DOCUMENT RESUME

ED 173 099 SE 027 906

AUTHOR Herriot, Sarah T.; And Others.
TITLE Calculus of Elementary Functions, Part III, Student Text. Preliminary Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 68
NOTE 233p.; For related documents, see SE 027 905 and ED 143 516; Contains occasional small, light and broken type

EDRS PRICE MF01/PC10 Plus Postage.
DESCRIPTORS *Calculus; Curriculum; *Graphs; Inductive Methods; *Instruction; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS *Functions (Mathematics); *School Mathematics Study Group

ABSTRACT

This is part three of a three-part SMSG calculus text for high school students. The aim of the text is to develop some of the concepts and techniques which will enable the student to obtain important information about graphs of elementary functions. Chapter topics include area and the integral, differentiation theory and technique, mathematical induction, and further techniques of integration. (MP)

**SCHOOL MATHEMATICS STUDY GROUP**



# CALCULUS OF ELEMENTARY FUNCTIONS

## Part III

*Student Text*

(Preliminary Edition)

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*Student Text*

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*Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.*



Printed in the United States of America

FOREWORD TO VOLUME III

As we saw in Volume I and II (Chapters 1-6) many properties of the graph of a function can be analyzed from knowledge of the derivatives of the function, the value of the derivative being the slope of the tangent line at a point. For polynomial, circular, exponential and related functions we were able to find derivatives, which were then used to analyze rise and fall, convexity, velocity or acceleration. These methods are extended in Chapter 8 of this volume to functions which are sums, products, composites, powers, reciprocals, quotients or inverses of known functions. In principle we shall then be able to analyze the properties of various algebraic combinations of the functions discussed in volume one and two.

This volume is begun with the study of area under the graph of a function, a concept which, at first glance, seems to be unrelated to that of tangent line. The fact that these two concepts are related is one of the great discoveries in mathematics, first noted by Borrow (1630-1677). He showed that the area bounded by the graph of $f$, the x-axis and vertical lines at $a$ and $b$ is given by $F(b) - F(a)$ where $F$ is a function whose derivative is $f$. This result is appropriately known as the Fundamental Theorem of Calculus. The first three sections of Chapter 7 are devoted to developing a geometric understanding of this result. The final three sections concentrate on notation and techniques for finding areas by using the Fundamental Theorem.

Chapter 8 is primarily a discussion of methods of differentiating algebraic combinations of functions. Where appropriate, integration concepts (that is, the area concepts of Chapter 7) are also discussed, as these provide a further geometric interpretation for analyzing the behavior of functions.

These integration concepts are explored further in Chapter 9, which contains an important method for finding antiderivatives, an interpretation of the Fundamental Theorem in terms of average value and volumes of solids of revolution, as well as numerical integration methods and a discussion of remainder estimates for Taylor approximation.

Some elementary differential equations are discussed in Chapter 10, with applications to motion and growth and decay. In addition it is shown how the expression of the logarithm as an integral can be used to obtain the properties of the logarithm and exponential functions.

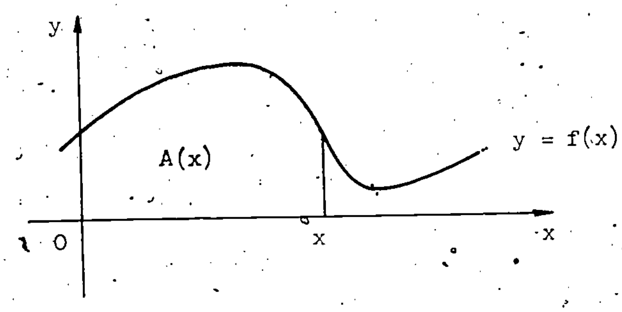
The appendices are intended to fill logical gaps in the intuitive development of the text and to extend the material of the text.

TABLE OF CONTENTS

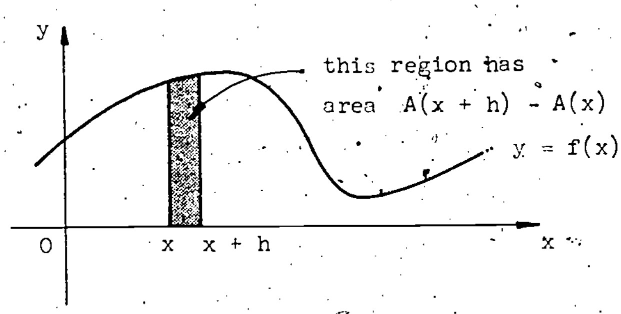
Chapter 7

# AREA AND THE INTEGRAL

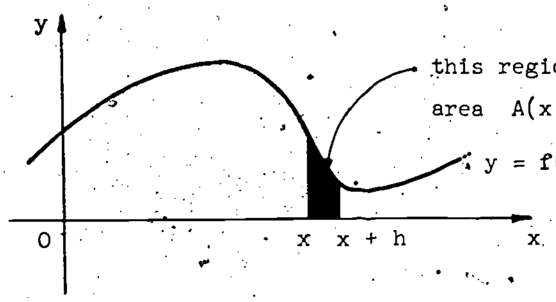
This chapter begins a discussion of the concept of area of a region bounded by the graph of a function. At first glance, this area appears to be entirely unrelated to our discussions of derivatives in volume one. Upon closer inspection, however, we shall discover that these two ideas must be related. Suppose $A(x)$ represents the area of the shaded region shown in the following figure.




As we move $x$ along the horizontal axis the area $A(x)$ of the shaded region changes. A measure of the rate of change in $A(x)$ is, of course, $A'(x)$, the value of the derivative of the area function at $x$. This change in area is also related to the height of the graph of $f$ at $x$; that is, to the value $f(x)$. Consider for example, the case when $f(x)$ is large.




If we move a small amount, say $h$ units, to the right, the area $A(x + h)$ increases fairly quickly, so that the additional area $A(x + h) - A(x)$ is fairly large. If, however, $f(x)$ is close to the x-axis




then the additional area $A(x + h) - A(x)$ will be fairly small.

These considerations lead us to suspect that there must be some relationship between the rate of change of the area function $x \to A(x)$ and the values of $f$, that is $A'(x)$ must be related to $f(x)$. In this chapter we shall show that for most of the functions of interest to us in this text, the derivative $A'$ of the area function is $f$; that is, $A'(x) = f(x)$.

Of course, it is not immediately obvious what the area bounded by a graph should be, particularly if $f$ is not a constant or linear function. Therefore, in the first section, after considering constant and linear cases, we deal with an approximation procedure for obtaining the area of a region bounded by the graph of a nonlinear function (Section 7-1). A useful notation for this area is introduced and various intuitive properties of area are then discussed (Section 7-2). A proof of the relation $A'(x) = f(x)$ is given in Section 7-3 where we establish the so-called Fundamental Theorem of Calculus, which states that the area bounded by the graph of $f$, the x-axis and vertical lines at $a$ and $b$ is given by the difference $F(b) - F(a)$ where $F$ is any antiderivative of $f$ (that is, $F' = f$). Further notation is introduced in Section 7-4, and the results are extended to signed area in Section 7-5.

The final section discusses the use of antiderivative formulas in calculating areas. Further antidifferentiation methods are discussed in Section 9-1 and Appendix 4. This basic connection between the area function and $f$ is also discussed in Section 8-2, where we use the Fundamental Theorem to discuss the relationship between the derivatives of a function and the shape of its graph.

## 7-1. Area Under a Graph

We first attack the general problem of finding the area of a region located in the first quadrant, bounded by the graph of a nonnegative function $f$, the x-axis, the y-axis and a second vertical line, as in Figure 7-1a. We shall not specify the value of the coordinate $x$ at which the second vertical line cuts the x-axis. This will allow us to find general formulas rather than particular numbers. We shall denote the desired area by $A(x)$.

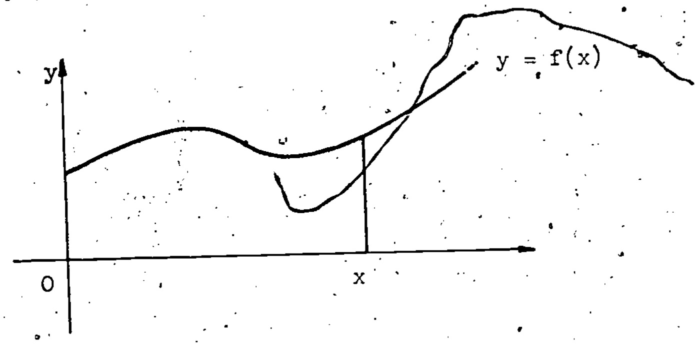



Figure 7-1a

Area under a graph

Frequently the first step a mathematician takes in attacking a new problem is to investigate a few special cases of the problem. He often finds this initial investigation very helpful in setting his mind working towards a general solution. In this spirit we begin with the simplest of polynomial functions and examine the area under the graph of the constant function

$$f : x \to c,$$

where $c$ is a fixed positive number. This case is very easy to handle. In fact, since we know that the area of a rectangle is equal to the product of its base and its height, we see that the desired area is

$$A(x) = cx.$$

(See Figure 7-1b.)

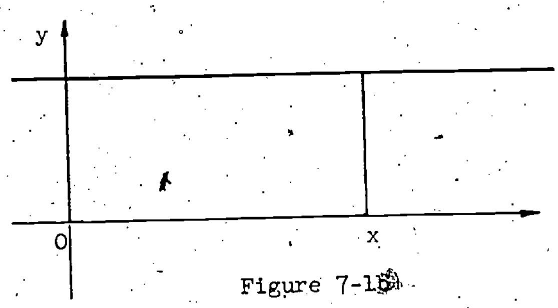



Figure 7-1b

The area of the shaded region is $cx$.

Note that the "area function"

$$A : x \to cx$$

is a linear function whose derivative $A'$ is

$$f : x \to c .$$

The next case we examine is that of a linear function

$$f : x \to mx + b.$$

The area we wish to find is that of the shaded region in Figure 7-1c.

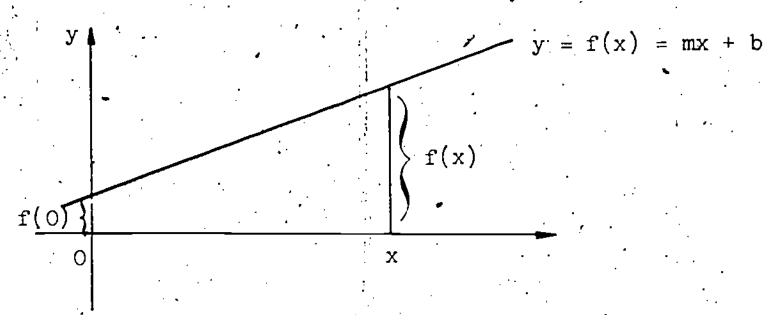



Figure 7-1c

Area under $f : x \to mx + b$

This case is also easy to handle since the shaded region is a trapezoid. We recall that the area of a trapezoid is $\frac{1}{2}$ the sum of the parallel bases times the height. In Figure 7-1c the trapezoid is lying on its side, its "bases" have lengths $f(0)$ and $f(x)$, its "height" is $x$. Therefore, the desired area is

$$\begin{aligned} A(x) &= \frac{f(0) + f(x)}{2} \cdot x \\ &= \frac{(m \cdot 0 + b) + (mx + b)}{2} \cdot x \\ &= \frac{mx + 2b}{2} \cdot x \\ &= \frac{mx^2}{2} + bx. \end{aligned}$$

We observe that the derivative $A'$ of the "area function"

$$A : x \to \frac{mx^2}{2} + bx$$

is the linear function

$$f : x \to mx + b.$$

After the constant functions and the linear functions, the next simplest polynomial functions are the quadratic functions. Even though these functions seem to be but a step removed from the linear functions, we shall see that they introduce an entirely new order of complexity. The reason for this is that the graphs of quadratic functions are curves, and we have no formulas for calculating areas of regions bounded by curves (except, of course, when the curves are circles). Hence, it will be wise to move more slowly, and first study a very special case--say the function $f : x \to x^2$. (See Figure 7-1d.)

