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ABSTRACT

This text presents lessons relating specific mathematical concepts to the ideas, skills, and tasks pertinent to the health care field. Among other concepts covered are linear functions, vectors, trigonometry, and statistics. Many of the lessons use data acquired during science experiments as the basis for exercises in mathematics. Lessons present mathematical exercises in the context of biomedical problems and situations. (RE)

BIOMEDICAL MATHEMATICS

UNIT I

MEASUREMENT, LINEAR FUNCTIONS AND DIMENSIONAL ALGEBRA

STUDENT TEXT

REVISED VERSION, 1975

THE BIOMEDICAL INTERDISCIPLINARY CURRICULUM PROJECT

SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION

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SECTION 1:

1-1 What Kind of Course Is This?

The mathematics part of the Biomedical Curriculum could be many things. It could be a conventional mathematics course, which emphasizes the learning of mathematical ideas and skills. Or it could concentrate on the mathematics needed for specific biomedical tasks. A third possibility is that you could do in mathematics class the computations required by the science part of the curriculum.

None of these three approaches alone seems adequate. The mathematics course you are now beginning is therefore a combination of all three.

You will be introduced to several mathematical topics. During this this course you will encounter linear functions, vectors, trigonometry and statistics, among others.

The mathematical ideas and skills you learn will be used to solve problems specifically related to biomedical careers. For instance, you will learn how to make the calculations required to prepare a given amount of a certain concentration of a chemical or drug. This is a type of calculation done frequently by both nurses and pharmacists. You will use trigonometry to determine the magnification of X-rays, as an X-ray technician must be able to do.

Your mathematical skills will also be used to solve problems which arise in science class. You will often do an experiment in science class and take the data you obtain to mathematics class, where you will learn how to treat the data mathematically.

While studying respiration, for example, you will examine the properties of gases in science class. You will do experiments in which you determine how the volume of a gas changes when it is compressed or when it is heated. Your laboratory data will be used to obtain an equation which expresses mathematically the relation between the pressure, the temperature and the volume of a gas.

You will use your knowledge of statistics to determine whether a class of elementary school children has adequate vision.

The mathematical skills that you learn, however, will not be applied only to biomedical problems and science data. We recognize that you are a person, and have included mathematical problems related to your own life. Vector techniques will be used to solve a variety of nutritional problems. You will use vectors to determine the number of calories in a food from the quantities of protein, fat and carbohydrate it contains. You will use a technique called linear programming to solve such problems as finding the least expensive combination of foods which has the same protein content as a pound of beef.

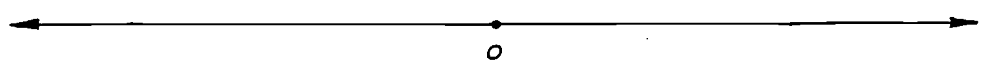
If we limited our mathematical skills to those needed to do current biomedical computations, we would be unprepared for problems that might arise in the future, requiring other mathematical techniques. Thus, later on in the course you will meet complex numbers. Complex numbers have a real part and an imaginary part. As far as biomedical mathematics is concerned, the imaginary part is still imaginary. But electronics engineers have found a use for complex numbers, and it is possible that in the future some biomedical mathematician will find he can solve a problem most easily using these same partly imaginary numbers.

1-2 The Number Line

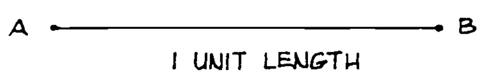
We will start our biomathematical journey with a familiar mathematical idea, the number line. Its relation to anything biomedical will not be obvious immediately. Be patient. It will be obvious shortly.

We start our review of the number line by asking the question, "How can we give an address to each location on the number line?"

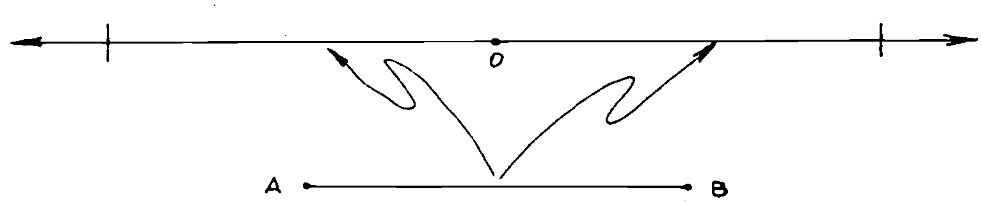
First we choose an arbitrary starting point and call it zero. This point is also called the origin.



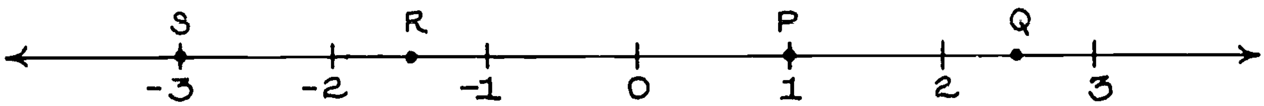
Next we choose a line segment $\overline{AB}$ of arbitrary length.



Line segments of length AB are marked off along the line starting from the origin in both directions.



Each point can now be assigned a number which corresponds to the number of lengths AB, or fractions of lengths AB, between the point and the origin. Points to the right of zero are assigned positive numbers, while points to the left of zero are assigned negative numbers.



The address of P is 1. The address of Q is 2.5, while the address of R, is -1.5. The address of S is -3.

What is the distance from P to Q? We may obtain the answer by counting the number of unit lengths between P and Q, in which event our answer is 1.5. Or we may be more elegant and subtract 1 from 2.5, in which case our answer is also 1.5.

$$2.5 - 1 = 1.5$$

To find the distance from R to Q we may again count the number of unit lengths between the two points. The number is 4. We may also find the distance by subtracting -1.5 from 2.5. But be careful to remember how negative numbers are subtracted.

$$2.5 - (-1.5) = 2.5 + 1.5$$

You can see that this must be so, because

$$2.5 + 1.5 = 4,$$

and 4 is the answer we obtained by counting unit lengths.

In the same manner, we may find the distance from R to P.

$$1 - (-1.5) = 1 + 1.5$$
$$= 2.5$$

The distance from S to R may also be determined in this way. We subtract the address of S from the address of R.

$$-1.5 - (-3) = -1.5 + 3$$
$$= 1.5$$

Observe that all of the distances we have determined have been distances going toward the right; that is, in the positive direction. When moving to the left, we go in the negative direction. If we calculate the distance we travel, we obtain a negative number.

We may determine the distance from Q to P by subtracting the address of Q from the address of P.

$$P - Q = \text{distance from Q to P}$$

$$1 - 2.5 = -1.5$$

We may interpret this to mean that P is 1.5 units in the negative direction from Q.

To find the distance from Q to any other point to the left we follow a similar pattern. For example for Q to R,

$$R - Q = \text{distance from Q to R.}$$

$$-1.5 - 2.5 = -4,$$

or 4 unit lengths in the negative direction from Q, which is toward the left.

The distance from R to S is calculated in the same way.

$$S - R = \text{distance from R to S.}$$

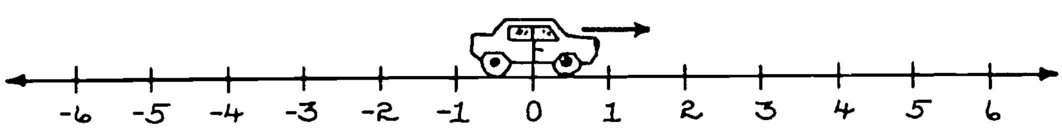
$$-3 - (-1.5) = -3 + 1.5$$

$$= -1.5$$

You may remember that the formal rules for addition and subtraction of positive and negative numbers are as follows.

Addition	Subtraction
$a + b = a + b$	$a - b = a - b$
$a + (-b) = a - b$	$a - (-b) = a + b$
$(-a) + b = b - a$	$(-a) - b = -a - b$, or $-(a + b)$
$(-a) + (-b) = -a - b$, or $-(a + b)$	$(-a) - (-b) = b - a$

While we are on the subject of the arithmetic of positive and negative numbers, we will also review the rules for multiplication and division.



Suppose we drive a car along the number line at a speed of two units per hour. Where would we be three hours after passing point 0?

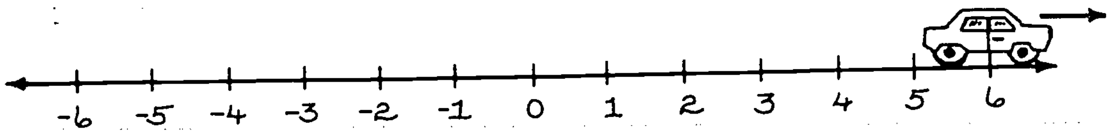
The distance traveled is given by the formula

$$\text{distance} = \text{speed} \times \text{time}$$

Therefore the distance traveled is

$$2 \times 3 = 6 \text{ units}$$

After three hours we would be at the point 6.



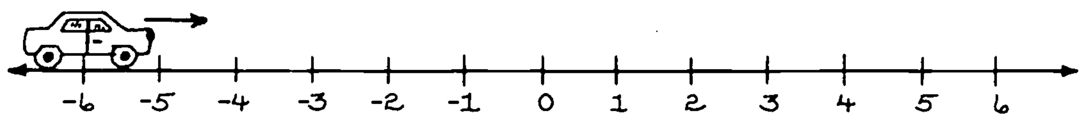
Another question is where we were three hours before we passed point 0. We may consider time in the past to be negative and express the time three hours ago as -3. We may justify assigning negative times to the past because we assigned positive times to the future. Our location three hours ago can then be found by the formula

$$\text{distance} = \text{speed} \times \text{time}$$

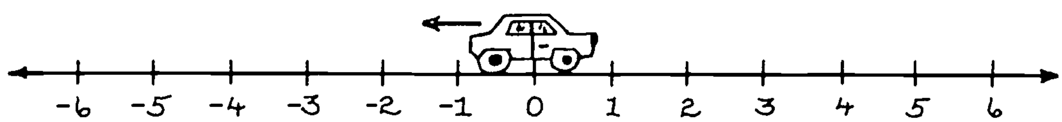
$$= 2 \times (-3)$$

$$= -6 \text{ units}$$

Three hours before passing the origin we passed the point -6.



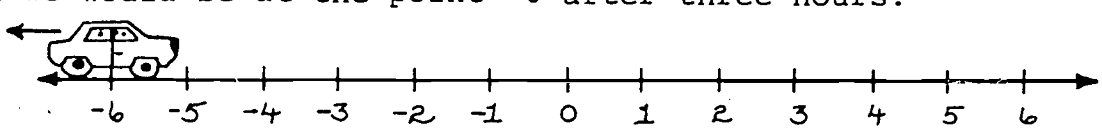
Suppose now that we put the car in reverse and backed up at a speed of 2 units per hour.



Where would we then be after three hours? Since we would be moving in a negative direction, we may state our speed as -2 units per hour. The distance traveled would be

$$-2 \times 3 = -6 \text{ units},$$

and we would be at the point -6 after three hours.



Let us consider one final situation. Imagine us backing up at two units per hour along the segment of the number line shown in the drawings. Consider the question of where we were three hours before passing the origin. You no doubt know without calculating that we were passing the point 6 at that time. But let us see how we would arrive at that number by arithmetic calculation. We decided times in the pass were negative, so the time three hours ago was -3. Since we were going backwards, our speed was -2 units per hour. The product of two negative numbers is a positive number. Therefore, our location three hours before passing the origin was

$$(-2) \times (-3) = 6$$

The situations described on the previous page, in which cars were driven back and forth along the number line illustrate the multiplication of negative numbers. The equations below express the rules governing those operations.

$$(x)\ (y) = xy$$
$$(-x)(y) = -xy$$
$$(x)(-y) = -xy$$
$$(-x)(-y) = xy$$

The rules governing division correspond to those for multiplication and we will merely summarize them by the following equations.

$$\frac{x}{y} = \frac{x}{y}$$
$$\frac{-x}{y} = -(\frac{x}{y})$$
$$\frac{x}{-y} = -(\frac{x}{y})$$
$$\frac{-x}{-y} = \frac{x}{y}$$

1-3 The Metric System

Our discussion of length has been based on a unit length, but we did not specify what the unit length was. In the Biomedical Curriculum, as in the scientific community in general, the metric system is almost always used.

Metric units of length are given in the following table. Also given are factors for converting from the metric system to the British system and from the British system to the metric system.

METRIC UNITS OF LENGTH

"nit	Meaning of Prefix	Abbreviation	Relation to 1 Meter
millimeter	one-thousandth	mm	1 m = 1000 mm
centimeter	one-hundredth	cm	1 m = 100 cm
decimeter	one-tenth	dm	1 m = 10 dm
meter		m	
dekameter	ten	dkm	1 dkm = 10 m
hectometer	one hundred	hm	1 hm = 100 m
kilmeter	one thousand	km	1 km = 1000 m

RELATION OF METRIC UNITS TO BRITISH UNITS

1 inch = 2.54 centimeters
1 meter ≃ 39.37 inches
1 meter ≃ 3.28 feet
1 mile ≃ 1.61 kilometers

The millimeter, the centimeter, the meter and the kilometer are the common units of length. The decimeter, the dekameter and the hectometer are rarely used and are given only to show the decimal nature of the metric system.

Note that 1 centimeter is equal to 10 millimeters. Also observe that 1 meter is slightly longer than 1 yard. Since you will be using the metric system throughout the Biomedical Program and since you may not be familiar with the units, you should try to develop a picture of the approximate size of the units. Remember that a meter is slightly greater than a yard and that about 2.5 centimeters are equivalent to an inch.

One reason the metric system is used by scientists is that its decimal nature makes conversion between units easy. Converting 180 millimeters to 18.0 centimeters or 0.180 meters is simpler than converting 180 inches to 15 feet or 5 yards. Another reason is that use of the metric system allows scientists of the United States and Canada, the only major industrial countries still using the British system, to communicate with scientists of other countries. It seems only a matter of time until the U.S. converts to the metric system as its official system of measure.

PROBLEM SET 1:

Consider the following pairs of numbers:

a. +8, +5   e. -3, -11

b. +7, -2   f. -12, -8

c. +4, +10   g. -9, +13

d. +6, -9   h. -11, +4

1. Find the sum of each pair.
2. Subtract the second number from the first one in each pair.
3. Find the algebraic sums of the following:

   a. $8 + 2 - 3 - 4 + 6 - 1 - 9 + 5$

   b. $- 3 + 7 + 2 - 6 - 5 + 1 - 8$

   c. $4 + 10 - 3 + 7 - 6 - 9 + 2$

   d. $9 + 7 - 8 + 4 - 5 - 2 + 7$

   e. $10 - 11 + 12 - 13 + 14 - 15$

   f. $1 - 9 + 2 - 8 + 3 - 7$

   g. $6 - 12 + 4 + 7 - 13 + 8$

4. Multiply:

a. (8)(5)
b. (-6)(7)
c. (-12)(-4)
d. (9)(-6)
e. (2)(-3)(4)
f. (-5)(3)(-4)
g. (-4)(-7)(-2)
h. (4)(6)(-11)
i. (13)(-1)(-3)(-10)
j. (-2)(-3)(-4)(-5)

5. Divide. Show your answers as reduced fractions.

a. $\frac{20}{5}$
b. $\frac{24}{-8}$
c. $\frac{-30}{6}$
d. $\frac{-35}{-7}$
e. $\frac{-28}{(-7)(3)}$
f. $\frac{-39}{(13)(2)(-3)}$
g. $\frac{91}{(-7)(13)}$
h. $\frac{(64)(-12)}{(-3)(4)(5)(-16)}$
i. $\frac{(-27)(-36)}{(-9)(-3)(-16)}$

SECTION 2:

2-1 Applied Math

This course is firmly in the applied branch of mathematics. You will learn how to use your math skills to get personally useful information. We will always answer the question, "What is the use of this technique?" You will learn more than just mathematics from this course because it is applied. For example, in the alcohol lessons you will learn how long it takes alcohol to leave the bloodstream. You will also get practice in using the mathematical tool called linear functions. In a truly applied course you get two pieces of knowledge for the price of one. We call it a bargain. However, to take advantage of the bargain you will have to solve many "word" problems. We realize that many of you find this to be a difficult task. Consequently, in many instances we have attempted to simplify the process by writing introductory "programmed" problems.

2-2 Completely Programmed Word Problems

To apply math to real life situations we will have to translate written English into mathematical terms. We can then get a mathematical solution. We illustrate with an example.

SITUATION:

A patient is brought into the emergency room. He has a high fever. It is 5° C (= 9° F) above normal. The doctor has the patient packed in ice to break the fever. One hour later his temperature is down to 7° C (= 12.6 °F) below normal.

PROBLEM:

Our target question is, "What was the patient's change in body temperature?" However, we will not try to answer this question immediately. We will first ask questions that require no computation at all. They require only that specific information be found in the statement of the situation. For example,

a. How much above normal was the patient's initial temperature?

Answer: 5° C above normal.

We might next ask you to translate the answer to part a into mathematical symbolism.

b. State the patient's temperature elevation as a signed number.

Answer: +5°.

Doing the same for the temperature depression, we have parts c and d.

c. How much below normal was the patient's temperature after one hour in an ice bath?

Answer: 7° C below normal

d. State the patient's temperature depression as a signed number.

Answer: -7°.

Notice that no computation has been required so far. Finally we are ready for it.

e. State the patient's change in body temperature as the difference of two signed numbers, then solve the equation.

Answer: Following the pattern of Section 1, we find the difference between T = +5 and t = -7.

$$t - T = \text{distance from } T \text{ to } t$$

$$-7 - (+5) = -12$$

f. Interpret your results, that is, tell how much the temperature changed and in what direction.

Answer: The patient's temperature changed 12° C. Since the sign is negative, the patient lost 12 degrees.

This temperature change problem is very simple. We will seldom break down such a simple problem into so many steps. We did it with this problem to illustrate a common pattern in our word probelms.

2-3 Semi-programmed Word Problems

Commonly the first time we present a new type of problem, we will completely program it. This was illustrated in the preceding section. As the problem set moves along, we will expect you to supply more of the steps on your own. If you have trouble, you can refer to a similar problem which has been completely programmed. The pattern of solution will often be the same. We illustrate with an example.

PROBLEM:

A patient is brought into the emergency room with a high fever. It is 6° C above normal. He is immediately put into an ice bath. After one hour his temperature is reduced to 8° C below normal.

a. Write and solve an equation which describes the patient's temperature change.

b. Interpret the results.

SOLUTION:

Your first task in this problem is to recognize that it is similar to the first problem. The exact same procedure that we used for the first problem may be used to solve this problem.

a. $-8 - 6 = -14$

b. The patient's temperature dropped 14° C.

PROBLEM SET 2:

1. Elmo liked to jog at night. He particularly liked to jog along country roads on moonless nights. He was jogging along in pitch darkness one night. His activity had raised his body temperature 1° C above normal. He tried to jog over a washed out bridge. Since Elmo could neither defy gravity nor walk on water, he fell into the creek. After 5 minutes in the creek his temperature had fallen to 2° C below normal.

a. Write and solve an equation which describes Elmo's change in body temperature.

b. How many degrees did Elmo's body temperature change and in what direction?

2. A kidney is an organ that removes waste products from body fluids. After the waste is removed it makes a solution (urine). If there is little extra water in the body it makes a concentrated urine (small water, much waste). If there is a great deal of extra water available

it makes a very dilute urine (much water, little waste). A sick kidney cannot handle the extremes, that is, very concentrated or very dilute urines. Therefore, a close watch is kept of the water intake and output of a person with kidney trouble. The objective of this watch is to have the intake and output equal each other. A chart similar to the one below is used to record the intake and output of body fluids for each 8-hour period.

	Oral (By Mouth)	IV (Injection into veins)	Clysis (Injection under skin)	Other	Total
Intake (ml)	400	400	250		

	Urine (#1)	Emesis (Vomit)	Feces (#2)	Sweat	Breathing and Respiration (Estimate)	Total
Output (ml)	350	200	375	150	100	

a. What does the kidney do?

b. What two conditions might hurt a sick person with kidney trouble?

c. What is the objective of this close watch of fluid balance?

d. Suppose that intake is more than output in one time period. During the next time period the intake will be (increased, decreased).

e. How much water was taken by mouth? (Include units.)

f. Give all intake numbers a positive value. Add them to find the total intake.

g. How much water was lost in the form of sweat? (Include units.)

h. Give negative values to the output quantities. Add them to find the total output.

i. Find the algebraic sum of the total intake and total output.

j. Interpret the result, that is, state whether the amount of fluid in the patient's body increased or decreased and by how much.

k. During the next time period the patient's intake should be (increased, decreased) by _______ ml.

3. Below is the record of another patient suffering from kidney trouble.

	Oral (By Mouth)	IV (Injection into veins)	Clysis (Injection under skin)	Other	Total
Intake (ml)	600	500	200	15	

	Urine (#1)	Emesis (Vomit)	Feces (#2)	Sweat	Breathing and Respiration (Estimate)	Total
Output (ml)	600	500	600	150	100	

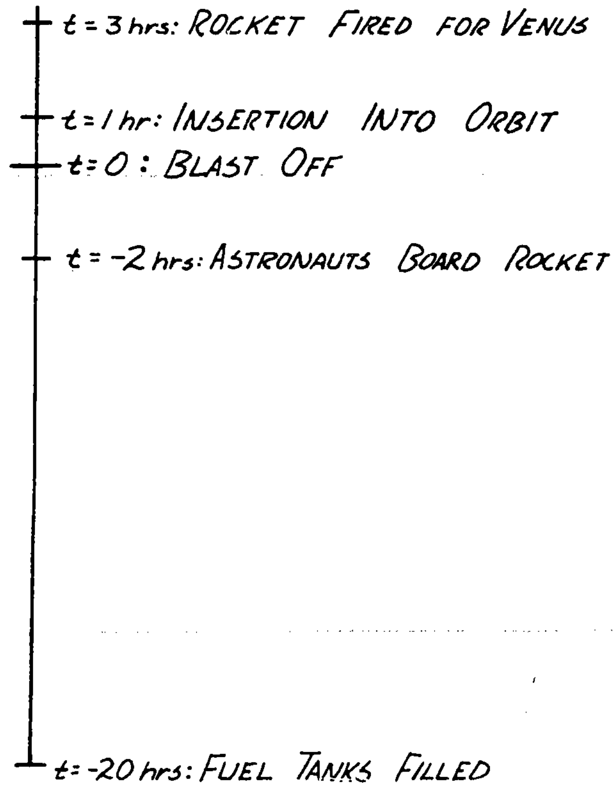
a. Give positive values to all inputs and sum them.

b. Give negative values to all outputs and sum them.

c. Find the algebraic sum of the inputs and outputs.

d. Tell how much the patient's water intake should be increased or decreased in the next 8-hour period.

4. When a rocket is launched there is a "countdown." The moment of blastoff is called time zero. In the jargon of the rocketeers time before blastoff is called "t minus." Time after blastoff is called, "t plus." On the following page we have a schedule for the blastoff of the Taurus III rocket launch to Venus.



a. How long before blastoff did the astronauts board the rocket?

b. How long were the astronauts in the space capsule before they were inserted into orbit?

c. Write a subtraction equation that expresses the time that passed between t = -20 hr and t = +200 hr.

d. Which came first, t = -40 hr or t = -2 hr?

5. The city of New Orleans lies about 2 meters below the average level of the Mississippi River. The levees which surrounded the city can contain flood waters which rise to 4 meters about the average level. How high are the levees?

6. Elmo had a net worth of -$53.14 on January 1. He had a negative net worth because he owed and spent more money than he had. When figuring net worth, debts and expenditures are given negative values. Money on hand or received is given a positive value. The algebraic sum of these positive and negative values is the net worth of an individual. Below we have a record of Elmo's financial fluctuations during the month of January.

Elmo made $16.33 mowing lawns and $24.53 selling lemonade at "Uncle Elmo's Fresh Squozen Lemonade Stand." Furthermore, he picked up $47.19 by panhandling. Best of all Elmo's father decided to forget about the $117.74 that he had loaned Elmo in December. In other words, Elmo could now take away a debt of $117.74 from his net worth.

On the negative side Elmo spent $96.14 on food and shelter.

a. What kinds of quantities are assigned negative values?

b. What kinds of quantities are assigned positive values?

c. The effect of taking away a debt is to (increase, decrease) Elmo's net worth.

d. Find Elmo's net worth at the end of the month.

e. Do you think Elmo's business will pick up when the weather warms up? (Hint: Is the month of January generally associated with lawn mowing and lemonade stands?)

7. Darius, the Persian king, was born in the year 558 B.C. Xerxes, also a Persian king, was born in the year 519 B.C.

a. The two kings were father and son. Which king was the father?

b. Write and solve a subtraction problm to determine the age of the father when his son was born. Use signed numbers.

8. The highest point in the connected United States is Mt. Whitney. It is 14,495 feet above sea level. The lowest point is located in Death Valley. The elevation of this point is 282 feet below sea level.

a. Write a subtraction problem which expresses the difference between these two elevations.

b. If you were standing on top of Mt. Whitney, how many feet would you be above the floor of Death Valley. (Solve the problem you wrote in Part a.)

SECTION 3: PATHS AND SURFACES

3-1 One Dimensional Quantities and Length

A path is a one dimensional space. Length is a measure of one dimensional space. Consider a train moving along its tracks. It may move freely either backwards or forwards. Movement right and left off the tracks is not a real option for a train. It is forced to accept either backwards or forwards along its tracks. Therefore, the world of trains is one dimensional. The space available to a train is measured by the length of track available to it. For example, there is a train which runs from Willits, California to Ft. Bragg, California. The size of this railroad may be meaningfully measured by the length of track needed to connect these two towns. Railroads are one dimensional and their sizes may be meaningfully measured in terms of length.

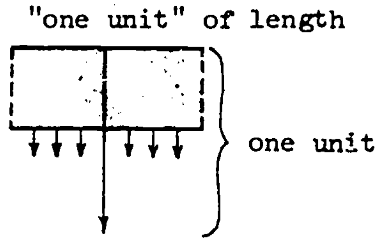
3-2 Two Dimensional Quantities and Area

A surface is a two dimensional space. Area is a measure of two dimensional space. Consider a ship moving about on Lake Michigan. Unlike the train, the ship may move either right or left as well as back and forth. A ship is a surface vehicle. It moves in a two dimensional space. The size of the space available to the ship is measured in terms of the area of the body of water that it is on. For example, suppose our ship always remains on Lake Michigan. The size of the space available to the ship is approximately the area of Lake Michigan. It is impossible to measure the surface of the lake in terms of length.

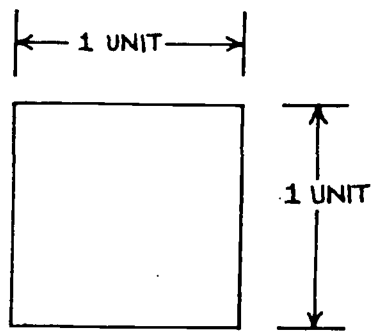
An automobile is another vehicle which operates in two-space, or, in other words, on surfaces. In an automobile, it is also possible to move right and left as well as backwards and forwards. Therefore the size of the space available to an automobile may be measured in terms of area. However, the case of an automobile is not as clear-cut as a ship. Most of the time automobiles operate on surfaces which restrict right and left type movements. Common sense restricts movement too far to the left into the opposing lane of traffic. Drainage ditches and so forth restrict movement too far to the right. Consequently, the surface available to a car often times resembles a path. This is the reason that you will hear, "Miles (kilometers in the future) of roadway," instead of "Square miles of roadway." Whenever this situation arises then we will assume that the auto is following a path. Whenever the surface is not long and skinny, then we will call it a true surface. For example when a car is moving about in an empty parking lot, then we say it is on a surface. Consequently we would measure the size of the parking lot in terms of area.

3-3 Area: The Measure of Two Space

Surfaces cannot be measured in units of length, so a different kind of reference unit is necessary. This unit is the unit square. A unit square may be generated by passing a unit line segment through a distance of one unit in a direction perpendicular to the line segment.

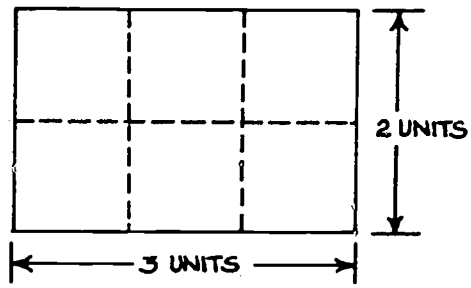


A unit square is a square whose sides are of unit length.



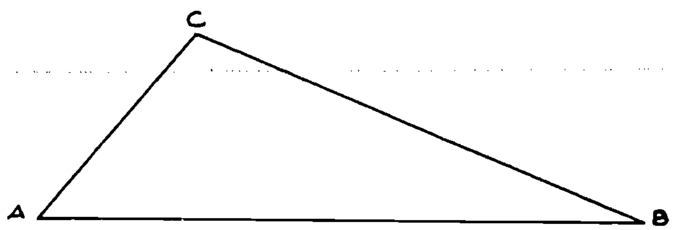
The unit square was chosen as the unit of area rather than some other figure (for example, the unit circle) because of the ease with which unit squares can be fitted together to find the area of other figures.

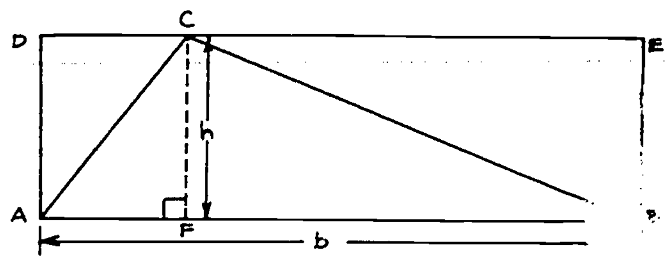
The area of a rectangle, you will remember, is the product of the number of rows of unit squares and the number of columns of unit squares that can be packed into the rectangle.



A RECTANGLE WITH AN AREA OF 6 UNIT SQUARES

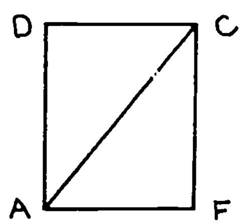
To find the formula for the area of a triangle one may start by constructing a rectangle as shown in the drawing.



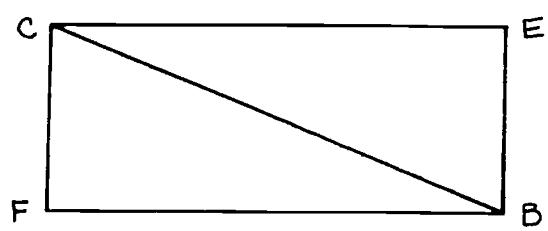


One side of the rectangle is side $\overline{AB}$ of the triangle, and the opposite side of the rectangle passes through point C of the triangle. $\overline{AB}$ is called the base (b) of the triangle and $\overline{CF}$ the height (h). The area of the rectangle is bh.

The area of the triangle is half the area of the rectangle, or $\frac{1}{2}$bh. We may show this to be true by noting that the area of triangle ACF is half the area of rectangle ADCF and the area of triangle BCF is half the area of rectangle BECF.

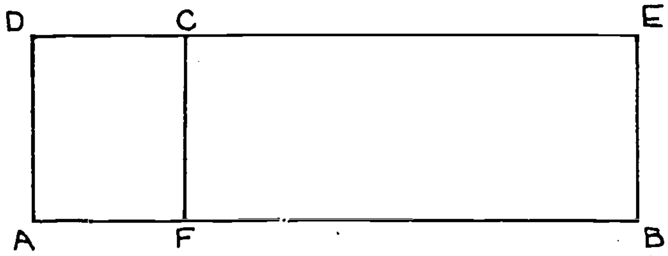


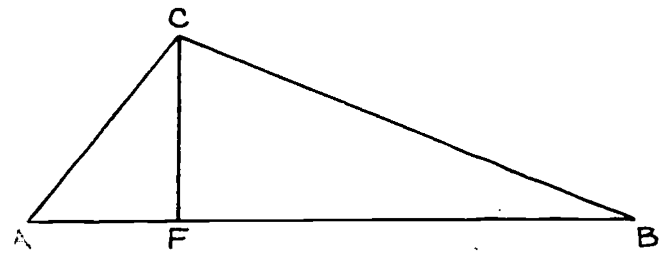
AREA OF △ ACF = ½ AREA OF ▭ADCF



AREA OF △BCF = ½ AREA OF ▭BECF

The area of triangle ACF plus the area of triangle BCF equals the area of triangle ABC. The area of rectangle ADCF plus the area of rectangle BECF is the area of rectangle ABED. It follows that the area of triangle ABC is half the area of rectangle ABED.





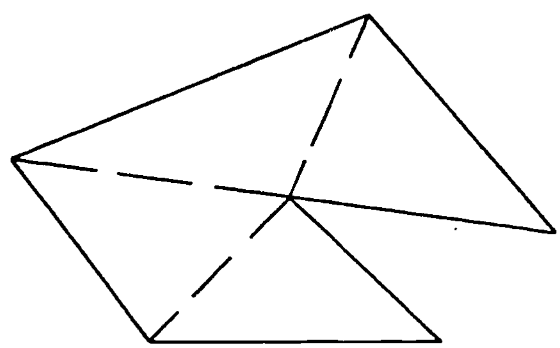
AREA OF ▭ABED =
AREA OF ▭ADCF +
AREA OF ▭BECF .

AREA OF ΔABC =
AREA OF ΔACF +
AREA OF ΔBCF .

Thus the area of a triangle is one-half the length of the base times the height.

$$A = \frac{1}{2} bh$$

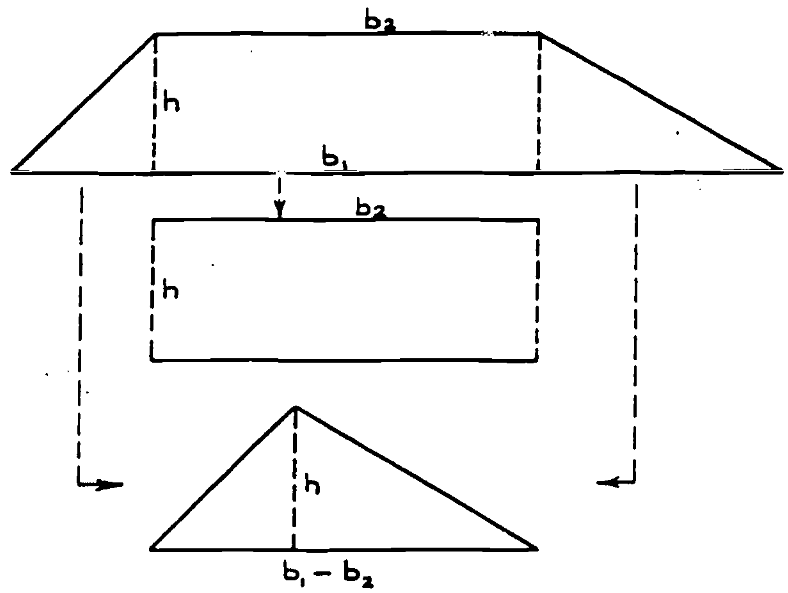
The area of any other polygon can be found by dividing the figure into a number of triangles, finding the area of each triangle and summing these areas.



When we work with complicated figures, and want to find a formula for area, we naturally have to use more letters to label all the dimensions. For example, it would take nine letters to fully label the figure above. In order to keep things from getting out of hand, mathematicians have taken to using letters with subscripts to represent variables or dimensions. A subscript is just a number written at the lower right hand corner of a letter. For example, we can form a_0, a_1, a_2, and so on. These quantities are read "a sub zero", "a sub one", "a sub two" and so on. The quantity "r sub ten" would be written as r_{10}. The handy thing about subscripts is that they allow us to form several quantities, all using the same letter.

Now we return to our discussion of area.

The formula for the area of a trapezoid is derived by dividing the figure into a rectangle and two triangles.



The area of the rectangle is b_2h.

The two triangles can be put together to form a large triangle with base $b_1 - b_2$ and height h. The area of this triangle is

$$\frac{1}{2}(b_1 - b_2)h.$$

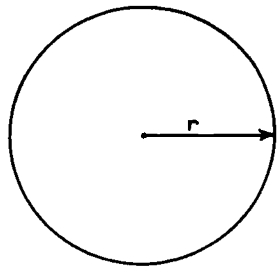
The area of the trapezoid is the sum of the area of the rectangle and the area of the larger triangle.

$$\begin{aligned}\text{Area of trapezoid} &= b_2h + \frac{1}{2}(b_1 - b_2)h \\ &= b_2h + \frac{1}{2}b_1h - \frac{1}{2}b_2h \\ &= \frac{1}{2}b_1h + \frac{1}{2}b_2h \\ &= \frac{1}{2}(b_1 + b_2)h\end{aligned}$$

For at least 4000 years mankind tried to express the area of one of the simplest of all geometric figures in terms of an exact number of unit squares. The figure is the circle, and no one was ever successful. Mathematicians now know better than to try.

The area of a circle can be given in terms of its radius and an irrational number called π. If we call the area A and the radius r, the area is given by the formula

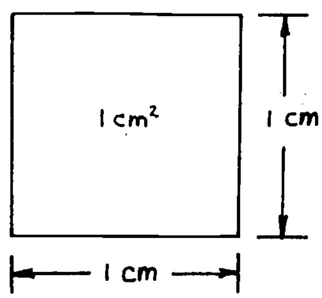
$$A = \pi r^2 .$$



Computers can give π to increasing degrees of accuracy by calculating additional decimal places, but there are always more digits beyond the last one given by the computer. To seventeen digits π is approximately equal to 3.1415926535897932...

3-4 Metric Area Units

The metric system is used for area as well as for length. We stated that a unit square is a square whose sides are unit lengths. If the unit length is 1 centimeter, for instance, the unit square has sides of 1 centimeter. We say that such a square has an area of 1 square centimeter. The abbreviation for "square centimeter" is "cm^2."



Other units of area, and the conversions between units, are given in the following table.

METRIC UNITS OF AREA

Unit	Abbreviation
square millimeter	mm^2
square centimeter	cm^2
square meter	m^2
square kilometer	km^2

Lengths	Areas
1 cm = 10 mm	1 cm^2 = 100 mm^2
1 m = 100 cm	1 m^2 = 10,000 cm^2
1 km = 1000 m	1 km^2 = 1,000,000 m^2

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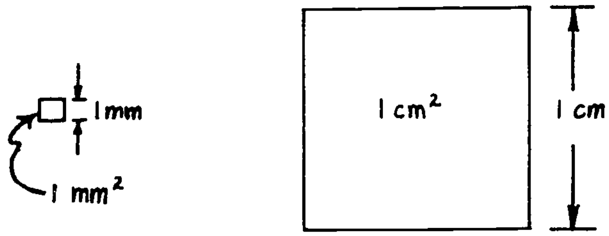
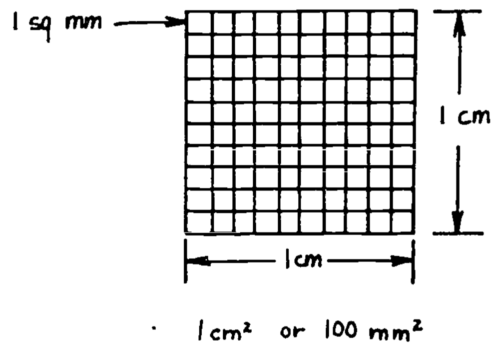
Note that one square centimeter is not equal to 10 square millimeters, but is equal to 100 square millimeters. The reason for this can be seen by considering a square millimeter and a square centimeter:



Ten rows and ten columns of square millimeters can be fit into the square centimeter.



Therefore the area of the square centimeter in units of square millimeters is 10 x 10 = 100 square millimeters.

PROBLEM SET 3:

1. Divide the following set of phrases into two categories by writing either a 1 for one-space or a 2 for two-space.

a. path

b. surface

c. a train

d. a ship on the Atlantic

e. a caboose

f. length

g. a four-wheel-drive vehicle on the Sahara desert

h. a ship in the Panama Canal

i. a car on a two lane road

j. a bicycle in a parking lot

k. an oil spill

l. a telephone wire

m. an artery

n. a leaf

o. a page in a book

p. a sentence on a page in a book

q. a phonograph record

r. the groove in a phonograph record

2. Write the subscripted variable indicated by each of the following phrases.

a. b sub one

b. A sub triangle

c. x sub i

d. m sub s

e. L sub mo

f. P sub zero

3. Write an Engish phrase equivalent to the following subscripted variables.

a. $A_{trapezoid}$

b. x_2

c. m_3

d. a_i

e. $Yellow_{marine}$

f. A_{set}

g. $commie_{versive}$

h. $hard_{ject}$

i. $fat_{contract}$

j. $very_{tle}$

k. add_{tract}

4. Select and correct the false statements.

a. $b_1 + b_2 = 3b$

b. $a_o + a_1 + a_1 = a_o + 2a_1$

c. $a_1 + a_1 = a_2$

d. $A_\Delta + A_\Delta = 2A_\Delta$

e. $a_1 + a_1 + a_1 = 3a_1$

5. If $b_1 = 3$, $b_2 = 7$, and $h = 5$, find the numerical values of the following:

a. $b_1 + b_2$

b. $b_1 \cdot h$

c. $b_2 \cdot h$

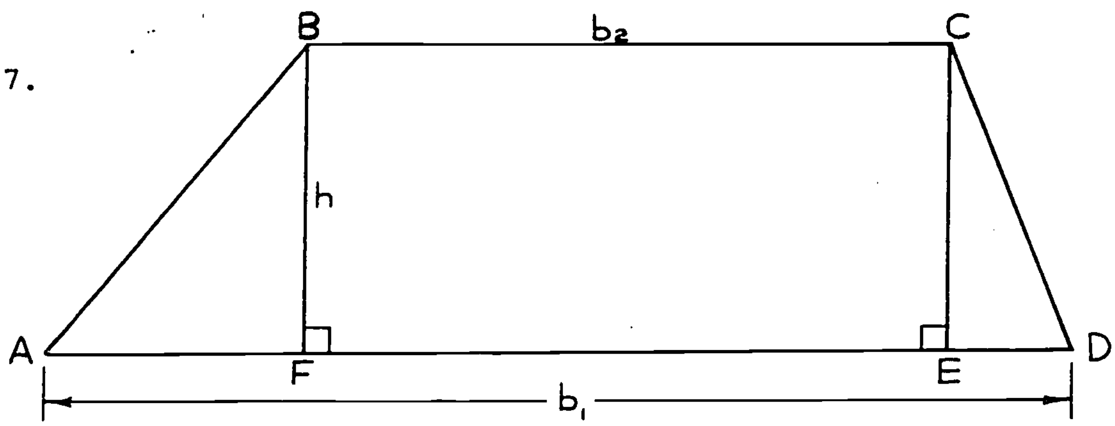
d. $\frac{1}{2} b_1 \cdot h$

e. $\frac{1}{2}(b_1 + b_2)h$

f. $\frac{1}{2}(b_2 - b_1)h$

6. Suppose b_1, b_2, and h each are in units of centimeters. Express the units of the answers for Problem #5.

7.



Trapezoid ABCD may be divided into sections by line segments $\overline{BF}$ and $\overline{CE}$ which are perpendicular to $\overline{AD}$.

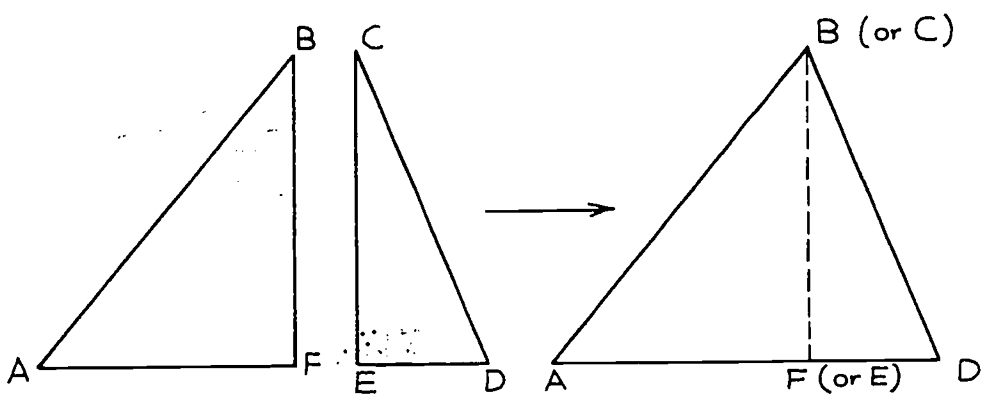
$$h = 5 \text{ cm}, \; b_1 = 16 \text{ cm}, \; b_2 = 10 \text{ cm}$$

Give numerical answers and units for the following.

a. What length is the base, $\overline{EF}$, of the resulting rectangle?

b. What is the area of the rectangle?

If the two triangles are joined along $\overline{CE}$ and $\overline{BF}$,



c. What is the length of the base of the new triangle?

d. Find the area of this triangle (ABD).

e. Find the area of the trapezoid by adding the areas of the rectangle and triangle.

f. Find the area of the trapezoid using the formula Area $= \frac{h}{2}(b_1 + b_2)$.

g. Are the areas found in part e and part f the same?

8. Find the areas of the following figures.

a. A rectangle; base = 5 cm, height = 2 cm.

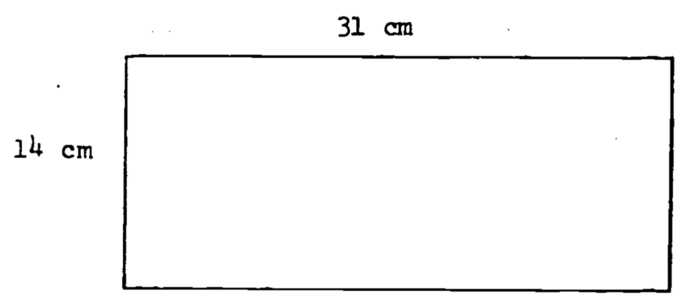
b. A triangle; base = 5 cm, height = 2 cm.

c. A trapezoid; bases = 100 cm and 80 cm, height = 40 cm.

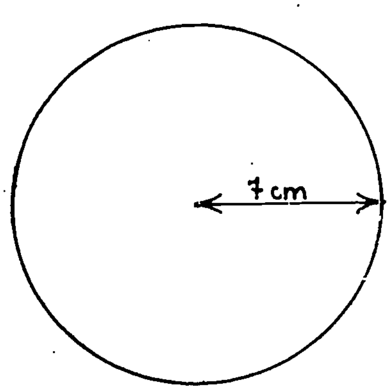
d. A trapezoid; bases = 1 m and 0.8 m, height = 0.4 m.

e. A circle; radius = 7 cm, $\pi \simeq \frac{22}{7}$.

9. a. Determine the area of the rectangle below.



b. Determine the area of the circle.



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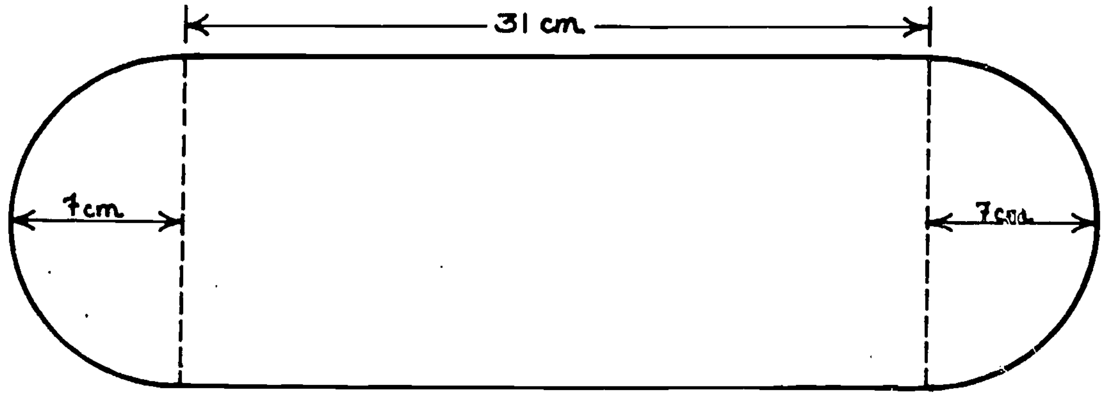
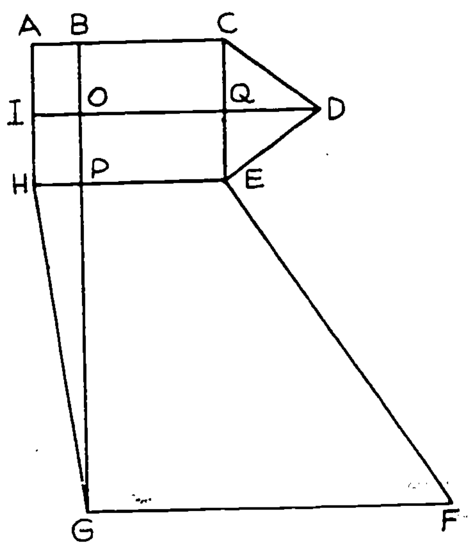
c. The figure below is a rectangle with two attached semi-circles. Determine the area of the figure. Use $\pi \approx \frac{22}{7}$.



10.



$\overline{AC} = \overline{HE} = 8$ cm

$\overline{CE} = \overline{AH} = 6$ cm

$\overline{ID} = 12$ cm

$\overline{GF} = 15$ cm

$\overline{BG} = 20$ cm

$\overline{CQ} = \overline{AI} = 3$ cm

From the above diagram and measurements, compute the area of:

a. rectangle ACEH

b. triangle CDE

c. trapezoid HEFG

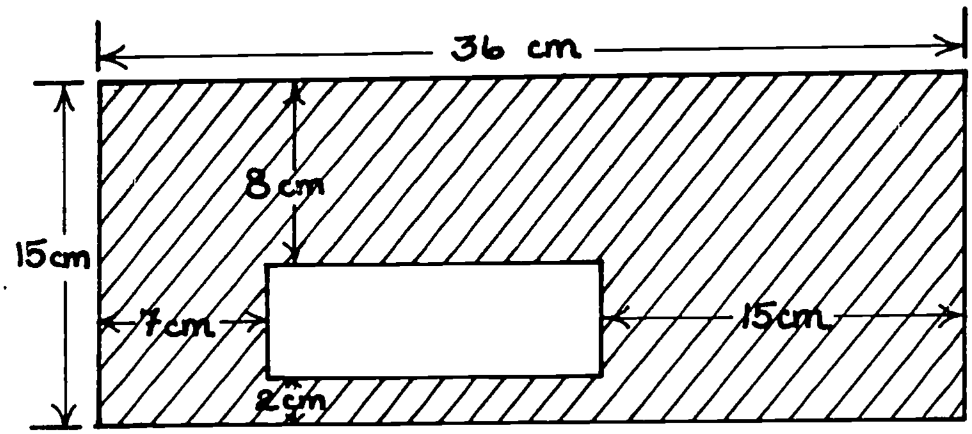
d. rectangle AIQC

e. triangle CQD

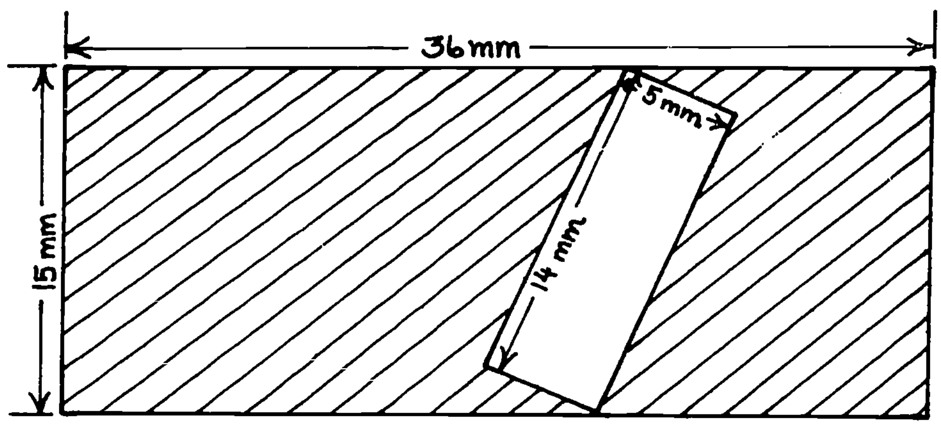
f. trapezoid ACDI

g. polygon ACDEFGH

11. a. Determine the area of the shaded region. Both figures are rectangles.



b. Determine the area of the shaded region. Both figures are rectangles.



12. The trapezoid, opposite, has a rectangle, a triangle and a circle cut out of it. h_3 is the height of the triangle, r is the radius of the circle and h_1 is the height of the trapezoid. Determine the area of the shaded region.

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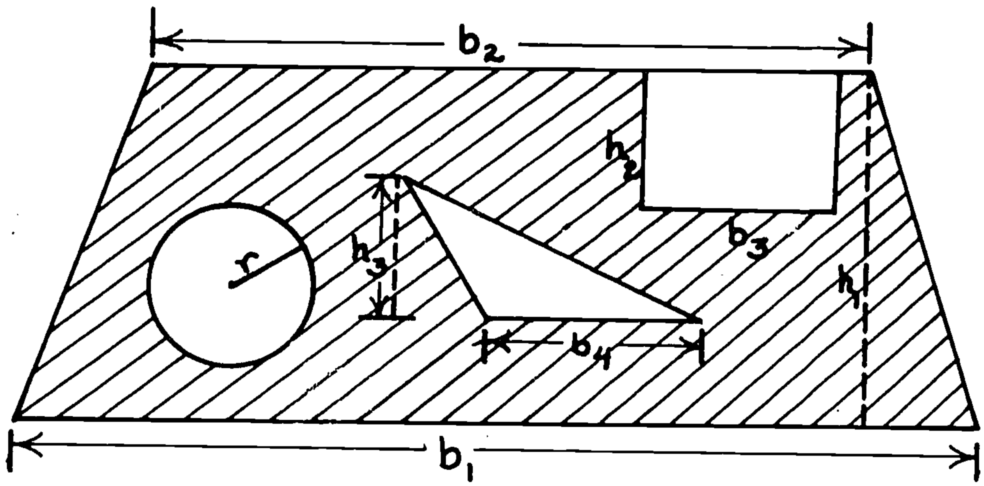


$b_1 = 40$ cm

$b_2 = 30$ cm

$b_3 = 8$ cm

$b_4 = 9$ cm

$h_1 = 15$ cm

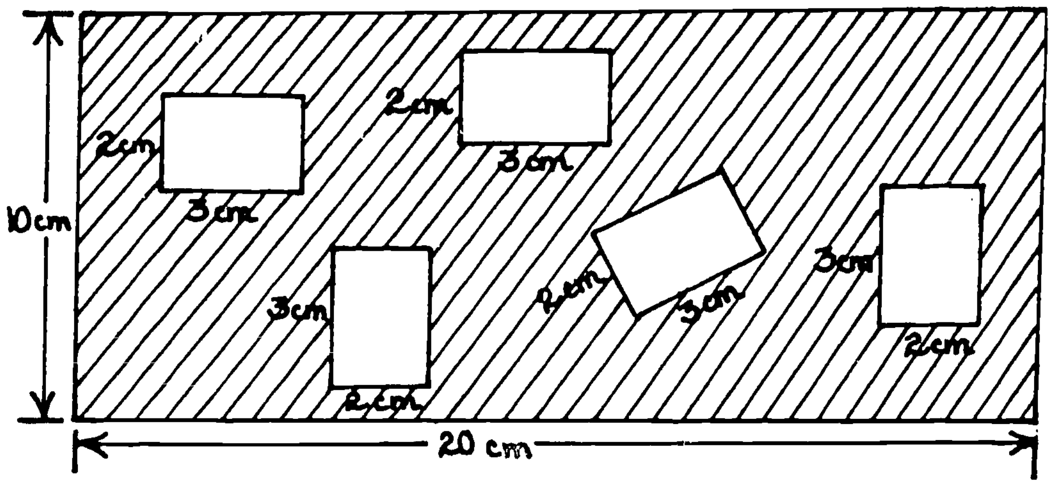
$h_2 = h_3 = 6$ cm

$r = \frac{7}{2}$ cm

$\pi \simeq \frac{22}{7}$

SECTION 4:

4-1 Collecting Like Terms



Finding the area of the shaded region in the drawing is not a difficult problem. It is equal to the area of the large rectangle minus the areas of the five small rectangles. The area of the large rectangle is $10 \cdot 20$ cm^2, while the area of each small rectangle is $2 \cdot 3$ cm^2. The area of the shaded region is therefore

$$10 \cdot 20 - 2 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 \text{ cm}^2$$

We could carry out the six multiplications and add the six terms, but we can make things easier for ourselves by noting that

$$- 2 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 = -5(2 \cdot 3)$$

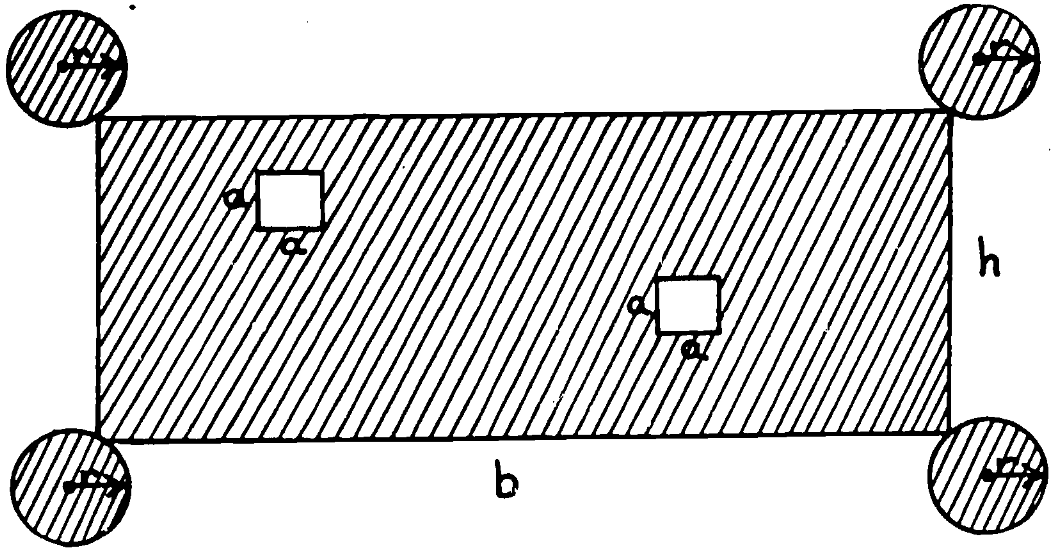
Mathematicians call this process "grouping" or "collecting of like terms." The area of the shaded region may be written as

$$A = 10 \cdot 20 - 5(2 \cdot 3)$$

We may now simplify this expression to

$$\begin{aligned} A &= 200 - 5 \cdot 6 \\ &= 200 - 30 \\ &= 170 \text{ cm}^2 \end{aligned}$$

Finding the area of the shaded region in the drawing on the next page is another problem in which collecting like terms simplifies an expression.



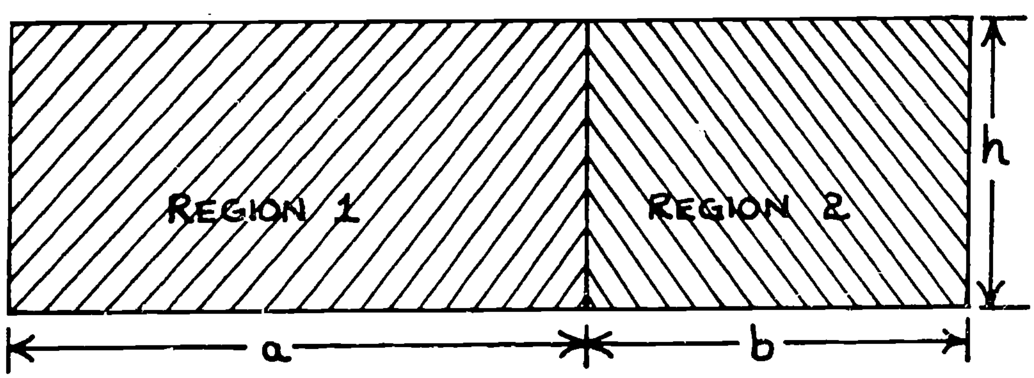
The area of the shaded region is the area of the rectangle minus the areas of the two squares plus the areas of the four circles. The area of each square is a^2, while the area of each circle is πr^2. The area of the shaded region is then

$$bh - a^2 - a^2 + \pi r^2 + \pi r^2 + \pi r^2 + \pi r^2$$

Since $- a^2 - a^2 = -2a^2$ and $\pi r^2 + \pi r^2 + \pi r^2 + \pi r^2 = 4\pi r^2$, we may collect terms to obtain a simplified expression.

$$A = bh - 2a^2 + 4\pi r^2$$

Collecting like terms is seen to be a useful technique when determining the areas of certain geometric figures. It is also an illustration of a more general law of algebra, the distributive law. Consider the rectangles below.



The area of Region 1 is ah. The area of Region 2 is bh. The sum of the areas of Regions 1 and 2 is ah + bh.

The area of the large rectangle including both Region 1 and Region 2 is (a + b)h. However, the area of the large rectangle is equal to the sum of the areas of Region 1 and Region 2. Therefore,

$$ah + bh = (a + b)h.$$

This equation is a general expression of the distributive law. The distributive law is an extremely useful tool throughout the realm of algebra. It may be used to factor expressions.

$$88 + 99 = 11(8 + 9)$$
$$= 11(17)$$
$$2ax + 8xz - 14cx = 2x(a + 4z - 7c)$$

It may also be used to simplify expressions.

$$3(y - x) + 2(2x - y) = 3y - 3x + 4x - 2y$$
$$= x + y$$
$$3a + 4b - 2(a + 2b) = 3a + 4b - 2a - 4b$$
$$= a$$

4-2 The FOIL Method and Distributivity

Many of you are already familiar with the FOIL method of multiplying two binomial expressions. Briefly, the lettters in the word FOIL describe the pattern of multiplication (First, Outside, Inside, Last). For example,

$$(a + b)(c + d) = \underbrace{ac}_{\text{First}} + \underbrace{ad}_{\text{Outside}} + \underbrace{bc}_{\text{Inside}} + \underbrace{bd}_{\text{Last}}$$

The same result may be obtained by applying the distributivity principle twice in succession.

[Once] $(a + b)(c + d) = (a + b)c + (a + b)d$

[Twice] $= ac + bc + ad + bd$

Except for the order of the terms, this result is exactly the same as our FOIL method result.

If you are indeed familiar with the FOIL method, then you may be inclined to ask, "Why bother to learn the distributive method?" The answer is that the distributive method is more general. For example, consider this example

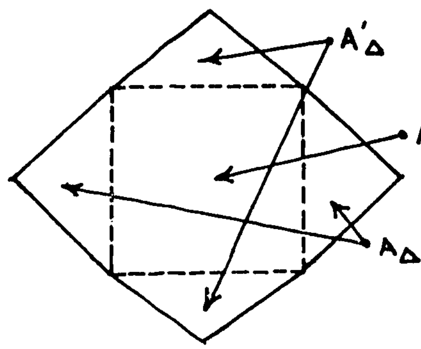
$$(a + b)(c + d + e) = ?$$

The FOIL method will not work for this problem, but the distributive method will.

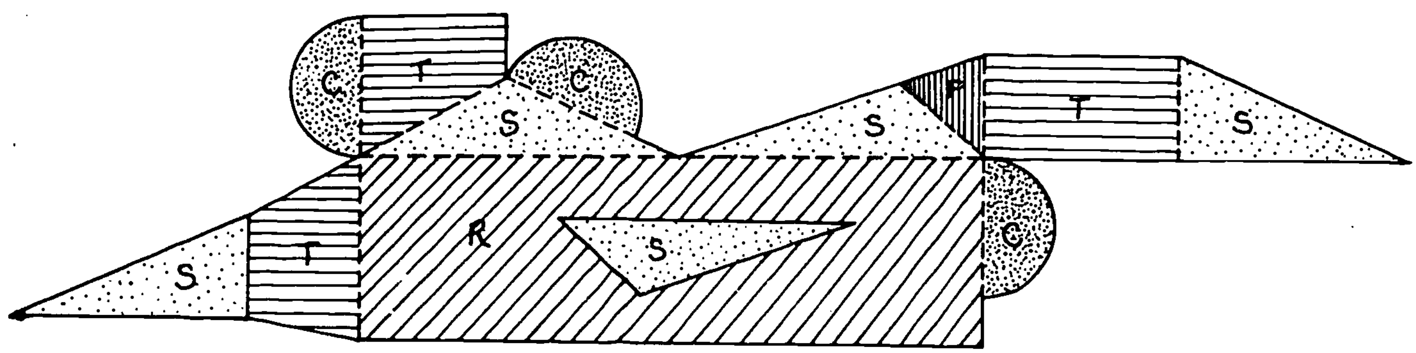
$$(a + b)(c + d + e) = (a + b)c + (a + b)d + (a + b)e$$
$$= ac + bc + ad + bd + ae + be$$

PROBLEM SET 4:

1. Write an expression for the area of the polygon. Group like terms.



2. The areas of the regions of the polygon below are represented by the letters within the region. If two regions have the same letter they have the same area. Write an expression for the area of the figures below. Group like terms.



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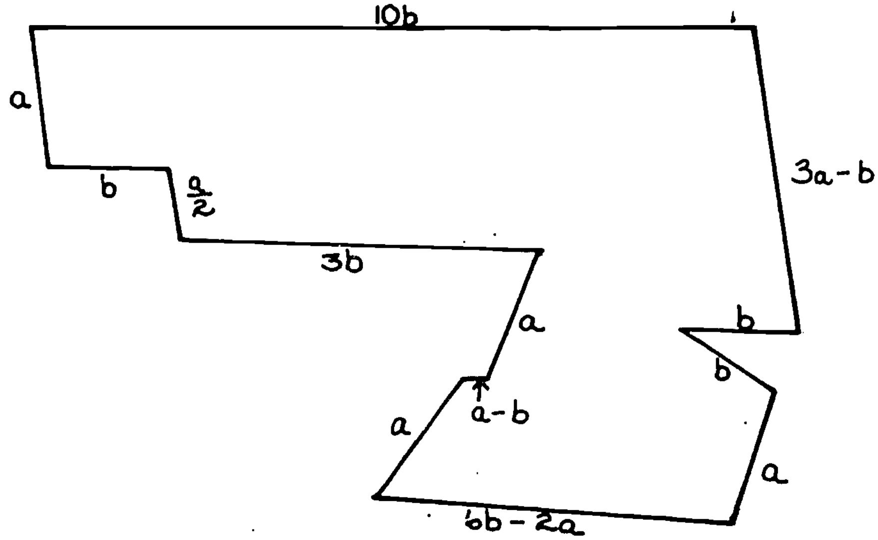
3. The perimeter of a polygon is the sum of the lengths of its edges. Write an expression for the perimeter of this polygon. Group like terms.



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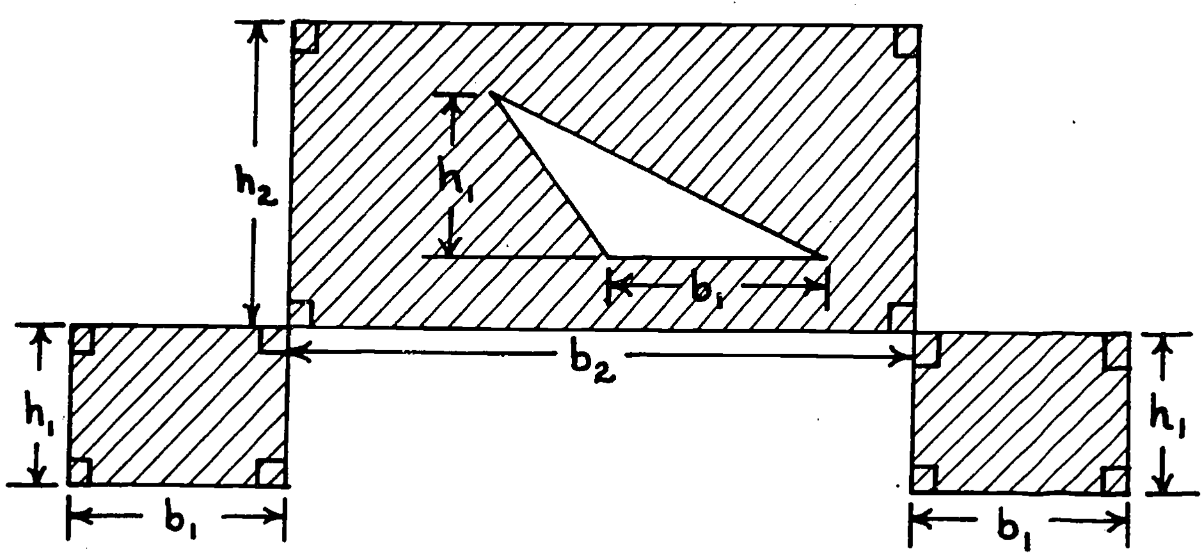
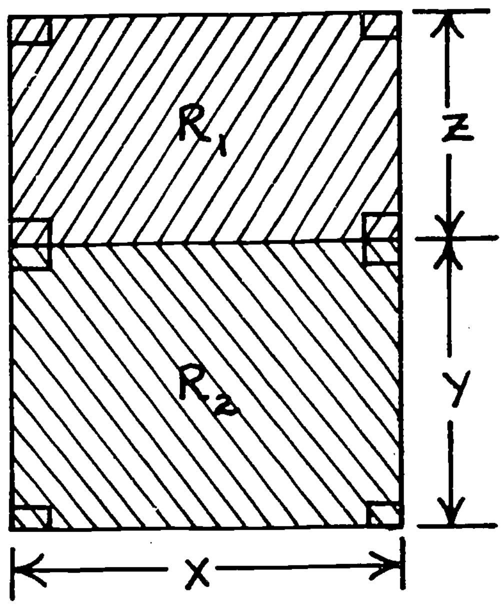


Write an expression for the area of the shaded region. Group like terms.

5.



a. Write an expression for the area R_1.

b. Write an expression for the area R_2.

c. The area $R_1 + R_2$ may be expressed as the sum of the answers to parts a and b. It may also be expressed as the product of the length and the width of the larger rectangle. Write an equation that shows these two equivalent representations of the area $R_1 + R_2$.

One form of the distributive law is $a(b + c) = ab + ac$. Apply this relation to problems 6 through 10.

6. $\frac{h}{2}(b_1 + b_2) =$

7. $-x(y - 2) =$

8. $2(z - 6) =$

9. $(-b - 4)(-8) =$

10. $-m(n + 4) =$

Apply the reverse of the distributive process to problems 11 through 15.

11. $b_1 \frac{h}{2} + b_2 \frac{h}{2} =$

12. $12z + 4y =$

13. $-xa + bx =$

14. $-2a - 8b =$

15. $64a - 32b =$

Simplify the following expressions. Group like terms and apply the distributive law.

