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AUTHOR Dubisch, Roy, Ed.
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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include periodicity, graphs, angles, vectors, formulas, tables, waves, and applications. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-2

SUPPLEMENTARY and ENRICHMENT SERIES

CIRCULAR FUNCTIONS

Edited by Roy Dubisch

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

Professor R. D. Anderson, Louisiana State University

Mr. M. Philbrick Bridgess, Roxbury Latin School, Westwood, Massachusetts

Professor Jean M. Calloway, Kalamazoo College, Kalamazoo, Michigan

Mr. Ronald J. Clark, St. Paul's School, Concord, New Hampshire

Professor Roy Dubisch, University of Washington, Seattle, Washington

Mr. Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Okla.

Mr. Karl S. Kalman, Lincoln High School, Philadelphia, Pennsylvania

Professor Augusta L. Schurrer, Iowa State Teachers College, Cedar Falls

Mr. Henry W. Syer, Kent School, Kent, Connecticut

CIRCULAR FUNCTIONS

This pamphlet is essentially the major portion of Chapter 5 of the SMSG text, Elementary Functions. A few minor changes have been made for clarity and to make the material self contained.

It is intended for use as a supplement to a standard trigonometry text that emphasizes the solution of triangles or as a unit in a course on elementary functions.

No previous knowledge of trigonometry is assumed but a background of a course in plane geometry and two years of algebra are prerequisites for the study of this material. In addition it is assumed that the student is familiar with the concept of a function as presented, for example, in the SMSG pamphlet on functions or in the SMSG text, Elementary Functions.

CIRCULAR FUNCTIONS

CIRCULAR FUNCTIONS

1. Circular Motions and Periodicity.

Introduction. From your earliest years you have been aware of motion and of change in the world around you. The rolling of a marble along a crack in the sidewalk, the flight of a ball tossed by a boy at play, the irregular rise and fall of a piece of paper fluttering in the breeze, the zig-zag course of a fish swimming erratically in a tank of water are a few of the varied patterns of movement you can observe. Very often, however, the motions you see have a quality not shared by the few just mentioned. The succession of day and night, the changing of the seasons, the rise and fall of the tides, the circulation of blood through your heart, the passage of the second hand on your watch over the 6 o'clock mark are patterns each having the characteristic quality that the motion involved repeats itself over and over at a regular interval. The measure of this interval is called the *period* of the motion, while the motion itself is called *periodic*.

The simplest periodic motion is that of a wheel rotating on its axle. Each complete turn of the wheel brings it back to the position it held at the beginning. After a point of the wheel traverses a certain distance in its path about the axle, it returns to its initial position and retraces its course again. The distance traversed by the point in a complete cycle of its motion is again a period, a period measured in units of length instead of units of time. If it should happen that equal lengths are traversed in equal times, the motion becomes periodic in time as well and the wheel can be used as a clock.*

The mathematical analysis of periodic phenomena is a vast and growing field, yet even in the most far-flung applications of the subject, such phenomena are analyzed essentially in terms of the simple periodicity of the path of a point describing a circle. In the treatment of the most intricate of periodicities, wheel motions always lie under the surface. An extended development of the theory of periodic phenomena is far beyond the scope of this pamphlet, but the study of the fundamental circular periodicities is certainly within our reach.

Circular Motions. Let us consider first the mathematical aspects of the motion of a point P on a circle. For convenience we take the circle

*The concept of time itself is inextricably tied up with that of clock, a periodic device which measures off the intervals. It would seem then that periodicity lies at the deepest roots of our understanding of the natural universe. How one decides that a repetitive event recurs at equal intervals of time and can therefore be considered a clock is a profound and difficult problem in the philosophy of physics and does not concern us here. (See *Physics*, Vol. 1, pp. 9-17, Physical Science Study Committee, Cambridge, Massachusetts, 1957.)

$u^2 + v^2 = 1$, which has its center at the origin of the uv-plane, radius 1 and consequently circumference 2π. Now we consider a moving point P which starts at the point (1,0) on the u-axis and proceeds in a counterclockwise direction around the circle. We can locate P exactly by knowing the distance x which it has traveled along the circle from (1,0). The distance x is the length of an arc of the circle. Since every point on the circle $u^2 + v^2 = 1$ has associated with it an ordered pair of real numbers (u,v) as coordinates, we may say that the motion of the point P defines a function* ρ. With each non-negative arc length x, we associate an ordered pair of real numbers (u,v), the coordinates of P (Figure 1), that is,

$$\rho: \ x \rightarrow (u,v).$$

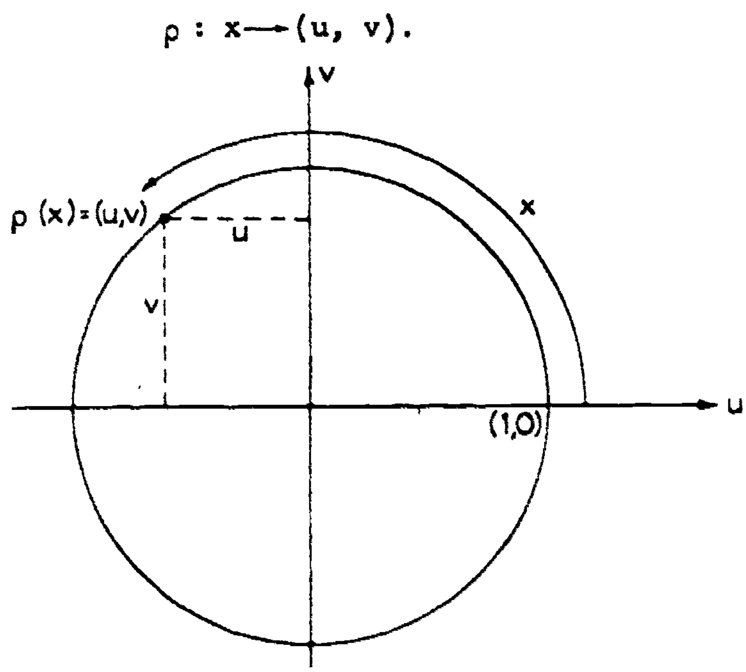

Figure 1. The function ρ.

However, it is inconvenient to work with a function whose range is a set of ordered pairs rather than single numbers. We shall instead define two functions as follows:

cos: $x \rightarrow u$, where u is the first component of $\rho(x)$;
sin: $x \rightarrow v$, where v is the second component of $\rho(x)$.

The terms cos and sin are abbreviations for cosine and sine. It is customary to omit parentheses in writing cos(x) and sin(x) and write simply

*See, for example, the SMSG pamphlet, Functions.

$\cos x$ and $\sin x$. For instance,

$$\begin{array}{lll} \rho(0) = (1,0) & : & \cos 0 = 1, \quad \sin 0 = 0 \\ \rho(\frac{\pi}{2}) = (0,1) & : & \cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1 \\ \rho(\pi) = (-1,0) & : & \cos \pi = -1, \quad \sin \pi = 0 \\ \rho(\frac{3\pi}{2}) = ? & : & \cos \frac{3\pi}{2} = ?, \quad \sin \frac{3\pi}{2} = ? \end{array}$$

(You should supply the proper symbols in place of the question marks.) From their mode of definition, the sine and cosine are called circular functions. These circular functions are related to but not identical with the familiar functions of angles studied in elementary trigonometry. We shall discuss the difference in Section 3, but we should notice now that when we write $\sin 2$, the 2 represents the real number 2 which can be thought of as the measure of the length of a circular arc and not 2 degrees.

Periodicity. From the definition of ρ, it follows that $\rho(x) = \rho(x + 2\pi)$ and consequently, $\cos x = \cos (x + 2\pi)$ and $\sin x = \sin (x + 2\pi)$. Functions which have this property of repeating themselves at equal intervals are said to be periodic. More generally, the function f is said to be periodic with period a, $a \neq 0$, if, for all x in the domain of f, $x + a$ is also in the domain and

$$f(x) = f(x + a). \qquad (1)$$

We usually consider the period of such a function as the smallest positive value of a for which (1) is true. The smallest positive period is sometimes called the fundamental period. From this definition we note that each successive addition or subtraction of a brings us back to $f(x)$ again. We may show this by first considering $f(x + 2a)$ where $a > 0$. We have

$$\begin{aligned} f(x + 2a) &= f\big((x + a) + a\big) \\ &= f(x + a) \\ &= f(x), \end{aligned}$$

and further

$$\begin{aligned} f(x + 3a) &= f\big((x + 2a) + a\big) \\ &= f(x + 2a) \\ &= f(x). \end{aligned}$$

In general, we have

$$f(x + na) = f(x) \quad \text{where} \quad n = 1, 2, 3, \ldots .$$

To show that this holds for negative n, we note that

$$\begin{aligned} f(x - a) &= f([x - a] + a) \\ &= f(x), \end{aligned}$$

$$
\begin{aligned}
f(x - 2a) &= f\big((x - 2a) + a\big) \\
&= f(x - a) \\
&= f(x).
\end{aligned}
$$

In general

$$f(x + na) = f(x) \quad \text{where} \quad n = -1, -2, -3, \ldots .$$

We may express these two ideas by

$$f(x + na) = f(x) \quad \text{where} \quad a > 0 \quad \text{and} \quad n \quad \text{is any integer.} \tag{2}$$

In other words, to determine all values of f, we need only know its values on the interval $0 \leq x \leq a$. Thus, suppose the period of f is $a = 2$ so that for all x in the domain of f

$$f(x + 2) = f(x).$$

Then to find $f(7.3)$ we write

$$
\begin{aligned}
f(7.3) &= f(1.3 + 3 \times 2) \\
&= f(1.3).
\end{aligned}
$$

To find $f(-7.3)$ we write

$$
\begin{aligned}
f(-7.3) &= f(0.7 - 4 \times 2) \\
&= f(0.7).
\end{aligned}
$$

Now returning to the unit circle, we observe that the functions cos and sin behave in exactly this way. From any point P on the circle, a further movement of 2π units around the circle $(a = 2\pi$ in Equation (2)$)$ will return us to P again. Thus the circular functions are periodic with period 2π, and consequently

$$
\left.
\begin{aligned}
\cos\,(x + 2n\pi) &= \cos x \\
\sin\,(x + 2n\pi) &= \sin x
\end{aligned}
\right. \tag{3}
$$

where n is any integer. To give meaning to these formulas for negative n, we interpret any clockwise movement on the circle as negative.

So now if we can determine values of cos and sin for $0 \leq x < 2\pi$, we shall have determined their values for all real x.

Exercises 1

1. Give five examples of periodic motion, and specify an approximate period for each. (For instance, the rotation of the earth about its own axis is periodic with period 24 hours.)

2. If $\rho(x + 2n\pi) = \rho(x)$, express each of the following as $\rho(b)$, where $0 \leq b < 2\pi$. (For example,

$$\rho(\frac{5\pi}{2}) = \rho(\frac{\pi}{2} + 2\pi) = \rho(\frac{\pi}{2}).$$

(a) $\rho(-\frac{\pi}{2})$ (c) $\rho(-\frac{3\pi}{2})$

(b) $\rho(3\pi)$ (d) $\rho(4076\pi)$

3. Give the coordinates of $\rho(x)$ for each part of Exercise 2 above.

4. Given that ρ has the period 2π, find two values of x where $0 \leq x < 4\pi$, such that

(a) $\rho(-\frac{\pi}{2}) = \rho(x)$; (c) $\rho(12\pi) = \rho(x)$;

(b) $\rho(13\pi) = \rho(x)$; (d) $\rho(-\pi) = \rho(x)$.

5. For what values of x, where $0 \leq x < 2\pi$, do the following relations hold?

(a) $\cos x = \sin x$ (b) $\cos x = -\sin x$

Hint: Use the fact that $(\cos x, \sin x)$ represents a point on the unit circle.

*6. We know that the functions represented by $\cos x$ and $\sin x$ have period 2π. Find the period of the functions represented by

(a) $\sin 2x$; (c) $\cos 4x$;

(b) $\sin \frac{1}{2}x$; (d) $\cos \frac{1}{2}x$.

*7. Let f and g be two functions with the same period a. Prove that:

(a) $f + g$ has a period a (not necessarily the fundamental period);

(b) $f \cdot g$ has a period a.

*8. Let f be a function with period a. Prove that the composition gf also has period a for any meaningful choice of a.

*9. Show that the functions sine and cosine have no positive period less than 2π.

---

2. Graphs of Sine and Cosine.

We wish now to picture the behavior of the two functions

$$\cos: x \to u = \cos x$$
$$\sin: x \to v = \sin x$$

for all real values of x. To do this we shall first look at some of the general properties of these functions, find some specific values of the functions at given values of x, and finally construct their graphs.

We already know that the sine and cosine functions are periodic with period 2π, and so we may restrict our attention to values of x where $0 \leq x < 2\pi$. Now by noting that u and v are the coordinates of a point on a unit circle, we have

$$u^2 + v^2 = 1. \qquad (1)$$

But since $u = \cos x$ and $v = \sin x$, we have

$$\cos^2 x + \sin^2 x = 1. \qquad (2)$$

If we write (2) as

$$\sin^2 x = 1 - \cos^2 x$$

and as

$$\cos^2 x = 1 - \sin^2 x$$

it is apparent that neither $\sin x$ nor $\cos x$ can exceed 1 in absolute value, that is

$$-1 \leq \sin x \leq 1$$
$$-1 \leq \cos x \leq 1.$$

Another property of sin and cos derives from the symmetry of the circle with respect to the u-axis. Two symmetric points on the circle are obtained by proceeding the distance x in both the clockwise and the counter-clockwise senses along the circle. In other words, if $\rho(x) = (u,v)$, then $\rho(-x) = (u,-v)$ (Figure 2). From this we obtain the important symmetric properties

$$\begin{aligned} \cos(-x) &= \cos x \\ \sin(-x) &= -\sin x. \end{aligned} \qquad (3)$$

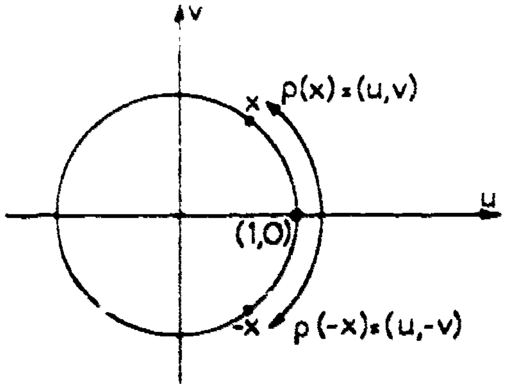

Figure 2. Symmetry relations.

Since we are ultimately interested in graphing $y = \sin x$ and $y = \cos x$, we have managed to narrow our attention to a rectangle of length 2π and of altitude 2 in the xy-plane* as in Figure 3.