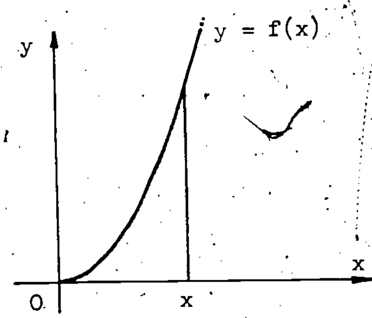



Figure 7-1d

Area under $f : x \to x^2$

If it were possible to cut the region up into a finite number of rectangular or triangular parts we could add the areas of the parts to obtain the total area. By this method the best we can do is to approximate the area. We can cover the region with rectangles and obtain as the sum of their areas a value that is somewhat larger than the one we seek. On the other hand, we can pack rectangles into the region without overlapping, and obtain in the sum of their areas a value that is somewhat too small. In this way we may at least hope to arrive at an approximate value that we might be able to use in constructing our area function.

Our procedure is to subdivide the line segment from $0$ to $x$ into a large number of equal parts, then to use the subintervals as bases of rectangles interior and exterior to the region. To illustrate this procedure we examine a case where the number of subdivisions is small.

Suppose we divide the line segment from $0$ to $x$ into $5$ equal subintervals. Each of these subintervals will be the base of an interior rectangle, the largest rectangle that can be drawn under the curve with this subinterval as base (Figure 7-1e). Each of these subintervals will also be the base of an exterior rectangle, the smallest rectangle that can be drawn above the curve with this rectangle as base (Figure 7-1f).

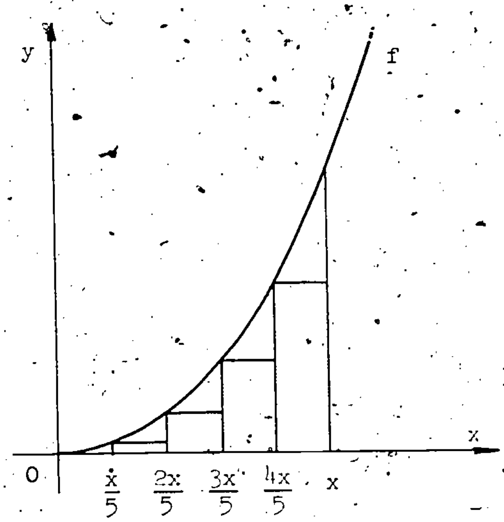



Figure 7-1e

Area approximated by interior rectangles.

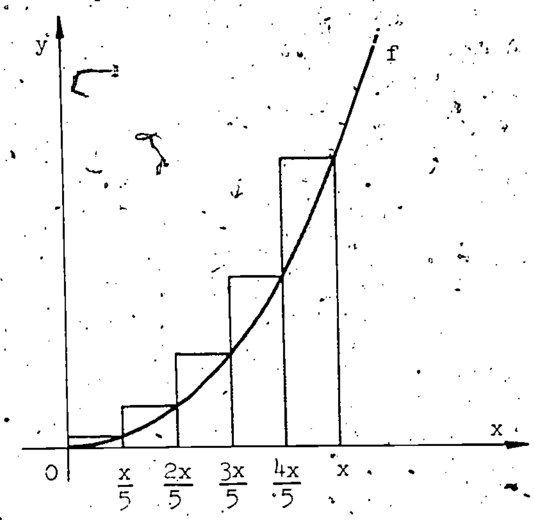



Figure 7-1f

Area approximated by exterior rectangles.

We see from these figures that our desired area $A(x)$ satisfies the two inequalities

(1) $A(x) >$ the sum of the areas of the interior rectangles,

(2) $A(x) <$ the sum of the areas of the exterior rectangles.

Let us calculate the sums of the areas of the interior and exterior rectangles. If we split the segment from $0$ to $x$ into $5$ equal parts, the length of each part will be $\frac{x}{5}$ and the endpoints of the parts will be

(3) $$0, \frac{x}{5}, \frac{2x}{5}, \frac{3x}{5}, \frac{4x}{5}, \frac{5x}{5}.$$

From Figure 7-1g we see that the height of an interior rectangle is $f(a)$, where $a$ is the left endpoint of its base; the height of an exterior rectangle is $f(b)$, where $b$ is the right endpoint of its base.

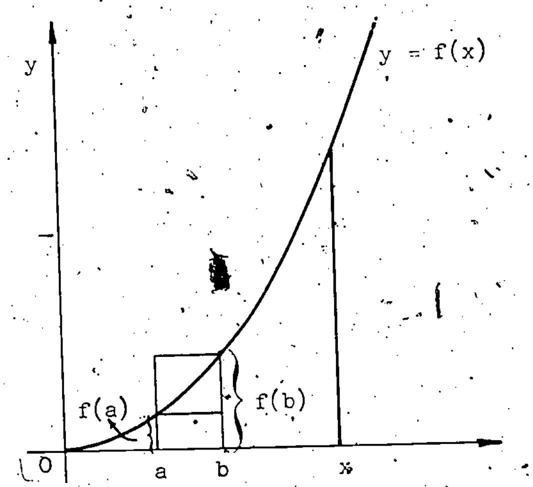



Figure 7-1g

Heights of interior and exterior rectangles.

Using the subdivisions (3) we know that the heights of the (five*) interior rectangles are

$$f(0),\ f(\tfrac{x}{5}),\ f(\tfrac{2x}{5}),\ f(\tfrac{3x}{5}),\ f(\tfrac{4x}{5});$$

the heights of the corresponding exterior rectangles are

$$f(\tfrac{x}{5}),\ f(\tfrac{2x}{5}),\ f(\tfrac{3x}{5}),\ f(\tfrac{4x}{5}),\ f(\tfrac{5x}{5}).$$

Multiplying each of these heights by the common base length $\frac{x}{5}$, we obtain the area of the corresponding rectangles. The sum of the area of the interior rectangles is

$$\frac{x}{5}[f(0) + f(\tfrac{x}{5}) + f(\tfrac{2x}{5}) + f(\tfrac{3x}{5}) + f(\tfrac{4x}{5})].$$

The sum of the areas of the exterior rectangles is

$$\frac{x}{5}[f(\tfrac{x}{5}) + f(\tfrac{2x}{5}) + f(\tfrac{3x}{5}) + f(\tfrac{4x}{5}) + f(\tfrac{5x}{5})].$$

Since $f : x \to x^2$ we have

*The leftmost "rectangular region" has zero area.



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$$f(0) = 0,\ f(\tfrac{x}{5}) = \frac{x^2}{25},\ f(\tfrac{2x}{5}) = \frac{4x^2}{25},\ f(\tfrac{3x}{5}) = \frac{9x^2}{25},$$

$$f(\tfrac{4x}{25}) = \frac{16x^2}{25} \text{ and } f(\tfrac{5x}{5}) = \frac{25x^2}{25}.$$

The sum of the areas of the interior rectangles

$$= \frac{x}{5}\left[0 + \frac{x^2}{25} + \frac{4x^2}{25} + \frac{9x^2}{25} + \frac{16x^2}{25}\right]$$

$$= \frac{x^3}{5}\left[\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right]$$

$$= \frac{6x^3}{25}.$$

The sum of the areas of the exterior rectangles

$$= \frac{x^3}{5}\left[\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{25}{25}\right]$$

$$= \frac{11x^3}{25}.$$

Our desired area $A(x)$ lies between these two quantities; that is,

$$\frac{6x^3}{25} < A(x) < \frac{11x^3}{25}.$$

This is certainly not a very accurate estimate of our desired area. If, however, we use a larger number of subdivisions we may hope to improve our estimate.

To obtain a general estimation formula, we let $n$ denote the number of subdivisions of the segment from $0$ to $x$. The length of each part will be $\frac{x}{n}$ and the endpoints will be

$$0,\ \frac{x}{n},\ 2(\tfrac{x}{n}),\ 3(\tfrac{x}{n}),\ \ldots,\ (n-1)(\tfrac{x}{n}),\ n(\tfrac{x}{n}).$$

The heights of the interior rectangles will be

$$f(0),\ f(\tfrac{x}{n}),\ f(\tfrac{2x}{n}),\ \ldots,\ f(\tfrac{(n-1)x}{n}).$$

The heights of the exterior rectangles will be

$$f(\tfrac{x}{n}),\ f(\tfrac{2x}{n}),\ \ldots,\ f(\tfrac{nx}{n}).$$

The sums of the areas of the interior and exterior rectangles will be, respectively

(4) $$\frac{x}{n}\left[f(0) + f\left(\frac{x}{n}\right) + f\left(\frac{2x}{n}\right) + \dots + f\left(\frac{(n-1)x}{n}\right)\right]$$

and

(5) $$\frac{x}{n}\left[f\left(\frac{x}{n}\right) + f\left(\frac{2x}{n}\right) + \dots + f\left(\frac{nx}{n}\right)\right].$$

Since $f : x \to x^2$, we have

$$f(0) = 0,\ f\left(\frac{x}{n}\right) = \frac{x^2}{n^2},\ f\left(\frac{2x}{n}\right) = \frac{4x^2}{n^2}$$

and, in general

$$f\left(\frac{kx}{n}\right) = \frac{k^2x^2}{n^2};\ k = 0, 1, 2, \dots, n.$$

The interior sum (4) can then be rewritten as

$$\frac{x}{n}\left[0 + \frac{x^2}{n^2} + \frac{4x^2}{n^2} + \dots + \frac{(n-1)^2x^2}{n^2}\right] = \frac{x^3}{n^3}\left[0 + 1 + 4 + \dots + (n-1)^2\right]$$

To simplify this we use the formula* for the first $(n - 1)$ squares

$$1 + 4 + \dots + (n-1)^2 = \frac{1}{6}(n-1)(n)(2n-1) = n^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right).$$

We can thus rewrite the interior sum (4) as

$$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

A similar process applied to the exterior sum (5) gives the sum of the areas of the exterior rectangles

$$\frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

Our desired area $A(x)$ lies between these two quantities; that is,

(6) $$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2} < A(x) < \frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

---

*See Appendix 3.

This must be true for each positive integer $n$. If $x$ is fixed and $n$ is very large compared to $x$ each of the terms

$$-\frac{x^3}{2n}, \quad \frac{x^3}{2n} \quad \text{and} \quad \frac{x^3}{6n^2}$$

must be very close to zero. This process suggests that the only value that the area $A(x)$ can have is $\frac{x^3}{3}$.

We summarize: if $f : x \to x^2$ and $A(x)$ is the area of the region bounded by the x-axis, the y-axis, the graph of $f$ and the vertical line $x$ units to the right of the origin, then

$$A : x \to \frac{x^3}{3}.$$

Note that the derivative of the area function is

$$f : x \to \frac{3x^2}{3} = x^2;$$

that is, $A' = f$.