16. $A_{total} = A_{\Delta} + A_{\Delta} + A_{\Delta}'$

17. $A_{total} = \frac{1}{2} A_{\square} + A_{\Delta} + A_{\Delta} + \frac{1}{2} A_{\square}'$

18. $a - 3b + 4(a + b)$

19. $3a + 4b + (2a - 3b)$

20. $a + 3b + \frac{1}{2}(a + c)$

21. $a - b + c - 4a - 3b$

22. $8a - [4b - (-3a + 4b) + a]$

23. $a - [-a(2 + b) + b(a - 2)]$

24. $7x - 2[-3(x - y) + 4(x + y)]$

25. $a - \{2x + [3a - 5x - (4a + x)] - 2a\}$

26. $A_{total} = \frac{1}{2} b_1 h + \frac{1}{2} b_1 h + \frac{1}{2} b_1 h + \frac{1}{2} b_2 h$

27. $ab - c - 3ab + (bc - c)$

28. $a - bc - (c + 3bc)$

29. $2ab - 3(c + ab) + c + d$

30. $ab + c + a(2b - 3c)$

Apply the distributive law twice in succession to determine the result of the following multiplications:

Example: $(a + b)(a + 2b) = (a + b)a + (a + b)2b$

$$= a^2 + ab + 2ab + 2b^2$$
$$= a^2 + 3ab + 2b^2$$

31. $(x - 4)(x + 2) =$

32. $(a + b)(2a + b) =$

33. $(80 + 4)(80 - 4) =$

34. $(M - 10)(N + 2) =$

35. $(x + y)(x^2 - xy + y^2) =$

36. $(x - y)(x^2 + xy + y^2) =$

SECTION 5:

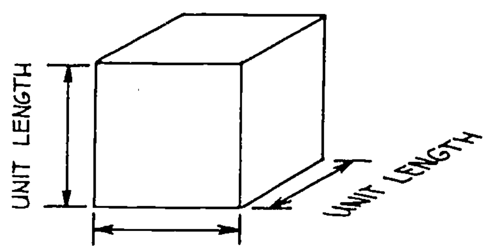
5-1 Volume

In diagnosing certain lung diseases, such as emphysema, the concept of volume is important. In administering drugs by injection, in giving transfusions of blood, the concept of volume is essential. What is volume?

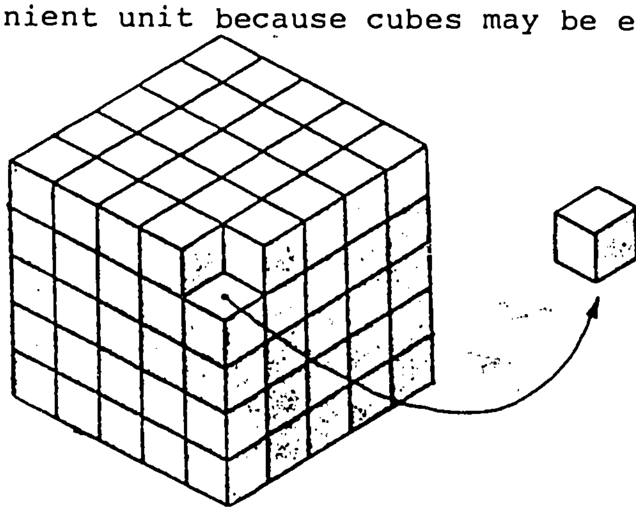
Volume is the measure of three-space, just as area is the measure of two-space and length of one-space. Helicopters and submarines are examples of vehicles which operate in three space. They have the capability of moving up and down in addition to right-left and forward. They operate in three space and the size of their medium is measured in the units of volume.

You have learned that the length of a line segment is the number of unit lengths that can be fitted onto the line segment, and the area of a surface is the number of unit areas that can be fitted into the surface. The volume of a space is the number of unit volumes that can be fitted into the space.

The unit volume most commonly used is the unit cube. The unit cube is a cube whose sides have a length of one unit.



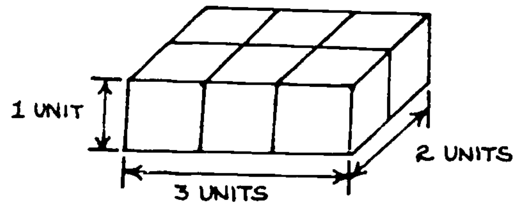
The cube is a convenient unit because cubes may be easily arranged to fill a space.



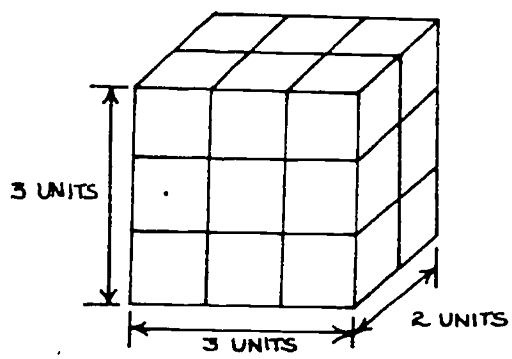
The volume of a space is the number of unit cubes needed to fill the space.

Counting the number of cubes stacked in some space is, of course, not a convenient process. The unit cube will be most useful as a unit volume if we have a formula for calculating the number of unit cubes in a given space. We will now find the formula for the volume of one type of solid figure, the rectangular solid.

Consider a rectangular solid with a height of one unit length. How many unit cubes can be fitted into this solid?



The number of unit cubes contained in the solid is equal to the number of unit squares contained in the base. The area of the base of the solid in the drawing is 6 square units; the volume of the solid is 6 cubic units. If the rectangular solid were not one unit high but h units high, the volume of the solid would be $h \cdot 6$ or $6h$.



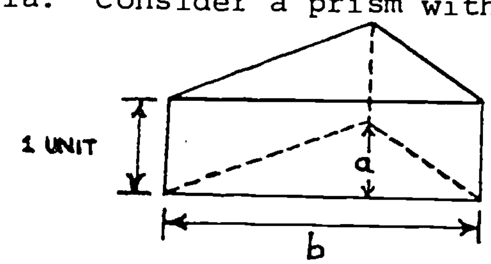
We can let A_b represent the area of the base and write the formula for the volume of a rectangular solid.

$$V = A_b \cdot h$$

Since the area of the base is equal to the length (ℓ) times the width (w), the formula can alternatively be written

$$V = \ell \cdot w \cdot h.$$

The triangular prism is another type of solid for whose volume we can obtain a formula. Consider a prism with a height of one unit.



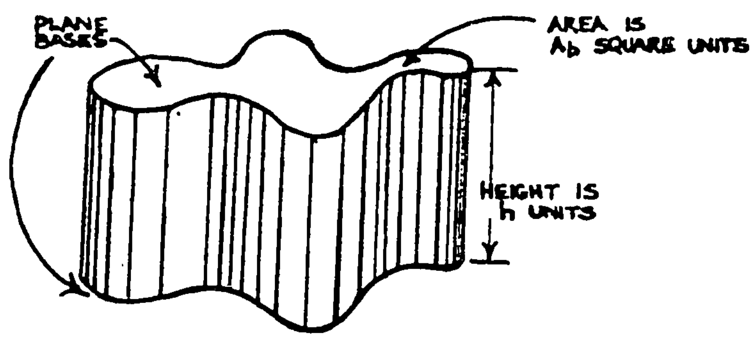
The number of unit cubes, even though some are fractional, that will fit into the prism is equal to the number of unit squares that will fit into the base: the same result we obtained for the rectangular solid. If the prism were h units high, the volume would be h times the area of the base. Letting A_b represent the area of the base, the formula for the volume of a triangular prism is

$$V = A_b \cdot h.$$

Since the area of a triangle is equal to one-half the altitude (a) times the base (b), the formula for the volume of a prism may also be written

$$V = \frac{1}{2} a \cdot b \cdot h.$$

The same reasoning we have used to obtain formulas for the volumes of rectangular solids and triangular prisms may be used to obtain formulas for the volumes of other solid figures which have plane bases and sides perpendicular to the bases.

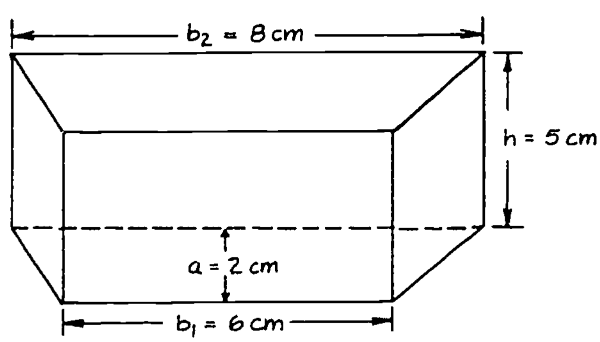


In every case we would find the volume of the solid to be given by the formula

$$V = A_b \cdot h.$$

Example:

Find the volume of the solid with the trapezoidal base shown in the drawing.



Solution:

The formula for the volume of a solid with plane bases and sides perpendicular to the bases is $V = A_b \cdot h$.

In this example $h = 5$ cm.

A_b is the area of the base. The formula for the area of a trapezoid is $A_b = \frac{1}{2}(b_1 + b_2)a$.

In this example $b_1 = 6$ cm, $b_2 = 8$ cm and $a = 2$ cm.

$$\text{Therefore } A_b = \frac{1}{2}(6 + 8) \cdot 2$$

$$= \frac{1}{2}(14) \cdot 2$$

$$= 14 \text{ cm}^2$$

Substituting 14 sq cm for A_b and 5 cm for h in the formula for volume, we obtain

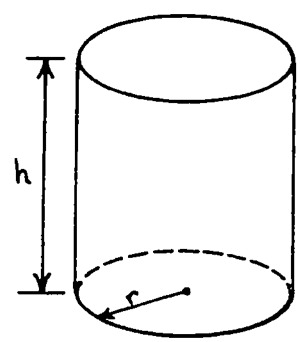
$$V = 14 \text{ cm}^2 \cdot 5 \text{ cm}$$

$$= 70 \text{ cm}^3$$

This answer may be interpreted as a volume equivalent to 70 unit cubes having sides of 1 cm.

Example:

Derive the formula for the volume of a cylinder in terms of its radius r and its height h.



Solution:

The formula for the volume is

$$V = A_b \cdot h$$

The base of a cylinder is a circle, and its area is given by

$$A_b = \pi r^2$$

Therefore the formula for the volume of the cylinder is

$$V = \pi r^2 h$$

It is essential to include in the answer the units in which the volume is measured.

5-2 Metric Volume Units

Units of volume are given in the following table.

METRIC UNITS OF VOLUME

Unit	Abbreviation
cubic centimeter	cm^3
cubic meter	m^3

Length	Volume
1 m = 100 cm	$1\ m^3 = 1{,}000{,}000\ cm^3$

METRIC UNITS OF CAPACITY

Unit	Abbreviation
milliliter	ml
liter	l

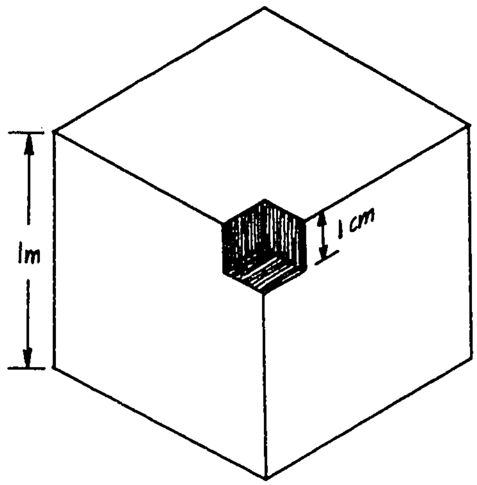
1 liter = 1000 ml

(We will not use "l" to stand for the liter, because it looks like the numeral for "one.")

RELATIONS BETWEEN UNITS OF VOLUME AND CAPACITY

$1\ ml \approx 1\ cm^3$
1 liter ≈ 1.06 quarts

Observe that one cubic meter is not equal to one hundred cubic centimeters but to 100 x 100 x 100, or 1,000,000 cubic centimeters. The reason is that one million cubes with sides of one centimeter can be fitted into a cube whose sides are one meter.



Also given in the table are units of capacity. The basic unit of capacity is the liter, which is approximately equal to a quart. The milliliter is one-thousandth of a liter. When the metric system was established, the milliliter was equivalent to the cubic centimeter; but by modern definitions, one milliliter is equivalent to 1.000028 cubic centimeters.

PROBLEM SET 5:

In Questions 1 through 4 be sure to include appropriate cubic units in your answers.

1. $V = \ell wh$ $\qquad w = 1.5$ cm

   $\ell = 10$ cm $\qquad h = 2$ cm

   a. What is the volume? (Substitute the given numbers into the formula to get a numerical answer.)

   b. What is the solid?

2. $V = \frac{1}{2}abh$ $\qquad b = 1.3$ m

   $a = 20$ m $\qquad h = 9.1$ m

   a. What is the volume?

   b. What is the solid?

3. $V = \frac{1}{2}(b_1 + b_2)ah$

   $b_1 = 6$ cm $\qquad a = 5$ cm

   $b_2 = 5$ cm $\qquad h = \frac{1}{10}$ cm

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a. What is the volume?

b. What is the solid?

4. $V = \pi r^2 h$

$\pi \simeq 3.14$

$r = 9$ m

$h = \frac{1}{3}$ m

a. What is the volume?

b. What is the solid?

5. Using the following formulas, compute volumes when $b_1 = 8$ cm, $b_2 = 14$ cm, $a = 5$ cm, $h = 10$ cm. Be sure to include the proper units along with the numerical part of the answer.

a. $(b_1 \cdot b_1)a$

b. $(b_1 \cdot b_2)a$

c. $\frac{1}{2}(b_2 \cdot a)h$

d. $\frac{1}{2}(b_2 \cdot h)a$

e. $\frac{1}{2}(b_1 + b_2)a \cdot h$

f. $\frac{1}{2}(b_1 + b_2)a \cdot h - (b_1 \cdot a)h$

g. $(b_2 \cdot a)h + \frac{1}{2}(b_2 - b_1)a \cdot h$

6. Assume for the following problems that volume equals areas of the base times the height ($V = A_b \ h$). Using this formula, find the unknown quantities.

a. $A_b = 30$ cm^2; $h = 10$ cm; $V = ?$

b. $A_b = 15$ in^2; $h = 5$ in; $V = ?$

c. $V = 45$ cm^3; $h = 5$ cm; $A_b = ?$

d. $V = 90$ mm^3; $A_b = 45$ mm^2; $h = ?$

e. $V = 1500$ m^3; $A_b = 100$ m^2; $h = ?$

f. $V = 360$ in^3; $h = 12$ in; $A_b = ?$

g. $A_b = 10.5$ m^2; $h = 0.5$ m; $V = ?$

h. $A_b = 0.36$ cm^2; $h = 5.0$ cm; $V = ?$

7. Given that $V = 1200$ cm^3, solve for the unknown quantity.

EXAMPLE: $V = \frac{1}{2}(b_1 + b_2)ah$; $b_1 = 10$ cm, $b_2 = 30$ cm, $h = 30$ cm, $a = ?$

$$1200 \text{ cm}^3 = \frac{1}{2}(10 \text{ cm} + 30 \text{ cm})a \cdot 30 \text{ cm}$$

$$1200 \text{ cm}^3 = \frac{1}{2}(40 \text{ cm})a \cdot 30 \text{ cm}$$

$$1200 \text{ cm}^3 = (600 \text{ cm}^2)a$$

$$a = \frac{1200 \text{ cm}^3}{500 \text{ cm}^2}$$

$$a = 2 \text{ cm}$$

a. $V = b_1 \cdot b_2 \cdot a$; $b_1 = 60$ cm, $b_2 = 2$ cm, $a = ?$

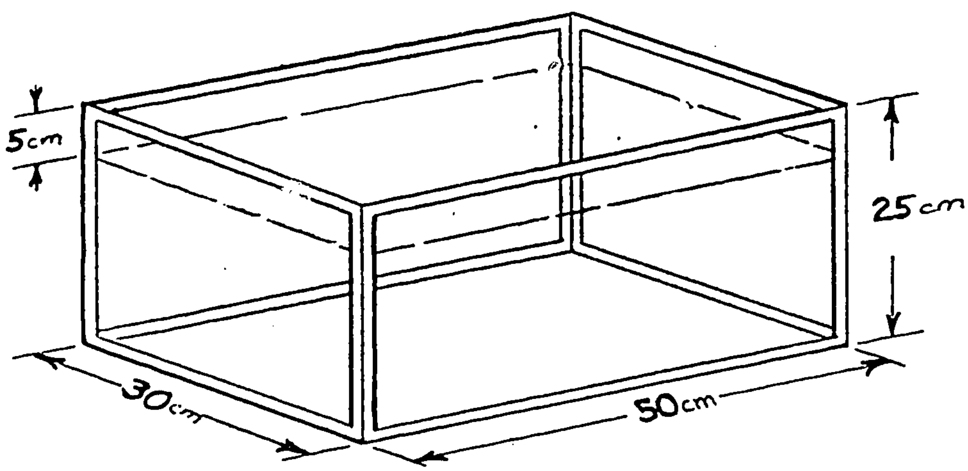
b. $V = \frac{1}{2}(b_1 \cdot a)h$; $b_1 = 40$ cm, $a = 10$ cm, $h = ?$

c. $V = \frac{1}{2}(b_1 \cdot a)h$; $a = 12$ cm, $h = 5$ cm, $b_1 = ?$

d. $V = (b_1 \cdot b_1)a$; $b_1 = 10$ cm, $a = ?$

e. $V = \frac{1}{2}(b_1 + b_2)a \cdot h$; $h = 10$ cm, $a = 20$ cm; $(b_1 + b_2) = ?$

8. A fish tank is in the shape of a simple rectangular solid with length 50 cm, width 30 cm, and height 25 cm. Water is filled to 5 cm from the top.



a. Find the total capacity of the tank.

b. Find the volume of water the tank now holds.

Suppose that 750 cm^3 of sand are placed in the tank.

c. What happens to the water level?

d. How far is the water level from the top of the tank now?

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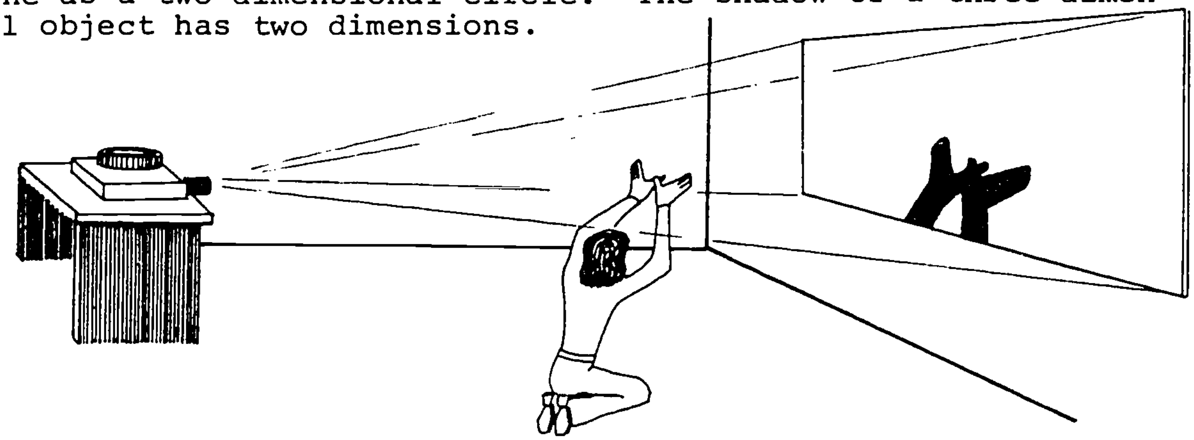
SECTION 6:

6-1 A Two-Dimensional Universe

You are a three-dimensional object and you live in three-dimensional space. You are familiar with how things are in a three-dimensional universe: objects have height, width and thickness. You can move forward and back, up and down, and right and left. When you look at a three-dimensional sphere, you see its outline as a two-dimensional circle. The shadow of a three-dimensional object has two dimensions.



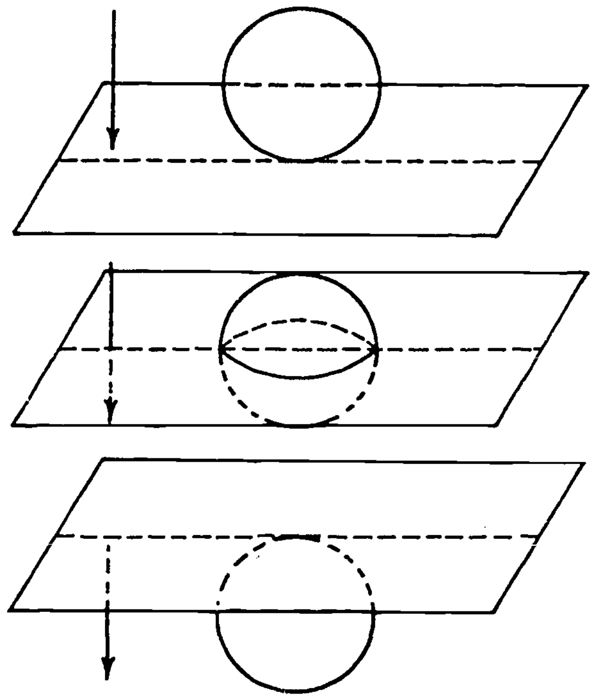
Imagine how life would be in Flatland, a two-dimensional universe. The surface of this page is an example of a portion of a two-dimensional universe. (The entire universe would extend infinitely in the two dimensions.) The letters on this page are two-dimensional objects: They have height and width but no thickness (if the thickness of the ink is ignored).

Suppose these letters to be creatures in Flatland. What would one letter look like to another? If you view a piece of paper from its edge, you will see a line segment. Likewise, one letter would appear to another to be a line segment.

(Our Flatland analogy is not perfect. A true two-dimensional object, such as a shadow, has no thickness at all and is invisible when viewed from the side. The same imperfection will occur when we consider lines, which have no thickness, and points, since a true point has no dimensions and is therefore not visible.)

What would inside and outside mean to the letter-creatures? Would one letter be able to see the inside of another letter? Being three-dimensional, we are able to see the area within the letter "O", but a Flatland neighbor "N" could not. N would have to be able to see "over" O and "over" is not a direction available to him. N could no more see the inside of O than you can see the lungs of a friend.

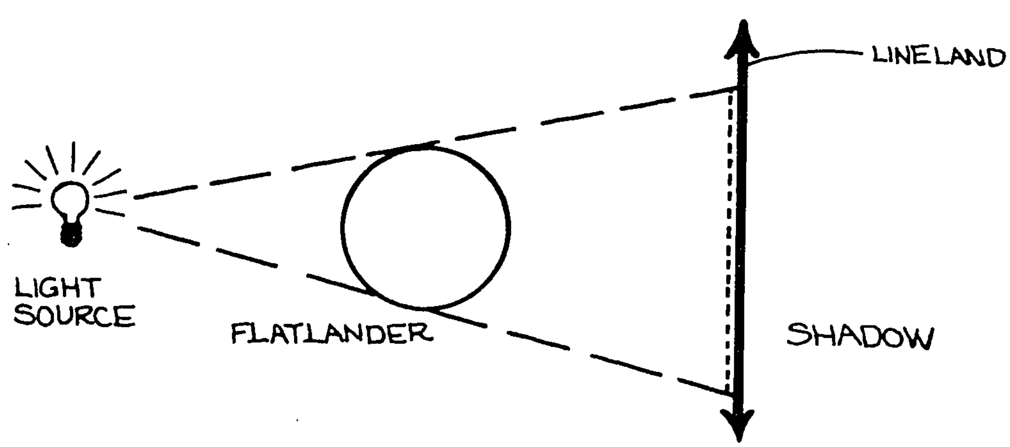
If a sphere passed through the plane of Flatland, it would appear to a letter-creature briefly as a point, then as a line, then as a point again, and finally disappear.



Flatlanders would not be able to communicate by writing letters as we do. Can you suggest a method of written communication they could use?

6-2 A One-Dimensional Universe

The shadow of a three-dimensional object has two dimensions, but the shadow of a creature in two-dimensional Flatland has one dimension.



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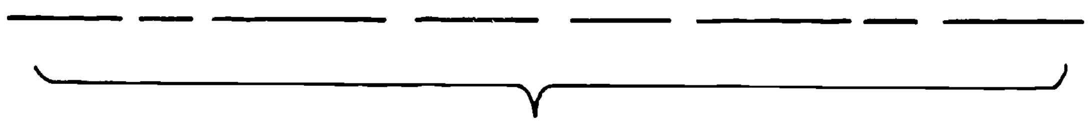
Consider now a universe confined to one dimension. Everything would take place along a line. We will call this universe Lineland. The only shapes possible for creatures in this world would be line segments and points.

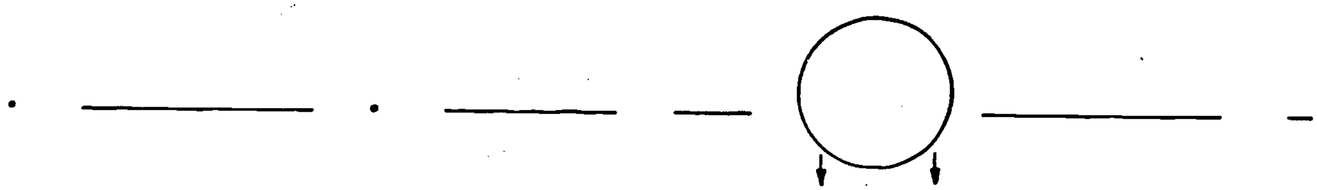


INHABITANTS OF LINELAND

Many aspects of Lineland would seem odd to us. If the line-segment creatures who inhabited Lineland had an eye at each end-point, they could see at most two other creatures. Each of the neighboring Linelanders would appear to be a point. The outside of a Linelander would be the end points of his line segment. The rest of his line segment would be his insides, which would not be visible to his friends along the line. If Linelanders wished to communicate with one another, they could not send written messages, but they might communicate by touch.

Linelanders would not be able to amuse themselves with our sports of football or basketball or tennis. Our one sport which they might enjoy is tug-of-war.

If a letter O from Flatland passed through Lineland, to the two nearby line-segment creatures he would appear as a point.



While he was passing through the line, he would appear to these creatures to be no different from another Linelander. When O had passed through, he would disappear as suddenly as he had appeared.

If a sphere passed through Lineland, it would also appear as a point, then disappear. A Linelander would be unable to distinguish between a sphere and a circle.

6-3 Our Three-Dimensional Universe

As inhabitants of Spaceland, our three-dimensional universe, we smile at the simple-minded ways of Flatlanders and Linelanders. We think that if they could be in three-dimensions, they could see things as they really are.

However, there is a disturbing question which science-fiction writers are fond of asking. Could there be four- or five-dimensional creatures, who laugh at us and our simple-minded three-dimensional ways?

How would a four-dimensional creature appear to us as he passed through Spaceland? A circle passing through Lineland appears to have only one dimension. A sphere passing through Flatland appears as a point, which grows to a circle, then shrinks to a point and disappears as it goes past. In the same way, a four-dimensional "sphere" passing through our universe would appear as a point, which would grow into a three-dimensional sphere, then shrink to a point and disappear. The "shadow" of a four-dimensional being would appear to us as three-dimensional.

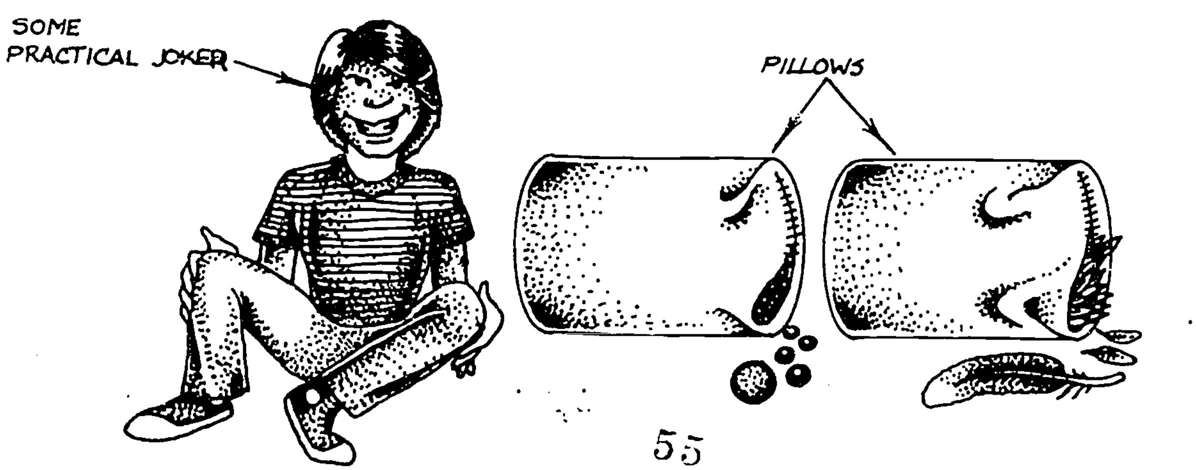
If you saw a point appear suddenly, grow to a sphere, shrink and finally disappear, would you believe it to be a four-dimensional creature just passing through? Do you think you could convince anyone else that what you saw was a visitor from the land of four dimensions?

No physical scientist has suggested that there is a fourth dimension in space. No theory of the universe requires a fourth spatial dimension. However, mathematics need not restrict itself to what a physical scientist considers to be reality.

6-4 Units Which Are Ratios

We are paying much attention to the three spatial measures length, area and volume. These are among the most basic of measures. It is important that you recognize what each one measures. Each is an example of a simple measure. Many things cannot be measured by means of a simple measure. Perhaps the most common examples of non-simple measures are measures of speed. For example, kilometers per hour and miles per hour are both measures of speed. Both are ratios. They compare length (either kilometers or miles) to time. Many other measures are of the non-simple variety. For example, density and pressure are non-simple measures.

Let's consider density for a while. Suppose we have two pillows which have the same volume. Furthermore, suppose that some practical joker has filled one of the pillows with buckshot and the other with feathers.



How could we tell them apart? Obviously, we could try lifting both pillows. The heavier pillow would be the one with the lead in it. Does this mean that lead is heavier than feathers? No, it does not. We could, if we wanted to, round up enough feathers to outweigh the lead pillow. If we want to say something about the relative weights of a given amount of lead or feathers, then we must also say something about the volumes of lead and feathers in realtion to their weights. Scientists use the idea of density to describe the volume related differences in weight between lead and feathers. Density is the ratio of the mass (or weight) of an object to its volume,

$$\rho = \frac{M}{V}$$

where ρ (the Greek letter "rho") equals density, M = mass, and V = volume.

Lead has a density of about 11 grams per cubic centimeter. Feathers have a density of about .1 gram per cubic centimer when they are completely fluffed up. The density of solids or liquids does not change with the amount being considered. A ton of lead has the same density as a gram of lead. We will illustrate this point with a couple of numerical examples.

<u>EXAMPLE 1</u>:

Mass = 2260 grams of lead

Volume = 200 cubic centimeters

ρ = ?

<u>SOLUTION</u>:

$$\rho = \frac{M}{V}$$

$$\rho = \frac{2260\ g}{200\ cm^3}$$

$$\rho = \frac{11.3\ g}{cm^3}$$

<u>EXAMPLE 2</u>:

Mass = 3.36 g of lead

Volume = .3 cm^3

ρ = ?

<u>SOLUTION</u>:

$$\rho = \frac{M}{V}$$

$$\rho = \frac{3.39\ g}{.3\ cm^3}$$

$$\rho = \frac{11.3\ g}{cm^3}$$

The density of a solid or liquid substance is a constant physical property of the substance. Tables of densities may be found in various places. Most physics books will have a density table. To find it, look up "density" in the index at the end of the book. The same procedure may be used to find the density table in the <u>CRC Handbook of Chemistry and Physics</u>. Feel free to use these references to answer any problem in the problem set. For example, suppose we give you the mass and volume of a sample of copper and ask you to find its density. Instead of calculating it from the numerical information, you can feel free to look it up in a book.

PROBLEM SET 6:

1. To a Flatlander a circle looks like a (point, line, circle).

2. Describe what a Flatlander sees as a sphere passes through his world.

3. If a Linelander had an eye on each end, how many other Lindlanders could he see? (or, what is the maximum number of other railroad cars that may be attached to a single railroad car?"

4. What sport might Linelanders enjoy?

5. If a four dimensional creature made an appearance to us Spacelanders, how many dimensions would he appear to have?

For Problems 6 through 16 find the density

$$\rho = \frac{M}{V}$$

where ρ = density

M = mass

V = volume

6. Aluminum: mass = 81 g
   volume = 30 cm^3

7. Silver: mass = 105 g
   volume = 10 cm^3

8. Silver: mass = 714 g
   volume = 68 cm^3

9. Gold: mass = 47.3 kg
   volume = 2450 cm^3

10. Copper: mass = 547 g
    volume = 61.2 cm^3

11. Copper: mass = 48.0 g
    volume = 5.37 cm^3

12. Chromium: mass = 97.3 g
    volume = 13.63 cm^3

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13. Chromium: mass = 28.2 g

Volume = 3.95 cm^3

14. Titanium: mass = 157.2 g

volume = 34.93 cm^3

15. A lead weight is dropped into a graduated cylinder. The level of the water in the cylinder increases from 50.0 to 52.5 ml. The mass of the weight is 28.25 g. What is the density of the lead weight?

16. A piece of wood is weighed. It's mass is found to be 48 g. When it is held under water in a graduated cylinder, it increases the water level from 30 ml to 90 ml. What is the density of the piece of wood?

SECTION 7:

7-1 Conversion Factors and Dimensional Algebra

One of the practical necessities of doing numerical work with units of measure is the ability to change from one measure to another. For example, you should be able to convert centimeters to meters, or kilograms to grams. In this section you will see how such conversions can be performed.

Consider the problem of expressing a length of 6 feet in units of centimeters. We begin by recalling the relation between feet and inches, and between inches and centimeters.

$$1 \text{ ft} = 12 \text{ in}$$

$$1 \text{ in} = 2.54 \text{ cm}$$

We may divide both sides of the first equation by 1 ft.

$$\frac{1 \text{ ft}}{1 \text{ ft}} = \frac{12 \text{ in}}{1 \text{ ft}}$$

Since the ratio of 1 ft to 1 ft is 1, we may rewrite the equation

$$1 = \frac{12 \text{ in}}{1 \text{ ft}}$$

We may similarly divide both sides of the equation 1 in = 2.54 cm by 1 in.

$$\frac{1 \text{ in}}{1 \text{ in}} = \frac{2.54 \text{ cm}}{1 \text{ in}}$$

$$1 = \frac{2.54 \text{ cm}}{1 \text{ in}}$$

Since

$$6 \text{ ft} \cdot 1 \cdot 1 = 6 \text{ ft},$$

we may write

$$6 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} \cdot \frac{2.54 \text{ cm}}{1 \text{ in}} = 6 \text{ ft}.$$

Observe that the ratio of 6 feet to 1 foot is 6.

$$\frac{6 \text{ ft}}{1 \text{ ft}} = 6$$

The equation suggests that we are allowed to cancel the "ft" in the numerator and the "ft" in the denominator.

$$\frac{6\ \cancel{ft}}{1\ \cancel{ft}} = 6$$

Canceling units is an important part of the method being developed. We return to the conversion of 6 feet to centimeters.

$$6\ ft \cdot \frac{12\ in}{1\ ft} \cdot \frac{2.54\ cm}{1\ in} = 6\ ft.$$

We may cancel the "ft" on the top with the "ft" on the bottom, and the "in" on the top with the "in" on the bottom.

$$6\ \cancel{ft} \cdot \frac{12\ \cancel{in}}{1\ \cancel{ft}} \cdot \frac{2.54\ cm}{1\ \cancel{in}} = 6\ ft.$$

Removing the canceled units reduces the equation to

$$6 \cdot 12 \cdot 2.54\ cm = 6\ ft.$$

We perform the multiplications on the left side of the equation to obtain

$$183\ cm = 6\ ft.$$

Terms such as $\frac{12\ in}{1\ ft}$ and $\frac{2.54\ cm}{1\ in}$ are called <u>conversion factors</u>. It is essential to write the conversion factors in such a way that the units may be canceled. In the previous problem the conversion factor used to convert feet to inches was written $\frac{12\ in}{1\ ft}$ and not $\frac{1\ ft}{12\ in}$. If we had tried to use $\frac{1\ ft}{12\ in}$ as a conversion factor, we would have obtained the equation

$$6\ ft \cdot \frac{1\ ft}{12\ in} \cdot \frac{2.54\ cm}{1\ in} = 6\ ft.$$

The units "ft" occur twice on the top and the units "in" twice on the bottom, so that no cancellation is possible. There is a rule for choosing a conversion factor. <u>If the unit to be canceled is on the top, the same unit should appear on the bottom of the conversion factor, and vice versa.</u>

A second rule to remember is that a conversion factor must be equal to 1. If units of length are being converted, the length represented by the top of the conversion factor must be equivalent to the length represented by the bottom. $\frac{12\ in}{1\ ft}$ is a conversion factor because a length of 12 inches is equivalent to a length of

1 foot. $\frac{2.54 \text{ cm}}{1 \text{ in}}$ is a conversion factor because 2.54 centimeters is equivalent to 1 inch.

The processes of multiplying by conversion factors and cancellation of units is often referred to as dimensional algebra. The advantage of using these techniques is illustrated by the following examples.

EXAMPLE:

Convert 4 pounds into grams, given that 2.2 lb ≃ 1 kg and that 1 kg = 1000 g.

SOLUTION:

Since the unit "lb" appears on the top, we require a conversion factor with "lb" on the bottom.

$$\frac{1 \text{ kg}}{2.2 \text{ lb}}$$

is a conversion factor because

$$\frac{1 \text{ kg}}{2.2 \text{ lb}} \simeq 1.$$

A second conversion factor is required, in which the unit "kg" is on the bottom. Since

$$\frac{1000 \text{ g}}{1 \text{ kg}} = 1$$

$\frac{1000 \text{ g}}{1 \text{ kg}}$ is the desired conversion factor.

We may thus write

$$4 \text{ lb} \simeq 4 \text{ lb} \cdot \frac{1 \text{ kg}}{2.2 \text{ lb}} \cdot \frac{1000 \text{ g}}{1 \text{ kg}}$$

$$\simeq 4 \cancel{\text{lb}} \cdot \frac{1 \cancel{\text{kg}}}{2.2 \cancel{\text{lb}}} \quad \frac{1000 \text{ g}}{1 \cancel{\text{kg}}}$$

$$\simeq 4 \cdot \frac{1}{2.2} \cdot 1000 \text{ g}.$$

The last equation is further simplified.

$$4 \text{ lb} \simeq 1.818 \times 1000 \text{ g} \quad \text{or} \quad 1818 \text{ g}$$

EXAMPLE:

Convert 60 miles per hour into feet per second, given that 1 hr = 60 min, 1 min = 60 sec and 1 mi = 5820 ft.

SOLUTION:

The following three conversion factors are required.

$$\frac{1 \text{ hr}}{60 \text{ min}}$$

$$\frac{1 \text{ min}}{60 \text{ sec}}$$

$$\frac{5280 \text{ ft}}{1 \text{ mi}}$$

We use these conversion factors to write

$$60 \frac{\text{mi}}{\text{hr}} = 60 \frac{\cancel{\text{mi}}}{\cancel{\text{hr}}} \cdot \frac{1 \cancel{\text{hr}}}{60 \cancel{\text{min}}} \cdot \frac{1 \cancel{\text{min}}}{60 \text{ sec}} \cdot \frac{5280 \text{ ft}}{1 \cancel{\text{mi}}}$$

$$= 88 \frac{\text{ft}}{\text{sec}} .$$

7-2 Another Use of Dimensional Algebra

In this section we will examine a type of problem which is similar to a problem in converting units. However, the conversion factors are terms which do not always represent a conversion between two units of measurement.

Consider the following problem. Each park in a certain town has 18 oak trees, and each oak tree has 200 acorns. How many acorns are in a park?

Given the quantities $\frac{18 \text{ oak trees}}{1 \text{ park}}$ and $\frac{200 \text{ acorns}}{1 \text{ oak tree}}$, we may easily determine the number of acorns in a park.

$$\frac{18 \text{ oak trees}}{1 \text{ park}} \cdot \frac{200 \text{ acorns}}{1 \text{ oak tree}} = 3600 \frac{\text{acorns}}{\text{park}}$$

Observe that the conversion factor

$$\frac{200 \text{ acorns}}{1 \text{ oak tree}}$$

is equivalent to 1 only for the purposes of this particular problem. In this particular case, it is possible to cancel units in the same way as in the previous unit conversion problems.

$$\frac{\cancel{\text{oak tree}}}{\text{park}} \cdot \frac{\text{acorns}}{\cancel{\text{oak tree}}} = \frac{\text{acorns}}{\text{park}}$$

PROBLEM SET 7:

1. The teacher will assign you one of the units from the following list to define in terms of metric units.

a. furlong	n. league	aa. micron
b. rod	o. gill	ab. grain
c. link	p. palm	ac. pennyweight
d. chain	q. imperial gallon	ad. scruple
e. pole	r. minim	ae. troy ounce
f. cord-foot	s. drop	af. stone
g. cord	t. fluid dram	ag. barn
h. fortnight	u. centare	ah. pottle
i. astronomical year	v. hectare	ai. skein
j. fathom	w. quintal (metric)	aj. span
k. cable	x. stere (pronounced "STEER")	ak. ell
l. hand	y. are (pronounced, "AIR")	al. noggin
m. nautical mile	z. myriameter	am. butt

Problems 2 through 10 are expressions converting one set of units to another. In some cases the conversion factors are written correctly; in other cases, a conversion factor is inverted. If the conversion factors of an expression are written correctly, write "correct." If a conversion factor is inverted, identify the factor and write it correctly.

2. $600{,}000\ \frac{\text{drops}}{\text{cloud}} \cdot \frac{2\ \text{clouds}}{\text{sky}} = 1{,}200{,}000\ \frac{\text{drops}}{\text{sky}}$

3. $80\ \frac{\text{heartbeats}}{\text{minute}} \cdot \frac{60\ \text{minutes}}{\text{hour}} \cdot \frac{\text{day}}{24\ \text{hours}} = 115{,}200\ \frac{\text{heartbeats}}{\text{day}}$

4. $600\ \frac{\text{kernals}}{\text{cob}} \cdot \frac{\text{plant}}{12\ \text{cobs}} = 7200\ \frac{\text{kernals}}{\text{plant}}$

5. $300\ \frac{\text{ml}}{\text{breath}} \cdot \frac{16\ \text{breaths}}{\text{min}} \cdot \frac{60\ \text{min}}{\text{hr}} \cdot \frac{\text{day}}{24\ \text{hr}} \cdot \frac{\text{yr}}{365\ \text{days}} \simeq 2{,}520{,}000{,}000\ \frac{\text{ml}}{\text{yr}}$

6. $12.9\ \frac{\text{grains}}{\text{minim}} \cdot \frac{16.2\text{ minims}}{\text{ml}} \cdot \frac{\text{gram}}{15.4\text{ grain}} \simeq 13.6\ \frac{\text{grams}}{\text{ml}}$

7. $2\ \frac{\text{rats}}{\text{cage}} \cdot \frac{14\text{ cages}}{\text{laboratory}} \cdot \frac{6\text{ laboratories}}{\text{school}} = 168\ \frac{\text{rats}}{\text{school}}$

8. $14.9\ \frac{\text{drams}}{\text{fluid oz}} \cdot \frac{29.6\text{ ml}}{\text{fluid oz}} \cdot \frac{1.77\text{ grams}}{\text{dram}} \simeq .89\ \frac{\text{grams}}{\text{ml}}$

9. $6.6\ \frac{\text{inches}}{\text{hr}} \cdot \frac{24\text{ hr}}{\text{day}} \cdot \frac{14\text{ days}}{\text{fortnight}} \cdot \frac{\text{rod}}{198\text{ inches}} = 11.2\ \frac{\text{rods}}{\text{fortnight}}$

10. $90{,}000{,}000\text{ sec} \cdot \frac{\text{min}}{60\text{ sec}} \cdot \frac{\text{hr}}{60\text{ min}} \cdot \frac{\text{day}}{24\text{ hr}} \cdot \frac{365\text{ days}}{\text{year}} \cdot \frac{100\text{ years}}{\text{century}}$

$\simeq .0285$ centuries

In Problems 11 through 17, supply the missing units and, if necessary, the entire conversion factor. Be sure the proper unit is on top and the proper unit is on the bottom. It is not necessary to perform the indicated multiplications and divisions (unless you wish to check your work).

11. $.18\text{ hr} \cdot \frac{60\text{ min}}{\text{hr}} \cdot (?) = 648\text{ sec}$

12. $88\ \frac{\text{ft}}{\text{sec}} \cdot \frac{\text{mile}}{5280\text{ ft}} \cdot (?) = 60\ \frac{\text{miles}}{\text{hr}}$

13. $4{,}000{,}000\ \frac{\text{mg}}{\text{liter}} \cdot (?) \cdot \frac{\text{kg}}{1000\text{ g}} = 4\ \frac{\text{kg}}{\text{liter}}$

14. $15.1\ (?) \cdot \frac{\text{gallons}}{3.8\text{ liters}} \cdot \frac{\text{kg}}{2.2\text{ lb}} \simeq 1.8\ \frac{\text{kg}}{\text{liter}}$

15. $1.31\ \frac{\text{g}}{\text{ml}} \cdot (?)\ \frac{\text{kg}}{1000\text{ g}} = 1.31\ \frac{\text{kg}}{\text{liter}}$

16. $1200\ (?) \cdot 23\ \frac{\text{breaths}}{\text{min}} \cdot \frac{\text{liters}}{1000\text{ ml}} = 27.6\ \frac{\text{liters}}{\text{min}}$

17. $30\ \frac{\text{students}}{\text{class}} \cdot \frac{18\text{ classes}}{\text{school}} \cdot \frac{1}{2}\ (?) = 270\ \frac{\text{girls}}{\text{school}}$

Perform the following unit conversions. Show all work.

18. 3 miles to km.

Use 1 mile = 5280 ft

1 ft $\simeq$ .305 m

1000 m = 1 km

64

19. 2 lb to kg.

Use 1 lb = 16 oz.

1 oz ≃ 28.35 g

1000 g = 1 kg

20. $720 \frac{mi}{day}$ to $\frac{km}{hr}$

Use 1 day = 24 hr

1 mi ≃ 1.61 km

21. 960 minims to pints.

Use 60 minims = 1 fluid dram

8 fluid drams = 1 fluid ounce

16 fluid ounces = 1 pint

22. $8.36 \frac{lb}{gallon}$ to $\frac{kg}{liter}$

Use 2.2 lb ≃ 1 kg

1 gal ≃ 3.8 liters

23. Nancy Nicotine smokes a pack of cigarettes a day. A pack contains 20 cigarettes. How many cigarettes does Nancy smoke during a year?

24. A diver is using air at a rate of 300 gallons per hour. What is his rate in liters per minute? (1 gallon ≃ 3.79 liters.)

25. Records show that during 1970 one industry in the San Francisco Bay Area released pollutants into the atmosphere at the rate of $2760 \frac{kg}{hr}$. What was the rate in tons per day? (1 ton = 2000 lb, 1 kg ≃ 2.2 lb.) Round your answer to the nearest ton.

SECTION 8:

8-1 Measurement

In this unit we occasionally define words and examine concepts which are so basic as to seem unnecessary or impossible to define. For example, we discussed earlier the meaning of the word "length."

In this section we consider the concept of measurement. The word "measure" was used in the discussion of length and area, and you will soon be using measurements as ordered pairs to be graphed. This is therefore an appropriate point at which to discuss measurement.

First, it should be pointed out that the word "measurement" has two different meanings. The word is used both to describe the act of measuring and the numerical result that is obtained. Thus determining the length of a line segment involves the act of measurement, while 6.25 centimeters is referred to as a measurement.

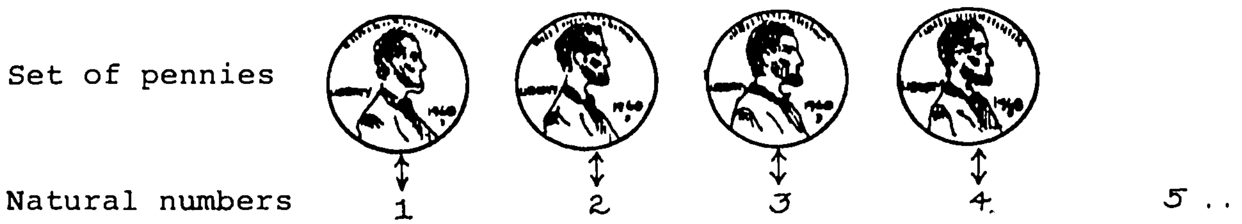
The most rudimentary type of measurement involves answering a question that requires an answer of yes or no. The idea that answering such a question as "Does Leroy have a fever?" or "Is George dead?" involves measurement may seem strange to you. We will discuss the idea after considering two other types of measurements.

A second type of measurement is counting. Counting is required to answer a question of the type, "How many pennies are in Mark's pocket?" or "How many towns are in California?" This type of question requires that the answer be a whole number. Mark could not have a fraction of a penny in his pocket, nor could California have a fractional number of towns. A feature of this type of question is that the correct answer is exact.

The measurements required to answer the third type of question cannot be exact. Questions such as, "How tall is Anne?" call for numbers that lie on a continuous scale, like points on a number line. Anne's height may be 1.75 meters, it may be 1.7515 meters, or it may be 1.751575836 meters. There is no limit to the precision with which we could state Anne's height. There is a limit, however, to the precision with which we can measure her height: we are limited by the precision of the device we use for measuring.

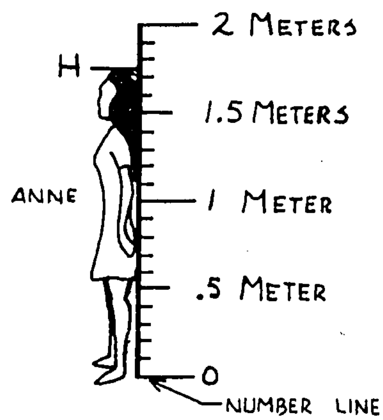
The process of measuring involves comparing the object being measured to a standard. To answer the question, "Does Leroy have a fever?", we need a standard definition of "fever" with which to compare Leroy's condition. To answer the question, "Is George dead?", we need a standard definition of "dead" with which to compare George's condition. Such a comparison is a qualitative measurement.

To answer the question, "How many pennies are in Mark's pocket?", a different type of standard is required. This standard is the set of natural numbers. Comparison is made by setting each penny in one-to-one correspondence with a number.



Counting objects is one type of quantitative measurement.

Another type of quantitative measurement is required to answer the question, "How tall is Anne?" The word "tall" calls for a comparison of lengths. Anne's length is compared to a number line which is scaled so that one unit is a standard length, such as a foot or a meter. However, unlike counting, our measurement will commonly contain a fraction of a unit.



Look at the diagram above. Try to determine Anne's height in meters. Notice that her height does not fall exactly on a divison. It is between 1.7 m and 1.8 m.

If you look very closely, you will see that the "H" line appears to be about halfway between 1.7 m and 1.8 m. 1.75 m is halfway between 1.7 and 1.8 meters [1.75 = (1.8 + 1.7) ÷ 2]. We will call 1.75 m the most precise measurement possible in this given situation.

This process for finding the most precise measurement of Anne's height is common. In this course you will often be called on to read scales. It is impossible to do good lab work in a hurry. To do it well you must take your time. Especially at first. Haste in scale reading will cause your answers to be more imprecise than is necessary. The quality of your lab work is greatly dependent on the habits you have. Sloppy, hasty, scale reading is a bad habit. Do not develop it now with the thought of changing it later when the results are more important. As you know, bad habits are hard to break. It is always a good policy to develop good habits right from the start. They will never have to be broken.

In summary, there are three kinds of measurement. One kind is qualitative (yes-no) and the other two are quantitative. Of the quantitative kinds of measurement, one kind is exact (counting) and the other (comparison to a continuous scale) requires careful scale reading.

8-2 Precision and Imprecision

Precision is a measure of the limits of the reproducibility of a measurement. In this case the reproducibility is closely related to the limits of careful scaling reading. 1.75 m is the most precise measurement of Anne's height that we can make from the drawing. However, it is extremely unlikely that her height is exactly 1.75 m. For example, suppose Anne's true height was closer to 1.751 m than 1.750 m. Would we be able to tell the difference just by a closer look at the diagram? The answer is no. The precision that we are able to achieve is limited by the measuring instrument we are using.

How far away from 1.75 m could Anne's true height be and yet be impossible to detect on the scale? To answer this question we must first determine the smallest and largest possible scale readings. Could Anne's height be as little as 1.74 m? Look at the diagram now. Look as closely as you can without developing eyestrain. The question is, "Could 'H' be located as little below halfway between 1.7 m and 1.8 meters?

The safest answer is that 'H' is at least 1.74 m. In other words:

$$H \geq 1.74 \text{ m}$$

The next logical step in establishing the bounds of possibility is to find a upper boundary for 'H'. In other words, how large might 'H' be? Look once again at the diagram.

For the sake of argument we are going to say that we are certain that 'H' couldn't possibly be more than 1.76 m. In mathematical symbolism we write:

$$H \leq 1.76 \text{ m}$$

By combining these two observations we can say that we are certain that 'H' falls between 1.74 and 1.76 on the scale. One way to represent this information is:

$$1.74 \leq H \leq 1.76$$

There is a much more compact way to say the same thing. It is:

$$H = 1.75 \pm .01$$

We call this statement a range of imprecision. It is one way of saying that we are reasonably certain that the position of 'H' on the scale does not vary more than .01 m (1 cm) from 1.75 m. The diagram below describes the new terminology associated with the idea of precision.

$$\underbrace{\underbrace{1.79}_{\text{midpoint}} \pm \underbrace{.01}_{\text{imprecision}}}_{\text{range of imprecision}}$$

For algebraic purposes a range of imprecision is often stated

$$x \pm \Delta x$$

where "x" represents the midpoint and "Δx" represents the imprecision. The small triangle, Δ, is the Greek letter <u>delta</u>. The "<u>d</u>" in <u>d</u>elta is supposed to remind people that "<u>d</u>ifferences" are involved. In this case it is the distance between an endpoint (either $x + \Delta x$ or $x - \Delta x$) and the midpoint (x).

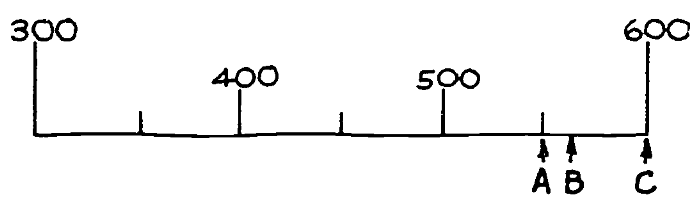
In summary, the precision attainable with a given measuring instrument will always be limited. We may make more and more precise measurements (less and less imprecise) by using better and better instruments. But we can never measure anything exactly in a continuous scale type measurement. Therefore, we write a range of imprecision, (for example $1.75 \pm .01$) to specify the precision attainable with a given measuring instrument.

8-3 More About Estimating Imprecision

We'll give you another example of careful scale reading. Keep one thing in mind; this procedure calls for human judgment. Therefore, one person's best estimate will vary somewhat from another's.

<u>EXAMPLE</u>:

Find the range of imprecision for the position of the arrow B.



<u>SOLUTION</u>:

First, we discover the meaning of the unmarked divisions. Notice that they lie halfway between the 100's divisions; therefore, the little divisions are 50 away from a big division.

Second, we visually estimate the fraction of a division indicated by the arrow. A reasonable guess is that B (the arrow) lies about one third of the way from point A to point C (that is, from 550 to 600). The numerical value of this distance is one third of 50.

$$\frac{1}{3} \times .50 = 16.666 \ldots$$

Since we're estimating in the first place, we won't be fussy about all those trailing 6's. We round off as shown below.

$$16.666\ldots \approx 17$$

Why 17? 17 is a more convenient number than $16\frac{2}{3}$ and we certainly couldn't tell the difference between $16\frac{2}{3}$ and 17 on the scale. To find the scale reading represented by the arrow, we add 550 and 17 to get 567. 567 is now a candidate for the midpoint of a range of imprecision.

There is another way to estimate the midpoint which involves less guess work. We take out our metric ruler and measure $\overline{AB}$ and $\overline{AC}$.

length of AB ≃ 4mm

length of AC ≃ 14 mm

Then we reason that B lies $\frac{4}{14}$ths of the way from A to C; therefore, the distance represented by $\overline{AB}$ is $\frac{4}{14}$ths of 50.

$$\frac{4}{\cancel{14}_{7}} \times \cancel{25}^{50} = \frac{100}{7}$$

$$\simeq 14.28$$

$$\simeq 14$$

Now, we get our estimate of the midpoint by adding 14 and 550 to get 564. This is our second candidate for a midpoint of a range of imprecision.

Either one (567 or 564) is okay. Two independent estimates will seldom agree exactly. Since we must choose only one before we go on, we choose the 567 estimate. People who read scales rarely pull out a ruler and measure. Therefore, we'll choose 567 because this figure was actually obtained by eyeball judgment.

Thirdly, we estimate the imprecision of our estimate (567). Five is one-tenth of the distance between the smallest divisions. This is generally a good method for choosing the numerical value of the imprecision. That is, it should be one-tenth of the distance between the smallest divisions. This rule of thumb becomes impossible to apply if the smallest divisions are very small or very large. However, it will work in the majority of cases.

Finally we put our estimated midpoint (567) and our estimated imprecision (5) together to get a range of imprecision.

$$567 \pm 5$$

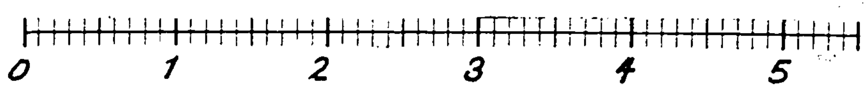
Notice that our more accurately estimated midpoint (564) lies within this range of imprecision.

8-4 Graphs of Ranges of Imprecision

Graphs help people visualize new ideas. With this in mind we'll show you how to graph and interpret graphs of ranges of imprecision.

PROBLEM:

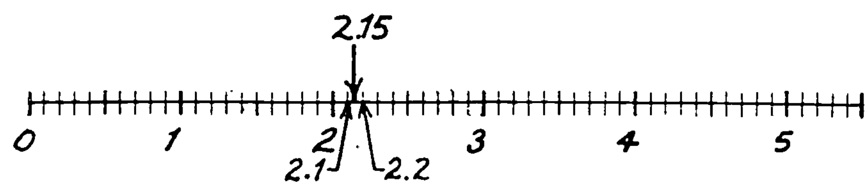
Graph 2.15 ± .25 on the number line given below.



SOLUTION:

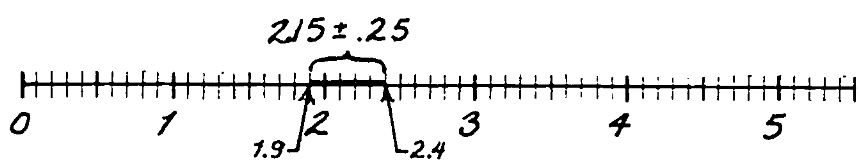
First we note that each small division means one-tenth (.1).

Second we see that we are asked to graph 2.15 and that 2.15 is not a whole tenth. However, we do know that 2.15 is between 2.10 and 2.20. We can find these two points on the given number line. When these two points are located we place 2.15 halfway between them.



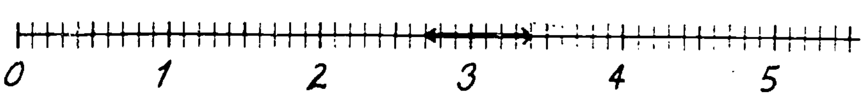
Thirdly, we take care of the ±.25 expression. Since 2.15 + .25 = 2.40 and 2.15 - .25 = 1.90, the points 2.4 and 1.9 will be the endpoints of our range.

Finally, since a range of imprecision represents all of the points between the endpoints, we shade in this region.



PROBLEM:

Interpret the range of imprecision graphed below.



SOLUTION:

To solve this problem we reverse the procedure of the previous problem. We first find the endpoints of the range. They are 3.4 and 2.7. The midpoint is found by adding these two numbers and dividing by two (averaging).

$$\text{midpoint} = \frac{2.7 + 3.4}{2}$$

$$= \frac{6.1}{2}$$

$$= 3.05$$

The imprecision is the distance from the midpoint to either one of the endpoints.

$$\text{imprecision} = 3.4 - 3.05$$

$$= .35$$

$$\text{or, imprecision} = 3.05 - 2.70$$

$$= .35$$

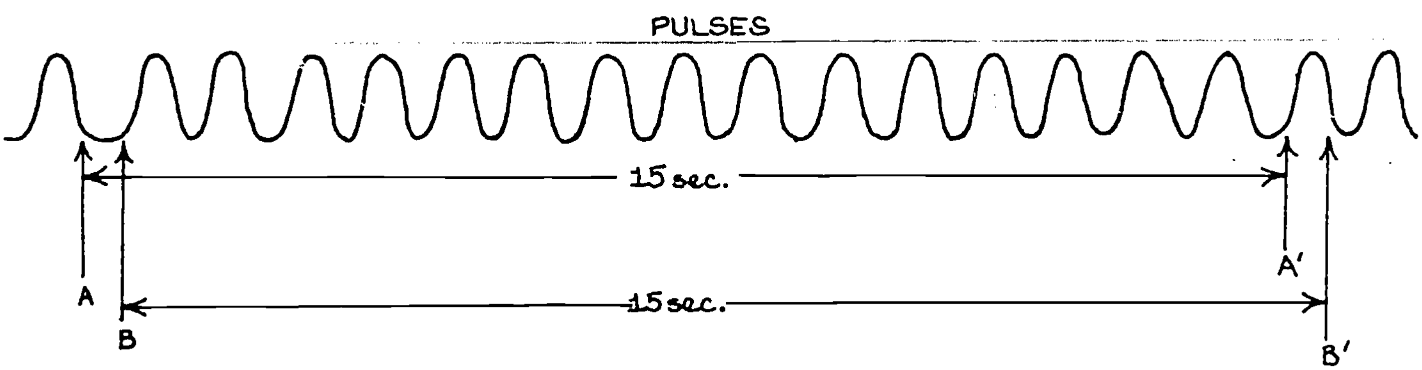
Finally, we write:

$$3.05 \pm .35$$

for the range of imprecision.

8-5 Measurement of Pulse Rate

One type of measurement that involves a range of uncertainty is the measurement of pulse rate.



The drawing shows heartbeats as they might be represented by a recording device. The distances from A to A' and from B to B' each represent a time interval of 15 seconds.

To determine a pulse rate you could count the number of beats in a 15-second interval. However, if you count the beats in the interval between A and A', you will find the number to be 15; while if you count the beats between B and B', you will get 16. Apparently the number of beats in an interval depends upon when the interval begins. This problem may be accounted for by expressing the number of beats as a range of uncertainty. Since the count can never vary by more than one beat, the error is ±1.

If we measured a pulse rate for 15 seconds and counted 17 beats, we would know that the true number could be no greater than 18 nor less than 16. We would report our measurement as 17 ± 1 beats per 15 seconds.

If we were requested to report the pulse rate in beats per minute, we would note that the most the rate could be is

$$4(17 + 1) = 72 \text{ beats per minute}$$

and the least the rate could be is

$$4(17 - 1) = 64 \text{ beats per minute.}$$

We would therefore report the pulse rate as 68 ± 4 beats per minute.

PROBLEM SET 8:

For Questions 1 through 11 write

Y-N for a qualitative type measurement

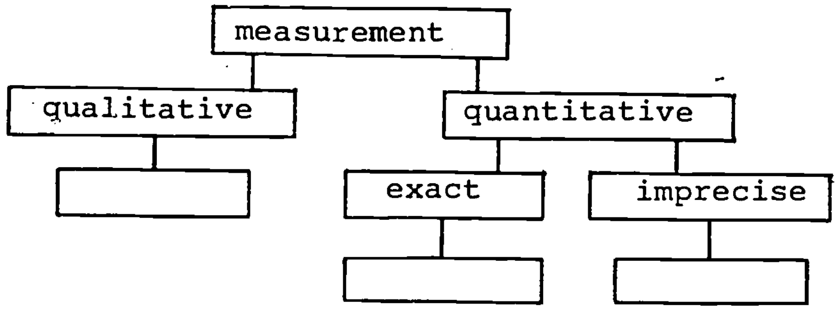
C for a counting type measurement

CS for a continuous scale type measurement

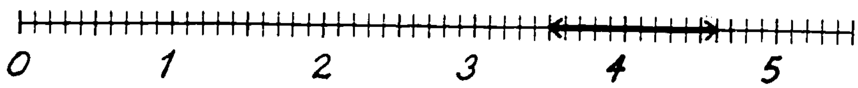
depending on what it requires to answer the question.

1. How many ducks does Elmo have in his pocket?
2. How far is it to the moon?
3. Is Elmo drunk?
4. Is Norbert Numnitz awake?
5. How high is this tree?
6. How long is a class period? ("Too long" is an unacceptable answer.)
7. How many periods are there in a school day? (Ditto for "too many.")
8. Is a class period longer than 20 minutes?
9. Does that gas pump have its nozzle in the wrong slot?
10. How many gas pumps are there in the U.S.A.?
11. How much gas does a gas pump pump in a day?

12. Copy the diagram and fill in Y-N, C, and CS in the empty boxes.



13.



a. What is the low end point (L) of the range of imprecision (L = ?).

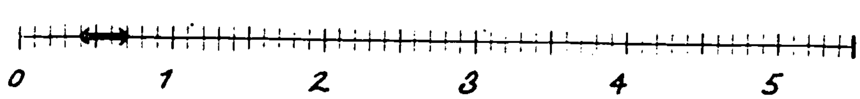
b. What is the high endpoint (H) of the range of precision (H = ?).

c. Average your answers to parts a and b. This is the midpoint (m). ($m = \frac{L + H}{2}$.)

d. What is the distance from the midpoint to an endpoint? This is the imprecision of the range of imprecision.

e. Write the range of imprecision which describes the shaded region.

14.



a. What is the low endpoint?

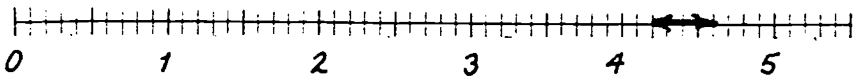
b. What is the high endpoint?

c. What is the midpoint?

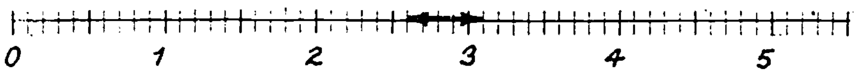
d. Write the appropriate range of imprecision.

For Problems 15 through 18 write the range of imprecision which describes the shaded region.

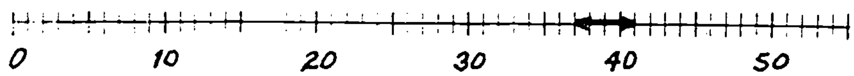
15.



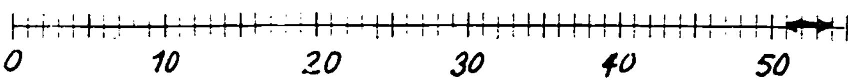
16.



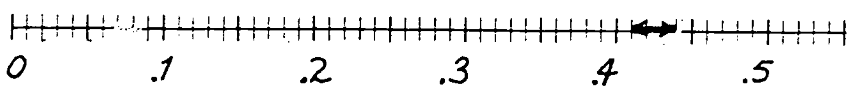
17. Noitce the change of scale



18.



19.



For Problems 20 through 24 sketch in ranges of imprecision on your own sheet of graph paper.

20. /3 ± 2

21. .4 ± .01

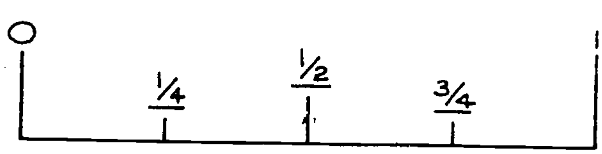
22. 4.35 ± .15

23. 38 ± 2.5

24. .46 ± .01

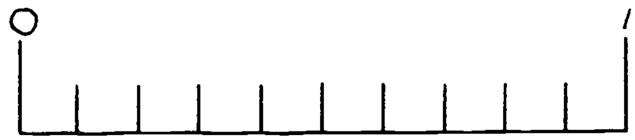
For Problems 25 through 33 determine the distance between the closest divisions.

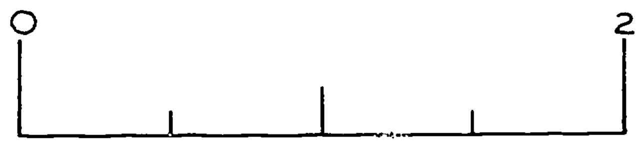
EXAMPLE:

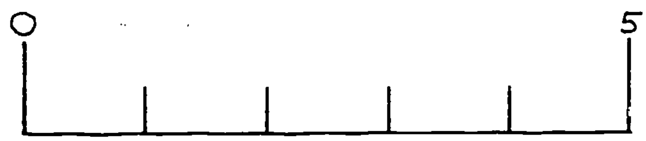


$d = \frac{1}{4}$

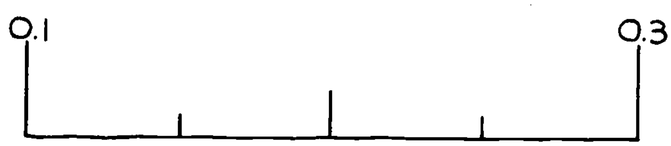
25. 

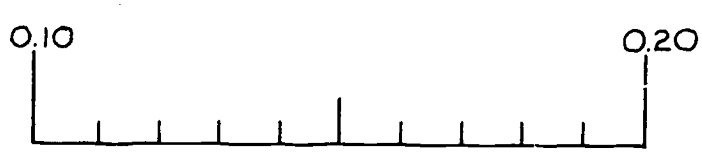
26. 

27. 

28. 

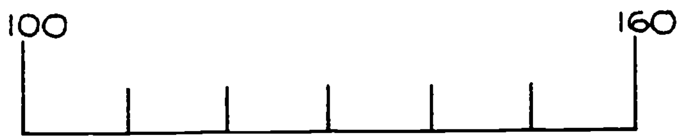
29. 

30. 

31. 

32.

33.



See Section 8-3 for more details relating to the following problem.

34.

a. What is the distance between the closest division?

b. The arrow is located between two divisions. What are the meanings of these two divisions?

c. What fraction of a division does the arrow cut off? (Use your own judgment or measure, as you wish.)

d. What distance does the fraction of part c represent?

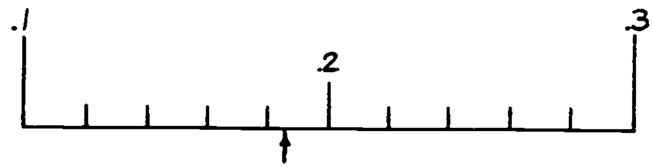
e. What is your "best guess" for the value of the midpoint.

f. Use the "one tenth of the smallest division" rule of thumb to find the imprecision.

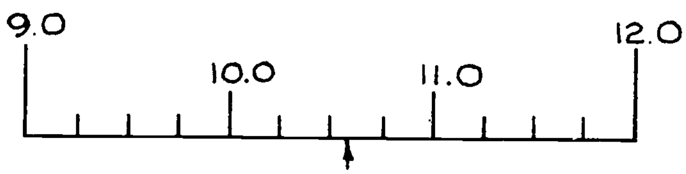
g. Write a range of imprecision which describes the position of the arrow.

For Problems 35 through 40 write a range of imprecision which describes the location of the arrow. Use the "one tenth of the closest divisons" rule for the imprecision.

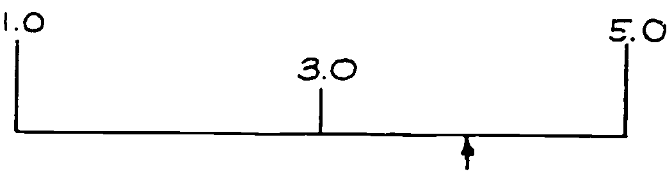
35.



36.

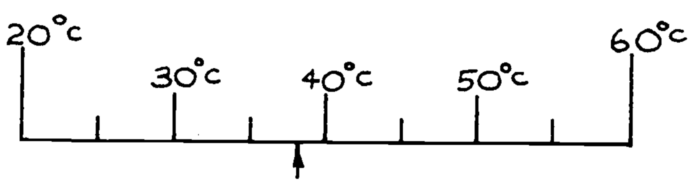


37.



