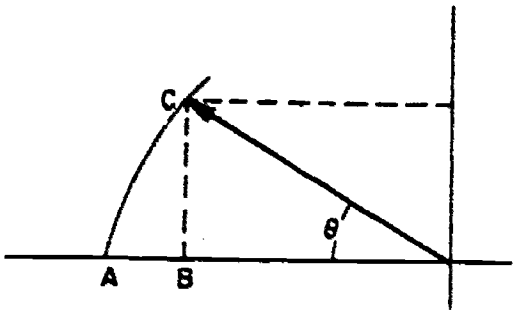
$$\overline{BC} = \frac{1}{2} \tan \alpha = \frac{1}{2} \cdot (.88) = 0.44 \text{ miles.}$$

Therefore, the boat must be carried <u>at least</u> 0.44 miles downstream. Another way of saying the same thing is that C is the <u>farthest</u> point <u>upstream</u> at which the man can land the boat.

The intuitive meaning of this problem is quite subtle. Let us consider the effect of different values of θ . Evidently if $\theta < 0$, then the man is using a component of his rowing <u>to help the current sweep him downstream</u>. This is the very opposite of what he wishes to do.

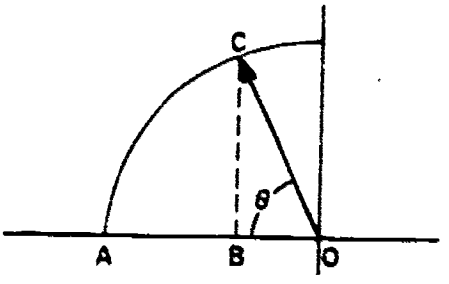
Hence, a wise choice of θ requires $0 \leq \theta \leq \frac{\pi}{2}$.

It might seem sensible to head straight for the opposite shore; i.e., to choose $\theta = 0$. Let us examine this possibility carefully.



If θ is chosen so that it is about 49°, then the man will have sacrificed a component $\overrightarrow{BA}$ which would carry him to the opposite shore, but he will have gained a much larger component $\overrightarrow{BC}$ which is keeping him from being swept downstream. For $\theta \approx 49^\circ$ he is crossing almost as fast as he would be for $\theta = 0$, but he is not being swept downstream so rapidly. It is a good bargain.

What would happen if the man sacrificed even <u>more</u> of the crossing component in order to gain a larger component working against the current? Suppose he chooses $\theta = 70^\circ$. In doing so he sacrifices a crossing component of $\overrightarrow{BA}$ in order to gain the component $\overrightarrow{BC}$ which opposes the current. The price is too great, however.

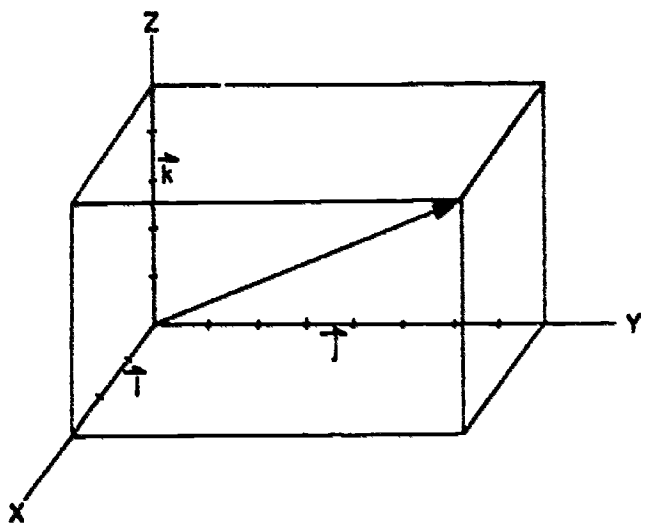


Even though the man is not being swept downstream <u>so rapidly</u>, he will actually be swept farther downstream. This is true because the crossing component $\overrightarrow{OB}$ is now very small; consequently, it takes him a long time to cross. During this time, he is swept, slowly but surely, a long way down the stream.

Finding the optimum value of θ is, therefore, a matter of compromise; it is motivated by a desire to oppose the current as much as possible, without slowing progress toward the opposite shore more than a little.