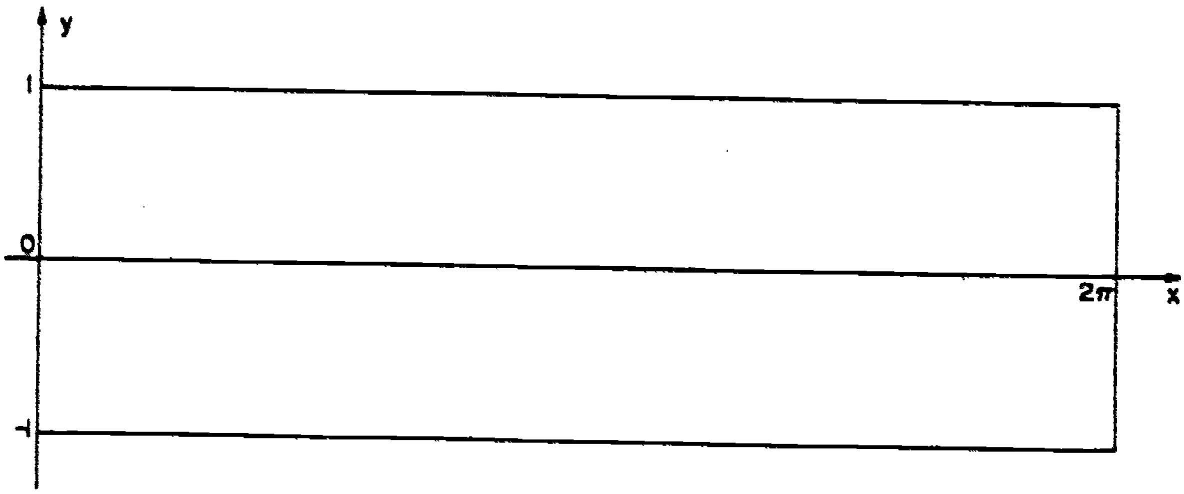

Figure 3. Rectangle to include one cycle of sin or cos.

If we can picture the graph of the functions in the interval $0 \leq x < 2\pi$, the periodicity properties of cos and sin will permit us to extend the graph as far as we like by placing the rectangles end to end along the x-axis as in Figure 4.

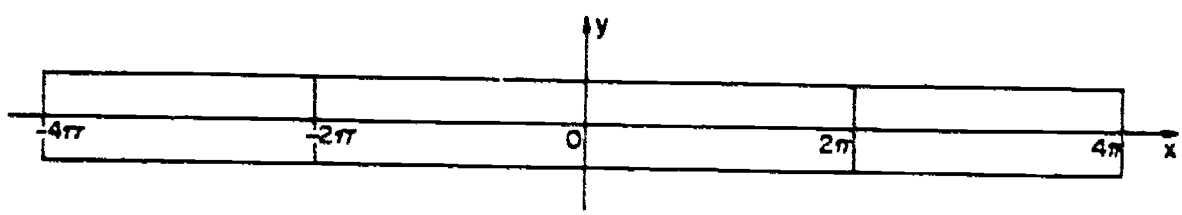

Figure 4. Rectangles of periodicity.

We therefore direct our attention to values of x such that $0 \leq x < 2\pi$. To begin with, the unit circle in the uv-plane is divided into four equal arcs by the axes; each arc is of length $\frac{\pi}{2}$, and the division points correspond to lengths of $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, with central angles of 0°, 90°, 180°, and 270°, respectively. The corresponding points on the circle will be $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, as in Figure 5.

*Since we shall have occasion to refer to two coordinate planes for points (u,v) and (x,y), we wish to point out the distinction between them. The uv-plane contains the unit circle with which we are dealing. This is the circle onto which the function ρ maps the real number x as an arc length. The xy-plane is the plane in which we take the x-axis as the real number line and examine not the point function $\rho(x)$ but the functions $\cos: x \to y = \cos x$ and $\sin: x \to y = \sin x$, each of which maps the real number x into another real number.

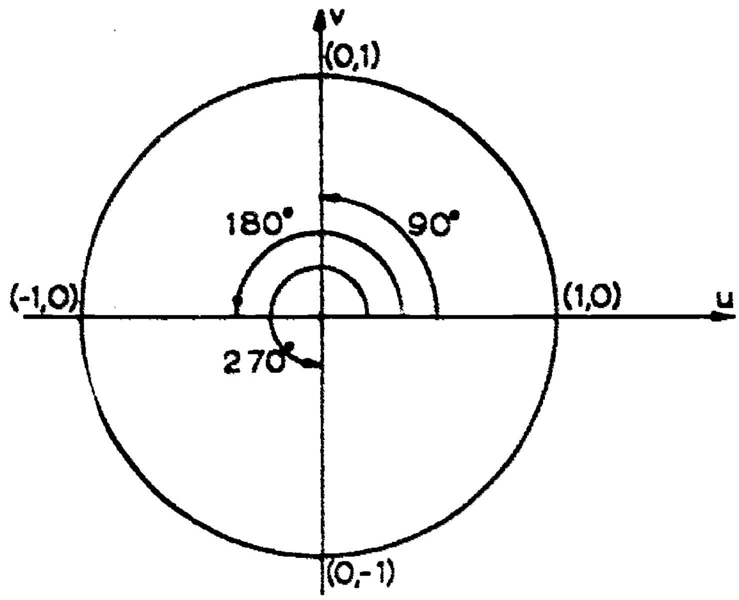

Figure 5. $\rho(x)$ for $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Since $\cos x = u$ and $\sin x = v$, we have

$$\cos 0 = 1, \qquad \sin 0 = 0,$$
$$\cos \frac{\pi}{2} = 0, \qquad \sin \frac{\pi}{2} = 1,$$
$$\cos \pi = -1, \qquad \sin \pi = 0,$$
$$\cos \frac{3\pi}{2} = 0, \qquad \sin \frac{3\pi}{2} = -1.$$

We next consider the midpoint of each of the quarter circles in Figure 6. These correspond to arc lengths of $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$, with central angles 45°, 135°, 225°, 315°.

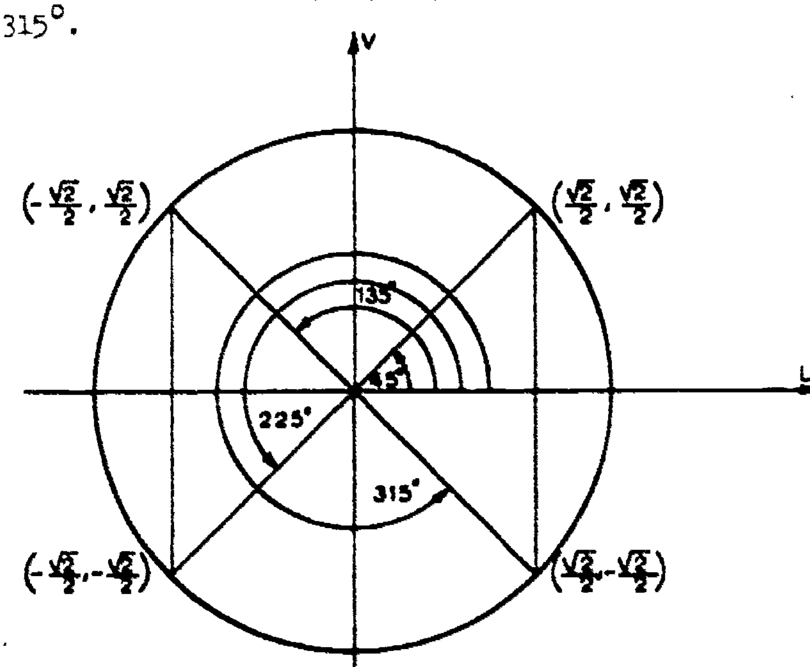

Figure 6. $\rho(x)$ for $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

If we drop perpendiculars to the u-axis from these points as in Figure 6, we note that radii to the points form angles of 45° with the u-axis. From geometry we know that for a 45° right triangle with hypotenuse 1, the sides are of length $\frac{\sqrt{2}}{2}$ and hence that the coordinates of the midpoints of the quarter circles are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, and $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, respectively. We may therefore add the following to our list of values:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$$

We can find the coordinates of the trisection points of the quarter circles by a similar method. In Figure 7, we show only two of the triangles, but the procedure is essentially the same in each quadrant.

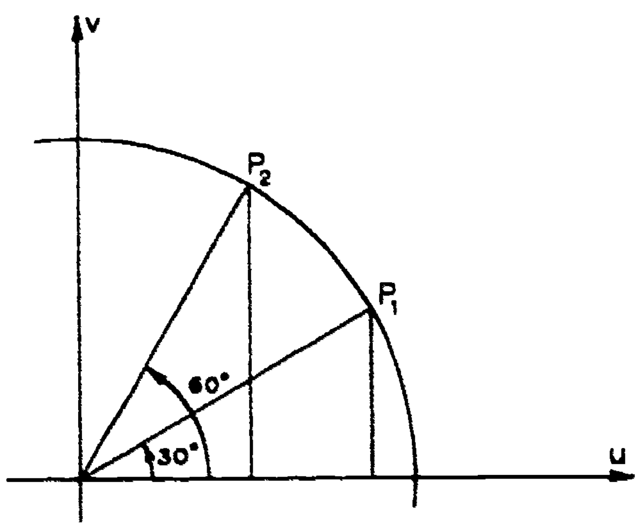

Figure 7. $\rho(x)$ for $x = \frac{\pi}{6}, \frac{\pi}{3}$.

From the properties of the 30°-60° right triangle, we note that P_1 and P_2 have coordinates $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, respectively. We may fill in the coordinates of all of these points of trisection as in Figure 8, from which we can find eight new values for cos and sin. Collecting in one table all of the values which we have so far determined, we have Table 1.

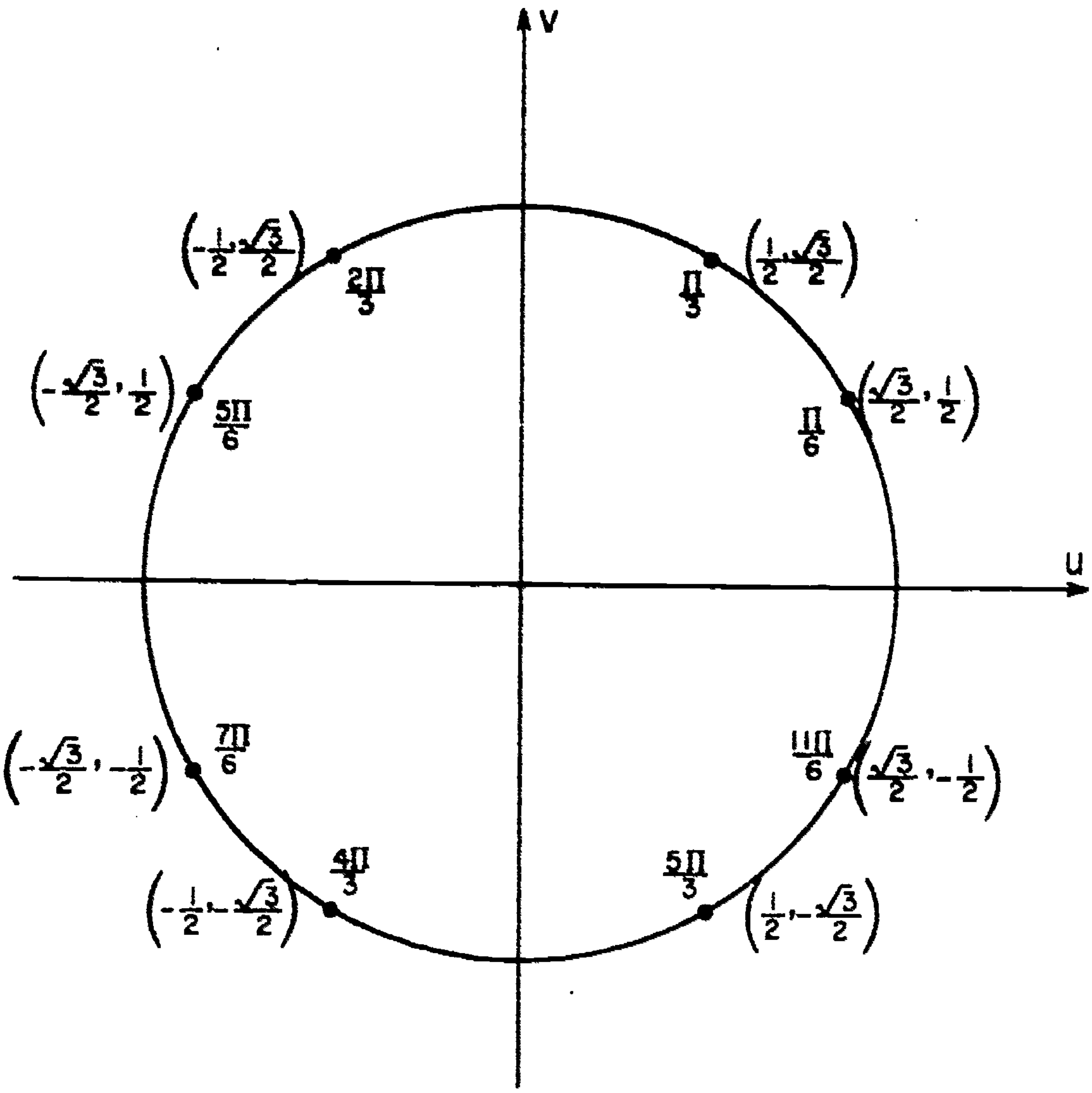

Figure 8. Further values of $\rho(x)$.

Table 1

Values for cos and sin for one period.

x	cos x	sin x
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2} \approx .87$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .71$	$\frac{\sqrt{2}}{2} \approx .71$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2} \approx .87$
$\frac{\pi}{2}$	0	1
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2} \approx .87$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -.71$	$\frac{\sqrt{2}}{2} \approx .71$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2} \approx -.87$	$\frac{1}{2}$
π	-1	0
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2} \approx -.87$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -.71$	$-\frac{\sqrt{2}}{2} \approx -.71$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2} \approx -.87$
$\frac{3\pi}{2}$	0	-1
$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2} \approx -.87$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .71$	$-\frac{\sqrt{2}}{2} \approx -.71$
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2} \approx .87$	$-\frac{1}{2}$
2π	1	0

With this table we are now in a position to begin graphing sin and cos. Because we wish to look at the graph of these functions over the real numbers, we shall use an xy-plane as usual and work with the points (x,y) where $y = \cos x$ or $y = \sin x$. We shall deal separately with each function, taking first $y = \cos x$. From Table 1 we can now plot some points in the rectangle in Figure 3, obtaining Figure 9.

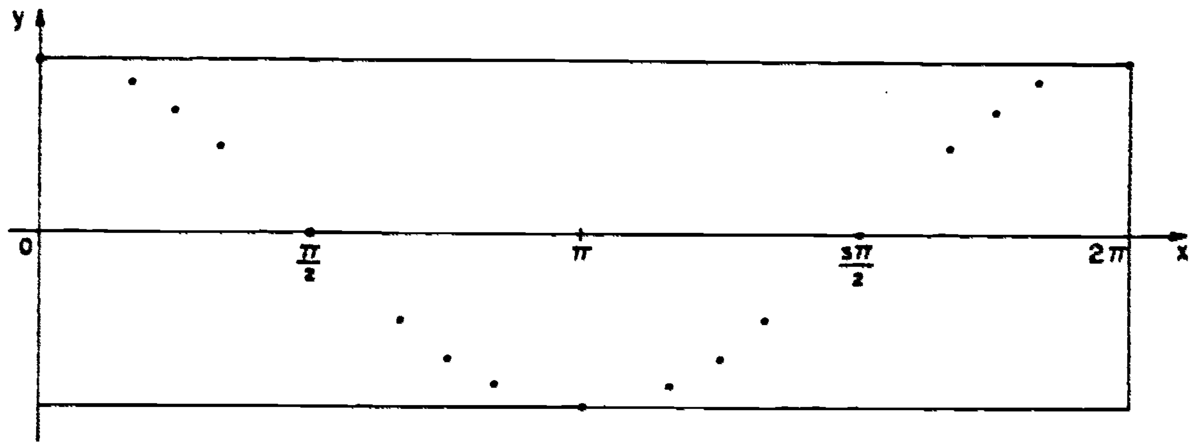

Figure 9. Values of cos: $x \to \cos x$.

By connecting these points by a smooth curve we should obtain a reasonable picture of the function

$$\cos: x \to \cos x$$

as in Figure 10.