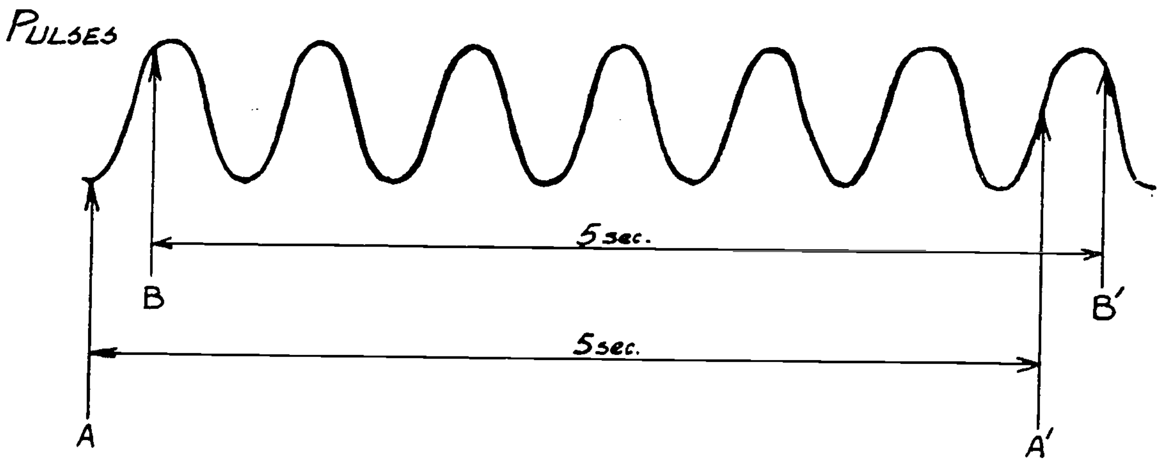
38.

39.



40.

41.



a. Measure the lines segments $\overline{AA'}$ and $\overline{BB'}$. State the length of these line segments in cm.

b. The two lines represent (equal, unequal) lengths of time.

c. Count the peaks between A and A'.

d. Count the peaks between B and B'.

e. What is the difference between the answers to parts c and d?

f. After you take a pulse reading, do you have any way of knowing whether your answer is one pulse high or one pulse low.

g. What is the imprecision of a pulse count? In other words, after you count pulses you can be sure that your count is within ______ of the true count?

SECTION 9:

9-1 The Fair Packaging and Labeling Act

During the next two days, you will study the regulations of The Federal Trade Commission (FTC) under the Fair Packaging and Labeling Act of 1966. Among other things, these regulations say that every package or container must be clearly labeled as to how much of the product it contains--how many ounces, gallons, square feet, or whatever.

In particular, the regulations say that the letters and numbers in this statement of contents must be at least a certain minimum size. This minimum size depends on the size of the container--not the size of the label.

This is the part of the FTC regulations which you will concentrate on in class. You will bring your own package to class, and check it to see whether it satisfies the regulations.

You will be measuring lengths and calculating certain areas of these packages. These areas may be used to see whether the packages are legally labeled.

9-2 Deciphering the Regulations

As you read through the regulations (they may be found at the end of this section), you will see that they are difficult reading and not very easy to understand. Although they may look like Greek, they are actually written in a strange lingo called "bureaucratese"--the native tongue of bureaucrats the world over.

These regulations would be easier to understand if we translated them for you into plain English. However, in the real world outside the classroom it is often the bureaucrats who run the show, and this may be true in some biomedical fields, as well. Therefore, it will be good training for you to struggle with these regulations in their original form and try to figure out what they mean.

In order to provide some guidelines to clarify some of the fuzzier points to keep you from getting too frustrated, here are some things you should watch for.

9-3 Cylindrical Containers

The word "circumference" is misused in the definition of the "principal display panel," although it is used correctly in Examples 1 and 4. The word "circumference," of course, means the distance around the edge of a circle or disc.

So the circumference of a cylindrical container (such as a soup can or a wine bottle) is a circular distance around the side of the container. What we are really interested in is not the circumference of the container but the cylindrical surface around the side of the container.

For a can, this is the entire surface <u>except</u> for the top and bottom lids. For a cylindrical bottle, this is the surface <u>not</u> including the bottom, the shoulder and the neck. Figure 1 illustrates this.

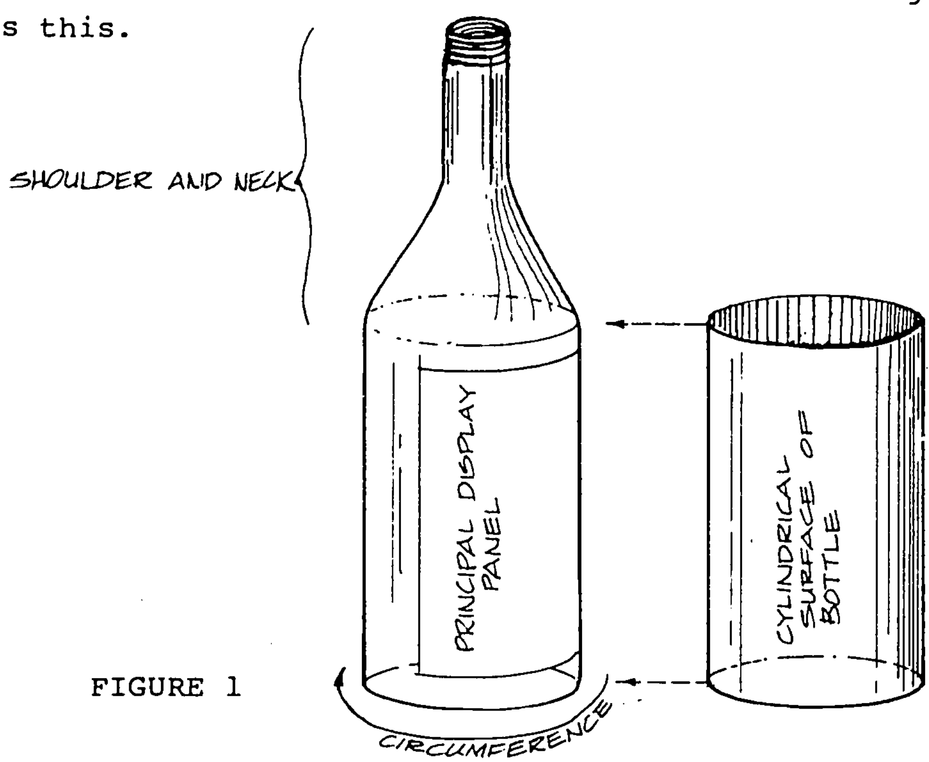


FIGURE 1

Where does the 40% come in? When a bottle or can stands on a display shelf, 50% of the cylindrical surface is invisible--the backside of the container. Another 10% or so along the sides is visible, but you can't read anything on this part of the surface because the line of sight is wrong. This leaves about 40% of the cylindrical surface that can be clearly seen from the front. This 40% is the principal display area of the container (see Figure 1).

9-4 Minimum Type Size

If the statement of contents is all in capital letters, then these letters, and the numerals, must all be at least as tall as the minimum type size. But if the statement contains any small letters, then the minimum type size refers to the body size of the small letters, that is, the height of the small letter "o" or "x", not the height of a letter such as "h" or "p."

Also notice that the minimum type size depends on the size of the container. In particular, it depends on the size of that part of the container which you are most likely to see on a store shelf, that is, the "principal display area." The type size does not depend on the size of the label. This prevents the manufacturer from using tiny type on a tiny label on a huge box.

Finally, all the fuss and bother about type sizes boils down to the following simple statements.

Let A = area of the principal display panel in inches^2.

If $A \leq 5$ then the minimum type size is $\frac{1}{16}$in.

If $5 < A \leq 25$ then the minimum type size is $\frac{1}{8}$in.

If $25 < A \leq 100$ then the minimum type size is $\frac{3}{16}$in.

If $100 < A \leq 400$ then the minimum type size is $\frac{1}{4}$in.

If $400 < A$ then the minimum type size is $\frac{1}{2}$in.

These are the minimum sizes when the statement of net contents is printed on a label, or directly on the container. However, if this statement is blown or embossed directly on the surface of the container (such as the raised letters molded on the surface of a glass bottle), then the minimum sizes are $\frac{1}{16}$ inch larger than the ones shown above.

RULES AND REGULATIONS FOR PACKAGING*

Definition of "principal display panel":

(h) The term "principal display panel" means that part of a label that is most likely to be displayed, presented, shown, or examined under normal and customary conditions of display for retail sale. The principal display panel must be large enough to accommodate all the mandatory label information required to be placed thereon by this part without obscuring designs, vignettes, or crowding. This definition does not preclude utilization of alternate principal display panels on the label of a package, but alternate principal display panels must duplicate the information required to be placed on the principal display panel by this part. This definition does not preclude utilization of the container closure as the surface bearing the principal display panel if that label location is the one most likely to be displayed, presented, shown, or examined under normal and customary conditions of display for retail sale. The principal display panel of a label appearing on a cylindrical surface is that 40 percent of the circumference which is most likely to be displayed, presented, shown, or examined under normal and customary conditions of display for retail sale.

☆ ☆ ☆

Regulations regarding type size in the statement of quantity of contents:

§ 500.18 Type size in relationship to the area of the principal display panel.

(a) The statement of net quantity of contents shall be in letters and numerals in a type size established in relationship to the area of the principal display panel of the package or commodity and shall be uniform for all packages or commodities of substantially the same size. For this purpose, "area of the principal display panel" means the area of the side or surface that bears the principal display panel, exclusive of tops, bottoms, flanges at tops and bottoms of cans, and shoulders and necks of bottles and jars. This area shall be:

(1) In the case of a rectangular package or commodity where one entire side properly can be considered to be the principal display panel side, the product of the height times the width of that side;

(2) In the case of a cylindrical or nearly cylindrical container or commodity, 40 percent of the product of the height of the container or commodity times the circumference; and

(3) In the case of any otherwise shaped container or commodity, 40 percent of the total surface of the container or commodity: *Provided, however,* That where such container or commodity presents an obvious "principal display panel" such as the top of a triangular or oval shaped container, the area shall consist of the entire top surface.

(b) With area of principal display panel defined as above, the type size in relationship to area of that panel shall comply with the following specifications:

(1) Not less than 1/16 inch in height on packages the principal display panel of which has an area of 5 square inches or less.

(2) Not less than 1/8 inch in height on packages the principal display panel of which has an area of more than 5 but not more than 25 square inches.

(3) Not less than 3/16 inch in height on packages the principal display panel of which has an area of more than 25 but not more than 100 square inches.

(4) Not less than ¼ inch in height on packages the principal display panel of which has an area of more than 100 square inches, except not less than ½ inch in height if the area is more than 400 square inches.

(c) Where the statement of net quantity of contents is blown, embossed, or molded on a glass or plastic surface rather than by printing, typing, or coloring, the lettering sizes specified in paragraph (b) of this section shall be increased by 1/16 of an inch.

(d) Letter heights pertain to upper case or capital letters. When upper and lower case or all lower case letters are used, it is the lower case letter "o" or its equivalent that shall meet the minimum standards.

(e) The ratio of height to width of a letter shall not exceed a differential of 3 units to 1 unit (no more than 3 times as high as it is wide).

(f) When fractions are used, each component shall meet one-half the minimum height standards.

*From "Rules, Regulations, Statement of General Policy or Interpretation and Exemptions Under the Fair Packaging and Labeling Act." Federal Trade Commission, Oct. 1, 1971.

BULLETIN No. 6

May 4, 1970

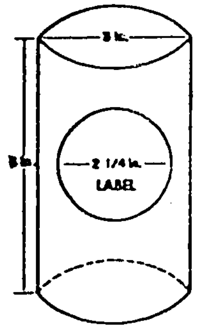
"Type Size to Express Statement of Quantity of Con Section 500.18."

It has become apparent that many packagers are misinterpreting the requirements of Section 500.18 of the FPLA regulations. Section 500.18 requires the statement of net quantity of contents to be in letters and numerals of a type size based on the area of the *principal display panel of the package or commodity*. The area of the principal display panel of the *package* is not the same as the area of the principal display panel of the *label*. Section 500.18 also expresses how the area of the principal display panel of the package or commodity is measured.

A review of many labels revised by packagers or labelers to comply with the requirements of the regulations indicated that some packagers or labelers are using the area of the principal display panel of the label as a determinant of the minimum type size to be used to express quantity of contents. The purpose of this Bulletin is to emphasize the correct method to use to determine proper type size. The following illustrations should accomplish this purpose. Included are examples of the minimum type size requirements as they affect declarations of quantity expressed in upper and lower case. Section 500.18(d) requires that when lower case letters are used, or when upper and lower case letters are used, the minimum type size relates to the lower case letters.

Example 1

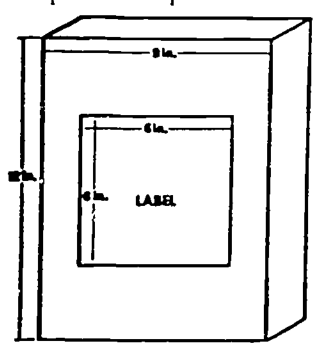
A spot label is to be applied to a cylindrical can. The spot label is circular and has a diameter of 2 1/4 inches, and an area of less than 5 square inches. The cylindrical can is 5 inches high and has a diameter of 3 inches. What is the minimum size type to be used on the label to properly express quantity of contents?



ANSWER: While the area of the principal display panel of the label is less than 5 square inches the area of the principal display panel of the can is 40 percent of the product of the can height X can circumference. Thus, 40 percent of (5 in. X 9.42 in.) is 18.84 square inches. (Circumference is 2 pi-r or 2 X 3.14 X 1.5.) Therefore, the required minimum type size to express quantity of contents on the label is 1/8 inch, in spite of the label itself having an area of less than 5 square inches. When the expression takes the form of "Net Weight 12 ozs.", the lower case letters must meet the 1/8 inch requirement, making the upper case "N" and "W" somewhat higher than 1/8 inch.

Example 2

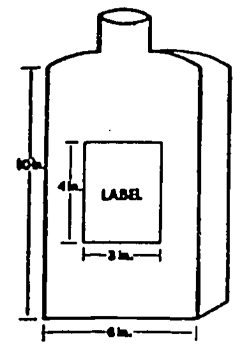
A rectangular package is 12 inches high and 9 inches wide. Printed directly on the package is labeling which begins 3 inches from the package top and extends to 3 inches from the bottom, and extends 6 inches across the width of the package. What is the required type size to express quantity of contents? The area of the printed space is 36 square inches.



ANSWER: Since the package is rectangular, the area of the principal display panel of the package is 9 in. X 12 in. or 108 square inches. Therefore, the minimum type size to express quantity of contents is 1/4 inch. The fact that the printed portion of the package (label) is 36 square inches has no bearing on the minimum type size. The statement "NET WEIGHT 48 OZS. (3 LBS.)" in this instance, must have all letters and numerals 1/4 inch high. If printed in the form of "Net Weight 48 Ozs. (3 lbs.)", the lower case letters must be 1/4 inch, thus requiring the upper case "N", "W", and "O" to be somewhat higher.

Example 3

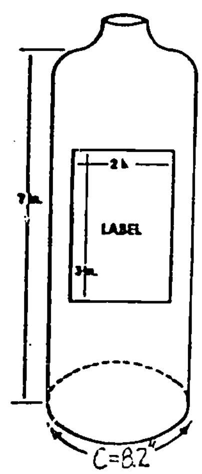
A liquid cleaner is bottled in a "flat" plastic bottle. From the bottom of the flask to the start of the shoulder of the flask, there is a 10 inch height. The flask is 6 inches in width. The labeler at times applies a 4 in. X 3 in. rectangular paper label to the face of the flask, but at other times intends to pack the cleaner fluid in the same flask but with all labeling molded on the plastic surface. In what minimum size type should the quantity of contents appear (1) on the paper label? (2) in the molded label information?



ANSWER: Excluding the shoulder of this "flat" plastic bottle, the area of the principal display panel of the bottle is 10 in. X 6 in., or 60 square inches. A minimum type size of 3/16 inch is required to express quantity of contents on the paper label which itself occupies only 12 square inches. However, since the regulations require the minimum type size of molded labeling to be increased by 1/16 inch, the minimum type size on the molded flask must be 1/4 inch.

Example 4

A glass bottle, cylindrical in shape, has a wrap aroung label. The label has a principal display panel of 3 in. X 2 in. or 6 square inches. The bottle has a height of 7 inches exclusive of shoulders and neck, and a circumference of 8.2 inches. What is the minimum size print of the quantity of contents statement?



ANSWER: Since the area of the principal display panel of the bottle is 40 percent of the product of height (7 inches) times circumference (8.2 inches), the area is 23 square inches. The minimum type size of the quantity of contents statement is 1/8 inch. Notice that the area of the label also happens to be less than 25 square inches and more than 5 square inches, but this is only a coincidence and does not govern the minimum type size to be used for this container.

PROBLEM SET 9:

Read the definition of "principal display panel" carefully and then answer Questions 1 through 5 true or false.

1. The "principal display panel" is a billboard.

2. The "principal display panel" is part of a label on a product likely to be seen by customers.

3. The word "preclude" means "permit."

4. The word "circumference" is used incorrectly in the last sentence.

5. The words "lateral surface area" should have been used instead of "circumference."

Read 500.18 of the FTC regulations carefully and then answer Questions 6 through 14 true or false.

6. The "area of the principal display panel" is the area of the label.

7. For a rectangular package, the area of the "panel" is the same as the area of the side with the largest area.

8. For a cylindrical container, the area of the "panel" is 40 per cent of the product of the height and the circumference of the cylinder.

9. For otherwise shaped containers, the "area of the panel" is 50% of the total surface area of the container.

10. If the area of the panel is less than 5 square inches, the minimum type size is $\frac{1}{16}$ inch.

11. If the area of the panel is greater than 5 in^2 but less than or equal to 25 in^2, the minimum type size is $\frac{1}{16}$ inch. ("A" is the "area of the panel.")

12. If 100 $in^2 < A$ and also $A \leq 400$ in^2, then the minimum type size is 1 inch.

13. If 25 $in^2 < A$ and also $A \leq 100$ in^2, then the minimum type size is $\frac{3}{16}$ inch.

14. Read Example 1 carefully and then answer the following.

a. What is the circumference of the top surface of the cylinder?

b. What is the lateral surface area of the cylinder? [A = (circumference) x (height)].

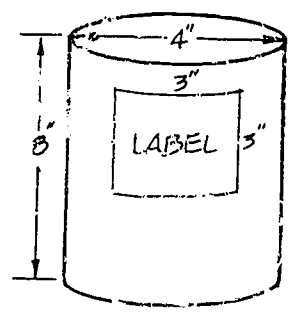
c. "The area of the display panel" of the can is 40% of the area found in Part b. Determine the area of the panel.

d. The answer to the example states that the minimum type size is $\frac{1}{8}$in. Where does $\frac{1}{8}$in come from?

e. Is the area of the label used at all is problem? Should it be?

Use the accompanying chart which summarizes type sizes to help answer the remaining problems

15. A cylindrical package of "puffed oats" has the dimensions shown.



a. Determine the circumference of the cylinder's top surface.

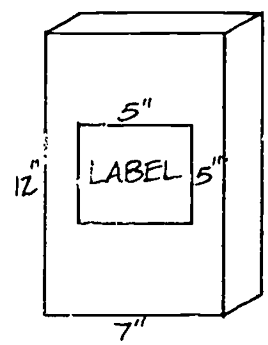
b. For a cylindrical package the "area of the principal display panel" is defined as 40% of the package height times the circumference. Determine that area.

c. Look up the area found in Part b. on the following chart and determine the minimum type size necessary, according to FTC regulations, to express the quantity of contents.

A, Area of Principal Display in Square Inches	Minimum Type Size
$A \leq 5$	$\frac{1}{16}$"
$5 < A \leq 25$	$\frac{1}{8}$"
$25 < A \leq 100$	$\frac{3}{16}$"
$100 < A \leq 400$	$\frac{1}{4}$"
$400 < A$	$\frac{1}{2}$"

Read Example 2 <u>before</u> doing the next problem.

16. A package of corn flakes is 7" wide and 12" high. The label is 5" by 5" and is placed in the center of one side. (See illustration.)

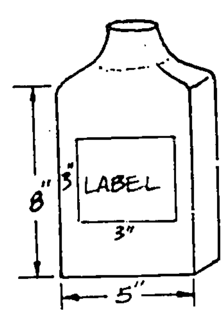


a. What is the area of the "principal display panel" of the package? (Ignore the dimensions of the label.)

b. Look up the area you found in Part a on the accompanying chart; and determine the minimum type size, according to FTC regulations, to express the quantity of contents.

Read Example 3 <u>before</u> doing the next problem.

17. A bottle of ammonia has the dimensions shown.



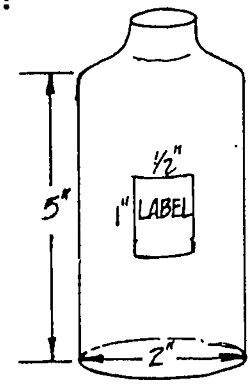
a. Since the flat display area is approximately rectangular, the area is ________ in^2.

b. In what minimum type size should the quantity of contents appear on the label?

c. According to FTC regulations, if the quantity is to appear on the bottle itself (molded), the type must be $\frac{1}{16}$ in higher than that in Part b. How high then must the type on the bottle be?

Read Example 4 <u>before</u> doing the next problem.

18. Given the glass bottle at the right, determine:



a. the area of the principle display panel

b. the minimum type size of the quantity of contents statement.

The following problem refers to <u>your</u> package or container. If you don't have one, ask your teacher for one.

19. a. Determine whether your container is cylindrical, rectangular or otherwise shaped. Try to match it with one of the illustrated containers in the examples.

b. Determine the area of the principal display panel of your container. Remember that the method of computing this area will depend on the shape of your container. Remember also that the area of the label itself is to be ignored.

c. Look up the area of the accompanying chart and determine the minimum type size required to express the quantity of contents.

d. Find the quantity of contents statement printed on your container and measure the size of the type.

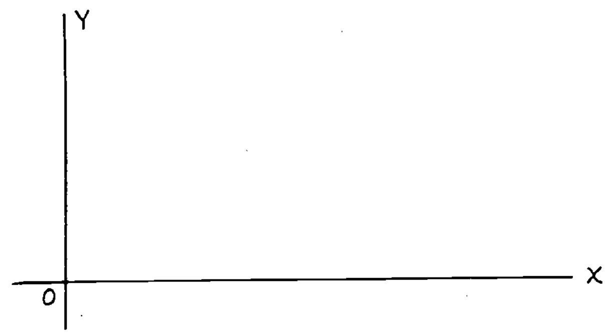
e. Is the type size as large as it is supposed to be?

SECTION 10:

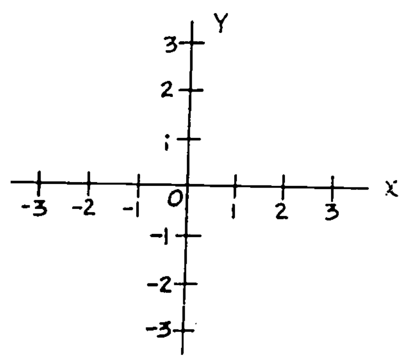
10-1 Graphs

It is often important to show the relationship between two measurable quantities. We will use the "Cartesian" method of constructing graphs. A description of this method follows.

First, intersecting lines are drawn perpendicular to one another. One line is horizontal and by convention is called the x-axis. The other line is vertical and is called the y-axis.

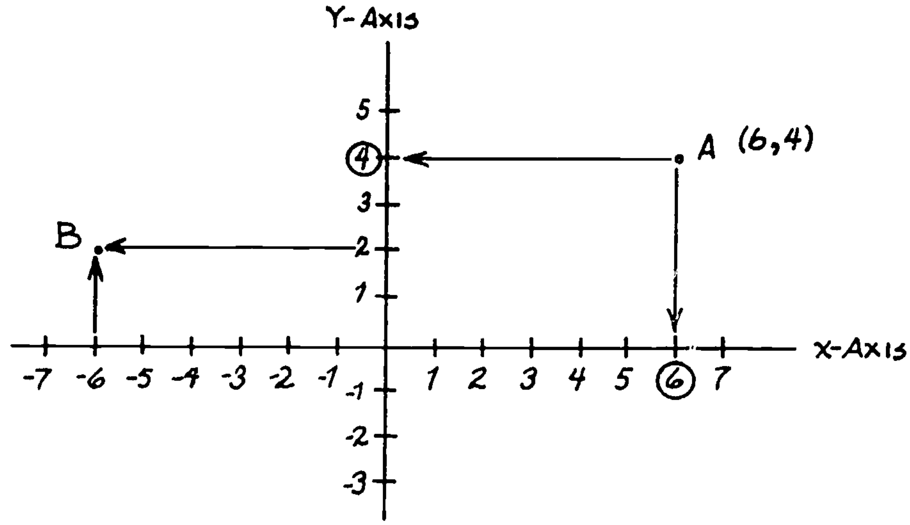


Second, each axis is divided into units of length and numbered such that on the x-axis positive numbers are to the right of the origin and negative numbers to the left and on the y-axis positive numbers are above the origin and negative numbers below.



In this case the scales used for the x and y-axis are identical. In many cases they won't be the same; however, we won't burden you with this problem now.

These two scaled, perpendicular axes may be used to give an address to any point located in the plane. Look at the graph on the following page. We want to give an address to point A. First we find the x-coordinate by starting at A and going in a direction perpendicular to the x-axis. We find that A corresponds to "6" on the x-axis.



Similarly, we find that A corresponds to "4" on the y-axis.

Next, we put these two bits of information together in a compact way. All mathematicians and scientists have agreed to do it in the same way so as the lessen the general level of confusion in the world. They have agreed to write the x-coordinate first and then the y-coordinate. A comma goes between the two numbers and parentheses go around the whole thing. Below we have listed three different pairs of terms used to describe ordered pairs.

Point A =	(6,	4)
	"horizontal coordinate"	"vertical coordinate"
or	"x-coordinate"	"y-coordinate"
or	"abscissa"	"ordinate"

The expression <u>(6, 4) is called an ordered pair</u>. Remember that the order of the numbers in an ordered pair is very important. (6, 4) and (4, 6) are not the same point. For practice you should try to find the ordered pair for point B.

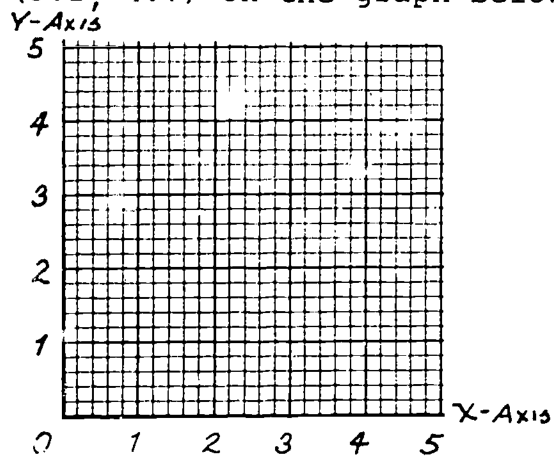
The x-coordinate (horizontal coordinate or abscissa) is -6. The y-coordinate (vertical coordinate or ordinate) is +2. When we write the ordered pair, the x-coordinate comes first; therefore, point B = (-6, 2). This is the only way to write the address as an ordered pair and be understood correctly by the rest of the world. (2, -6) would work only if you said that you were going to put the y-coordinate first in all your ordered pairs. But why march out of step with the rest of the world? Always put your x-value first just like everybody else. This is one situation where conformity is the way to go.

In this section we started with a couple of points and found their addresses. More often we will start with the address and want to locate the point. The process for locating points from ordered pairs is just the reverse of what we discussed above. We take it up next.

10-2 Going From Ordered Pair to Point

PROBLEM:

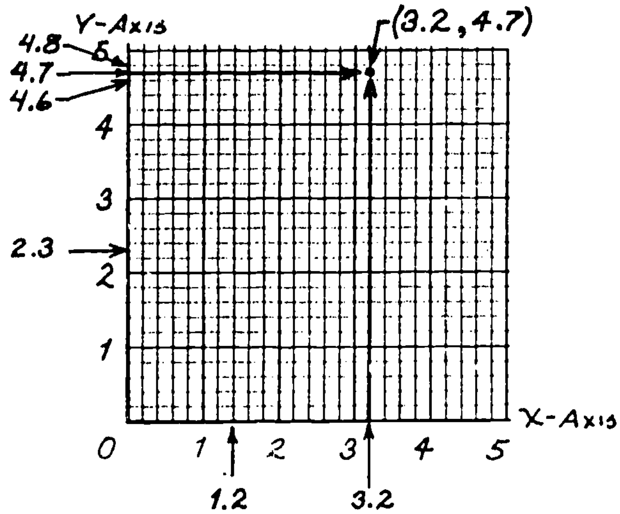
Plot the point (3.2, 4.7) on the graph below



SOLUTION:

Our first task here is to see that neither 3.2 nor 4.7 will fall on a dark black line because neither is a whole number. 3.2 will lie somewhere between 3 and 4 on the x-axis and 4.7 will lie somewhere between 4 and 5 on the y-axis.

Second we find out what each small division means. We need to know this to plot points between the dark lines. Notice that there are 5 little lines between the big ones. Therefore each little division represents one fifth of the distance between big divisions. Therefore, each little division represents .2 because $\frac{1}{5} = .2$. This means that 3.2 is located one little division to the right of 3 on the x-axis. Locating 4.7 is a little trickier; it won't fall exactly on a line. 4.7 will fall halfway between 4.6 and 4.8 on the y-axis.



Finally, we plot the point. It is found by going in a perpendicular direction from each axis to the point of intersection.

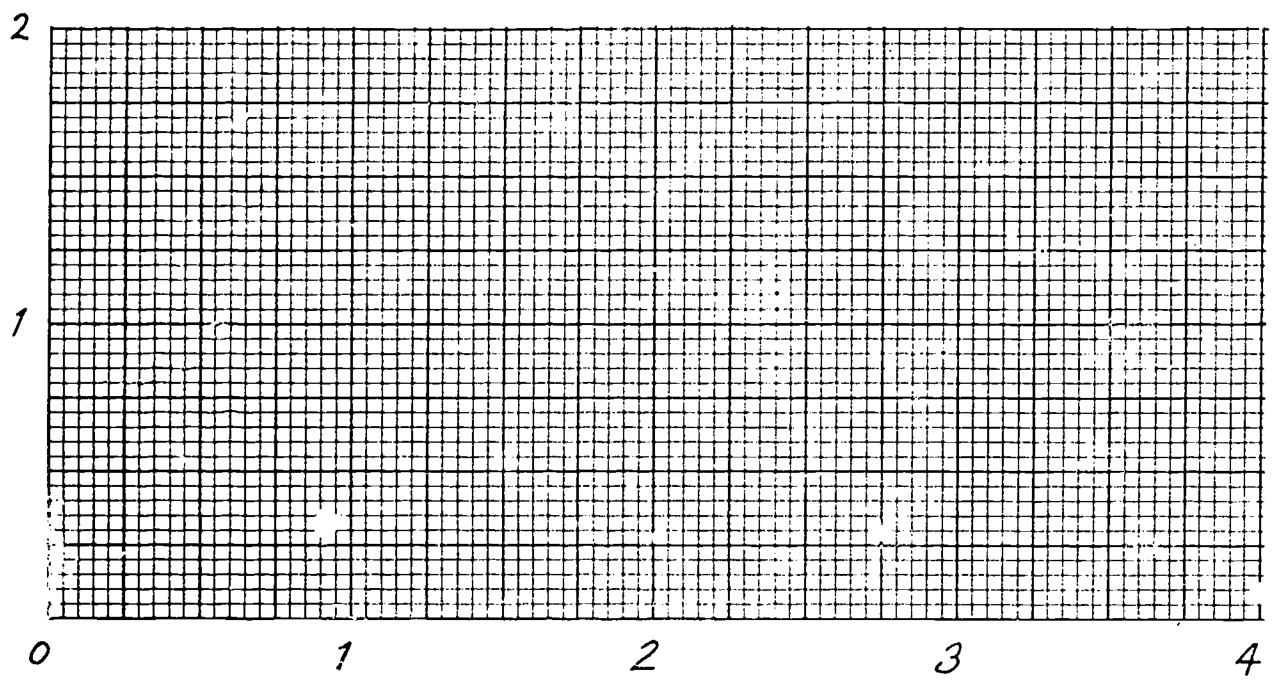
For practice you can try to locate the point (1.4, 2.3) on the graph.

10-3 <u>Regions of Imprecision</u>

Whenever both coordinates are stated in terms of ranges of imprecision then the plot of the two ranges will be a rectangular "region of imprecision."

<u>PROBLEM</u>:

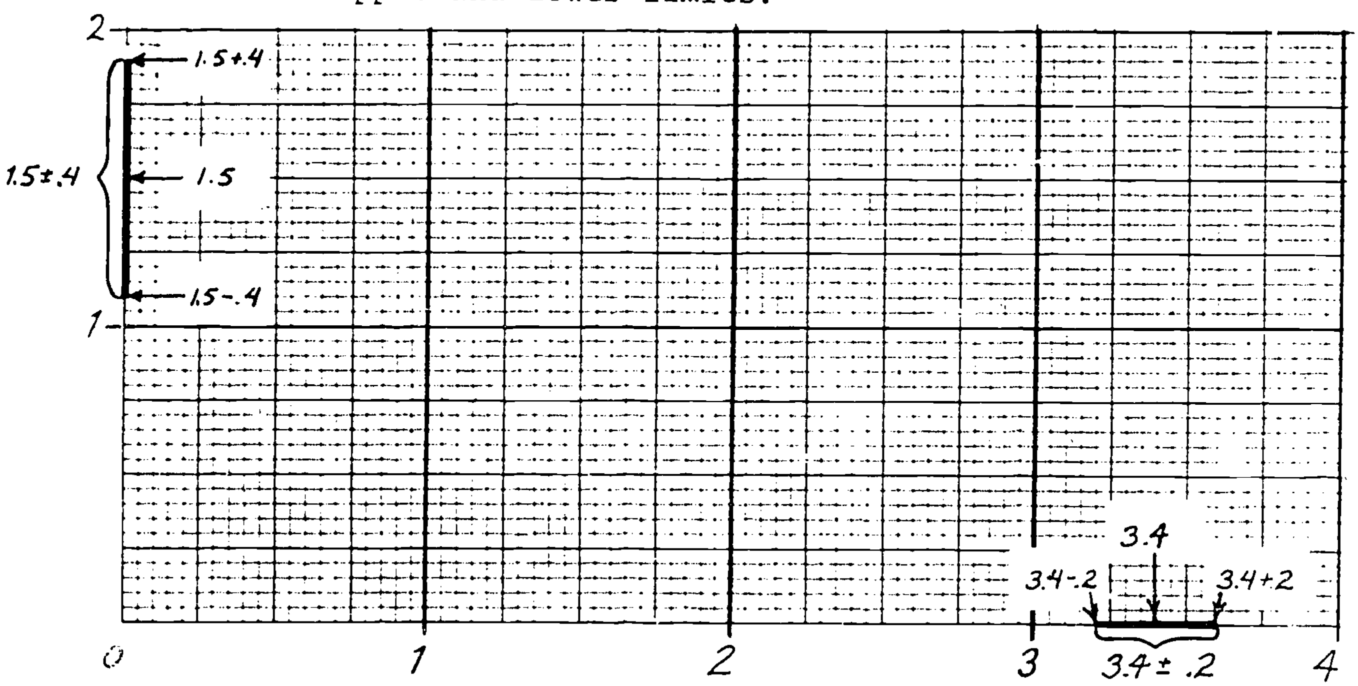
Plot the region of imprecision represented by the ordered pair (3.4 ± .2, 1.5 ± .4) on the graph below.



SOLUTION:

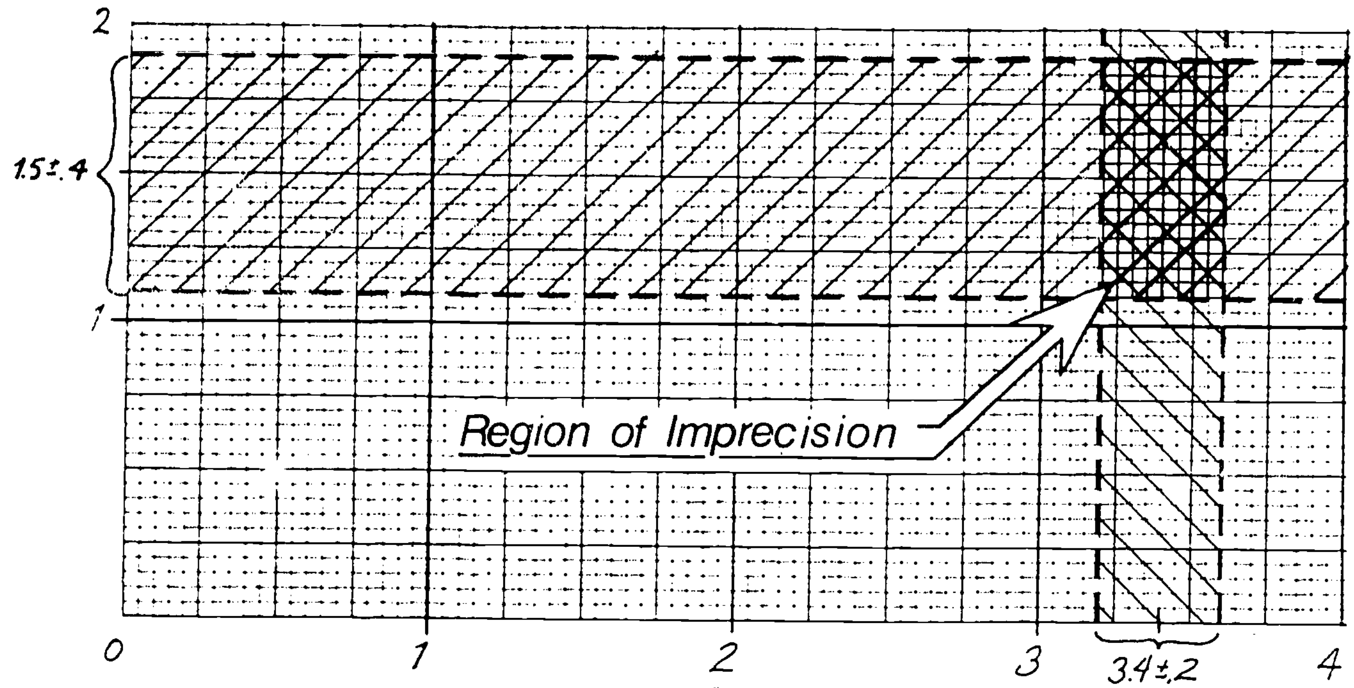
Our first task, as always, is to figure out what the divisions on the scales mean. Notice that there are four big divisions per unit. This implies that each big division is one fourth or .25 of a unit. Next we deduce the meaning of the little divisions. There are five little divisions in each big division. Therefore, a little divison is one fifth of a big division. This means that the little divisons are .05 units each ($\frac{1}{5}$ of .25 = .05).

Second, we locate each range of precision on the appropriate axis. This is done in the same way that we graphed ranges of imprecision on number lines earlier. First we locate the midpoint and then the upper and lower limits.



Finally, we find the region determined by the two ranges of imprecision. This is done by moving out perpendicularly from each of the 6 points on the two axes. The region of imprecision is the intersection of the two ranges of imprecision.

In the graph on the following page, there are lines drawn in for the purpose of illustration. Later on we will leave out these lines and only show the region of imprecision.



PROBLEM SET 10:

1. (2, 4) is an ordered pair.

   a. List three names for the 2.

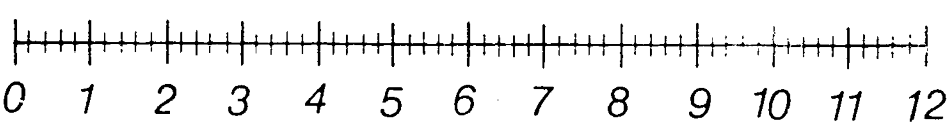
   b. List three names for the 4.

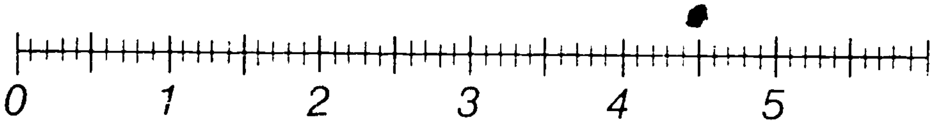
   c. Fill in the blanks. The _____ comes first and then the _______. A _______ goes in the middle and ______ go around the whole thing.

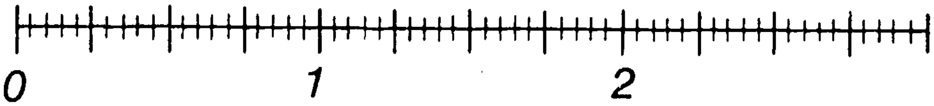
2. In the text it tells you how you can march out of step with the rest of the world. How?

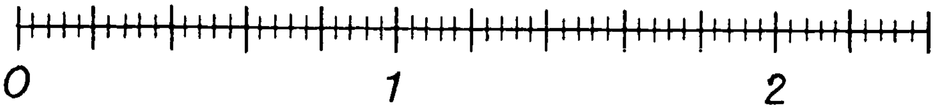
3. What is the mathematical name for the graphing technique that we are using?

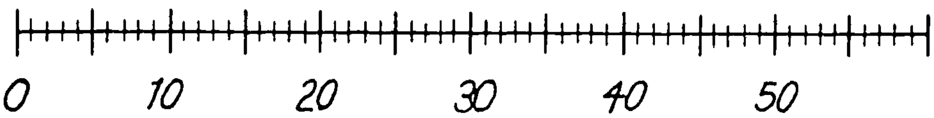
In Problems 4 through 10 determine the value of the smallest division.

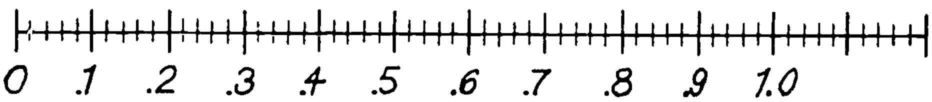
4. 

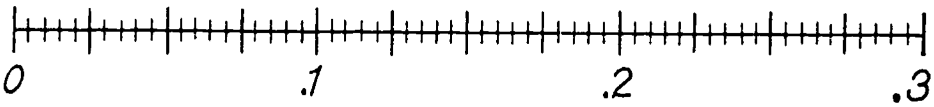
5. 

6. 

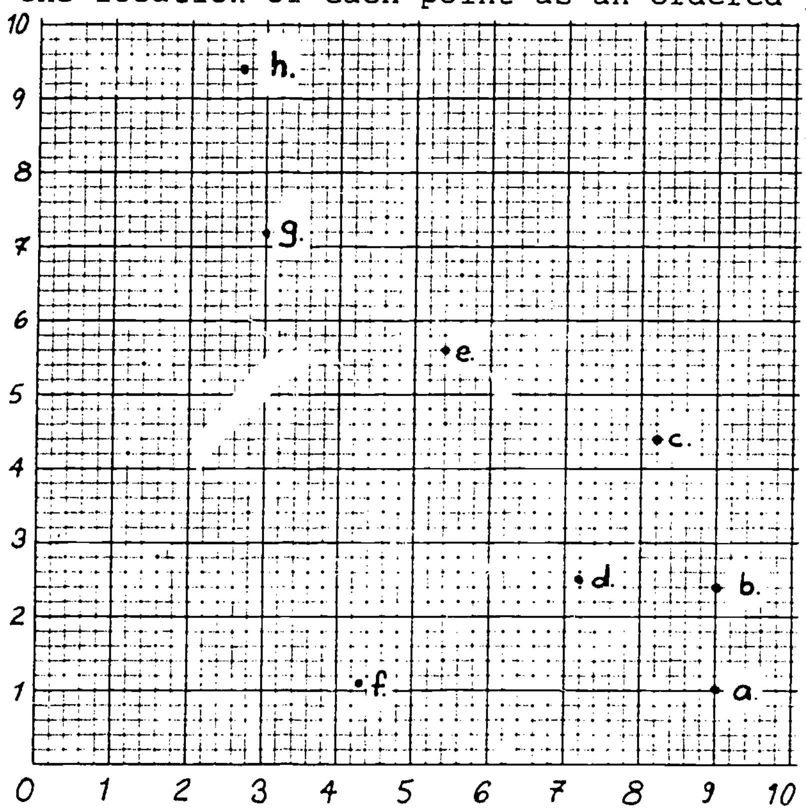
7. 

8. 

9. 

10. 

11. State the location of each point as an ordered pair.

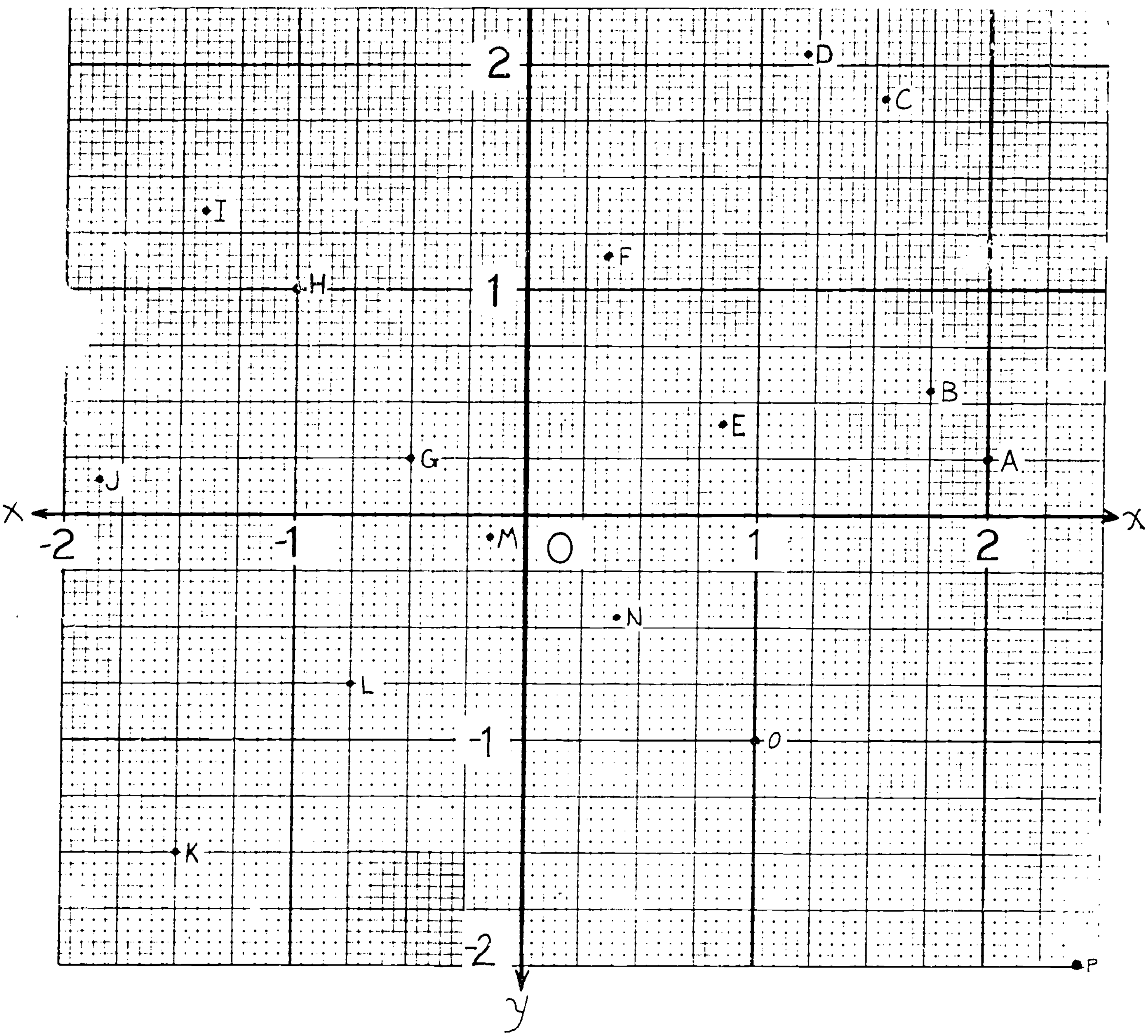


12. Graph each ordered pair listed below. Use the same scaling as the graph in Problem 11.

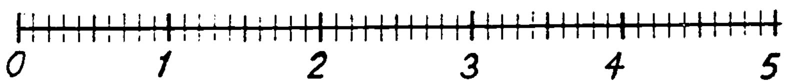
a. (3.2, 4.6) c. (.2, 4) e. (7.2, 1.4) g. (2.8, 6.6)

b. (9.1, 1) d. (4.7, 3.5) f. (6.3, 8.9) h. (5.5, 9.9)

13. State the location of each point as an ordered pair.



For Problems 14 through 17 graph the indicated regions of imprecision. Use this scale on each graph. 1 unit = 2 cm.



14. $(3.0 \pm .5, 5.0 \pm .5)$

15. $(2.7 \pm .5, 4.5 \pm .5)$

16. $(3.0 \pm .1, 5.0 \pm .5)$

17. $(4.75 \pm .05, 4.5 \pm .1)$

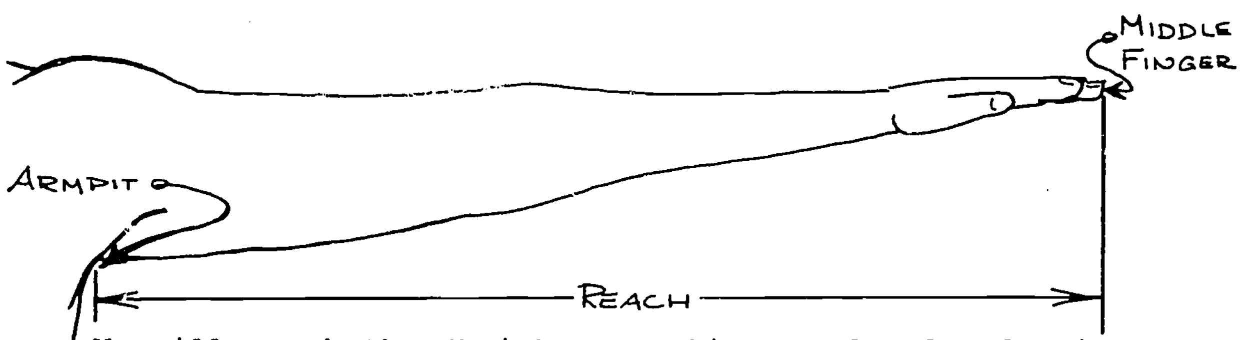
SECTION 11:

11-1 The Requirements for a Good Graph

A graph is a picture, of sorts. It is a visual representation of information. If a graph is well constructed, then it is a great help in seeing patterns and relationships between variables.

As part of the activities connected with this topic, the lengths of different parts of your body will be measured. We will then use graphical analysis to see relationships between the different lengths.

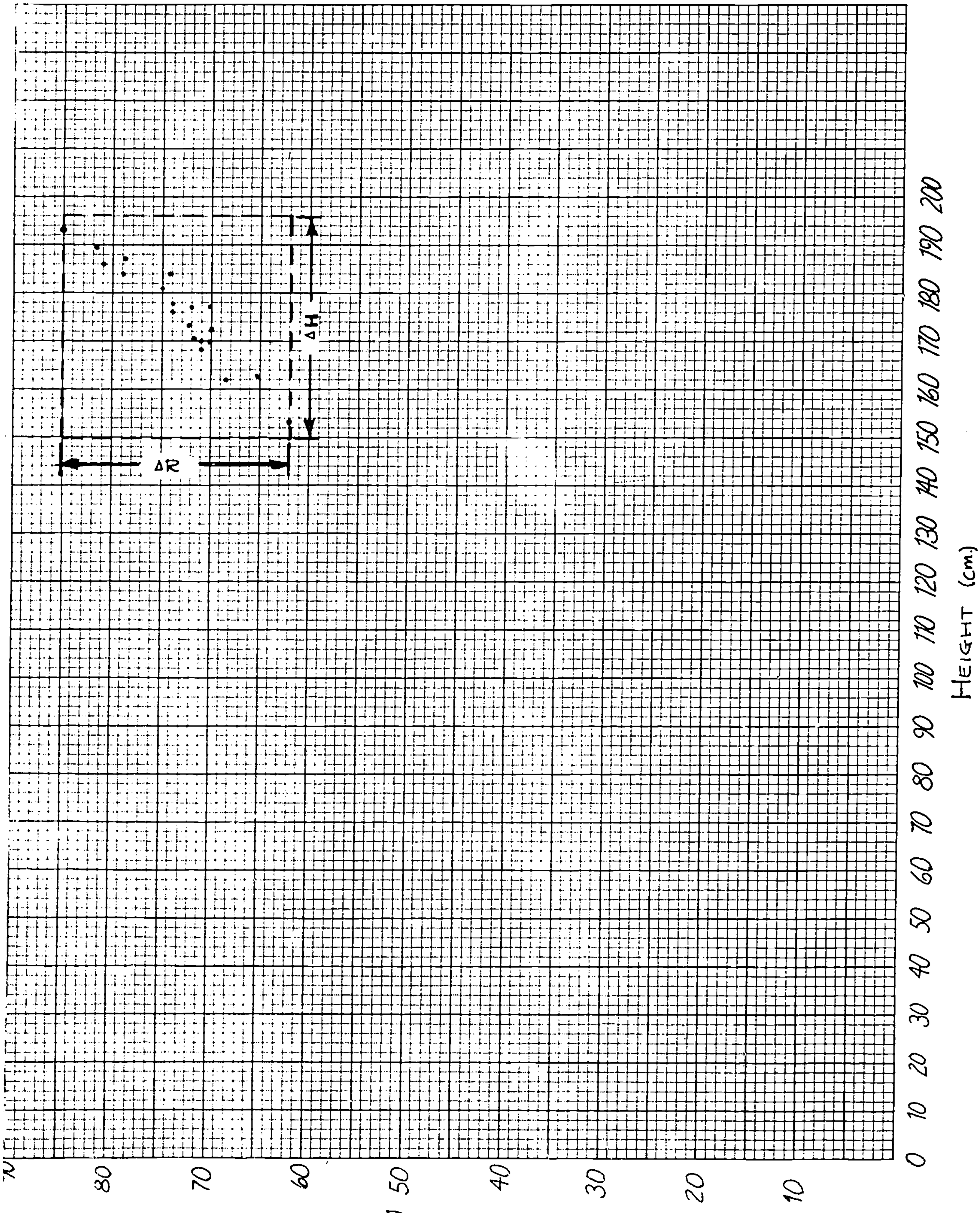
Below we have a sample of the kind of information that will be collected and analyzed. It records the height and "reach" for the entire Biomedical Project staff as of April, 1975. We have defined "reach" to mean the distance from the armpit to the tip of the middle finger.



We will graph the (height, reach) set of ordered pairs. After that we will try to see a pattern in the arrangement of the points. Any observable pattern can be described mathematically. More about this later.

Staff Person	Height(cm) (x)	Reach(cm) (y)
Karen	152.8	62.0
Al	163.4	65.6
Dick	172.3	72.1
Skip	175.8	74.6
Leonard	184.0	74.5
Mike	169.9	71.0
Glenna	170.5	71.4
Renita	162.0	68.7
Ron	176.5	70.6
Chris	168.0	71.0
Jon	192.8	85.0
Ken	176.7	72.0
Gail	169.8	70.4
Glenn	186.5	81.1
Dave	187.0	77.9
Jay	180.6	75.0
Bill	176.8	73.7
Elmer	183.8	79.0
Sandy	172.7	72.3
Joe	188.7	81.9

Our first step in the development of graphing techniques is to graph the points in a way that follows our previous pattern of including the origin. This graph certainly improves our ability to see a pattern in the points, but it has a serious drawback. Notice that all of the points are located within the region bounded by the dotted lines. The rest of the graph might as well not even be there. Furthermore, by cramping all of the points into the box, we decrease the precision of the location of the points. Our objective in this section is to demonstrate a technique for expanding the region within the dotted lines to fill a complete sheet of graph paper. In this particular instance we will increase the length of each side of the box about four times. This will have the effect of reducing the imprecision of locating a point on an axis to about one fourth of what it is in this graph. In summary, we say that a "good" graph


ΔH
ΔR
HEIGHT (cm.)
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200
REACH (cm.)
10 20 30 40 50 60 70 80

should use most of the available space, because this increases the precision of the scales on the axes.

This is certainly one feature of a good graph, but it is possible to imagine a graph that fills most of the available space but still has a defect. Can you think of such a graph? It is so obvious that you have probably never seen a graph with this kind of defect because it would be hard to produce.

Suppose that we had a graph where a big unit meant $2\frac{2}{3}$ cm of height. What would two units be? What would one little unit (one fifth of a big unit) be? Clearly an inconvenient scale like $2\frac{2}{3}$ cm per big unit is something to avoid. So, we say that another requirement of a good graph is that its scales be easy to read and interpret. In fact this last requirement is even more important than the first one we mentioned.

Below we state our two requirements for a good graph.

1. The scales should be easy to read and interpret.
2. The plotted points should fill the available space as much as possible while also maintaining convenient, readable scales (Requirement #1).

In graph construction the meeting of these two objectives we define as heaven. There are many possible mathematical paths to heaven. We will offer only one in the following text. However, any method that meets these two objectives is OK. It is not how you get there that is important, only that you get there, somehow.

11-2 Ground Rules

In the game of graph construction the playing field is a piece of graph paper. Most often we will be graphing on a grid 24 cm long and 18 cm wide, with five divisions to the centimeter. We will describe how to fit sets of data onto this kind of grid. However, remember that there are many other kinds of graph paper and there is certainly no law against taping two pieces of graph paper together if you want to. We will use the 18 cm by 24 cm format in our discussions because it is the most common, but we will expect you to be flexible enough to make good graphs on other formats.

11-3 The Game

We wish to "blow up" the dotted box on the first graph to fill one complete sheet of graph paper. We will first give you a procedural step and follow it with an example from our (height, reach) data.

1. Find the difference between the maximum and minimum numbers in both columns.

EXAMPLE:

An examination of the table reveals tha Jon and Karen are the extremes of both height and reach.

ΔH = (Jon's height) - (Karen's height)
= 192.8 - 152.8
= 40 cm

ΔR = (Jon's reach) - (Karen's reach)
= 85.0 - 62.0
= 23 cm

2. Find the most convenient choice of scale for each combination of variable and axis. (Examples: ΔH - long axis, ΔH - short, ΔR - long, ΔR - short.)

We find the most convenient scale for a particular variable and axis combination by following steps 2a and 2b, below.

a. We define "x" to be our unknown scale for the big units (1 cm divisions). We require that:

$$x \geq \frac{\Delta H \text{ or } \Delta R}{\text{axis length}}$$

Explanation for ΔH on the short axis:

ΔH is 40 cm and there are 18 big units on the short axis. Therefore, we need to find "x" such that 18 times "x" $\geq$ 40. In other words, we want to be sure that 18 big units (x's) will cover the entire range of heights (ΔH).

The smallest "x" which satisfies this requirement is the one that satisfies the equality:

$$(18) \text{ times } (x) = 40$$

$$\text{or, } x = \frac{40}{18}$$

$$= 2.222\ldots$$

This is the smallest possible size for a big unit that will allow us to cover the entire range. Notice, however, that 2.222... is an inconvenient size for a big unit; therefore, we increase the value of "x" according to constraint b.

b. We require that "x" be the smallest number greater than the (range ÷ axis length) which allows (x ÷ 5) to be "convenient." By "convenient" we mean that the decimal expression of (x ÷ 5) will contain only one digit. This digit may not be seven or nine.

Explanation and Example:

First we mention that we want (x ÷ 5) to be convenient because we want the little (.2 cm) units to be easy to remember and add. Therefore we are going to require that (x ÷ 5) have only one digit in its decimal expression. Furthermore, this digit cannot be either a seven or a nine.

If we increase x to 2.5, then both x = 2.5 and (x ÷ 5) = .5 are convenient numbers.

A QUICK AND DIRTY EXAMPLE OF STEP 2:

This time we find a convenient scale for ΔH on the long axis. We are looking for an "x" which satisfies these two conditions:

a. $x \geq \frac{\Delta H}{\text{(length of long axis)}}$

b. $\frac{x}{5}$ is "convenient"

By convenient we mean that the decimal expression of $\frac{x}{5}$ contains only one digit and this digit is neither a nine nor a seven.

$$\frac{\Delta H}{\text{(length of long axis)}} = \frac{40}{24}$$

$$= 1\tfrac{2}{3}$$

Constraint 2a requires that $x \geq 1\frac{2}{3}$.

Constraint 2b requires that $(x \div 5)$ be convenient. 2.0 is the first number greater than $1\frac{2}{3}$ such that $(x \div 5)$ is convenient.

Finally,

$x = 2$ cm per big unit

$\frac{x}{5} = .4$ cm per little unit

3. Assemble the information on each variable-axis combination in one place. Choose the best variable-axis combination.

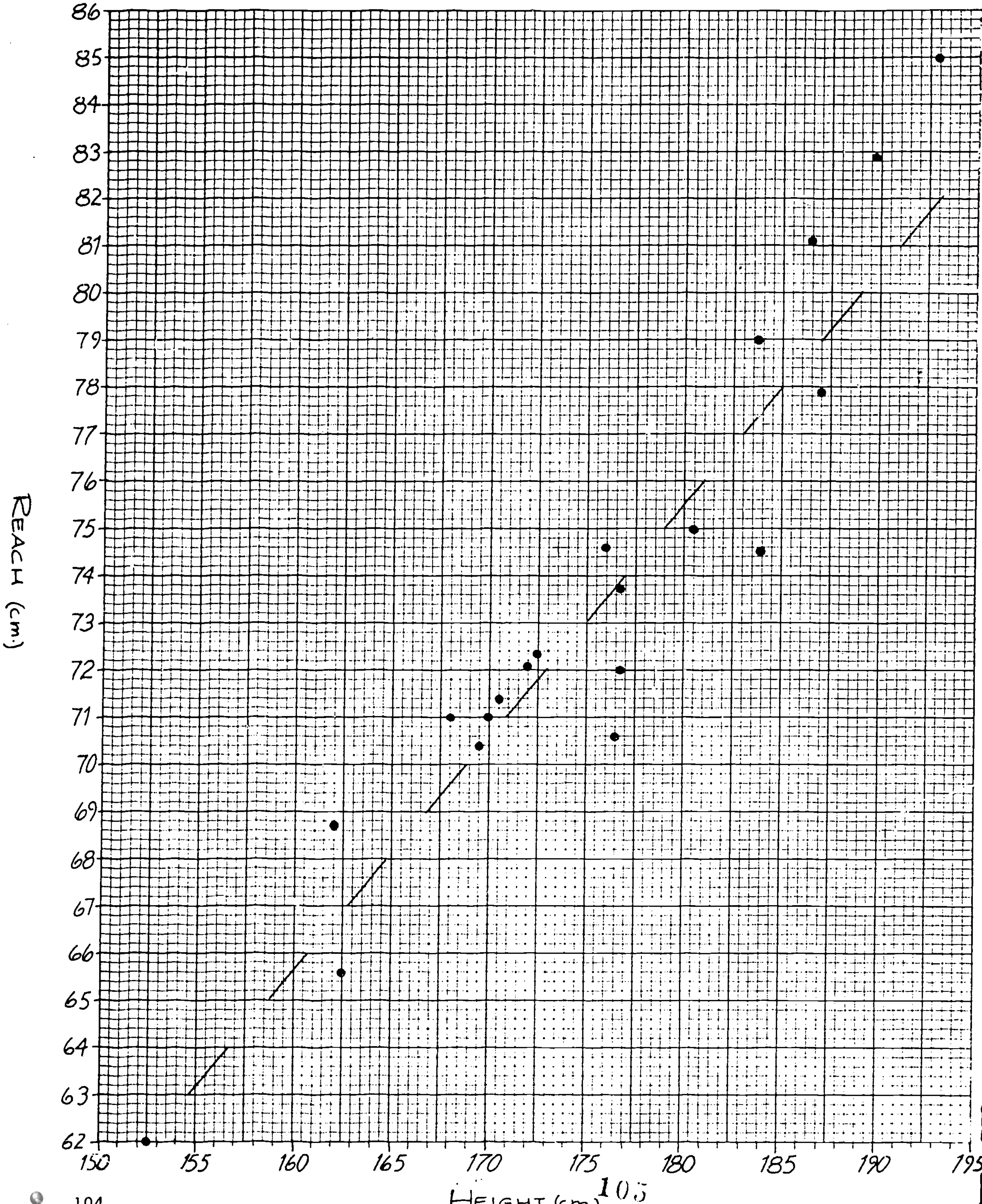
<u>EXAMPLE</u>:

ΔH - short		ΔL - short
2.5 cm - big		1.5 cm - big
.5 cm - little	BEST CHOICE	.3 cm - little
ΔH - long		ΔL - long
2.0 cm - big		1.0 cm - big
.4 cm - little		.2 cm - little

4. Scale your graph paper according to the variable-axis combination of part 3. <u>Start each axis with a whole number multiple of the big division scale</u>.

<u>EXAMPLE</u>:

The smallest height is 152.8 cm. The next lowest whole multiple of 2.5 is 152.5. However, the whole range will still fit on the axis if we go one step farther back to 150; therefore, we can start at 150 if we like. This is what we did on the following graph.



REACH (cm.)
86
85
84
83
82
81
80
79
78
77
76
75
74
73
72
71
70
69
68
67
66
65
64
63
62
150
155
160
165
170
175
180
185
190
195
HEIGHT (cm.)

5. Label each axis. Include what is being measured and the units being used.

EXAMPLE:

"Reach (cm)"

6. Plot the points.

11-4 More Examples

EXAMPLE 1:

Find the most convenient scale for reach on the short axis.

SOLUTION:

$\Delta R = 23$ cm

axis length = 18 cm

Constraint 2a requires that:

$$x \geq \frac{23}{18}$$

$$\geq 1.277....$$

Constraints 2a and 2b together require that:

$$x = 1.5$$

because

$$\frac{x}{5} = .3$$

Finally, the scales are,

$$x = 1.5 \text{ cm reach per big unit}$$

$$\frac{x}{5} = .3 \text{ cm reach per little unit.}$$

EXAMPLE 2:

Find the most convenient scale for reach on the long axis.

SOLUTION:

$\Delta R = 23$ cm

axis length = 24 cm

Constraint 2a requires that:

$$x \geq \frac{23}{24}$$

$$\geq .95833...$$

Constraints 2a and 2b together require that:

$$x = 1$$

because

$$\frac{x}{5} = .2$$

Finally, the scales are:

$$x = 1 \text{ cm reach per big unit}$$

$$\frac{x}{5} = .2 \text{ cm reach per little unit}$$

11-5 About the Usefulness of Graphs

Early in this section we said that graphs help us see relationships between variables. This particular graph will help us see the relationship between height and reach. One thing is obvious from the graph. We can see that as height increases, reach tends to increase. This is probably no surprise to anyone. However, this is not all the information available. Notice the dotted line on the graph. Most of the points lie close to this line. Now suppose that you were a clothes manufacturer. How long would you make the sleeve of a suit for a person 172.5 cm tall?

We can use the graph to find a common reach (not sleeve length) for this height. We find 172.5 on the horizontal axis and go up to the line and then over to the vertical axis. The number there is about 71.7 cm. Based on this graph we expect people 172.5 cm tall to have a reach within a couple of cm of 71.7 cm. Sleeve length might be deduced from this figure.

This is not the last you'll see of graphical analysis in this course. In the material which follows we will develop a technique for finding an equation for the dotted line. Also, we will expect you to find a range of imprecision for graphical predictions. For example, later on when we ask, "What is the expected reach of a

175 cm tall person?", we will expect an answer in the form, "The expected arm length is 71.7 ± 3 cm." However, figuring out how to scale graphs is all you need to be concerned with now.

PROBLEM SET 11:

1. What are the two requirements of a good graph?

2. Why is the graph in Section 11-1 unacceptable?

3. Describe the grid that will be most used in this course. Give the dimensions (in cm) and the number of divisions per big unit.

4. (True or False) One rule that we have established is, "Do <u>not</u> tape two pieces of graph paper together."

5. (True or False) In this course you will be expected to follow our method of graphing as well as meet the two requirements of a good graph.

6. Let ΔH = 70 cm. Find the most convenient scale for:

   a. the short axis.

   b. the long axis.

State the meanings of both the big units and the little units for both answers.

7. Let ΔL = 31 cm. Find the most convenient scale for:

   a. the short axis.

   b. the long axis.

8. Let ΔH = 35 cm. Find the most convenient scale for:

   a. the short axis.

   b. the long axis.

9. Now suppose we are graphing pressure (P) and volume (V) for a trapped quantity of gas. Let ΔP = 100 mm of mercury. Find the most convenient scale for:

   a. the short axis.

   b. the long axis.

10. Let "V" have the meaning of Problem 9 and $\Delta V = 3.5\ cm^3$. Find the most convenient scale for:

a. the short axis.

b. the long axis.

11. Let $\Delta P = 150$ mm of mercury. Find the most convenient scale for:

a. the short axis.

b. the long axis.

12. Let $\Delta V = 4.5\ cm^3$. Find the most convenient scale for:

a. the short axis.

b. the long axis.

13. What two things should be in the label of an axis?

14. Describe how to begin numbering an axis.

Use the final graph in the Text to answer Problems 15 through 18.

15. What reach would you expect a 160 cm tall person to have?

16. What reach would you expect a 190 cm tall person to have?

17. What height would you expect a person with a reach of 77 cm to be?

18. What height would you expect a person with a reach of 65 cm to be?

19. What kinds of people would you expect to be very interested in the relative proportions of the human body?

SECTION 12:

12-1 Functions

Certain sets of ordered pairs are called functions. You will find that functions are used to describe quantitatively many biomedical phenomena. In this course many types of functions will be studied. The idea of a function is a basic concept in mathematics.

A function may be thought of as a table. The table below gives the mass of each of a group of students.

Student	Student's Mass (kg)
Lynne	44
Chad	73
Bill	77
Belinda	48
Pam	55

One set contained in the table is the set of students. This set is called the domain of the function. By convention the domain is on the left side of a table unless otherwise specified. We may write the domain

$$D = \{\text{Lynne, Chad, Bill, Belinda, Pam}\}$$

The second set is the set of numbers representing the mass of each student. This set is called the range of the function. It may be written

$$R = \{44, 73, 77, 48, 55\}$$

The function itself is the set of ordered pairs. The function may be written

$$f = \{(\text{Lynne}, 44), (\text{Chad}, 73), (\text{Bill}, 77), (\text{Belinda}, 48), (\text{Pam}, 55)\}$$

A special terminology is connected with functions. You will often encounter statements written in the form

$$f(\text{Bill}) = 77$$

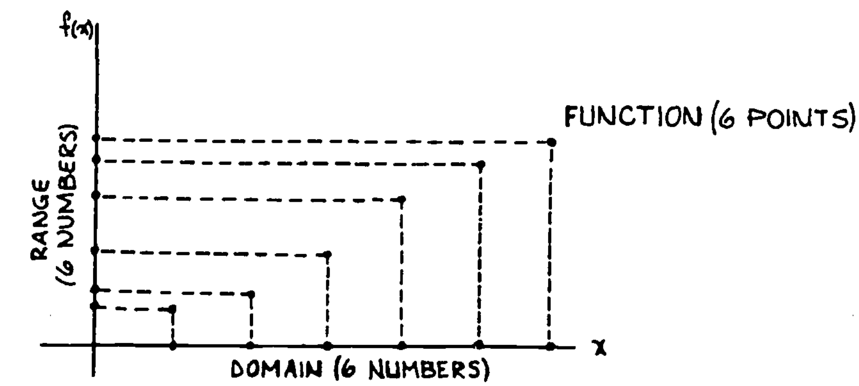
The statement is read "the function of Bill is 77," or, for short, "eff of Bill is 77." We can only put domain elements within the parentheses in function notation. For example,

OK	NOT OK
f(Bill) = 77	f(77) = Bill
F(Chad) = 73	f(73) = Chad
DOMAIN ELEMENTS	RANGE ELEMENTS

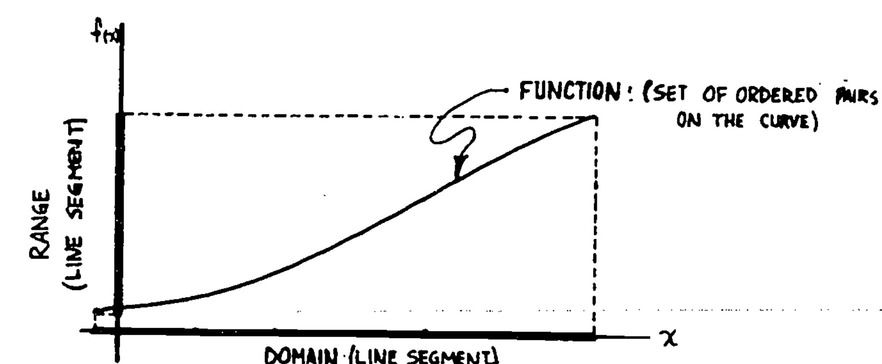
Statements like f(Bill) = 77 may be manipulated in the same ways as ordinary equations. For example,

$$f(\text{Bill}) + f(\text{Chad}) = 77 + 73$$
$$= 150$$

Functions are often represented by graphs.