This same relationship $A' = f$ was true in the case of constant and linear functions. We might conjecture that it is always true. In Section 7-3 we shall show that it is indeed true for a wide class of functions $f$, a class which includes most of the functions of interest to us in this book.

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## Exercises 7-1

1. We showed in this section that the region bounded by the coordinate axes, $y = x^2$, and a vertical line at $\underline{x}$, has an area which is between the sum of the interior and the exterior rectangles. This inequality (6) was

$$x^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(x) < x^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

(a) It follows that

$$1^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(1) < 1^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

Express this relationship when

(i) $n = 5$

(ii) $n = 100$

(b) From (6) we know that

$$2^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(2) < 2^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

Using directly the results of part (a), i.e., with minimum computation, express this relationship when

(i) $n = 5$

(ii) $n = 100$

(c) Using $A : x \to \frac{1}{3} x^3$ for the area function associated with the function, $f : x \to x^2$, find the area in the first quadrant of the region bounded by the coordinate axes, $y = x^2$, and the vertical line at

(i) $x = \frac{1}{2}$

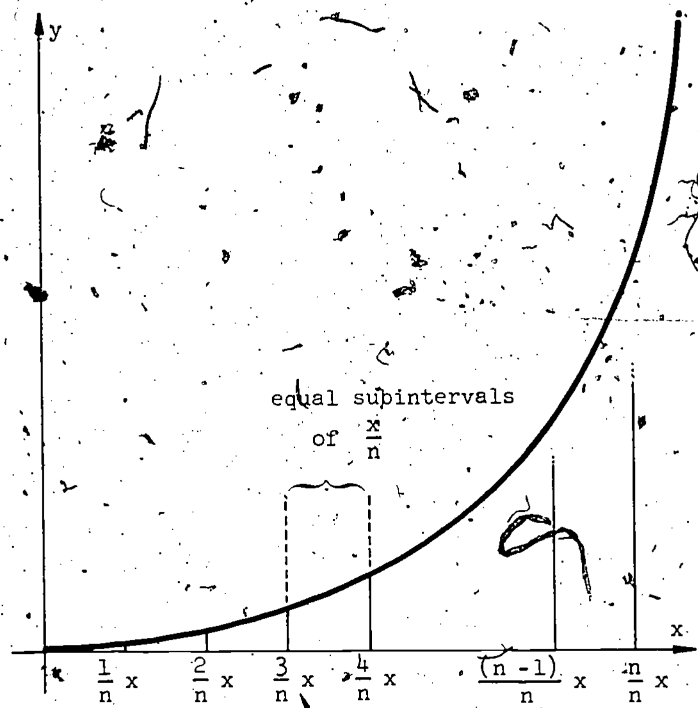
(ii) $x = 3\sqrt{3}$

2. If $f : x \to x^3$, and $A(x)$ is the area of the region depicted in the sketch to the right, show that the area function is

$$A : x \to \frac{x^4}{4},$$

using the method of this section for finding the area function of $x \to x^2$.

[Hint: The sum of $(n - 1)$ cubes is $\left(\frac{(n-1)n}{2}\right)^2$.]




Equal sub-intervals of $\frac{x}{n}$.

(a) First, show that the sum of the areas of the interior rectangles is

$$\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right).$$

(b) Second, find the sum of the areas of the exterior rectangles, showing that

$$\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(x) < \frac{x^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right),$$

and as $n \to \infty$, $A : x \to \frac{1}{4}x^4$.

(c) Next, using the inequality of part (b) above, and letting $x = 1$, find an expression for $A(1)$, when

(i) $n = 5$

(ii) $n = 100$.

(d) From the expressions found for $A(1)$ in part (c) above, find, with minimum computation, an expression for $A(2)$, when $n = 100$.

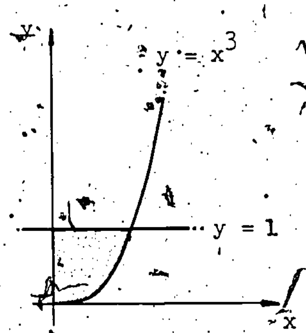
(e) Using $A : x \to \frac{1}{4} x^4$ for the area function associated with the function, $f : x \to x^3$, find the area in the first quadrant of the region bounded by the coordinate axes, $y = x^3$, and the vertical line at

(i) $x = 0.4$

(ii) $5\sqrt{2}$

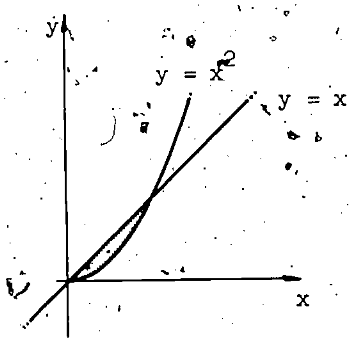
3. Find the area of the region in the first quadrant bounded by $x = 0$, $y = 1$, and $y = x^3$.

[Hint: $y = 1$ and $y = x^3$ intersect at $(0,0)$ and $(1,1)$. The shaded area equals the area under $y = 1$ minus the area under $y = x^3$ (between the intersection points).]




4. Find the area of the region in the first quadrant bounded by $y = x$ and $y = x^2$.

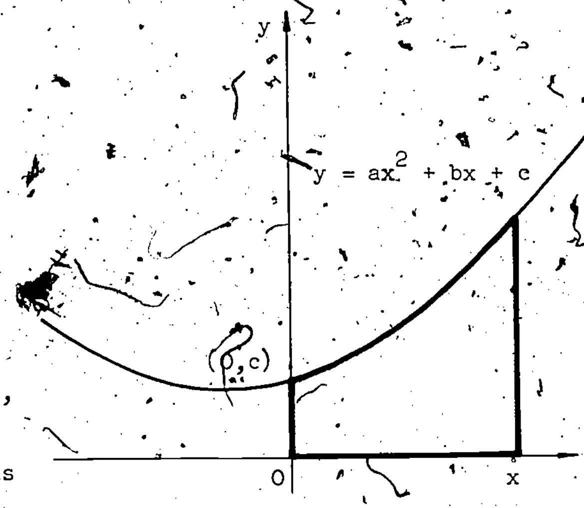
[Hint: Find the intersection points; find the area under each curve between intersection points; find the difference between these areas.]




5. Sketch $y = x^3$ and $y = x^2$. $[-\frac{1}{2} < x < \frac{3}{2}]$
In a similar manner to that of Number 3 and Number 4, find the area between the two curves.

6. Assume that $y = ax^2 + bx + c$ is in the position of the sketch to the right. By summing the areas of interior rectangles and exterior rectangles, find the area of the region bounded in the first quadrant by the coordinate axes, the curve $y = ax^2 + bx + c$, and the vertical line at $x$. That is, if $f : x \to ax^2 + bx + c$, show that the area function is

$$A : x \to \frac{1}{3} ax^3 + \frac{1}{2} bx^2 + cx.$$




7. What conditions on $a$, $b$, $c$ will guarantee that there are some positive numbers $x$ such that on the interval from $0$ to $x$ the graph of $y = ax^2 + bx + c$, $a \neq 0$, will lie in the first quadrant?

8. Using the results of Number 7, determine which of the following have non-empty regions in the first quadrant bounded by the coordinate axes, the graph of the function and some vertical line to the right of the origin

(a) $f : x \to x^2 + 1$

(b) $f : x \to x^2 - 2x$

(c) $f : x \to 2x - 3x^2$

(d) $f : x \to x - 1 + x^2$

9. For each of the following use Number 6 to find an expression for $A(x)$, and then the area of the region bounded by the coordinate axes, the indicated curve and the indicated vertical lines.

(a) $f : x \to 2x^2 + 6x + 3$

(i) $x = 1$ (ii) $x = 3$

(b) $f : x \to 12x^2 + 38x + 16$

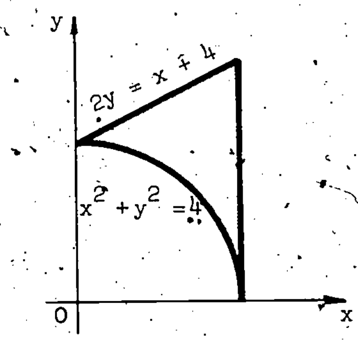
(i) $x = \frac{1}{2}$ (ii) $x = 1$

(c) $f : x \to 12 + 18x - 3x^2$

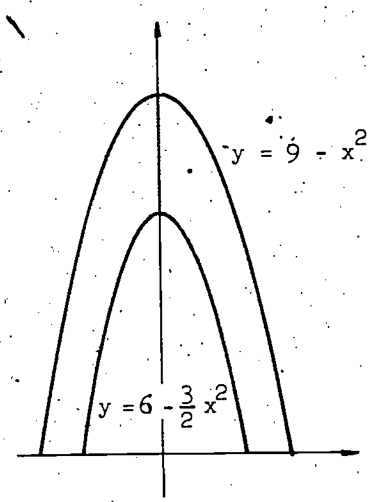
(i) $x = 0$ (iii) $x = 2$

(ii) $x = 1$ (iv) $x = 4$

10. Find the area in quadrant one bounded by the quarter circle (with center at origin and radius 2), the line $x - 2y + 4 = 0$, and the vertical line tangent to the circle.
[Hint: Find intersection points; find area of quarter circle by geometry; subtract areas.]




11. Find area of region bounded by $y = 0$, $y = 9 - x^2$, and $y = 6 - \frac{3}{2}x^2$.
[Hint: Use symmetry.]




12. (a) For the function $f : x \to x^2$, we developed in this section an inequality for the area function:

$$\frac{x^3}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right) < A(x) < \frac{x^3}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right)$$

Show that if we average these sums of areas of interior and exterior for $n = 5$, we have $A(x) \approx \frac{17}{50}x^3$.

(b) Now estimate $A(x)$ for the same function by connecting $(0, f(0))$ to $\left(\frac{x}{5}, f\left(\frac{x}{5}\right)\right)$; $\left(\frac{x}{5}, f\left(\frac{x}{5}\right)\right)$ to $\left(\frac{2x}{5}, f\left(\frac{2x}{5}\right)\right)$, ..., and summing the resulting trapezoids.



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(c) As a third estimate, sum 5 rectangles with equal widths along the the x-axis, and heights erected at the midpoint of each interval; i.e., the width of each rectangle would be $\frac{x}{5}$, and the heights would be $\frac{x}{10}$, $\frac{3x}{10}$, ...

(d) Which of these three estimates above is the closest to the exact area of $\frac{1}{3}x^3$.

## 7-2. Integral Notation

Let us introduce some common notation. Suppose that $a \leq b$, that $f(x)$ is defined for $a \leq x \leq b$, and that the graph of $f$ does not go below the x-axis in this interval; that is, $f(x) \geq 0$ for $a \leq x \leq b$. The symbol

$$\int_a^b f$$

is read "the integral of $f$ from $a$ to $b$" and denotes the area of the region bounded by the x-axis, the graph of $f$ and the two vertical lines given by $x = a$ and $x = b$. (See Figure 7-2a.) The terminology and the symbol "$\int$" (which is the Roman letter summa) arise from the procedure (described in the previous section) of approximating sums for finding areas. The

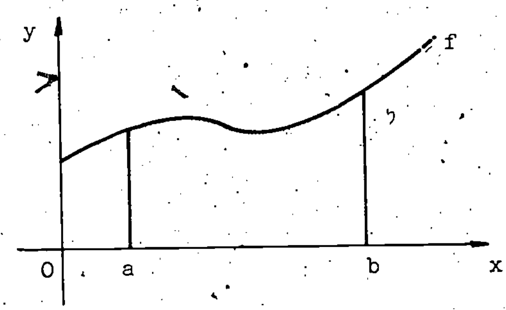


Figure 7-2a

Area under a graph.

numbers $a$ and $b$ are, respectively, called the lower and upper limits of integration.