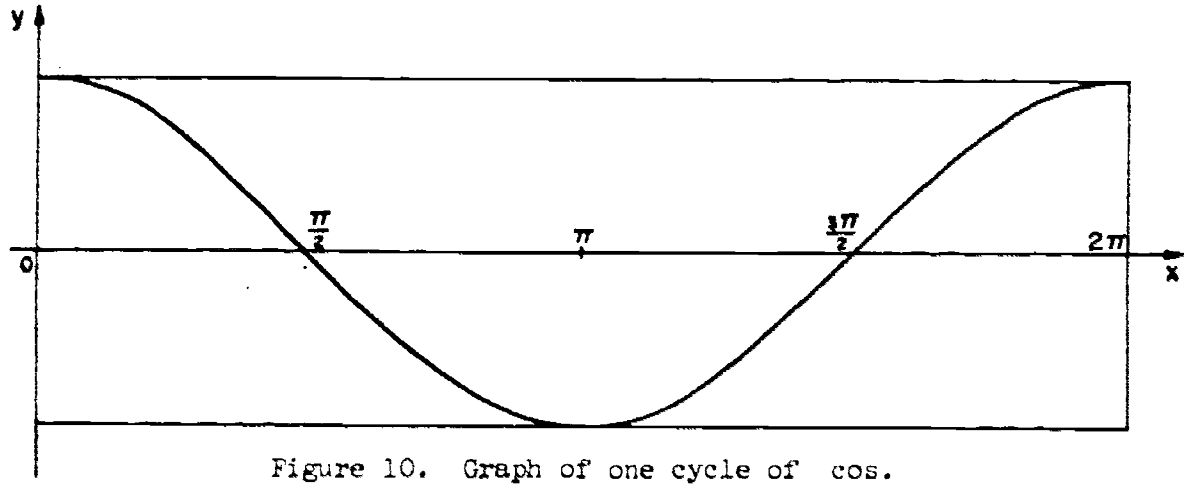

Figure 10. Graph of one cycle of cos.

If we wish to extend our picture to the right and left, we use the periodicity property to obtain Figure 11.

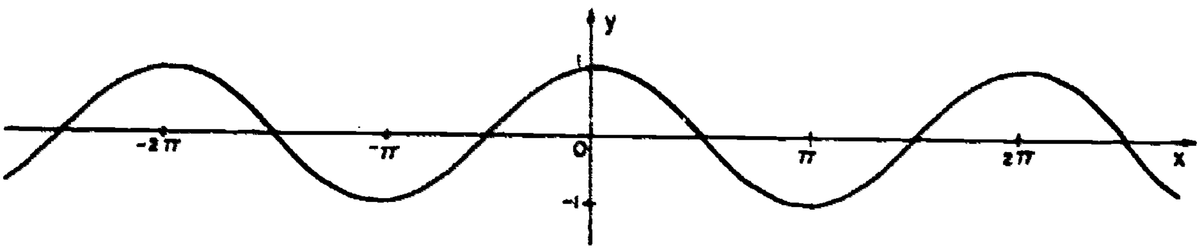

Figure 11. Graph of cos.

A similar treatment of $y = \sin x$ leads to Figures 12, 13, and 14.

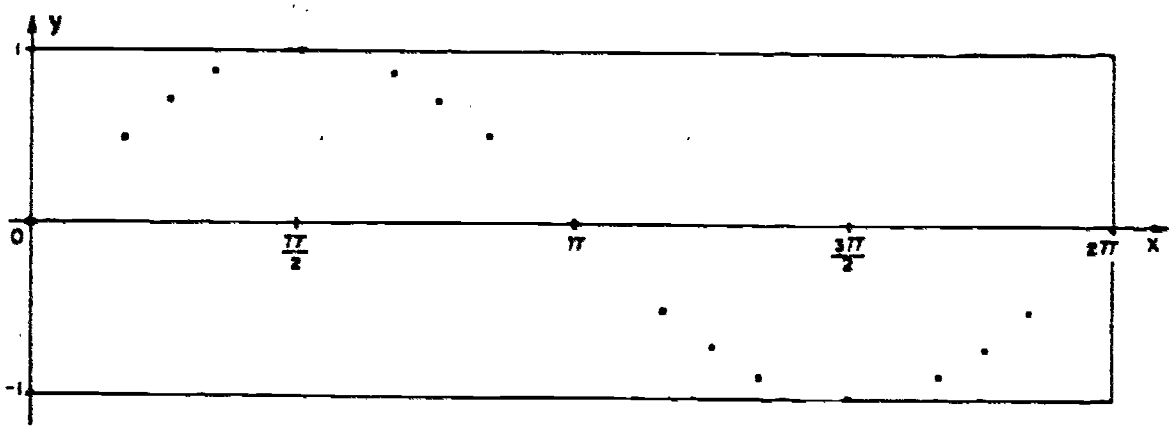

Figure 12. Values of sin x: $x \rightarrow \sin x$.

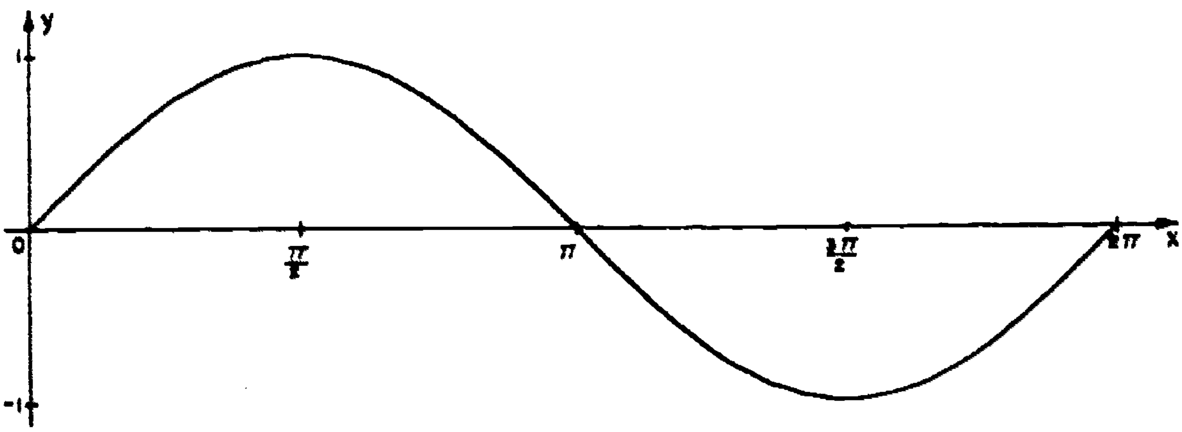

Figure 13. Graph of one cycle of sin.

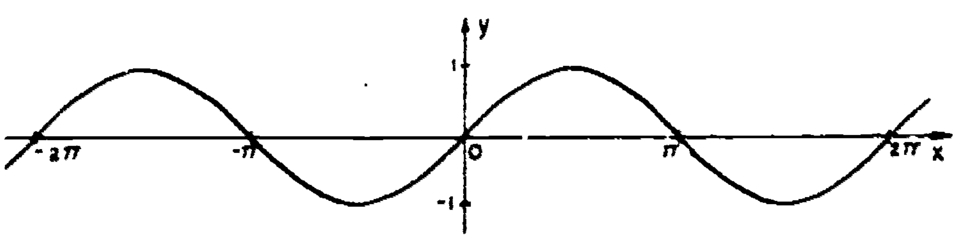

Figure 14. Graph of sin.

Since it is often necessary to work with

$$y = A \cos x$$
$$y = \cos Bx \qquad (4)$$
$$y = \cos (x + C) \quad (A,\ B,\ \text{and}\ C\ \text{constants})$$

or some combination of these expressions, it is worthwhile to inquire into the effect that these constants have on the behavior of y. In case of

$$y = A \cos x \quad (A > 0),$$

the A simply multiplies each ordinate of $y = \cos x$ by A, and the graph of $y = A \cos x$ would appear as in Figure 15.

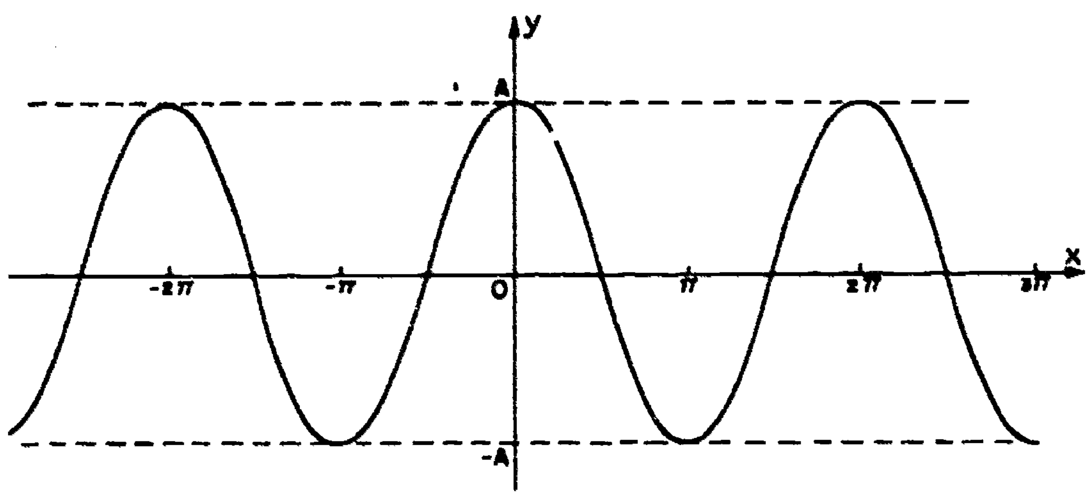

Figure 15. Graph of $y = A \cos x$.

In Exercises 5, 6, and 7 you are asked to determine for yourself the effects of B and C in Equations (4).

Exercises 2

1. Using $f(x + 2n\pi) = f(x)$, and $f\colon\ x \to \cos x$, find
   (a) $f(3\pi)$;
   (b) $f(\frac{7\pi}{3})$;
   (c) $f(\frac{9\pi}{2})$;
   (d) $f(\frac{25\pi}{6})$;
   (e) $f(-7\pi)$;
   (f) $f(-\frac{10\pi}{3})$.

2. If $f\colon\ x \to \sin x$, find the values of f in Exercise 1 above.

3. For what values of x (if any) will

(a) $\sin x = \cos x$? (c) $\sin x = \sin(-x)$?

(b) $\sin x = -\cos x$? (d) $\cos x = \cos(-x)$?

4. Graph on the same set of axes the functions $f: x \to y$ defined by the following, using Table 1 to find values for the functions.

(a) $y = 2 \cos x$ (b) $y = 3 \cos x$ (c) $y = \frac{1}{2} \cos x$

5. Repeat Exercise 4 using

(a) $y = \cos 2x$; (b) $y = \cos 3x$; (c) $y = \cos \frac{1}{2}x$.

6. Repeat Exercise 4 using

(a) $y = \cos (x + \frac{\pi}{2})$; (b) $y = \cos (x - \frac{\pi}{2})$; (c) $y = \cos (x + \pi)$.

7. From the results of Exercises 4, 5, and 6 above, what effect do you think the constant k will have on the graph of

(a) $y = k \cos x$? (b) $y = \cos kx$? (c) $y = \cos (x + k)$?

8. From the results of Exercise 6(b) above and Figure 14, what can you say about $\cos (x - \frac{\pi}{2})$ and $\sin x$?

9. As explained in the text, symmetric points with respect to the u-axis on the unit circle $u^2 + v^2 = 1$ are obtained by proceeding a distance x in the clockwise and counterclockwise senses along the circle. In other words, if $\rho(x) = (u,v)$, then $\rho(-x) = (u,-v)$. If follows that

$$\cos x = \cos (-x)$$
$$\sin x = -\sin (-x).$$

What relations between the circular functions can you derive in similar fashion from the following symmetries of the circle?

(a) The symmetry with respect to the origin.

(b) The symmetry with respect to the v-axis.

---

3. Angle and Angle Measure.

As we remarked in Section 1, the circular functions are closely related to the functions of angles studied in elementary trigonometry. In a sense, all that we have done is to measure angles in a new way. To see precisely what the difference is, let us recall a few fundamentals.

An angle is defined in geometry as a pair of rays with a common end point. (Fig. 16) Let R_1 and R_2 be two rays originating at the point O. Draw any circle with O as center; denote its radius by r. The rays R_1 and R_2 meet the circle in two points P_1 and P_2 which divide the circle into two parts. Here we consider directed angles and distinguish between the angles defined by the pair R_1, R_2 according to their order. Specifically, we set $\alpha = \measuredangle(R_1, R_2)$ and $\beta = \measuredangle(R_2, R_1)$ where each angle includes the arc of the circle which is obtained by passing counterclockwise along the circle from the first ray of the pair to the second (Figure 16).

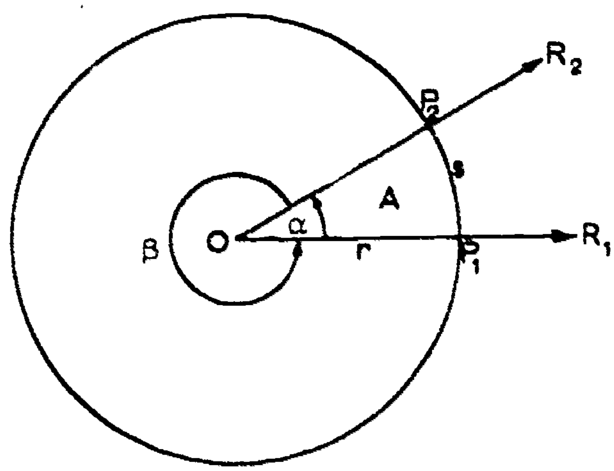

Figure 16. Angles α and β.

In establishing degree measure, we divide a circle into 360 equal units and measure an angle α by the number of units of arc it includes. For instance, if we found that an angle included $\frac{1}{3}$ of the circumference, we would say that the angle measured $\frac{1}{3} \times 360^\circ$ or 120°. In general, if we divide the circumference of a circle into k equal parts, each of length $\frac{2\pi r}{k}$, then this length could be our unit of angle measure. Since the numerical factor $\frac{2\pi}{k}$ appears in many important formulas, it is useful to choose k so that the factor is 1. In order to do this, it is clear that k must equal 2π. In this case, $\frac{2\pi r}{k}$ will be equal to r, the radius of the circle. When $k = 2\pi$ we call the resulting unit of angle measure a radian. Radian measure is related to degree measure by

$$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^\circ = \left(\frac{180}{\pi}\right)^\circ \qquad (1)$$

and

$$1^\circ = \frac{\pi}{180} \text{ radians}. \qquad (2)$$

You should note that this definition of the radian measure of an angle implies that an angle of 1 radian intercepts an arc of length s equal to r, the radius of the circle. In general, an angle of x radians intercepts an arc of length xr. That is, $s = xr$ where x is the measure of the central angle in radians while s and r are the lengths of the arc and the radius measured in the same linear units.

In working with radian measure, it is customary simply to give the measure of an angle α as, say, $\frac{\pi}{2}$, rather than $\frac{\pi}{2}$ radians. If we use degree measure, however, the degree symbol will always be written, as, for example, 90°, 45°, etc.

It is also possible to measure an angle α by the area A of the sector it includes (Figure 16). Specifically, we have that the area A is the same fraction of the area of the interior of the circle as the arc s is of the circumference, that is,

$$\frac{A}{\pi r^2} = \frac{s}{2\pi r}. \qquad (3)$$

We saw above that the arc length s on a circle included by an angle α may be expressed as $s = xr$ where r is the radius of the circle and x is the radian measure of α. It follows from (3) that

$$\frac{A}{\pi r^2} = \frac{x}{2\pi}$$

or

$$x = \frac{2A}{r^2}. \qquad (4)$$

That is, the measure x of α in radians is twice the area of the included sector divided by the square of the radius.