By convention the horizontal coordinates represent the domain and the vertical coordinates represent the range. The function is the set of ordered pairs on the graph. A set of a finite number of ordered pairs is called a finite function. The table of students and the mass of each student is an example of a finite function.



This graph is an example of an <u>infinite function</u>. An infinite function is a set of an infinite number of ordered pairs. The graph is an infinite function because the curve represents an infinite number of ordered pairs.

With a finite function it is possible to list all of the ordered pairs that belong to the function. Listing all of the ordered pairs of an infinite function is impossible.

12-2 An Additional Requirement for Functions

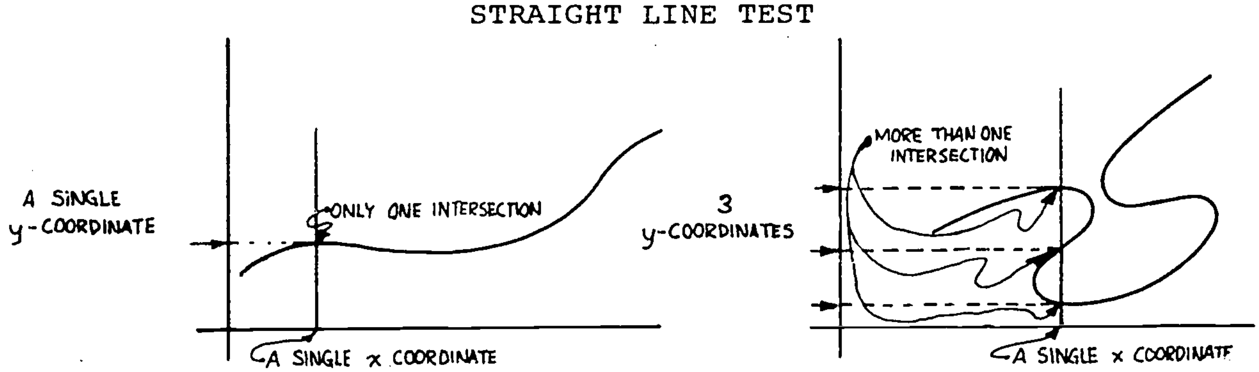
Some sets of ordered pairs are functions. Others are not. The sets of ordered pairs that are not functions fail to meet a condition that is designed to reduce confusion. It says that we cannot pair two different y-coordinates with a single x-coordinate. It is easy to see how this constraint works. Consider the set

$$g = \{(2, 3), (2, 4)\}$$

Now we ask the question, "What is the y-coordinate when x = 2?" We cannot answer this question with a single number. This is a potential source of confusion; therefore, the set g cannot be a function.

There is a simple test for infinite graph-functions to see whether they meet the anti-confusion standard. It is called the "straight-line" test. If a line perpendicular to the x-axis intersects the graphed curve more than once, then the curve cannot be a function.

STRAIGHT LINE TEST



FUNCTION NON-FUNCTION

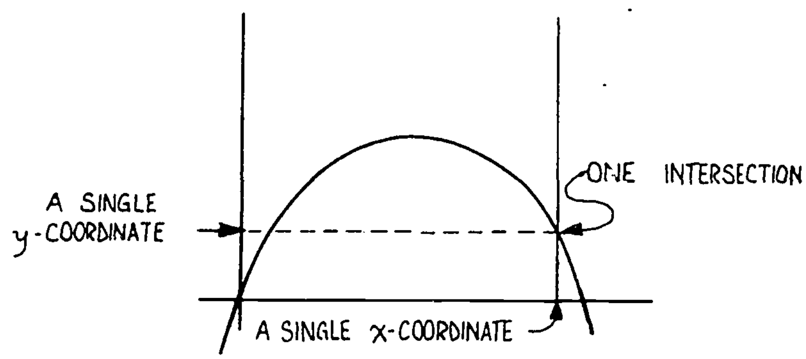
Notice that the straight line test detects situations where more than one y-coordinate is paired with an x-coordinate.

There is a potential source of confusion about the anti-confusion standard. It is easy to get the wrong idea that a function cannot pair two different x-values to a single y-value. However, this is perfectly all right. Consider the set

$$h = \{(2, 3), (4, 3)\}$$

Now we ask the question, "What is y when x = 2?" There is only one possible answer; y = 3. Similarly there is only one possible y when x = 4. Therefore, h is a function.

To reinforce this idea, let's look at an infinite graph function and apply the straight line rule.



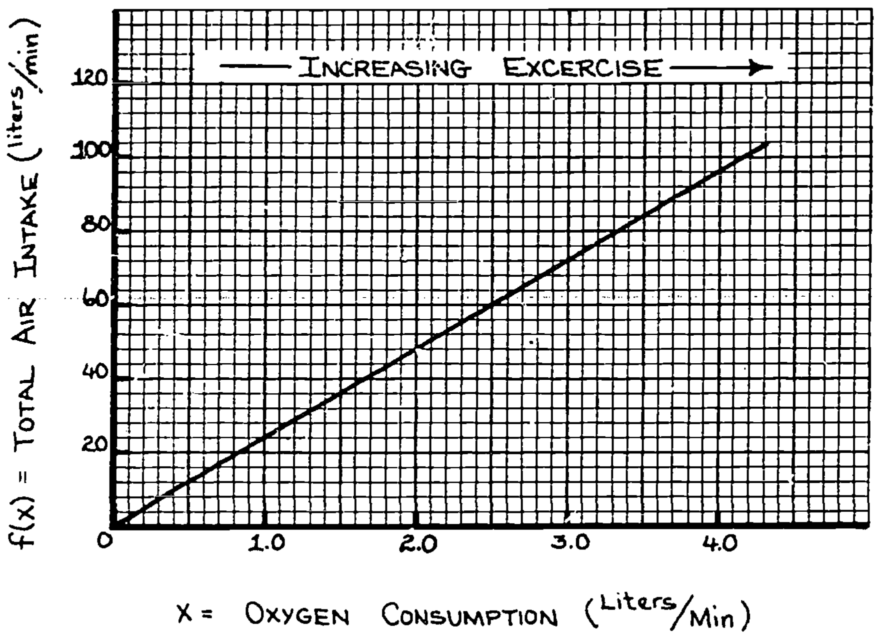
FUNCTION

No matter where we put the perpendicular line, it will only cross the curve once. Therefore, it passes the straight line test and is a function. Notice, however, that the curve may pair a particular y-coordinate with two different x-coordinates. This feature of the curve cannot prevent the curve from representing a function.

Why is there a double standard? In other words, why are the x's required to have only one mate while a y may have many? It reflects the uses of functions. Traditionally, the x's are used to predict y's. Therefore, we want a function to set up a situation which produces only one prediction (i.e., a "y") for a particular x. On the other hand, it often does not matter that a y may predict (in reverse) two different x's.

PROBLEM SET 12:

Data relating oxygen consumption (as an indicator of exercise level) to total air intake show the relationship to be a function. A graph representing this is shown below.



Refer to this graph when answering Problems 1-4.

1. Construct a table relating five elements of the domain (oxygen consumption) to five elements of the range (total air intake).

2. a. Is the function represented by the graph finite or infinite?

   b. Is the table you have constructed a finite or infinite function?

3. Approximate f(x) when:

   a. x = 1.0   c. x = 1.7

   b. x = 3.0   d. x = 3.5

4. Approximate x when:

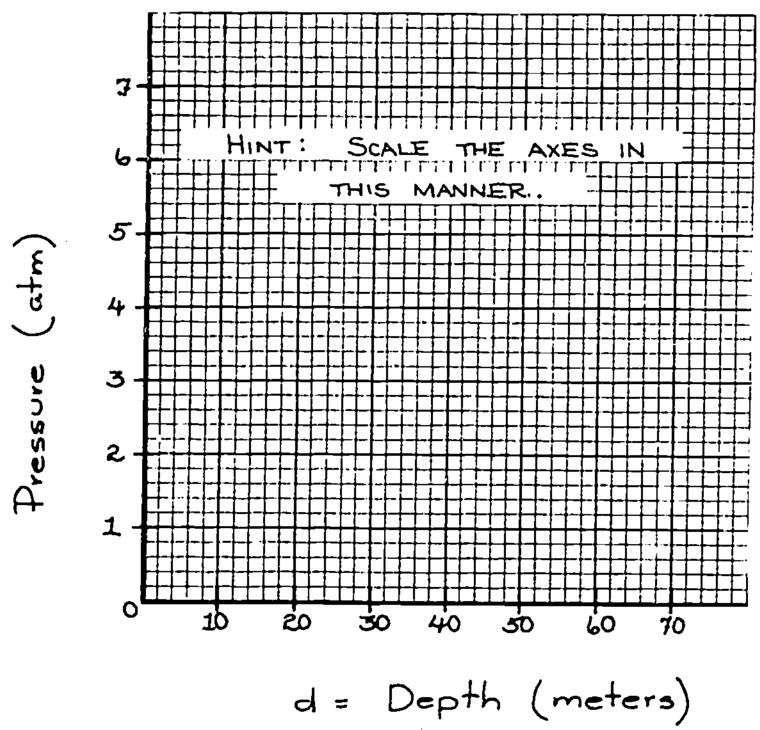
   a. f(x) = 41   c. f(x) = 84

   b. f(x) = 100   d. f(x) = 54

In general, pressure experienced by an underwater diver is a function, "f," of his depth. Refer to the following table-function in answering Problems 5-9.

d = Depth (meters)	f(d) = Pressure (atm)
0	1
10	2
20	3
30	4
40	5
50	6

5. Graph the table-function. Connect the points with a straight line. Call the function represented by the line "g."



6. Which column is conventionally associated with the domain of the table-function, "f"?

7. a. Write the table-function, f, as a set of ordered pairs.

   b. Is the table-function, f, finite or infinite?

   c. Is the graph-function, g, finite or infinite?

8. For the graph, approximate g(d) when:

   a. d = 15   c. d = 28

   b. d = 45   d. d = 36

9. For the graph, approximate d when:

   a. g(d) = 2.5   c. g(d) = 7.0

   b. g(d) = 5.8   d. g(d) = 4.3

10. Identify the sets of ordered pairs which are functions.

   a. $a = \{(2, 3), (2, 4)\}$

   b. $b = \{(47, 3), (6, 3)\}$

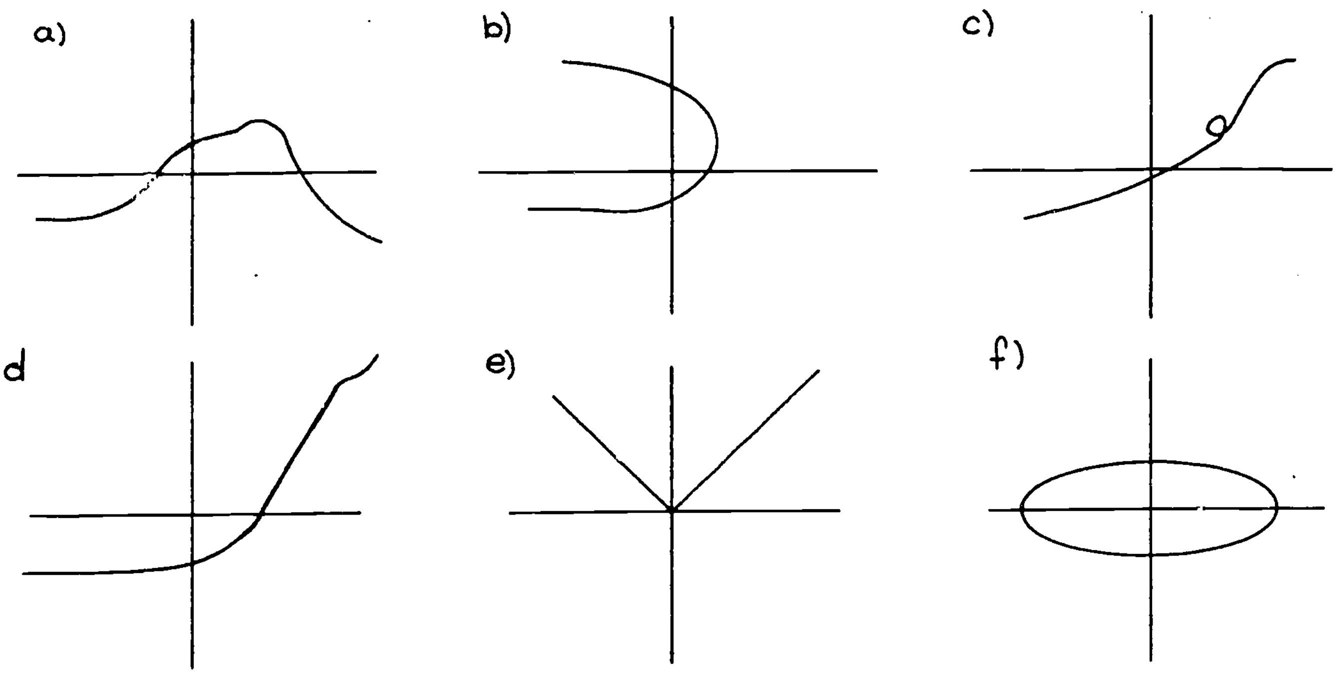
   c. $c = \{(\pi, \frac{\pi}{2}), (4, 6)\}$

   d. $d = \{(2, 4), (3, 8), (9, 16), (2, 17)\}$

   e. $e = \{(9, 6), (10, 6), (18, 6)\}$

   f. $f = \{(1, 1), (-1, 1), (2, 4), (-2, 4)\}$

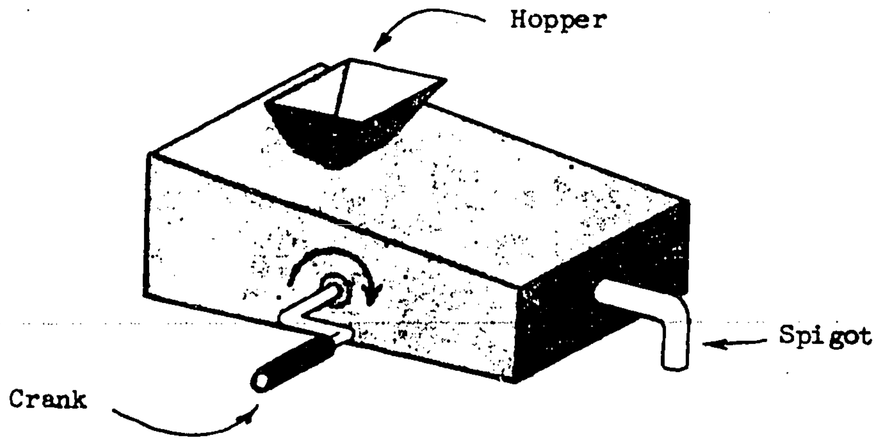
11. Use the "straight line rule" to detect non-functions. Identify the <u>functions</u>.



SECTION 13:

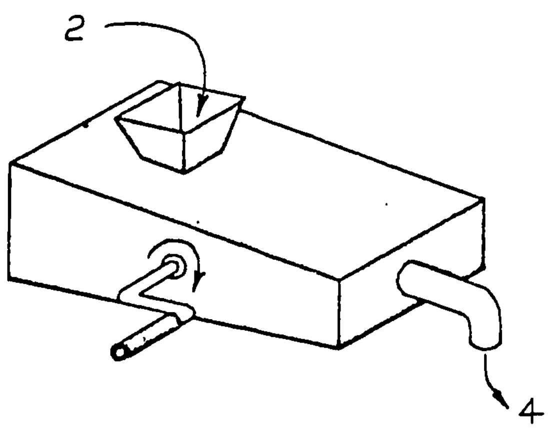
13-1 Function Machines

To continue to develop our understanding of functions, we will pretend to have a device called a function machine.

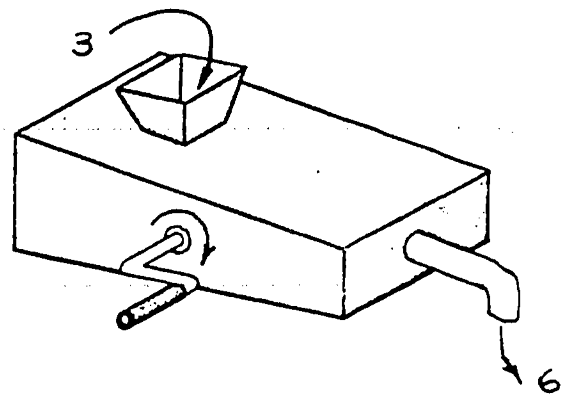


If a number that belongs to the domain of a function is dropped into the hopper, and the crank is turned clockwise, the machine operates on the number, and out of the spigot comes a number in the range of the function. We cannot see the wheels and gears inside a machine, but we may be able to guess how the machine operates by observing what numbers come out when certain numbers are dropped in the hopper.

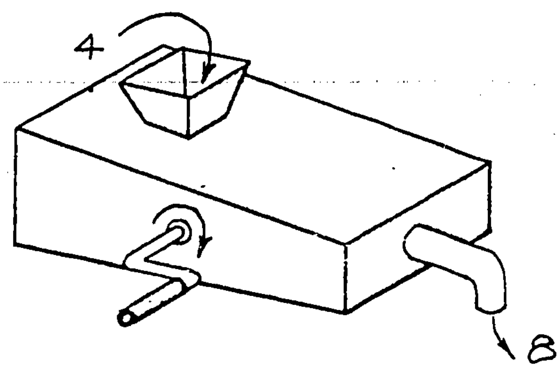
Let us drop the number 2 into one of the machines.



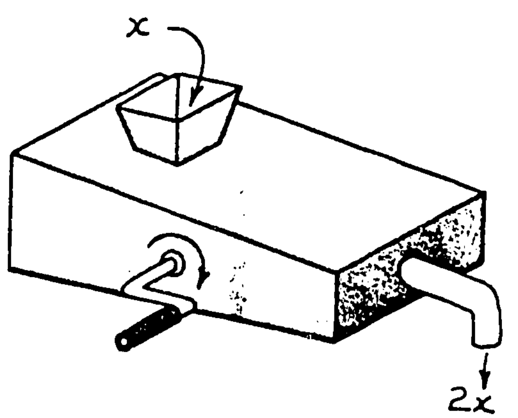
We turn the crank and out comes 4. We then drop in 3 and turn the crank, and this time out comes 6.



When we drop 4 into the hopper, 8 comes out of the spigot.

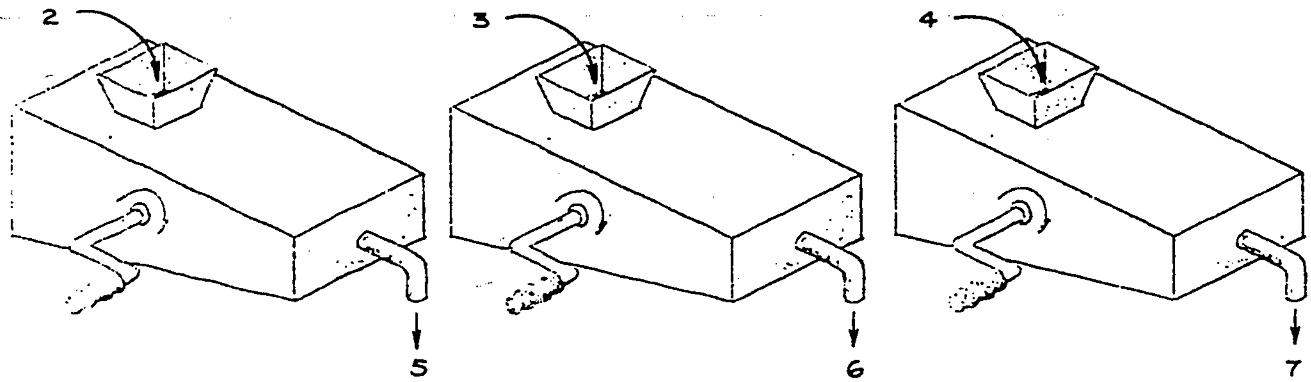


You have no doubt guessed that this particular machine multiplies each number by 2. We are not restricted to feeding the machine numbers; it operates equally well on literal algebraic statements. For example, if we feed the machine x, we get 2x from the spigot.



Notice that this function machine operates on numbers in a predictable fashion. Every number we feed to the machine is multiplied by 2. If we dropped the number 4 into the hopper repeatedly, we could count on obtaining 8 every time.

Let us now look at another function machine and feed it numbers to see if we can guess how it operates.



When we feed the machine 2, we get 5; when we feed it 3, we get 6; and when we feed it 4, we get 7. This machine apparently adds 3 to a number. What would we get if we fed the machine x?

When we fed x to the first machine, 2x came out of the spigot. Mathematicians call this 2x the <u>function of x</u>, which may abbreviated f(x). This terminology may be expressed by the statement

$$f(x) = 2x \ .$$

If f(x) = 2x, it follows that

$$\begin{aligned} f(5) &= 2 \cdot 5 \\ &= 10 \end{aligned}$$

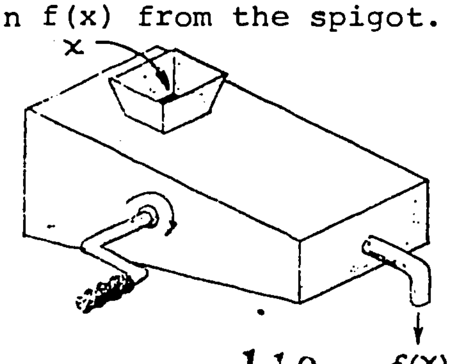
and

$$\begin{aligned} f(721) &= 2 \cdot 721 \\ &= 1442 \end{aligned}$$

When we fed x to the second function machine we obtained x + 3 from the spigot. Therefore, in this case, x + 3 is the function of x.

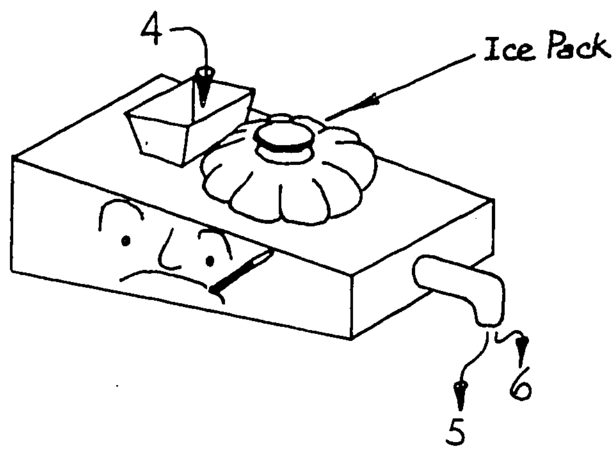
$$f(x) = x + 3$$

No matter how a function machine operates, if we feed x into the hopper, we obtain f(x) from the spigot.



13-2 Dysfunction Machines

We can imagine a function machine that gets sick or breaks down or something and stops behaving like a proper function machine. We can tell that this has happened by observing the ordered pairs that the machine produces. If the machine pairs more than one y with a particular x, then we have a sick function machine on our hands. It's sick because it can no longer produce ordered pairs that are elements of a function. Consider the following machine



It's sick. It's trying to pair both 5 and 6 to a single input, 4. This is not a function machine. It's a dysfunction machine.

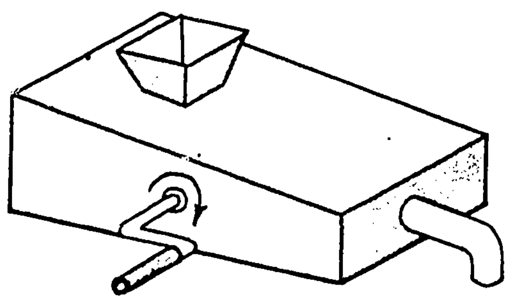
PROBLEM SET 13:

1. Match the function machine to the function.

Choose $f(x)$ from:

(1) $f(x) = x$	(5) $x + 2$	(9) $f(x) = \frac{1}{2}x$
(2) $3x$	(6) $2x + 2$	(10) $x - 7$
(3) $2x$	(7) $2x + 1$	
(4) 3	(8) $\frac{1}{3}x - 1$	

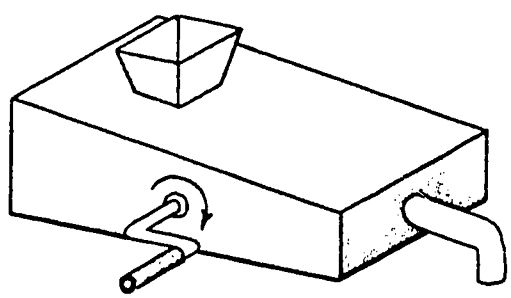
a. Domain: $\{1, 2, 3\}$



Range: $\{2, 4, 6\}$

$f = \{(1, 2), (2, 4), (3, 6)\}$ $f(x) =$

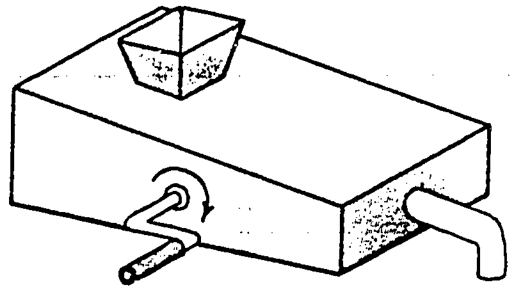
b. Domain: $\{1, 2, 3\}$



Range: $\{3, 5, 7\}$

$f = \{(1, 3), (2, 5), (3, 7)\}$ $f(x) =$

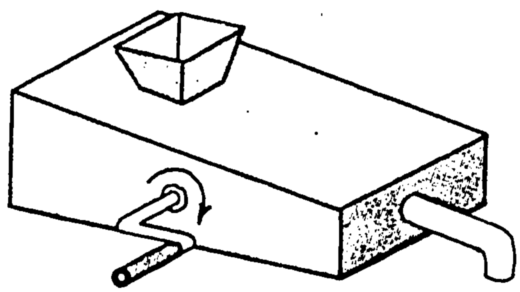
c. Domain: {1, 2, 3}



Range: {3, 6, 9}

f = {(1, 3), (2, 6), (3, 9)}

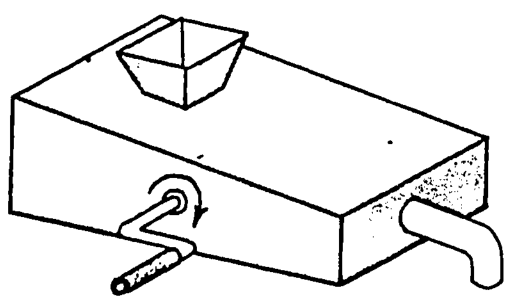
d. Domain: {1, 2, 3}



Range: {3, 4, 5}

f = {(1, 3), (2, 4), (3, 5)}

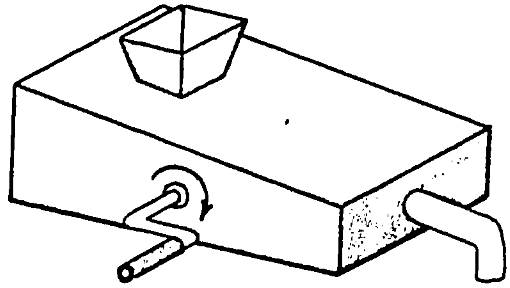
e. Domain: {0, 3, 4}



Range: {2, 8, 10}

f = {(0, 2), (3, 8), (4, 10)}

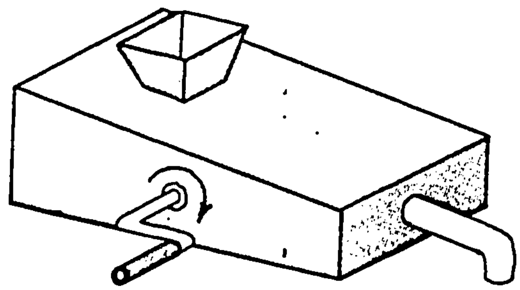
f. Domain: {5, 7, -8}



Range: {5, 7, -8}

f = {(5, 5), (7, 7), (-8, -8) }

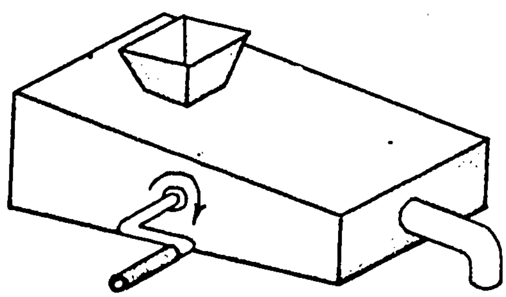
g. Domain: {5, 7, 8}



Range: {3}

f = {(5, 3), (7, 3), (8, 3)}

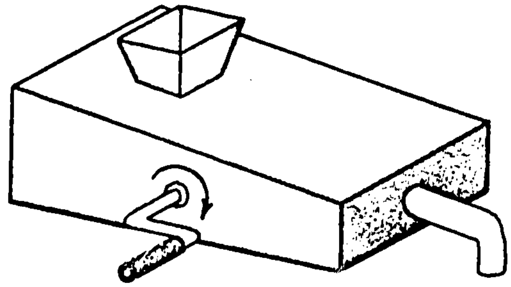
h. Domain: {7, 17, -7}



Range: {0, 10, -14}

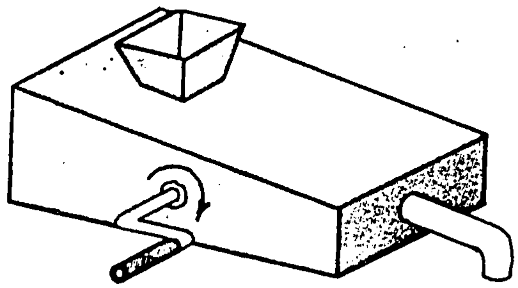
f = {(7, 0), (17, 10), (-7, -14)}

i. Domain: {4, 6, -18}



f = (4, 2), (6, 3), (-18, -9) Range: {2, 3, - 9}

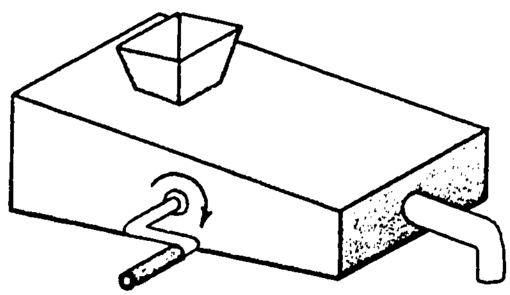
j. Domain: {6, 15, -9}



f = {(6, 1), (15, 4), (-9, -4)} Range: {1, 4, - 4}

2. Given the domain and a rule, "crank" out the range:

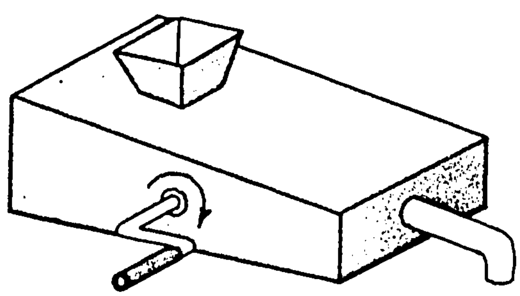
a. Domain: {0, 4, -5}



f(x) = 2x + 1

Range: { }

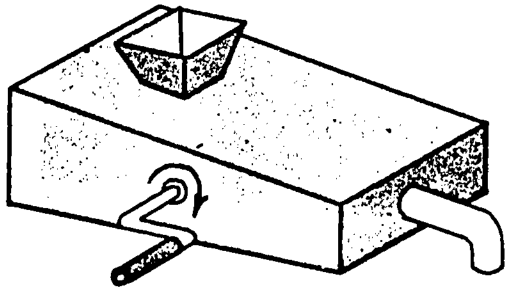
b. Domain: {3, 8, -13}



f(x) = 3x - 7

Range: { }

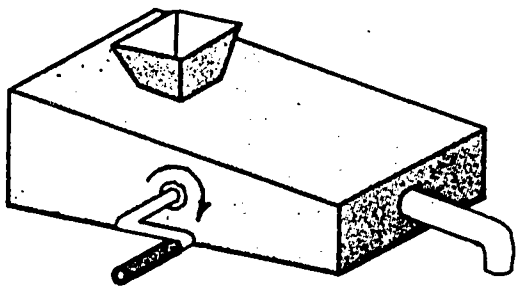
c. Domain: {17, 23, 28}



$f(x) = 29$

Range: { }

d. Domain: {3, 7, 9, -5}



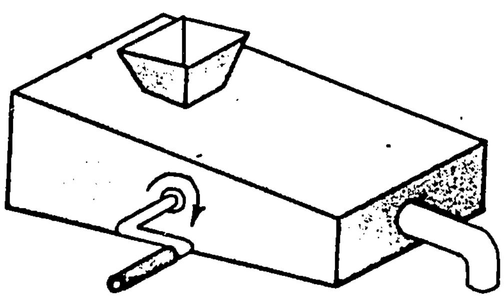
$f(x) = 11 - x$

Range: { }

3. What's my rule?

Example:

Domain: {3, 8, 11, - 4}

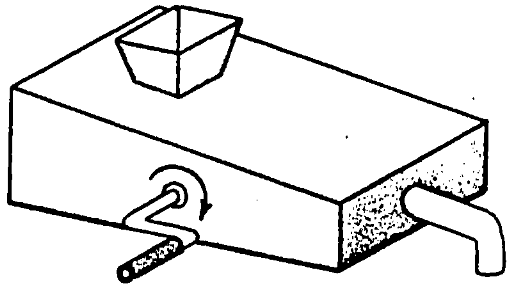


$f(x) = 2x - 2$

Range: {4, 14, 20, - 10}

f = {(3, 4), (8, 14), (11, 20), (-4, -10)}

a. Domain: {1, 2, -3}

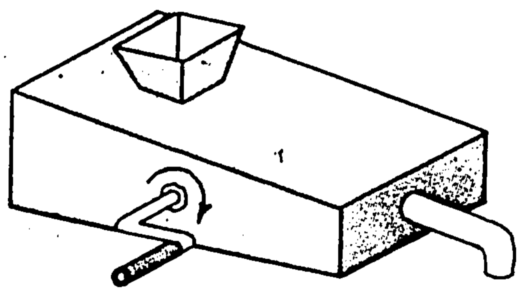


$f(x) = ?$

Range: {4, 8, -12}

f = {(1, 4), (2, 8), (-3, -12)}

b. Domain: {4, 7, -8}

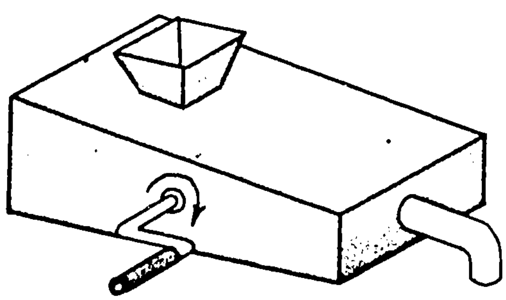


$f(x) = ?$

Range: {4, 1, 16}

f = {(4, 4), (7, 1), (-8, 16)}

c. Domain: {11, 13, 17}

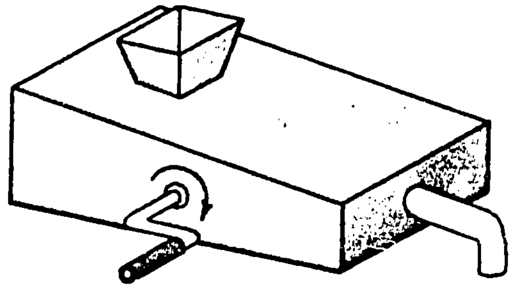


$f(x) = ?$

Range: {7}

f = {(11, 7), (13, 7), (17, 7)}

d. Domain: {3, 9, 15}

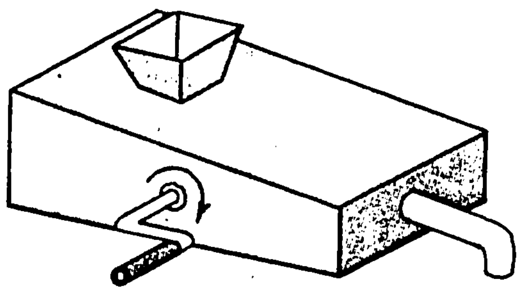


f(x) = ?

Range: {0, 2, 4}

f = {(3, 0), (9, 2), (15, 4)}

e. Domain: {2, 3, -4}

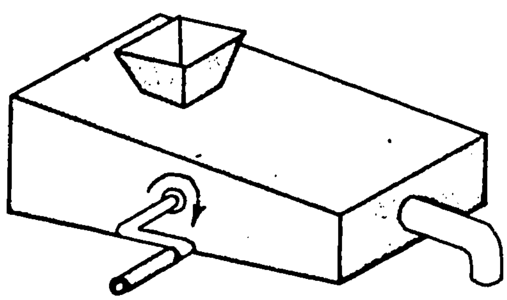


f(x) = ?

Range: {12, 16, -12}

f = {(2, 12), (3, 16), (-4, -12)}

f. Domain: {3, 7, 11, -14}



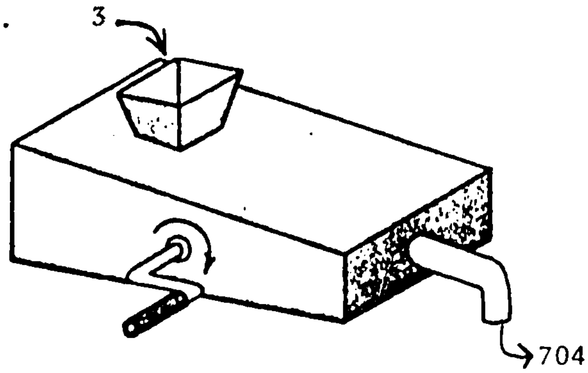
f(x) = ?

Range: {6, 10, 14, -11}

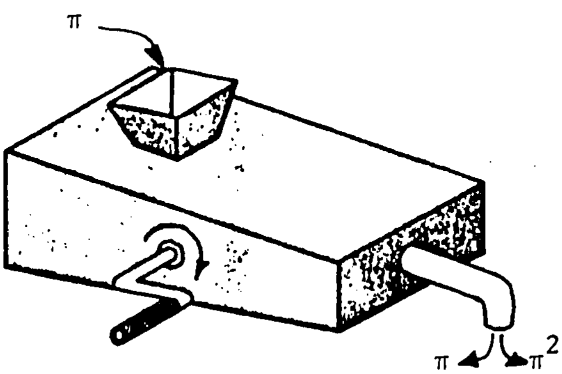
f = {(3, 6), (7, 10), (11, 14), (-14, -11)}

4. Diagnose the sick function machines

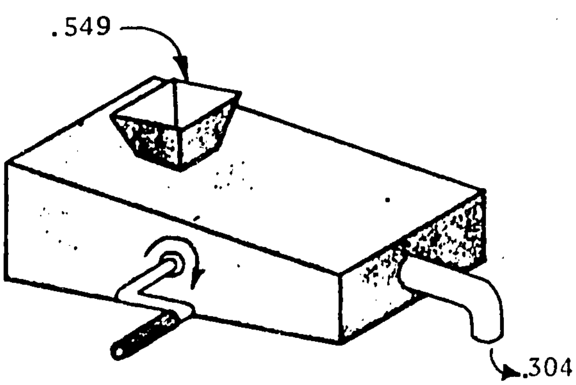
a.



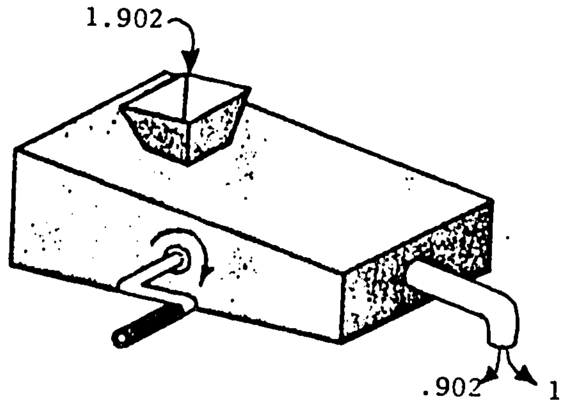
b.



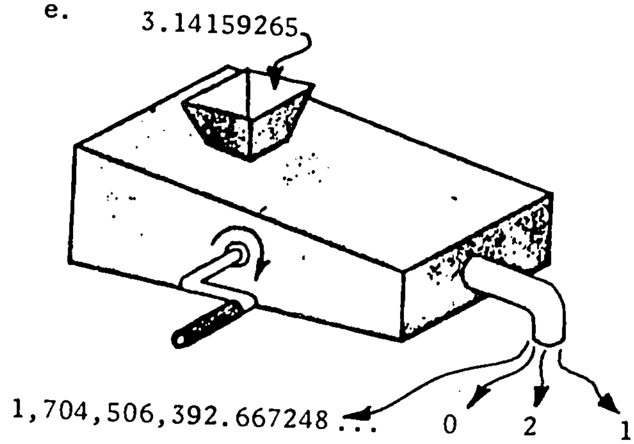
c.



d.



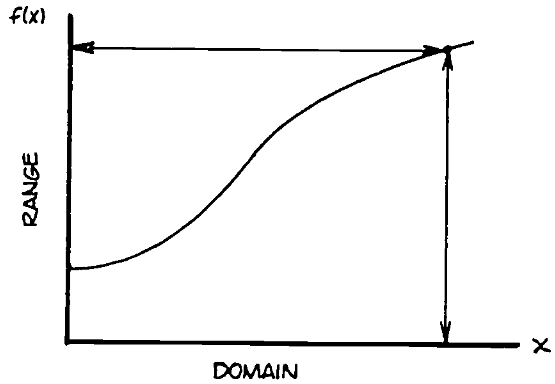
e.



SECTION 14:

14-1 Functions and Nonfunctions, A Review

In the previous section we decided that a properly working function machine must operate predictably. If we feed the machine some number, we can predict with certainty what the output number will be. Each number in the domain is paired to exactly one number in the range.



The pairing of an element in the domain to one element in the range is basic to the whole concept of a function.

Let us further examine pairing by considering another type of function, that represented by the table below.

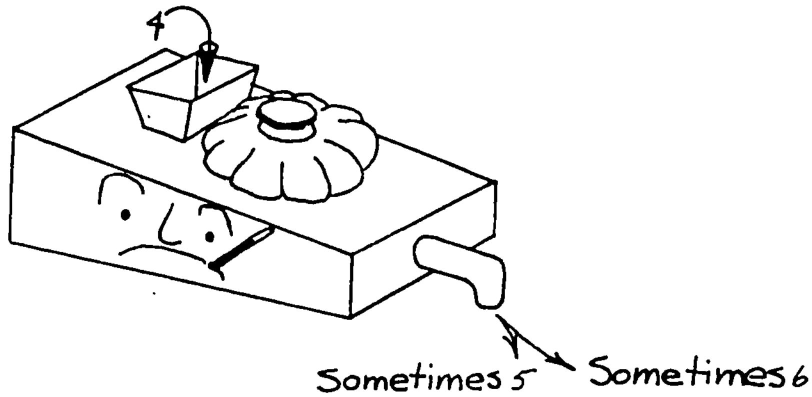
Student	Height in cm
Puchinsky	150
Smith	173
Fosdick	180
Wellburne	192
Brown	165
Nederlick	188

The domain of the function is the set D = {Puchinsky, Smith, Fosdick, Wellburne, Brown, Nederlick}. The range is the set R = {150, 173, 180, 192, 165, 188}. Again we observe that the function has the property that each element in the domain is paired to exactly one element in the range. Puchinsky, for example, cannot be both 150 cm and 170 cm tall.

At this point we will define the term function in rather formal language. Then we will discuss several cases to help you understand the meaning of the definition.

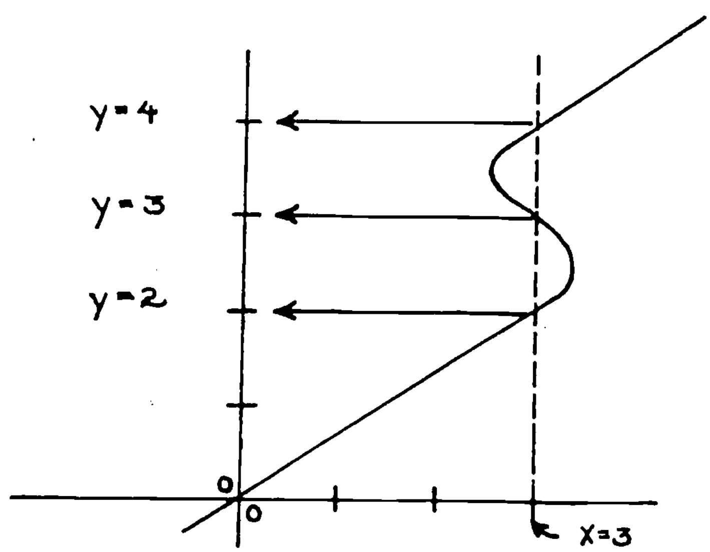
Let D and R be any two sets. A function from D to R is a set of ordered pairs with first coordinates from D and second coordinates from R with the further property that if (x, y) and (x, y') belong to the function, then y = y'.

Suppose we had a set of ordered pairs which included (x, y) and (x, y'), and y, was not equal to y'. Remember our sick function machine, which when fed 4, would sometimes give us 6 and sometimes 5.



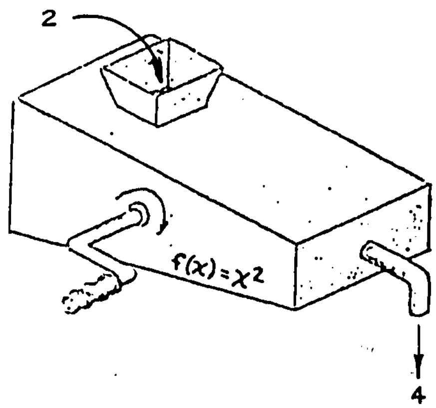
The definition of a function does not allow (4, 5) and (4, 6) in the same function. Therefore, this machine does not represent a healthy function. It's a sick function machine, sick, sick, sick. Tch. Tch.

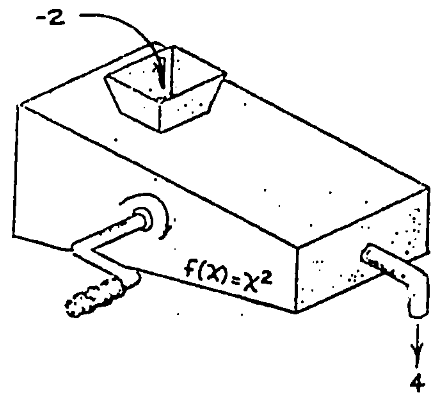
Does the graph below represent a function? We apply the "straight line rule" to find out.



When x = 3, y can be 2, 3 or 4. Three of the ordered pairs represented by the graph are thus (3, 2), (3, 3) and (3, 4). The definition of a function states that if a function includes (x, y) and (x, y') y must equal y'. Therefore, the graph does not represent a function and we see how the straight line rule agrees with the formal definition.

Nothing in the definition of a function requires that an element of the range be paired with only one element of the domain. If (x, y) and (x', y) belong to a function, x need not be equal to x'. In a table listing a set of students as the domain and their heights as the range it is quite possible two students will have the same height. It is also possible that the same number will pop out of a function machine when different numbers are fed to it.





As a final example to contrast the domain of a function with the range of a function, consider the following tables, which list pairs of husbands and wives.

Table 1

Husband	Wife
Joe	Joan
John	Jeri
Sam	Sue
Sam	Cheryl

Table 2

Husband	Wife
Joe	Sue
John	Sue
Sam	Sue

The set of ordered pairs in Table 1 is S = {(Joe, Joan), (John, Jeri), (Sam, Sue), (Sam, Cheryl)}. S is not a function because Sam is married to both Sue and Cheryl. The set of ordered pairs in Table 2 is T = {(Joe, Sue), (John, Sue), (Sam, Sue)}. T is a function, although Sue is married to three men.

It may seem most unfair that a woman be allowed three husbands and a man only one wife, and males will be glad to know that the concept of a function is more widely used in mathematical sciences than by marriage counselors.

PROBLEM SET 14:

1. Some of the following tables represent sets of ordered pairs that are not functions. Identify nonfunctional sets of ordered pairs and list two ordered pairs from the set which prohibit the set from being a function.

Example.

z.

x	y
2	19
2	21
3	14
6	17
8	-4

Ans. z. (2, 19), (2, 21)

a.

Student	Test Scores (%)
A	73
B	28
C	93
D	64
E	93
F	98

b.

Height (cm)	Weight (kg)
139	41
142	48
142	49
145	50
150	59
165	65

c.

Age (years)	Weight (kg)
1	12
6	21
19	49
19	51
30	64

d.

Student	Weight (kg)
A	43
B	43
C	52
D	71

e.

Patient	Temperature (degrees F)
A	98.6
B	100.1
C	98.6
D	101.0
E	28.3

f.

Object	Object's Volume (cm^3)
A	23
A	29
A	37
B	14
B	17

g. $x_1 \neq x_2 = x_3 \neq x_4 \neq x_1$

y_1, y_2, y_3, y_4 all distinct

x	f(x)
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

h.

Magazine	Number of Pages
Snort	47
Star	98
Glurk	64
Beige	53
Bugle	47

i.

Food	Price of Food
loaf of bread	27¢
dozen eggs	53¢
dozen eggs	55¢
lb bacon	$1.01

j.

Student	Birthday
A	Dec. 2
B	June 26
C	Sept. 30
D	Feb. 16

2. Which of the following blood kindship relations will be a functional relationship for all cases? In each case, the domain is stated first.

a. mothers to children

b. children to mothers

c. brothers to sisters

d. sons to fathers

3. Given a domain and a rule, express the function as a set of ordered pairs.

EXAMPLE: Domain: $\{1, 2, 3, 4\}$

Rule: $f(x) = 2x + 1$

Function: $f = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$

a. Domain: $\{0, 2, 4\}$

Rule: $f(x) = x$

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b. Domain: $\{-1, 0, 1\}$
Rule: $f(x) = 2x + 4$

c. Domain: $\{-3, -1, 0, 1, 3\}$
Rule: $f(x) = 3$

d. Domain: $\{2, 3, 7, 9\}$
Rule: $f(x) = 8 - x$

e. Domain: $\{-4, 2, 8\}$
Rule: $f(x) = \frac{1}{2}x$

f. Domain: $\{-4, 2, 8\}$
Rule: $f(x) = 8 - \frac{1}{2}x$

g. Domain: $\{0, 1, 2, 3\}$
Rule: $f(x) = x + 7$

h. Domain: $\{-2, -1, 0, 1, 2\}$
Rule: $f(x) = 2x + 4$

4. Professor Myopia has spent 14 years in an attempt to list <u>all</u> of the ordered pairs possible for the function defined by

Domain = All possible values of x such that
$x \geq 0$ and $x \leq 1$

Range = All possible values of f(x) such that
$f(x) \geq 1$ and $f(x) \leq 3$

$f(x) = 2x + 1$

A portion of Professor Myopia's list follows. Notice that we have listed the elements of the set vertically instead of horizontally.

$$f = \left\{ \begin{array}{l} (0, 1), \\ (.00000000001, 1.00000000002), \\ (.00000000002, 1.00000000004), \\ (.00000000003, 1.00000000006), \\ \cdot \\ \cdot \\ \cdot \\ (.99999999997, 2.99999999994), \\ (.99999999998, 2.99999999996), \\ (.99999999999, 2.99999999998), \\ (1.00000000000, 3.00000000000) \end{array} \right\}$$

a. Unfortunately Professor Myopia has failed. Find one ordered pair that he left out of his list.

b. If Professor Myopia added your ordered pair to the list would it then be complete?

c. Should Professor Myopia turn his back on 14 years of work and take up a new project?

5. Recall from the lesson the distinction made between finite and infinite functions. If all of the ordered pairs contained in a function may be listed, then the function is a finite function.

Determine whether the following functions are finite or infinite:

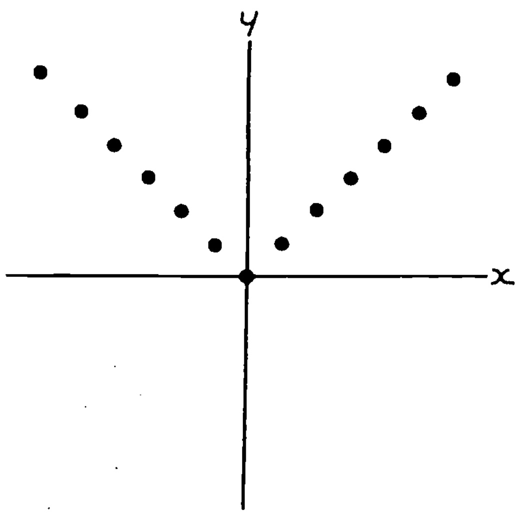
a. A function whose domain is {1, 2, 3, 4} and range is {3, 4, 5, 6}.

b. A function whose domain is 27 students and range is their weights.

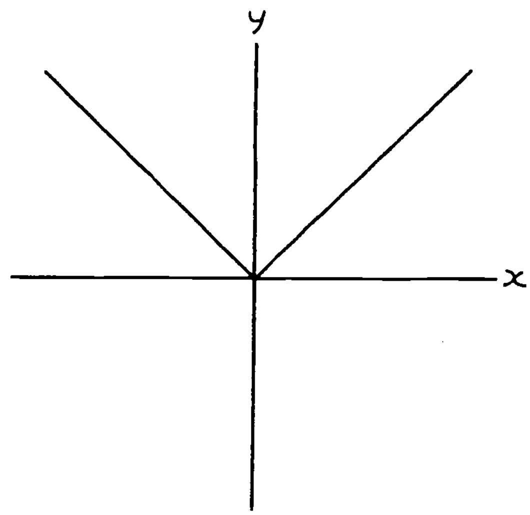
c. A function whose domain is all the natural numbers, and $f(x) = 2x$.

d. A function whose domain is all numbers between 0 and 10, and $f(x) = 8$.

e.



f.



g. Rising temperature of a car engine as a function of time.

h.

Student	Grade
Ron	42%
Sue	94%
Carol	78%
Todd	87%
Elmo	3%

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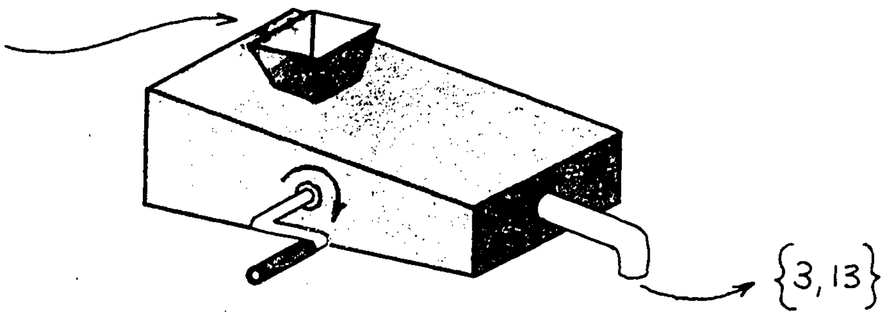
i. Height as a function of the students in a classroom.

j. {3, 8}



6. Given the following sets of ordered pairs, pick out the functions:

a. {(2, 8), (3, 10), (4, 12)}

b. {(a, b), (c, d), (e, f), (g, h)}

c. {(2, 7), (3, 8), (5, 8), (7, 13)}

d. {(1, 3), (1, 4), (1, 5)}

e. {(a, b), (a, c), (a, d)} where b ≠ c ≠ d

f. {(a, k), (b, k), (c, k)}

g. {(1, 2), (3, 4), (5, 6)}

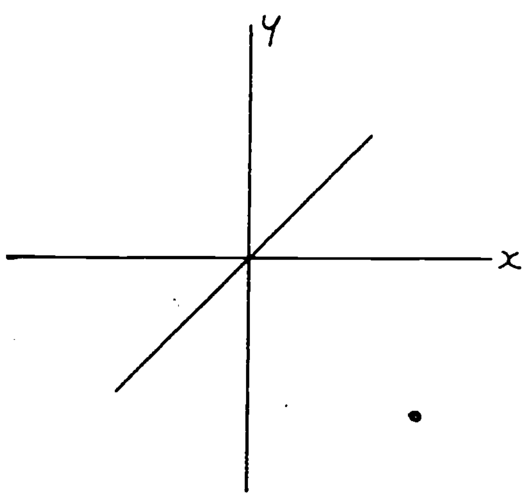
h. {(Joe, Sue), (Bill, Mary), (Ann, Mike)}

i. {(Joe, Carol), (Joe, Sue)}

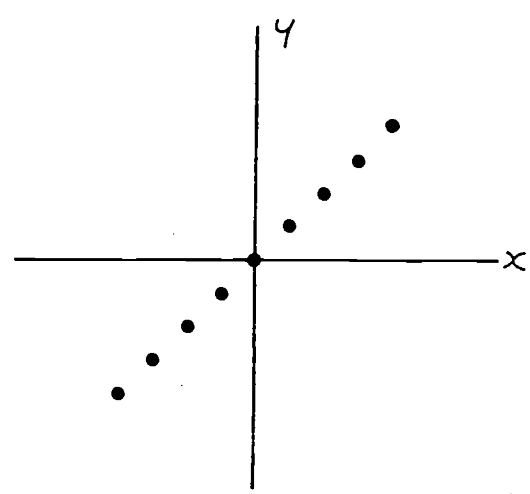
j. {(Joe, Carol), (Ron, Carol), (Todd, Carol)}

7. Which of the following graphs represent functions? Identify which functions are finite and which are infinite.

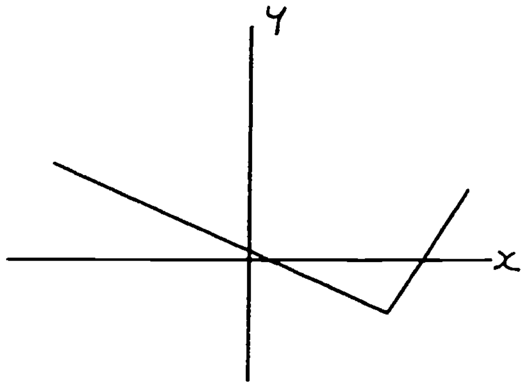
a.



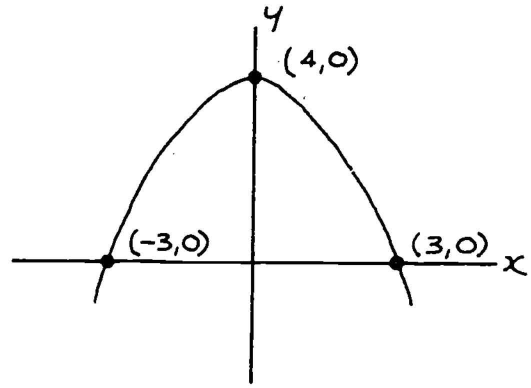
b.



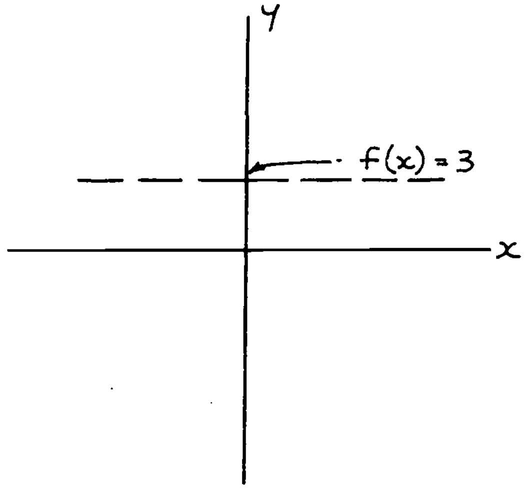
c.



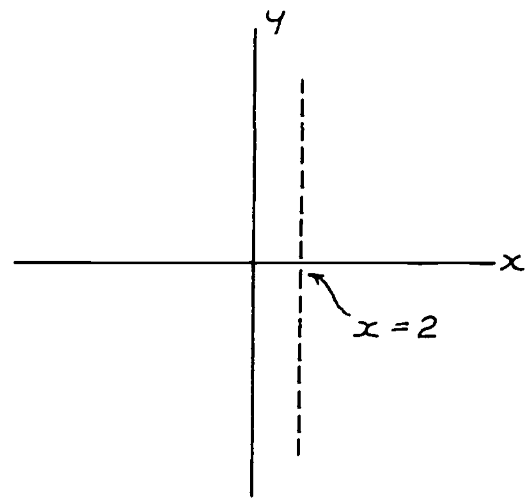
d.



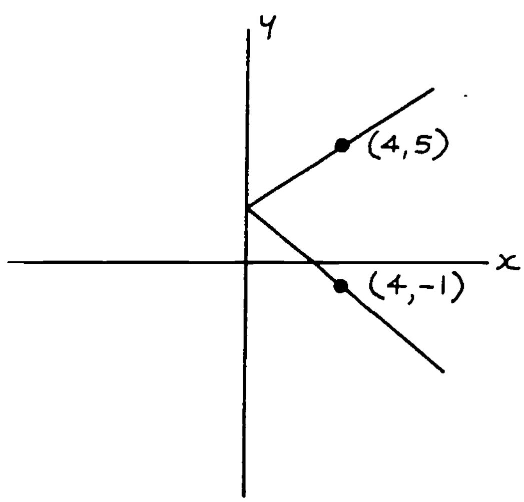
e.



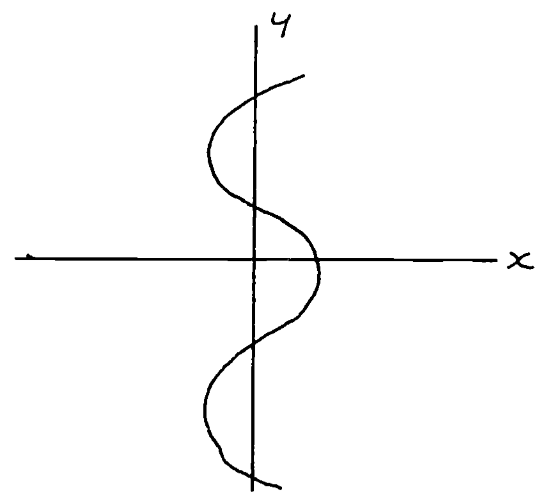
f.



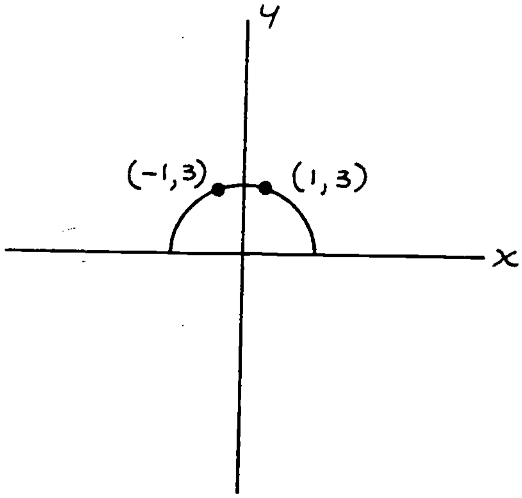
g.



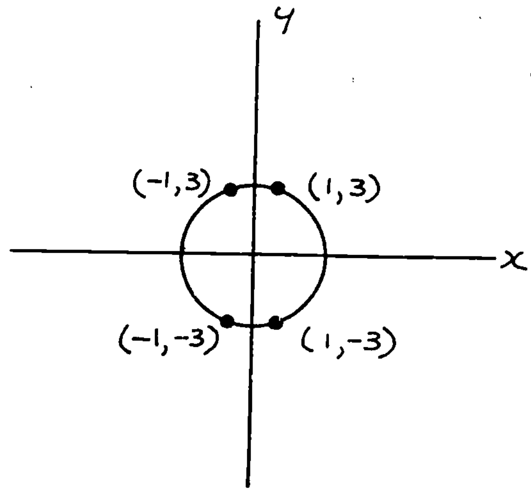
h.



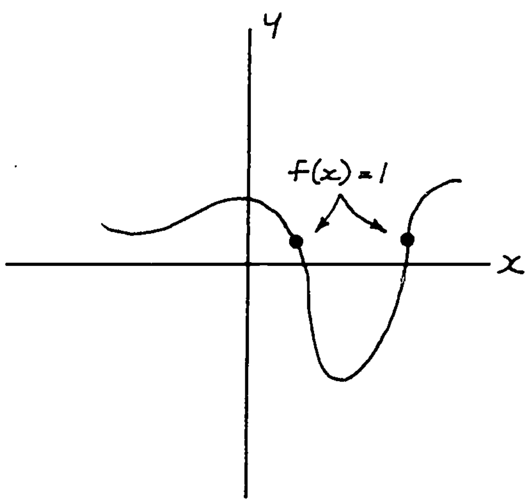
i.



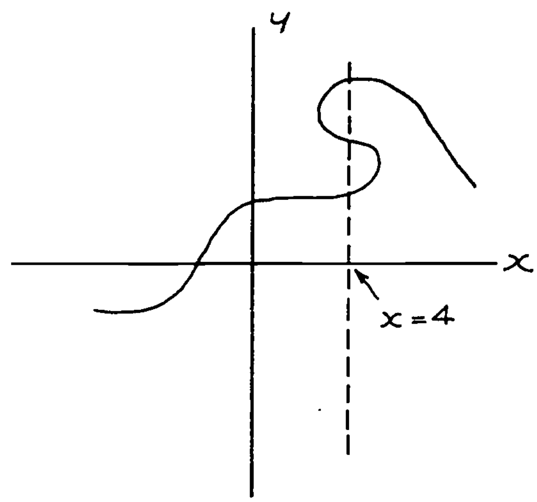
j.



k.



l.



SECTION 15: SLOPE

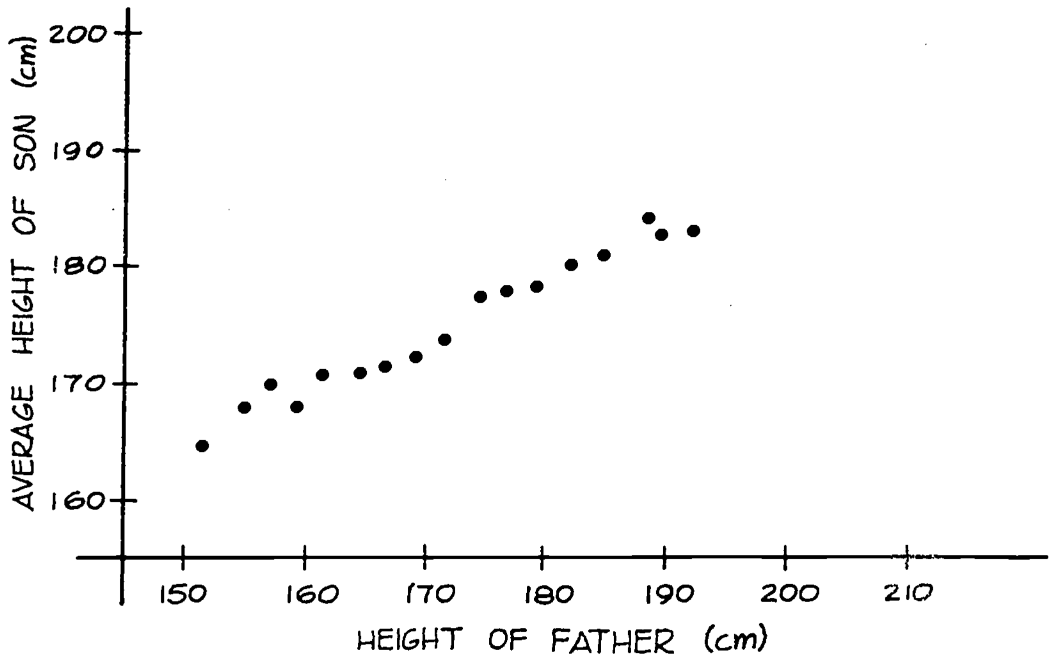
15-1 Why We Need Equations for Straight Lines

Often in biomedical investigations, a set of ordered pairs is obtained similar to the one in the following table.

Height of father (cm)	Average height of son (cm)
151	165
154	168
156	170
159	168
161	171
164	171
166	172
169	173
171	174
174	177
176	178
179	178
182	180
184	181
187	184
189	183
192	183

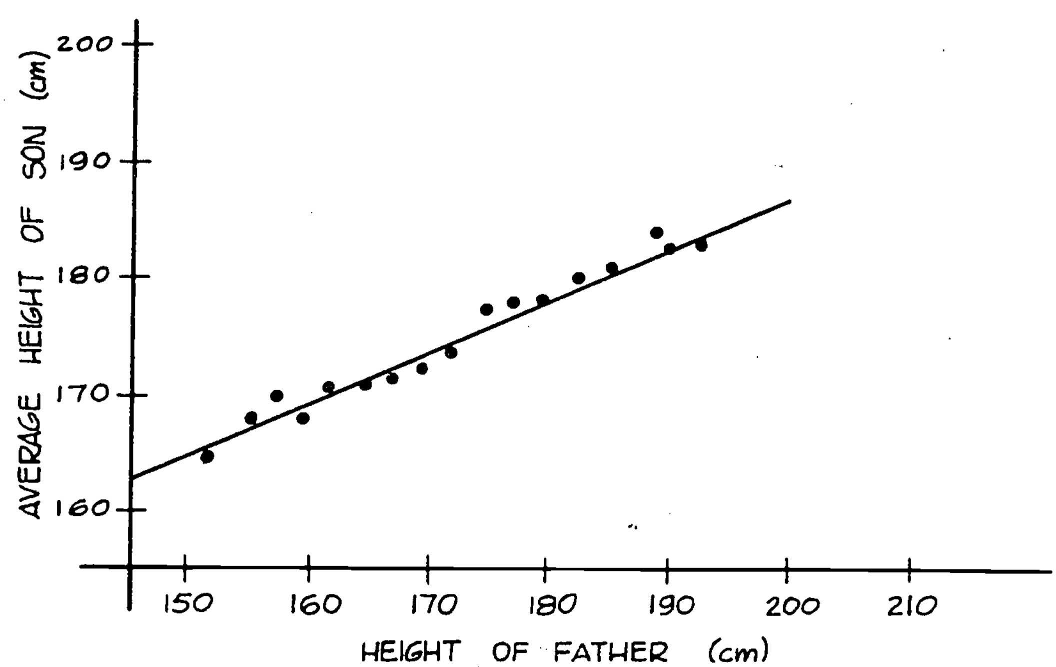
(These data were obtained from a survey. The heights of all sons of fathers of a given height were averaged.)

This set may also be shown as a graph.



In this section and the next three, you will be shown how to find the equations which describe many relationships similar to this one.