In the first section for convenience we took the lower limit $a = 0$ and denoted the upper limit $b$ by $x$, obtaining formulas for

$$A(x) = \int_0^x f$$

for certain simple functions $f$. Using elementary geometry we found that

(1) if $f : x \to c$, then $\int_0^x f = cx$;

and

(2) if $f : x \to mx + b$, then $\int_0^x f = \frac{mx^2}{2} + bx$.

We also approximated with interior and exterior rectangles to conclude tentatively that

$$(3) \qquad \text{if } f : x \to x^2, \text{ then } \int_0^x f = \frac{x^3}{3}.$$

In order to become more familiar with the integral notation (1), we shall discuss here some properties which we expect area to possess. This discussion will argue from intuition -- that is, we shall suppose that the desired areas $\int_a^b f$ can be found in such a way as to be consistent with elementary area principles. In the appendices we shall show that the process of approximation by sums of areas of interior and exterior rectangles will, for the elementary functions which concern us, indeed give a concept of area which is consistent with these principles.

The area of a region, such as that shown in Figure 7-2a, should be a non-negative number; that is,

$$(4) \qquad \text{if } f(x) \geq 0 \text{ for } a \leq x \leq b, \text{ then } \int_a^b f \geq 0.$$

We expect that the area of a region should not exceed the area of any larger region; a useful formulation of this idea:

$$(5) \qquad \text{if } f(x) \leq g(x), \text{ for } a \leq x \leq b, \text{ then } \int_a^b f \leq \int_a^b g.$$

(See Figure 7-2b.)

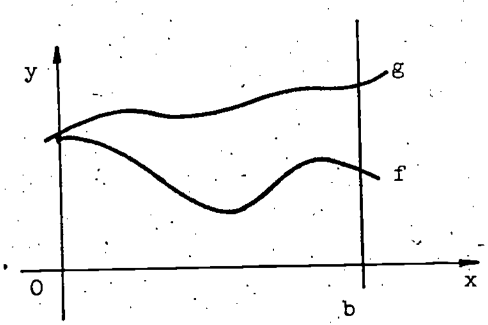


Figure 7-2b

The area under $f$ does not exceed the area under $g$.

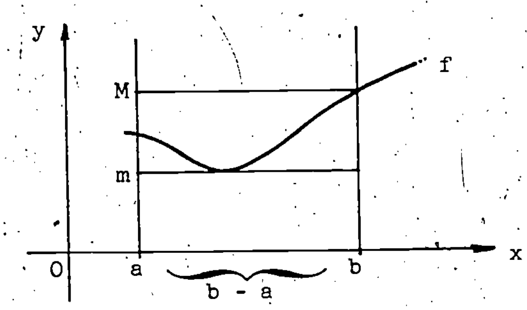


Figure 7-2c

$$m(b - a) \leq \int_a^b f \leq M(b - a)$$

An application of the inequality of (5) gives bounds for area in terms of bounds for f. Suppose M is a constant and $f(x) \leq M$ for $a \leq x \leq b$. With $g: x \to M$ we can apply (5) to obtain

$$\int_a^b f \leq \int_a^b g = M(b - a).$$

Similar arguments can be applied if $m \leq f(x)$ to obtain $m(b - a) \leq \int_a^b f$. (See Figure 7-2c.) In summary:

(6) if $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

A line has no width and hence zero area. Thus, if we take $b = a$, we should expect the area to be zero, that is,

(7)
$$\int_a^a f = 0$$

This is consistent with our result (6), for if we take $b = a$ we obtain

$$0 = m \times 0 \leq \int_a^a f \leq M \times 0 = 0.$$

If we choose new horizontal or vertical scales then we expect the area to be changed by a corresponding factor. One useful consequence of this:

(8) If $g(x) = \alpha f(x)$, for $a \leq x \leq b$, where $\alpha$ is a positive constant, then

$$\int_a^b g = \alpha \int_a^b f.$$

(See Figure 7-2d.)

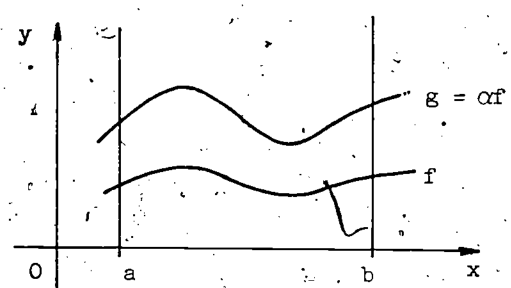


Figure 7-2d

The area under $g$ is $\alpha$ times the area under $f$.

In one region is the union of two non-overlapping regions we expect the area of the first region to be the sum of the areas of the subregions. This additivity principle has two useful consequences, (9) and (10).

(9) If $c$ lies between $a$ and $b$, then

$$\int_a^b f = \int_a^c f + \int_c^b f;$$

—that is, if we cut the region under $f$ by a vertical line, then the area is the sum of the two resulting areas. (See Figure 7-2e.)

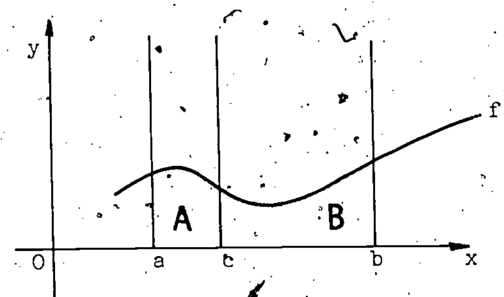


Figure 7-2e

The area of the region under the graph of $f$ between $a$ and $b$ is the sum of the areas of regions A and B.

A second useful formulation of additivity is obtained for the sum of two graphs. The sum $f + g$ is defined as the function whose value at $x$ is $f(x) + g(x)$; that is, the graph of $f + g$ is obtained by adding the ordinates of the graphs of $f$ and $g$. We have

(10) $$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(See Figure 7-2f).

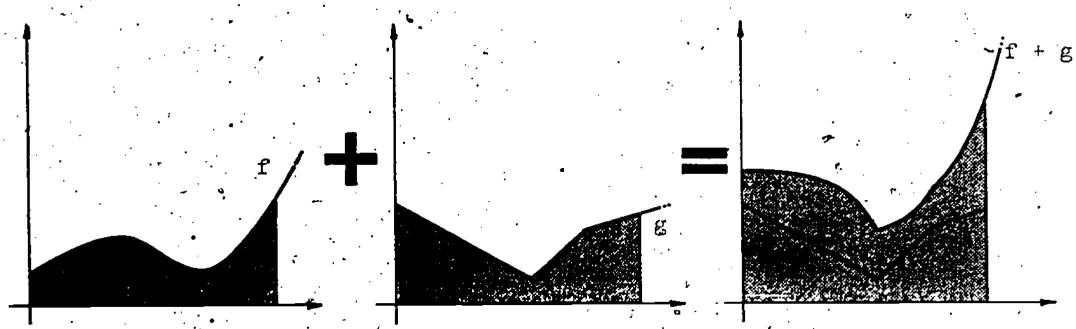


Figure 7-2f

The area of the region under the graph of $f$ plus
the area of the region under the graph of $g$ is
the area of the region under the graph of $f + g$.

These principles are not independent; that is, some are consequences of others. Other useful principles will be introduced as need arises. The following examples show how we can combine area principles with our knowledge of particular areas to find other areas.

Example 7-2a. Find the area (of the region*) under the graph of $f : x \to 1 + x + x^2$, between $a = 0$ and $b = 4$.

We need to evaluate

$$\int_0^4 f.$$

The function $f$ can be expressed as the sum of the two functions

$$f_1 : x \to 1 + x \quad \text{and} \quad f_2 : x \to x^2.$$

Formulas (2) and (3) give

$$\int_0^x f_1 = x + \frac{x^2}{2} \quad \text{and} \quad \int_0^x f_2 = \frac{x^3}{3}.$$

so that with $x = 4$ we have:

$$\int_0^4 f_1 = 4 + \frac{16}{2} = 12$$

$$\int_0^4 f_2 = \frac{64}{3}.$$

*By "area under the graph" we shall mean "area of the region under the graph," as described in the opening paragraph of this section.

The additivity principle (10) then gives

$$\int_0^4 f = \int_0^4 f_1 + \int_0^4 f_2 = 12 + \frac{64}{3} = \frac{100}{3}.$$

Example 7-2b. Find $\int_1^5 f$, where

$$f : x \to 5 - x + 2x^2.$$

We seek to find the area of the region bounded by the graph of $f$, the x-axis and the lines $x = 1$, $x = 5$. Let us first graph the function $f$. The integral $\int_1^5 f$ is the area of the shaded region of Figure 7-2g.

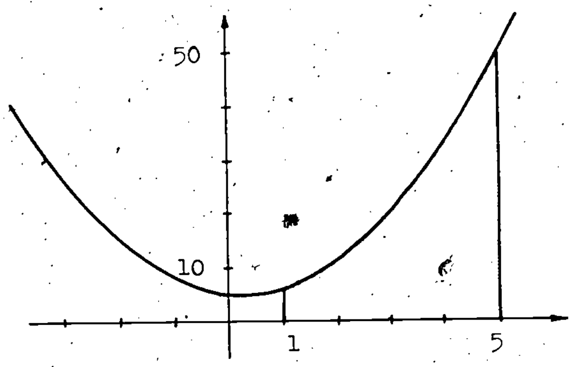


Figure 7-2g

$f : x \to 5 - x + 2x^2$

To calculate $\int_1^5 f$ we first express it in terms of integrals with lower limit 0. We have

$$\int_0^5 f = \int_0^1 f + \int_1^5 f$$

so that

(11) $$\int_1^5 f = \int_0^5 f - \int_0^1 f.$$

Now we write

$$f_1(x) = 5 - x \text{ and } f_2(x) = x^2,$$

so that with $f = f_1 + 2f_2$, (8) and (10) give

$$\int_0^5 f = \int_0^5 f_1 + 2\int_0^5 f_2; \quad \int_0^1 f = \int_0^1 f_1 + 2\int_0^1 f_2.$$

The results (2) and (3) give

$$\int_0^x f_1 = 5x - \frac{x^2}{2} \quad \text{and} \quad \int_0^x f_2 = \frac{x^3}{3}.$$

Therefore, we have

$$\begin{aligned}\int_1^5 f &= \int_0^5 f - \int_0^1 f \\ &= \left(\int_0^5 f_1 + 2\int_0^5 f_2\right) - \left(\int_0^1 f_1 + 2\int_0^1 f_2\right) \\ &= \left(5\cdot 5 - \frac{5^2}{2} + 2\cdot \frac{5^3}{3}\right) - \left(5.1 - \frac{1^2}{2} + 2\cdot\frac{1^3}{3}\right) \\ &= \frac{272}{3}.\end{aligned}$$

<u>Example 7-2c</u>. Find the area of the region between the graphs of the functions $f$ and $g$ defined by

$$f(x) = x^2 - 6x + 7 \quad \text{and} \quad g(x) = -x^2 + 7x - 11.$$

Figure 7-2h indicates the region whose area is sought (The points of intersection are found by solving $x^2 - 6x + 7 = -x^2 + 7x - 11$ for $x$).