Exercises 3

1. Change the following radian measure to degree measure.
   (a) $\frac{2\pi}{3}$
   (b) $\frac{\pi}{6}$
   (c) $-\frac{2\pi}{3}$
   (d) $\frac{7\pi}{6}$
   (e) 2π
   (f) $\frac{5\pi}{6}$
   (g) $\frac{8\pi}{3}$
   (h) $\frac{18\pi}{5}$
   (i) $\frac{13\pi}{4}$

2. Change the following degree measure to radian measure.
   (a) 270°
   (b) -30°
   (c) 135°
   (d) 480°
   (e) 195°
   (f) -105°
   (g) 810°
   (h) 190°
   (i) 18°

3. What is the measure (in radians) of an angle which forms a sector of area 9π if the radius of the circle is 3 units?

4. What is the area of the sector formed by an angle of $(\frac{1}{2})\pi$, if the radius of the circle is 2 units?

5. Suppose that we wish to find a unit of measure so that a quarter of a circle will contain 100 such units.
   (a) How many such units will be equivalent to 1°?
   (b) How many such units will be equivalent to 1 radian?
   (c) How many of these units will a central angle contain, if the included arc is equal in length to the diameter of the circle?

4. Uniform Circular Motion.

Let us again consider the motion of a point P around a circle of radius r in the uv-plane, and suppose that P moves at the constant speed of s units per second. Let $P_0(r,0)$ represent the initial position of P. After one second, P will be at P_1, an arc-distance s away from P_0. After two seconds, P will be at P_2, an arc-distance 2s from P_0, and similarly after t seconds P will be at arc-distance ts. (Figure 17.)

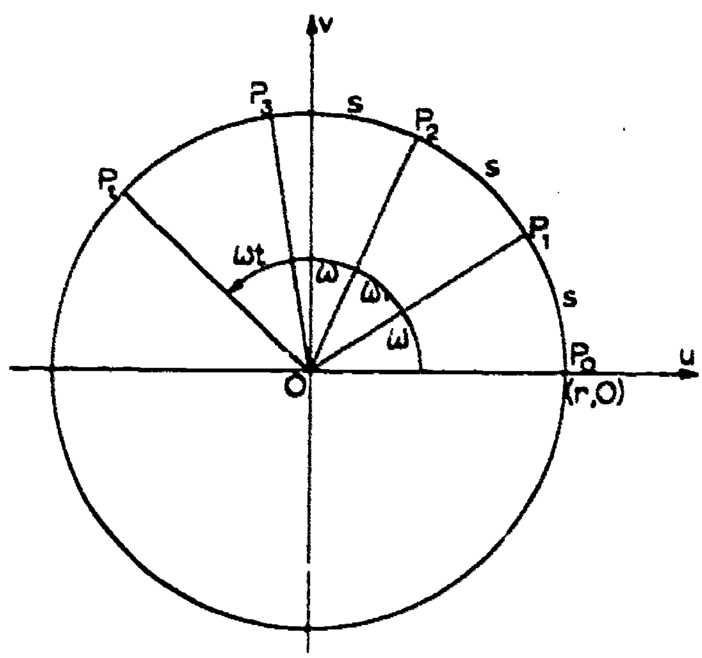

Figure 17. Uniform motion of P on circle O.

Clearly, $\angle P_0OP_1 = \angle P_1OP_2 = \angle P_2OP_3 = \ldots$ and likewise for each additional second, since these central angles have equal arcs, each of length s. Each of these central angles may be written as $\omega = \frac{s}{r}$. After 2 seconds, OP will have rotated through an angle 2ω into position OP_2; after 3 seconds through an angle 3ω; and, in general, after t seconds through an angle of $t\omega$ or ωt. In other words, after t seconds, P will have moved from (r,0) an arc-distance st, and OP_0 will have rotated from its initial position through an angle of ωt into the position OP. If we designate the coordinates of P by (u,v) we have

$$u = r \cos \omega t$$
$$v = r \sin \omega t. \qquad (1)$$

When $\omega t = 2\pi$, P will again be in the position P_0. This motion of the point from P_0 back into P_0 again is called a <u>cycle</u>. The time interval during which a cycle occurs is called the <u>period</u>; in this case, the period is

$\frac{2\pi}{\omega}$. The number of cycles which occur during a fixed unit of time is called the frequency. Since we refer to the alternating current in our homes as "60-cycle", an abbreviation for "60 cycles per second", this notion of frequency is not altogether new to us.

To visualize the behavior of the point P in a different way, consider the motion of the point Q which is the projection of P on the v-axis. As P moves around the unit circle, Q moves up and down along a fixed diameter of the circle, and a pencil attached to Q will trace this diameter repeatedly--assuming that the paper is fixed in position. If, however, the strip of paper is drawn from right to left at a constant speed, then the pencil will trace a curve, something like Figure 18.

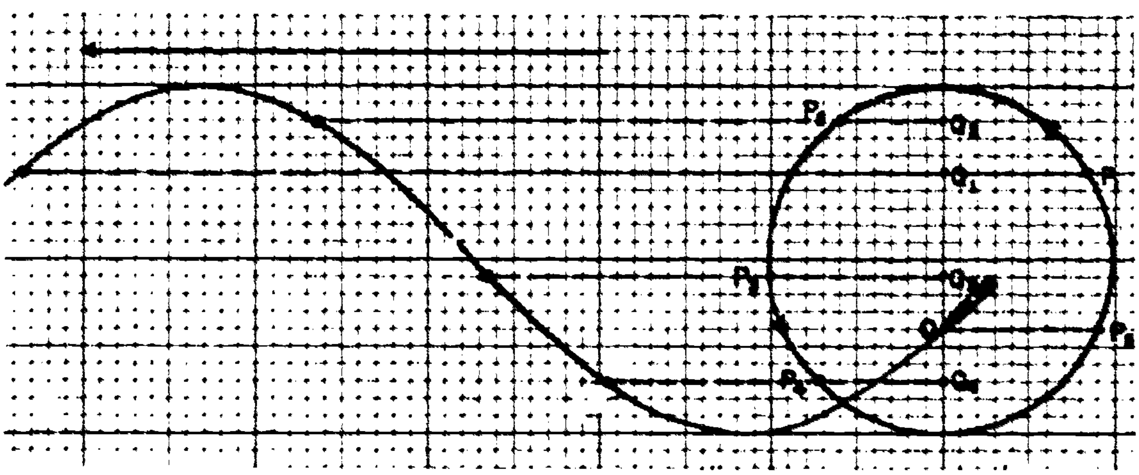

Figure 18. Wave motion.

An examination of this figure will show why motion of this type is called wave motion. We note that the displacement y of Q from its central position is functionally related to the time t, that is, there is a function f such that $y = f(t)$. By suitably locating the origin of the ty-plane, we may have either $y = \cos \omega t$ or $y = \sin \omega t$; thus either of these equations may be looked upon as describing a pure wave or, as it is sometimes called, a simple harmonic motion. The surface of a body of water displays a wave motion when it is disturbed. Another familiar example is furnished by the electromagnetic waves used in radio, television, and radar, and modern physics has even detected wave-like behavior of the electrons of the atom.

One of the most interesting applications of the circular functions is to the theory of sound (acoustics), A sound wave is produced by a rapid alternation of pressure in some medium. A pure musical tone is produced by any pressure wave which can be described by a circular function of time, say:

$$p = A \sin \omega t \qquad (2)$$

where p is the pressure at time t and the constants A and ω are positive. The equation (2) for the acoustical pressure, p, is exactly in the form of one of the equations of (1) even though no circular motion is involved: all that occurs is the fluctuation of the pressure at a given point of space.* Here the numbers A and ω have direct musical significance. The number A is called the <u>amplitude</u> of the wave; it is the peak pressure and its square is a measure of the loudness. The number ω is proportional to the frequency and is a measure of pitch; the larger ω the more shrill the tone.

The effectiveness of the application of circular functions to the theory of sound stems from the principle of superposition. If two instruments individually produce acoustical pressures p_1 and p_2, then together they produce pressures $p_1 + p_2$. If p_1 and p_2 have a common period then the sum $p_1 + p_2$ has the same period. This is the root of the principle of harmony; if two instruments are tuned to the same note, they will produce no strange new note when played together.

Let us suppose, for example, that two pure tones are produced with individual pressure waves of the same frequency, say

$$u = A \cos \omega t \tag{3}$$

$$v = B \sin \omega t \tag{4}$$

where A, B, and ω are positive. According to the principle of superposition, the net pressure is

$$p = A \cos \omega t + B \sin \omega t.$$

What does the graph of this equation look like? We shall answer this question by reducing the problem to two simpler problems, that is, of graphing (3) and (4) above. For each t, the value of p is obtained from the individual graphs, since

$$p = u + v.$$

To illustrate these ideas with specific numerical values in place of A, B, and ω, let

$$A = 3, \quad B = 4, \quad \omega = \pi.$$

Then we wish to graph

$$p = 3 \cos \pi t + 4 \sin \pi t. \tag{5}$$

Equations (3) and (4) become

$$u = 3 \cos \pi t, \tag{6}$$

$$v = 4 \sin \pi t. \tag{7}$$

*The acoustical pressure is defined as the difference between the gas pressure in the wave and the pressure of the gas if it is left undisturbed.

By drawing the graphs of (6) (Figure 19) and (7) (Figure 20) on the same set of axes, and by adding the corresponding ordinates of these graphs at each value of t, we obtain the graph of (5) shown in Figure 21. You will notice that certain points on the graph of p are labeled with their coordinates. These are points which are either easy to find, or which have some special interest.

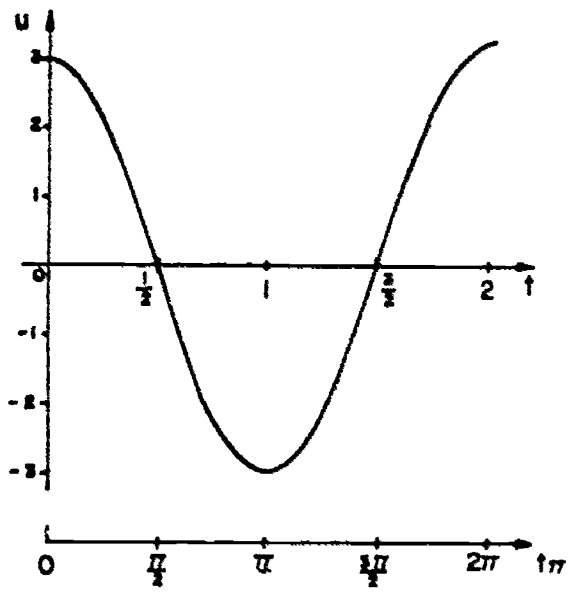

Figure 19.
Graph of $u = 3 \cos \pi t$.

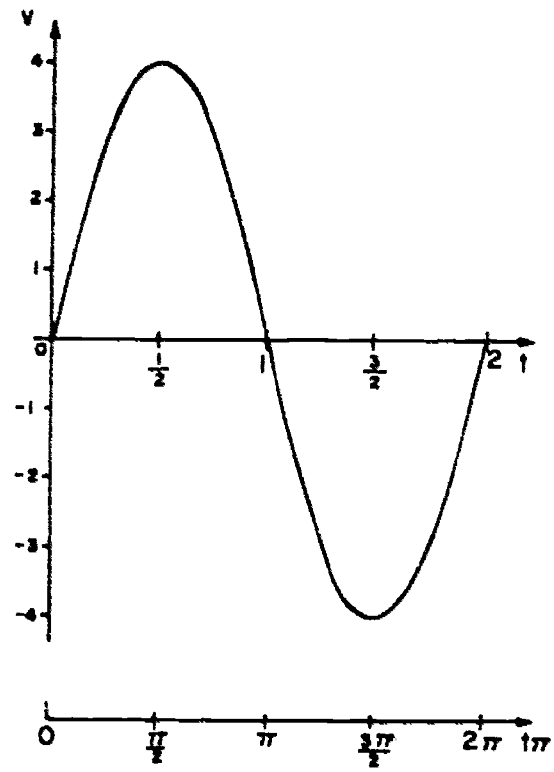

Figure 20.
Graph of $v = 4 \sin \pi t$.

The points (0,3), (0.5,4), (1,-3), (1.5,-4), and (2,3) are easy to find since they are the points where either $u = 0$ or $v = 0$. The points (0.29,5) and (1.29,-5) are important because they represent the first maximum and minimum points on the graph of p, while (0.79,0) and (1.79,0) are the first zeros of p. To find the maximum and minimum points and zeros of p involves the use of tables, and hence we shall put off a discussion of this matter until Section 7, although a careful graphing should produce fairly good approximations to them.

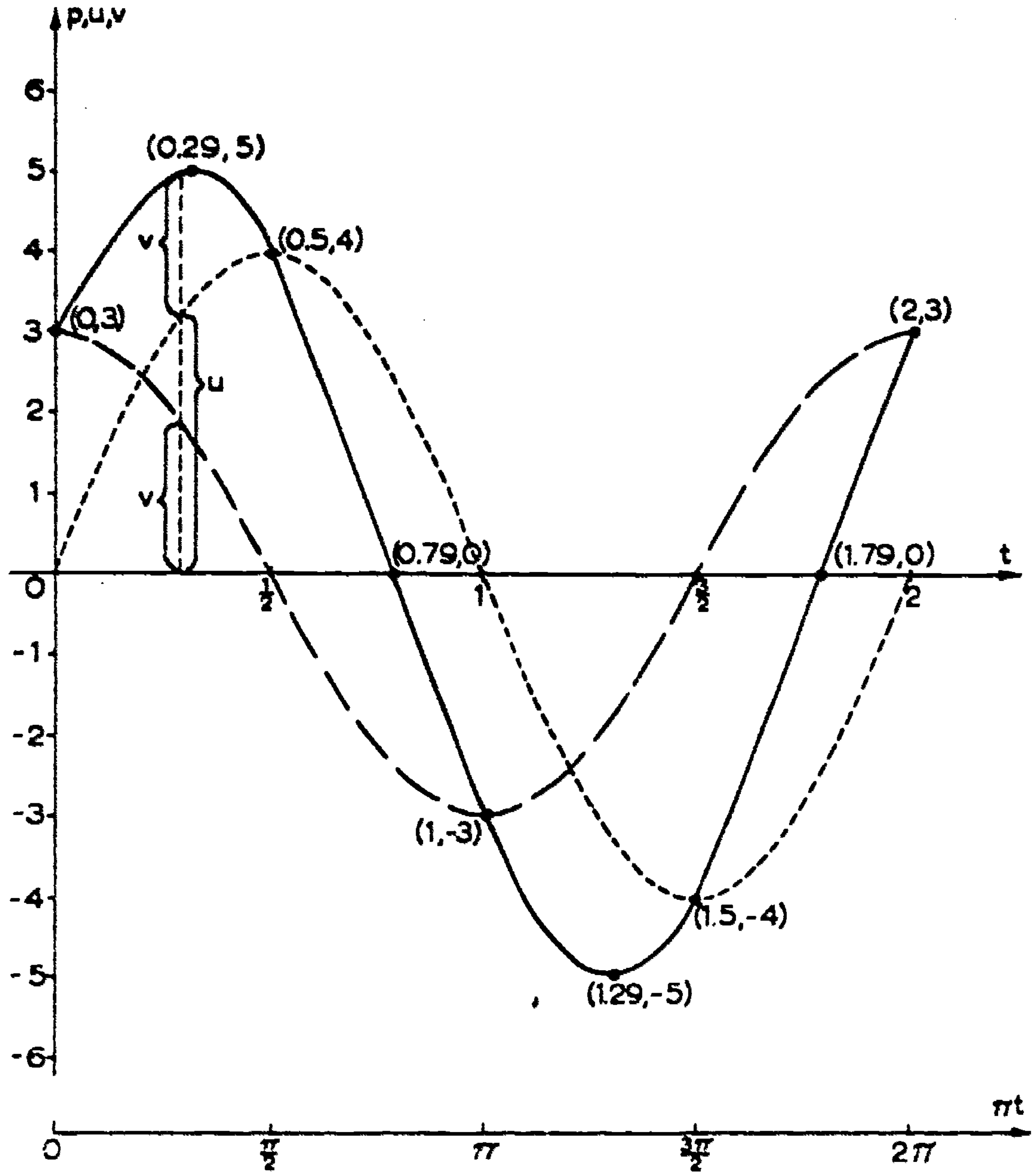

Figure 21. The sum of two pure waves of equal period.