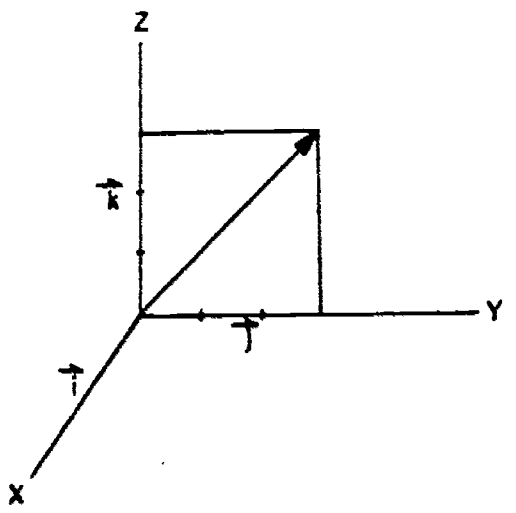
Exercises 5d. Answers.

1. (a)



$3\vec{i} + 8\vec{j} + 5\vec{k}$

(b)



$3\vec{j} + 3\vec{k}$

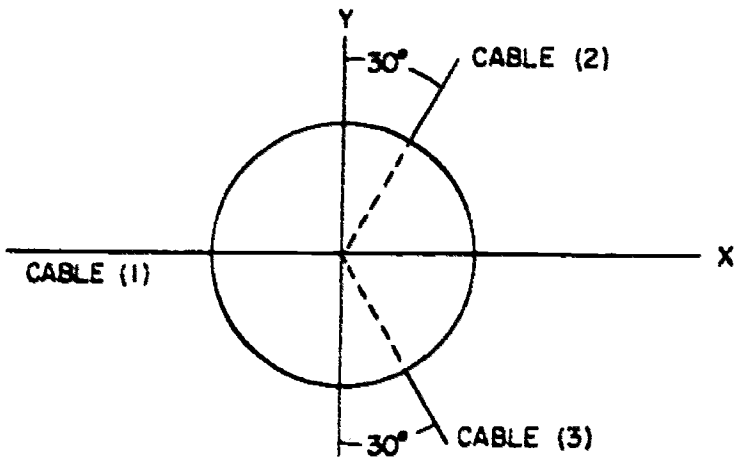
Since the graphs for each of the remaining parts of this problem are similar to (a) and (b), they have been omitted.

2. (a) 16 .
   (b) 10 .
   (c) 0 .
   (d) 0 .
   (e) 0 .

3. (a) $\frac{16}{3\sqrt{29}}$.
   (b) $\frac{10}{\sqrt{29}\,\sqrt{12}}$.
   (c) 0 .
   (d) 0 .
   (e) 0 .

4. 0 .

5. We shall give two solutions to this problem.
   <u>First solution</u>: Let us first find vectors having the directions of the suspending cords. By orienting axes appropriately we obtain the top view represented in the following diagram.



Let $\vec{A}$ be a vector which is parallel to cable (1). Then the vector $-\vec{i}$ makes an angle of 30° with $\vec{A}$, an angle of 60° with $\vec{k}$, and an angle of 90° with $\vec{j}$. Let $\vec{A} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$. If we "dot" $\vec{j}$ into both sides of this equation, we get

$$\begin{aligned}\vec{j}\cdot\vec{A} &= \vec{j}\cdot(a_x\vec{i} + a_y\vec{i} + a_z\vec{k})\\ &= a_x(\vec{j}\cdot\vec{i}) + a_y(\vec{j}\cdot\vec{j}) + a_z(\vec{j}\cdot\vec{k})\\ &= a_x(0) + a_y(1) + a_z(0)\\ &= a_y\ .\end{aligned}$$

Now, if we choose $|\vec{A}| = 1$, we get

$$a_y = \vec{j} \cdot \vec{A} = |\vec{j}| \cdot |\vec{A}| \cos 90^\circ$$
$$= 0 .$$

Proceeding similarly, we have

$$\vec{i} \cdot \vec{A} = a_x = \cos 150^\circ = - \cos 30^\circ$$
$$= -\frac{\sqrt{3}}{2} .$$

$$\vec{k} \cdot \vec{A} = a_z = \cos 60^\circ = \frac{1}{2} .$$

Hence,

$$\vec{A} = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k} .$$

We now seek a vector $\vec{B}$ which is parallel to cable (2). Let us first find a vector $\vec{u}$ of length one, which lies in the xy-plane directly under cable (2); (i.e., $\vec{u}$ points along the noon-day shadow of cable (2)).