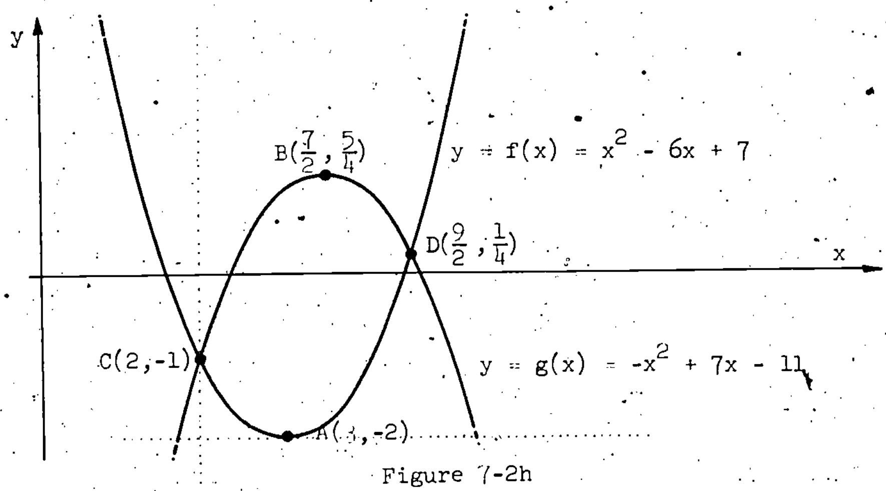


Figure 7-2h

A and B are, respectively, the minimum of $f$ and maximum of $g$, while C and D are the points of intersection.

At this early stage in our development we solve this problem by using formulas (2) and (3) and various area principles. First we choose new axes so our curves will be in the first quadrant and our integrals can be taken with lower limit 0. One way to do this is to choose our vertical axis through $C$ and horizontal axis through $A$. (See Figure 7-2h.) Call these the $s$ and $t$ axes, respectively. Thus,

$$s = y + 2, \qquad y = s - 2;$$
$$t = x - 2, \qquad x = t + 2.$$

For the graphs of $f$ and $g$ we obtain the new equations

$$s - 2 = (t + 2)^2 - 6(t + 2) + 7$$
$$s - 2 = -(t + 2)^2 + 7(t + 2) - 11,$$

which are respectively

$$s = t^2 - 2t + 1 \quad \text{and} \quad s = -t^2 + 3t + 1.$$

Our desired area is the difference of the areas of the shaded regions shown in Figure 7-2i.

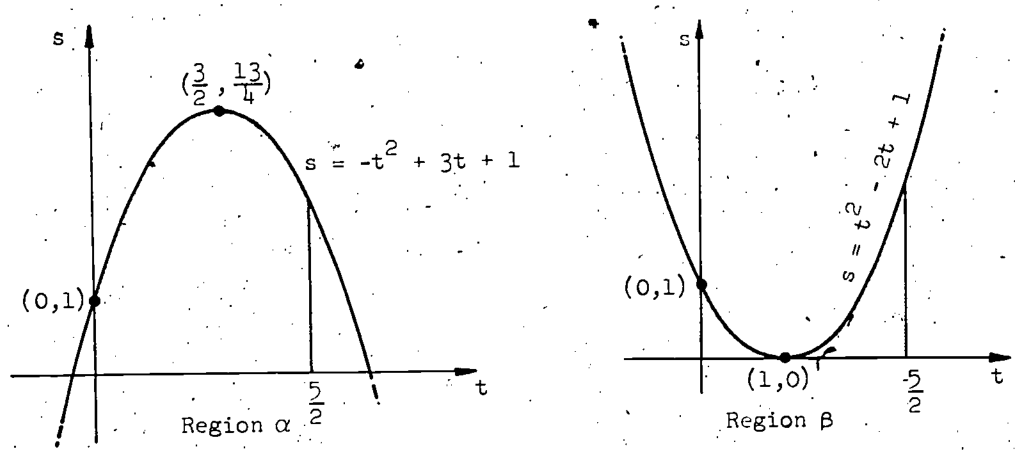


Figure 7-2i

The negative signs in these expressions causes some difficulties in calculating the desired areas. We could resort to approximations by upper and lower sums (that is, use the formula of Exercises 7-1, No. 6). Instead, let us continue using area principles to reduce our problem to the known integral forms of (2) and (3). First we find the area of $\alpha$. We replace $s$ by $-s$ to obtain (a reflection in the t-axis) the graph of $s = t^2 - 3t - 1$ (Figure 7-2j).

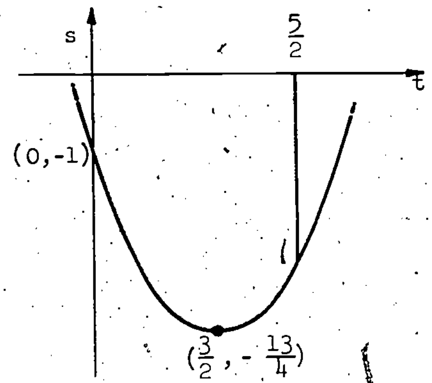


Figure 7-2j

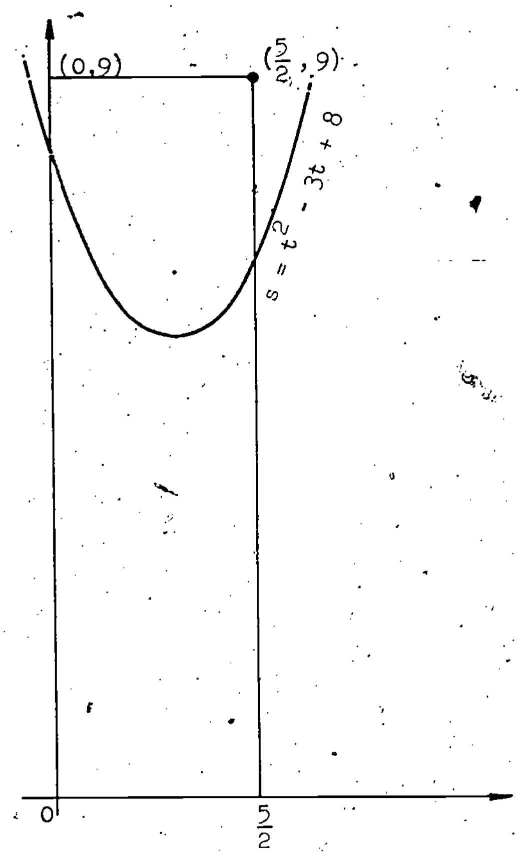


Figure 7-2k

Now replace $s$ by $s - 9$ (which shifts the graph 9 units upward) to obtain

$$s = t^2 - 3t + 8 \qquad \text{(Figure 7-2k)}.$$

The area of $\alpha$ is then the same as the area of the shaded region of Figure 7-2k, and we have

$$\text{area of } \alpha = 9 \times \frac{5}{2} - \int_0^{5/2} s, \quad \text{where } s = t^2 - 3t + 8.$$

Thus, the additivity principle (10) can be applied to obtain

$$\text{area of } \alpha = \frac{45}{2} - \left(\int_0^{5/2} s_1 + \int_0^{5/2} s_2\right),$$

where

$$s_1 = t^2 \text{ and } s_2 = -3t + 8$$

since each of these functions is nonnegative for $0 \leq t \leq \frac{5}{2}$ and $s = s_1 + s_2$.

We have

$$\int_0^{5/2} s_1 = \frac{(\frac{5}{2})^3}{3} \qquad \text{(from (3))}$$

$$\int_0^{5/2} s_2 = \frac{-3}{2}(\frac{5}{2})^2 + 8(\frac{5}{2}) \qquad \text{(from (2))}$$

so that

$$\text{area of } \alpha = \frac{45}{2} - (\frac{125}{24} + \frac{85}{8}) = \frac{20}{3} .$$

A similar calculation gives

$$\text{area of } \beta = \frac{35}{24} ,$$

where $\beta$ is the second region of Figure 7-21. The area which we seek is

$$\text{area of } \alpha - \text{area of } \beta = \frac{125}{24} .$$

In Section 7-5 we shall develop methods which will simplify this problem considerably.

## Exercises 7-2

1. Suppose $f : x \to x^2$, $g : x \to 2x + 3$.

   (a) Graph each.

   (b) Show that $f(x) \le g(x)$ for $0 \le x \le 3$.

   (c) Show that $\int_0^3 f \le \int_0^3 g$.

2. Over the indicated interval for the following functions: graph the function; find the maximum (M) value of the function; find the minimum (m) value of the function; and, using these, express with an inequality the lower and upper bounds of the integral expression for the area. [Hint: See Figure 7-2c.]

   (a) $f : x \to x + 1$, $0 \le x \le 1$

   (b) $f : x \to x^2 - 2x + 3$, $0 \le x \le 3$

3. For $f : x \to 3x - 2$ and $g = \sqrt{2}\, f$ find $\int_5^{10} f$, $\int_5^{10} g$ and verify that $\int_5^{10} g = \sqrt{2}\int_5^{10} f$.

4. For $f : x \to -2x + 20$ and $g : x \to -2(x - h) + 20$.

   (a) Find a suitable translation such that $f(3) = g(0)$ and $f(7) = g(4)$. Graph $f$ and $g$.

   (b) Find $\int_0^3 f$, $\int_0^4 g$, $\int_0^7 f$ and verify that $\int_0^7 f = \int_0^3 f + \int_0^4 g$. Thus $\int_0^7 f = \int_0^3 f + \int_3^7 f$.

5. For $f : x \to 3x + 5$, $g : x \to x$ and $h : x \to 1$ verify that $\int_a^b f = 3\int_a^b g + 5\int_a^b h$ by using (1) and (2) to find each integral.

6. Find each of the following integrals, after first graphing the given function over the interval.

(a) $\int_1^3 x^2 + x$

(b) $\int_1^4 x^2 - 4x + 5$

(c) $\int_1^3 -x^2 + 2x + 3$

(d) $\int_2^4 \frac{1}{4} x^2 + \frac{1}{2} x - 1$

7. Suppose $f : x \to px^2 + qx + r$ where $p, q$ and $r$ are nonnegative constants.

(a) Put $F : x \to \frac{p}{3} x^3 + \frac{q}{2} x^2 + rx$ and show that $F' = f$.

(b) Show that if $0 \leq a \leq b$ then

$$\int_a^b f = F(b) - F(a)$$

(Hint: $\int_a^b f = \int_0^b f - \int_0^a f$)

8. In Exercises 7-1, Number 2 it was shown that for $f : x \to x^3$

$$\int_0^x f = \frac{1}{4} x^4 \quad \text{for} \quad x \geq 0.$$

Suppose $g : x \to px^3 + qx^2 + rx + s$, where $p, q, r$ and $s$ are nonnegative constants. Suppose also that

$$G : x \to \frac{p}{4} x^4 + \frac{q}{3} x^3 + \frac{r}{2} x^2 + sx.$$

(a) Show that $G' = g$.

(b) Show that if $0 \leq a \leq b$ then $\int_a^b g = G(b) - G(a)$.