Dashed curve: $u = 3 \cos \pi t$.

Dotted curve: $v = 4 \sin \pi t$.

Full curve: $p = 3 \cos \pi t + 4 \sin \pi t$; $0 \leq t \leq 2$.

(The scales are not the same on the two axes; this distortion is introduced in order to show the details more clearly.)

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Exercises 4

1. Extend the three curves in Figure 21 to the interval $|t| \leq 2$. To the interval $|t| \leq 3$. What do you observe about the graph of $p = 3 \cos \pi t + 4 \sin \pi t$ over $|t| \leq 3$? Is it periodic? What is its period? Give reasons for your answers.

2. Sketch graphs of each of the following curves over one complete cycle; and state what the period is, and what the range is, if you can.

   (a) $y = 2 \sin 3t$

   (b) $y = -3 \sin 2t$

   (c) $y = 4 \cos (\frac{x}{2})$

   (d) $y = 3 \cos (-x)$

   (e) $y = 2 \sin x - \cos x$

---

5. Vectors and Rotations.

In the next section, we shall develop the important formulas for $\sin (x + y)$ and $\cos (x + y)$. Because our development will rely on certain properties of plane vectors, we give, in this section, an informal summary of those properties.

You have probably encountered vectors in your earlier work in mathematics and science. The physicist uses them to represent quantities such as displacements, forces, and velocities, which have both magnitude and direction. Some examples of vector quantities are the velocity of a train along a track or of the wind at a given point, the weight of a body (the force of gravity), and the displacement from the origin of a point in the Cartesian plane.

In a two-dimensional system, it is often convenient to represent vectors by arrows (which have both a length, representing magnitude, and a direction) and to use geometrical language. We shall do this, and we shall restrict ourselves to vectors all of which start from a single point; in our discussion we shall take this point to be the origin. If $\vec{S}$ and $\vec{T}$ are vectors, we define the sum $\vec{S} + \vec{T}$ to be the vector $\vec{R}$ represented by the diagonal of the parallelogram which has sides $\vec{S}$ and $\vec{T}$, as shown in Figure 22. If $\vec{T}$ is a vector and a is a number not equal to 0, then we define the product $a\vec{T}$ to be a vector whose magnitude is $|a|$ times that of $\vec{T}$ and whose direction is the same as $\vec{T}$ if $a > 0$ and opposite to $\vec{T}$ if $a < 0$; in either case, $\vec{T}$ and $a\vec{T}$ are collinear. Figure 23 illustrates this for $a = 2$ and $a = -2$. It is an experimental fact that these definitions correspond to physical reality; the net effect of two forces acting at a point, for example, is that of a single force determined by the parallelogram law of addition.

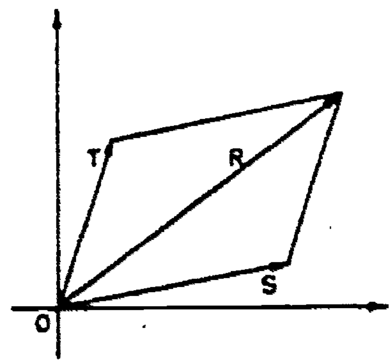

Figure 22.
The sum of two vectors.

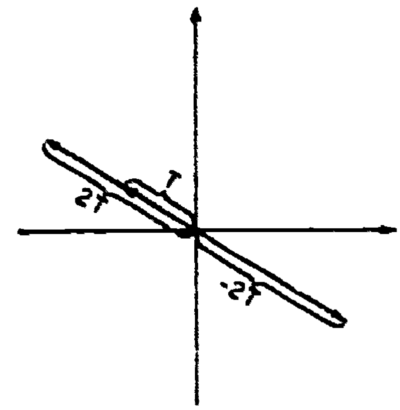

Figure 23.
A vector multiplied by a number.

These definitions of vector sum and of multiplication by a number make it possible to express all plane vectors from the origin in terms of two basic vectors. It is convenient to take as these basic vectors the vector $\vec{U}$ from the origin to $(1,0)$ and the vector $\vec{V}$ from the origin to $(0,1)$. Then, for any vector $\vec{R}$, there exist unique numbers u and v such that

$$\vec{R} = u\vec{U} + v\vec{V}; \qquad (1)$$

in fact, the numbers u and v are precisely the coordinates of the tip of the arrow representing $\vec{R}$ (Figure 24). To take a specific example, the vector $\vec{S}$ from the origin to the point $P(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ can be expressed in terms of the basic vectors $\vec{U}$ and $\vec{V}$ as

$$\vec{S} = -\tfrac{1}{2}\vec{U} + \tfrac{\sqrt{3}}{2}\vec{V},$$

as shown in Figure 25.

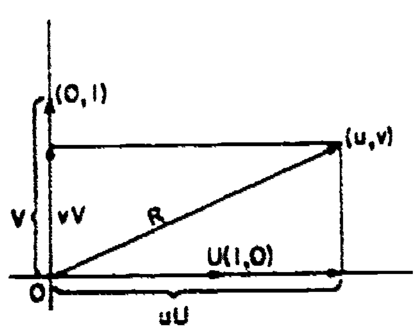

Figure 24.
A vector in terms of the basic vectors $\vec{U}$ and $\vec{V}$.

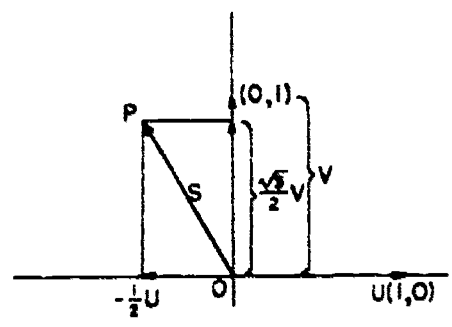

Figure 25.
$\vec{S} = -\frac{1}{2}\vec{U} + \frac{\sqrt{3}}{2}\vec{V}$.

We now introduce the idea of a rotation of the whole plane about the origin O. Such a rotation carries each vector into a unique vector, and we may therefore regard it as a function whose domain and range are sets of vectors. Most of the functions you have met are functions which map numbers into numbers, but it will be useful, in this section, to think of a rotation as a new kind of function which maps vectors into vectors.

Any rotation of the sort we are considering is completely specified by the length x of the arc AP of the unit circle through which the rotation carries the point $A(1,0)$. Let f be the rotation (function) which maps the vector $\overrightarrow{OA}$ (that is, $\overrightarrow{U}$) into the vector $\overrightarrow{OP}$ whose tip P has coordinates (u,v). As we have seen above, $\overrightarrow{OP}$ can be expressed in terms of the basic vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ as $u\overrightarrow{U} + v\overrightarrow{V}$. Hence

$$f(\overrightarrow{U}) = \overrightarrow{OP} = u\overrightarrow{U} + v\overrightarrow{V}, \qquad (2)$$

as pictured in Figure 26. The same rotation f carries the point $B(0,1)$ into the point $Q(-v,u)$, as can be shown by congruent triangles, (see Figure 26), so we also have

$$f(\overrightarrow{V}) = \overrightarrow{OQ} = -v\overrightarrow{U} + u\overrightarrow{V}. \qquad (3)$$

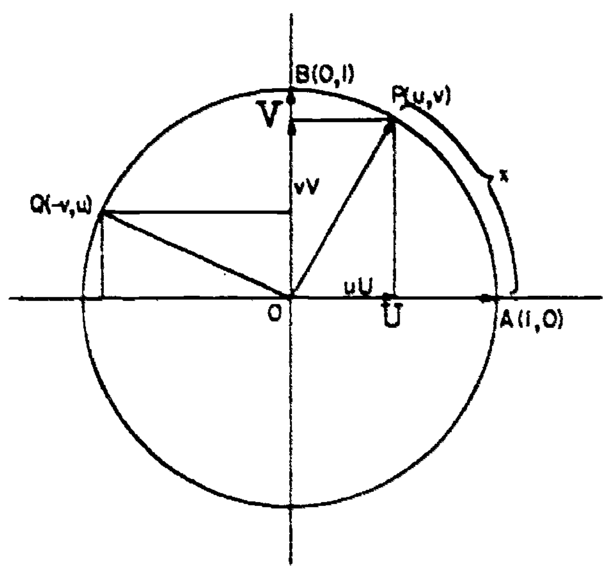

Figure 26.

The effect of a rotation on the basic vectors $\overrightarrow{U}$ and $\overrightarrow{V}$.

(The figure is valid only when $0 < x < \frac{\pi}{2}$. The result, however, is true for any real x; for a more general derivation, see Exercises 8 and 9.)

Now suppose that we subject the plane to a second rotation g, in which points on the unit circle are displaced through an arc of length y. Since g also is a function, we may regard the successive applications of the rotations f and g as a composite function gf (see, for example, the SMSG pamphlet, Functions). From Equation (2) and the definition of composition, we have

$$(gf)(\vec{U}) = g\left(f(\vec{U})\right) = g(\overrightarrow{OP}) = g(u\vec{U} + v\vec{V}). \qquad (4)$$

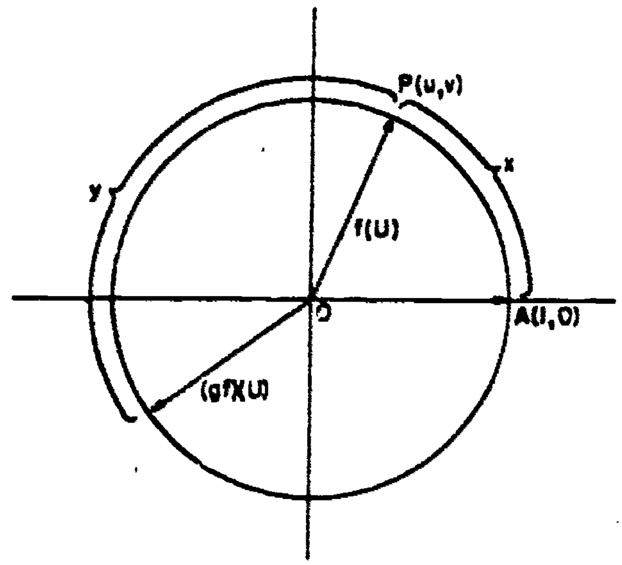

Figure 27.

We must now pay some attention to two important properties of rotations. First, a rotation does not change the angle between two vectors, and collinear vectors will therefore be rotated into collinear vectors. Second, a rotation does not change the length of any vector. Now, if a is a number $(a \neq 0)$ and $\vec{T}$ is a vector, then the vector $a\vec{T}$ is collinear with $\vec{T}$. If f is a rotation, the two stated properties ensure that $\vec{T}$ and $f(\vec{T})$ have the same length, that $a\vec{T}$ and $f(a\vec{T})$ have the same length, and that $f(a\vec{T})$ is collinear with $f(\vec{T})$. We will therefore get the same vector from $\vec{T}$ if we first multiply by a and then rotate, or first rotate and then multiply by a:

$$f(a\vec{T}) = af(\vec{T}). \qquad (5)$$

The same two properties of rotations also ensure that a parallelogram will not be distorted by a rotation. Since the addition of vectors is defined in terms of parallelograms, it follows that rotations preserve sums; that is, if f is a rotation, and if $\vec{S}$ and $\vec{T}$ are vectors, then

$$f(\vec{S} + \vec{T}) = f(\vec{S}) + f(\vec{T}). \qquad (6)$$

From (5) and (6),

$$g(u\vec{U} + v\vec{V}) = ug(\vec{U}) + vg(\vec{V}),$$

and we may therefore rewrite (4) as

$$(gf)(\vec{U}) = ug(\vec{U}) + vg(\vec{V}). \qquad (7)$$

Exercises 5

1. Let T be the vector $\overrightarrow{OP}$ where P is the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Write $\vec{T}$ in the form $u\vec{U} + v\vec{V}$. If $\vec{T} = f(\vec{U})$, find the arc on the unit circle which specifies the rotation f.

2. In Exercise 1, replace P by
   (a) the point $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$;
   (b) the point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$.

3. Find $f(\vec{U})$ if the rotation f is specified by an arc of the unit circle which is
   (a) $\frac{3\pi}{2}$ units long;
   (b) 2π units long.

4. Write $f(\vec{U})$ in the form $u\vec{U} + v\vec{V}$ if f corresponds to an arc of the unit circle which is
   (a) $\frac{\pi}{4}$ units long;
   (b) $\frac{\pi}{3}$ units long.

5. Do Exercise 4 for an arc $\frac{3\pi}{4}$ units long.

6. Let f correspond to a rotation of $\frac{\pi}{2}$ units and g to a rotation of $\frac{\pi}{4}$ units. Show that, since $\vec{V} = f(\vec{U})$, the result in Exercise 5 is equivalent to $g(\vec{V})$.

7. If f and g are any two rotations of the plane about the origin, show that $fg = gf$.

8. If the rotation f corresponds to an arc x and the rotation g to an arc $\frac{\pi}{2}$, show that $f(\vec{V}) = (fg)(\vec{U}) = (gf)(\vec{U})$.

9. In Exercise 8, put $f(\vec{U}) = u\vec{U} + v\vec{V}$, and hence show that $f(\vec{V}) = g(u\vec{U}) + g(v\vec{V}) = ug(\vec{U}) + vg(\vec{V}) = u\vec{V} - v\vec{U}$.

---

6. The Addition Formulas.

We are now ready to bring the circular functions into the picture. Since f maps the vector $\overrightarrow{OA}$ onto $\overrightarrow{OP}$ so that $A(1,0)$ is carried through an arc x of the unit circle to $P(u,v)$, it follows from the definitions of Section 1 that

$$u = \cos x \quad \text{and} \quad v = \sin x.$$

Hence Equations (2) and (3) of Section 5 can be written

$$f(\vec{U}) = (\cos x)\vec{U} + (\sin x)\vec{V} \tag{1}$$

and

$$f(\vec{V}) = (-\sin x)\vec{U} + (\cos x)\vec{V}. \tag{2}$$

Since, moreover, the rotation g differs from the rotation f only in that the arc length involved is y instead of x, we may similarly write

$$g(\vec{U}) = (\cos y)\vec{U} + (\sin y)\vec{V} \tag{3}$$

and

$$g(\vec{V}) = (-\sin y)\vec{U} + (\cos y)\vec{V}. \tag{4}$$

Substituting these results in (7) of Section 5 gives us

$$\begin{aligned}(gf)(\vec{U}) &= (\cos x)\left((\cos y)\vec{U} + (\sin y)\vec{V}\right) + (\sin x)\left((-\sin y)\vec{U} + (\cos y)\vec{V}\right)\\ &= (\cos x \cos y - \sin x \sin y)\vec{U} + (\sin x \cos y + \cos x \sin y)\vec{V}.\end{aligned} \tag{5}$$

Furthermore, the composite rotation gf can be regarded as a single rotation through an arc of length $x + y$, and we may therefore write, by analogy with (1),

$$(gf)(\vec{U}) = \left(\cos (x + y)\right)\vec{U} + \left(\sin (x + y)\right)\vec{V}. \tag{6}$$

We now have, in (5) and (6), two ways of expressing the vector $(gf)(\vec{U})$ in terms of the basic vectors $\vec{U}$ and $\vec{V}$. Since there is essentially only one such way of expressing any vector, it follows that the coefficient of $\vec{U}$ in (5) must be the same as the coefficient of $\vec{U}$ in (6), or

$$\cos (x + y) = \cos x \cos y - \sin x \sin y, \tag{7}$$

and a similar comparison of the coefficients of $\vec{V}$ in the two expressions yields

$$\sin (x + y) = \sin x \cos y + \cos x \sin y. \tag{8}$$

These are the desired addition formulas for the sine and cosine.