Evidently,

$$\vec{u} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j} .$$

Now, $\vec{B}$ lies in the plane of $\vec{k}$ and $\vec{u}$. Hence, we can use $\vec{k}$ and $\vec{u}$ as basis vectors. Thus,

$$\vec{B} = b_1\vec{k} + b_2\vec{u} .$$

To find b_1 and b_2 , we proceed as before. Let $|\vec{B}| = 1$.

$$\vec{k} \cdot \vec{B} = b_1 = \cos 60^\circ = \frac{1}{2} .$$

$$\vec{u} \cdot \vec{B} = b_2 = \cos 30^\circ = \frac{\sqrt{3}}{2} .$$

Consequently,

$$\vec{B} = \frac{1}{2}\vec{k} + \frac{\sqrt{3}}{2}\vec{u}$$
$$= \frac{1}{2}\vec{k} + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right)$$
$$= \frac{\sqrt{3}}{4}\vec{i} + \frac{3}{4}\vec{j} + \frac{1}{2}\vec{k} .$$

By symmetry, we can see that $\vec{C}$, the unit vector parallel to cable (3), must be

$$\vec{C} = \frac{\sqrt{3}}{4}\vec{i} - \frac{3}{4}\vec{j} + \frac{1}{2}\vec{k} .$$

The forces are equal in each cable. Let one unit of length of vector represent one pound of force. Then, since the cables are flexible and can transmit only forces parallel to themselves, we have

$$\vec{F}_1 = c\vec{A} ,$$
$$\vec{F}_2 = c\vec{B} ,$$
$$\vec{F}_3 = c\vec{C} .$$

We can now find the scalar c . The total upward component is

$$c(\tfrac{1}{2}\vec{k} + \tfrac{1}{2}\vec{k} + \tfrac{1}{2}\vec{k}) = c\,\tfrac{3}{2}\vec{k} .$$

But the total upward component must balance the downward force of gravity. Consequently,

$$\frac{3}{2}c = 15 ,$$
$$c = 10 .$$

Thus, $\vec{F}_1 = 10\,\vec{A}$.

Since $|\vec{A}| = 1$, it follows that $|\vec{F}_1| = 10$. Therefore, there is a tension of 10 pounds in each cable.

<u>Second Solution</u>: Begin exactly as you did in the first solution, but notice that once we have

$$\vec{A} = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k} ,$$

we know that

$$\vec{B} = b_x\vec{i} + b_y\vec{j} + \frac{1}{2}\vec{k} ,$$
$$\vec{C} = c_x\vec{i} + c_y\vec{j} + \frac{1}{2}\vec{k} .$$

<u>We do not need to find x and y components</u>. It suffices to work only with vertical components.

As before,

$$\vec{F}_1 = c\vec{A} ,$$
$$\vec{F}_2 = c\vec{B} ,$$
$$\vec{F}_3 = c\vec{C} ;$$

and we get

$$\tfrac{3}{2}c = 15 ,$$
$$c = 10 ,$$
$$|\vec{F}_1| = 10 .$$

Hence, we again find that each cord exerts a force of 10 pounds on the lighting fixture.

6. Let us choose axes so that the xy-plane is horizontal, with $\vec{j}$ pointing north and $\vec{i}$ pointing east. The three vectors we need to consider are as follows:

   $\vec{A}_G$ (representing the velocity of the airplane with respect to the ground);

   $\vec{A}_W$ (representing the velocity of the airplane with respect to the air); and

   $\vec{W}_G$ (representing the velocity of the wind with respect to the ground).

We know from physics that

$$\vec{A}_G = \vec{A}_W + \vec{W}_G .$$

Now, $\vec{A}_W = 100[(\cos 30^\circ)\vec{j} + (\cos 60^\circ)\vec{k}]$

$= 50\sqrt{3}\,\vec{j} + 50\vec{k}$;

also, $\vec{W}_G = 30\vec{i}$.