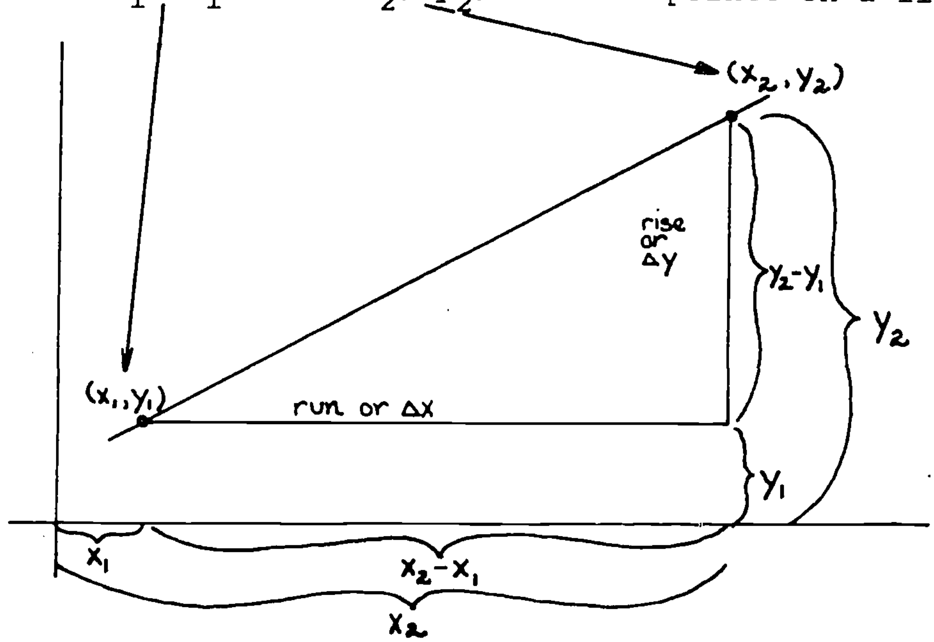
Finding an equation which fits all of the points on the graph is not easy, so we draw a line which approximates the positions of the points just as we did earlier on our graph of height vs. reach.



This line probably represents the general relationship between the heights of fathers and the average heights of their sons better than the specific data in the table.

15-2 Slope, Notation and Formulas

The first step in determining the equation of a line is finding the slope. If (x_1, y_1) and (x_2, y_2) are two points on a line,



the slope is defined by the formula

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{where } x_2 \neq x_1$$

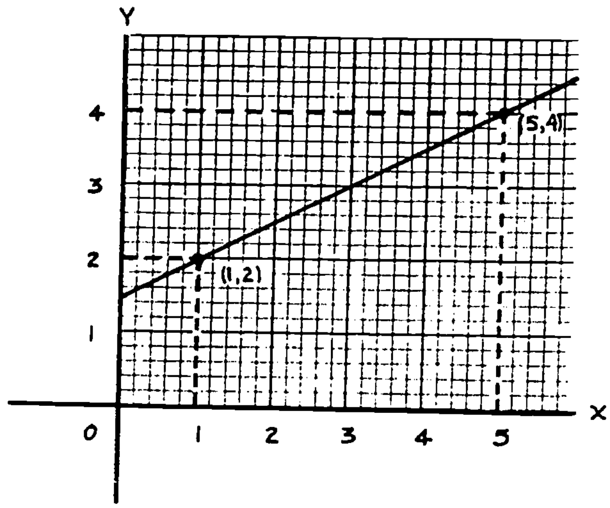
The numerator $y_2 - y_1$ is commonly called the rise. It is also referred to as the change in y, which is written Δy. The denominator $x_2 - x_1$ is called the run or the change in x, and may be written Δx.

The formula for the slope may thus be written

$$\text{slope} = \frac{\text{rise}}{\text{run}} \quad \text{or} \quad \text{slope} = \frac{\Delta y}{\Delta x} .$$

Mathematicians generally use the letter "m" to signify the slope.

The following example illustrates the calculations involved in determining the slope of a line.



Two points on the line are (1, 2) and (5, 4), so we set $x_1 = 1$, $y_1 = 2$, $x_2 = 5$ and $y_2 = 4$. The slope is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

$$= \frac{4 - 2}{5 - 1}$$

$$= \frac{2}{4}$$

Slopes are expressed as fractions in reduced form so we reduce the fraction $\frac{2}{4}$ and write $m = \frac{1}{2}$. Since $m = \frac{1}{2} = \frac{\Delta y}{\Delta x}$, this result means that there is a change of 1 in the y-coordinate for every change of 2 in the x-coordinate.

15-3 Does the Slope Change When A Different Point is Chosen First?

Does the choice of which point is the first point make a difference? Let us find out by setting $x_1 = 5$, $y_1 = 4$, $x_2 = 1$, and $y_2 = 2$.

$$m = \frac{2 - 4}{1 - 5}$$

$$= \frac{-2}{-4}$$

$$= \frac{2}{4}$$

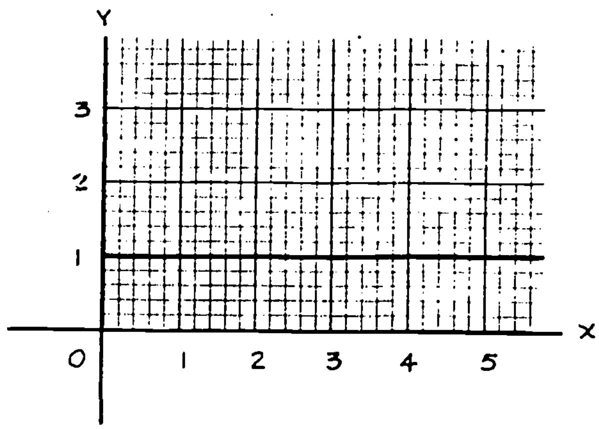
$$= \frac{1}{2}$$

The order of points makes no difference.

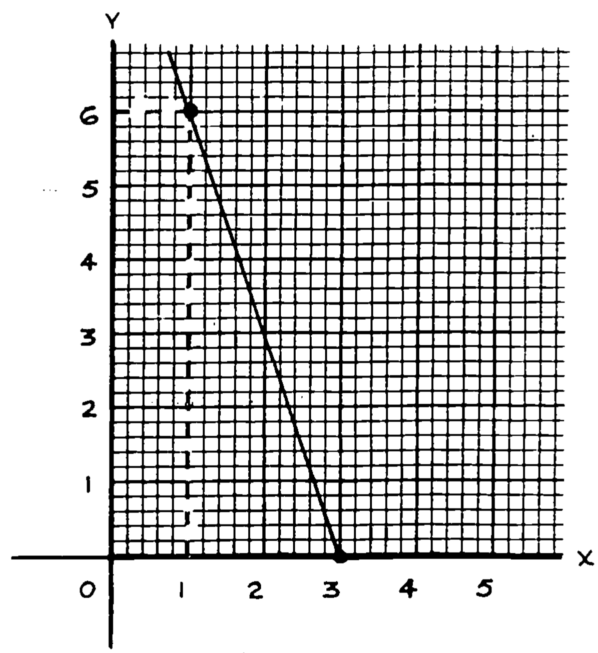
15-4 Positive, Zero, Negative and Undefined Slopes

The slope of any line that rises to the right is a positive number.

The slope of a horizontal line is zero, because $y_1 = y_2$, or because the rise is zero.



If a line rises to the left, the slope is a negative number.



The slope of this line may be found by setting $x_1 = 3$, $y_1 = 0$, $x_2 = 1$ and $y_2 = 6$.

$$m = \frac{6 - 0}{1 - 3}$$

$$= \frac{6}{-2}$$

$$= -3$$

The line could alternatively be said to fall to the right, which would imply that $x_1 = 1$, $y_1 = 6$, $x_2 = 3$ and $y_2 = 0$. The slope is calculated to be

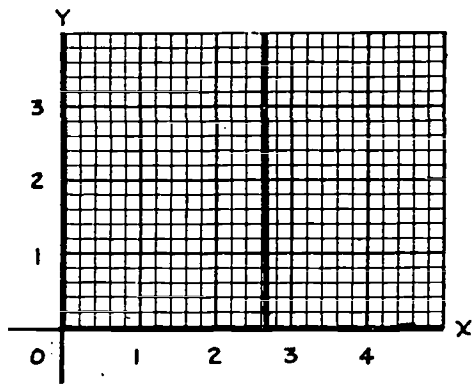
$$m = \frac{0 - 6}{3 - 1}$$

$$= \frac{-6}{2}$$

$$= -3$$

the same result as before. However, notice that we used a rise of -6. It is perfectly okay to have a negative rise.

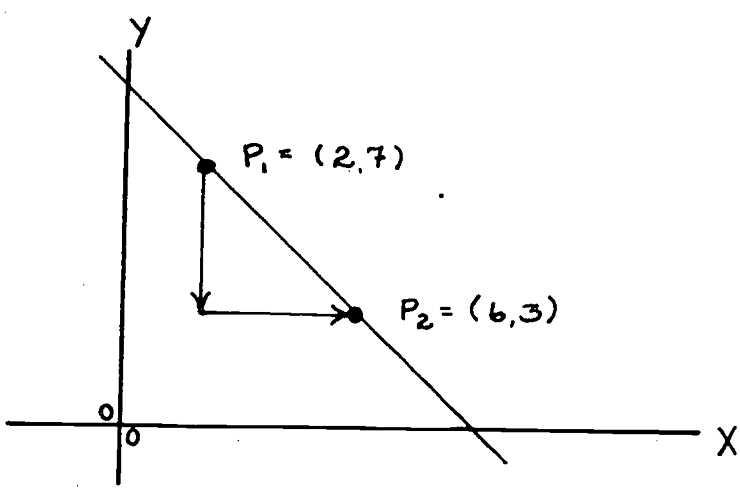
The slope of a vertical line is undefined, because $x_1 = x_2$, or because the run is zero.



Since the run is zero, the denominator of the expression $\frac{\text{rise}}{\text{run}}$ is zero. As you know, the result of division by zero is undefined.

PROBLEM SET 15:

1.



All answers are numbers.

a. $y_2 - y_1 =$ b. $\Delta x =$ c. rise =

d. $x_2 - x_1 =$

e. $\Delta y =$

f. the change in x is ______.

g. run =

h. the change in y is ______.

i. $\frac{\Delta y}{\Delta x} =$

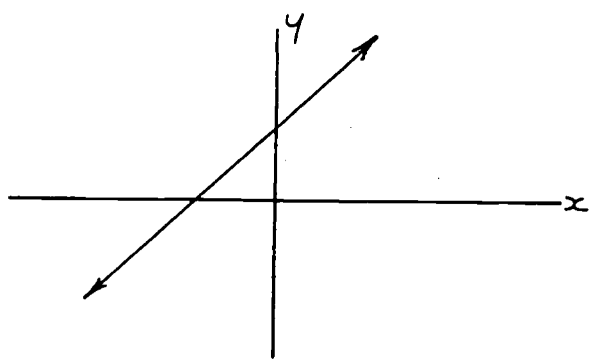
j. $\frac{\text{rise}}{\text{run}} =$

k. The ratio of the change in y to the change in x is ______.

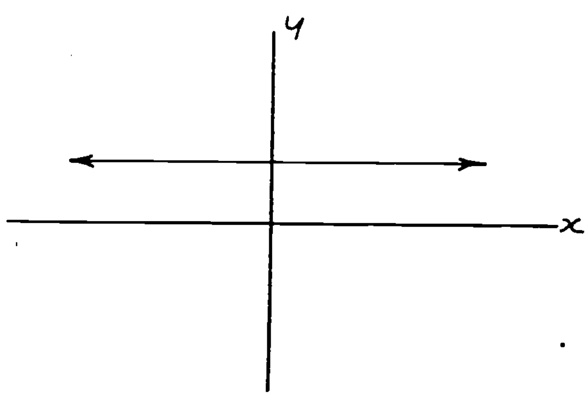
Find the slope of the line connecting the following pairs of points:

2. (8, 6) and (13, -4)
3. (1, 0) and (-1, 0)
4. (0, 2) and (0, -3)
5. (-3, 1) and (-2, 8)
6. (3, -2) and (-5, -4)
7. (8, 13) and (5, -13)
8. Determine from the graphs whether the slope is +, -, 0, or undefined.

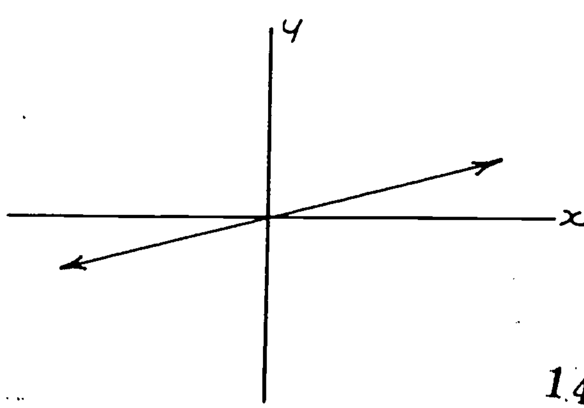
a.



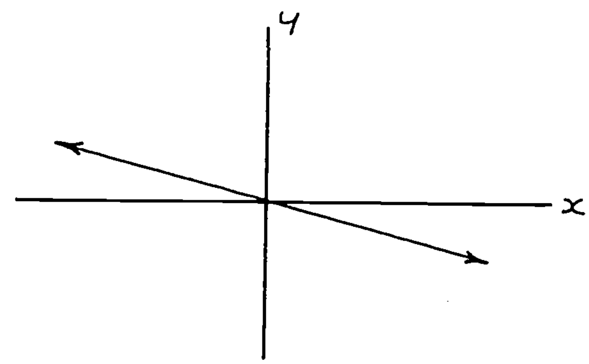
b.



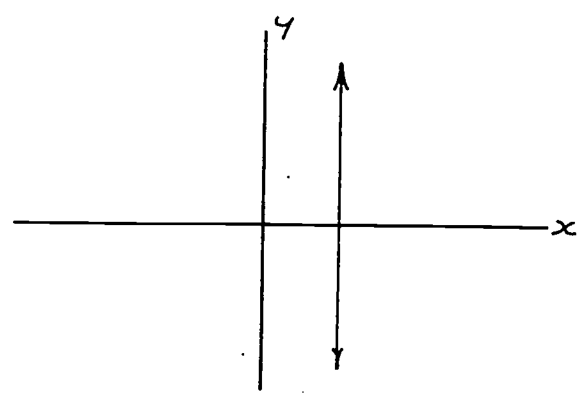
c.



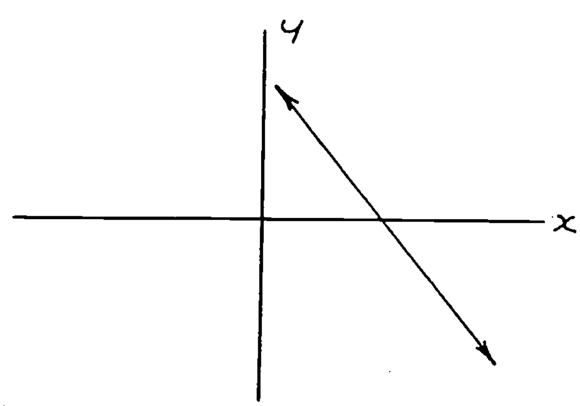
d.



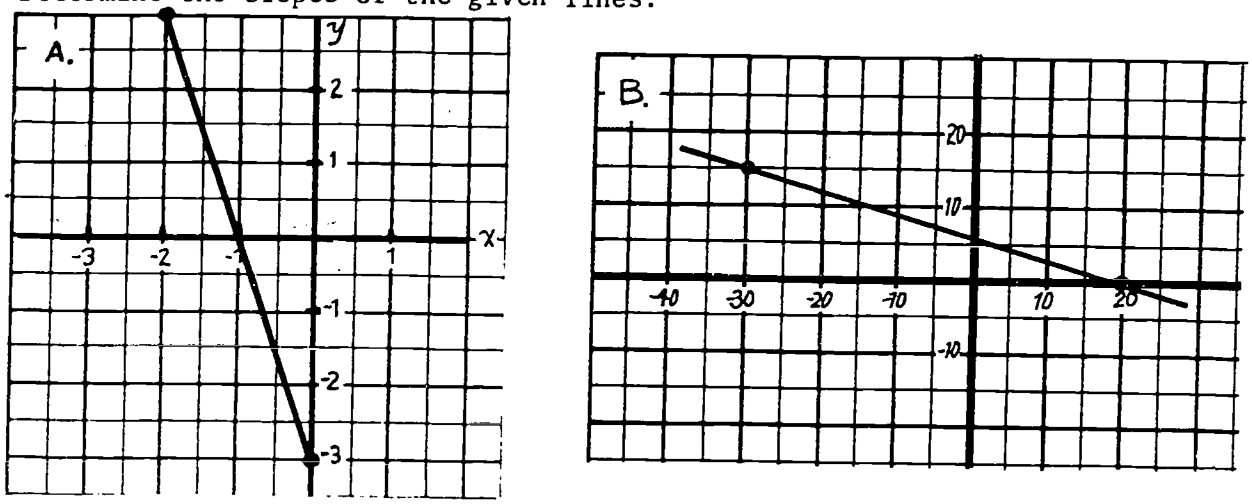
e.



f.



9. Determine the slopes of the given lines.



10. On a single graph draw lines through the point (7, 5) having the following slopes:

$\frac{1}{4}$, -4, 0, -1 HINT: remember that $-4 = \frac{-4}{1}$ rise = Δy, run = Δx

a. What is the slope of the line that descends most rapidly?

b. If the numerical value of the slope is increased, but the sign remains the same, what is the effect upon the line?

c. What appears to be the relationship between the line having a slope of -4 and the line having a slope of $\frac{1}{4}$?

11. Find the slopes of the sides of a triangle having the following vertices:

A = (3, 2), B = (6, 5), C = (3, 8)

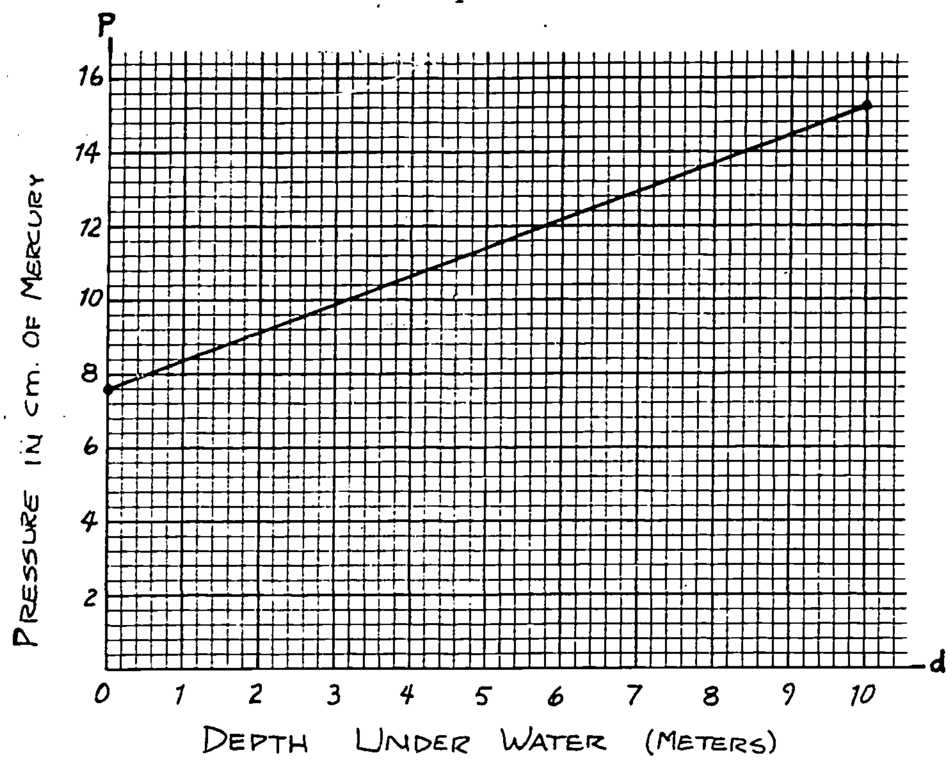
12. Use the definition of slope to show that the points A = (-4, -6), B = (1, 0), and C = (11, 12) lie on a straight line.

13. Points A = (0, 0), B = (2, 3), and C = (10, 15) all lie on the same line.

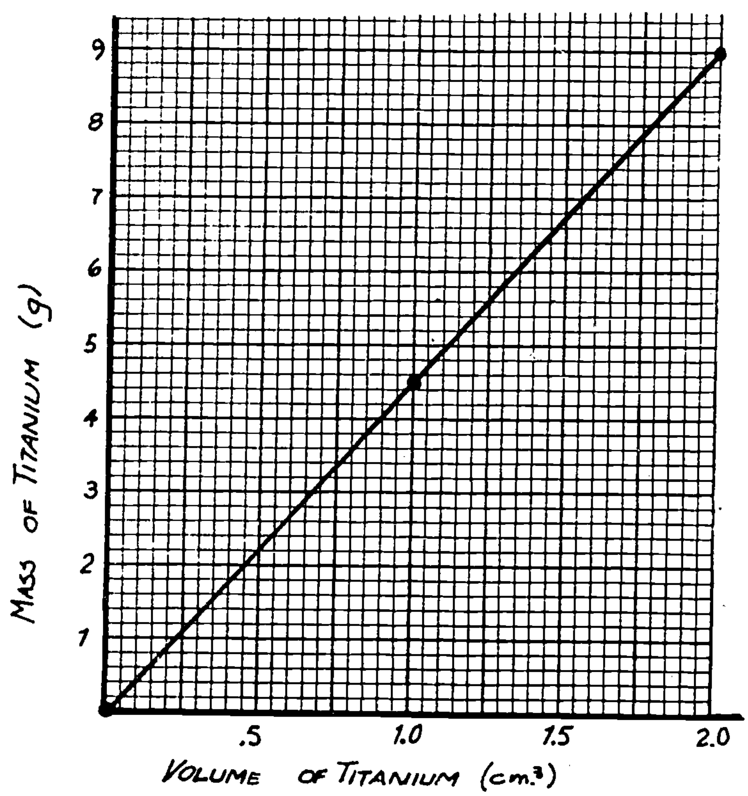
a. Find m for $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

b. What would be the slope of $\overline{AP}$ for any point P on the line?

14. The graph below shows pressure (P) in cm of mercury as a function of depth under water (d) in meters. Find the slope of the line.



15. The graph below shows the mass of the element titanium as a function of its volume. Find the slope of the line. In this case the slope is the density of titanium in units of grams per cm^3.

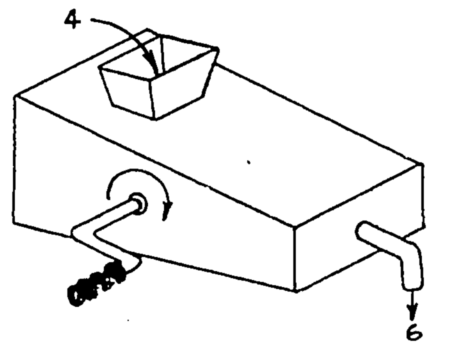


SECTION 16:

16-1 Linear Functions of a Special Type

If the ordered pairs which make up a function graph as points along a straight line, the function is called a linear function. The purpose of this section is to obtain the equations of lines which pass through the origin.

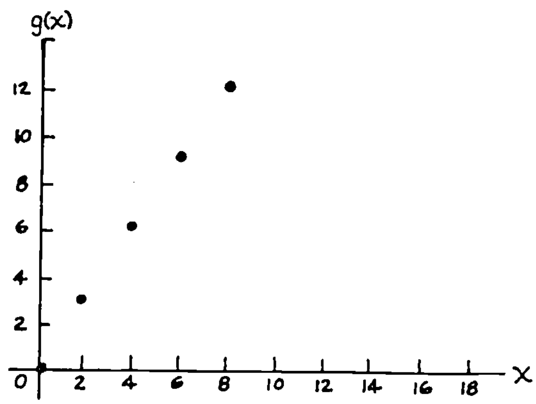
Once again, we turn to that useful device, the function machine. When we feed a certain machine 0, it puts out 0; when we feed it 2, it gives us 3; when we put in 4, we get back 6.



If the input is 6, the output is 9; and if 8 is put in, 12 comes out.

Let us call the input x and the output g(x). The domain is {0, 2, 4, 6, 8} and the range is {0, 3, 6, 9, 12}. The function may be listed as a table or plotted as a graph.

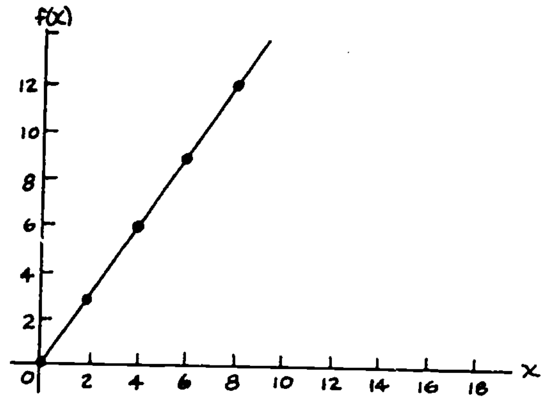
x	g(x)
0	0
2	3
4	6
6	9
8	12



The operation this machine performs is multiplication of the input number by $\frac{3}{2}$. This fact can be expressed as an equation:

$$g(x) = \frac{3}{2}x$$

Consider now the line which passes through the points of the graph of the function g. This line represents the function of f. The domain and range of f are all real numbers.



Recall that we defined the slope by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or

$$= \frac{\Delta x}{\Delta y}$$

Since we may choose any two values of x to use in the formula, we will let $x_1 = 0$, and $x_2 = 8$. Therefore $y_1 = 0$ and $y_2 = 12$. We now calculate the slope.

$$m = \frac{12 - 0}{8 - 0}$$

$$= \frac{12}{8}$$

$$= \frac{3}{2}$$

Observe that the value of the slope is the same number as the coefficient of x in the equation of the function machine,

$$g(x) = \frac{3}{2}x$$

The observation that the slope of the line is equal to the coefficient of x in the equation can be made for any line passing through the origin. This will always be true for linear functions written in this form. This is a very useful property. It allows us to determine the slope of a line by merely looking at its equation.

EXAMPLE:

What is the slope of the line described by the equation $y = (714\pi^2)x$.

SOLUTION:

The coefficient of x is the slope.

$$m = 714\pi^2$$

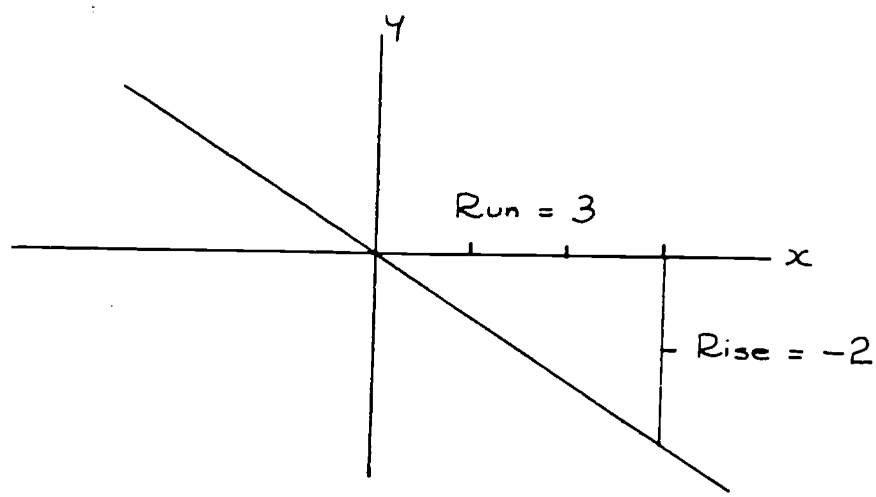
In summary, we can say three things about any equation of the form $y = mx$.

1. The graph of the equation is a line.
2. The line passes through the origin.
3. The slope of the line is the coefficient of x.

16-2 Finding an Equation From a Graph

EXAMPLE:

Find the equation of the line.



SOLUTION:

Since the line passes through the origin, the equation is of the form

$$y = mx$$

The slope m is obtained by the equation

$$m = \frac{\text{rise}}{\text{run}}$$

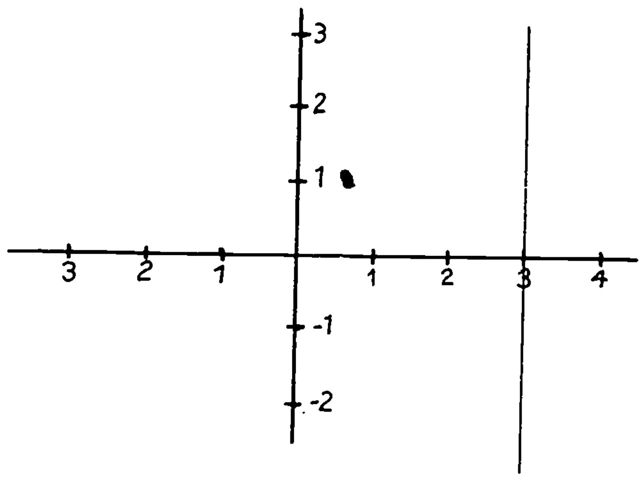
$$= \frac{-2}{3}$$

Substituting this value for m, we obtain

$$y = -\frac{2}{3}x$$

EXAMPLE:

Find the equation of the line



SOLUTION:

An inspection of the graph reveals that the run is zero. Consequently, the slope is undefined because division by zero makes fractions undefined. A further consequence is that we have no value to substitute for m in the equation $y = mx$. We must find another method for writing the equation.

Fortunately there is a simple alternative. We leave y out of the equation entirely and write

$$x = 3$$

An equation of this type may be used to describe any line perpendicular to the x-axis. The constant is simply the x-coordinate of the intersection of the line with the x-axis (see Section 15-4).

16-3 Graphing a Line From Its Equation

EXAMPLE:

Draw the graph of the equation

$$y = -2.5x$$

SOLUTION:

First we recognize that this equation is in the form

$$y = mx$$

Therefore, the graph is a line which goes through the origin and has a slope of -2.5. To graph a line it is necessary to know two points on the line. We already know one, the origin. We can use the information about slope to generate another point.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

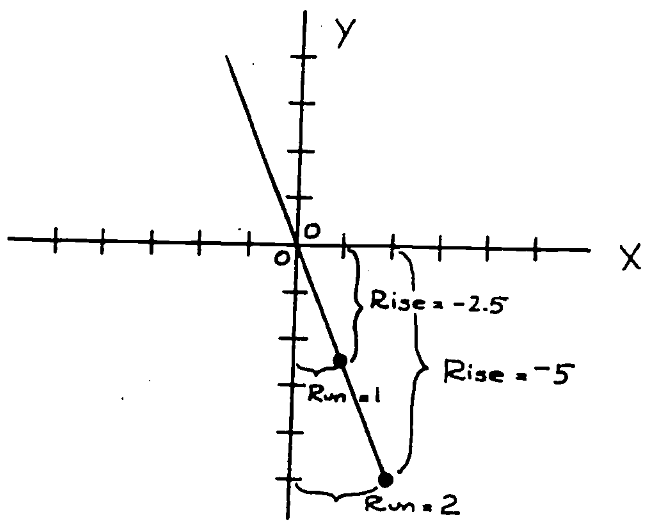
$$= \frac{-2.5}{1}$$

We see that a rise of -2.5 and a run of 1 will give a slope of -2.5. We are now ready to graph the line.

1. We place a point on the origin.

2. Starting from the origin, we go down 2.5 (rise = -2.5) and then right one unit (run = 1). We place our second point at this spot.

3. We draw a line through these points.



The diagram shows that we could have used a rise of -5 and a run of 2 to end up with the same line.

16-4 Simplifying an Equation to Find Its Slope

EXAMPLE:

What is the slope of the line represented by the equation

$$y = 3x - 7x + x(3 - 1.5)$$

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SOLUTION:

$$y = 3x - 7x + x(3 - 1.5)$$
$$= -4x + x(3 - 1.5)$$
$$= -4x + x(1.5)$$
$$= -2.5x$$
$$m = -2.5$$

16-5 Direct Proportion

The idea of direct proportion is an important one. It may surprise you to find out that you have already used the idea in this course. When you learned the technique of dimensional algebra, you learned how to set up direct proportions. The only new thing in this section will be the language connected with direct proportion.

Let's get started with a simple dimensional algebra unit conversion. We will then compare the old terminology of dimensional algebra to the new terminology of direct proportion.

EXAMPLE:

Convert 17 meters to cm.

SOLUTION:

We use the identity

$$100 \text{ cm} = 1 \text{ m}$$

to set up the conversion factor

$$\frac{100 \text{ cm}}{1 \text{ m}} = 1$$

We put the 100 cm on top because we want to end up with cm on top. Next we write

$$(\text{length in cm}) = \frac{100 \text{ cm}}{1 \text{ m}} \ (\text{length in m})$$

When we substitute 17 m for the length in meters we get

$$(\text{length in cm}) = \frac{100 \text{ cm}}{1 \ \cancel{\text{m}}} \cdot 17 \ \cancel{\text{m}}$$
$$= 1700 \text{ cm}$$

The direct proportion statement in the example is

$$(\text{length in cm}) = 100 \frac{\text{cm}}{\text{m}} (\text{length in m})$$

In the language of direct proportion this sentence is read, "Length in cm is directly proportional to length in meters." The conversion factor is called the, proportionality constant. Commonly, proportionality constants are stated without units, although it is okay to include units if you want to.

We will be asking you to translate statements in the language of direct proportion into equations.

EXAMPLE:

Translate the statement, "Volume in cm^3 (c) is directly proportional (k) to volume in m^3 (m).

SOLUTION:

$$c = km$$

In summing up this section we want to point out that simple unit conversions and direct proportions are both examples of linear functions of the type

$$y = mx$$

For example, if we were to graph the relationship between cm and meters, the graph would be linear. The slope would be 100 and the line would go through the origin.

PROBLEM SET 16:

1. List three things that can always be said about an equation of the form

$$y = mx$$

2. Find the slope of the following lines by inspection:

a. $y = \frac{3}{2}x$

b. $y = x$

c. $y = \frac{-21}{5}x$

d. $y = -5x$

e. $y = \frac{103}{62}x$

f. $y = 103x$

3. Reduce the following equations to standard ($y = mx$) form and find the slopes.

a. $y = 3x - \frac{1}{4}x$

b. $y = x(3 - \frac{1}{4}) + 2x$

c. $x - \frac{27}{4}x = y$

d. $y = \frac{81}{3}x - 28x + 3x$

e. $y = x - \frac{41}{7}x$

f. $3x + 7x = y$

g. $y = \frac{2}{3}x - \frac{1}{6}x$

h. $y = (3 - 7)x - \frac{1}{2}x$

i. $30x + 2x = y$

j. $y = x(41 - 11) + 17x$

4. Find the equation of the line passing through the origin (0, 0) and:

a. (3, 3)

b. $(-\frac{1}{2}, 8)$

c. (2, 7)

d. (−3, −10)

e. $(\frac{7}{2}, \frac{3}{2})$

f. $(-\frac{3}{5}, \frac{3}{5})$

g. (3, 0)

h. (0, 3)

i. $(\frac{1}{7}, -\frac{8}{7})$

j. $(\frac{8}{3}, \frac{3}{8})$

5. Graph the following equations.

a. $y = x$

b. $y = \frac{3}{2}x$

c. $y = \frac{1}{4}x$

d. $y = -5x$

e. $y = 3x - 3(x + 2x)$

f. $y = x - \frac{3}{2}x$

6. The equation $d = rt$ (distance is directly proportional to time) is a linear function in the form $y = mx$.

a. d corresponds to __________ in the equation $y = mx$.

b. r corresponds to __________.

c. t corresponds to __________.

7. a. Translate the statement, "The mass (m) of a sample of an element is directly proportional (k) to the number (N) of atoms in the sample," into an equation.

b. What variable corresponds to f(x) or y?

c. What variable corresponds to m?

d. What variable corresponds to x?

For Problems 8 through 21 write equations which describe the given statements.

8. The volume (V) of a quantity of gas at constant pressure is directly proportional (k) to its temperature (T).

9. The volume (V) of pine wood is directly proportional (k) to its mass (m).

10. The barometric pressure (B) is directly proportional (k) to the height (H) of a column of mercury in an evacuated tube.

11. The gauge pressure (g) experienced by an underwater diver is directly proportional (k) to his depth (d).

12. Length in feet (f) is directly proportional to length in meters (m).

13. Force in pounds (l) is directly proportional to mass (m) in kg on the surface of the earth.

14. Distance in miles (M) is directly proportional to distance in kilometers (K).

15. Force (F) is directly proportional to acceleration (a).

16. Pressure (P) is directly proportional to force (F) as long as the area remains constant.

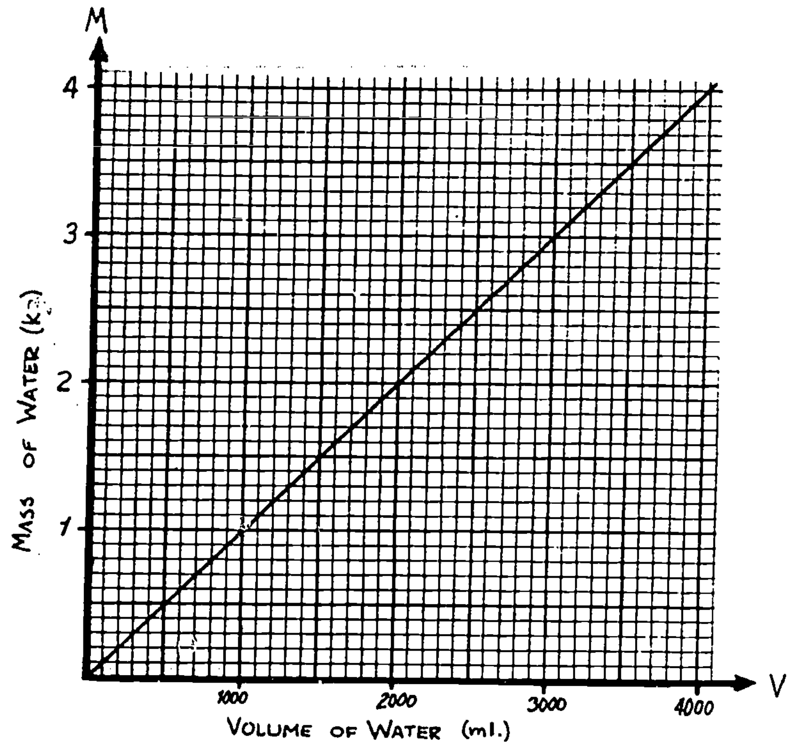
17. The interest (I) earned by a savings account is directly proportional to the amount (A) in the account.

18. Distance in furlongs (f) is directly proportional to distance in decimeters (d).

19. The volume (V) of a cubical object is directly proportional to the cube of one side (S^3).

20. The mass (m) of water is directly proportional to the volume of water (V).

21. The graph below relates the mass of water (m) in kg to its volume (v) in ml. Notice that the scales for the two axes are different.



a. Write a sentence in the language of direct proportions. which describes the graph.

b. Find the slope of the line. Hint: It is not 1 because the scales of the two axes are different.

c. What is the proportionality constant?

d. Write an equation of the form y = mx which describes the graph.

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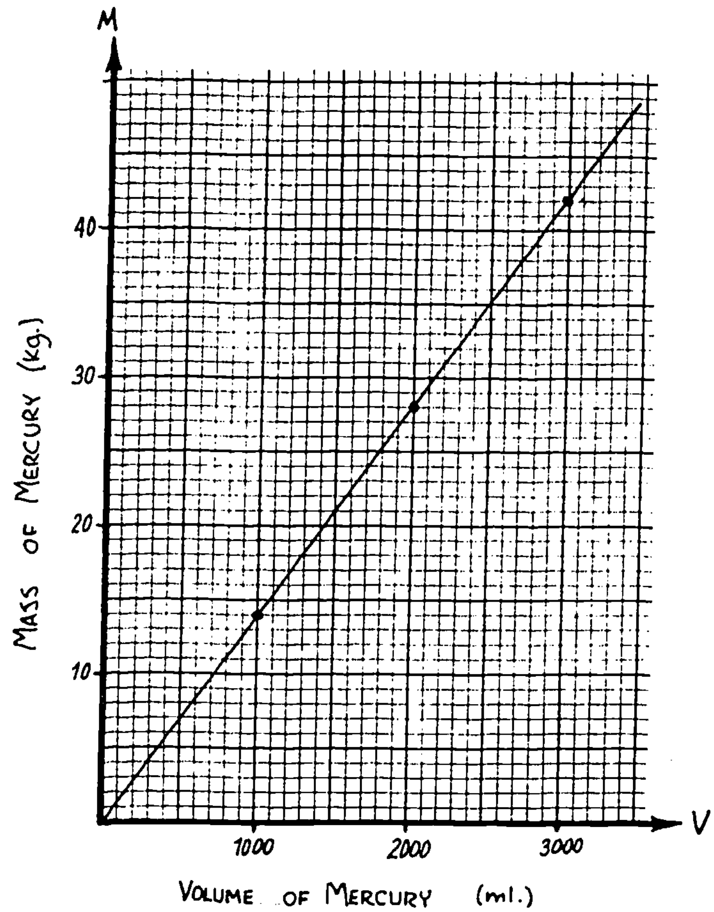
22. The graph below relates the mass of mercury (m) to its volume (v).



a. Write a sentence which uses the language of direct proportions to describe the graph.

b. Find the slope of the line.

c. What is the proportionality constant?

d. Write an equation of the form y = mx which describes the graph.

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SECTION 17:

17-1 More Linear Functions

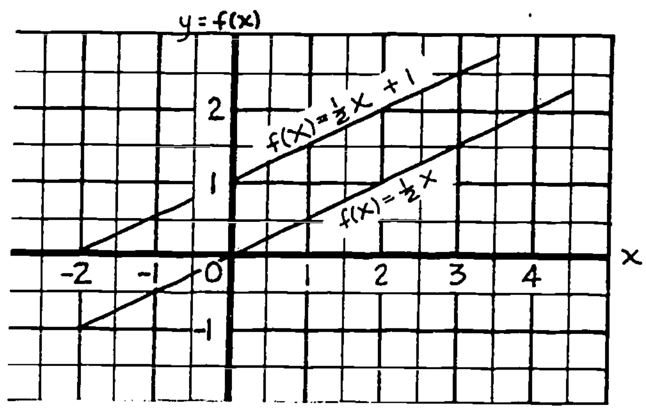
The graphs of many linear functions do not pass through the origin. Consider the function

$$f(x) = \frac{1}{2}x + 1 .$$

The function is shown below as a table and as a graph. Also shown on the graph is the function

$$f(x) = \frac{1}{2}x .$$

x	$f(x) = \frac{1}{2}x + 1$
0	1
1	$1\frac{1}{2}$
2	2
-1	$\frac{1}{2}$
-2	0



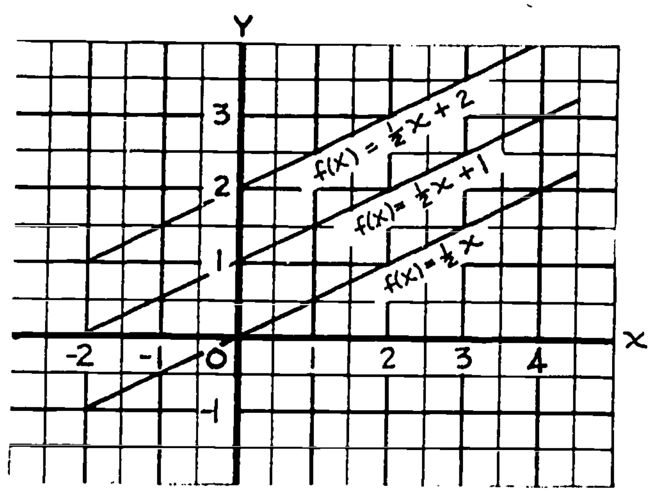
How does the graph of $f(x) = \frac{1}{2}x + 1$ differ from the graph of $f(x) = \frac{1}{2}x$? The slope of both lines is $\frac{1}{2}$, as can be seen by determining the rise and run of each. However, the graph of $f(x) = \frac{1}{2}x + 1$ crosses the y-axis at the point (0, 1) instead of at the origin.

Consider another function,

$$f(x) = \frac{1}{2}x + 2.$$

This function is tabulated and graphed on the next page. The graphs of $f(x) = \frac{1}{2}x + 1$ and $f(x) = \frac{1}{2}x$ are shown again for comparison.

x	$f(x) = \frac{1}{2}x + 2$
0	2
1	$2\frac{1}{2}$
2	3
-1	$1\frac{1}{2}$
-2	1



The line representing $f(x) = \frac{1}{2}x + 2$ has a slope of $\frac{1}{2}$, the same slope as the other two lines; but it crosses the y-axis at the point (0, 2).

The observations we have made about the functions graphed above enable us to make general statements about equations of the form $f(x) = mx + b$ (or $y = mx + b$). The coefficient of x is the slope and has the same meaning as in equations of the form $y = mx$. The term b is called the vertical intercept or the y-intercept, because it is the value of y at the point where the line crosses the y-axis. This can be shown by noting that $x = 0$ where the line crosses the y-axis and substituting $x = 0$ into the equation

$$\begin{aligned} y &= mx + b \\ &= m \cdot 0 + b \\ &= b \end{aligned}$$

EXAMPLE:

Determine the slope and y-intercept of the equation

$$y = -7x + 3.4$$

without drawing a graph.

SOLUTION:

The equation $y = -7x + 3.4$ is of the form $y = mx + b$. The slope is m, which is -7. The y-intercept is b, which is 3.4.

The next example illustrates how to draw the graph of an equation of the form y = mx + b.

EXAMPLE:

Draw the graph of the equation

$$y = \frac{1}{4}x + 2$$

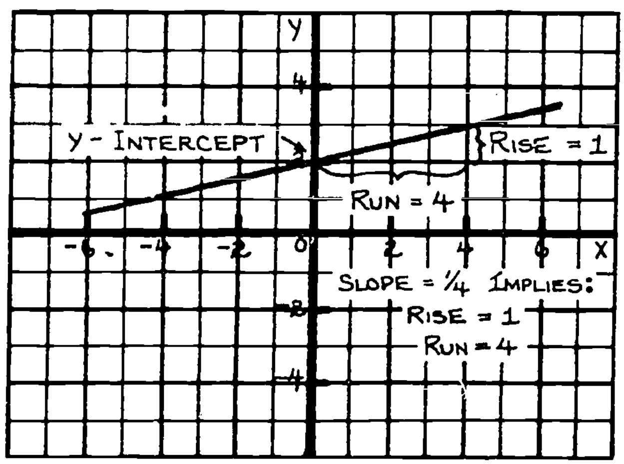
using the slope and y-intercept.

SOLUTION:

The equation is of the form y = mx + b, and therefore represents a linear function.

The y-intercept is 2, meaning the graph passes through the point (0, 2).

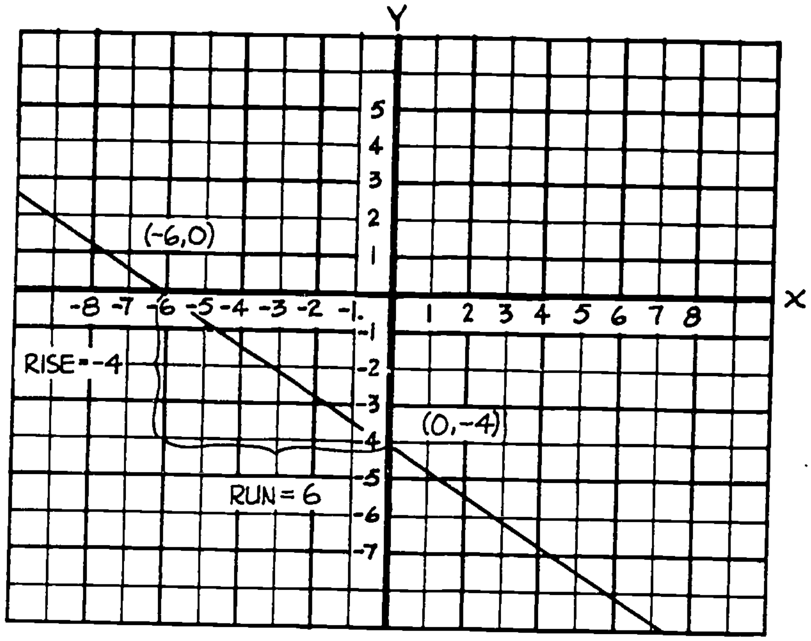
The slope is $\frac{1}{4}$, so another point on the line may be located by measuring from the point (0, 2) a rise of 1 and a run of 4. The two points are connected to give the desired line.



Frequently we are required to find the equation represented by a graph. If the graph is a straight line, we may find the equation using the sort of reasoning used to solve the following problem.

EXAMPLE:

Determine the equation of the line graphed on the following page.



SOLUTION:

Since the graph is a straight line, its equation may be written in the form y = mx + b.

From the intersection of the line with the x-axis (-6, 0) to the intersection with the y-axis (0, -4) the rise is -4 and the run is 6.

The slope is found from the equation

$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{-4}{6}$$

$$= -\frac{2}{3}.$$

The y-intercept is b, which is -4. Therefore, the equation of the line is

$$y = -\frac{2}{3}x - 4.$$

PROBLEM SET 17:

1. Determine the slope (m) and the y-intercept (b) for the lines by inspection.

   a. $y = -\frac{3}{2}x + 3$

   b. $y = 4x - 7$

   c. $y = 7x + \frac{8}{3}$

   d. $y = -\frac{12}{7}x - \frac{12}{7}$

   e. $y = 3x$

   f. $y = \frac{18}{5}$

2. Convert to the form $y = mx + b$ and determine the slope and y-intercept by inspection.

   a. $y = 3(x - 7)$

   b. $y = \frac{3}{2}(2x + 3) - \frac{1}{2}$

   c. $y = 17x - 8$

   d. $y = \frac{8}{3}(x - 3) + 8$

   e. $y = \frac{7}{2}(2x + 1) - 7x$

   f. $y = -16(\frac{1}{4} - x) + 3x - 2$

3. Graph the following lines.

   a. $y = 2x + 2$

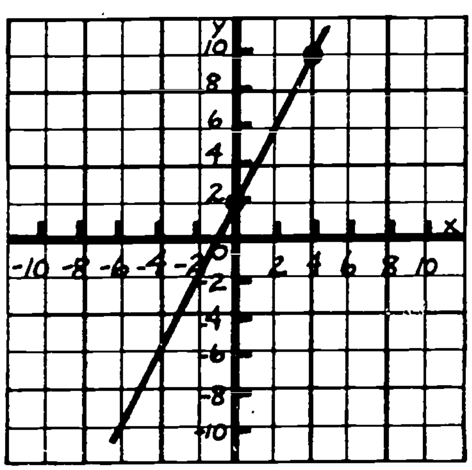
   b. $y = -\frac{1}{2}x + 1$

   c. $y = x - 2$

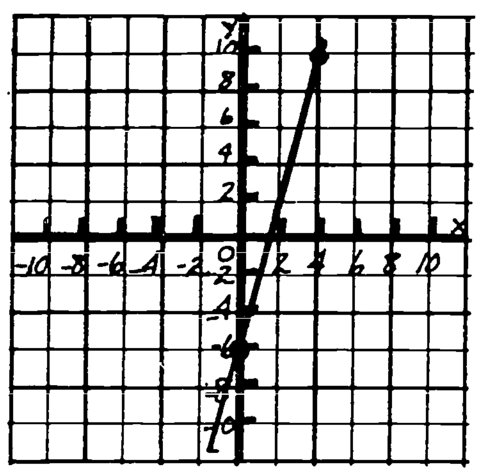
   d. $y = -\frac{3}{2}x - 3$

4. Determine the equations describing the following lines.

   a.

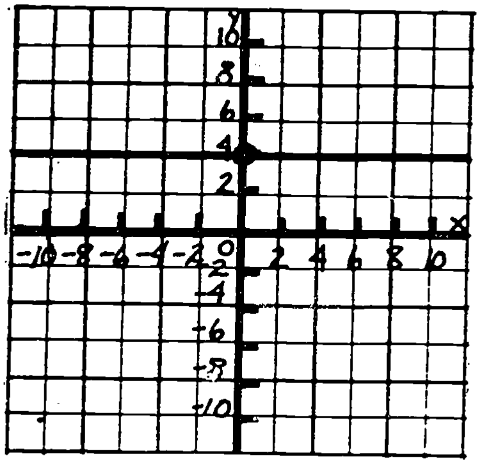


   b.

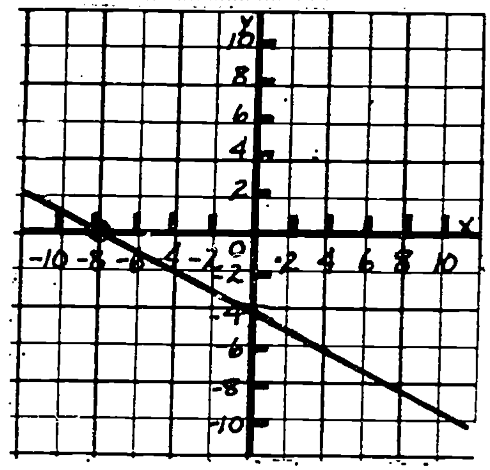


16.i

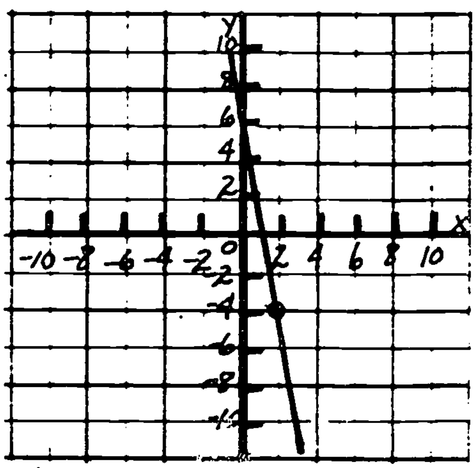
c.


y
10
8
6
4
2
x
-10 -8 -6 -4 -2 0 2 4 6 8 10
-2
-4
-6
-8
-10

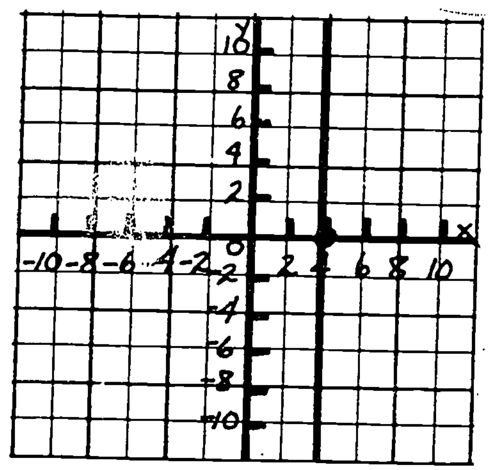
d.


y
10
8
6
4
2
x
-10 -8 -6 -4 -2 0 2 4 6 8 10
-2
-4
-6
-8
-10

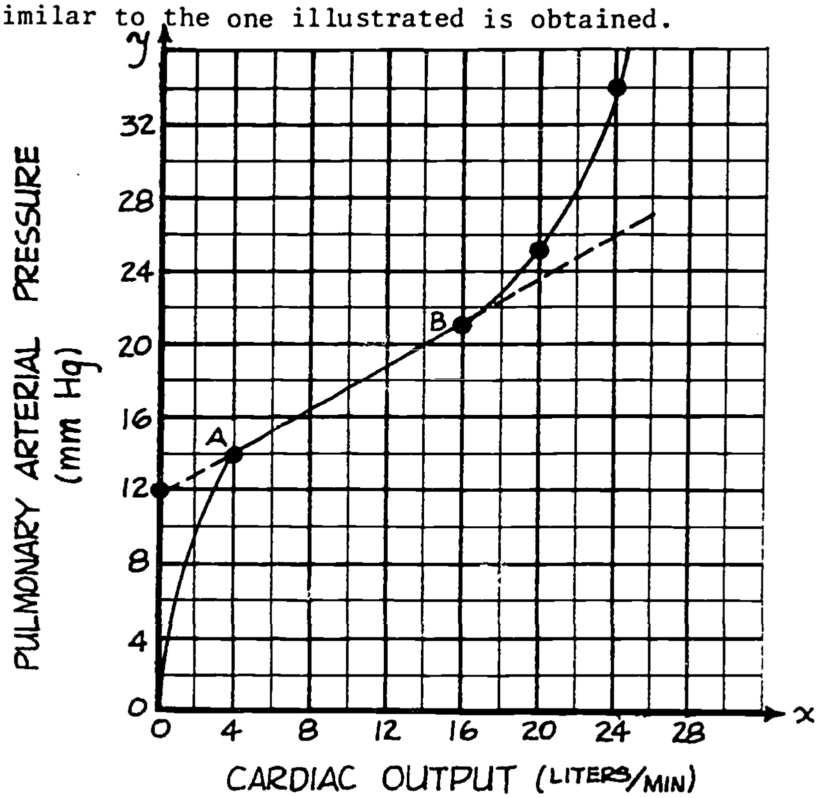
e.


y
10
8
6
4
2
x
-10 -8 -6 -4 -2 0 2 4 6 8 10
-2
-4
-6
-8

f.


y
10
8
6
4
2
x
-10 -8 -6 -4 -2 0 2 4 6 8 10
-2
-4
-6
-8
-10

5. When pulmonary arterial pressure is plotted as a function of cardiac output, a graph similar to the one illustrated is obtained.



Notice that the graph approximates a straight line between 4 and 16 liters/min.

a. Give the coordinates for the two end points (A and B) of this straight segment of the graph.

b. What sign would you expect the slope for the line through these two points to have? Why?

c. Find the slope of $\overline{AB}$.

d. From the graph, find the coordinates of the y-intercept of the approximating straight line.

e. Write an equation for $\overline{AB}$ in the form $y = mx + b$.

f. Using the equation found in Part e, find the value of pulmonary arterial pressure when cardiac out put equals 6, 10, 12 and 15 liters/min. Compare your results with the graph as a check.

g. For what domain does the line found in Part e approximate the cardiac output-pulmonary arterial pressure relationship?

h. Find the range associated with the domain in Part g.

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SECTION 18:

18-1 Finding the Equation of a Line from Two Points

In the preceding sections, you learned how to determine the equation of any line by finding the slope and y-intercept from its graph. It is also possible to determine the equation, without drawing a graph, from a knowledge of just two points that lie on the line. This technique is often useful in solving biomedical problems.

Suppose we know that two points on a line are

$$(x_1, y_1) = (3, 4) \quad \text{and}$$

$$(x_2, y_2) = (8, 9)$$

and we are asked to find the equation of the line. We recall that the equation of the line is of the form

$$y = mx + b,$$

and that the slope m is defined by the equation

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Knowing (x_1, y_1) and (x_2, y_2), we can calculate m.

$$m = \frac{9 - 4}{8 - 3}$$

$$= \frac{5}{5}$$

$$= 1$$

$$\text{and} \quad y = 1 \cdot x + b$$

$$\text{or} \quad y = x + b$$

The y-intercept, b, may be found by substituting either pair of coordinates into the last equation. We substitute the pair $x_1 = 3$ and $y_1 = 4$.

$$4 = 3 + b$$

$$4 - 3 = 3 - 3 + b$$

$$1 = b$$

Therefore the equation of the line is

$$y = x + 1.$$

If we had substituted the other pair of coordinates, the result would have been the same.

$$x_2 = 8 \quad \text{and} \quad y_2 = 9$$

$$y = x + b$$

$$9 = 8 + b$$

$$9 - 8 = 8 - 8 + b$$

$$1 = b$$

$$y = x + 1$$

PROBLEM SET 18:

1. Refer to the accompanying graph. Answer true or false for each of the following statements.

   a. Line A and line B have equal slopes but different y-intercepts.

   b. Line A and line B have equal y-intercepts but different slopes.

   c. Line A and line B have different slopes and different y-intercepts.

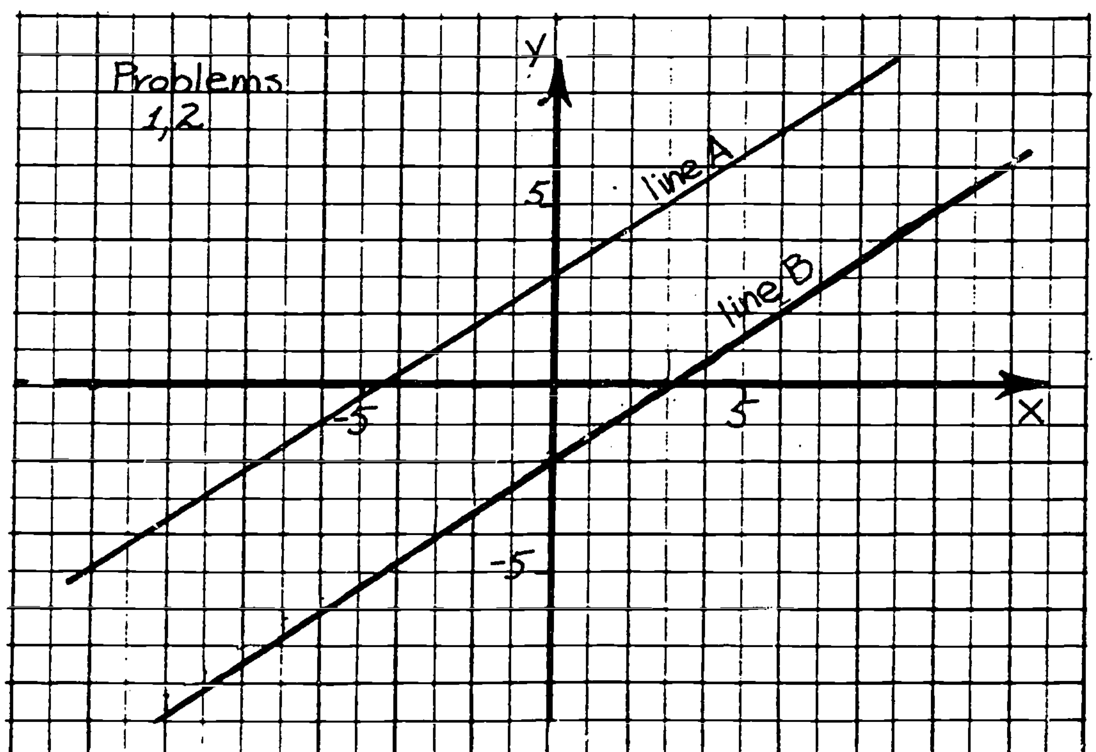
2. Refer to the accompanying graph.

   a. What is the slope of line A?

   b. What is the slope of line B?

   c. What is the y-intercept of line A?

   d. What is the y-intercept of line B?



3. Refer to the accompanying graph. Answer true or false for each of the following statements.

a. Line A and line B have equal slopes but different y-intercepts.

b. Line A and line B have equal y-intercepts but different slopes.

c. Line A and line B have different slopes and different y-intercepts.

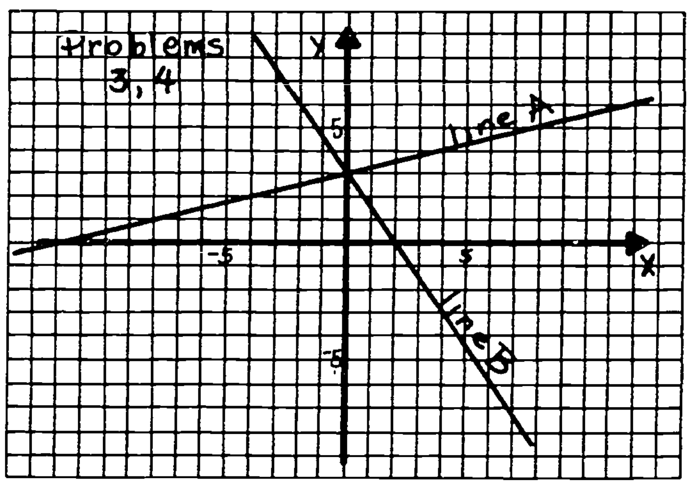
4. Refer to the accompanying graph.

a. What is the slope of line A?

b. What is the slope of line B?

c. What is the y-intercept of line A?

d. What is the y-intercept of line B?



Find the equation of the line passing through each pair of points.

5. (4, 6) and (20, 30)

6. (8, 8) and (10, 8)

7. (1, 5) and (2, 6)

8. (3, 4) and (9, 22)

9. (-5, 3) and (5, 18)

10. If a line has a slope of 6 and passes through (1, 9), what is its equation? (Hint: First write the equation as $y = 6x + b$.)

11. If a line has a slope of -3 and passes through $(4, 7)$, what is its equation?

12. If a line has a y-intercept of -4 and passes through $(8, 12)$, what is its equation? (Hint: First write the equation as $y = mx - 4$.)

13. If a line has a y-intercept of 7 and passes through $(7, 21)$, what is its equation?

14.

x	y
-3	-4
-2	-1
-1	2
0	5
1	8
2	11
3	14

The accompanying table lists the coordinates of some of the points on a line.

a. Graph the line.

b. What is the y-intercept of the line?

c. What is the formula for the slope of a line?

d. Use the formula to determine the slope of this line.

e. What is the equation of this line?

SECTION 19:

19-1 The Expansion of Air With an Increase in Temperature

The manufacturers of rubber rafts warn that their rafts not be allowed to sit around in the hot sun for extended periods. They make this warning because the heat of the sun is sufficient to rupture the raft due to the tendency of air to expand when heated.

On the other hand, an inflated inner tube which is left outside overnight will appear to be less inflated in the morning. This happens because air tends to contract as it is cooled.

By now you should have performed or witnessed an investigation into the relationship between the volume of a quantity of trapped air and its temperature. In math you will graph the data and compare your results with "Charles Law." Charles Law is the name given to the formula which describes the relationship between volume and temperature for a quantity of trapped gas at constant pressure.

19-2 Graphing the Data

Below is a sample set of data similar to that which you have.

TEMPERATURE (°C)	LENGTH OF AIR COLUMN (cm)
56	18.0
49.5	17.6
43.5	17.3
38.5	16.95
33	16.6
27	16.3
22.5	16.0
17	15.7
10	15.3
2.5	14.9

Almost certainly, your data will not have a single ordered pair identical to one in the table above. This is a result of the very small likelihood of starting out with the same quantity of trapped air. Room temperature is in the neghborhood of 22.5° C.

The length of the air column at 22.5° C was 16.0 cm when we ran the experiment. This was our initial amount of air. If your initial quantity of air differs at all from this, then none of your ordered pairs will match ours.

How then are we to make sense out of all of the different sets of data? We will do it by way of Charles Law. It makes two predictions about the appearance of the graph of every set of data.

1. The graph of the points will be linear.

2. The line will go through the point (-273° C, 0).

We will talk more about the significance of -273° C later, but now we wish to see if the predictions of Charles Law are accurate. To do so, we will need to graph the data. To insure that the point (-273° C, 0 cm) will appear on the graph, we add this ordered pair to the sample data set. Next we follow the procedure described in Section 11 for scaling the graph, using our sample data. You will be following the same procedure, using your own data.

1. We find ΔT and ΔV for each column.

$$\begin{aligned} \Delta T &= 56 - (-273) \\ &= 329 \\ \Delta V &= 18 - 0 \\ &= 18 \end{aligned}$$

2. We find the best scale for each axis.

<u>ΔT: Long Axis</u>

$\frac{329}{24} = 13.7 \ldots$

Scales: 15° C per big unit

3° C per little unit

<u>ΔV: Short Axis</u>

$\frac{18}{18} \approx 1$

Scales: 1 cm per big unit

.2 cm per little unit

<u>ΔT: Short Axis</u>

$\frac{329}{18} = 18.27 \ldots$

Scales: 20° C per big unit

4° C per little unit

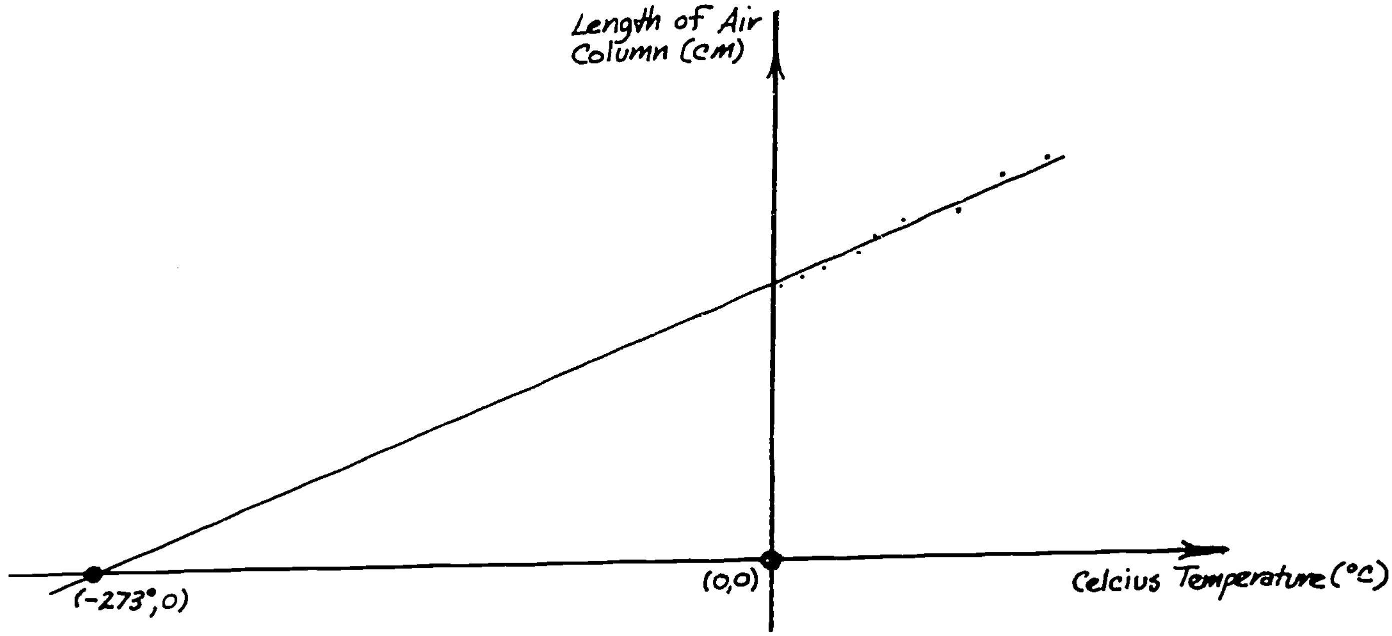
<u>ΔV: Long Axis</u>

$\frac{18}{24} = .75$

Scales: 1 cm per big unit

.2 cm per little unit

3. An examination of the possibilities reveals that putting temperature on the long axis and volume on the short axis is the best choice. Of course the scaling system you use on your own graph will depend on your particular data. It may turn out to be the same as the one presented here or it may not. What we can say is that your graph should resemble the following one in its general appearance.



As you can see it is possible to draw a line through (-273, 0) which fits the data points fairly closely.