We also obtain the subtraction formulas very quickly from Equations (7) and (8). Thus

$$\cos (x - y) = \cos \left(x + (-y)\right) = \cos x \cos (-y) - \sin x \sin (-y). \tag{9}$$

Since, however, (Section 2, Equations (3))

$$\cos (-y) = \cos y$$

and

$$\sin (-y) = -\sin y,$$

we may write (9) as

$$\cos (x - y) = \cos x \cos y + \sin x \sin y. \tag{10}$$

In the Exercises, you will be asked to show similarly that

$$\sin (x - y) = \sin x \cos y - \cos x \sin y. \qquad (11)$$

From formulas (7) and (8) and (10) and (11), it is easy to derive a large number of familiar trigonometric formulas.

Example. Find $\cos (x + \pi)$ and $\sin (x + \pi)$.

Solution. By (7), with $y = \pi$,

$$\cos (x + \pi) = \cos x \cos \pi - \sin x \sin \pi.$$

Now, $\cos \pi = -1$ and $\sin \pi = 0$. Hence $\cos (x + \pi) = -\cos x$.

Similarly, from (8), $\sin (x + \pi) = \sin x \cos \pi + \cos x \sin \pi$

$$= \sin x(-1) + \cos x(0)$$

$$= -\sin x.$$

Exercises 6

1. By use of the appropriate sum or difference formula show that

   (a) $\cos (\frac{\pi}{2} - x) = \sin x$;

   (b) $\sin (\frac{\pi}{2} - x) = \cos x$;

   (c) $\cos (x + \frac{\pi}{2}) = -\sin x$;

   (d) $\sin (x + \frac{\pi}{2}) = \cos x$;

   (e) $\cos (\pi - x) = -\cos x$;

   (f) $\sin (\pi - x) = \sin x$;

   (g) $\cos (\frac{3\pi}{2} + x) = \sin x$;

   (h) $\sin (\frac{3\pi}{2} + x) = -\cos x$;

   (i) $\sin (\frac{\pi}{4} + x) = \cos (\frac{\pi}{4} - x)$.

2. Prove that $\sin (x - y) = \sin x \cos y - \cos x \sin y$.

*3. Show that formulas (7), (8), and (11) may all be obtained from formula (10), and, hence, that all of the relationships mentioned in this section follow from formula (10).

4. Prove that the function tangent (abbreviated tan) defined by

$$\tan\colon\ x \to \frac{\sin x}{\cos x} \qquad (x \neq \pm \frac{\pi}{2} + 2n\pi)$$

   is periodic, with period π. Why are the values $\pm \frac{\pi}{2} + 2n\pi$ excluded from the domain of the tangent function?

5. Using the definition of the function tangent in Exercise 4 and the formulas (7), (8), (10), (11), develop formulas for $\tan (x + y)$ and $\tan (x - y)$ in terms of $\tan x$ and $\tan y$.

6. Using the results of Exercise 5, develop formulas for $\tan (\pi - x)$ and $\tan (\pi + x)$. Also, show that $\tan (-x) = -\tan x$.

7. Express $\sin 2x$, $\cos 2x$, and $\tan 2x$ in terms of functions of x. (Hint: Let $y = x$ in the appropriate formulas.)

8. Express $\sin 3x$ in terms of functions of x.

9. In Exercise 7 you were asked to express $\cos 2x$ in terms of functions of x. One possible result is $\cos 2x = 1 - 2\sin^2 x$. In this expression substitute $x = \frac{y}{2}$ and solve for $\sin \frac{y}{2}$.

10. In Exercise 9, $\cos 2x$ may also be written as $2\cos^2 x - 1$. Use this formula to get a formula for $\cos \frac{y}{2}$.

11. Using the definitions of the function tan and the results of Exercises 9 and 10, derive a formula for $\tan \frac{y}{2}$. This will be an expression involving radicals, but by rationalizing in succession the numerator and the denominator you can get two different expressions for $\tan \frac{y}{2}$, not involving radicals.

In Section 5 we developed the algebra of rotations, and in this section we have applied this algebra to derive the addition formulas for the sine and cosine functions. As we shall now indicate, there is a close parallel between the algebra of rotations and the algebra of complex numbers.

If two complex numbers are expressed in polar form, as are

$$z_1 = r_1(\cos x_1 + i \sin x_1)$$

and

$$z_2 = r_2(\cos x_2 + i \sin x_2)$$

then their product can be found by multiplying their absolute values r_1 and r_2, and adding their arguments, x_1 and x_2:

$$z_1 z_2 = r_1 r_2\big(\cos (x_1 + x_2) + i \sin (x_1 + x_2)\big).$$

Multiplying any complex number z by the special complex number

$$\cos x + i \sin x = 1(\cos x + i \sin x)$$

is therefore equivalent to leaving the absolute value of z unchanged and adding x to the argument of z. Hence, if we represent z by a vector in the complex plane, then multiplying by $\cos x + i \sin x$ is equivalent to rotating this vector through an arc x, as in Section 5.

Let us replace the vector $\vec{U}$ of Section 5 by the complex number

$$1 = \cos 0 + i \sin 0.$$

Then the product

$$(\cos x + i \sin x) \cdot 1 = \cos x + i \sin x$$

represents the vector formerly called $f(\vec{U})$ (see Figure 28), and $(gf)(\vec{U})$ becomes

$$(\cos y + i \sin y)(\cos x + i \sin x) \cdot 1 =$$
$$\big((\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y)\big) \cdot 1.$$

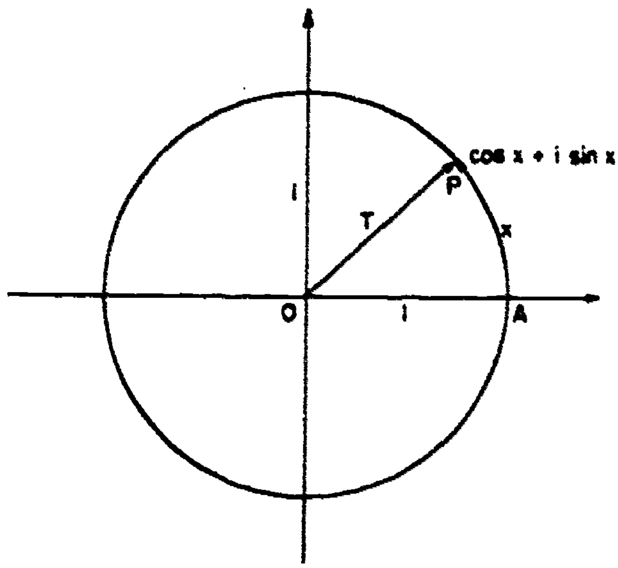

Figure 28. Representation of $\vec{T} = \cos x + i \sin x$.

If we replace $(gf)(\vec{U})$ by

$$\big(\cos (x + y) + i \sin (x + y)\big) \cdot 1$$

we have

$$\cos (x + y) + i \sin (x + y) = (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y).$$

By equating real and imaginary parts we obtain the addition formulas (7) and (8).

The subtraction formulas may be derived equally simply. Since g^{-1} is equivalent to rotating through an angle $-y$, we have $(g^{-1}f)(\vec{U}) = g^{-1}f(\vec{U})$ and therefore

$$\big(\cos (x - y) + i \sin (x - y)\big) \cdot 1.$$

Hence

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

and

$$\sin (x - y) = \sin x \cos y - \cos x \sin y.$$

7. Construction and Use of Tables of Circular Functions.

It is difficult to give in a short space an indication of the enormous variety of ways in which the addition formulas of Section 6

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \quad (1)$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad (2)$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad (3)$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \quad (4)$$

turn up in mathematics and in the application of mathematics to the sciences. In this section and in Sections 8 and 9, we shall describe some of the more common applications. The first of these is their use in the construction of a table of values of the sine and cosine functions.

In Exercise 1 of Section 6, you used the difference formulas to show that

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

and

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x.$$

These formulas permit the tabulation of $\sin x$ and $\cos x$ in a very neat way. If we had a table of cosines for $0 \leq x \leq \frac{\pi}{2}$, this would, in effect, give a table of sines in backward order. For example, from the table of special values in Section 2, we obtain the sample table shown, where $y = \frac{\pi}{2} - x$.

x	$\cos x$	$\frac{\pi}{2} - x$
0	1	$\frac{\pi}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{\pi}{2}$	0	0
$\frac{\pi}{2} - y$	$\sin y$	y

In this table the values of the cosine are read from the top down, and the values of the sine from the bottom up. Since it is a very inefficient use of space to put so few columns on a page, the table is usually folded in the middle about the value $x = y = \frac{\pi}{4}$ and is constructed as in the following sample:

x	$\cos x$	$\sin x$	$\frac{\pi}{2} - x$
0	1	0	$\frac{\pi}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\pi}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$\frac{\pi}{2} - y$	$\sin y$	$\cos y$	y

At the end of this pamphlet we give three tables:

I. A table of $\sin x$ and $\cos x$ for decimal values of x up to 1.57 (slightly less than $\frac{\pi}{2}$).

II. A table of $\sin \frac{\pi x}{2}$ and $\cos \frac{\pi x}{2}$ in decimal fractions of $\frac{\pi}{2}$ up to 1.00.

III. A table of $\sin^{\circ} x$, $\cos^{\circ} x$, and $\tan^{\circ} x$, in degrees up to 90°.

(We define $\sin^{\circ}: x^{\circ} \to \sin x^{\circ}$, with similar definitions for $\cos^{\circ}$ and $\tan^{\circ}$. It is usual to write $\sin x$ in place of $\sin^{\circ} x$, etc., when the context makes it clear what is intended. We shall follow this practice.)

Exercises 7a

1. Why is Table I not folded as are Tables II and III?

2. Find from Table I $\sin x$ and $\cos x$ when x is equal to
   (a) 0.73;
   (b) -5.17;
   (c) 1.55;
   (d) 6.97 (Hint: $2\pi \approx 6.28$).

3. From Table I, find x when $0 \leq x \leq \frac{\pi}{2}$ and
   (a) $\sin x \approx 0.1099$;
   (b) $\cos x \approx 0.9131$;
   (c) $\sin x \approx 0.6495$;
   (d) $\cos x \approx 0.5403$.

4. From Table II, find $\sin \omega t$ and $\cos \omega t$ if $\omega = \frac{\pi}{2}$ and
   (a) $t = 0.31$;
   (b) $t = 0.79$;
   (c) $t = 0.62$;
   (d) $t = 0.71$.

5. From Table II, find t (interpolating, if necessary), if $\omega = \frac{\pi}{2}$, $0 \leq t \leq 1$ and
   (a) $\sin \omega t \approx 0.827$;
   (b) $\cos \omega t \approx 0.905$;
   (c) $\sin \omega t \approx 0.475$;
   (d) $\cos \omega t \approx 0.795$.

6. From Table III, find $\sin x$ and $\cos x$ (interpolating, if necessary) when

(a) $x = 45^\circ$; (c) $x = 36.2^\circ$;

(b) $x = 73^\circ$; (d) $x = 81.5^\circ$.

7. From Table III, find x when $0 \leq x \leq 90^\circ$ and

(a) $\sin x \approx 0.629$; (c) $\sin x \approx 0.621$;

(b) $\cos x \approx 0.991$; (d) $\cos x \approx 0.895$.

Extending the scope of the tables. Table I, at the end of this pamphlet, gives values of the circular functions $\cos: x \to \cos x$ and $\sin: x \to \sin x$ only for $0 \leq x \leq \frac{\pi}{2}$, but we can extend its scope to the set of all real numbers by using (a) equations (1)-(4), (b) our knowledge of the circular functions of all multiples of $\frac{\pi}{2}$ (see, for example, Table 1), and (c) the fact that any real number can be expressed as the sum (or difference) of two numbers of which one is a multiple of $\frac{\pi}{2}$ and the other is in the interval $\{x: 0 \leq x \leq \frac{\pi}{2}\}$. Similar remarks apply to Tables II and III. The technique is best explained through examples.

Example 1. Find $\sin 2$.

Solution. Since $\frac{\pi}{2} \approx 1.57$, we write $2 = 1.57 + 0.43$, and, using equation (2), we then have

$$\begin{aligned}\sin 2 &= \sin(1.57 + 0.43)\\ &\approx \sin(\tfrac{\pi}{2} + 0.43)\\ &= \sin\tfrac{\pi}{2}\cos 0.43 + \cos\tfrac{\pi}{2}\sin 0.43\\ &= \cos 0.43\\ &\approx 0.9090.\end{aligned}$$

Alternatively, $2 = 3.14 - 1.14 \approx \pi - 1.14$, and therefore

$$\begin{aligned}\sin 2 &\approx \sin(\pi - 1.14)\\ &= \sin\pi\cos 1.14 - \cos\pi\sin 1.14\\ &= \sin 1.14\\ &\approx 0.9086.\end{aligned}$$

Example 2. Find $\cos 4.56$.

Solution. Since $4.56 = 3.14 + 1.42 \approx \pi + 1.42$, we have

$$\begin{aligned}\cos 4.56 &\approx \cos(\pi + 1.42)\\ &= \cos\pi\cos 1.42 - \sin\pi\sin 1.42\\ &= -\cos 1.42\\ &\approx -0.1502.\end{aligned}$$

This technique can be used to simplify expressions of the form $\sin\,(n\frac{\pi}{2} \pm x)$ and $\cos\,(n\frac{\pi}{2} \pm x)$.

Exampl` 3. Simplify $\cos\,(\frac{5\pi}{2} + x)$.

Solution.
$$\begin{aligned} \cos\,(\tfrac{5\pi}{2} + x) &= \cos \tfrac{5\pi}{2} \cos x - \sin \tfrac{5\pi}{2} \sin x \\ &= \cos \tfrac{\pi}{2} \cos x - \sin \tfrac{\pi}{2} \sin x \\ &= -\sin x. \end{aligned}$$

Example 4. Find $\cos 0.82\pi$.

Solution. In this case, it is easier to use Table II. Since $0.82\pi = 0.50\pi + 0.32\pi$, we have

$$\begin{aligned} \cos 0.82\pi &= \cos\,(\tfrac{\pi}{2} + 0.32\pi) \\ &= \cos \tfrac{\pi}{2} \cos 0.32\pi - \sin \tfrac{\pi}{2} \sin 0.32\pi \\ &= -\sin 0.32\pi \\ &= -\sin 0.64(\tfrac{\pi}{2}) \\ &\approx -0.844. \end{aligned}$$

Exercises 7b

Using the table that you think most convenient, find

1. $\sin 1.73$;
2. $\cos 1.3\pi$;
3. $\sin\,(-.37)$;
4. $\sin\,(-.37\pi)$;
5. $\cos 2.8\pi$;
6. $\cos 1.8\pi$;
7. $\cos 3.71$;
8. $\sin 135^\circ$;
9. $\cos\,(-135^\circ)$;
10. $\sin 327^\circ$;
11. $\cos\,(-327^\circ)$;
12. $\cos 12.4\pi$;
13. $\sin 12.4$;

*14. $\cos\,(\sin .3\pi)$;

*15. $\sin\,(\sin .7)$.