9. In Number 7 put $G(x) = F(x) + 1000$ and show that $\int_a^b f = G(b) - G(a)$.

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10. Find $\int_0^5 f$ where $f : x \to |x - 2|$.

(Hint: A graph is, of course, helpful.)

11. For $f : x \to x^2$, show how to find $\int_{-10}^{-3} f$.

(Hint: Translate along the x-axis using $g(x) = f(x - h)$ for some $h$.)

12. Verify that Number 7(b) still holds if $a \leq b \leq 0$ or if $a \leq 0 \leq b$.
(Hint: Use a translation.)

13. Suppose $a \leq c \leq b$ and that

$$f(x) = \begin{cases} h(x), & a \leq x \leq c \\ 0, & c < x \leq b \end{cases}$$

$$g(x) = \begin{cases} 0, & a \leq x \leq c \\ h(x), & c < x \leq b \end{cases}$$

Show that $f + g = h$ and use this to show that (9) is a consequence of (10).

14. Find the area and graph of the region bounded by $f(x) = y = 2(x - 5)^2 - 2$ and $y = 0$. (Hint: Translate and graph the area into the first quadrant.)

15. Find the area of the region bounded by $f(x) = y = -(x + 1)^2 + 1$ and $g(x) = y = x$.

16. Suppose $A(x) = \int_a^x f$, where $f$ is nonnegative. Show that if $a \leq x_1 \leq x_2$ then $A(x_1) \leq A(x_2)$. (Hint: Use (4) and (9)).

## Exercises 7-2

1. Find the area of the regions represented by the following integrals, where $f : x \to x^3$.

   (a) $A(3) = \int_0^3 f$

   (b) $A(2) = \int_0^2 f$

   (c) Find the area represented by $\int_2^3 f$ by use of the relationship,

   $$\int_2^3 f = \int_0^3 f - \int_0^2 f$$

2. (a) By dividing the interval into appropriate subintervals, find the area over the interval $x = 0$ to $x = 2$ under the graph of the function

   $$f : x \to |x - 1|$$

   by use of integration.

   (b) Sketch the graph and check your solution to part (a) by geometry.

3. (a) Find $A(2)$ if $f : x \to x^4$.

   (b) Using Formula (8), find the area, over the same interval, under the graph of the following functions:

   (i) $x \to \frac{1}{2} x^4$ (iii) $x \to 5x^4$

   (ii) $x \to \frac{1}{8} x^4$ (iv) $x \to 10x^4$

4. Over the indicated interval for the following functions: find the maximum $(M)$ value of the function; find the minimum $(m)$ value of the function; and, using these, express with an inequality the lower and upper bounds of the integral expression for the area.

   [Hint: See Figure 7-2c.]

   (a) $f : x \to x + 1, \quad 0 \leq x \leq 1$

   (b) $f : x \to x^2 - 2x + 3, \quad 0 \leq x \leq 3$

5. Given: $f : x \to \cos x$, where $0 \leq x \leq \frac{\pi}{2}$, and the corresponding area function $A : x \to \sin x$, where $0 \leq x \leq \frac{\pi}{2}$.

   (a) Find the area of the region bounded by the graph of $f$, and the coordinate axes.

(b) What value of $\underline{a}$ will make the vertical line $x = a$ divide this region into two equal parts?

(c) If this region (of part (a)) is divided into three regions of equal area by vertical lines, what are the equations of these lines?

## 7-3. The Fundamental Theorem of Calculus

In Section 7-1 we found some formulas for the area of the region in the first quadrant bounded by the graph of a function $f$, the x-axis, the y-axis and a second vertical line, $x$ units to the right of the origin, such as that shown in Figure 7-3a.

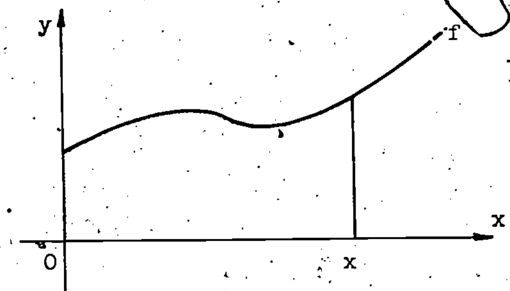


Figure 7-3a

Area Under a Graph

Calling the indicated area $A(x)$ we obtained a function $x \to A(x)$, which we called the "area function." Using the integral notation we have

$$A(x) = \int_0^x f.$$

The results obtained in Section 7-1 can be tabulated as follows:

| Function $f$ | Area function $A$ | Derivative of area function $A'$ |
|---|---|---|
| $x \to c$ | $x \to cx$ | $x \to c$ |
| $x \to mx + b$ | $x \to \frac{mx^2}{2} + bx$ | $x \to mx + b$ |
| $x \to x^2$ | $x \to \frac{x^3}{3}$ | $x \to x^2$ |

It is impossible to miss the similarity between the first and third columns of this table. Since these two columns are identical except for heading we are practically compelled to suspect that there must be some relationship between $f$ and the derivative $A'$ of its area function $A$. We conjecture:

(1) $\boxed{\text{If } A \text{ is the area function associated with a function } f, \text{ then } A' = f.}$

In order to establish this result we need some conditions on the function $f$. Let us prove this with the following assumptions on $f$:

(a) $f$ is an increasing function; that is,

(2) $$f(c) < f(d) \quad \text{if} \quad 0 \leq c < d,$$

(b) the graph of $f$ has no "gaps" for $x \geq 0$.

These two conditions imply

(3) If $x \geq 0$ and $c$ is close enough to $x$ then $f(c)$ is close to $f(x)$.

This result will be established in the appendices. As it seems plausible we shall assume it to be true at this point. (The same assumption was used for $x \to e^x$ in the discussions of Section 6-7).

In order to prove (1) we wish to show that if $|h|$ is small then

$$\frac{A(x + h) - A(x)}{h} \approx f(x);$$

that is, the slope of the line connecting $P(x,A(x))$ to $Q(x + h, A(x + h))$, approximates $f(x)$ for $|h|$ small. Since this slope will also approximate $A'(x)$, the slope of the tangent line at $P(x,A(x))$, we shall then know that $A'(x) = f(x)$. (See Figure 7-3b.)

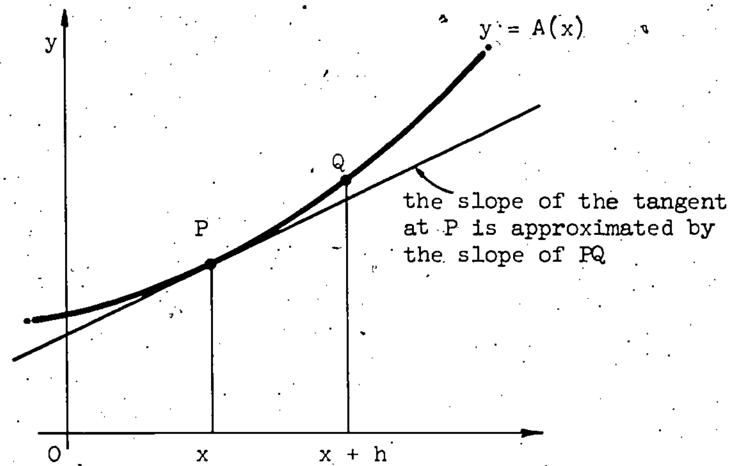


Figure 7-3b

Graph of the Area Function

Let us first suppose that $h > 0$, so that the graph of $f$ is something like that shown in Figure 7-3c. The two quantities $A(x)$ and $A(x + h)$ are the areas of the regions bounded by the y-axis, the x-axis, the graph of $f$, and the vertical lines which are respectively $x$ and $x + h$ units to the right of the origin. Hence, the difference

$$A(x + h) - A(x)$$

represents the area of the shaded region shown in Figure 7-3c.

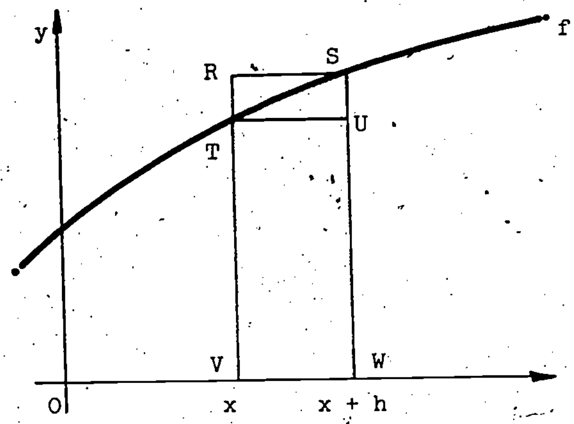


Figure 7-3c

$$A(x + h) - A(x) = \text{Area of the shaded region}.$$

Since we have assumed that $f$ is increasing, the shaded region of Figure 7-3c includes the smaller rectangle $TUWV$ and is included in the larger rectangle $RSWV$. These rectangles have base length $h$ and the respective heights $f(x)$ and $f(x + h)$. Thus

$$hf(x) < \text{area of shaded region} < hf(x + h);$$

that is,

$$hf(x) < A(x + h) - A(x) < hf(x + h).$$

This inequality used the assumption that $h > 0$. If we divide by $h$ we obtain

(4) $$f(x) < \frac{A(x + h) - A(x)}{h} < f(x + h).$$

Here is where we use (3), for if $h$ is small, then $x + h$ is close to $x$ so that $f(x + h)$ is close to $f(x)$. Hence, if $h$ is small and positive then

$$\frac{A(x + h) - A(x)}{h} \approx f(x).$$

Comparable arguments will give the same result if $h < 0$, so that, indeed $A' = f$, if the assumptions (2) hold. We can, of course, replace the assumption that $f$ is increasing by the assumption that $f$ is decreasing. This will invert the inequality signs in (4) but not change the conclusion.

In the above proof we used the fact that

$$A(x + h) - A(x)$$

is the area of the shaded region shown in Figure 7-3c. This will also be true if the lower limit is taken to be any number $a \le x$. In other words, if we put

$$A(x) = \int_a^x f, \quad x \ge a.$$

This represents the area of the shaded region shown in Figure 7-3d. The difference

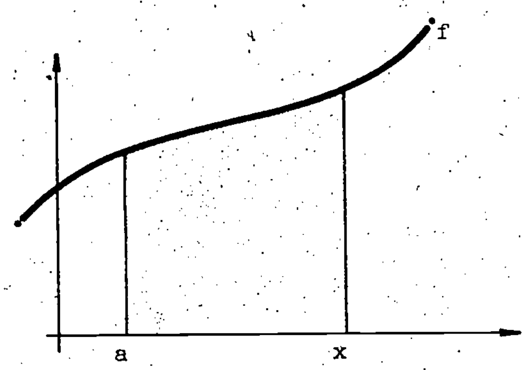


Figure 7-3d

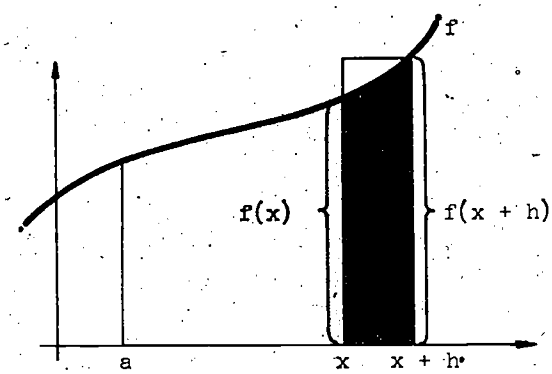


Figure 7-3e

$$A(x + h) - A(x)$$

will be the area of the darkly shaded region shown in Figure 7-3e.