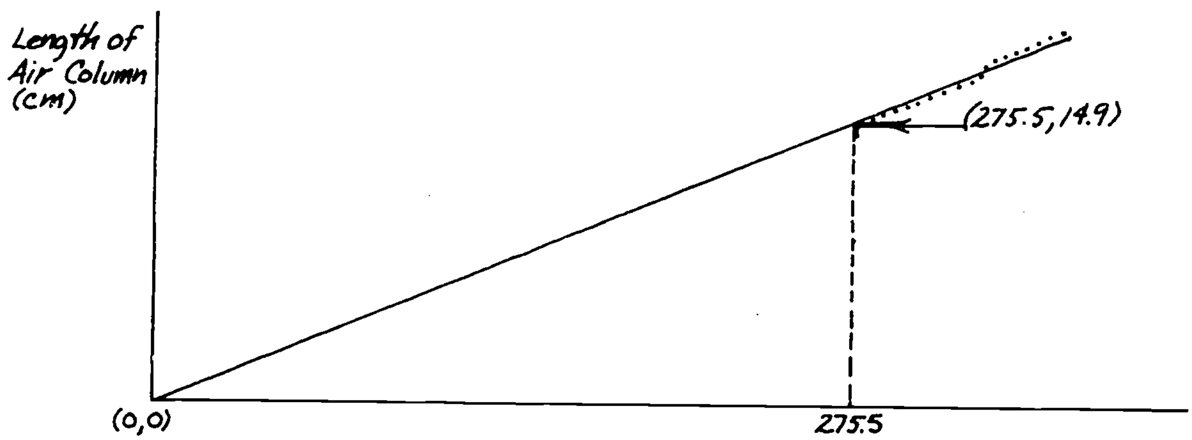
19-3 More About the Significance of -273° C

Negative 273° C or more exactly -273.16° C is called "absolute zero." It is the temperature at which all molecular motion would be completely stopped if it were possible to cool anything to this temperature. Curiously, it is impossible to chill anything to absolute zero. Unfortunately, we cannot explain why this is true. The reasons are too complex for you to understand with your present background. As you can see from the length of the temperature axis on the graph, absolute zero is very much colder than the temperatures we commonly experience. In fact, the lowest temperature ever recorded in the Antartic is only -88 ° C. In the neighborhood of absolute zero everything is frozen solid except the inert gas helium, which is liquid.

19-4 The Kelvin or Absolute Temperature Scale

Degrees above absolute zero is the natural way to measure the temperature of gases. Temperature Celcius is converted to absolute temperature by adding 273 (or sometimes more precisely 273.16) to temperature Celcius. For example 0° C, the freezing point of water, is 273° absolute. Absolute temperature is also known as "Kelvin" temperature, after Lord Kelvin, a famous British physicist.

Notice what happens when the temperatures in our data set are converted to degrees Kelvin or absolute. -273° C becomes 0° K. Consequently, the line on the graph will go through the origin (0° K, 0 cm) when the absolute temperature scale is used.



Notice that our original data point (2.5° C, 14.9 cm) has now become (275.5 °K, 14.9 cm). This is true because 2.5° C = 2.5 + 273° K = 275.5° K. The conversion of Celcius temperatures to absolute greatly simplifies the equation used to describe the relationship between volume and temperature. As you know any line which goes through the origin may be described by an equation of the form

$$y = mx$$

Since we are concerned with temperature (T) and volume (V), we will use the letter T for x and V for y to write

$$V = mT$$

This is an algebraic statement of Charles Law.

17:

PROBLEM SET 19:

In all problems use -273° C for absolute zero.

1. Express 0° C as a Kelvin temperature.

2. Express 0° K as a Celcius temperature.

3. 10° C = ____ ° K.

4. 250° K = ____°C.

5. -40° C = ____°K.

6. 400° K = _____°C.

7. a. Graph the results of your own Charles Law investigation. Include absolute zero on your graph.

   b. Draw a line through absolute zero and the rest of your points.

SECTION 20:

20-1 The Relationship Between Pressure and Volume for Gases

In this Section we will show how skills associated with linear functions may be used to analyze a nonlinear relationship. Our pattern of attack will be similar to what we have just been through for Charles Law.

However, before beginning this math section you must complete all of the calculations associated with the "Effect of Pressure Change on the Volume of Confined Gas" Laboratory Activity 16 in Science. When this is done, your data will be similar to that shown in the tables below.

SAMPLE DATA:

PLUNGER/SYRINGE ACTIVITY

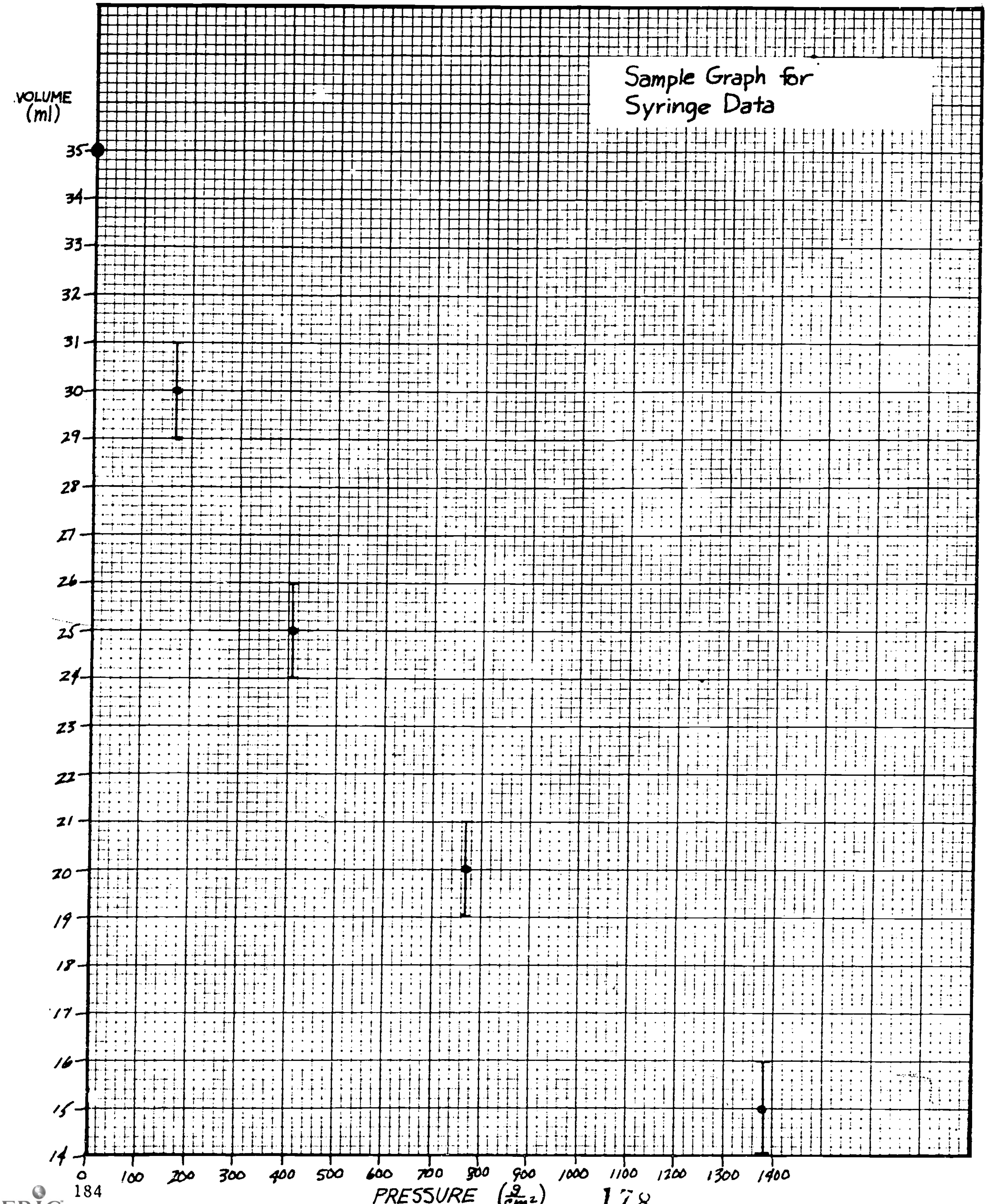
Pressure (g/cm^2)	Volume (ml)
0	35.0
170	30 ± 1
410	25 ± 1
770	20 ± 1
1380	15 ± 1

LIQUID FILLED TUBE ACTIVITY

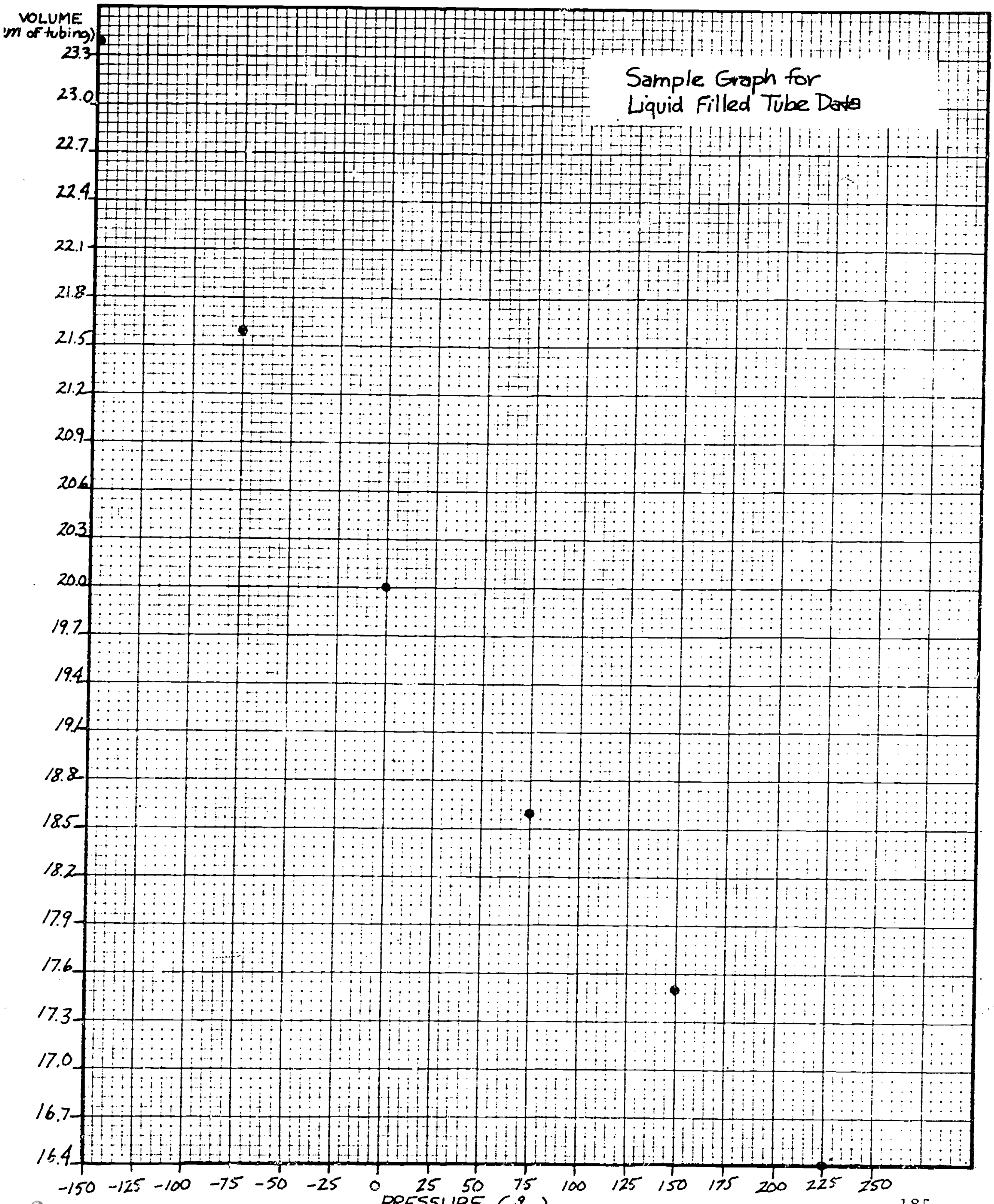
Pressure (g/cm^2)	Volume (cm of tube length
225	16.4
150	17.5
75	18.6
0	20.0
-75	21.6
-150	23.4

All pressures should be converted to g/cm^2. The units used for volume depend upon the particular lab activity performed. If the plunger-syringe activity was done, then volume should be in ml. Also, each volume, other than the initial one of 35 ml, should be stated as a range of imprecision. If you performed the activity which invovles a liquid filled tube, then the units of volume should be "cm of tubing." This is not a standard unit of volume to be sure, but it saves us the trouble of converting all of the tube lengths to cm^3.

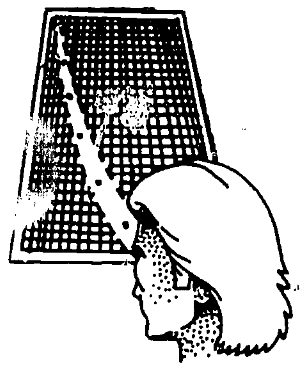
On the following pages we have sample graphs of P vs V for each experiment. Notice that the points do not lie on a straight line. This feature of the graphs may be best seen by looking at them from a low angle.



Sample Graph for
Syringe Data
VOLUME
(ml)
35
34
33
32
31
30
29
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
PRESSURE (g/cm²)



VOLUME
(cm of tubing)
23.3
23.0
22.7
22.4
22.1
21.8
21.5
21.2
20.9
20.6
20.3
20.0
19.7
19.4
19.1
18.8
18.5
18.2
17.9
17.6
17.3
17.0
16.7
16.4
Sample Graph for
Liquid Filled Tube Data
-150 -125 -100 -75 -50 -25 0 25 50 75 100 125 150 175 200 225 250
PRESSURE ($\frac{g}{cm^2}$)



Looking at Graphs From a Low Angel to Best See Curvature

Since we don't know how to derive equations for any other kinds of functions but linear ones, it would appear (superficially) that we are stuck, stymied and stopped. However, this isn't so.

First of all we stop pretending that nobody knows the equation which relates pressure and volume at constant temperature. In fact, it is well known. It can be found in many places later on in this book and in any other chemistry or physics text that you might care to look in. Discovery is certainly an important part of science. However, no scientist who wants to uncover new facts will spend all of his time rediscovering already known facts. He makes good use of a library to make sure that he begins where others have stopped. His insurance policy is a skeptical attitude. He always reserves the right to reproduce somebody else's results. However, he will take many things on faith. One of those things is Boyle's Law, the law which relates pressure and volume for gases at a constant temperature. It states that

$$PV = k$$

or, in other words, the product of pressure (P) and volume (V) is a constant (k).

20-2 Straightening Out a Curve

We are now in the position of checking out the work of others. There is a way to do this which is more informative than multiplying all of the P's and V's. We will use our newly acquired skills with linear functions.

Suppose that we first find $\frac{1}{V}$ for each volume. Then we plot $\frac{1}{V}$ as a function of P. We assert that if $PV = k$, then the plot of P vs $\frac{1}{V}$ will be linear. Furthermore the extent to which the points deviate from linearity is an indication of the degree of disagreement between observation and theory.

20-3 A Derivation of the Assertions in Section 20-2

We feel an obligation as mathematicians to demonstrate, mathematically, the truth of the assertion in the previous section.

If

$$k = PV$$

then the equality will remain true when both sides are divided by Vk.

$$\frac{\cancel{k}}{V\cancel{k}} = \frac{P\cancel{V}}{\cancel{V}k}$$

When the canceling is done, we have

$$\frac{1}{V} = \frac{P}{k}$$

or equivalently

$$\frac{1}{V} = \left(\frac{1}{k}\right)P$$

because multiplying by $\frac{1}{k}$ is equivalent to dividing by k. Now recall that we will plot $\frac{1}{V}$ on the y-axis. We substitute y for $\frac{1}{V}$ in the equation to get

$$y = \left(\frac{1}{k}\right)P$$

Similarly we substitute x for P to get

$$y = \left(\frac{1}{k}\right)x$$

This is a linear equation of the form y = mx where m in this case is $\frac{1}{k}$. So we see that Boyle's Law predicts that a plot of $\frac{1}{V}$ as a function of pressure should be a straight line.

PROBLEM SET 20:

1. What units should all pressure data be stated in?

2. What units should volume data from the syringe activity be stated in?

3. What units should volume data from the liquid filled tube activity be stated in?

4. How does a scientist make sure that he doesn't spend all of his time rediscovering already known facts?

5. How does a scientist protect himself from the possibility of placing false trust in somebody else's results?

6. (True or False) If the relationship between variables is nonlinear, then skills associated with linear functions cannot be used to analyze the relationship.

7. Graphs of P vs V are (linear, nonlinear)

8. We expect a graph of P vs $\frac{1}{V}$ to be (linear, nonlinear).

9. What is the name given to the relationship between pressure and volume at constant temperature?

10. Where else is it possible to find the mathematical formula which relates pressure and volume at constant temperature?

11. Calculate $\frac{1}{V}$ for each of your volume measurements. If you performed the syringe activity calculate $\frac{1}{V}$ only for the midpoint of your range of imprecision.

12. Scale and graph your own set of $(P, \frac{1}{V})$ ordered pairs.

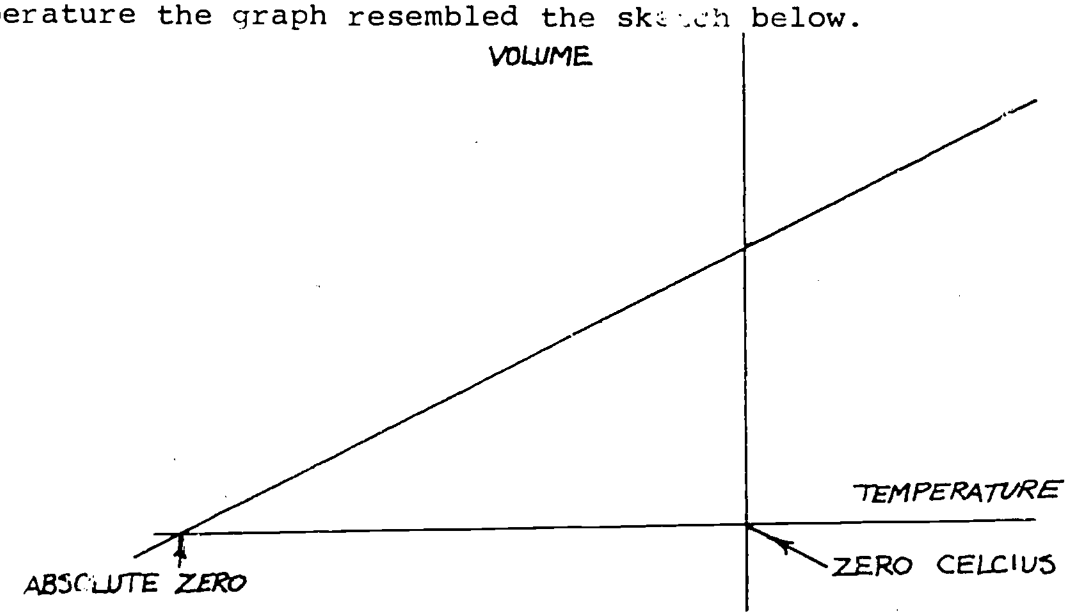
SECTION 21:

21-1 Another Feature of Functions of the Form $y = mx$

In Section 20-3 we mentioned one characteristic of a function of the form $y = mx$. It's graph will always be linear. Furthermore, we claimed that a graph of P vs $\frac{1}{V}$ would also be linear. This is due to Boyle's Law which claims that $PV = k$. This equation may be rewritten in the form $y = mx + b$ as $\frac{1}{V} = \frac{1}{k} P$. In other words if $\frac{1}{V}$ is allowed to be the y-coordinate and P is allowed to be the x-coordinate then we expect the graph to be linear. By now you should have completed the graph of $\frac{1}{V}$ as a function of P. If everything was done properly, then you have by now, confirmed the truth of this prediction.

Now we are in a position to deal with another feature common to all functions of the form $y = mx$. The line will always go through the origin. The line on the graph you constructed should not go through the origin. Can you think of a reason?

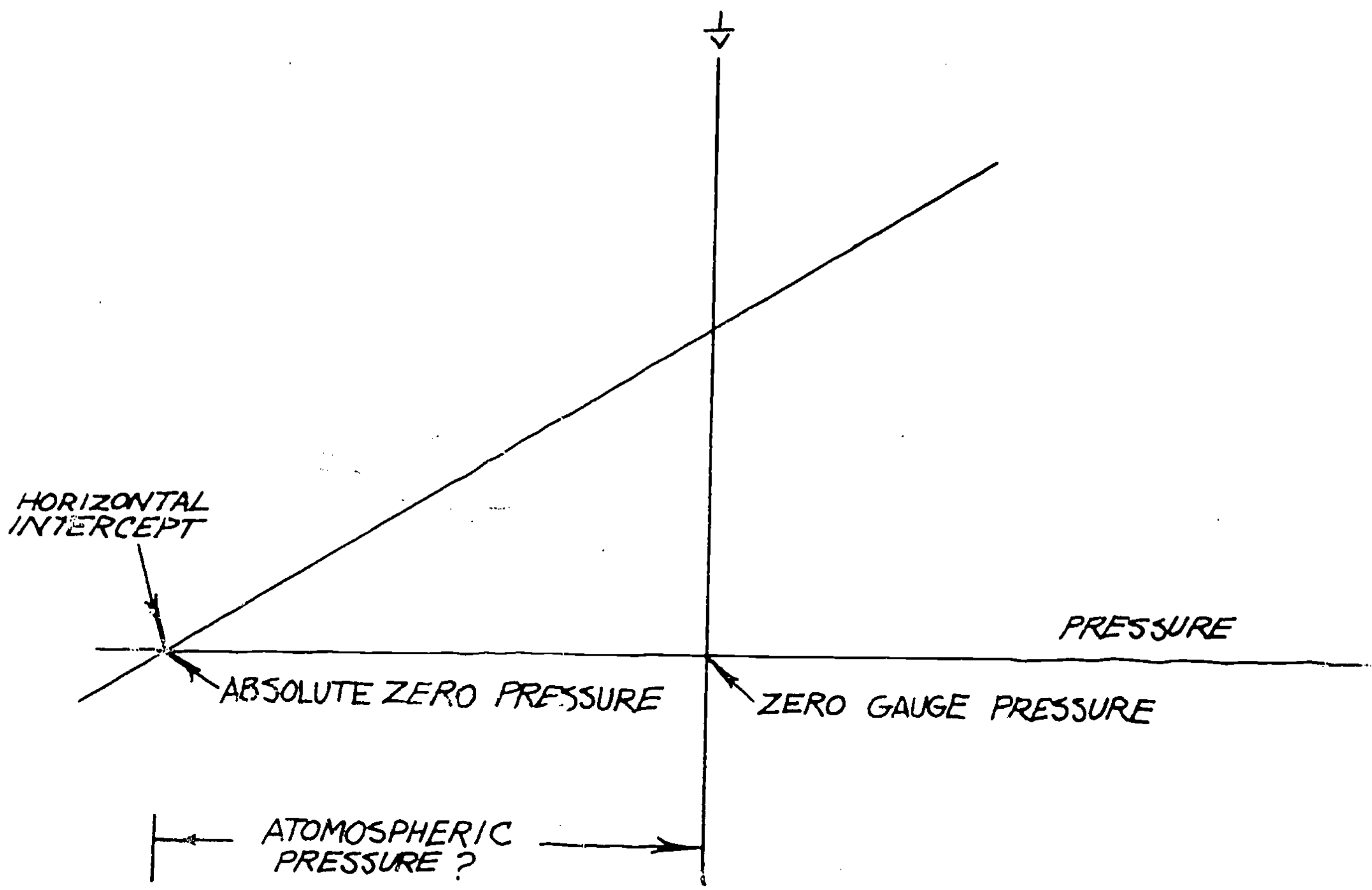
Before we tell you the answer outright we will try to lead you to deduce the correct answer based upon your experience with Charles Law. Recall that when we first graphed volume as a function of temperature the graph resembled the sketch below.



Notice that the line did not go through the origin. This is because 0° Celcius is about 273° C above absolute zero. In order to write an equation of the form $y = mx$ which described the relationship, we had to measure temperature in absolute terms. This meant that we had to add 273° to each temperature datum. When this was done we could write $V = mT$ where T was stated in terms of absolute temperature.

We are now faced with a parallel situation with our graph of P vs $\frac{1}{V}$. The line does not go through the origin, even though we expect the relationship between P and $\frac{1}{V}$ to be described by an equation of the form y = mx. Our experience with Charles Law leads us to suspect that our zero pressure is not absolute zero pressure. You have probably heard of the term "atmospheric pressure." It seems reasonable to suspect that our zero pressure is atmospheric pressure above absolute zero pressure.

We can check our suspicions by finding the horizontal intercept of the line which goes through the graphed points. We have sketched the situation below.



We can find the horizontal intercept either graphically or analytically. Then we can compare the distance between the horizontal intercept and zero to atmospheric pressure. Your teachers should have a record of atmospheric pressure for the day that the activity was performed. The two figures should agree fairly closely.

21-2 A Graphical Method of Finding the Horizontal Intercept

You have a graph of the data which is not large enough to graphically extrapolate the line back to the horizontal intercept. This difficulty might be overcome by taping additional pieces of graph paper to your existing graph. We indicate how this might be done in the diagram on the following page. The main idea is to keep taping (or glueing) on additional pieces of graph paper while keeping the original scaling. The process stops when enough pieces have been added to allow the line to cross the horizontal axis. The horizontal coordinate of this intersection may then be found.

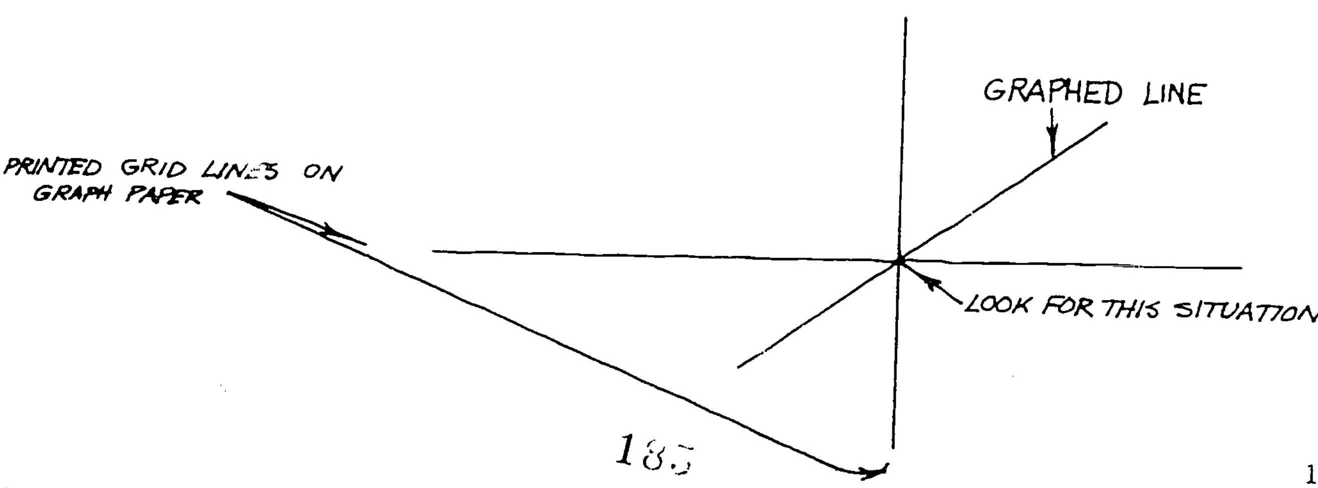
21-3 An Analytical Method of Determining the Horizontal Intercept

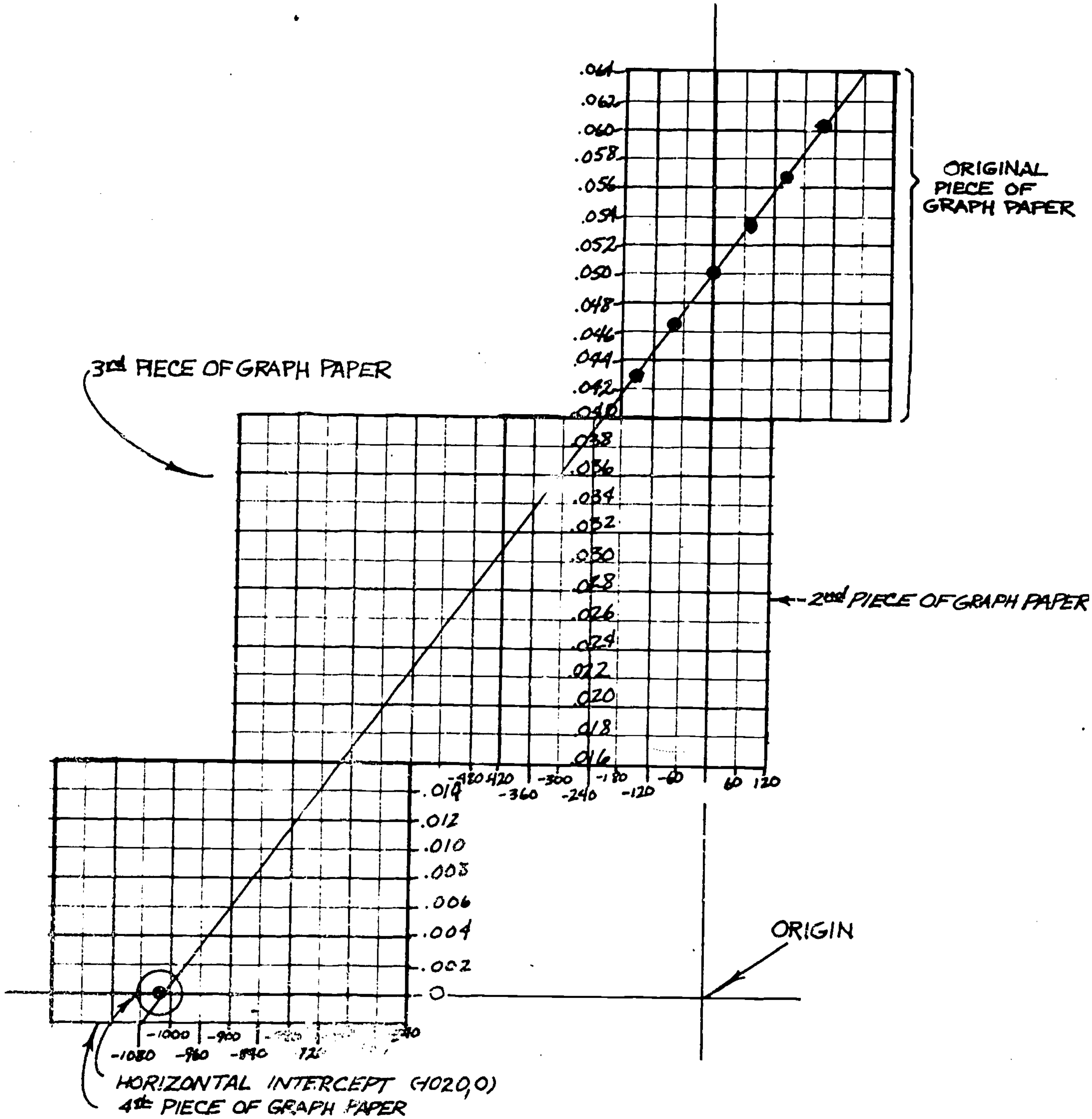
This technique may be conceptually more difficult but saves considerable time.

First we find the equation of the line on the graph in standard $y = mx + b$ form.

The vertical intercept, b, may be read directly off the graph. For the sample liquid filled tube data it is .050.

The slope of the line should be calculated from two points which are widely separated. This will tend to reduce the imprecision of our final result by reducing the effect of scale reading error on the slope calculation. In a further effort to reduce scale reading error you should try to find a printed grid intersection that the graphed line intersects exactly.




ORIGINAL PIECE OF GRAPH PAPER
3rd PIECE OF GRAPH PAPER
2nd PIECE OF GRAPH PAPER
ORIGIN
HORIZONTAL INTERCEPT (-1020,0)
4th PIECE OF GRAPH PAPER
.064
.062
.060
.058
.056
.054
.052
.050
.048
.046
.044
.042
.040
.038
.036
.034
.032
.030
.028
.026
.024
.022
.020
.018
.016
.014
.012
.010
.008
.006
.004
.002
0
-480 -420 -360 -300 -240 -180 -120 -60 60 120
-1080 -1000 -960 -900 -840

You should look for a situation like this as close to the extreme right or left of the graph paper as possible. On the sample graph on the following page we have chosen the point (288, .064) as our first point.

Now, we are ready to choose the second point. Foreknowledge of the pattern of solution tells us that the rise should only have a single nonzero digit in its decimal expression. This is because we will have to divide by it before we are through and we want an easy division task. Therefore we use two criteria to choose the second point.

1. The rise should only have a single nonzero digit in its decimal expression.

2. The second point should be as far away from the first as possible while at the same time it

   a. fulfills #1 above

   b. it is still on the graph paper.

For example, the y-coordinate of our first point is .064. The maximum rise on the graph paper which uses this point is .024. In other words, from .040 (the smallest y-value) to .064. Since .024 has more than one nonzero digit in its decimal expression it fails to meet criteria #1 above. .020 is the largest possible rise which satisfies #1. We use a rise of .020 to select our second point. The y-value .044 is .020 below .064 on the graph. Finally, we find the x-coordinate which corresponds to a y-value of .044. A close inspection of the graph reveals that it is about -123. Hence our second point is (-123, .044).

We can now calculate the slope

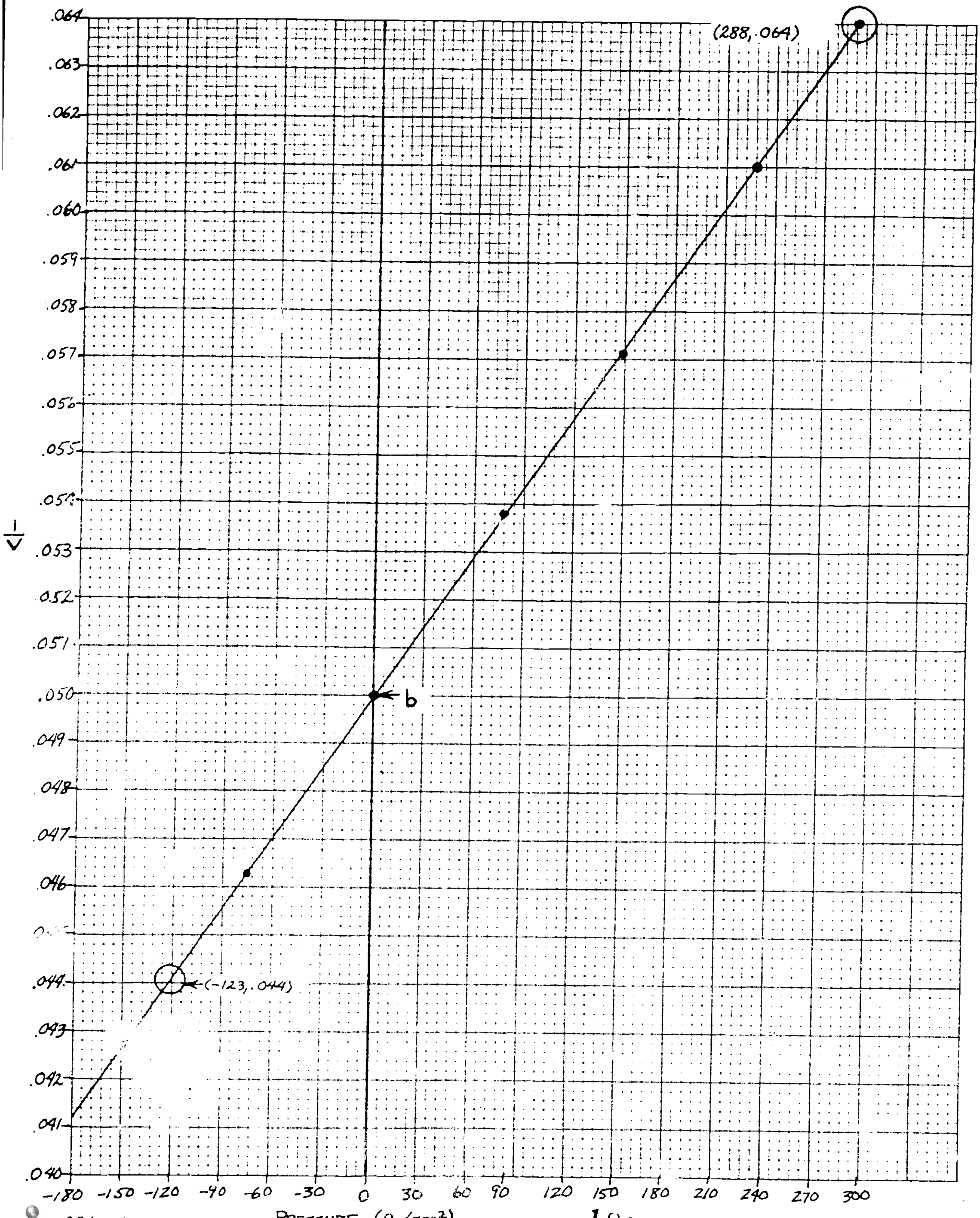
$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{.064 - .044}{288 - (-123)}$$

$$= \frac{.02}{411}$$

We leave this expression in quotient form in the hope that we can simplify it later on. By combining our information on slope and intercept, we get the equation for the line.

$$y = \frac{.02}{411}x + .050$$


(288, .064)
b
(-123, .044)
1/V
.064
.063
.062
.061
.060
.059
.058
.057
.056
.055
.054
.053
.052
.051
.050
.049
.048
.047
.046
.044
.043
.042
.041
.040
-180 -150 -120 -90 -60 -30 0 30 60 90 120 150 180 210 240 270 300
PRESSURE (g/cm²)

The calculation of the horizontal intercept is now straightforward. We want to find out what x is when y is zero. To do this we simply substitute zero for y and solve for x.

$$0 = \frac{.02}{411}x + .05$$

We subtract .05 from each side to get

$$-.05 = \frac{.02}{411}x$$

Now we multiply both sides by $\frac{411}{.02}$ to get

$$(-.05)\ \frac{411}{.02} = x$$

It should now be clear why we wanted only a single nonzero digit in the decimal expression of the rise. Division is greatly simplified. After a common factor of .01 is cancelled we get

$$-\frac{5}{2}(411) = x$$

Since $\frac{5}{2}$ is 2.5 we see that

$$-(2.5)(411) = x$$

or

$$-1027.5 = x$$

In fact 1027.5 g/cm^2 is a reasonable figure for atmospheric pressure.

PROBLEM SET 21:

1. List two characteristics of the graph of any function of the form $y = mx$.

2. Why didn't our graph of volume as a function of Celcius temperature go through the origin?

3. What had to be done to each temperature datum in order to make the line go through the origin.

4. When Charles Law is written in the form $V = mT$, what are the units of T?

5. Absolute zero temperature is about ______ degrees below the freezing point of water (0° C).

6. We say that the equation

$$\frac{1}{V} = \frac{1}{k}P$$

is in the form

$$y = mx$$

a. What variable ($\frac{1}{V}$, $\frac{1}{k}$, or P) is plotted on the y-axis?

b. What variable ($\frac{1}{V}$, $\frac{1}{k}$, or P) corresponds to the slope?

c. What variable ($\frac{1}{V}$, $\frac{1}{k}$, or P) is plotted on the x-axis?

7. Why doesn't the line on the graph of $\frac{1}{V}$ as a function of P go through the origin?

8. How far should the horizontal intercept be from the origin for the line on the graph of P vs $\frac{1}{V}$?

9. Ideally, what should be added to each pressure datum in order to make the line go through the origin?

10. (True or False) The vertical intercept or "b" may be read directly from the graph of P vs $\frac{1}{V}$.

11. Why should the slope of the line on the P vs $\frac{1}{V}$ graph be calculated from two <u>widely separated</u> points?

12. a. Sketch the relation between printed grid lines and your graphed line that should exist at the location of the first point.

b. What other condition should this first point fulfill?

13. Why should the rise have only a single nonzero digit in its decimal expression?

14. In the example in the text, why couldn't a rise of .030 be used to locate the second point?

15. Determine the slope and vertical intercept of the line on your own graph of P vs $\frac{1}{V}$.

16. Write the equation of your line in standard y = mx + b form. Substitute your numerical values for m and b in the equation.

17. Substitute y = 0 into your equation and solve for x to find the horizontal intercept.

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SECTION 22:

22-1 Vital Capacity (FEV)

Vital capacity is the maximum amount of expirable air. This quantity is used to aid in the diagnosis of certain lung ailments such as emphysema and chronic bronchitis. So, what is a normal vital capacity for a given individual? We will start out by pretending that we don't know the answer to this question. Our purpose in this pretense is to examine a method that scientists use to answer these kinds of questions. It turns out that a method quite similar to the one we used to "discover" the gas laws will lead to an equation that predicts vital capacity which is based on other measurable quantities.

Recall how you found Charles Law. First you plotted measurements of volume (vertical axis) as a function of temperature (horizontal axis). Next we saw that a straight line which went through the point (absolute zero, zero volume) fit the data points quite well.

Now think about Boyle's Law for a moment. First you plotted measurements of volume (vertical axis) as a function of pressure (horizontal axis). Once this was done it was obvious that the points did not lie on a straight line. This created some problems. First of all you don't presently have the background to write a descriptive equation for anything but a straight line. It is conceivable that this problem could be taken care of with more study. However, there is a more basic difficulty. It turns out that it is very difficult to see the difference between different kinds of curves. On the other hand, it is relatively easy to see whether or not a set of points lies on a straight line. In the case of Boyle's Law it was possible to get a linear set of points by plotting the reciprocal of volume (vertical axis) as a function of pressure (horizontal axis). We chose the reciprocal of volume instead of volume squared, for example, because we had foreknowledge that it would lead to a linear set of points.

Now we are pretending that we don't know what will work for vital capacity. How can we go about finding out what other body measurements most accurately predict vital capacity? The first step is one of imagination. We dream up all other body measurements that might have some relationship to vital capacity. We then plot vital capacity (vertical axis) as a function of any one of these other measurements (horizontal axis). Finally we examine the graphs. We find the point set which appears most linear. We decide which point set is most linear by judging each graph on the basis of two criteria.

1. The degree to which the points appear to be linear.

2. The "tightness" with which the points cluster around the line.

On the following pages we have graphs of vital capacity as a function of age, weight, body surface area, height and $height^3$. You should closely examine each graph and judge each one on the basis of the two criter[illegible] listed above. Then make a choice about which variable should be u[illegible] predict a person's vital capacity. Try not to look forward to f[illegible] ut our choice before you make your own.

Statisticians have developed techniques for evaluating a set of points for linearity and compactness. According to their techniques, he[illegible] cubed is the best predictor of vital capacity for people in y[illegible] age group. For smoothly aged adults, the best method of predicting vital capacity uses a combination of height and age.

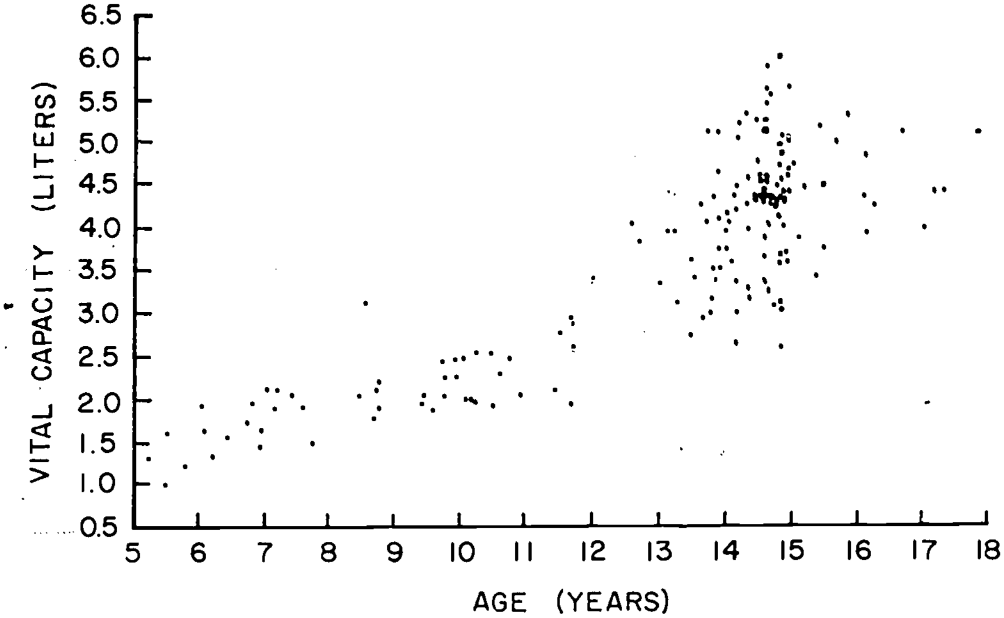
Notice that none of the graphs exhibit the tight linearity that we saw with Charles Law and Boyles Law. This is typical of relationships between biological variables. This implies that for a given height we will be forced to state the expected vital capacity in terms of a range of imprecision.

Another point that ought to be mentioned is the need for a large number of points. Without a large sample it would be difficult to decide which predictor would give the best results.

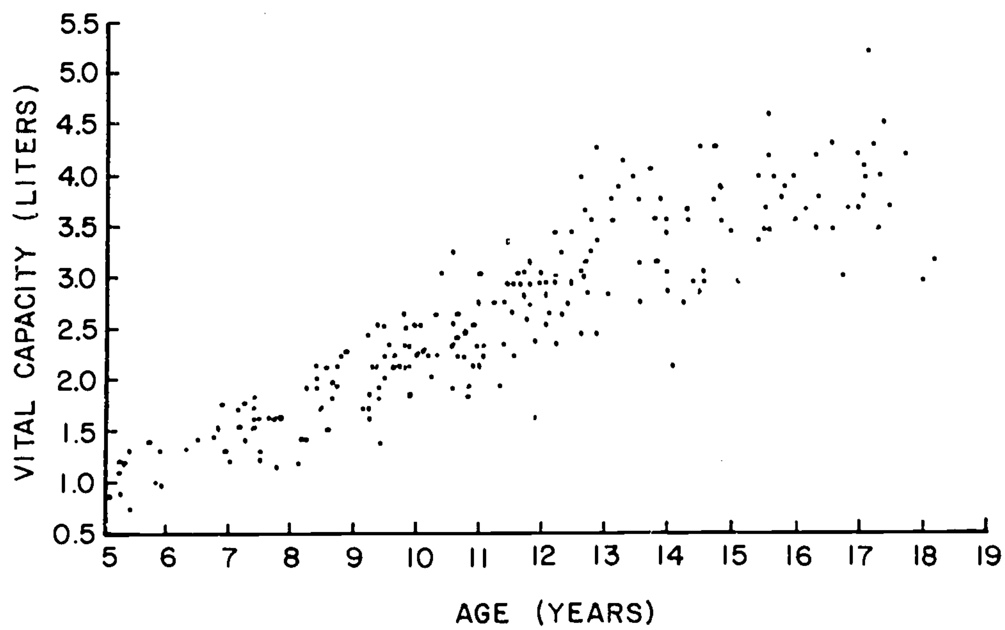
22-2 Your Data and You

Your teacher will assign a graphing task to you. You will be asked to plot either data for females or males. $Height^3$ will be plotted on the horizontal axis and vital capacity on the vertical axis. The data you plot will be that which you collected on each other in science class. In the following section your graphs will be compared to the statistically derived results mentioned earlier.

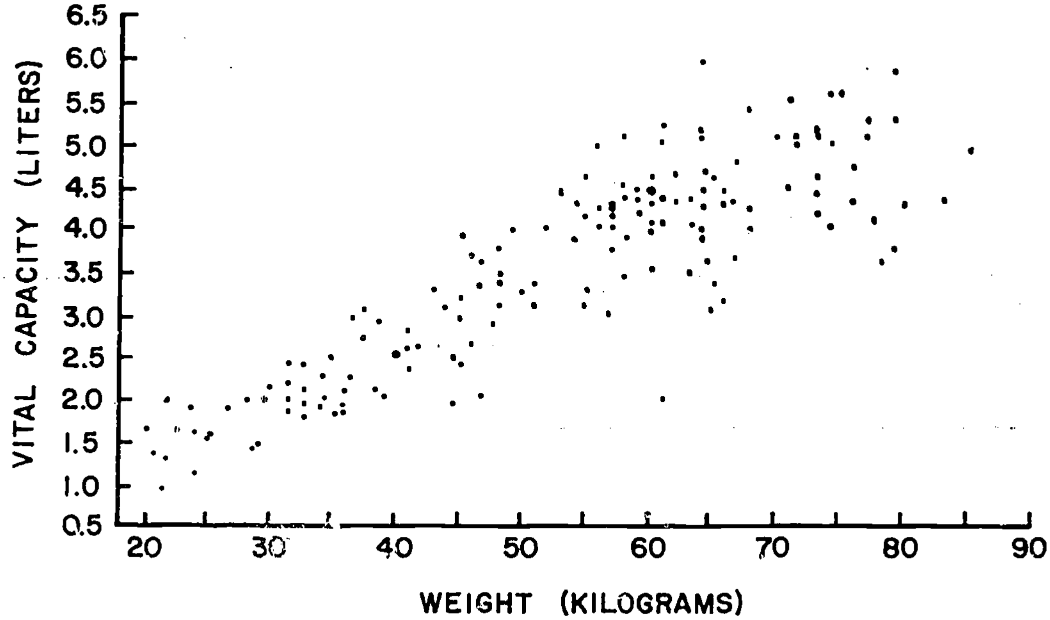
VITAL CAPACITY VS. AGE
IN MALE CHILDREN AND ADOLESCENTS



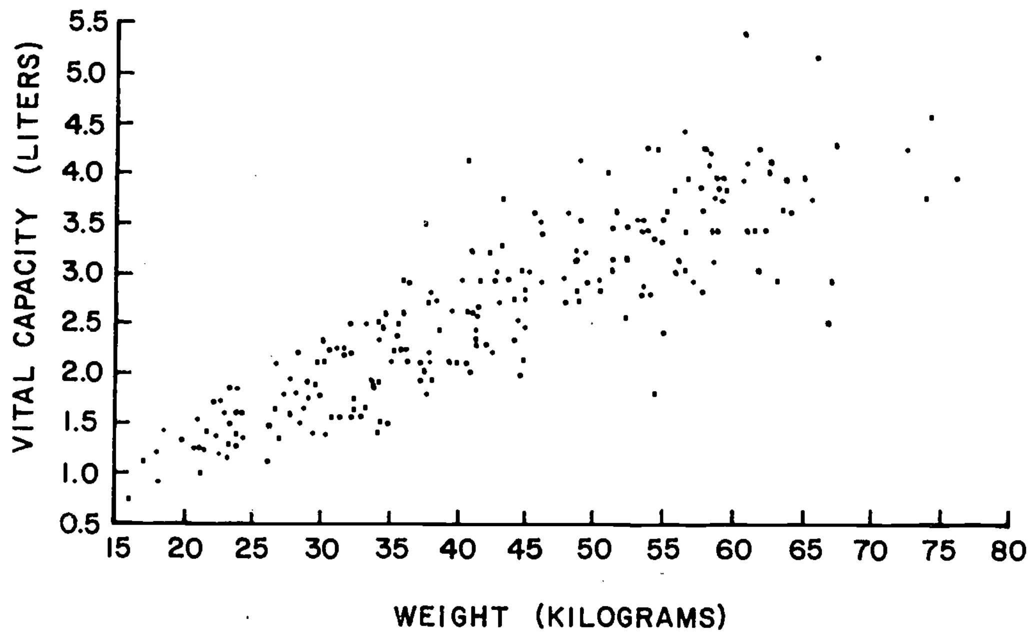
VITAL CAPACITY VS. AGE
IN FEMALE CHILDREN AND ADOLESCENTS



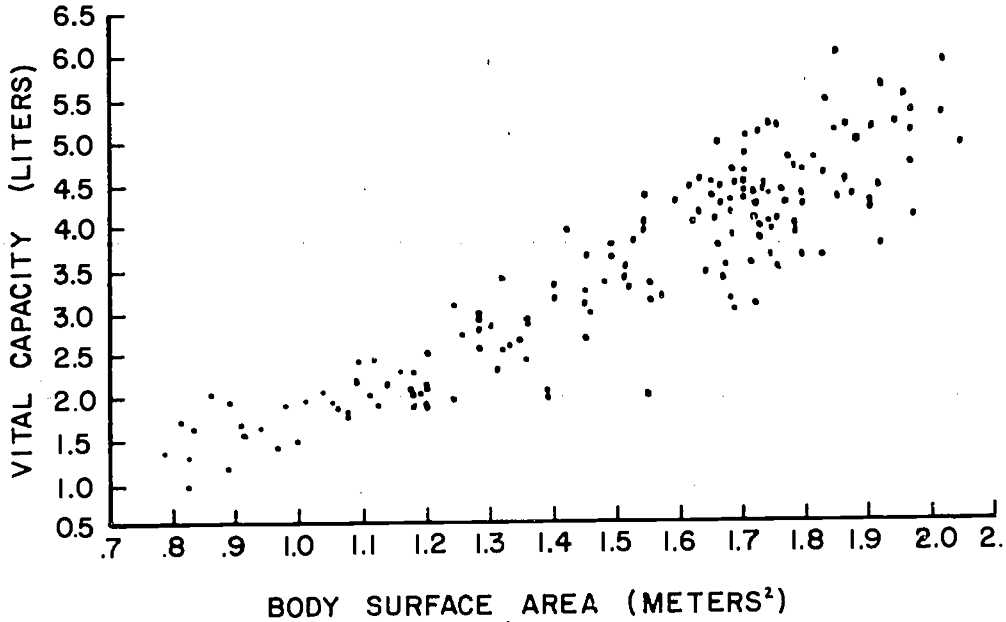
VITAL CAPACITY VS. WEIGHT IN MALE CHILDREN AND ADOLESCENTS



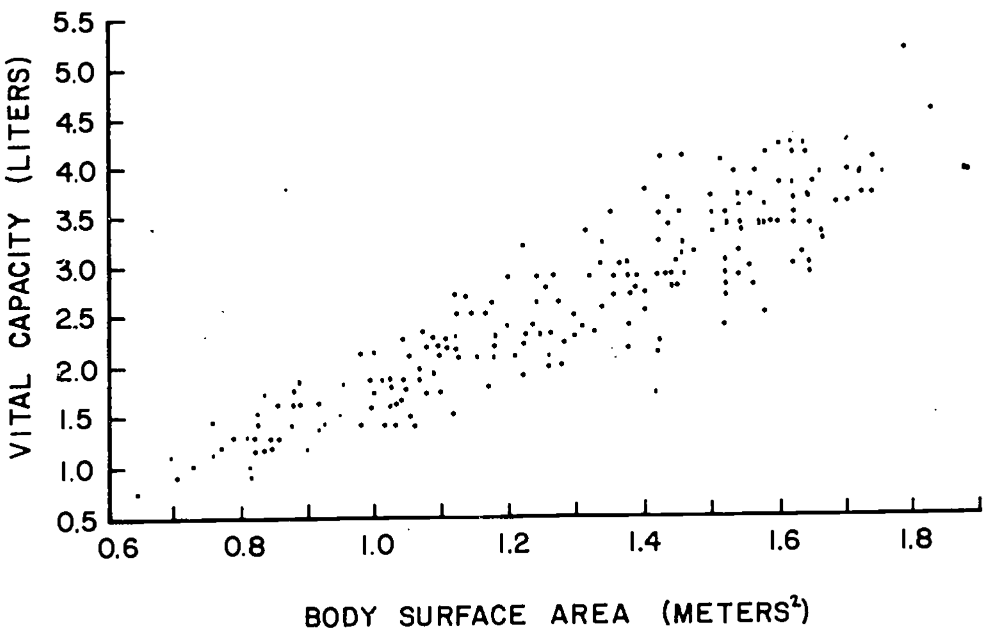
VITAL CAPACITY VS. WEIGHT IN FEMALE CHILDREN AND ADOLESCENTS



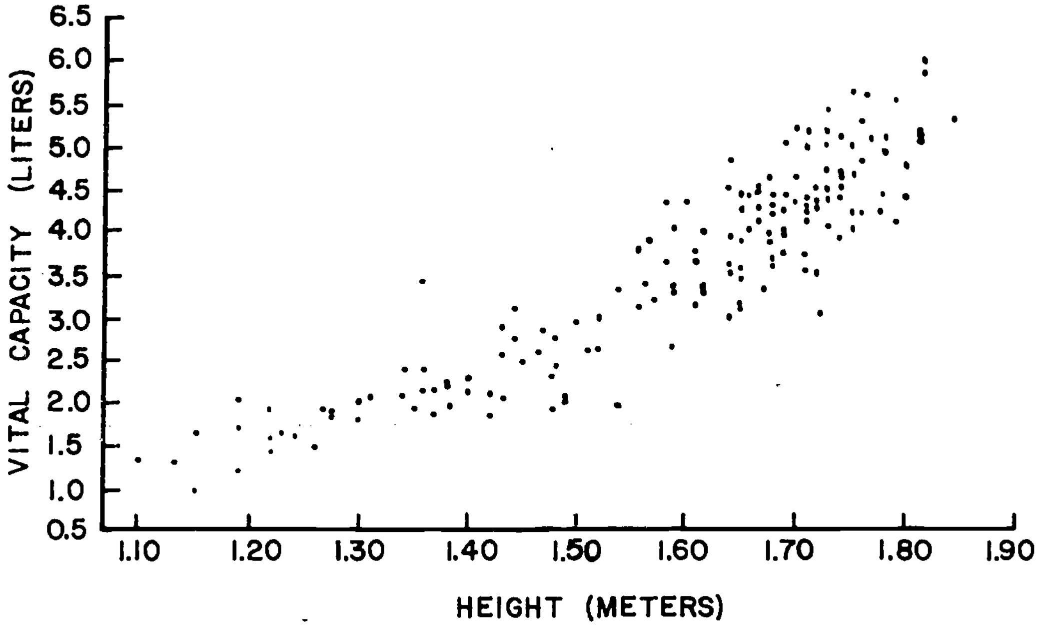
VITAL CAPACITY VS. BODY SURFACE AREA IN MALE CHILDREN AND ADOLESCENTS



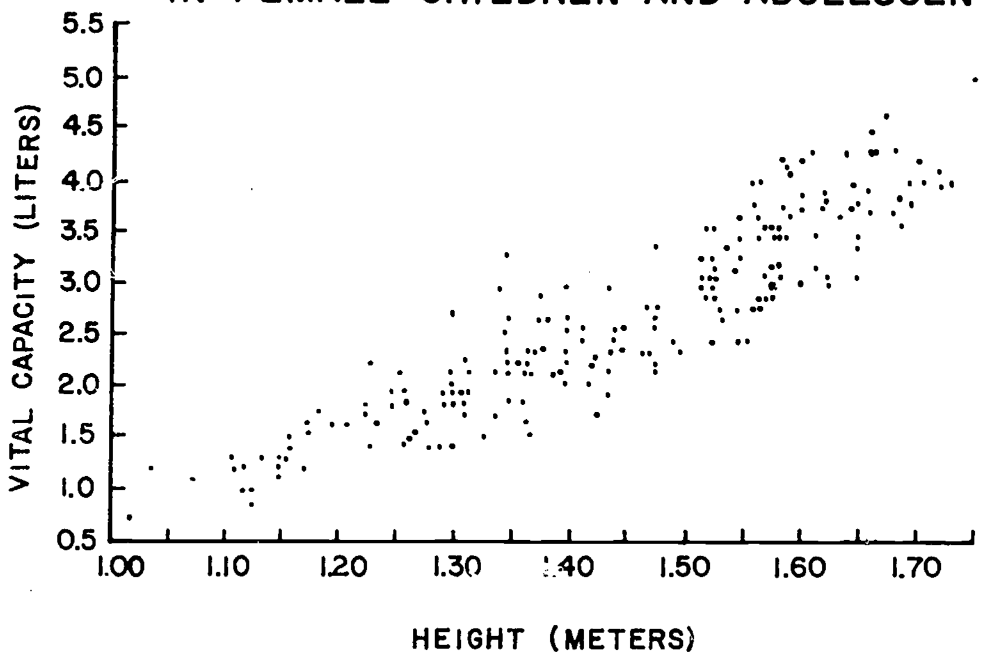
VITAL CAPACITY VS. BODY SURFACE AREA IN FEMALE CHILDREN AND ADOLESCENTS



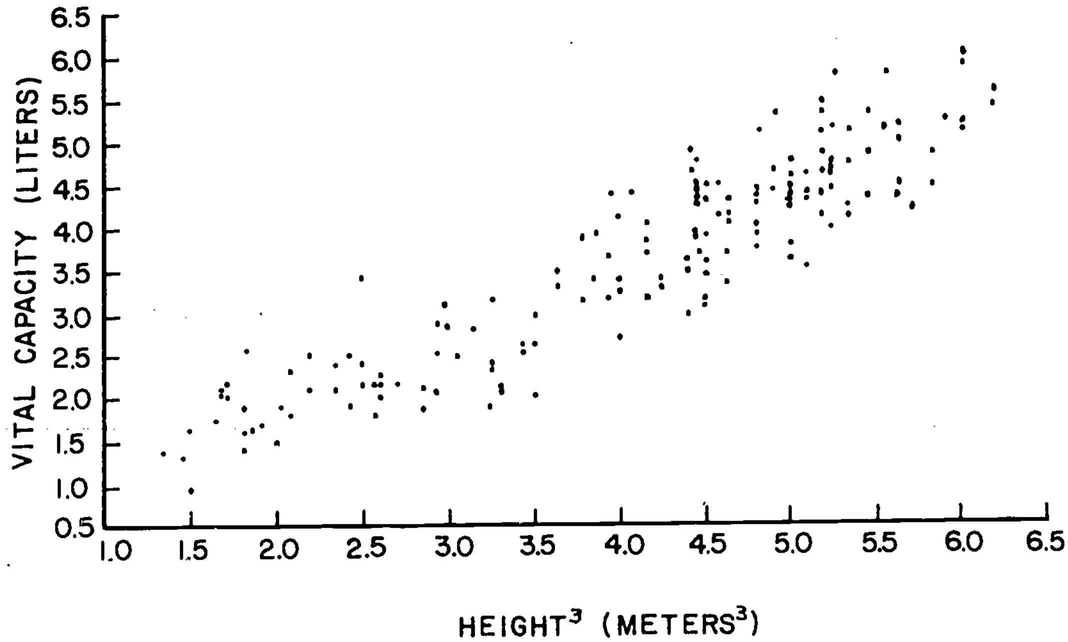
VITAL CAPACITY VS. HEIGHT IN MALE CHILDREN AND ADOLESCENTS



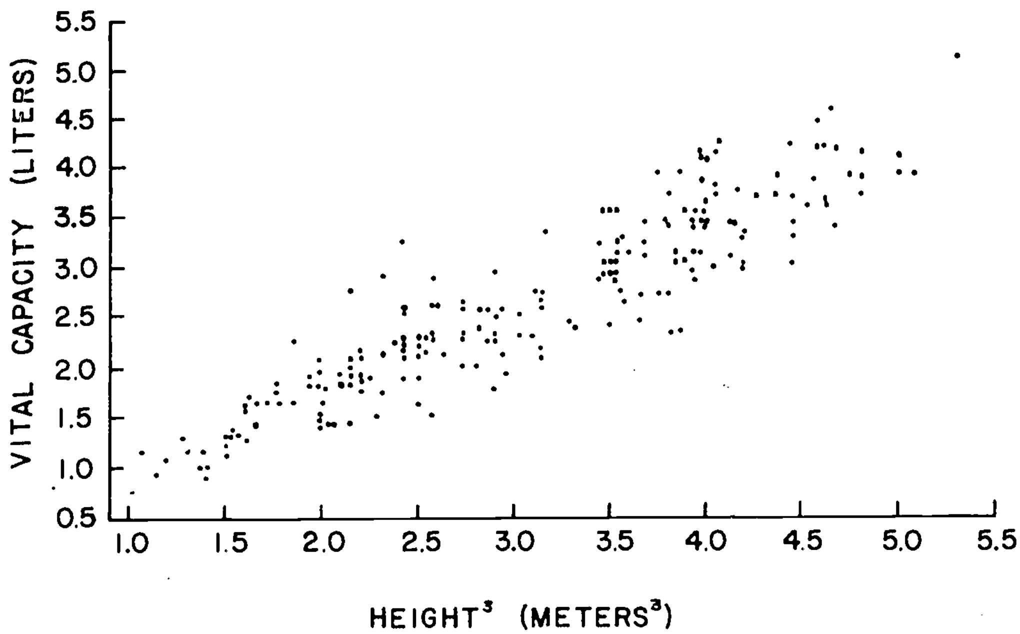
VITAL CAPACITY VS. HEIGHT IN FEMALE CHILDREN AND ADOLESCENTS



VITAL CAPACITY VS. HEIGHT^3 IN MALE CHILDREN AND ADOLESCENTS



VITAL CAPACITY VS. HEIGHT^3 IN FEMALE CHILDREN AND ADOLESCENTS



Height in Meters	(Height in Meters)^3	Height in Meters	(Height in Meters)^3
1.34	2.41	1.69	4.83
1.35	2.46	1.70	4.91
1.36	2.52	1.71	5.00
1.37	2.57	1.72	5.09
1.38	2.63	1.73	5.18
1.39	2.69	1.74	5.27
1.40	2.74	1.75	5.36
1.41	2.80	1.76	5.45
1.42	2.86	1.77	5.55
1.43	2.92	1.78	5.64
1.44	2.99	1.79	5.74
1.45	3.05	1.80	5.83
1.46	3.11	1.81	5.93
1.47	3.18	1.82	6.03
1.48	3.24	1.83	6.13
1.49	3.31	1.84	6.23
1.50	3.38	1.85	6.33
1.51	3.44	1.86	6.43
1.52	3.51	1.87	6.54
1.53	3.58	1.88	6.64
1.54	3.65	1.89	6.75
1.55	3.72	1.90	6.86
1.56	3.80	1.91	6.97
1.57	3.87	1.92	7.08
1.58	3.94	1.93	7.19
1.59	4.02	1.94	7.30
1.60	4.10	1.95	7.41
1.61	4.17	1.96	7.53
1.62	4.25	1.97	7.65
1.63	4.33	1.98	7.76
1.64	4.41	1.99	7.88
1.65	4.49	2.00	8.00
1.66	4.57	2.01	8.12
1.67	4.66	2.02	8.24
1.68	4.74		

PROBLEM SET 22:

1. Scale and graph the vital capacity data for either the males or females in your class.

SECTION 23:

23-1 A Statistical Description of a Vital Capacity Point Scatter

On the following page we have a graph of height^3 vs vital capacity for the 15 female students in the Pudworthy High School Biomed class. Statisticians have developed a set of linear equations to describe vital capacity data such as this. They claim that 95% of all 17-year-old females will lie between the upper and lower lines decribed below.

Upper Limiting Line: $V = .75\ H^3 + 1.1$

Middle Line: $V = .75\ H^3 + .4$

Lower Limiting Line: $V = .75\ H^3 - .3$

We will demonstrate how to graph the lower limiting line on the graph. The other lines may be graphed in a similar fashion.

First we substitute 4.8 for H^3 in the equation

$$V = .75\ H^3 + 1.1$$

We chose an x-coordinate of 4.8 because this will insure that the corresponding y-coordinate will lie on the graph. In other words we expect a line that will be below the point scatter to be on the graph on the right side and not on the graph on the left.

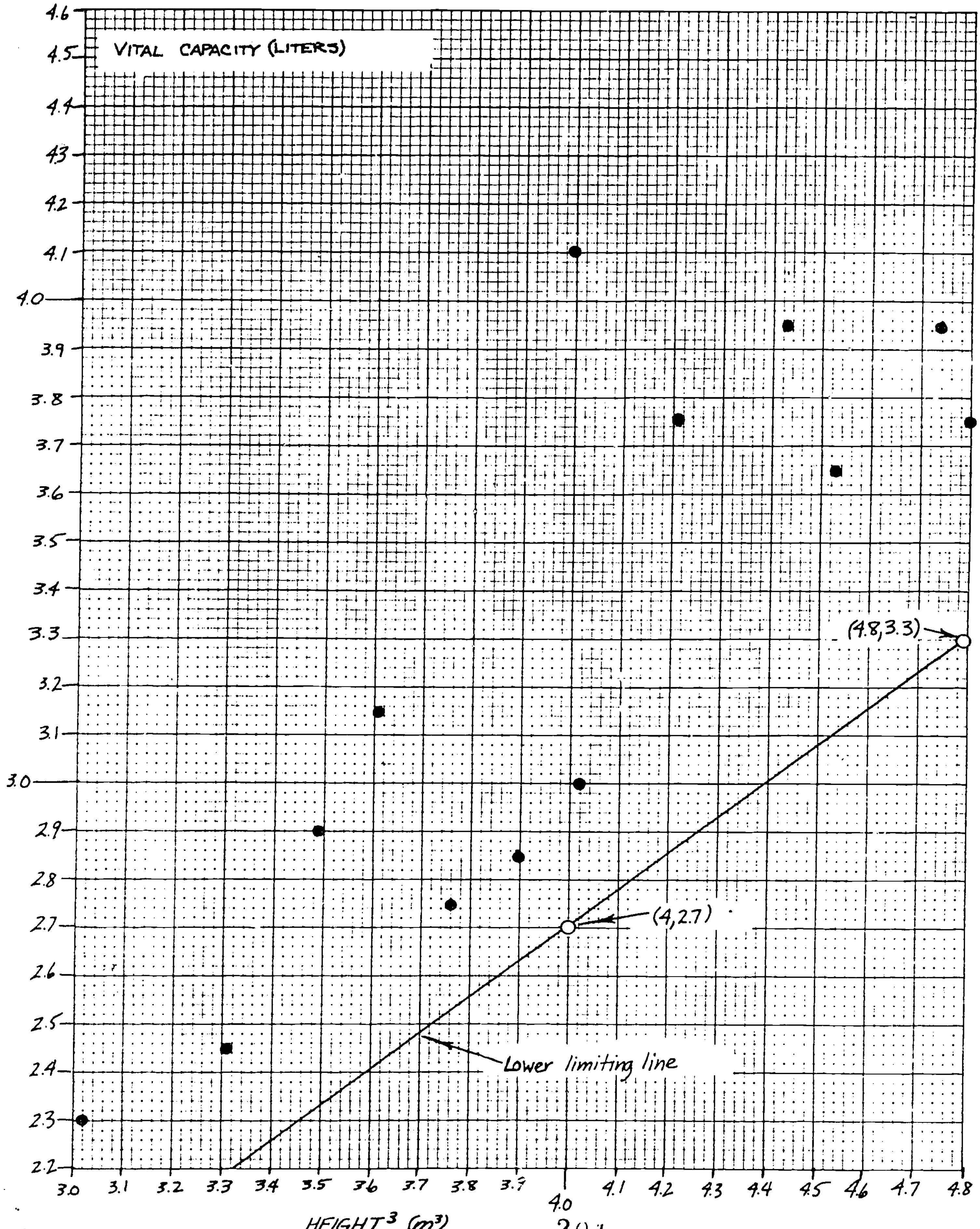
Substituting in $H^3 = 4.8$ we get

$$V = .75(4.8) - .3$$
$$= 3.6 - .3$$
$$= 3.3$$

This establishes the location of one point on the graph of the line $V = .75\ H^3 + 1.1$. It is (4.8, 3.3).

In a similar fashion we find the y-coordinate which corresponds to an x-value of 4. Without bothering you with the computational details, the point is (4, 2.7). The line which goes through these two points is drawn on the following graph.

An inspection of the graph shows that none of the data points falls below this line. This means that none of the female Pudworthy Biomed Students needs to worry about the size of her vital capacity


VITAL CAPACITY (LITERS)
4.6
4.5
4.4
4.3
4.2
4.1
4.0
3.9
3.8
3.7
3.6
3.5
3.4
3.3
3.2
3.1
3.0
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
(4.8,3.3)
(4,2.7)
Lower limiting line
3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4.0 4.1 4.2 4.3 4.4 4.5 4.6 4.7 4.8
HEIGHT^3 (m^3)

23-2 Stating a Predicted Vital Capacity as a Range of Imprecision

The set of three lines allow us to state a predicted vital capacity in terms of a range of imprecision. Suppose Miss Haley Tosis is 1.56 m tall. We want to find a range of imprecision for her expected vital capacity. First we find H^3. Conveniently, it is about 3.8 m.