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8. Pure Waves: Frequency, Amplitude, and Phase.

As we remarked in Section 4, the superposition of two pure waves of the same frequency yields a pure wave of the given frequency. Now we shall be able to prove this result. In order to be more specific, instead of assuming that either of equations (1) in Section 4 defines a pure wave, let us say that, by definition, a pure wave will have the form

$$y = A \cos\,(\omega t - \alpha), \qquad (1)$$

where A and ω are positive and $0 \leq \alpha < 2\pi$. The number α is called the phase of the pure wave. The sine function now becomes simply a special case of (1), and defines a pure wave with phase $\frac{\pi}{2}$,

$$y = \sin \omega t = \cos (\omega t - \tfrac{\pi}{2}). \qquad (2)$$

The phase of a pure wave has a simple interpretation. We will take the graph of

$$y = \cos \omega t \qquad (3)$$

as a standard of reference, and the cycle over the interval $(0 \le t < \frac{2\pi}{\omega})$ between two peaks of (3) as the standard cycle. Now the graph of $y = A \cos (\omega t - \alpha)$ reaches its peak, corresponding to the first peak of its standard cycle, at the point where $\omega t - \alpha = 0$, that is, at $t = \frac{\alpha}{\omega}$. Since $\frac{\alpha}{\omega}$ is positive, it is clear that the wave (1) reaches its first peak after the standard wave (3) reaches its first peak, since (3) has a peak at $t = 0$. That is, the wave (1) lags behind the wave (3) by an amount $\frac{\alpha}{\omega}$. Since the period of (3) is $\frac{2\pi}{\omega}$, this lag amounts to the fraction

$$\frac{\frac{\alpha}{\omega}}{\frac{2\pi}{\omega}} = \frac{\alpha}{2\pi}$$

of a period. (Figure 29.) We see from (2) that $\sin \omega t$ lags behind $\cos \omega t$ by a quarter period. (See Figures 19 and 20.)

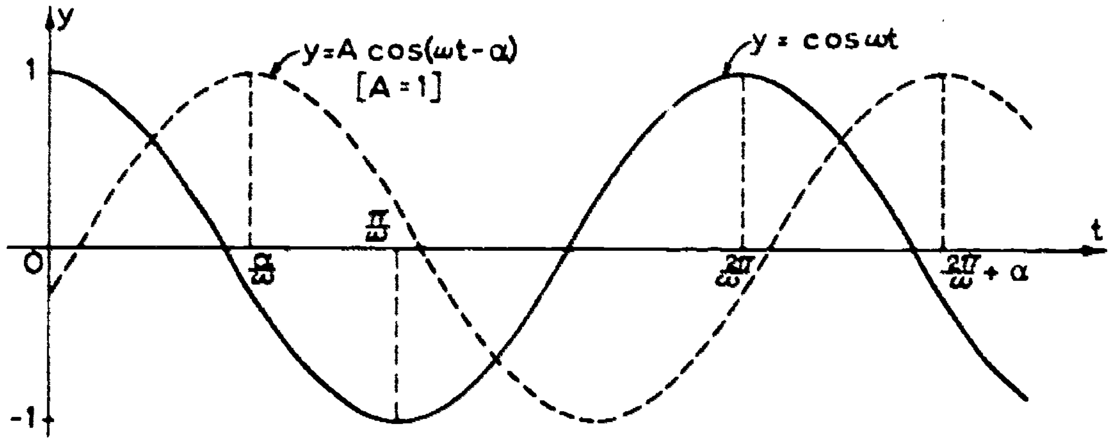

Figure 29. Graphs of two cosine curves.

We now wish to test the idea that the sum of two pure waves which have the same period but differ in amplitude and phase is again a pure wave of the same period with some new amplitude and phase. You will recall that in Section 4 we sketched the graph of

$$y = 3 \cos \pi t + 4 \sin \pi t \qquad (4)$$

by adding the ordinates of the graphs of $u = 3 \cos \pi t$ and $v = 4 \sin \pi t$. The graph supported this idea. At that time we also had to leave open the question of the exact location of the maximum and minimum points and the zeros of the graph.

We are now in a position to deal with these problems. Since finding the maximum and minimum points and finding the zeros involve essentially the same procedure, we shall confine our attention to the maximum and minimum points.

Our basic problem still is to express

$$y = 3 \cos \pi t + 4 \sin \pi t$$

in the form of

$$y = A \cos (\omega t - \alpha) \qquad (1)$$

that is, to show that y is a pure wave, but in the process we shall be able to obtain the exact location of the maximum and minimum points of the graph of the sum. If we write out (1) in terms of the formula

$$\cos (\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha \qquad (5)$$

we obtain

$$y = A \cos (\omega t - \alpha) = A(\cos \omega t \cos \alpha + \sin \omega t \sin \alpha)$$

or

$$y = A \cos \omega t \cos \alpha + A \sin \omega t \sin \alpha. \qquad (6)$$

In our case, $\omega = \pi$ we have

$$y = A \cos \pi t \cos \alpha + A \sin \pi t \sin \alpha. \qquad (7)$$

Upon comparing (7) with (4), we note that if

$$A \cos \alpha = 3 \quad \text{and} \quad A \sin \alpha = 4 \qquad (8)$$

then (7) and (4) will be identical. We shall therefore seek values of A and α which satisfy the equations (8). To do this, we may begin by squaring both sides of the equations (8) and adding them to obtain

$$3^2 + 4^2 = A^2 \cos^2\alpha + A^2 \sin^2\alpha$$

$$9 + 16 = A^2(\cos^2\alpha + \sin^2\alpha)$$

or

$$A^2 = 25.$$

Since A is positive, we have

$$A = 5, \qquad (9)$$

and consequently from (8),

$$\cos \alpha = \frac{3}{5} \quad \text{and} \quad \sin \alpha = \frac{4}{5}. \qquad (10)$$

From Table I

$$\alpha \approx 0.927. \qquad (11)$$

Now, by using (9) and (11), we may put (4) in the form

$$y = 3 \cos \pi t + 4 \sin \pi t \approx 5 \cos (\pi t - 0.927), \qquad (12)$$

showing that it is a pure wave with amplitude 5, period 2 (as before), and phase 0.927. We note that $\frac{0.927}{\pi} \approx 0.295$ is very close to the value 0.29 obtained graphically in Section 4. We are also in a position to locate the maximum and minimum points of our graph. From (12), y will be a maximum when

$$\cos (\pi t - 0.927) = 1,$$

that is,

$$\pi t - 0.927 = 0$$

$$t = \frac{0.927}{\pi} \approx 0.295,$$

and y will be a minimum when

$$\cos (\pi t - 0.927) = -1,$$

that is

$$\pi t - 0.927 = \pi$$

$$t = 1 + \frac{0.927}{\pi} \approx 1.295$$

where, in each case, we have taken the smallest positive value of t.

We now put the general equation

$$y = B \cos \omega t + C \sin \omega t \qquad (13)$$

in the form (1). If we proceed exactly as before, using (6) and (13), we find that for specified B and C, $A = \sqrt{B^2 + C^2}$ and a solution of the equations

$$\cos \alpha = \frac{B}{A} \quad \text{and} \quad \sin \alpha = \frac{C}{A} \qquad (14)$$

will determine a unique α in the interval from 0 to 2π, from which the form (1) follows. (See Exercise 3 below.)

Exercises 8

1. What is the smallest positive value of t for which the graph of equation (4) crosses the t-axis? Compare your result with the data shown in Figure 21.

2. Sketch each of the following graphs over at least two of its periods. Show the amplitude, period, and phase of each.

   (a) $y = 2 \cos 3t$

   (b) $y = 2 \cos (\frac{3t}{2})$

   (c) $y = 3 \cos (-2t)$

   (d) $y = -2 \sin (\frac{t}{3})$ (Remember that the phase is defined to be positive.)

   (e) $y = -2 \sin (2t + \pi)$

   (f) $y = 5 \cos (3t + \frac{\pi}{6})$

3. Express each of the following equations in the form $y = A\cos(\pi t - \alpha)$ for some appropriate real numbers A and α.

(a) $y = 4\sin\pi t - 3\cos\pi t$

(b) $y = -4\sin\pi t + 3\cos\pi t$

(c) $y = -4\sin\pi t - 3\cos\pi t$

(d) $y = 3\sin\pi t + 4\cos\pi t$

(e) $y = 3\sin\pi t - 4\cos\pi t$

4. Without actually computing the value of α, show on a diagram how A and α can be determined from the coefficients B and C of $\cos\omega t$ and $\sin\omega t$ if each of the following expressions of the form $B\cos\omega t + C\sin\omega t$ is made equal to $A\cos(\omega t - \alpha)$. Compute α, and find the maximum and minimum values of each expression, and its period. Give reasons for your answers.

(a) $3\sin 2t + 4\cos 2t$

(b) $2\sin 3t - 3\cos 3t$

(c) $-\sin(\frac{t}{2}) + \cos(\frac{t}{2})$

5. Verify that the superposition of any two pure waves $A\cos(\omega t - \alpha)$ and $B\cos(\omega t - \beta)$ is a pure wave of the same frequency, that is, that there exist real numbers C and γ such that

$$A\cos(\omega t - \alpha) + B\cos(\omega t - \beta) = C\cos(\omega t - \gamma).$$

6. Solve for all values of t:

(a) $3\cos\pi t + 4\sin\pi t = 2.5$

[Method: From equation (12) we see that this equation is equivalent to $5\cos(\pi t - 0.927) = 2.5$. For every solution, we have

$$\cos(\pi t - 0.927) = 0.5,$$

which is satisfied only if the argument of the cosine is $\frac{\pi}{3} + 2n\pi$ or $-\frac{\pi}{3} + 2n\pi$. It follows that the equation is satisfied for all values of t such that

$$\pi t - 0.927 = \pm\frac{\pi}{3} + 2n\pi$$

or such that

$$t \approx \frac{0.927}{\pi} \pm \frac{1}{3} + 2n.$$

Question: What is the smallest positive value of t for which equation (a) is satisfied?]

(b) $3\cos\pi t + 4\sin\pi t = 5$

(c) $\sin 2t - \cos 2t = 1$

(d) $4\cos\pi t - 3\sin\pi t = 0$

(e) $4\cos\pi t + 3\sin\pi t = 1$

*7. Show that any wave of the form

$$y = B \cos(\mu t - \beta), \quad (\mu \neq 0),$$

can be written in the form (1), that is,

$$y = A \cos(\omega t - \alpha)$$

where A is non-negative, ω positive and $0 \leq \alpha < 2\pi$.

---

9. Analysis of General Waves.

In Sections 4 and 8 we considered the superposition of two pure waves of the same period (or frequency). We found that the superposition of such waves is again a pure wave of the given frequency. Next we ask what conclusion we can draw about the superposition of two waves with different periods. Suppose,

$$y = 2 \sin 3x - 3 \cos 2x.$$

Unfortunately, $\sin 3x$ and $\cos 2x$ have different fundamental periods, $\frac{2\pi}{3}$ and π, so they cannot be combined into a single term, the way we could if we had only $\cos 3x$ and $\sin 3x$, say, or $\cos 2x$ and $\sin 2x$. However, any multiple of a period can be looked upon as a period. That is, we can consider $y = 2 \sin 3x$ as having a period of $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, 2π, $\frac{8\pi}{3}$, or any other integral multiple of $\frac{2\pi}{3}$. Similarly, $y = 3 \cos 2x$ can be considered as having a period of π, 2π, 3π, etc. Now, comparing these values, we note that both expressions can be considered as having a period of 2π, and hence their difference will also have a period of 2π. In effect, we simply find the least common multiple of the periods of two dissimilar expressions of this form and we have the period of their sum or difference. There is little else that we can conclude in general. About all we can do to simplify matters is to sketch separately the graphs of

$$u = 2 \sin 3x, \quad v = 3 \cos 2x,$$

and $y = u - v$. The result is shown by the three curves in Figure 30.

The superposition of sine and cosine waves of different periods can produce quite complicated curves. In fact, with only slight restrictions, any periodic function can be approximated arbitrarily closely as a sum of a finite number of sines and cosines. The subject of harmonic analysis or Fourier series is concerned with approximating periodic functions in this way. The principal theorem, first stated by Fourier, is that a function f of period a can be approximated arbitrarily closely by sines and cosines for each of which some multiple of the fundamental period is a.

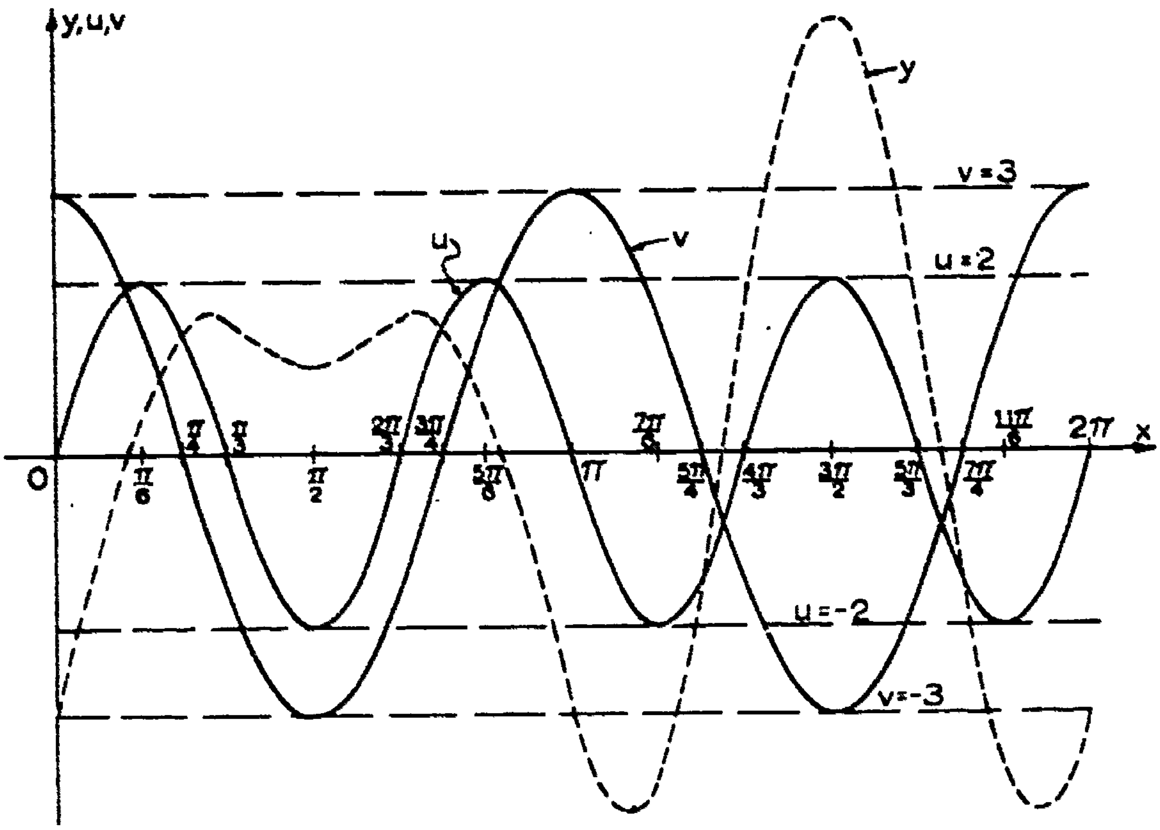

Figure 30.