Assuming that $f$ is increasing for $x > a$ we could repeat the foregoing arguments to conclude that

$$f(x) < \frac{A(x + h) - A(x)}{h} < f(x + h), \quad \text{if } h > 0$$

and

$$f(x) > \frac{A(x + h) - A(x)}{h} > f(x + h), \quad \text{if } h < 0.$$

If we assume that the graph of $f$ has no "gaps" we thus arrive at the result

$$\frac{A(x + h) - A(x)}{h} \approx f(x) \text{ if } |h| \text{ is small,}$$

and hence conclude that $A' = f$.

This fact that the derivative of the area function is $f$ will be referred to as the Area Theorem.*

---

*This is also sometimes known as the Fundamental Theorem of Calculus, a subsequent theorem which can be established analytically without area arguments.

AREA THEOREM. Suppose $f$ is nonnegative and increasing on the interval $a \le x \le b$ and that the graph of $f$ has no "gaps." For each $x$ in this interval, if we put

$$A(x) = \int_a^x f,$$

then

$$A'(x) = f(x).$$

The same result will hold if $f$ is assumed to be decreasing on the interval. In the appendices it will be shown that the theorem remains true if only the continuity condition (3) holds.

The Area Theorem doesn't yet tell us how to find the area function $x \to A(x)$; it only tells us that the derivative $A'$ must be $f$. Consider, for example, the problem of finding the area function

$$(5) \qquad A(x) = \int_0^x f, \quad \text{for } f : x \to x^3.$$

We know that the derivative of

$$x \to x^4$$

is the function $x \to 4x^3$, so if we divide by 4 then the derivative of

$$x \to \frac{1}{4}x^4 \text{ is } x \to x^3.$$

Thus a good candidate for $A$ is

$$A : x \to \frac{1}{4}x^4.$$

Note, however, that the derivative of

$$x \to \frac{1}{4}x^4 + 10$$

is also $x \to x^3$. In fact, if $C$ is any constant then the derivative of

$$x \to \frac{1}{4}x^4 + C \text{ is } x \to x^3,$$

so that any function of the type $x \to \frac{1}{4}x^4 + C$ is a candidate for $A$. Fortunately, there are no other possibilities for $A$. This is a consequence of the following theorem.

THE CONSTANT DIFFERENCE THEOREM.* If $G'(x) = F'(x)$, $a \leq x \leq b$, then there is a constant $C$ such that

$$G(x) = F(x) + C, \quad a \leq x \leq b.$$

Proof. A rigorous proof of this result is surprisingly complicated, making use of the fact that the real line has no "gaps." (See the appendices.) We give here an intuitive argument.

Put $C = G(a) - F(a)$, so that the graphs of $y = G(x)$ and $y = F(x)$ are $C$ units apart at the point where $x = a$. Since $G' = F'$, each graph is rising (or falling) at the same rate at each point and hence the two graphs must remain $C$ units apart (see Figure 7-3f); that is,

$$G(x) = F(x) + C \quad \text{for } a \leq x \leq b.$$

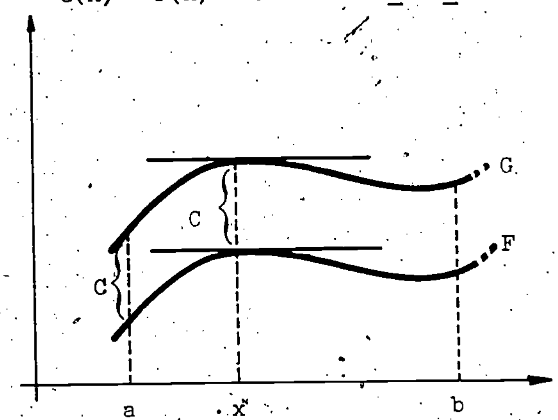


Figure 7-3f

Consider again the problem of finding the area function of (5); that is,

$$A(x) = \int_0^x f, \quad \text{where } f : x \to x^3.$$

We noted that if $F : x \to \frac{1}{4}x^4$, then $F' = f$. Furthermore, the Area Theorem tells us that $A' = f$. Therefore, the Constant Difference Theorem tells us that there must be a constant $C$ such that

$$A(x) = F(x) + C.$$

*For ease of reference we have given this commonly untitled result a name.



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To determine $C$, we need only calculate $A(x)$ and $F(x)$ for one value of $x$, say $x = 0$; $A(0) = F(0) + C$, so $C = A(0) - F(0)$. Recall that

$$A(0) = \int_0^0 f = 0.$$

(See (7) of Section 7-2.) Note that $F(0) = 0$. It must, therefore, be true that $C = 0$, so that $A$ and $F$ are the same function; that is:

$$\text{if } f : x \to x^3, \text{ then } \frac{1}{4}x^4 = \int_0^x f.$$

The following theorem summarizes this method for finding area functions. This theorem is generally referred to as the <u>Fundamental Theorem of Calculus</u>*, and provides a basic technique for calculating areas by using antiderivatives.

<u>THE FUNDAMENTAL THEOREM OF CALCULUS</u>. If $f$ is nonnegative, increasing and its graph has no gaps on the interval $a \le x \le b$, and if $F$ is any function whose derivative is $f$ on this interval, then

$$\int_a^x f = F(x) - F(a), \quad a \le x \le b.$$

<u>Proof</u>. The area function

$$A(x) = \int_a^x f$$

is a function whose derivative is $f$ (from the Area Theorem). Furthermore, $A(a) = 0$ so that

$$\int_a^x f = A(x) - A(a).$$

The idea now is to show that if $F' = f$, then

$$F(x) - F(a) = A(x) - A(a).$$

Since the functions $F$ and $A$ have the same derivative $f$ the Constant Difference Theorem implies that there is a constant $C$ such that

$$A(x) = F(x) + C, \quad a \le x \le b.$$

---

*It relates differentiation and integration.

A simple calculation then gives

$$\begin{aligned} A(x) - A(a) &= (F(x) + C) - (F(a) + C) \\ &= F(x) - F(a). \end{aligned}$$

Thus we indeed have

$$F(x) - F(a) = A(x) - A(a) = \int_a^x f.$$

Remark. This theorem will still be true if $f$ is assumed to be decreasing on the interval, for the Area Theorem will remain true and the above proof can be repeated verbatim. The theorem is easily extended to the case when the interval can be subdivided into smaller intervals, on each of which $f$ increases or decreases. For example, suppose that $F' = f$ and that $f$ increases for $a \le x \le c$ and decreases for $c \le x \le b$. (See Figure 7-3g.) Now recall that

$$\int_a^b f = \int_a^c f + \int_c^b f$$

and apply the Fundamental Theorem to each term to obtain

$$\int_a^c f = F(c) - F(a), \quad \int_c^b f = F(b) - F(c).$$

When we add the two integrals the term $F(c)$ drops out.

We have

$$\begin{aligned} \int_a^b f &= F(c) - F(a) + F(b) - F(c) \\ &= F(b) - F(a). \end{aligned}$$

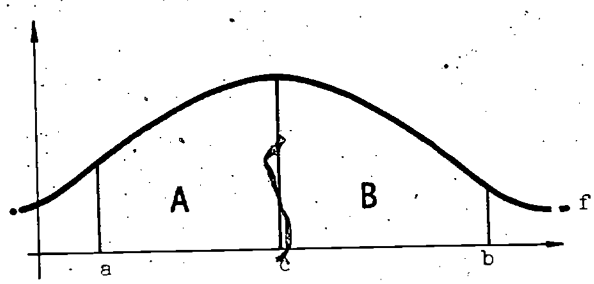


Figure 7-3g

Area of Shaded Region = Area of A + Area of B

**Example 7-3a.** Find $A(x) = \int_0^x f$, where $f : x \to e^x$.

We know that $f' = f$. The Fundamental Theorem then gives:

$$A(x) = \int_0^x f = f(x) - f(0)$$
$$= e^x - e^0$$
$$= e^x - 1.$$

**Example 7-3b.** Find $A(x) = \int_2^x f$, where $f : x \to x^4$.

The derivative of $x \to x^5$ is $x \to 5x^4$ so that the derivative of

$$F : x \to \frac{1}{5} x^5 \text{ is } f : x \to x^4.$$

The Fundamental Theorem gives:

$$A(x) = \int_2^x f = F(x) - F(2) = \frac{1}{5} x^5 - \frac{32}{5}.$$

**Example 7-3c.** Find $\int_{-\pi/2}^{\pi/2} f$, where $f : x \to \cos x$.

The sine function $F : x \to \sin x$ is a function whose derivative is $f$. The interval can be subdivided into two subintervals (namely $-\frac{\pi}{2} \le x \le 0$ and $0 \le x \le \frac{\pi}{2}$) so that $f$ increases on the first subinterval and decreases on the second interval (see Figure 7-3h). We can, therefore, apply the remark following Fundamental Theorem to conclude that

$$\int_{-\pi/2}^{\pi/2} f = F(\tfrac{\pi}{2}) - F(-\tfrac{\pi}{2})$$
$$= \sin \frac{\pi}{2} - \sin(-\frac{\pi}{2})$$
$$= 1 - (-1)$$
$$= 2.$$

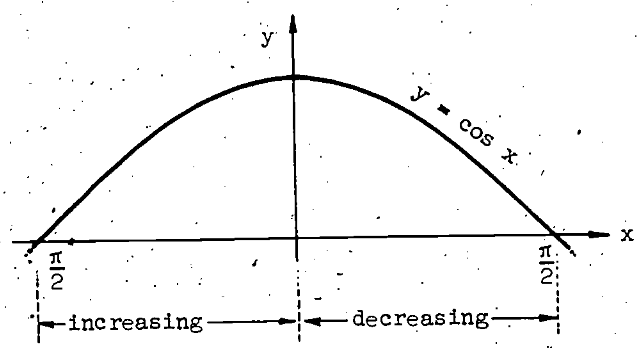


Figure 7-3h

## Exercises 7-3

1. In Section 7-1 we obtained the estimates

$$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2} < A(x) < \frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}$$

for each positive integer $n$, where

$$A(x) = \int_0^x f;\ f : x \to x^2.$$

Average these to obtain the general estimate

$$A(x) \approx \frac{x^3}{3} + \frac{x^3}{6n^2}.$$

Use this estimate for $A(x)$ in order to calculate approximations of the following quantities when $n = 10$.

(a) $A(2)$

(b) $A(2.1)$

(c) $\dfrac{A(2.1) - A(2)}{0.1}$

(d) $\dfrac{A(x + h) - A(x)}{h}$ for general positive $x$, $h$.

(e) Let $h$ approach $0$ in (d) and use this to estimate $A'(x)$.

2. Suppose $f$ is increasing and nonnegative for $a \le x \le b$. Show that

$$f(a)(b - a) \le \int_a^b f \le f(b)(b - a).$$

3. Suppose $f : x \to x^2 + 1$. Find

(a) $\lim\limits_{h \to 0} \displaystyle\int_1^{1+h} f$

(b) $\lim\limits_{h \to 0} \dfrac{1}{h} \displaystyle\int_1^{1+h} f$

(c) Did you need to calculate $\displaystyle\int_1^{1+h} f$ in order to answer (a) and (b)? Explain.