Next we substitute $H^3 = 3.8$ into the equation for the middle line.

$$\begin{aligned} V &= .75\ H^3 + .4 \\ &= .75(3.8) + .4 \\ &= 2.85 + .4 \\ &= 3.25 \end{aligned}$$

This number, 3.25 liters, will be the midpoint of our range of imprecision. We could find the upper and lower bounds of the range in a similar fashion. But there is a more straightforward way to find the range. Notice that the intercept ("b") for the upper line is .7 more than the intercept for the middle line. Similarly, b for the lower line is .7 less than b for the middle line. Since all the equations are identical but for the variation in intercept, this suggests that the imprecision of the predicted vital capacity is .7. Hence Haley's predicted vital capacity is 3.25 + .7 liters.

PROBLEM SET 23:

1. Graph the upper, middle and lower lines on the vital capacity graph you constructed.

MALES:		FEMALES:	
Upper:	$V = .95\ H^3 + .6$	Upper:	$V = .75\ H^3 + 1.1$
Middle:	$V = .95\ H^3 - .125$	Middle:	$V = .75\ H^3 + .4$
Lower:	$V = .95\ H^3 - .85$	Lower:	$V = .75\ H^3 - .3$

State predicted vital capacities as ranges of imprecision for the female students in Problems 2 through 5. Include units.

2. Ima Gas: H = 1.36 m
3. Digit Alice: H = 1.41 m
4. Kate L. Chip: H = 1.59 m
5. Lynn E. R. Funk-Shun: H = 1.71 m

Do the same for the male students in Problems 6 through 8.

6. Albert Ross: $H = 2.00$ m

7. J. K. Flipflop: $H = 1.71$ m

8. Phil Ling: $H = 1.77$ m

9. a. State your own height.

   b. State your own expected vital capacity as a range of imprecision

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SECTION 24:

24-1 Open Sentences and Solution Sets

An equation is a sentence which may be true, false or open. An example of a true sentence is

$$7 + 9 = 14 + 2.$$

An example of a false sentence is

$$3 = 4.$$

An example of an open sentence is

$$16x = 32.$$

An open sentence is neither true nor false until the variable is replaced by a number from a specified domain. In the example above, if x is replaced by 2, the sentence is true; if x is replaced by 3, the sentence is false.

The solution set of an open sentence is the set of numbers which make the sentence true. The solution set of the equation $16x = 32$ is $\{2\}$.

EXAMPLE:

Determine the solution set of the open sentence

$$12 - x = 9$$

when the domain is the set of all positive integers $\{1, 2, 3, \ldots\}$.

Answer: $\{3\}$

EXAMPLE:

Determine the solution set of the open sentence

$$4x = 10$$

(a) when the domain is the set of all positive integers and
(b) when the domain is the set of all real numbers.

Answers: (a) ϕ , the empty set
(b) $\{\frac{5}{2}\}$

24-2 Solving Equations

Equations which have the same solution set are called equivalent equations. For example the equations

$$x = 3$$

and $$x + 7 = 10$$

are equivalent because the solution set of each equation is {3}.

In the example above, 7 has been added to each side of the first equation to give the second equation. It is always true that when the same quantity is added to both sides of an equation, the resulting equation is equivalent to the initial equation.

It is also true that when the same quantity is subtracted from both sides of an equation, or both sides are multiplied by the same quantity, or divided by the same quantity (with the exception of zero), the resulting equation is equivalent to the initial equation.

As an example consider the equation

$$3x = 12$$

and the equation which is obtained by multiplying both sides by 2

$$6x = 24.$$

The solution set of each equation is {4}; the equations are equivalent.

If two or more operations are combined in succession, the final equation is still equivalent to the first. If 6 is added to each side of the equation

$$6x = 24$$

to give

$$6x + 6 = 30$$

the solution set of the final equation is {4}, the same as for the original equation.

$$3x = 12.$$

The operations of addition, subtraction, multiplication and division are used to derive equivalent equations in the process of solving an equation. An equation is said to be solved when an equation equivalent to the original equation is derived that has a variable isolated on one side of the equation. Once an equation is in this form, the choice of solution set is either obvious or greatly simplified.

EXAMPLE:

Solve the equation

$$2x + 7 = 25.$$

SOLUTION:

Subtract 7 from each side to obtain an intermediate equivalent equation with one less term than the original equation.

$$2x + 7 - 7 = 25 - 7$$
$$2x = 18$$

Divide each side by 2 to isolate x on the left side of the equation.

$$\frac{2x}{2} = \frac{18}{2}$$

$$x = 9$$

The solution set of $x = 9$ is easily seen to be $\{9\}$.

Other examples of equations with a variable isolated on one side are

$$x = 15$$

$$y = 16 + \sqrt{6}$$

and $$u = 2v - 5w.$$

The first two equations are examples of a numerical solution. A numerical solution is an equation with a variable isolated on one side and a number on the other side. The third equation is an example of a literal solution. A literal solution has the variable solved for isolated on one side and an expression that includes one or more other variables on the other side.

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EXAMPLE:

Is the equation

$$x = 5$$

a solution to the equation

$$x + 9 = 14?$$

If so, is the solution numerical or literal?

Answers: Yes; because the solution set of each equation is {5}. (The equation $x = 5$ is derived by subtracting 9 from each side of the equation $x + 9 = 14$.)

The solution is numerical because only a number appears on the right side of the equation.

EXAMPLE:

Is the equation

$$x = 16$$

a solution to the equation

$$\frac{x}{4} = 16?$$

ANSWER: No, because the solution set of the equation $x = 16$ is {16}, and the solution set of the equation $\frac{x}{4} = 16$ is {64}.

We will illustrate the role of numerical solutions and literal solutions in solving equations by solving a Boyle's Law problem using two different methods.

EXAMPLE:

Given that $PV = k$ and that $k = 5100$ when P is measured in g/cm^2 and V is measured in cm^3, what is the volume of the gas when $P = 1020\ g/cm^2$?

SOLUTION:

Method 1: Substitute the numerical values of P and k into the equation $PV = k$.

$$1020\ V = 5100$$

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Divide both sides by 1020.

$$V = \frac{5100}{1020}$$

$$= 5\ cm^3$$

Method 2: Solve the equation PV = k literally for V by dividing both sides by P.

$$\frac{PV}{P} = \frac{k}{P}$$

$$V = \frac{k}{P}$$

Now substitute the numerical values of k and P.

$$V = \frac{5100}{1020}$$

$$= 5\ cm^3$$

In Method 1 numerical values are substituted and a numerical solution is found. In Method 2 a literal solution is first derived and then numerical values are substituted to obtain a numerical solution. Each method gives a solution set of $\{5\ cm^3\}$ and either may be used to solve this type of problem.

24-3 A Less Obvious Example

We will not always have simple equations to solve. Many times there will be more than one intermediate equivalent equation between the original equation and the solution equation.

PROBLEM:

Solve the equation

$$\frac{100}{x} - 20 = 90$$

for x.

SOLUTION 1:

We first remove the (-20) term from the left side.

$$\frac{100}{x} - 20 + 20 = 90 + 20$$

$$\frac{100}{x} = 110$$

Next we multiply both sides by x.

$$\frac{100}{x} \cdot x = 110x$$

$$100 = 110x$$

The solution is in sight now. All we have to do is divide both sides by 110 to isolate x,

$$\frac{100}{110} = \frac{110x}{110}$$

$$\frac{10}{11} = x$$

and we are done.

SOLUTION 2:

Instead of removing the (-20) term first, we can get x out of the denominator first. We do it by multiplying both sides of the original equation by x.

$$\frac{100}{x} - 20 = 90$$

$$x(\frac{100}{x} - 20) = 90x$$

We next apply the distributive property to the left side to get,

$$x \cdot \frac{100}{x} - 20x = 90x$$

$$100 - 20x = 90x$$

We can now collect the x's on the right by adding 20x to each side

$$100 - 20x + 20x = 90x + 20x$$

$$100 = 110x$$

And finally,

$$\frac{10}{11} = x$$

as before.

In the cases of more complicated equations there will always be a variety of ways to solve the equation. The most efficient path to the solution will not always be obvious. But it doesn't matter. The important thing is to begin. If you follow the simple rules of finding simpler equivalent equations, you will inevitably end up at the solution equation. This is another situation in mathematics where there may be many possible ways to get to an answer. It doesn't matter which path you choose, because all paths lead to the same place, the solution equation.

PROBLEM SET 24:

1. Use the following information to solve the problems below.

$D_1 = \{0, 1, 2, \ldots\}$

D_2 = the set of all real numbers

$12x = 6$ $\qquad$ $2x = 5$

$9 + 6 = 96$ $\qquad$ $4x = 2$

$2x + 3 = 2x - 3$ $\qquad$ $2 \cdot 7 = 10 + 4$

a. Select a true sentence.

b. Select a false sentence.

c. Select an open sentence.

d. Select an open sentence that has a solution set of ϕ for D_1 and $\{2\frac{1}{2}\}$ for D_2.

e. Select an open sentence that has a solution set of ϕ for D_2.

f. Select two equations that are equivalent when D_2 is the domain for both.

Determine the solution set for Problems 2 through 11.

SHOW INTERMEDIATE EQUIVALENT EQUATIONS

Example: $9 + x = 14$

Intermediate equivalent equation:

$9 - 9 + x = 14 - 9$ (subtract 9 from each side)

$x = 5$

$\{5\}$

2. $9x = 18$

3. $4 + x = 87$

4. $4V = 12 \cdot 7$

5. $31 = N - 8$

6. $64 = 4b$

7. $37 - x = -42$

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8. $x - 10 = -78$

9. $\frac{P \cdot 48}{6} = 104$

10. $97 + 12 = 12 - x$

11. $\frac{V}{250} = \frac{4}{300}$

For Problems 12 through 21 show at least two intermediate equivalent equations and a numerical solution.

Example: $\frac{4}{x} = 5$

Intermediate equivalent equations: $\frac{4}{x} \cdot x = 5 \cdot x$ (multiply both sides by x)

$4 = 5x$

$\frac{4}{5} = \frac{5x}{5}$ (divide both sides by 5)

$\frac{4}{5} = x$ (numerical solution)

12. $\frac{5}{x} = 6$

13. $3x + 4 = 10$

14. $\frac{22.4}{T} = .5$

15. $36N - 9 = -81$

16. $\frac{m}{4} + 6 = 0$

17. $16(N + 2) = 32$

18. $-23.5 = \frac{141}{x}$

19. $.3x + 2 = 14$

20. $4x = x + 132$

21. $19 = \frac{z}{3} - 4$

Ignore uncertainty considerations in Problems 22 through 30.

22. Use the equation $PV = P'V'$ and the information:

$P =$ 800 mm Hg

$V' =$ 22.4 liters

$P' =$ 720 mm Hg

to find V.

23. Use the equation $PV = k$ and the information:

$V = 4.8$ liters

$k = 14,400$ when V is in liters and P is in mm Hg

to determine P.

24. Using $PV = k$, determine the constant k for a gas that occupies 25.3 liters at 315 mm Hg.

25. Use the relation $PV = P'V'$ to find the new volume of gas V' when

$P = 740$ mm Hg

$V = 450$ ml

$P' = 750$ mm Hg.

26. A gas has a volume of 10 liters under a pressure of 775 mm Hg. What will be the volume at 750 mm Hg?

27. A quantity of oxygen occupies 380 ml when measured under a pressure of 740 mm Hg. What would the volume of the same quantity of gas be at 760 mm Hg if the temperature were held constant?

28. Use Charles' Law, $\frac{V}{T} = \frac{V'}{T'}$, to determine V' when

$V = 175$ ml

$T = 300°$ K

$T' = 273°$ K.

29. Use Charles' Law, $\frac{V}{T} = \frac{V'}{T'}$, to determine V' when

$V = 51$ liters

$T = 16°$ C

$T' = -1°$ C

Remember to convert degrees Celsius to degrees Kelvin.

30. 30 ml of hydrogen are measured at a temperature of 27° C. What would the volume of the same gas be at 0° C if pressure were held constant?

SECTION 25:

25-1 The Combined Gas Law

Charles Law relates the volume of a gas to the temperature of the gas at constant pressure. The law states that the volume of a gas divided by the absolute temperature of the gas at constant pressure is constant. Charles' Law may be written as an equation:

$$\frac{V}{T} = k \qquad \text{at constant P}$$

or

$$\frac{V}{T} = \frac{V'}{T'}$$

V and T are the volume and absolute temperature at one set of conditions, and V' and T' are the volume and absolute temperature at another set of conditions.

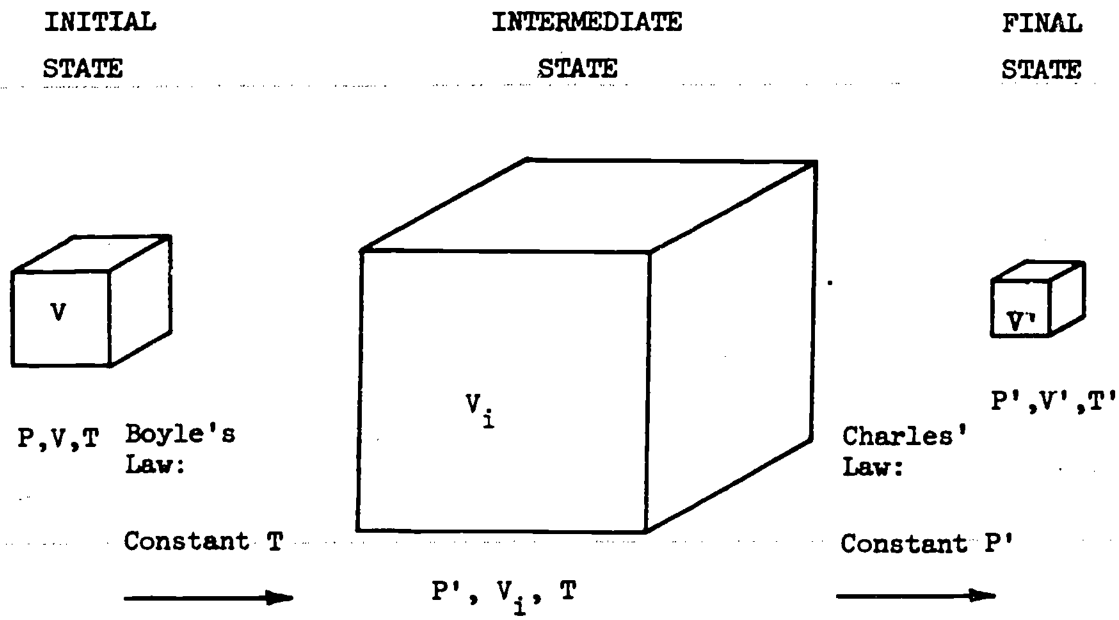
Boyle's Law may be written as a similar equation.

$$PV = P'V' \text{ at constant T}$$

P and P' are the pressures at two sets of conditions.

In this section we will combine Boyle's Law and Charles' Law into one combined gas law, which may be expressed as an equation relating the three variables volume, pressure and temperature. The equation will relate P, V and T, which are the pressure, volume and temperature of an initial state of a gas, to P', V' and T', which are the final pressure, volume and temperature of the gas after undergoing some change.

The derivation of the desired equation is simplified by creating a hypothetical intermediate state. The intermediate state is attained by keeping the temperature constant at T and changing the pressure to P'. The volume of the intermediate state we will call V_i. The gas is converted from the intermediate state to the final state by holding the pressure constant at P' and changing the temperature from T to T'.



We may apply Boyle's Law to the transformation from the initial state to the intermediate state because temperature is constant. The equation we obtain is

$$PV = P'V_i .$$

This equation may be solved for V_i to give the equation

$$V_i = \frac{PV}{P'} .$$

Pressure is constant in the transformation from the intermediate state to the final state, so we may apply Charles' Law, which gives us the equation

$$\frac{V_i}{T} = \frac{V'}{T'} .$$

V_i may be eliminated by substituting its literal solution from the Boyle's Law equation into the Charles' Law equation. The result is

$$\frac{\left(\frac{PV}{P'}\right)}{T} = \frac{V'}{T'}$$

Notice that the left-hand side is equivalent to

$$\frac{PV}{P'} \div T \quad \text{or} \quad \frac{PV}{P'} \cdot \frac{1}{T} .$$

This allows the equation to be rewritten as

$$\frac{PV}{P'T} = \frac{V'}{T'} .$$

Both sides of the equation may now be multiplied by P' to obtain

$$\frac{PV}{T} = \frac{P'V'}{T'} .$$

This final equation is the combined gas law.

The gas law may be used to calculate the final conditions of a gas which has undergone a change. The following problem is an example.

EXAMPLE:

A gas initially at a pressure of 2 atm, a volume of 12 liters and a temperature of 300° K is heated to 400° K and its pressure decreased to 1 atm. What is the final volume of the gas?

SOLUTION:

The problem may be solved using either of the two methods discussed in the preceding section.

Method 1: For each of the variables whose numerical value is known substitute a number.

$$\frac{PV}{T} = \frac{P'V'}{T'}$$

$$\frac{2(12)}{300} = \frac{1 \cdot V'}{400}$$

Multiply both sides by 400 to isolate V'.

$$V' = \frac{2(12)(400)}{300}$$

$$= \frac{9600}{300}$$

$$= 32 \text{ liters}$$

Method 2: Solve the gas law equation literally for V'.

$$\frac{PV}{T}\left(\frac{T'}{P'}\right) = \frac{P'V'}{T'}\left(\frac{T'}{P'}\right)$$

$$\frac{PVT'}{TP'} = V'$$

Substitute numerical values for the variables on the left side of the equation to obtain a numerical solution for V'.

$$\frac{2(12)(400)}{300(1)} = V'$$

$$32 \text{ liters} = V'$$

It is often possible to check whether your result is reasonable by considering the qualitative aspects of the gas laws. In the preceding example the temperature increase caused the volume to increase according to Charles' Law. According to Boyle's Law the decrease in pressure also caused a volume increase. This qualitative prediction of an overall volume increase is consistent with our calculated final volume of 32 liters.

2i7

PROBLEM SET 25:

For Problems 1 through 10, solve the equation.

1. $\frac{60}{273} = \frac{V}{364}$

2. $\frac{760 \cdot 42}{280} = \frac{720 \cdot 38}{T'}$

3. $2(380 - N) = 190$

4. $r \cdot 121 \div 33 = 517$

5. $594 \cdot 780 = 11t - 594$

6. $1287 - 3m = 9873$

7. $43(10 - z) = 473$

8. $-6(3 - 2A) = -108$

9. $\frac{113}{T} + 12 = -1005$

10. $3x + 1 = 13$

11. a. Use the relation $PV = P'V_i$ to determine V_i when

$P = 300$ mm Hg

$V = 6$ liters

$P' = 360$ mm Hg.

Temperature is constant at 250° K.

b. Use the relation $\frac{V_i}{T} = \frac{V'}{T'}$ to determine V' when

V_i = The answer to part a

$T = 250°$ K

$T' = 300°$ K.

12. a. Use the relation $\frac{V}{T} = \frac{V_i}{T'}$ to determine V_i when

$V = 6$ liters

$T = 250°$ K

$T' = 300°$ K.

Pressure is constant at 300 mm Hg.

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b. Use the relation $PV_i = P'V'$ to determine V' when

V_i = the answer to part a

P = 300 mm Hg

P' = 360 mm Hg.

13. Using $\frac{PV}{T} = \frac{P'V'}{T'}$, determine the value of T' if

P = 300 mm Hg

V = 9.1 liters

V' = 5.1 liters

P' = 330 mm Hg

$T = 273° K$ (0° C).

14. Initially a gas has

P = 360 mm Hg

$V = 400\ cm^3$

$T = 300° K$.

Then the conditions are changed so that

$P' = 2P$ and $T' = 3T$.

Use the combined gas law to determine the new volume of the gas, V'.

15. Initially a quantity of gas had

$P = 6\ kg/cm^2$

V = 20 liters

$T = 350° K$.

Later the same quantity of gas had the following properties.

$P' = 24\ kg/cm^2$

V'= 40 liters

What is T'?

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16. For a given quantity of helium,

$P = P'$

$V = 4$ liters

$T = 300^\circ$ K

$T' = 25^\circ$ K.

What is the final volume?

17. A quantity of xenon gas was subjected to a change of temperature and pressure. In its final state, the gas had a temperature of -56° C, a pressure of 4 atmospheres and a volume of 0.7 liters. The initial temperature and pressure were 6° C and 1 atmosphere. What was the initial volume?

SECTION 26:

26-1 Pressure, Depth, and Aqualungs

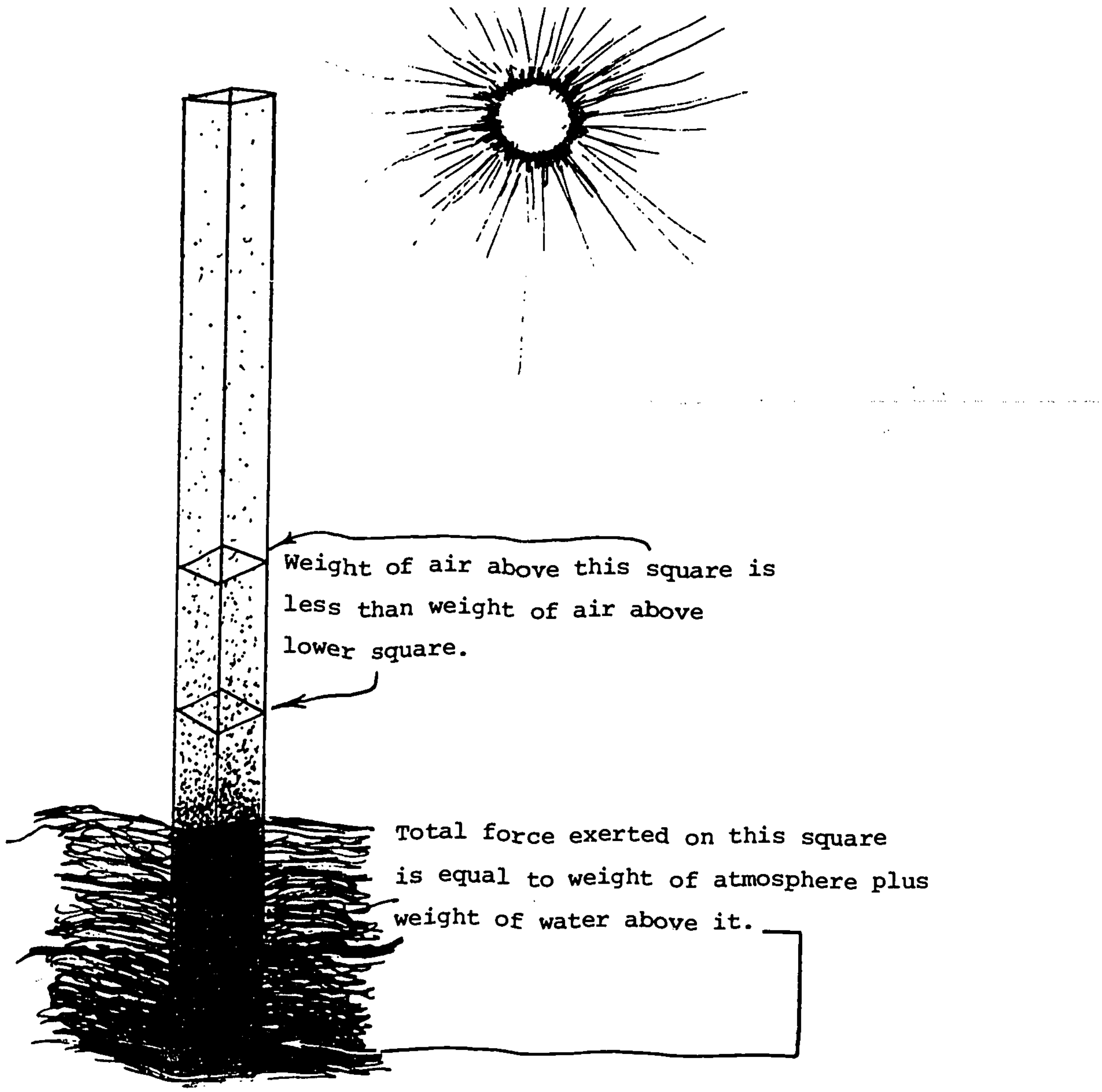
How many situations can you think of in which knowledge of the gas laws is important for the support of life?

In the present section we will be concerned with the application of the gas laws to one such situation--underwater diving. If the techniques used to solve the respiration problems of divers are mastered, you will be able to use the gas laws to solve life-support problems in other biomedical applications.

A diver is dependent for his air upon an aqualung. Aqualung air-tank capacities are in the neighborhood of 12 to 15 liters. They are strapped to the diver's back. Tubes connect the tanks to his mouth. When the diver inhales, a valve held in his mouth permits passage of air through the tubes. He expels air from his lungs into the water. This device is called an open-circuit breathing apparatus.

Pressure gauges commonly used by divers show the difference between pressure within the tank and atmospheric pressure. For instance, when pressure within the tank equals atmospheric pressure, the guage reads 0 lb/sq in. The pressure shown by a guage is called guage pressure. Consequently, true pressure, or absolute pressure, is the sum of guage pressure and atmospheric pressure. In all of the examples and problems that follow, the pressures given are absolute pressures, even though one would be reading guage pressures in a real situation.

Atmospheric pressure is the effect of the weight of the air above. As a person goes to a higher elevation the weight of the air above becomes less, and consequently the pressure of the atmosphere decreases. This idea is illustrated in the following diagram.



The higher the altitude, the less the pressure. As the pressure decreases, the density of the air decreases. In other words, the number of air molecules in a given volume decreases.

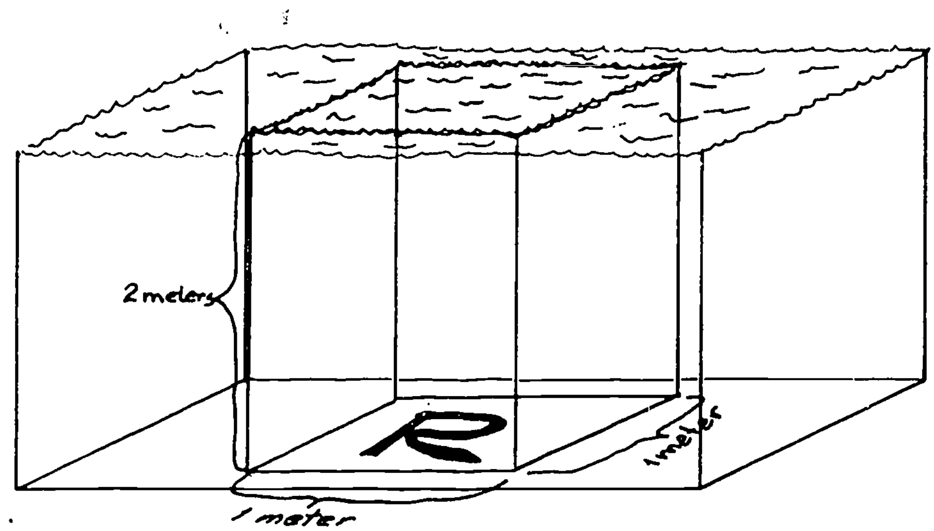
The situation under water is different with respect to the relationship between density and pressure. The density of water does not change significantly with variations in pressure. In other words, the number of water molecules in a given volume remains about the same, no matter what the pressure is. The constant density of water makes the calculation of pressure due to water simple. On the other hand, the nonconstant density of air makes the calculation of the pressure due to the air so difficult that we cannot demonstrate it to you now.

26-2 The Calculation of Pressure Due to Water

We will start out by finding the volume of water above whatever we're interested in. The mass of this water is directly related to its volume by its density, a constant. The pressure on our object is the mass of the water above divided by the horizontal cross section area of the object.

EXAMPLE:

Find the pressure due to water above region R.



SOLUTION:

First we find the volume of water above region R. The volume of the water is the (length) x (width) x (height).

$$V = \ell w h$$
$$= (1)(1)(2)$$
$$= 2 \text{ cubic meters}$$

Since each cubic meter of water has a mass of 1000 kg, the mass of the water above R is (2)(1000 kg) or 2000 kg.

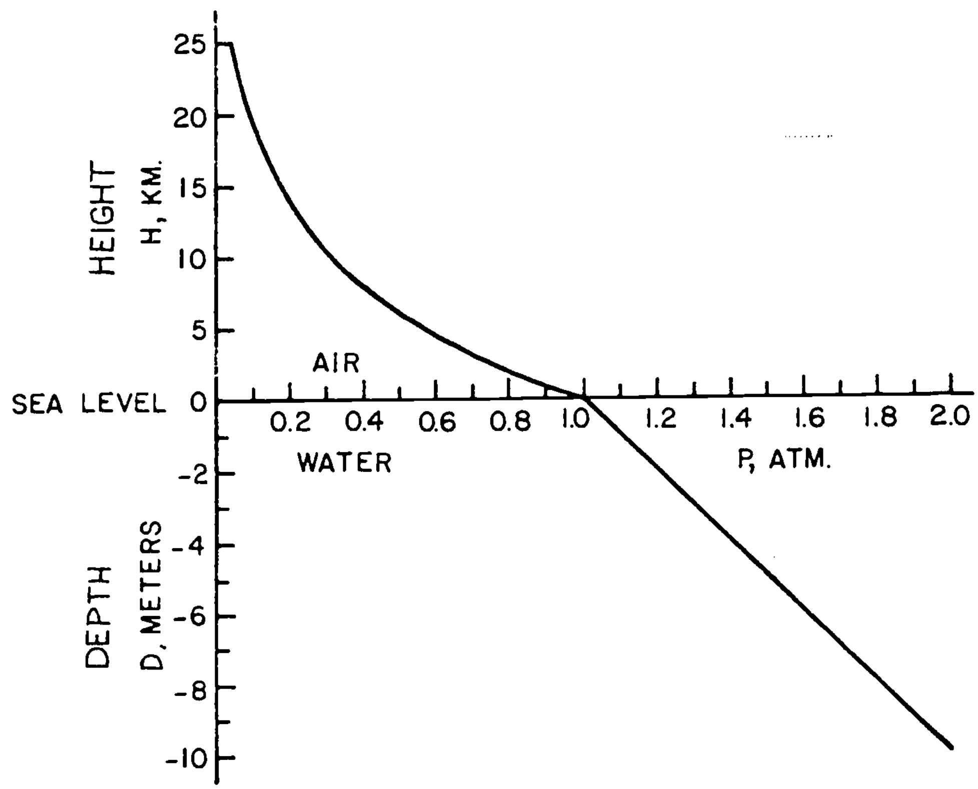
This mass is resting on an area of 1 square meter.

The pressure due to water is the mass of water resting on R (2000 kg) divided by the area of R (1 square meter). The pressure is 2000 kg per m^2.

Notice that to find the mass of water we multiplied the volume of water above R by the density of water, a constant. This technique will not work to find the pressure due to air. We cannot find the mass of a column of air by multiplying its volume by a constant density because the density of a column of air is not constant.

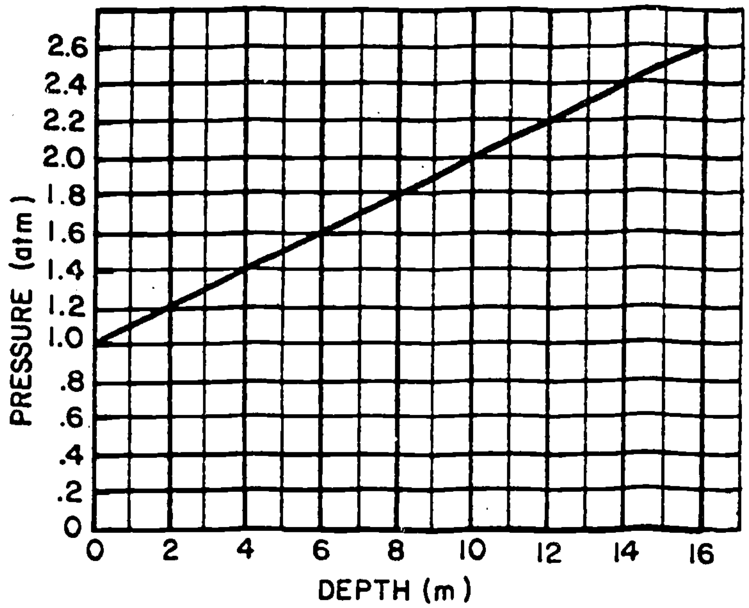
26-3 Pressure and Depth Under Water

It is the difference in the density behavior of air and water that accounts for the graph below. Pressure is a linear function of depth under water and a nonlinear function of depth under air. Notice that the scales for height and depth are different.



The graph on the following page shows the same straight line as before, except that the axes have been interchanged and the depths are shown as positive numbers. The graph shows the pressure as a function of depth under water.

22?



Note that the y-intercept of the graph is 1.0 atm. This is because at the surface of the water ($d = 0$) the pressure equals atmospheric pressure. The pressure on a diver is the pressure due to the water plus atmospheric pressure.

Since pressure (P) is a linear function of depth (d), the relationship may be written as an equation,

$$P = md + b.$$

The slope m is the change in pressure per unit change in depth, or about $\frac{1}{10}$ atm/m. The y-intercept b we noted to be 1 atm. We may rewrite the equation as

$$P = \frac{1}{10}d + 1 \quad \text{or} \quad P = \frac{d}{10} + 1.$$

The relationship is an approximate one, so we will write it as

$$P \approx \frac{d}{10} + 1.$$

This equation is useful in solving many types of problems pertaining to divers.

<u>EXAMPLE</u>:

What is the pressure on a diver at a depth of 8 m?

SOLUTION:

$$P \approx \frac{d}{10} + 1$$

$$\approx \frac{8}{10} + 1$$

$$\approx \frac{18}{10}$$

$$\approx \frac{9}{5} \text{ atm} \quad \text{or} \quad 1.8 \text{ atm}$$

EXAMPLE:

A diver at a depth of 10 meters is using air at a rate of 40 liters per minute. If the pressure in his 12-liter capacity tank is 80 atmospheres, how long can he remain underwater?

SOLUTION:

The pressure on the diver at a depth of 10 m is calculated.

$$P \approx \frac{10}{10} + 1$$

$$\approx 1 + 1$$

$$\approx 2 \text{ atm}$$

The volume occupied by the air in his tank at $P \approx 2$ atm is found by using the equation

$$PV = P'V'$$

$$80(12) \approx 2V'$$

$$V' \approx \frac{80(12)}{2}$$

$$\approx 480 \text{ liters}$$

Since the diver uses air at a rate of 40 liters/min, a volume of ~480 liters will last

$$\frac{\sim 480}{40} \approx 12 \text{ min.}$$

26-4 About Actual Diving Conditions

Our goal in the problems that follow is to answer questions similar to, "How long can a diver stay down for a given tank size, tank pressure, water temperature, and rate of air use?" We have made an attempt to find actual, realistic numbers for each of these variables. Therefore, the answers to the questions are realistic.

The tank capacity will figure in all of our problems. When the tank is being filled up with air, the tank's capacity is the final volume occupied by the compressed air. When the tank is being emptied out under water, it will be the initial volume occupied by the compressed air before it is released. Typical tank capacities run in the neighborhood of 12 to 15 liters.

The rate of air use will also figure into our calculations. A diver uses air at a rate that is largely independent of pressure. The rate of air use in terms of liters per minute is mainly a function of activity level. Vigorous activity will require more liters of air per time unit than a restful state requires. A given activity level will require about the same number of liters per minute whether a person is out of the water or submerged 20 meters under the surface. However, a given number of liters of air is much more dense at a depth of 20 meters than it is on the surface. This denser air is noticeably more difficult to move air into and out of the lungs. Typical air use rates are in the range of 22 to 40 liters per minute.

To find the volume of air available for breathing, we add a step in the solution that does not occur in actual diving situations. We occasionally release all of the air in the tank into an imaginary diving bell. We only do this to find the volume that the air would occupy at a particular pressure and depth. It is a step that helps us find a mathematical solution, but rarely, if ever, occurs. The length of time we get by supposing that a diver is breathing from an imaginary diving bell is the actual breathing time available to the diver. This is true even though the diver is breathing directly from his tank.

There is one final detail to be taken care of before a truly realistic answer to our target question can be gotten. It has to do with the conditions controlling the ability to suck air from a tank. When the pressure in the tank is reduced to the pressure of the surrounding water, very little more air may be removed from the tank. You can perform a simple experiment to test the truth of this statement.

Get a glass gallon bottle of some sort. Try to suck air out of it. You won't get much. The surrounding air and the air in the milk bottle are both at the same pressure, one atmosphere. When you try to get air out of the milk bottle, you are attempting to lower the pressure inside the bottle to less than one atmosphere. It is very difficult.

A diver at a depth of 10 meters faces the same problem when his tank pressure gets down to 2 atmospheres. The surrounding water is also at a pressure of two atmospheres. Even though there is more air in his tanks, he cannot get it out as long as he stays at a depth of 10 meters. However, he can get air out by going upward toward the surface. As he does so, the pressure around him is reduced, the pressure in the tank becomes more than the surrounding pressure and he will find that a small amount of air will become available.

PROBLEM SET 26:

1. An aqualung tank has a capacity of 12 liters. If such a tank is filled until a pressure of 200 atm absolute is attained, how many liters of air at 1 atm were pumped in?

Initially: $P = 1$ atm

$V = ?$

Finally: $P' = 200$ atm

$V' = 12$ liters

Use Boyle's Law, $PV = P'V'$.

2. If 1500 liters of air at one atmosphere were pumped into a 12-liter tank, what would the pressure be inside the tank?

3. The temperature of the air is 304 K (~88° F). 1200 liters of air at one atmosphere are pumped into a 12-liter tank which is then submerged into water at 285 K (~54° F).

a. What is the initial volume of the air that eventually ends up inside the tank?

b. What is the initial temperature of the gas in degrees Kelvin?

c. What is the initial pressure of the air in atmospheres?

d. What is the final volume of the compressed air?

e. What is the final temperature of the air in degrees Kelvin?

f. Use the relationship

$$\frac{PV}{T} = \frac{P'V'}{T'}$$

to find P' or the final pressure of the air inside the tank.

4. The pressure under water is a linear function of the depth. The relation is approximately

$$P \simeq \frac{d}{10} + 1$$

where P = pressure in atmospheres and d = depth in meters. Note that when $d = 0$ m, $P = 1$ atm because the atmosphere exerts 1 atm of pressure.

a. Use the given formula to find the pressure exerted on a diver at 5 meters.

b. Find the pressure exerted on a diver at 25 meters.

5. The pressure valve on an aqualung reads 50 atm absolute. The capacity of the tank is 12 liters. The depth of the diver is 30 meters. The diver is using air at the rate of 30 liters/minute. Initially the air is inside the tank and finally it is expelled into the water.

a. What is the initial pressure of the air?

b. What is the initial volume of the air?

c. Use the equation in Problem 4 to determine the pressure exerted on the expired air.

d. Use Boyle's Law, $PV = P'V'$, to determine the total volume that the air in the tank would occupy at a depth of 30 meters if it were allowed to escape into a diving bell.

e. How long could the diver remain at this depth if all of the air in the tank were available for breathing?

*f. How many liters of air at this depth would be unavailable for breathing?

6. The absolute pressure in an aqualung tank is 169 atm. The capacity of the tank is 15 liters. The depth of the diver is 3 meters. The diver is using air at the rate of 39 liters/minute. Initially the air is in the tank and finally it is expelled into the water.

a. What equation relates the pressure and volume of a quantity of gas under two different sets of conditions with constant temperature?

b. What is the initial pressure of the air?

c. What is the initial volume of the air?

d. What is the pressure of the expired air? (Use equation in Problem 4.)

e. What would be the volume of all of the air in the tank if it were not confined to the tank but subjected only to the pressure of 3 meters of water and one atmosphere of air?

f. How long could the diver remain at this depth if all of the air in the tank were available for breathing?

*g. How much air would be unavailable for breathing?

7. If all of the air in a 14-liter capacity air tank which had an internal pressure of 200 atm were released into a diving bell at a depth of 90 meters, what would the new volume of the air be?

8. How long could a diver remain at a depth of 20 meters if the pressure in his 12-liter capacity tank was 150 atm and he was using air at the rate of 30 liters per minute? Assume that all of the air in the tank would be available for breathing.

9. An aqualung tank is filled to its maximum pressure of 200 atm absolute on the surface. The surface temperature is 33° C. The capacity of the tank is 14 liters. The diver plans to spend most of his underwater time at a depth of approximately 4 meters. The temperature of the water is 16° C. Consider the air to be in its initial state when it is inside the tank on the surface.

a. What is the initial pressure of the air?

b. What is the initial temperature of the air in ° K?

c. What is the initial volume of the air?

d. What is the pressure at 4 meters depth?

e. What is the absolute temperature at 4 meters depth?

f. Use the relation

$$\frac{PV}{T} = \frac{P'V'}{T'}$$

to find the volume that the air in the tank would occupy if it were all released into a diving bell at 4 meters depth.

g. If the diver is consuming air at the rate of 34 liters per minute, how long might he expect to be able to breathe from the tank?

10. A 15-liter capacity tank is filled to its maximum pressure of 200 atm at a temperature of 300° K on the surface. How long could a diver remain at a depth of 5 meters if the water temperature was 282° K and he was using air at the rate of $23\frac{1}{2}$ liters per minute?

11. A number of oxygen tanks are kept in the storeroom of a hospital, located next door to the boiler room in the basement. One afternoon Elmo, the orderly, was sent to get a tank of oxygen for a patient on the fourth floor. The temperature in the storeroom was 35° C. As he picked up the tank, Elmo noticed that the guage read 203 atm. When he wheeled it into the patient's room, he saw that the pressure had dropped and warned the doctor that the tank must be leaking and might be dangerous. The doctor commended Elmo for being safety conscious, but explained that the reason for the drop in pressure was that the temperature in the patient's room was only 24° C. Elmo wasn't sure he understood this, but you know how to calculate the new pressure.

a. What was the initial pressure in the tank?

b. What was the initial temperature of the tank in degrees Kelvin?

c. What was the final temperature of the tank in degrees Kelvin?

d. The combined gas law states $\frac{PV}{T} = \frac{P'V'}{T'}$. Is V = V' in this situation?

e. How can the combined gas laws be simplified for this situation?

f. Using the simplified form of the combined gas law obtained in Part e and the numerical values obtained in Parts a, b, and c, write an equation which can be solved for P'.

g. What is P'?

12. Suppose, in Problem 11, that the oxygen tank were stored in the unheated warehouse adjacent to the hospital, where the temperature was -12° C. If the guage had indicated 203 atm in the warehouse, what pressure would it indicate on the fourth floor of the hospital where the temperature was 24° C. Remember that the volume remains the same.

13. A patient receives oxygen from a 9-liter tank with 200 atm initial pressure. The oxygen is expended at the rate of 10 liters per minute, measured at 1 atm pressure.

a. Assuming all the gas in the tank is available for breathing, what is the final volume of the gas at 1 atm pressure?

b. How long will it be before all the oxygen is expended?

11. A number of oxygen tanks are kept in the storeroom of a hospital, located next door to the boiler room in the basement. One afternoon Elmo, the orderly, was sent to get a tank of oxygen for a patient on the fourth floor. The temperature in the storeroom was 35° C. As he picked up the tank, Elmo noticed that the guage read 203 atm. When he wheeled it into the patient's room, he saw that the pressure had dropped and warned the doctor that the tank must be leaking and might be dangerous. The doctor commended Elmo for being safety conscious, but explained that the reason for the drop in pressure was that the temperature in the patient's room was only 24° C. Elmo wasn't sure he understood this, but you know how to calculate the new pressure.

a. What was the initial pressure in the tank?

b. What was the initial temperature of the tank in degrees Kelvin?

c. What was the final temperature of the tank in degrees Kelvin?

d. The combined gas law states $\frac{PV}{T} = \frac{P'V'}{T'}$. Is $V = V'$ in this situation?

e. How can the combined gas laws be simplified for this situation?

f. Using the simplified form of the combined gas law obtained in Part e and the numerical values obtained in Parts a, b, and c, write an equation which can be solved for P'.

g. What is P'?

12. Suppose, in Problem 11, that the oxygen tank were stored in the unheated warehouse adjacent to the hospital, where the temperature was -12° C. If the guage had indicated 203 atm in the warehouse, what pressure would it indicate on the fourth floor of the hospital where the temperature was 24° C. Remember that the volume remains the same.

13. A patient receives oxygen from a 9-liter tank with 200 atm initial pressure. The oxygen is expended at the rate of 10 liters per minute, measured at 1 atm pressure.

a. Assuming all the gas in the tank is available for breathing, what is the final volume of the gas at 1 atm pressure?

b. How long will it be before all the oxygen is expended?

FROM DECIMAL FORM TO SCIENTIFIC NOTATION

EXAMPLE:

Write 64,000,000 in scientific notation.

SOLUTION:

First we write 6.4.

Second, we observe that 6.4 is smaller than our original number 64,000,000; therefore we must multiply it by a positive power of ten to compensate.

Third, we count the number of places that we moved the decimal point

$$6\underbrace{4{,}000{,}000}_{\text{7 places}}$$

This means that we must multiply 6.4 by ten raised to the +7 power.

Finally, we have

$$6.4 \times 10^7 = 64{,}000{,}000$$

EXAMPLE:

Write .00031 in scientific notation.

SOLUTION:

First we write 3.1.

Second, we observe that 3.1 is bigger than our original number, .00031; therefore, we must multiply 3.1 by a negative power of ten to compensate.

Third, we count the number of places that we moved the decimal point.

$$.\underbrace{0003}_{\text{4 places}}1$$

This means that we must multiply 3.1 by ten raised to the -4 power.

Finally we have

$$3.1 \times 10^{-4} = .00031$$

FROM SCIENTIFIC NOTATION TO DECIMAL FORM

EXAMPLE:

Write 9.87×10^4 in decimal form.

SOLUTION:

First, we interpret the meaning of 10^4. It means that we must move the decimal point 4 places to the right. It is the sign of the exponent which tells us the direction to move the decimal point. Positive means right. Negative means left. This is the same arrangement found in number lines.

98,700.

4 places right

Finally,

$$9.87 \times 10^4 = 98{,}700$$

EXAMPLE:

Write 5.4×10^{-7} in decimal form.

SOLUTION:

The exponent of ten tells us to move the decimal point to the left (negative sign) seven places.

.00000054

7 places left

Finally,

$$.00000054 = 5.4 \times 10^{-7}$$

27-2 A Law of Exponents

What is the product of $10^2 \cdot 10^3$? Is it 10^6 or 10^5? Let us find out by writing 10^2 as 100 and 10^3 as 1000, and carrying out the multiplication.

$$\begin{aligned} 10^2 \cdot 10^3 &= 100 \cdot 1000 \\ &= 100{,}000 \\ &= 10^5 \end{aligned}$$

The multiplication above illustrates a law of exponents:

$$a^x \cdot a^y = a^{x+y}$$

This law of exponents is useful for solving problems such as the following example.

EXAMPLE:

Transform 96.9×10^9 into the form $.969 \times 10^n$.

SOLUTION:

Our problem is to determine n. Observe that 96.9 can be expressed as $.969 \times 10^2$ since $.969 \times 10^2 = .969 \times 100 = 96.9$. Thus

$$96.9 \times 10^9 = (.969 \times 10^2) \times 10^9.$$

By applying the associative law of multiplication and regrouping the terms on the right, we have

$$96.9 \times 10^9 = .969 \times (10^2 \times 10^9).$$

According to the exponent law

$$10^2 \times 10^9 = 10^{11}.$$

Therefore

$$96.9 \times 10^9 = .969 \times 10^{11}.$$

PROBLEM SET 27:

Determine n in the following equations:

1. $587 = 5.87 \times 10^n$

2. $6835 = 68.35 \times 10^n$

3. $3.7 = 3.7 \times 10^n$

4. $.075 = 7.5 \times 10^n$

5. $.075 = .75 \times 10^n$

6. $.075 = 75 \times 10^n$

7. $.075 = .0075 \times 10^n$

8. $5.87 \times 10^n = .0587$

9. $5.87 \times 10^n = 58{,}700$

10. $.0587 \times 10^n = 58.7$

Write the following numbers in scientific notation leaving only one digit to the left of the decimal point.

11. 9648

12. 43

13. 5.867

14. 1000

15. 60.708

16. .002

17. .0576

18. .000018

19. -357.2

20. -.00909

21. $\frac{1}{5}$

Convert the following numbers into decimal notation.

22. 6.8×10^3

23. 7.6006×10^0

24. 5.76×10^{-2}

25. -3.001×10^4

26. -4.7×10^{-1}

27. 77.1×10^{-5}

28. 9.8×10^1

29. A "googol" is defined as 1 followed by 100 zeros.

a. Write out a googol in conventional notation only if you want to.

b. Write a googol in scientific notation.

30. The sun is about 93 million miles from the earth. Write this distance in scientific notation.

31. A "micron" is defined as 1 millionth of a meter. Express a micron in scientific notation.

32. Use scientific notation to express these units in terms of the metric system. Use references to find the meanings of these units.

a. light year

b. parsec

c. barn

d. nanosecond

SECTION 28:

28-1 Multiplication in Scientific Notation

One of the virtues of scientific notation is that it facilitates multiplication and division of very large and very small numbers. Indeed, if multiplication and division were not easier when numbers are expressed in scientific notation, the use of the notation would be extremely limited.

Consider the multiplication

$$30{,}000{,}000{,}000 \times 400{,}000 = 12{,}000{,}000{,}000{,}000{,}000.$$

The operation is easy, but counting the zeros offers an opportunity for error and writing the zeros is time consuming.

In scientific notation, the same multiplication is written

$$(3 \times 10^{10}) \times (4 \times 10^{5}).$$

Recall the associative property of multiplication.

$$(ab)c = a(bc)$$

The associative property enables us to write

$$(3 \times 10^{10}) \times (4 \times 10^{5}) = 3 \times (10^{10} \times 4) \times 10^{5}.$$

Recall also the commutative property of multiplication.

$$ab = ba$$

The commutative property permits us to write

$$3 \times (10^{10} \times 4) \times 10^{5} = 3 \times (4 \times 10^{10}) \times 10^{5}.$$

Again applying the associative property, we write

$$\begin{aligned} 3 \times (4 \times 10^{10}) \times 10^{5} &= (3 \times 4) \times (10^{10} \times 10^{5}) \\ &= 12 \times (10^{10} \times 10^{5}) \end{aligned}$$

In Section 27 we found that

$$10^{p} \times 10^{q} = 10^{p+q}.$$

Therefore

$$12 \times (10^{10} \times 10^{5}) = 12 \times 10^{15}.$$

We have found then that

$$(3 \times 10^{10}) \times (4 \times 10^{5}) = 12 \times 10^{15}.$$

PROBLEM SET 27:

Determine n in the following equations:

1. $587 = 5.87 \times 10^n$
2. $6835 = 68.35 \times 10^n$
3. $3.7 = 3.7 \times 10^n$
4. $.075 = 7.5 \times 10^n$
5. $.075 = .75 \times 10^n$
6. $.075 = 75 \times 10^n$
7. $.075 = .0075 \times 10^n$
8. $5.87 \times 10^n = .0587$
9. $5.87 \times 10^n = 58{,}700$
10. $.0587 \times 10^n = 58.7$

Write the following numbers in scientific notation leaving only one digit to the left of the decimal point.

11. 9648
12. 43
13. 5.867
14. 1000
15. 60.708
16. .002
17. .0576
18. .000018
19. -357.2
20. -.00909
21. $-\frac{1}{5}$

Convert the following numbers into decimal notation.

22. 6.8×10^3
23. 7.6006×10^0
24. 5.76×10^{-2}
25. -3.001×10^4
26. -4.7×10^{-1}
27. 77.1×10^{-5}
28. 9.8×10^1

Our result may be stated as a general formula by writing

$$(a \times 10^p) \times (b \times 10^q) = ab \times 10^{p+q}$$

EXAMPLE:

Within a container are 2×10^{21} molecules, each of which collides with the walls of the container an average of 4×10^4 times per second. Determine the total number of collisions between molecules and walls in one second. (This problem was originally posed in Section 27.)

SOLUTION:

$$\begin{aligned}\text{Total number} &= (2 \times 10^{21}) \times (4 \times 10^4) \\ &= (2 \times 4) \times (10^{21} \times 10^4) \\ &= 8 \times 10^{25} \text{ collisions/sec}\end{aligned}$$

Many multiplications are of a type in which one or both of the exponents are negative numbers; for example,

$$(3 \times 10^{10}) \times (4 \times 10^{-5}) = 12 \times (10^{10} \times 10^{-5})$$

The exponent laws are equally valid for positive and negative exponents.

$$a^p \times a^q = a^{p+q}$$

is true whether p and q are positive or negative. Therefore

$$\begin{aligned}10^{10} \times 10^{-5} &= 10^{10+(-5)} \\ &= 10^5\end{aligned}$$

and

$$(3 \times 10^{10}) \times (4 \times 10^{-5}) = 12 \times 10^5.$$

EXAMPLE:

If one milliliter of air has a mass of 1.3×10^{-3} grams, what is the mass of 2.24×10^4 milliliters of air?

SOLUTION:

$$\begin{aligned}(1.3 \times 10^{-3}) \times (2.24 \times 10^4) &= (1.3 \times 2.24) \times (10^{-3} \times 10^4) \\ &= 2.912 \times 10^1 \\ &= 29.12 \text{ grams}\end{aligned}$$

28-2 Division in Scientific Notation

We now turn our attention to the division of one number expressed in scientific notation by another number expressed in scientific notation.

An example is the division of 8×10^2 by 2×10^4. We rearrange the expression slightly.

$$\frac{8 \times 10^2}{2 \times 10^4} = \frac{8 \times 10^2}{2} \left(\frac{1}{10^4}\right)$$

Recall the exponent law for division we used in Section 31.

$$\frac{1}{x^p} = x^{-p}$$

We apply the law here to write

$$\frac{1}{10^4} = 10^{-4}.$$

Therefore,

$$\frac{8 \times 10^2}{2 \times 10^4} = \frac{8}{2} \times 10^2 \times 10^{-4}$$

$$= 4 \times 10^2 \times 10^{-4}.$$

We recall the exponent law of multiplication and write

$$\frac{8 \times 2}{2 \times 10^4} = 4 \times 10^{-2}.$$

The result of the problem above may be generalized by writing

$$\frac{a \times 10^p}{b \times 10^q} = \frac{a}{b} \times 10^{p-q}.$$

EXAMPLE:

What is the slope of a graph in which $y_2 - y_1 = 6 \times 10^4$ and $x_2 - x_1 = 4 \times 10^2$?

SOLUTION:

Slope is defined as $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Our result may be stated as a general formula by writing

$$(a \times 10^p) \times (b \times 10^q) = ab \times 10^{p+q}$$

EXAMPLE:

Within a container are 2×10^{21} molecules, each of which collides with the walls of the container an average of 4×10^4 times per second. Determine the total number of collisions between molecules and walls in one second. (This problem was originally posed in Section 27.)

SOLUTION:

$$\begin{aligned} \text{Total number} &= (2 \times 10^{21}) \times (4 \times 10^4) \\ &= (2 \times 4) \times (10^{21} \times 10^4) \\ &= 8 \times 10^{25} \text{ collisions/sec} \end{aligned}$$

Many multiplications are of a type in which one or both of the exponents are negative numbers; for example,

$$(3 \times 10^{10}) \times (4 \times 10^{-5}) = 12 \times (10^{10} \times 10^{-5})$$

The exponent laws are equally valid for positive and negative exponents.

$$a^p \times a^q = a^{p+q}$$

is true whether p and q are positive or negative. Therefore

$$\begin{aligned} 10^{10} \times 10^{-5} &= 10^{10+(-5)} \\ &= 10^5 \end{aligned}$$

and

$$(3 \times 10^{10}) \times (4 \times 10^{-5}) = 12 \times 10^5.$$

EXAMPLE:

If one milliliter of air has a mass of 1.3×10^{-3} grams, what is the mass of 2.24×10^4 milliliters of air?

SOLUTION:

$$\begin{aligned} (1.3 \times 10^{-3}) \times (2.24 \times 10^4) &= (1.3 \times 2.24) \times (10^{-3} \times 10^4) \\ &= 2.912 \times 10^1 \\ &= 29.12 \text{ grams} \end{aligned}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6 \times 10^4}{4 \times 10^2}$$

$$= \frac{6}{4} \times 10^4 \times 10^{-2}$$

$$= 1.5 \times 10^2$$

PROBLEM SET 28:

The answers to all problems in this set are to be stated in scientific notation.

Perform the indicated operations.

1. $10^2 \cdot 10^3$

2. $10^4 \cdot 10^{-1}$

3. $\frac{10^5}{10^2}$

4. $\frac{10^5}{10^8 \times 10^2}$

5. $10^{-7} \cdot 10^{-3}$

6. $\frac{10^0 \cdot 10^5}{10^3}$

7. $x^2 \cdot x^3$

8. $x^4 \cdot x^{-1}$

9. $\frac{x^5}{x^2}$

10. $\frac{x^2}{x^5}$

For Problems 11-20, perform the indicated operations, leaving <u>one</u> digit to the left of the decimal point in your answer.

11. $(2 \times 10^2)(3 \times 10^3)$

12. $(1.5 \times 10^1)(1.5 \times 10^3)$

13. $(3 \times 10^{-2})(6 \times 10^{-4})$

14. $(.05 \times 10^{-1})(.1 \times 10^0)$

15. $(1.07 \times 10^{17})(10.1 \times 10^{-3})$

16. $(2.20 \times 10^6)(10.3 \times 10^{-3})(10 \times 10^7)$

17. $\frac{6 \times 10^5}{3 \times 10^2}$

18. $\frac{(3.05 \times 10^8)}{1 \times 10^{12}}$

19. $(1.32 \times 10^{-18}) \div (2.2 \times 10^{-7})$

20. $\frac{8.32 \times 10^{-4}}{4 \times 10^{-9}}$

Perform the indicated operations.

21. $\frac{5 \times 6 \times 8}{3 \times 4}$

22. $\frac{(3 \times 10^3)(5 \times 10^5)}{10 \times 10^2}$

23. $\frac{(2.5 \times 10^{-3})(3.5 \times 10^2)}{(5 \times 10^{-8})}$

24. $\frac{(3.5 \times 10^{-7})(2.07 \times 10^3)}{(7.0 \times 10^3)(8.28 \times 10^{-4})}$

25. A googol is 1 followed by 100 zeros.

   a. Write a googol in scientific notation.

   b. Multiply a googol by a googol and express the answer in scientific notation.

26. Given the gas law $\frac{PV}{T} = \frac{P'V'}{T'}$, and that

   $P = 1 \times 10^2$ atmospheres

   $V = 2.5 \times 10^{-2}$ liters

   $P' = 5 \times 10^2$ atmospheres

   $T = 2.80 \times 10^2$ degrees Kelvin

   $T' = 2.80 \times 10^2$ degrees Kelvin,

   express V' in scientific notation.

*27. Light travels at 3×10^6 meters per second. Express the distance that light will travel in one year in scientific notation. (Assume 1 year equals 365 days.)

   Note: This distance is called a "light-year." The star closest to the sun, called Proxima Centauri, is "only" 4.3 light years away!

*28. The farthest object in the heavens visible to the naked eye is the Andromeda galaxy (M-31). It is about 2×10^6 light years away. How many meters away is the Andromeda Galaxy?

*29. As of May, 1975 the farthest object detectable by instruments was a quasar labeled OJ-471. Astronomers estimate that it is about 3×10^9 (3 billion) light years away.

   a. How many meters away is OJ-471?

   b. Geologists estimate that the earth is about 5×10^9 (5 billion) years old. How old was the earth when the light we now receive from OJ-471 left OJ-471?

SECTION 29:

29-1 Decimal Powers of Ten

Long before the math course deals with the properties of exponents in detail, you will encounter powers of ten which are not whole numbers. You will see them in connection with pH, a measure of acidity, in science. Consequently we will deal with the problem briefly now. We will postpone a more detailed investigation of the world of exponents until later.

One way to get started is to pose the following question for discussion.

QUESTION:

What is the natural meaning for $10^{.1}$?

ANSWER:

Since $10^{.1}$ is an exponential expression, we expect it to follow the general exponential property.

$$(10^x)(10^y) = 10^{x+y}$$

When we multiply $10^{.1}$ by itself 10 times,

$$\underbrace{(10^{.1})(10^{.1})(10^{.1}) \;\ldots\; (10^{.1})}_{\text{10 factors}} = 10^{\overbrace{.1+.1+.1+\;\ldots\;+.1}^{\text{ten .1's}}}$$

Or, in other words

$$(10^{.1})^{10} = 10^{(.1)\cdot(10)}$$

$$= 10^1$$

This says that $10^{.1}$ raised to the tenth power is 10. This is one way of saying that $10^{.1}$ is the tenth root of 10. We have arrived at the natural meaning for $10^{.1}$. Now we are ready for the numerical value of $10^{.1}$.

$$10^{.1} \simeq 1.2589$$

Since this number is difficult to square, cube and so forth, we will often sacrifice a little precision by using the approximation

$$10^{.1} \simeq \frac{5}{4}$$

$$\simeq 1.25$$

We can use our numerical value of $10^{.1}$ to assemble a table of values for powers of ten in steps of .1. For example,

$$10^{.2} = (10^{.1})(10^{.1})$$

$$\simeq (1.2589)(1.2589)$$

$$\simeq 1.5848$$

Or less exactly,

$$10^{.2} \simeq (\tfrac{5}{4})(\tfrac{5}{4})$$

$$\simeq \frac{25}{16}$$

$$\simeq 1.5625$$

In turn, we can use our numerical value of $10^{.2}$ to find a numerical value for $10^{.3}$.

$$10^{.3} = (10^{.2})(10^{.1})$$

$$\simeq (1.5848)(1.2589)$$

$$\simeq 1.9951$$

Or, less exactly

$$10^{.3} \simeq 2$$

In a similar fashion we can build up a table for the powers of ten in steps of .1.

$10^{.1} \simeq 1.25$	$10^{.6} \simeq 4.0$
$10^{.2} \simeq 1.6$	$10^{.7} \simeq 5.0$
$10^{.3} \simeq 2.0$	$10^{.8} \simeq 6.3$
$10^{.4} \simeq 2.5$	$10^{.9} \simeq 7.9$
$10^{.5} \simeq 3.2$	$10^{.10} \simeq 10$

It is easy to see that each succeeding power of ten is approximately $\frac{5}{4}$ times the preceding one.

EXAMPLE:

Show numerically that $10^{.8} \simeq (10^{.7})(10^{.1})$

Use the approximation $10^{.1} \simeq \frac{5}{4}$

SOLUTION:

First we recognize that the problem asks us to multiply $10^{.7}$ by $\frac{5}{4}$ and then check to see if this result agrees with the table.

We find the numerical value of $10^{.7}$ in the table

$$10^{.7} \simeq 5$$

$$(10^{.7})(10^{.1}) \simeq (5)(\tfrac{5}{4})$$

$$10^{.8} \simeq \frac{25}{4}$$

$$\simeq 6\tfrac{1}{4}$$

$$\simeq 6.25$$

We see that 6.25 is in close agreement to the value of 6.3 for $10^{.8}$ which is in the table.

EXAMPLE:

Find a numerical value for $\frac{10^{.4}}{10^{.2}}$

SOLUTION:

We use the exponential property which says

$$\frac{10^y}{10^x} = 10^{y-x}$$

to write

$$\frac{10^{.4}}{10^{.2}} = 10^{(.4\ -\ .2)}$$

$$= 10^{.2}$$

Next we go to the table to find

$$10^{.2} \simeq 1.6$$

Therefore

$$\frac{10^{.4}}{10^{.2}} \simeq 1.6$$

29-2 Expressing Decimal Exponents as Numbers in Scientific Notation

PROBLEM:

Express $10^{4.7}$ in scientific notation.

SOLUTION:

Our first task here is to recognize that the exponent 4.7 is the sum of a whole number and a decimal.

$$4.7 = 4 + .7$$

Therefore

$$10^{4.7} = 10^{(4+.7)}$$

Next we use the property of exponents to rewrite the right side as a product of two powers of ten.

$$10^{(4+.7)} = (10^4)(10^{.7})$$

Finally we refer to the list of equations in Section 29-1 to find that $10^{.7} \simeq 5$. Therefore

$$10^{4.7} \simeq 5 \times 10^4$$

PROBLEM:

Express $10^{-3.2}$ in scientific notation.

SOLUTION:

The key to this problem is expressing the exponent as the sum of a negative integer and a positive decimal. We need a positive intege because our table of values for 10^x uses positive tenths as exponents.

$$-3.2 = -4 + .8$$

Therefore

$$10^{-3.2} = 10^{(-4 + .8)}$$

From this point we follow the pattern of the previous problem.

$$10^{(-4 + .8)} = (10^{-4})(10^{.8})$$

Therefore

$$\frac{10^{.4}}{10^{.2}} \simeq 1.6$$

29-2 Expressing Decimal Exponents as Numbers in Scientific Notation

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SOLUTION:

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Therefore

$$10^{4.7} = 10^{(4+.7)}$$

Next we use the property of exponents to rewrite the right side as a product of two powers of ten.

$$10^{(4+.7)} = (10^{4})(10^{.7})$$

Finally we refer to the list of equations in Section 29-1 to find that $10^{.7} \simeq 5$. Therefore

$$10^{4.7} \simeq 5 \times 10^{4}$$

PROBLEM:

Express $10^{-3.2}$ in scientific notation.

SOLUTION:

The key to this problem is expressing the exponent as the sum of a negative integer and a positive decimal. We need a positive integer because our table of values for 10^{x} uses positive tenths as exponents.

$$-3.2 = -4 + .8$$

Therefore

$$10^{-3.2} = 10^{(-4 + .8)}$$

From this point we follow the pattern of the previous problem.

$$10^{(-4 + .8)} = (10^{-4})(10^{.8})$$

From the table we find that

$$10^{.8} \simeq 6.3$$

Therefore,

$$10^{-3.2} \simeq 6.3 \times 10^{-4}$$

PROBLEM SET 29:

1. $\underbrace{(10^{.1})(10^{.1}) \ldots (10^{.1})} = 10$

   How many factors?

2. $10^{.1}$ is the ____ root of 10.

3. $\underbrace{(10^{.2})(10^{.2}) \ldots (10^{.2})} = 10$

   How many factors?

4. $10^{.2}$ is the ____ root of ten.

5. $(10^{.5})(x) = 10^{1.0}$

   $x = ?$

6. $10^{.5}$ is the ____ root of ten.

7. Why will we sometimes use the approximation

$$10^{.1} \simeq \frac{5}{4}$$

instead of $10^{.1} \simeq 1.2589$?

Problems 8 through 40 refer to these approximate values for the decimal powers of ten.

$10^0 = 1$	$10^{.6} \simeq 4.0$
$10^{.1} \simeq 1.25$	$10^{.7} \simeq 5.0$
$10^{.2} \simeq 1.6$	$10^{.8} \simeq 6.3$
$10^{.3} \simeq 2.0$	$10^{.9} \simeq 7.9$
$10^{.4} \simeq 2.5$	$10^{1.0} = 10$
$10^{.5} \simeq 3.2$	

8. Show numerically that

$$10^{.2} \simeq (10^{.1})(10^{.1})$$

9. Show numerically that

$$10^{.4} \simeq (10^{.3})(10^{.1})$$

10. Show numerically that

$$10^{.5} \simeq (10^{.4})(10^{.1})$$

11. Show numerically that

$$10^{.6} \simeq (10^{.5})(10^{.1})$$

12. Show numerically that

$$10^{1.0} \simeq (10^{.9})(10^{.1})$$

Find numerical values for the quotients in Problems 13 through 20. Use the property

$$\frac{10^{y}}{10^{x}} = 10^{y-x}$$

13. $\frac{10^{.8}}{10^{.6}} \simeq$

14. $\frac{10^{.9}}{10^{.4}}$

15. $\frac{10^{1.3}}{10^{.9}} \simeq$

16. $\frac{10^{4.8}}{10^{3.9}} \simeq$

17. $\frac{10^{-3.8}}{10^{-4.4}} \simeq$

18. $\frac{10^{787.3}}{10^{786.7}} \simeq$

19. $\frac{10^{7.8}}{10^{7.4}} \simeq$

20. $\frac{10^{.7}}{10^{.6}} \simeq$

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Express the following powers of ten in scientific notation.

21. $10^{1.2} \simeq ?$

22. $10^{3.4} \simeq ?$

23. $10^{7.3} \simeq ?$

24. $10^{9.9} \simeq ?$

25. $10^{1.7} \simeq ?$

26. $10^{2.6} \simeq ?$

27. $10^{8.1} \simeq ?$

28. $10^{101.5} \simeq ?$

29. $10^{24.8} \simeq ?$

30. $10^{15.4} \simeq ?$

31. $10^{-1.5} \simeq ?$

32. $10^{-7.9} \simeq ?$

33. $10^{-8.4} \simeq ?$

34. $10^{-21.7} \simeq ?$

35. $10^{-17.2} \simeq ?$

36. $10^{-4.3} \simeq ?$

37. $10^{-7.6} \simeq ?$

38. $10^{-24.8} \simeq ?$

39. $10^{-.9} \simeq ?$

40. $10^{-7.2} \simeq ?$

SECTION 30:

30-1 Review Problems

The problems in the following problem set provide extra practice with the ideas in Sections 24 through 29. The following table tells the ideas reviewed in each problem.

PROBLEM	CONTENT	REVIEW OF SECTIONS
1	scientific notation	27 and 28
2	scientific notation and combined gas law	24, 25, 26, 27, 28
3	simple equations	24
4	scientific notation and Boyle's Law	24, 27, 28
5	diver problems	26
6	scientific notation and simple equations	24, 27, 28
7	diver problem	26
8	decimal powers of ten	29
9	decimal powers of ten	29

PROBLEM SET 30:

1. State each of the following expressions in scientific notation.

   a. $10^{-3} \times 10^{2} \times 10^{4}$

   b. $\dfrac{10^{0} \times 10^{-5}}{10^{+10}}$

   c. $\dfrac{(5 \times 10^{3})(2.50 \times 10^{-2})}{(5 \times 10^{-7})(25 \times 10^{0})}$

2. Express T in scientific notation, using the combined gas law $\frac{PV}{T} = \frac{P'V'}{T'}$ and the following data.

   $P = 1.60 \times 10^{2}$ atm

   $V = 8 \times 10^{-1}$ ml

   $T' = 2.70 \times 10^{2}$ °K

   $P' = 16 \times 10^{-1}$ atm

   $V' = 12 \times 10^{1}$ ml

3. Solve the following equations.

a. $3s - 6 = 18$

b. $2(t - 6) = t + 4$

c. $\frac{32}{x} = .4$

d. $15 - \frac{h}{2} = 2h$

4. A quantity of hydrogen occupies 2.82×10^2 ml at a pressure of 5 atm. The gas is compressed into a volume $\frac{2}{3}$ the size of the original volume.

a. What is the new volume in scientific notation?

b. Assuming constant temperature, what is the new pressure?

5. A diver is working at a depth of 15 meters with a 12-liter tank on his back containing air at 200 atm pressure.

a. What is the pressure on the inside of the tank?

b. What is the pressure on the outside of the tank? [Remember: pressure in atm $\approx \frac{\text{depth in meters}}{10} + 1$]

c. What is the volume of air in the tank?

d. What volume would the air occupy if <u>all</u> of the air escaped from the tank into a bell jar at this depth (use PV = P'V']?

e. If the diver used air at the rate of 30 liters a minute, how long could he breathe, assuming he could use all the air in the tank?

6. Find the solution sets to the following equations, and express the answers in scientific notation.

a. $y \times 10^4 = 10^{-7}$

b. $\frac{L \times 10^{-8}}{10^4} = 10^3$

c. $\frac{(5 \times 10^3)(S)}{.5 \times 10^4} = 10^2$

7. A 6-liter tank is filled with air to a pressure of 100 atm. It is then suspended by a cable at a depth of 490 meters.

a. What is the pressure on the outside of the tank? (pressure in atm $\approx \frac{\text{depth in meters}}{10} + 1$)

b. If all the air in the tank should escape, what would be the volume of air escaping? (use PV = P'V')

c. There is a leak in the tank and all the air escapes into the water at an average rate of .1 liter/minute, as measured outside the tank. How long will it take for the air to escape?