$u = 2 \sin 3x, \quad v = 3 \cos 2x$

$y = u - v = 2 \sin 3x - 3 \cos 2x, \quad 0 \leq x \leq 2\pi.$

Specifically,

$$f(x) \approx A_0 + (A_1 \cos \frac{2\pi x}{a} + B_1 \sin \frac{2\pi x}{a}) + (A_2 \cos \frac{4\pi x}{a} + B_2 \sin \frac{4\pi x}{a}) + \cdots$$
$$+ (A_n \cos \frac{2n\pi x}{a} + B_n \cos \frac{2n\pi x}{a}), \qquad (1)$$

and the more terms we use, the better is our approximation.

As an example, consider the function depicted in Figure 31. This function is defined on the interval $-\pi \leq x < \pi$ by

$$f(x) = \begin{cases} 0, & \text{if } x = -\pi \\ -1, & \text{if } -\pi < x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x < \pi. \end{cases} \qquad (2)$$

For all other values of x we define $f(x)$ by the periodicity condition

$$f(x + 2\pi) = f(x).$$

This function has a particularly simple approximation as a series of the form (1), namely,

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin (2n - 1)x}{2n - 1}\right). \qquad (3)$$

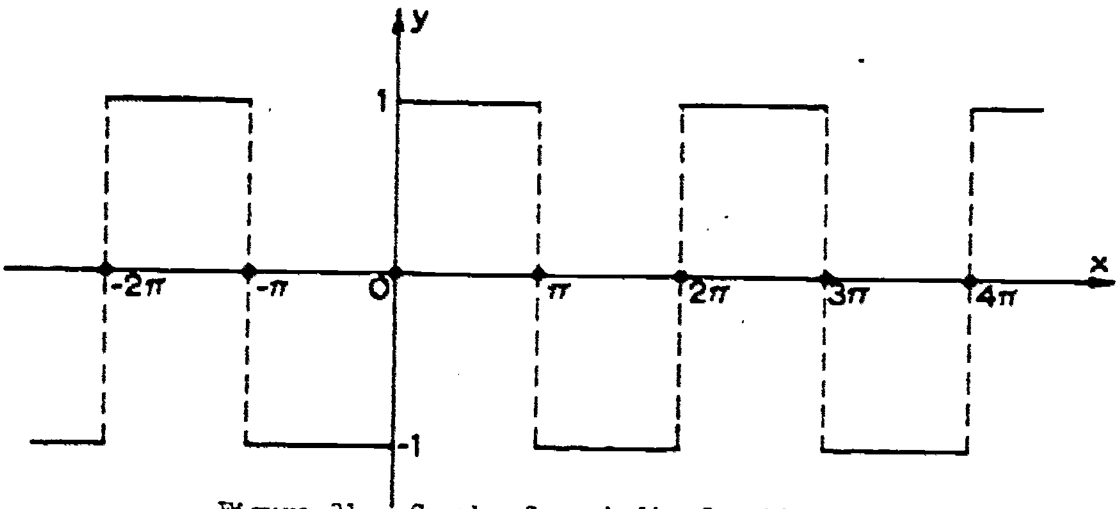

Figure 31. Graph of periodic function.

$$x \to f(x) = \begin{cases} 0, & \text{if } x = -\pi \\ 1, & \text{if } 0 < x < \pi \\ 0, & \text{if } x = 0 \\ -1, & \text{if } -\pi < x < 0 \end{cases}; \quad f(x + 2\pi) = f(x).$$

Fourier series: $\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots + \frac{\sin (2n-1)x}{2n-1}\right).$

As an exercise, you may graph the successive approximations to $f(x)$ by taking one, then two, then three terms of the series, and see how the successive graphs approach the graph of $y = f(x)$.

The problem of finding the series (1) for any given periodic function f is taken up in calculus.

Exercises 9

1. Sketch graphs, for $|x| < \pi$, for each of the following curves.

(a) $y = \frac{4}{\pi} \sin x$

(b) $y = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3}\right)$

(c) $y = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right)$

2. (a) Find the periods of each of the successive terms of the series (3), namely,

$$\sin x, \frac{\sin 3x}{3}, \frac{\sin 5x}{5}, \dots .$$

(b) What terms of the general series (1) are missing? From the symmetry properties of the function f defined by (2) can you see a reason for the absence of certain terms?

10. Further Applications of Circular Functions.

We have seen in Section 4 that the circular functions of real numbers arise naturally in the theory of sound and, more generally, in the study of simple harmonic motion. Here, however, it would be possible (although somewhat inconvenient) to work in terms of functions of angles rather than of real numbers. There are many applications of the circular functions, however, where one must use the real-number approach.

For example, in the study of vibrating membranes (e.g., a drum!) the equation

$$\cos x - \frac{\sin x}{x} = 0$$

arises. Clearly, $\sin x$ is a number and hence $\frac{\sin x}{x}$ is meaningful only if we are considering x as also being a real number.

Similarly in the study of the motion of an electron of mass m and charge e subjected to an electric field of intensity E and a magnetic field of intensity H we need to consider the expression

$$\frac{Em}{H^2e}\left(\frac{He}{m}t - \sin\frac{Het}{m}\right)$$

where, again, t must be considered as a real number.

In particular, the calculus abounds in situations where the real-number point of view must be used. Listed below are a few of the many expressions found in the calculus in which the trigonometry of real numbers must be used:

$\frac{\sin h}{h}$ (in finding the "derivative" of $\sin x$);

$\frac{3}{2}x - 2\sin x + \sin 2x$ (in finding the area under one arch of a "cycloid");

$x - \frac{\sin 2x}{2}$ (in "integrating" $\sin^2 x$);

$e^x(\cos x + \sin x)$ (in solving a "differential" equation).

Suggestions for Further Reading

Dubisch, Roy.

Trigonometry. New York: The Ronald Press Co., 1955.
Chapters 2 and 3.

Vance, E. P.

Modern Algebra and Trigonometry. Reading, Massachusetts: Addison-Wesley Publishing Co., 1962.
Chapter 6.

Table I

Values of sin x and cos x for $0 \leq x \leq 1.57$.

x	sin x	cos x	x	sin x	cos x
.00	.0000	1.0000	.40	.3894	.9211
.01	.0100	1.0000	.41	.3986	.9171
.02	.0200	.9998	.42	.4078	.9131
.03	.0300	.9996	.43	.4169	.9090
.04	.0400	.9992	.44	.4259	.9048
.05	.0500	.9988	.45	.4350	.9004
.06	.0600	.9982	.46	.4439	.8961
.07	.0699	.9976	.47	.4529	.8916
.08	.0799	.9968	.48	.4618	.8870
.09	.0899	.9960	.49	.4706	.8823
.10	.0998	.9950	.50	.4794	.8776
.11	.1098	.9940	.51	.4882	.8727
.12	.1197	.9928	.52	.4969	.8678
.13	.1296	.9916	.53	.5055	.8628
.14	.1395	.9902	.54	.5141	.8577
.15	.1494	.9888	.55	.5227	.8525
.16	.1593	.9872	.56	.5312	.8473
.17	.1692	.9856	.57	.5396	.8419
.18	.1790	.9838	.58	.5480	.8365
.19	.1889	.9820	.59	.5564	.8309
.20	.1987	.9801	.60	.5646	.8253
.21	.2085	.9780	.61	.5729	.8196
.22	.2182	.9759	.62	.5810	.8139
.23	.2280	.9737	.63	.5891	.8080
.24	.2377	.9713	.64	.5972	.8021
.25	.2474	.9689	.65	.6052	.7961
.26	.2571	.9664	.66	.6131	.7900
.27	.2667	.9638	.67	.6210	.7838
.28	.2764	.9611	.68	.6288	.7776
.29	.2860	.9582	.69	.6365	.7712
.30	.2955	.9553	.70	.6442	.7648
.31	.3051	.9523	.71	.6518	.7584
.32	.3146	.9492	.72	.6594	.7518
.33	.3240	.9460	.73	.6669	.7452
.34	.3335	.9428	.74	.6743	.7385
.35	.3429	.9394	.75	.6816	.7317
.36	.3523	.9359	.76	.6889	.7248
.37	.3616	.9323	.77	.6961	.7179
.38	.3709	.9287	.78	.7033	.7109
.39	.3802	.9249	.79	.7104	.7038

Table I -- Cont.

x	sin x	cos x	x	sin x	cos x
.80	.7174	.6967	1.20	.9320	.3624
.81	.7243	.6895	1.21	.9356	.3530
.82	.7311	.6822	1.22	.9391	.3436
.83	.7379	.6749	1.23	.9425	.3342
.84	.7446	.6675	1.24	.9458	.3248
.85	.7513	.6600	1.25	.9490	.3153
.86	.7578	.6524	1.26	.9521	.3058
.87	.7643	.6448	1.27	.9551	.2963
.88	.7707	.6372	1.28	.9580	.2867
.89	.7771	.6294	1.29	.9608	.2771
.90	.7833	.6216	1.30	.9636	.2675
.91	.7895	.6137	1.31	.9662	.2579
.92	.7956	.6058	1.32	.9687	.2482
.93	.8016	.5978	1.33	.9711	.2385
.94	.8076	.5898	1.34	.9735	.2288
.95	.8134	.5817	1.35	.9757	.2190
.96	.8192	.5735	1.36	.9779	.2092
.97	.8249	.5653	1.37	.9799	.1994
.98	.8305	.5570	1.38	.9819	.1896
.99	.8360	.5487	1.39	.9837	.1798
1.00	.8415	.5403	1.40	.9854	.1700
1.01	.8468	.5319	1.41	.9871	.1601
1.[illegible]	.8521	.5234	1.42	.9887	.1502
1.03	.8573	.5148	1.43	.9901	.1403
1.04	.8624	.5062	1.44	.9915	.1304
1.05	.8674	.4976	1.45	.9927	.1205
1.06	.8724	.4889	1.46	.9939	.1106
1.07	.8772	.4801	1.47	.9949	.1006
1.08	.8820	.4713	1.48	.9959	.0907
1.09	.8866	.4625	1.49	.9967	.0807
1.10	.8912	.4536	1.50	.9975	.0707
1.11	.8957	.4447	1.51	.9982	.0608
1.12	.9001	.4357	1.52	.9987	.0508
1.13	.9044	.4267	1.53	.9992	.0408
1.14	.9086	.4176	1.54	.9995	.0308
1.15	.9128	.4085	1.55	.9998	.0208
1.16	.9168	.3993	1.56	.9999	.0108
1.17	.9208	.3902	1.57	1.0000	.0008
1.18	.9246	.3809			
1.19	.9284	.3717			

Table II

Tables of sin and cos in decimal fractions of $\frac{\pi}{2}$.

x	$\sin x\frac{\pi}{2}$	$\cos x\frac{\pi}{2}$	
.00	.000	1.000	1.00
.01	.016	1.000	.99
.02	.031	1.000	.98
.03	.048	.999	.97
.04	.063	.998	.96
.05	.078	.997	.95
.06	.094	.996	.94
.07	.110	.994	.93
.08	.125	.992	.92
.09	.141	.990	.91
.10	.156	.988	.90
.11	.172	.985	.89
.12	.187	.982	.88
.13	.203	.979	.87
.14	.218	.976	.86
.15	.233	.972	.85
.16	.249	.969	.84
.17	.264	.965	.83
.18	.279	.960	.82
.19	.294	.956	.81
.20	.309	.951	.80
.21	.324	.946	.79
.22	.339	.941	.78
.23	.353	.935	.77
.24	.368	.930	.76
.25	.383	.924	.75
.26	.397	.918	.74
.27	.412	.911	.73
.28	.426	.905	.72
.29	.440	.898	.71
.30	.454	.891	.70
	$\cos y\frac{\pi}{2}$	$\sin y\frac{\pi}{2}$	y

Table II -- Cont.

x	$\sin x\frac{\pi}{2}$	$\cos x\frac{\pi}{2}$	
.30	.454	.891	.70
.31	.468	.884	.69
.32	.482	.876	.68
.33	.495	.869	.67
.34	.509	.861	.66
.35	.523	.853	.65
.36	.536	.844	.64
.37	.549	.836	.63
.38	.562	.827	.62
.39	.575	.818	.61
.40	.588	.809	.60
.41	.600	.800	.59
.42	.613	.790	.58
.43	.625	.780	.57
.44	.637	.771	.56
.45	.649	.760	.55
.46	.661	.750	.54
.47	.673	.740	.53
.48	.685	.729	.52
.49	.696	.718	.51
.50	.707	.707	.50
	$\cos y\frac{\pi}{2}$	$\sin y\frac{\pi}{2}$	y

Table III

x°	$\sin^\circ x$	$\cos^\circ x$	$\tan^\circ x$	x°	$\sin^\circ x$	$\cos^\circ x$	$\tan^\circ x$
0	0.000	1.000	0.000				
1	.018	1.000	.018	46	0.719	0.695	1.036
2	.035	0.999	.035	47	.731	.682	1.072
3	.052	.999	.052	48	.743	.669	1.111
4	.070	.998	.070	49	.755	.656	1.150
5	.087	.996	.088	50	.766	.643	1.192
6	.105	.995	.105	51	.777	.629	1.235
7	.122	.993	.123	52	.788	.616	1.280
8	.139	.990	.141	53	.799	.602	1.327
9	.156	.988	.158	54	.809	.588	1.376
10	.174	.985	.176	55	.819	.574	1.428
11	.191	.982	.194	56	.829	.559	1.483
12	.208	.978	.213	57	.839	.545	1.540
13	.225	.974	.231	58	.848	.530	1.600
14	.242	.970	.249	59	.857	.515	1.664
15	.259	.966	.268	60	.866	.500	1.732
16	.276	.961	.287	61	.875	.485	1.804
17	.292	.956	.306	62	.883	.470	1.881
18	.309	.951	.325	63	.891	.454	1.963
19	.326	.946	.344	64	.899	.438	2.050
20	.342	.940	.364	65	.906	.424	2.145
21	.358	.934	.384	66	.914	.407	2.246
22	.375	.927	.404	67	.921	.391	2.356
23	.391	.921	.425	68	.927	.375	2.475
24	.407	.914	.445	69	.934	.358	2.605
25	.423	.906	.466	70	.940	.342	2.747
26	.438	.899	.488	71	.946	.326	2.904
27	.454	.891	.510	72	.951	.309	3.078
28	.470	.883	.532	73	.956	.292	3.271
29	.485	.875	.554	74	.961	.276	3.487
30	.500	.866	.577	75	.966	.259	3.732
31	.515	.857	.601	76	.970	.242	4.011
32	.530	.848	.625	77	.974	.225	4.331
33	.545	.839	.649	78	.978	.208	4.705
34	.559	.829	.675	79	.982	.191	5.145
35	.574	.819	.700	80	.985	.174	5.671
36	.588	.809	.727	81	.988	.156	6.314
37	.602	.799	.754	82	.990	.139	7.115
38	.616	.788	.781	83	.993	.122	8.144
39	.629	.777	.810	84	.995	.105	9.514
40	.643	.766	.839	85	.996	.087	11.43
41	.658	.755	.869	86	.998	.070	14.30
42	.669	.743	.900	87	.999	.052	19.08
43	.682	.731	.933	88	.999	.035	28.64
44	.695	.719	.966	89	1.000	.018	57.29
45	.707	.707	1.000	90	1.000	.000	