Consequently,

$$\vec{A}_G = 30\vec{i} + 50\sqrt{3}\,\vec{j} + 50\vec{k} .$$

The upward component $50\vec{k}$ does not appear in the ground speed. In fact, the ground speed is

$$|\vec{A}_G - 50\vec{k}| = \sqrt{30^2 + (50\sqrt{3})^2}$$
$$= \sqrt{8400}$$
$$\approx 92 \text{ miles per hour.}$$

7. Evidently, the pilot will achieve the <u>fastest</u> ground speed if his heading is <u>with the wind</u>. Using the notation employed in Problem 6, we have

$$\vec{A}_W = 50\sqrt{3}\,\vec{i} + 50\vec{k}\ ,$$

$$\vec{A}_G = (30 + 50\sqrt{3})\vec{i} + 50\vec{k}\ ,$$

$$|\vec{A}_G - 50\vec{k}| = \sqrt{(30 + 50\sqrt{3})^2}$$

117 m.p.h.

Similarly, the <u>smallest</u> ground speed will be achieved if the pilot heads into the wind; in this case the ground speed will be

$$|\vec{A}_G - 50\vec{k}| = \sqrt{(50\sqrt{3} - 30)^2}$$

57 m.p.h.

8. The proof is analogous to the one for two dimensions.

9. $7\vec{i} - 3\vec{j} + 5\vec{k}$.

10. $\frac{1}{\sqrt{170}}$.

11. $\frac{1}{\sqrt{14}}$.

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6. <u>Vectors as a Formal Mathematical System</u>.

The main topic of this section is the solution of a problem. To teach this section successfully the teacher must do more than solve the problem. He must help the student understand what the problem is and also help him understand that which is offered as a solution of the problem really solves the problem.

First, let us consider what the problem is. We learned that vectors obey certain rules. We ask whether vectors are the only objects which obey these rules. The answer is certainly "no," since forces and velocities also obey them. The question which we propose is whether any system of objects which obeys these rules can be correctly treated as a system of vectors--whether it

is "essentially the same" as our system of vectors. We answer this question by proving that any system which obeys Rules 1 - 11 is <u>isomorphic</u> to our system of vectors.

<u>Exercises 6. Answers.</u>

1. Yes.

2. The system obeys the rules 1, 2, 3, 4, 5, 6, 8, 9, 11, but not 7 and not 10 .

   The left member of Rule 7 becomes

$$r \odot (s \odot (a,b)) = r \odot (\tfrac{sa}{2}, \tfrac{sb}{2})$$

$$= (\frac{r(\frac{sa}{2})}{2}, \frac{r(\frac{sb}{2})}{2})$$

$$= (\frac{rsa}{4}, \frac{rsb}{4})$$

   and the right member of Rule 7 becomes

$$(rs) \odot (a,b) = (\tfrac{rs}{2}a, \tfrac{rs}{2}b) .$$

   These are not equal.

   The left member of Rule 10 becomes

$$1 \odot (a,b) = (\tfrac{a}{2}, \tfrac{b}{2}) .$$

   The right member of Rule 10 becomes (a,b) . These are not equal.

3. This system obeys rules 1, 2, 5, 6, 7, 9, 10, 11, but not 3 , not 4 , and not 8 .

The left member of Rule 3 becomes

$$(a,b) \oplus ((c,d) \oplus (e,f)) = (a,b) \oplus \left(\frac{c+e}{2}, \frac{d+f}{2}\right)$$

$$= \left(\frac{2a+c+e}{4}, \frac{2b+d+f}{4}\right) .$$

The right member of Rule 3 becomes

$$((a,b) \oplus (c,d) + (e,f) = \left(\frac{a+c}{2}, \frac{b+d}{2}\right) \oplus (e,f)$$

$$= \left(\frac{a+c+2e}{2}, \frac{b+d+2f}{2}\right) .$$

These are not equal.

The left member of Rule 4 becomes

$$(a,b) \oplus (x,y) = \left(\frac{a+x}{2}, \frac{b+y}{2}\right) .$$

The right member of Rule 4 is (a,b) .

These two are equal if and only if $x = a$ and $y = b$.

Therefore there is no single (x,y) such that for all (a,b) ,

$$(a,b) \oplus (x,y) = (a,b) .$$

The left member of Rule 8 becomes

$$(r + s) \odot (a,b) = ((r + s)a , (r + s)b) .$$

The right member of Rule 8 becomes

$$(r \odot (a,b)) \oplus (s \odot (a,b)) = (ra,rb) \oplus (sa,sb)$$

$$= \left(\frac{ra+sa}{2}, \frac{rb+sb}{2}\right) .$$

These are not equal.