8. a. $(10^{4.7})(10^{3.2}) = ?$

b. $10^{.2}$ is the ____ root of ten.

c. $10^{.5}$ raised to the ______ power is ten.

d. $(10^{3.7})(10^{-5.1}) = ?$

9. Express the following decimal powers of ten in scientific notation.

a. $10^{4.9} \approx ?$

b. $10^{7.3} \approx ?$

c. $10^{-1.7} \approx ?$

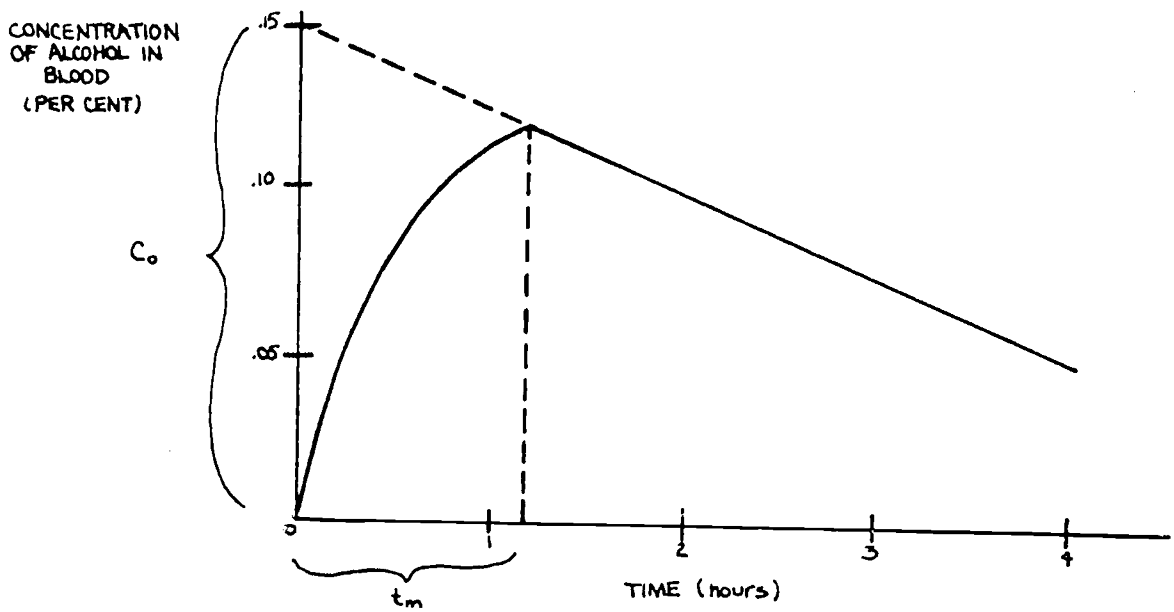
d. $10^{-99.1} \approx ?$

SECTION 31:

31-1 Alcohol Ingested on an Empty Stomach

In this unit you have studied several biomedical phenomena which may be described by linear functions. The phenomenon we will now examine is the concentration of alcohol in the blood, which over part of its domain is a linear function of time.

The relation between alcohol concentration in the blood and the time elapsed since its ingestion is shown in the following graph.



This graph represents the situation in which alcohol is taken on an empty stomach. We will later contrast this situation to that in which alcohol is taken on a full stomach.

The concentration of alcohol in the blood rises over a relatively short period of time to a peak value. This period of time is known as the distribution phase and is designated t_m. The subscript m is for "maximum."

After the alcohol concentration has reached its maximum value, the relationship between concentration and time is approximately linear. This period is called the post-absorptive phase. The slope of the line in the post-absorptive phase is negative, meaning that the concentration decreases by a constant amount each hour.

As you have learned, the linear portion of the graph can be described by an equation of the form $y = mx + b$. Researchers in this area of investigation generally write the equation in the form

$$c \simeq \beta t + c_o.$$

c is the concentration of alcohol in the blood given as a per cent, which is equivalent to giving the concentration as grams of alcohol per 100 grams of blood. β is the slope of the line and t is the elapsed time, in hours, since ingestion. c_o is the y-intercept and is obtained by extending the line to intersect the y-axis.

c_o is what the initial concentration would be if the alcohol diffused throughout the body immediately, and no time were required for distributuion. c_o is called the "initial" concentration, although the actual alcohol concentration in the blood is never that large. The "initial" concentration has been found to be directly proportional to the amount of alcohol ingested (A) and inversely proportional to the mass of the body (M). These relationships are expressed by the equation

$$\frac{A}{M} \simeq rc_o$$

r is a constant of proportionality. When A is given in milliliters, and M is given in kilograms, the numerical value of r has been empirically determined to be $9.04 \pm .71$.

The time of the distribution phase, t_m, has been found to be $1.43 \pm .49$ hours, and the slope β has been found to be $-.01338 \pm .00184 \frac{\text{g alcohol}}{\text{100 g blood}}$ per hour. β varies little when the same individual is tested on different days, but does vary from individual to individual.

The equations and the values of the constants were obtained under certain specific conditions.

1. The subjects had fasted overnight.

2. The total amount of alcohol was ingested in a ten-minute period.

3. The subjects who could tolerate it took the alcohol as straight vodka, which is about 50% alcohol.

These three conditions are not likely to be encountered in a social drinking situation, but they are of value in establishing basal values.

31-2 <u>The Alcohol Content of a Beverage</u>

Before the formula $c = \beta t + c_o$ may be used to find the blood alcohol concentration for a particular time t, we must calculate c_o, the initial concentration. It is found from the formula

$$c_o = \frac{1}{r}\frac{A}{M}$$

where A is the number of ml of <u>pure</u> alcohol. Since a person will rarely take pure alcohol it will often be necessary to determine the amount of alcohol in a beverage which contains less than 100% alcohol. Different beverages have different concentrations of alcohol. The table below shows typical concentrations of alcohol for different beverages.

Beverage	Alcoholic Concentration in "percent" $1 \text{ percent} = \frac{1 \text{ ml alcohol}}{100 \text{ ml beverage}}$
Beer	5%
Table Wine	12%
Port	19%
Whiskey	50%

For hard liquor such as whiskey the concentration is often stated in terms of "proof." The proof of a liquor is simply twice its percentage concentration. For example, 50 percent = 100 proof and pure alcohol is 100% or 200 proof.

Notice that beverage alcohol concentrations are in terms of <u>ml</u> alcohol per 100 <u>ml</u> beverage, while blood alcohol concentrations are stated in terms of <u>grams</u> alcohol per 100 <u>grams</u> of blood. It seems unnecessarily confusing doesn't it? We certainly think so. However, this is the way it is and we must make the best of it.

The following example demonstrates how to use the volume of a beverage and its concentration to find "A", the amount of pure alcohol.

EXAMPLE:

How many ml of alcohol are contained in 1050 ml of wine (12% alcohol)?

SOLUTION:

First, we interpret the 12%. It means that there are 12 ml of alcohol in 100 ml of wine.

Second, we set up a dimensional algebra statement that converts volume of beverage to volume of alcohol.

$$10\cancel{50}\ \cancel{\text{ml beverage}} \cdot \frac{12\text{ ml alcohol}}{\cancel{100\text{ ml beverage}}} = (10.5)(12)\text{ ml alcohol}$$

$$= 126\text{ ml alcohol}$$

We put "ml alcohol" on top because we want to end up with units of ml alcohol. Notice also that we must distinguish between "ml alcohol" and "ml beverage." If we didn't it would be easy to get confused. Furthermore, ml alcohol and ml beverage are not identical.

31-3 A More Complicated Example

EXAMPLE:

Old Idiot Rum is 140 proof.

a. How many ml of alcohol are in 400 ml of the rum?

b. If enough cola drink is added to the rum to make 1000 ml of solution, what is the percentage alcohol concentration of the solution?

SOLUTION:

a. First we interpret 140 proof. 140 is twice the percentage concentration; therefore, Old Idiot is 70% alcohol.

By following the pattern of the first example we find the volume of alcohol in 400 ml of the rum.

$$4\cancel{00}\ \cancel{\text{ml rum}} \cdot \frac{70\text{ ml alcohol}}{\cancel{100\text{ ml rum}}} = 280\text{ ml alcohol}$$

b. Enough cola is added to this 280 ml alcohol to make 1000 ml of solution. We are interested in the amount of alcohol in 100 ml of the drink.

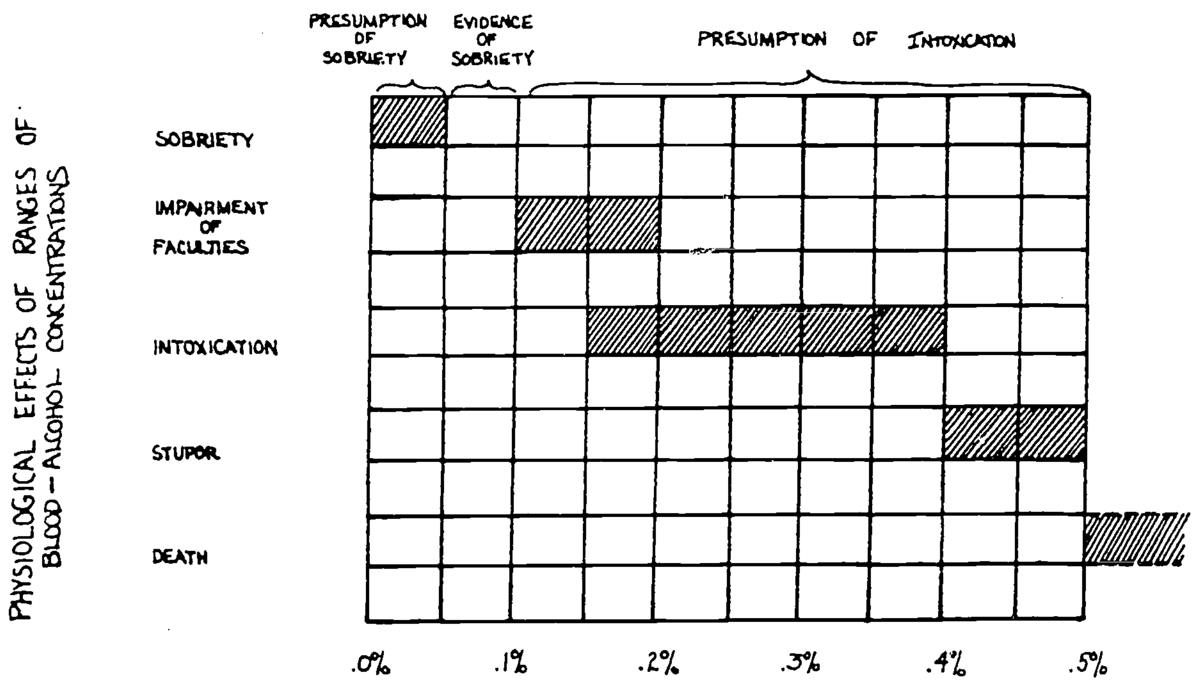
$$1\cancel{00}\text{ ml drink}\quad \frac{280\text{ ml alcohol}}{10\cancel{00}\text{ ml drink}} = 28\text{ ml alcohol}$$

Since there are 28 ml alcohol in 100 ml of the drink, the concentration of alcohol is 28%.

31-4 Alcohol and the Law

Every state has legislation concerning drinking and driving. There is a relationship between blood alcohol concentration and the physiological impairment. The following chart shows the relationship.

TYPICAL LEGAL IMPLICATIONS OF RANGES OF BLOOD-ALCOHOL CONCENTRATION



Different states have differences in the details of their laws. However, if you are stopped by a policeman and found to have a blood alcohol concentration greater than .1 percent, you will generally have to do some explaining to a judge. Your teacher will be able to supply you with the details which apply to your state. Another source of information on this matter is your local police department. It would be wise to double check the information your teacher gives you by giving city hall a call. Laws can change more quickly than textbooks can be revised.

PROBLEM SET 31:

1. Old Bloodshot Whiskey is 120 proof. How many ml of pure alcohol are in 500 ml of the whiskey?

2. Fannie Chartreuse Shingles wine is 15% alcohol. How many ml of pure alcohol are in 600 ml of the wine?

3. A 6.2-liter solution is made up of 75% alcohol. How many liters of alcohol are in the solution?

4. If 200 ml of alcohol are contained in a 1,800 ml solution, what percentage of the solution is alcohol?

5. If 200 ml of coconut oil are contained in a 4-liter solution of sun tan lotion, what percentage of the solution is coconut oil?

6. Raskolnikov's Vodka is 50% alcohol.

a. How many ml of alcohol are in 750 ml of the vodka?

b. If enough orange juice is added to make 1,000 ml of solution, what percentage of the solution is alcohol?

7. A 2-liter batch of cake batter is 0.5% vanilla extract.

a. How many ml of vanilla extract are in the batter?

b. If the vanilla extract is 40% alcohol, how many ml of alcohol are in the cake batter?

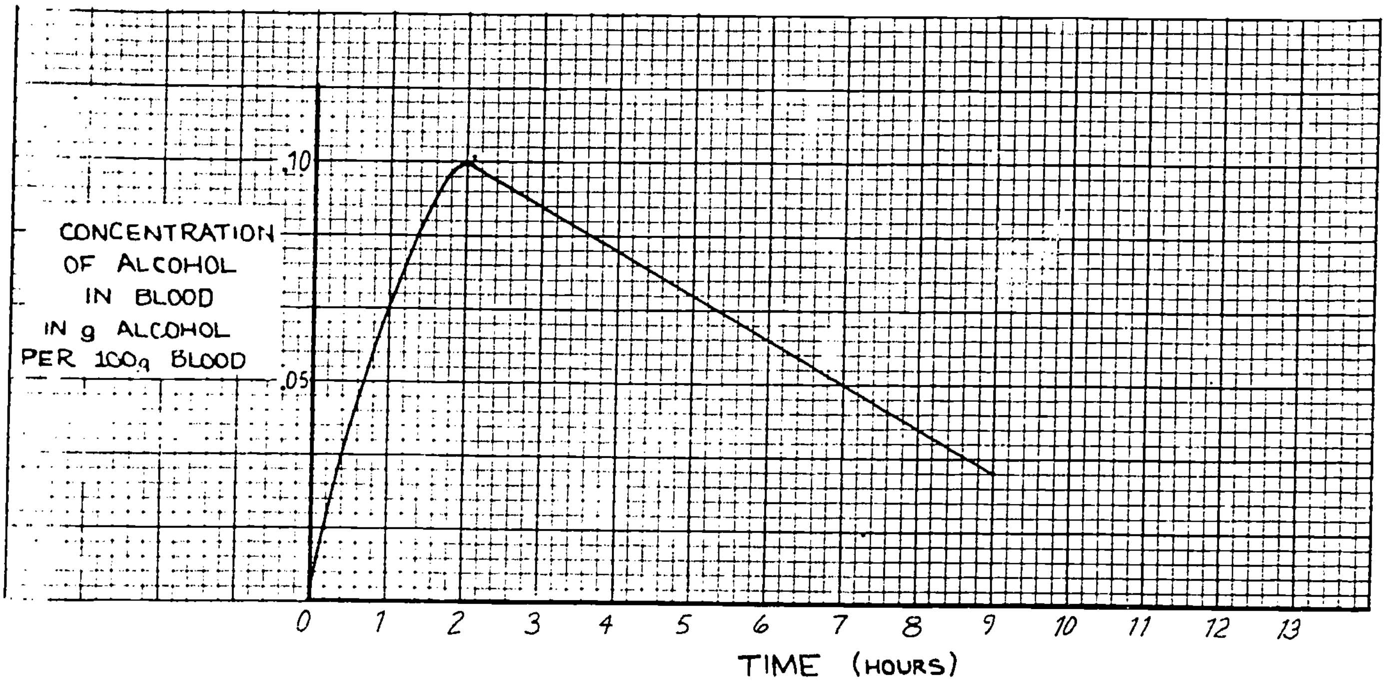
8. Dave drinks 200 ml of bourbon (45% alcohol) before he gets bored. He then drinks 200 ml of vodka (50% alcohol).

a. How many ml of alcohol does Dave drink when he drinks the bourbon?

b. How many ml of alcohol does Dave drink when he drinks the vodka?

c. What is the total number of ml of alcohol that Dave drinks?

d. What percentage of the 400 ml of bourbon and vodka is alcohol?



Problems 9 through 18 refer to the above graph. Give true or false answers for Problems 9 through 17.

9. At time t = 7 the % concentration of alcohol in the blood is .05%.

10. The slope of the straight-line portion of the graph is positive.

11. The slope may be interpreted as how much the % concentration of alcohol in the blood decreases per hour.

12. Between times t = 2 and t = 7, the rise is -.05 and the run is 5.

13. At time t = 0 the % concentration of alcohol in the blood is .01%.

14. The slope of the entire graph is constant.

15. The curved portion of the graph from time t = 0 to t = 2 represents the phase when alcohol is diffusing from the stomach into the rest of the body.

16. In the equation $c \approx \beta t + c_o$, β is the slope of the straight-line portion of the graph and can be thought of as the rate at which the body eliminates alcohol from the bloodstream.

17. In the equation $c \approx \beta t + c_o$ the quantity c_o may be thought of as the alcohol concentration in the blood if all the alcohol immediately diffused throughout the body.

18. a. The time it takes for the concentration to reach its highest value is called "t_m." Find t_m from the graph.

b. Notice that the straight-line portion of the graph does not reach the y-axis. If the straight-line portion were extended until it struck the y-axis, the y-intercept of the line could be determined. The y-intercept of the straight-line portion is called the "initial concentration" and is denoted by c_o. Use a straight-edge to determine c_o.

c. The rate of change in the % concentration of alcohol in the blood is called "β"; it is the slope of the straight-line part of the graph. Find it. Hint: Note that the x and y axes have different scales.

d. Substitute the values you found for β and c_o into the equation $c = \beta t + c_o$.

You now have a equation which relates the concentration of alcohol in the blood to elapsed time after ingestion.

e. Does the equation found in part d apply when $t = 1$ hour?

f. Does the equation apply for $t = 14$ hours?

g. By extending the straight-line part of the graph downward to the x-axis (time axis), you can find how long it will take to reach 0 concentration. Find that time.

h. Describe the domain of the linear function, or the values of t for which the linear function, $c \approx \beta t + c_o$, will give the same blood-alcohol concentration as that shown in the graph.

i. What should happen to the line for values of $t > 12$ hours? Why?

19. a. Using the same scales as in the illustrated graph, draw your own graph using the following information.

$\beta \simeq -.01$

$t_m \simeq 2$ hours

$c_o \simeq .20\%$

b. How long does it take for the concentration to reach .10%?

c. Use the graph or the formula $c \simeq \beta t + c_o$ to determine how long it will take for the concentration to equal zero.

d. Try to find the concentration for t = 1 hour using the equation $c \simeq \beta t + c_o$. Does your answer agree with the graph? If not, why not.

20. a. Use the same scales as the illustrated graph to draw your own graph using the following information.

$\beta = -.015$ (this is within the range $-.01338 \pm .00184$)

$t_m = 1.6$ hours (this is within the range $1.43 \pm .49$)

$c_o = .30\%$

b. How long does it take for the concentration to reach .10%?

c. How long does it take until c = 0?

21. a. Construct your own graph. Scale as in previous problems. Use the following information.

$\beta = -.014$ (this is within the range $-.01338 \pm .00184$)

$t_m = 1$ hour (this is within the range $1.43 \pm .49$ hr)

$c_o = .28\%$

b. How long does it take until c = 0?

c. What is the maximum blood alcohol concentration obtained?

SECTION 32

32-1 An Unfortunate Incident at the Winery

Pierre de la Peinteur is a professional winery painter of many years experience. One day he was applying a coat of pleasing pink to the ceiling of the ancient Chateau Acide when he stepped into a bucket of paint and fell off his scaffold.

Fortunately Pierre was working directly over an open vat of the chateau's finest wine, Acide Ordinaire. Otherwise he might have been killed. He plunged head first into the vat of wine.

Not knowing how to swim, Pierre, who was esteemed throughout the countryside for his quick thinking, decided to avoid drowning by allowing the wine to enter his stomach instead of his lungs.

Luckily, two strong vintage tramplers were nearby. They rapidly came to Pierre's assistance and pulled him feet first from the vat. By this time, however, Pierre had consumed 2.25 liters of Acide Ordinaire, which is 12% alcohol by volume.

It soon became clear that poor Pierre was in no condition to ply his trade. Nor would it be safe to allow Pierre to drive home, since the local magistrate had a tendency to sentence to the guillotine those caught driving with a blood-alcohol concentration of over .10 per cent.

The proprietor of the chateau prudently prepared to protect Pierre from such a painful potentiality. Being a man of great generosity, he extracted 21 francs from Pierre's pocket to pay for the wine he had drunk. In so doing, the proprietor placed himself at considerable disadvantage, since this was the wholesale price. He then dragged Pierre's 100-kilogram bulk to the bottle-corking machine and shackled him to it until such time as it would be legal for Pierre to start his vehicle.

We will analyze poor Pierre's condition in terms of the concentration of alcohol in his blood. Specifically, we will answer four questions, assuming the following values for the constants β, r and t_m.

$$\beta \simeq -.015\% \text{ per hour}$$

$$r \simeq 9$$

$$t_m \simeq 1.5 \text{ hours}$$

1. What was Pierre's "initial" concentration (c_o)?

Pierre consumed 2.25 liters, or 2250 ml, of wine which was 12% alcohol. The amount of alcohol (A) he consumed was therefore 2250(.12) = 270 ml. The "initial" concentration is obtained by the formula

$$\frac{A}{M} \simeq rc_o.$$

We substitute A = 270 ml, M = 100 kg and r ≃ 9 and write

$$\frac{270}{100} \simeq 9c_o$$

We divide both sides of the equation by 9 to obtain

$$\frac{270}{100(9)} \simeq c_o$$

$$c_o \simeq .3\%$$

2. What was the maximum blood-alcohol level actually attained by Pierre?

Recall that blood-alcohol concentration is related to elapsed time after ingestion by the equation

$$c \simeq \beta t + c_o$$

Maximum concentration is attained at t_m, the end of the distribution phase. Therefore,

$$c_{max} \simeq \beta t_m + c_o$$

We substitute $\beta \simeq -.015\%$ per hour, $t_m \simeq 1.5$ hours and $c_o \simeq .3\%$. The result is

$$\begin{aligned} c_{max} &\simeq (-.015)(1.5) + .3 \\ &\simeq -.0225 + .3 \\ &\simeq .2775\% \end{aligned}$$

3. How much time elapsed before Pierre's blood-alcohol concentration reached .10%?

In this case the unknown is time. The known quantities are $c \simeq .10\%$, $\beta \simeq -.015\%$ per hour and $c_o \simeq .3\%$. We substitute these quantities into the equation

$$c \simeq \beta t + c_o$$

to obtain

$$.10 \simeq (-.015)t + .3$$

Solving for t,

$$.10 - .3 \simeq (-.015)t$$

$$-.2 \simeq (-.015)t$$

$$\frac{-.2}{-.015} \simeq t$$

the result is

$$t \simeq 13\tfrac{1}{3} \text{ hours.}$$

4. After how many hours would Pierre's blood-alcohol concentration be 0%?

We substitute the quantities

$$c = 0\%$$

$$\beta \simeq -.015\% \text{ per hour}$$

$$c_o \simeq .3\ \%$$

into the equation

$$c \simeq \beta t + c_o.$$

The result is

$$0 \simeq -.015t + .3$$

$$-.3 \simeq -.015t$$

$$t \simeq \frac{-.3}{-.015}$$

$$\simeq 20 \text{ hours.}$$

PROBLEM SET 32:

1. Upon passing an important exam, a jubilant medical student consumed 1.05 liters (about 1.1 qt) of wine in a short period of time on an empty stomach. The wine was 12% alcohol by volume. The student's mass was 70 kg (approximately 154 lb). Assume the following values for the constants β, r and t_m.

$$\beta \simeq -.015\% \text{ per hour}$$

$$r \simeq 9$$

$$t_m \simeq 1.5 \text{ hours}$$

a. How many ml of alcohol did the student imbibe?

b. Calculate c_o or the "initial" concentration.

c. What was the maximum blood-alcohol level actually attained?

d. How long would it be before the blood-alcohol concentration reached the .15% level?

e. How long would it be before the blood-alcohol concentration reached the .05% level?

2. Dave likes to drink a mixture of 40% pure alcohol and 60% rain water as his breakfast.

a. If he drinks 226 ml of the mixture, how many ml of alcohol has he drunk?

b. Dave's body mass is 50 kg ($\simeq$110 lb). Dave ingested the amount of alcohol found in part a. Determine c_o from the equation

$$\frac{A}{M \cdot r} \simeq c_o$$

where A = amount of alcohol ingested in ml

M = body mass in kg

$r \simeq 9.04$

c. If it takes 90 minutes for the concentration of alcohol in Dave's blood to reach its highest level, find that concentration using the formula $c \approx \beta t + c_o$. Assume $\beta \approx -.013\%$ per hour.

d. How long will it take from the time of ingestion for the concentration of alcohol in Dave's blood to fall to .07%?

e. How long will it take for the concentration to fall to zero?

3. George's body mass is 75 kg (165 lbs). He drinks 1125 ml of wine (12% alcohol) on an empty stomach. Use the following constants.

$$r \approx 9$$

$$\beta \approx -.015\% \text{ per hr}$$

$$t_m \approx 1.5 \text{ hr}$$

a. What is George's c_o?

b. How long will it take for George's blood-alcohol concentration to reach .05%?

c. How long will it be before all of the alcohol is gone from George's system?

4. Given that the formula for the per cent concentration of alcohol in the blood is $c \approx \beta t + c_o$, write a formula that will give the time it takes for all alcohol to have disappeared from the body.

5. Dave claims that, since he has twice as much mass as Jack, he can drink twice as much as Jack and not be any drunker than Jack would be. Use the following data.

$$r \approx 9.04 \pm .71$$

$$t_m \approx 1.43 \pm .49 \text{ hours}$$

$$\beta \approx -.015\% \text{ per hour}$$

Dave's mass ≈ 100 kg

Jack's mass ≈ 50 kg

A for Dave ≈ 195 ml

A for Jack ≈ 97.5 ml

a. Assume that Jack's value for r is the maximum.

$$r \simeq 9.04 + .71$$

$$r \simeq 9.75$$

Calculate c_o from the equation

$$c_o \simeq \frac{A}{r \cdot M}.$$

b. Calculate c_o for Dave in the same manner as you did for Jack. Assume that r for Dave is the minimum.

$$r \simeq 9.04 - .71$$

$$r \simeq 8.33$$

$$r \simeq \frac{25}{3}$$

c. Assume that t_m for Jack is the maximum.

$$t_m \simeq 1.43 + .49 \text{ hours}$$

$$t_m \simeq 1.92 \text{ hours}$$

Calculate Jack's maximum blood-alcohol concentration from the equation

$$c \simeq \beta t_m + c_o.$$

d. Assume that t_m for Dave is the minimum.

$$t_m \simeq 1.43 - .49 \text{ hour}$$

$$t_m \simeq .94 \text{ hour}$$

Calculate Dave's maximum blood-alcohol concentration from the equation

$$c \simeq \beta t_m + c_o.$$

e. Was it true in this case that Dave could drink twice as much as Jack and not have a higher blood-alcohol concentration?

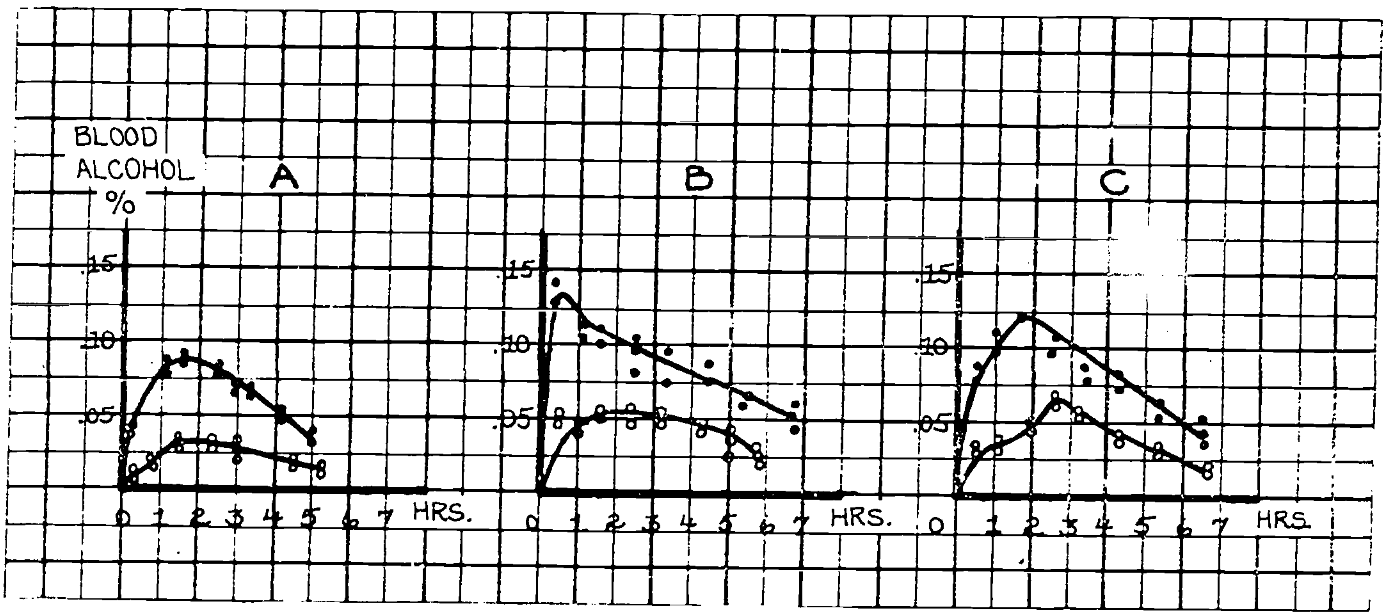
f. State the conditions under which it would be true that Dave could drink twice as much as Jack and have the same blood-alcohol concentration.

*6. Write a problem that involves your own body mass and a typical amount of alcohol that you might drink when you become of legal age. Include answers. Your body mass in kg ≈ (_____ lb) x 0.45.

SECTION 33:

33-1 Alcohol Ingested With a Meal

When alcohol is taken on a full stomach, the relationship between blood-alcohol concentration and time is not the same as when the subject has fasted. The graphs for three individuals who were tested twice with full stomachs and twice with empty stomachs, given the same quantity of alcohol, are shown below.



THE INFLUENCE OF A MEAL ON THE BLOOD-ALCOHOL CURVE

●–● ALCOHOL TAKEN ON AN EMPTY STOMACH
O–O ALCOHOL TAKEN WITH A MEAL

Notice that the curve for alcohol taken with a meal is always lower than the curve for alcohol taken on an empty stomach. At any given time the concentration of alcohol in the blood is less if the individual has food in his stomach.

Observe also that the pattern of blood-alcohol concentration is not consistent from individual to individual with food in the stomach. The curve of A rises to a distinct peak, the curve of B remains relatively level for two hours and the distribution phase of C does not resemble the distribution phases of either A or B.

The graphs further show that the time necessary for the blood-alcohol concentration to reach its maximum value (t_m) is longer with food in the stomach than without, and that the "initial" concentration (c_o) is lower with food in the stomach. The slope of the line (β) during the post-absorptive phase is less consistent than with an empty stomach.

It is easy to see how the presence of food in the stomach has the effect of lowering the initial concentration (c_o) and slowing the rate of elimination (β).

When there is food in the stomach, some of the alcohol is absorbed by the food. This decreases the amount of alcohol that initially passes through the wall of the stomach into the bloodstream. Consequently, c_o is lowered since there is less alcohol initially in the blood.

The absorbtion of alcohol by the food is also the reason why the rate of elimination is slowed. As the food is digested the absorbed alcohol is slowly released into the bloodstream. This slow addition of alcohol to the blood tends to reduce the rate of elimination of alcohol from the blood.

Recall that the initial concentration is obtained by the formula

$$\frac{A}{M} \approx rc_o.$$

We have just stated that the value of c_o when the subject has eaten is less than it is when the subject has fasted, for a given amount of alcohol ingested. Since the value of c_o is different, for the same given values of A and M, the proportionality constant r has a different value when the subject has eaten than when he has not. We will use the approximate value

$$r \approx 14$$

for cases in which the subject has a full stomach.

We will also use different values for β and t_m when the subject has food in his stomach. The approximate values are

$$t_m \approx 2.5 \text{ hours}$$

$$\beta \approx -.01\% \text{ per hour.}$$

No ranges of imprecision are included, because individual variation is so great that the numbers are extremely approximate. The numbers are given only to allow a comparison of full-stomach blood-alcohol concentrations to empty-stomach concentrations.

PROBLEM SET 33:

1. On a full stomach c_o is (higher, lower).

2. On a full stomach the rate of elimination is (faster, slower).

3. When alcohol is drunk on a full stomach, some of the alcohol is absorbed by the food. Therefore, a smaller amount of alcohol goes through the wall of the stomach. This has the immediate effect of ______________________.

4. When the food and the alcohol absorbed in it are digested, the absorbed alcohol is slowly released into the blood. This has the effect of __________.

5. Examine Graph A in the text. The empty stomach curve has a (higher, lower) c_o and the slope is (steeper, less steep) than the full stomach curve.

6. a. Select the fastest rate of elimination.

β = -.01% per hour

β = -.015% per hour

β = -.008% per hour

b. In this context the more negative β is, the (faster, slower) the rate of elimination.

7. Examine graph B in the text. The lowest blood-alcohol concentration for the (full, empty) stomach curve is greater than the highest concentration for the (full, empty) curve.

8. Examine all three graphs in the text. Each graph represents

a. a different person tested twice on both a full stomach and an empty stomach.

b. four different people, two on empty stomachs and two on full stomachs.

c. two people, one on a full stomach and one on an empty stomach with each person being checked twice.

9. a. We can see from the graphs that a particular person's response is (roughly r ucible, varies greatly each time he is tested).

b. The pattern for full stomach blood-alcohol concentrations (roughly the same, differs significantly) from person to person.

10. (True or False) The reason that the full stomach curves are lower is that the subjects were given less alcohol.

Use the following constants, as needed, for all of the problems in this set. Assume that they apply to animals as well.

Empty Stomach	Full Stomach
$r \approx 9$	$r \approx 14$
$\beta \approx -.015\%/hr$	$\beta \approx -.01\%/hr$
t_m 1.5 hr	$t_m \approx 2.5$ hr

11. Two hours after drinking 140 ml of bourbon (about 3 shots or jiggers), Cindy is arrested for speeding. Cindy's body mass is 45 kg. The bourbon was 90 proof, or 45% alcohol.

a. Cindy lives in North Carolina, where there is presumptive evidence of intoxication when $c > .10\%$. If she drank on an empty stomach, will she be held for drunken driving as well? Show your calculations.

b. If she drank on a full stomach, will she be legally drunk? Again, show your calculations. Remember that she was stopped 2 hours after drinking and t_m for a full stomach is 2.5 hours.

12. George drank 1890 ml of wine (12% alcohol) on a full stomach. His body mass is 75 kg.

a. What was George's c_o?

b. How long did it take for George's blood-alcohol concentration to reach the .15% level?

c. Determine t for c = .051%

d. Determine t for c = 0%

e. Return to part a, without passing GO, and repeat all four calculations (a, b, c and d) assuming that George's stomach was empty when he drank.

13. Saturday night, Elmo and a friendly hippopotamus stopped by the Dispensary of Dubious Delight and consumed a vast quantity of Old Popskull (50% alcohol). The hippopotamus ate as he drank, because hippopotami feel insecure when their stomachs aren't full.

When their thirsts were quenched, Elmo decided as a kindly gesture to drive his friend home to the zoo. This was mainly because the hippopotamas had been complaining of a touch of bursitis in his left front elbow.

As they approached Elmo's Locomobile, Elmo realized that he had somehow lost his driver's license. After thinking over the situation for 2.5 hours, Elmo made a decision. Being a law-abiding citizen, he concluded that the best thing was to let his friend do the driving.

A half-mile down the road, they were stopped by the authorities. Since the hippopotamus was the driver, he was subjected to a blood-alcohol test. The poor ungainly beast experienced great difficulty in attempting to blow up the balloon, so the troopers finally took a blood sample from his left front elbow. (Curiously enough, this seemed to relieve the bursitis a bit.)

A quick test established that the hippo's blood-alcohol concentration was .2%, and he was arrested for drunken driving. The authorities searched their statute book for a law on stupidity, but they could find none, so they arrested Elmo for loitering.

Being just an average-sized creature of his kind, the hippopotamus weighs in at 815 kg (about 1800 lb). Just how much Old Popskull did he drink?

14. Six hours after a full meal and much wine (12% alcohol), Pat, whose body mass is 50 kg, discovers the concentration of alcohol in her blood to be .06%. (Don't ask how she did this.) How much wine did she drink?

15. Fred used to take great delight in feeding beer (4% alcohol) to his pet hampster, Eloise. He thought it was great fun to watch her silly antics when she was drunk. Unfortunately, one day Eloise drank too much beer and dropped dead. Assume that death occurs when c reachs .5% and that Eloise was drinking on a full stomach. Her body mass was 0.1 kg. What is the minimum amount of beer that Eloise could have drunk?

*16. Ed drinks 1 liter of corn whiskey (70% alcohol) on a full stomach. After 25 hours his blood alcohol concentration is .25%. What is Ed's body mass?

SECTION 34:

34-1 Review of Sections 1 through 33

The following table lists the material covered by each problem.

PROBLEMS	CONTENT	REVIEW OF SECTIONS
1	length, area and ordered pairs	3, 4, 10
2-4	linear functions and function notation	12-17
5	determination of the equation of a line from two points	18
6	pressure	26
7	pressure	15, 16, 26
8	solving equations	24
9	scientific notation	27, 28
10-11	combined gas law	20,21,24,25,27,28
12	diver problem	26
13	blood-alcohol concentration	31-33
14	decimal powers of ten	29
15	decimal powers of ten	29

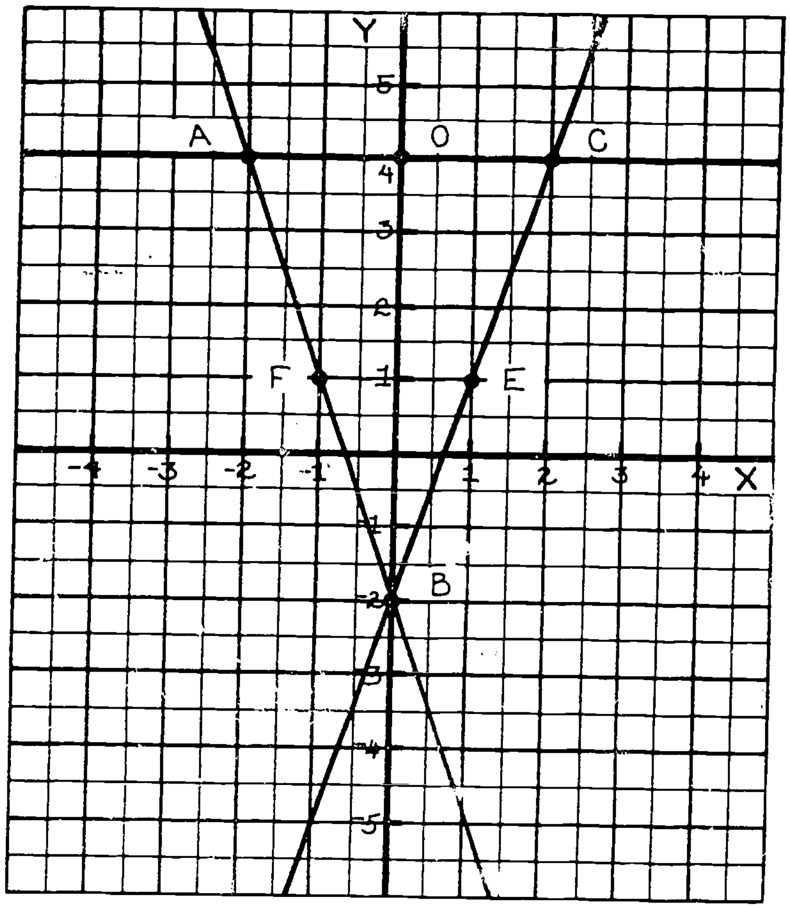
PROBLEM SET 34:

1. Refer to the graph.

   a. What are the coordinates of points A, B and E?

   b. What are the lengths of $\overline{AC}$ and $\overline{OB}$?

   c. What is the area of triangle ABC?



2. $f(x) = 2x - 3$.

a. What is $f(0)$?

b. Graph the function $f(x) = 2x - 3$ for $x \geq -1$ and $x \leq 4$.

c. What is the slope of the line?

d. What is the y-intercept?

3. Refer to the graph in Problem #1.

a. What is the slope of $\overline{CB}$?

b. What is the slope of $\overline{AB}$?

c. What is the slope of $\overline{AC}$?

d. What is the y-intercept of $\overline{CB}$?

e. What is the y-intercept of $\overline{AB}$?

f. What is the y-intercept of $\overline{AC}$?

g. Write the equation of the function whose subset is $\overline{CB}$.

h. Write the equation of the function whose subset is $\overline{AB}$.

i. Write the equation of the function whose subset is $\overline{AC}$.

4. A line passes through the following points: $(0,1)$, $(2, -5)$, $(-1, 4)$.

a. Plot the points and draw a line through them.

b. Write the equation of the line.

c. The line also passes through $(g, 10)$. What is g?

5. A line passes through the points $(4,6)$, $(-1, -4)$.

a. <u>Without</u> drawing a graph, calculate the slope.

b. Calculate the y-intercept.

c. Write the equation of the line.

d. If $(Q, 8)$ is on the line, what is Q?

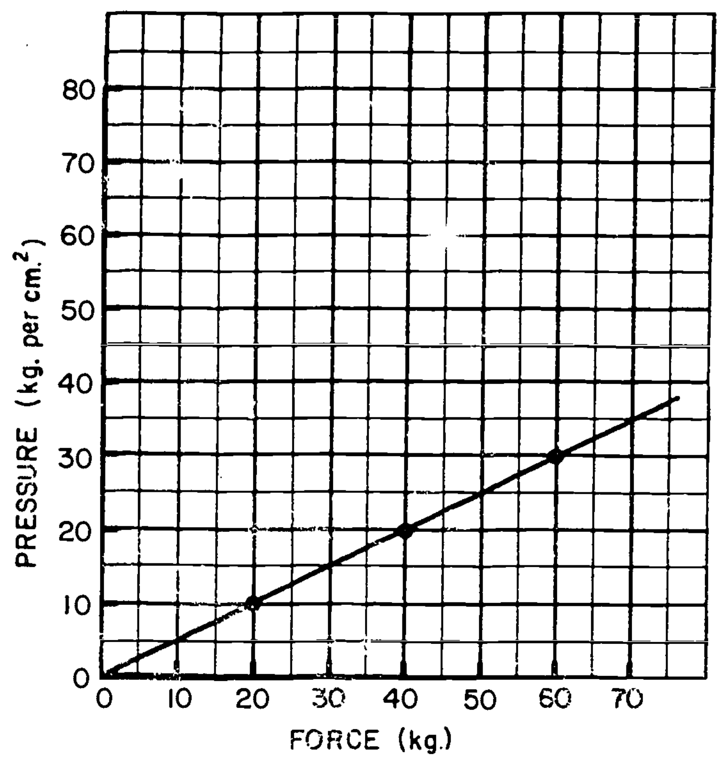
6. A man weighing 100 kg stands on one foot whose surface area is 20 cm^2 What is the pressure on his foot in kg per sq cm? [Remember: pressure = force/area]

7. Refer to the following graph relating pressure on a certain surface to the force exerted.

   a. Determine by inspection of the graph the force when the pressure is 25 kg per sq cm.

   b. What is the slope of the line?

   *c. What is the area of the surface acted on by the force?



8. Solve each of the following equations.

   a. $2h - 6 = 2$[illegible]

   b. $\frac{p}{2} = 3p - 10$

   c. $\frac{2s}{5} = 4$

   d. $.3 = \frac{56}{x}$

28[illegible]

9. Write each of the following expressions as a number in scientific notation. Allow only one digit to remain to the left of the decimal point in your answer.

a. $\dfrac{(4.3 \times 10^{-2}) \times (10^{5})}{10^{4}}$

b. $\dfrac{(6 \times 10^{2}) \times (2.6 \times 10^{-3})}{1.30 \times 10^{-7}}$

c. $\dfrac{(5.0 \times 10^{-6})\ (5.0 \times 10^{+6})}{10^{0}}$

10. $\dfrac{PV}{T} = \dfrac{P'V'}{T'}$

$P = 3.20 \times 10^{2}$ atm

$T' = 3.60 \times 10^{2}$ °K

$P' = 32 \times 10^{-1}$ atm

$V' = 12 \times 10^{1}$ ml

$T = 4.65 \times 10^{2}$ °K

What is V?

11. A quantity of oxygen occupies 2.82×10^{2} ml at a temperature of 2.70×10^{2} °K and a pressure of 5 atm. The temperature is lowered to $\frac{2}{3}$ of its former value, while the volume remains the same.

a. What is the new temperature?

b. What is the new pressure?

12. An 8-liter tank is suspended at a depth of 30 meters. The pressure of the gas in the tank is 100 atm.

a. What is the pressure on the outside of the tank?

[pressure in atm $\approx \dfrac{\text{depth in meters}}{10} + 1$]

b. What volume would the gas attain if all of it escaped into a diving bell at the same depth? [Use PV = P'V']

c. If the gas entered the diving bell at a rate of 10 liters per minute, how long would it take for all the gas to escape?

13. Mort, whose mass is 75 kg, drinks 6 glasses of beer (1.2 liters) on an empty stomach. A friend tells him that the beer is 4.5% alcohol and that Mort shouldn't drive home until the concentration of alcohol in his blood is down to .05%.

a. How many liters of alcohol has Mort drunk?

b. How many ml of alcohol has Mort drunk?

c. From the formula $c_o \simeq \frac{A}{Mr}$ find c_o, the "initial" blood-alcohol concentration in percent.

(Remember: A is amount of alcohol drunk in ml
M is Mort's mass in kg
$r \simeq 9$)

d. Using the formula $c \simeq \beta t + c_o$, how long should Mort wait before driving home?

(Remember: c is the percent concentration of alcohol in Mort's blood
t is the time in hours
c_o is the "initial" concentration found in part c
$\beta \simeq -.012\%$ per hour)

14. Express these decimal powers of ten as numbers in scientific notation.

a. $10^{7.5}$

b. $10^{-7.5}$

c. $10^{-13.3}$

15. Express these numbers in scientific notation as decimal powers of ten.

a. 5×10^{-7}

b. 6.3×10^{-11}

c. 2×10^{-3}

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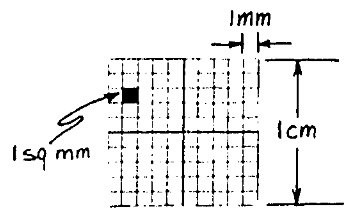
SECTION 35:

35-1 Conversion of Units of Area and Volume

We have not yet applied the method of conversion factors to problems involving units of area and volume such as the square foot or the cubic centimeter. The reason is that the conversion of units which are squared or cubed is made easier with an additional technique.

Suppose that we need to convert an area of 6.3 square centimeters into units of square millimeters. Earlier you learned that 1 square centimeter is equal to $10 \cdot 10 = 100$ square millimeters.



We may use as a conversion factor

$$\frac{100 \text{ mm}^2}{1 \text{ cm}^2}$$

and solve the problem by writing

$$6.3 \cancel{\text{cm}^2} \cdot \frac{100 \text{ mm}^2}{1 \cancel{\text{cm}^2}} = 630 \text{ mm}^2$$

A square centimeter is the area of a square whose sides have a length of one centimeter.

$$1 \text{ cm}^2 = 1 \text{ cm} \cdot 1 \text{ cm}$$

The preceding problem requires us to convert 6.3 cm^2 to mm^2. We begin by writing

$$6.3 \text{ cm}^2 = 6.3 \text{ cm} \cdot \text{cm}$$

Since 1 cm = 10 mm, we use $\frac{10 \text{ mm}}{1 \text{ cm}}$ as a conversion factor, and we write

$$6.3 \text{ cm}^2 = 6.3 \text{ cm} \cdot \text{cm} \cdot \frac{10 \text{ mm}}{1 \text{ cm}} \cdot \frac{10 \text{ mm}}{1 \text{ cm}}$$

We cancel the appropriate units and simplify the right side of the equation.

$$6.3\ cm^2 = 6.3\ \cancel{cm} \cdot \cancel{cm} \cdot \frac{10\ mm}{1\ \cancel{cm}} \cdot \frac{10\ mm}{1\ \cancel{cm}}$$

$$= 6.3 \cdot 10\ mm \cdot 10\ mm$$

$$= 630\ mm \cdot mm$$

Since $mm \cdot mm = mm^2$, we write finally

$$6.3\ cm^2 = 630\ mm^2$$

We have solved the same problem by two methods. With one method we used $\frac{100\ mm^2}{1\ cm^2}$ as a conversion factor. With the other method we used $\frac{10\ cm}{1\ mm} \cdot \frac{10\ cm}{1\ mm}$ to convert cm^2 to mm^2. There is yet a third method which uses $(\frac{10\ mm}{1\ cm})^2$ as a conversion factor.

$$6.3\ cm^2 = 6.3\ cm^2\ (\frac{10\ mm}{1\ cm})^2$$

$$= 6.3\ \cancel{cm^2}\ (\frac{100\ mm^2}{1\ \cancel{cm^2}})$$

$$= 630\ mm^2$$

For solving this problem, when we know the conversion factor for directly converting square centimeters to square millimeters, the latter two methods may seem unnecessarily complicated. However, often we do not know the factor converting one unit of area to another, and it is then that the latter methods are extremely helpful. Consider the following example.

EXAMPLE:

Express an area of 18.2 square feet in units of square meters, given that 1 meter ≈ 3.28 feet.

SOLUTION:

$$18.2\ ft^2 \simeq 18.2\ \cancel{ft} \cdot \cancel{ft} \cdot \frac{1\ m}{3.28\ \cancel{ft}} \cdot \frac{1\ m}{3.28\ \cancel{ft}}$$

$$\simeq 18.2 \cdot \frac{1}{3.28}\,m \cdot \frac{1}{3.28}\,m$$

$$\simeq \frac{18.2}{3.28 \cdot 3.28}\,m \cdot m$$

$$\simeq 1.69\ m^2$$

ALTERNATIVE SOLUTION:

$$18.2\ ft^2 \simeq 18.2\ ft^2 \left(\frac{1\ m}{3.28\ ft}\right)^2$$

$$\simeq 18.2\ \cancel{ft^2} \left(\frac{1\ m^2}{(3.28)^2\ \cancel{ft^2}}\right)$$

$$\simeq \frac{1.82}{(3.28)^2}\ m^2$$

$$\simeq 1.69\ m^2$$

Converting a unit of volume to another unit of volume is done in an analogous manner. The methods are demonstrated in the following example.

EXAMPLE:

Express a volume of .043 cubic meters in units of cubic centimeters.

SOLUTION:

$$.043\ m^3 = .043\ m \cdot m \cdot m$$

Since 1 m = 100 cm, we use $\frac{100\ cm}{1\ m}$ as a conversion factor and write

$$.043\ m^3 = .043\ \cancel{m} \cdot \cancel{m} \cdot \cancel{m} \cdot \frac{100\ cm}{1\ \cancel{m}} \cdot \frac{100\ cm}{1\ \cancel{m}} \cdot \frac{100\ cm}{1\ \cancel{m}}$$

$$= .043 \cdot 100\ cm \cdot 100\ cm \cdot 100\ cm$$

$$= .043 \cdot 10^2 \cdot 10^2 \cdot 10^2\ cm \cdot cm \cdot cm$$

Since cm · cm · cm = cm^3, we may write

$$.043 \text{ m}^3 = .043 \cdot 10^2 \cdot 10^2 \cdot 10^2 \text{ cm}^3$$
$$= .043 \cdot 10^6 \text{ cm}^3$$
$$= 4.3 \cdot 10^4 \text{ cm}^3 \quad \text{or} \quad 43{,}000 \text{ cm}^3$$

<u>ALTERNATIVE SOLUTION</u>:

$$.043 \text{ m}^3 = .043 \text{ m}^3 \left(\frac{100 \text{ cm}}{1 \text{ m}}\right)^3$$
$$= .043 \cancel{\text{m}^3} \left(\frac{100^3 \text{cm}^3}{1 \cancel{\text{m}^3}}\right)$$
$$= .043 \cdot 10^6 \text{ cm}^3$$
$$= 4.3 \cdot 10^4 \text{ cm}^3$$

PROBLEM SET 35:

Problems 1-9 are expressions converting a quantity from one set of units to another. In some expressions, every conversion factor is written correctly. Write "correct" for each of these. In other equations, there are one or more errors. In this case, rewrite the entire equation. <u>Caution</u>: in one problem the error involves a number.

1. $14.7 \frac{lb}{in^2} = 14.7 \frac{lb}{in \cdot in}$

$$14.7 \frac{lb}{in^2} \cdot \frac{1 \text{ in}}{2.54 \text{ cm}} \cdot \frac{1 \text{ in}}{2.54 \text{ cm}} \cdot \frac{1 \text{ kg}}{2.20 \text{ lb}} \simeq 1.04 \frac{kg}{cm^2}$$

2. $62.4 \frac{lb}{ft^3} = 62.4 \frac{lb}{ft \cdot ft \cdot ft}$

$$62.4 \frac{lb}{ft^3} \cdot \frac{3.28 \text{ ft}}{m} \cdot \frac{3.28 \text{ ft}}{m} \cdot \frac{3.28 \text{ ft}}{m} \cdot \frac{1 \text{ kg}}{2.20 \text{ lb}} \simeq 1001 \frac{kg}{m^3}$$

3. $1000 \frac{mg}{cm^3} \cdot \left(\frac{100 \text{ cm}}{m}\right)^3 \cdot \frac{1 \text{ kg}}{10^6 \text{ mg}} = 1000 \frac{kg}{m^3}$

4. $.022 \frac{moles}{cm^3} \cdot \left(\frac{1 \text{ cm}}{10 \text{ mm}}\right)^3 \cdot 6 \times 10^{23} \frac{molecules}{mole} = 1.32 \times 10^{19} \frac{molecules}{mm}$

5. $.91 \frac{g}{liter} \cdot \frac{1 \text{ liter}}{1000 \text{ ml}} \cdot \frac{1 \text{ ml}}{cm^3} \cdot \left(\frac{100 \text{ cm}}{m}\right)^3 \simeq 9.1 \times 10^2 \frac{g}{m}$

6. $89 \text{ in}^2 \cdot \left(\frac{1 \text{ ft}}{12 \text{ in}}\right)^2 \cdot \left(\frac{1 \text{ m}}{3.28 \text{ ft}}\right)^2 \simeq .057 \text{ m}^2$

7. $18 \frac{ft^3}{hr} \cdot \left(\frac{1 \text{ m}}{3.28 \text{ ft}}\right)^3 \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{100 \text{ cm}}{m} \simeq 8502 \frac{cm^3}{min}$

8. $32 \frac{ft}{sec^2} \cdot \left(\frac{3600 \text{ sec}}{hr}\right)^2 \cdot \frac{1 \text{ m}}{3.28 \text{ ft}} \simeq 1.26 \times 10^8 \frac{m}{sec}$

9. $4.7 \frac{m^3}{hr} \cdot \left(\frac{100 \text{ cm}}{m}\right)^3 \cdot \frac{1 \text{ hr}}{60 \text{ min}} = \frac{4.7 \cdot 100}{60} \frac{cm^3}{min}$

In Problems 10-19, a conversion factor is missing. In some problems you will have to supply both the number and the units; in other problems, only the units. You do not need to perform the multiplications, although multiplication may be used to check your work.

10. $28 \frac{kg}{m^2} \cdot (0.3)^2 \; (?) \cdot \frac{2.20 \text{ lb}}{kg} \simeq 5.54 \frac{lb}{ft^2}$

11. $1000 \frac{mg}{cm^3} \cdot (?) \cdot \frac{1 \text{ g}}{1000 \text{ mg}} = 10^6 \frac{g}{m^3}$

12. $14.7 \frac{lb}{in^2} \cdot (?) \simeq 2117 \frac{lb}{ft^2}$

13. $570 \frac{\text{grains}}{\text{fl dram}} \cdot \frac{\text{8 fl drams}}{\text{fl oz}} \cdot \frac{\text{1 fl oz}}{29.6 \text{ cm}^3} \cdot (?) \cdot (\frac{\text{1 gram}}{15.4 \text{ grains}}) \simeq 10^7 \frac{\text{grams}}{m^3}$

14. $1 \frac{\text{gram}}{\text{liter}} \cdot \frac{\text{1 liter}}{\text{1000 ml}} \cdot (?) \cdot (\frac{100 \text{ cm}}{m})^3 \simeq 1000 \frac{\text{grams}}{m^3}$

15. $3200 \frac{ml}{\text{breath}} \cdot 1500 \frac{\text{breaths}}{hr} \cdot 1 \frac{cm^3}{ml} \cdot (?) \simeq 4.8 \frac{m^3}{hr}$

16. $2 \text{ moles} \cdot \frac{22.4 \text{ liters}}{\text{mole}} \cdot \frac{1000 \text{ ml}}{\text{liter}} \cdot \frac{1 \text{ cm}^3}{ml} \cdot (?) \simeq 4.48 \times 10^{-2} \; m^3$

17. $6 \times 10^{-4} \frac{\text{moles}}{ml} \cdot \frac{1 \text{ ml}}{cm^3} \cdot (\frac{100 \text{ cm}}{m})^3 \cdot 32 \; (?) \simeq 1.92 \times 10^4 \frac{g}{m^3}$

18. $62.4 \frac{lb}{ft^3} \cdot \frac{1}{7.48} \; (?) \cdot \frac{1 \text{ kg}}{2.20 \text{ lb}} \simeq 3.79 \frac{kg}{\text{gallon}}$

19. $32 \frac{ft}{sec^2} \cdot (?) \cdot \frac{\text{1 mile}}{5280 \text{ ft}} \cdot \frac{1.61 \text{ km}}{\text{mile}} \simeq 1.26 \times 10^5 \frac{km}{hr^2}$

Perform the following unit conversions. Show all work. In any divisions that are required, do not carry your calculations beyond the hundredths column.

20. 9.9 ft^3 to m^3; use 1 m $\simeq$ 3.3 ft.

21. 24 ml to m^3; use 1 m = 10^2 cm; 1 ml $\simeq$ 1 cm^3.

22. 13.2 ft^2 to m^2; use 1 m $\simeq$ 3.3 ft.

23. 35.937 gal to m^3; use 1 m ≈ 3.3 ft; 1 ft^3 ≈ 7.5 gal.

24. 14.7 $\frac{lb}{in^2}$ to $\frac{kg}{cm^2}$; use 1 in ≈ 2.5 cm; 1 kg ≈ 2.2 lb.

25. $\frac{5.844 \times 10^{-4} \text{ g NaCl}}{\text{ml solution}}$ to $\frac{\text{moles NaCl}}{\text{liter solution}}$;

use 1 mole NaCl ≈ 58.44 g; 1 liter = 10^3 ml.

26. $\frac{600 \text{ ml}}{\text{breath}}$ to $\frac{\text{liters}}{\text{hr}}$; use 1 min ≈ 14 breaths; 60 min = 1 hr; 10^3 ml = 1 liter.

27. $\frac{1 \text{ mg}}{\text{liter}}$ to $\frac{g}{m^3}$; use 1 gram = 10^3 milligrams (mg); 1 liter ≈ 10^3 cm^3; 100 cm = 1 m.

28. $\frac{3.65 \times 10^{-3} \text{ g HCl}}{\text{ml solution}}$ to $\frac{\text{moles HCl}}{\text{liter solution}}$;

use 36.5 g HCl ≈ 1 mole HCl; 10^3 ml = 1 liter.

29. One form of the combined gas law states that, for a given quantity of gas, $\frac{PV}{T} = k$. Although k is a constant, it must have the same units as the left side of the equation. If P is given in atmospheres, V in liters and T in degrees Kelvin, the units of k are $\frac{\text{liter - atm}}{{}^\circ K}$. For one mole of a gas, k = .082 $\frac{\text{liter - atm}}{{}^\circ K}$. Find k for one mole of gas, if volume is given in cubic feet. (1 meter ≈ 3.28 ft) Terminate your answer at the ten-thousandths column.

30. Fred has a book that shows all of the Olympic records for track events. One day he began to wonder how the lengths of the Olympic races (in meters) compared to American races (in yards). At the bottom of the page he found a footnote that stated that 1 yard ≈ .9144 meters. Fred worked a few problems like the following one.

$$1500 \text{ m} \cdot \left(\frac{1 \text{ yd}}{.9144 \text{ m}}\right) \approx ?$$

He soon got tired of dividing by .9144 for each race. Then he realized that he could calculate a new conversion factor that would avoid division completely.

Fred's new conversion factor was of the form

$$\frac{x \text{ yards}}{1 \text{ meter}}.$$

a. Find the new conversion factor. Do not carry your calculations beyond the ten-thousandths column.

b. Use the new conversion factor to find the length of the 100-meter race in yards.

SECTION 36:

36-1 More About Dimensional Algebra

Consider a problem. If a person inhales .6 liter of air per breath and takes 14 breaths per minute, how many liters of air does he breathe in 2 hours?

First, we determine the number of liters breathed in one hour.

$$.6\frac{\text{liter}}{\cancel{\text{breath}}} \cdot \frac{14\ \cancel{\text{breaths}}}{\cancel{\text{min}}} \cdot \frac{60\ \cancel{\text{min}}}{\text{hr}} = 504\ \frac{\text{liters}}{\text{hr}}$$

It is now no problem at all to find the volume consumed in 2 hours; it will just be twice 504 liters, or 1008 liters. Using dimensional algebra we can write the solution like this.

$$508\ \frac{\text{liters}}{\cancel{\text{hr}}} \times 2\ \cancel{\text{hr}} = 1008\ \text{liters}$$

This problem is typical of many situations in which we are given a rate (504 $\frac{\text{liters}}{\text{hr}}$) and an amount of time (2 hr) and need to find a total (1008 liters). Such problems can be solved by using the formula

rate x amount = total

It is not always the case that the rate involves time. For example, a concentration may be involved instead. However the above formula still applies.

EXAMPLE:

An average adult male has 5.5 liters of blood. If the concentration of Vitamin C in the blood is 1.5 $\frac{\text{mg}}{100\ \text{ml}}$, what is the total amount of Vitamin C in the bloodstream (in mg)?

SOLUTION:

We first determine the concentration of Vitamin C in mg per liter.

$$1.5\ \frac{\text{mg}}{100\ \cancel{\text{ml}}} \times \frac{1000\ \cancel{\text{ml}}}{1\ \text{liter}} = 15\ \frac{\text{mg}}{\text{liter}}$$

We now apply the formula, Rate x Amount = Total.

$$15\ \frac{\text{mg}}{\cancel{\text{liter}}} \times 5.5\ \cancel{\text{liters}} = 82.5\ \text{mg}$$

Once you get the hang of it you can solve such problems in one step instead of two. It is just a matter of carefully keeping track of the units.

$$1.5\ \frac{\text{mg}}{100\ \cancel{\text{ml}}} \times \frac{1000\ \cancel{\text{ml}}}{1\ \cancel{\text{liter}}} \times 5.5\ \cancel{\text{liters}} = 82.5\ \text{mg}$$

36-2 Units Relevant to Air Pollution

Two units are commonly used when expressing the concentration of pollutants in the air. One is used to measure the concentration of solid pollutants and the other to measure the concentration of gaseous pollutants.

The concentration of solid matter in the air is generally expressed in the units micrograms per cubic meter. The prefix "micro-" means "one millionth" and its symbol is the Greek letter μ (mu).

$$1\ \mu\text{g} = 1 \times 10^{-6}\ \text{g}$$

Problems concerning solid pollutants lend themselves to the usual techniques of dimensional algebra. The following example is typical.

EXAMPLE:

Suppose that the concentration of dust in the atmosphere is $25\ \frac{\mu g}{m^3}$. What volume of atmosphere will contain half a gram of dust?

SOLUTION:

This is basically a problem of making sure that the numbers are arranged in such a way that units cancel properly. Since the answer must be a volume, we know ahead of time that there must be a "m^3" remaining after all the cancellations.

$$\frac{1\text{m}^3}{25\ \cancel{\mu\text{g}}} \times \frac{1\ \cancel{\mu\text{g}}}{10^{-6}\ \cancel{\text{g}}} \times .5\cancel{\text{g}} = 2 \times 10^4\ \text{m}^3$$

The concentration of gaseous pollutants, such as the oxides of carbon, nitrogen and sulfur, is commonly given in units of parts per million. Parts per million is abbreviated "ppm." Parts per million may be though of as a ratio of volumes or a ratio of numbers of molecules, but not as a ratio of masses. That is,

$$1 \text{ ppm } SO_2 = \frac{1 \text{ liter } SO_2}{10^6 \text{ liters air}}$$

and

$$1 \text{ ppm } SO_2 = \frac{1 \text{ molecule } SO_2}{10^6 \text{ molecules air}}$$

but

$$1 \text{ ppm } SO_2 \neq \frac{1 \text{ g } SO_2}{10^6 \text{ g air}}$$

There is a parallel between the units ppm and the familiar notion of percentage. One percent just means one part in a hundred, so percentage and parts per hundred have the same meaning. The only reason that percentage is not used to measure air pollution is that pollutants seldom reach a concentration of one part in a hundred. Therefore, a smaller unit is handier.

The equivalence of the ratio of volumes and the ratio of numbers of molecules st us from Avogadro's principle, which states that equal volumes of gas at the same temperature and pressure contain equal numbers of molecules. Therefore, if the volumes of two gases are related by a certain ratio; the numbers of molecules will be related by the same ratio.

EXAMPLE:

If the concentration of CO (carbon monoxide) in the air is 15 ppm (Federal Alert Level), how many cm^3 of CO are in 1 m^3 of air?

SOLUTION:

Recalling that parts per million is a ratio of volumes, we can write

$$15 \text{ ppm CO} = \frac{15 \text{ cm}^3 \text{ CO}}{10^6 \text{ cm}^3 \text{ air}}$$

At this point the problem can be solved by the use of dimensional algebra. Note the conversion factor between cubic centimeters and cubic meters.

$$\frac{15 \text{ cm}^3 \text{ CO}}{10^6 \cancel{\text{cm}^3 \text{ air}}} \cdot \frac{100^3 \cancel{\text{cm}^3 \text{ air}}}{1 \cancel{\text{m}^3 \text{ air}}} \cdot 1 \cancel{\text{m}^3 \text{ air}} = 15 \text{ cm}^3 \text{ CO}$$

Notice that in a problem of this kind it is necessary to name the substances involved. For example,

$$\frac{15 \text{ cm}^3 \text{ CO}}{10^6 \text{ cm}^3 \text{ air}}$$

Otherwise, an expression such as

$$\frac{15 \text{ cm}^3}{10^6 \text{ cm}^3}$$

would permit the cancellation of the units. Cancellation is not permitted in this case because cm^3 of CO is not equivalent to 1 cm^3 of air.

PROBLEM SET 36:

1. If a patient's heart rate is 80 $\frac{\text{beats}}{\text{minute}}$, what will be the total number of beats in 3 hours? Substitute the correct numbers for the letters in the following equation and solve for R.

$$80 \frac{\text{beats}}{\text{minute}} \cdot \frac{\text{P minutes}}{\text{1 hour}} \cdot \text{Q hours} = \text{R beats}$$

2. In an adult human, about 2.3×10^6 red blood cells die each second. How many will die in 2 hours?

3. In an adult human male the concentration of hemoglobin in the blood is about 16 $\frac{\text{g}}{100 \text{ ml}}$. Figuring that there are about 5.5 liters of blood in the body, how many grams of hemoglobin will there be altogether?

4. If the concentration of CO in the atmosphere is 10 ppm, how many cm^3 of CO are in 1000 cm^3 of air? Substitute the correct numbers for the letters in the following equation to find the answer.

$$\frac{10 \text{ cm}^3 \text{ CO}}{10^6 \text{ cm}^3 \text{ air}} \cdot \text{P cm}^3 \text{ air} = \text{Q cm}^3 \text{ CO}$$

5. If the concentration of SO_2 in the air is .3 ppm (Federal Alert Level), how many cm^3 of SO_2 are in 10 m^3 of air? Substitute the correct numerical values for the letters in the following equation and solve for P.

$$\frac{.3\ cm^3\ SO_2}{10^6\ cm^3\ air} \cdot \frac{M^3\ cm^3\ air}{1\ m^3\ air} \cdot N\ m^3\ air = P\ cm^3\ SO_2$$

6. If the concentration of O_3 in the air is .1 ppm (Federal Alert Level), how many cm^3 of O_3 are in 10 m^3 of air?

7. If the concentration of NO_2 in the air is 1.2 ppm (Federal Warning Level), how many m^3 of air will contain 1 cm^3 of NO_2? Substitute the correct numerical values for the letters in the following equation. Do not carry your calculations beyond the hundredths column.

$$\frac{P\ cm^3\ air}{Q\ cm^3\ NO_2} \cdot \frac{1\ m^3\ air}{100^3\ cm^3\ air} \cdot S\ cm^3\ NO_2 = T\ m^3\ air$$

8. If the concentration of SO_2 in the air is .6 ppm, how many m^3 of air will contain 1 cm^3 of SO_2? Do not carry your calculations beyond the hundredths column.

9. The Federal Emergency Level for the concentration of particulates in the air is 10^3 $\mu g/m^3$.

a. At this concentration, how many m^3 of air will contain 1 gram of solid matter? Substitute the correct numerical values for the letters in the conversion equation.

$$\frac{1\ m^3}{10^3\ \mu g} \cdot \frac{X\ \mu g}{g} \cdot Y g = T\ m^3$$

b. How many m^3 of air will contain 5 g particulates?

10. At rest, a person inspires about 600 ml of air per breath. His rate of respiration is about 12 breaths/min. The concentration of lead is 1 μg lead/m^3 air (this is a high but frequently observed level).

a. What is the rate at which lead enters the lungs in μg/day?

b. What is the rate at which lead enters the lungs in g/year?

11. A solution of chlorpromazine labeled "5 mg/ml" is at hand. A doctor has prescribed that .02 g be administered to a patient. How many ml of the solution on hand should be given to the patient?

12. How many cm^3 of a streptomycin solution labeled "500 mg/cm^3" should be given to a patient for whom a dose of .4 g has been prescribed?

13. How many ml of a penicillin solution labeled "5×10^5 units/ml" should be given to a patient for whom a dose of 400,000 units has been prescribed?

14. Mabel's blood-alcohol concentration is $\frac{.2 \text{ g alcohol}}{100 \text{ g blood}}$. The density of alcohol is approximately $\frac{.8 \text{ g alcohol}}{\text{ml alcohol}}$. The ratio of the volume of blood to body weight is about $\frac{8 \text{ ml blood}}{100 \text{ g body weight}}$. The density of blood is approximately $\frac{1 \text{ ml blood}}{\text{g blood}}$. How many ml of alcohol are there in Mabel's blood if she weighs 70 kg?

15. There are approximately 9×10^3 white blood cells in a mm^3 of blood. The ratio of the volume of blood to body weight is approximately $\frac{8 \text{ ml blood}}{100 \text{ g body weight}}$. How many white blood cells would there be in a 50 kg ($\approx$ 100 lb) woman?

16. Assume that the average heart pumps at the right of approximately 5 liters per minute. How long in days would it take a pump which pumped at the same rate to fill a swimming pool with dimensions of 30 meters x 10 meters x 2.5 meters? Recall that the volume of a rectangular solid is given by the equation V = length x width x height. Do not carry your calculations beyond the hundredths column.