4. Suppose $F(x) = \int_2^x f$, where $f : x \to x^3$

(a) What is $F(2)$?

(b) What is $F'(3)$?

(c) Did you need to find an antiderivative for $f$ in order to answer (a) or (b)?

5. Find the derivative of each of the following functions $F$.

(a) $F(x) = \int_{-1}^x f,\ x \geq -1;\ f : x \to x^4 + x^2$

(b) $F(x) = \int_{-100\pi}^x f,\ x \geq -100\pi;\ f : x \to \sin^3 x$

(c) $F(x) = \int_{5/2}^x f,\ x \geq \frac{5}{2};\ f : x \to e^{\sin x}$

(d) $F(x) = \int_0^x f,\ x \geq 0;\ f : x \to x^{100}$

6. Suppose $f : x \to \frac{1}{x}$ and

$$F(x) = \int_1^x f,\ x \geq 1;\ G(x) = \int_2^x f,\ x \geq 2$$

(a) What is $F(1)$? $G(2)$?

(b) What is $F'(x) - G'(x)$, $x \geq 2$?

(c) If $\alpha = \int_1^2 f$, what is $F(x) - G(x)$ equal to for $x \geq 2$?

7. Find $f'$ and $g'$ when

(a) $f : x \to x^2 - x + 3$

(b) $g : x \to x^2 - x + 18$

(c) What is the relationship between your answers to (a) and (b)? Why?

8. Find two distinct functions $g$ such that $g'$ is the function $x \to 3x^2$. How are your functions related to each other?

9. Find the area bounded by the coordinate axes, the line $x = 2$, and the graph of the function $f$, where

(a) $f : x \to x^2$

(b) $f : x \to 2x + 1$

(c) $f : x \to 4x^3 + x$

10. (a) Sketch the graph of $f : x \to x^2 + 1$.

(b) Mark the region bounded by this graph, the coordinate axes, and the line $x = 1$. Find the area of this region.

(c) Mark the region bounded by your graph, the coordinate axes, and the line $x = 2$. Find the area of this region.

(d) Mark the region bounded by your graph, the x-axis, and the lines $x = 1$ and $x = 2$. How is this region related to the regions you marked in (b) and (c)? Find its area.

11. (a) Sketch the graph and find the area bounded by the graph of $f : x \to 16 - x^2$, the x-axis, and lines $x = 2$ and $x = 3$.

(b) Sketch the graph and find the area bounded by the graph of $f : x \to 4x^3 - x$, the x-axis, and the lines $x = 1$ and $x = 2$.

12. For each of the following functions $f$ find a function $F$ such that $F' = f$. Then use the Fundamental Theorem to evaluate the given integral. (You will need to recall your differentiation formulas in order to construct $F$).

(a) $f : x \to x^6$, $\int_1^3 f$

(b) $f : x \to x^6 + x$, $\int_1^3 f$

(c) $f : x \to \frac{1}{x}$, $\int_4^5 f$

(d) $f : x \to \frac{1}{\sqrt{x}}$, $\int_2^4 f$

(e) $f : x \to e^x$, $\int_{-5}^0 f$

(f) $f : x \to e^{2x}$, $\int_{-5}^0 f$

(g) $f : x \to \sin x$, $\int_0^{\pi/2} f$

(h) $f : x \to \sin 2x$, $\int_0^{\pi/2} f$

13. For $f : x \to (x - 1)^2$ show how the interval $0 \le x \le 3$ can be subdivided so that on each subinterval $f$ is always increasing or always decreasing. Give a sketch.

14. For the function of Number 13,

(a) find $F$ so that $F' = f$

(b) find $\int_0^1 f$, $\int_1^3 f$ and $\int_0^3 f$ by using the Fundamental Theorem.

15. (a) Find two different functions $g$ such that $g' = f$, where $f : x \to 6x^2 + 2$, and for each of them find the value of $g(2) - g(0)$.

(b) What is the area bounded by the coordinate axes, the graph of $f : x \to 6x^2 + 2$, and the line $x = 2$?

16. (a) Find two different functions $g$ such that $g' = f$ where $f : x \to 4x + 3$, and for each of them find the value of $g(2) - g(1)$.

(b) What is the area bounded by the graph of $f : x \to 4x + 3$, the x-axis, and the lines $x = 1$ and $x = 2$?

17. If $g$ and $h$ are two different functions such that $g' = h'$, what is the relation between the number $g(5) - g(3)$ and the number $h(5) - h(3)$?

18. Find a function $F$ such that $f' : x \to x^3 - x^2$ and $F(0) = 0$. How many such functions are there?

19. Find a function $G$ such that $G' : x \to x^3 - x^2$ and $G(0) = 1$. (Hint: How will $G$ be related to the function $F$ of No. 18?)

20. Suppose $f$ is nonnegative and increasing, that $A(x) = \int_a^x f$ and that $a \leq c \leq d$.

(a) Show that $A(\frac{c + d}{2}) \leq A(c) + \frac{d - c}{2} f(\frac{c + d}{2})$

(Hint: Graph the areas and write $A(\frac{c + d}{2}) = A(c) + \int_c^{c+d/2} f$ and use No. 2.)

(b) Show that $A(\frac{c + d}{2}) \leq A(d) - (\frac{d - c}{2}) f(\frac{c + d}{2})$

(c) Deduce from (a) and (b) that $A(\frac{c + d}{2}) \leq \frac{A(c) + A(d)}{2}$.

## 7-4. Calculating Areas

Suppose we wish to find

$$\int_a^b f;$$

that is, the area of the shaded region shown in Figure 7-4a.

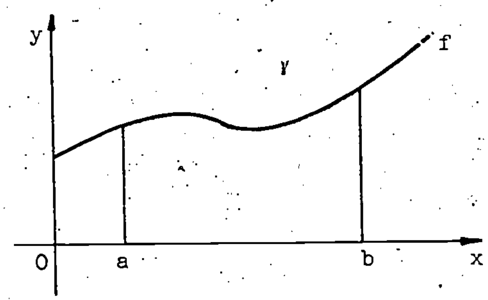


Figure 7-4a

Area of Shaded Region = $\int_a^b f$

The Fundamental Theorem of Calculus gives us a means for doing this. Suppose $F$ is a function whose derivative is $f$. The Fundamental Theorem then tells us that

(1) $$\int_a^b f = F(b) - F(a).$$

(Of course, we are assuming that $f$ satisfies the conditions of the Fundamental Theorem, or the remark following the theorem.)

In this section we shall introduce some further notation which is useful in finding areas and indicate some of the ways we can use (1). A more systematic discussion of the use of (1) will be given in Section 7-6.

It is convenient to have a notation for the integral in terms of the expressions used in defining the function $f$. A common notation for

$$\int_c^b f \text{ is } \int_c^b f(x)dx.$$

The symbol "dx" is a single symbol, meant to indicate that $f$ is to be taken as a function of $x$. For example,

(2) $$\int_a^b x^2\,dx, \text{ means } \int_a^b f, \text{ where } f : x \to x^2.$$

Of course, the letter $x$ used in $f : x \to x^2$ is a "dummy" letter. Any other letter not already in use will do just as well. Thus we could write $f$ as

$$f : t \to t^2 \quad \text{or} \quad f : \psi \to \psi^2.$$

In these cases we would write (2) as

$$\int_a^b t^2\,dt \quad \text{or} \quad \int_a^b \psi^2\,d\psi$$

A function $F$ whose derivative is $f$ is often called an antiderivative (or indefinite integral) of $f$. It is also common to use the notation

$$F(x)\Big|_a^b \quad \text{for} \quad F(b) - F(a).$$

The Fundamental Theorem of Calculus is often stated in the form:

(3) $$\boxed{\begin{array}{l} \displaystyle\int_a^b f(x)\,dx = F(x)\Big|_b^b = F(b) - F(a), \\ \text{where } F \text{ is an antiderivative* of } f. \end{array}}$$

For example, since the derivative of

$$x \to \frac{1}{3}x^3 \quad \text{is} \quad x \to x^2,$$

we say that $x \to \frac{1}{3}x^3$ is an antiderivative of $x \to x^2$ and write

$$\int_a^b x^2\,dx = \frac{1}{3}x^3\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

*$F' = f$

Example 7-4a. Find $\int_1^4 t^5\, dt$

First we find an antiderivative of $t \to t^5$. Differentiation of polynomials reduces degree by one, so antidifferentiation should raise degree by one. If we recall that the function

$$t \to t^6$$

has derivative $t \to 6t^5$, we can see that

$$t \to \frac{1}{6} t^6$$

is the antiderivative of $t \to t^5$. Therefore, we have

$$\int_1^4 t^5\, dt = \frac{1}{6} t^6 \Big|_1^4 = \frac{4^6}{6} - \frac{1^6}{6} = \frac{4095}{6}.$$

Example 7-4b. Find the area of the region between the x-axis and one arch of the sine curve given by $y = \sin x$. We want to find (Figure 7-4b).

$$\int_0^\pi \sin x\, dx.$$

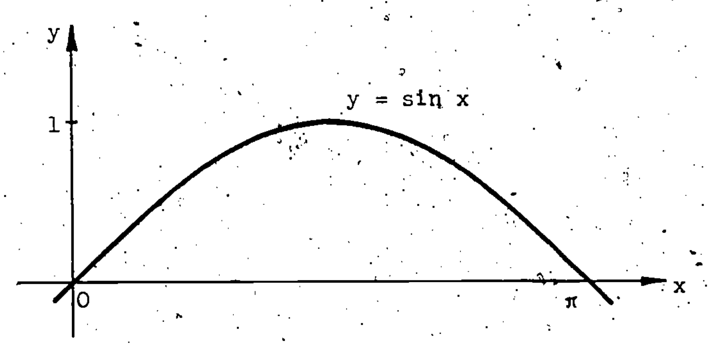


Figure 7-4b

$$\int_0^\pi \sin x\, dx = \text{area of shaded region.}$$

The derivative of the cosine function is the negative of the sine function so that

$$x \to -\cos x$$

is an antiderivative of $x \to \sin x$. We have

$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = -\cos \pi + \cos 0$$

$$= -(-1) + 1 = 2.$$

Example 7-4c. Find $\int_0^3 (x^2 + 2x + 4)dx$.

We could find an antiderivative of

$$x \to x^2 + 2x + 4$$

directly and use (3). An alternative approach (which amounts to the same thing) is to remember that the integral of a sum is the sum of the integrals, so that we can write

$$\int_0^3 (x^2 + 2x + 4)dx = \int_0^3 x^2 dx + \int_0^3 2x\,dx + \int_0^3 4dx.$$

The functions

$$x \to x^2,\ x \to 2x \ \text{and} \ x \to 4$$

have the respective antiderivatives

$$x \to \frac{1}{3}x^3,\ x \to x^2 \ \text{and} \ x \to 4x;$$

so we have

$$\int_0^3 (x^2 + 2x + 4)dx = \frac{1}{3}x^3 \Big|_0^3 + x^2 \Big|_0^3 + 4x \Big|_0^3$$

$$= \frac{1}{3}(3^3 - 0^3) + (3^2 - 0^2) + (4 \cdot 3 - 4 \cdot 0)$$

$$= 30.$$

Example 7-4d. Describe the area of the region between the graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ as the difference of two integrals and evaluate.

The area of region $A$ in Figure 7-4b is

$$\int_0^1 \sqrt[3]{x}\,dx - \int_0^1 \sqrt{x}\,dx.$$

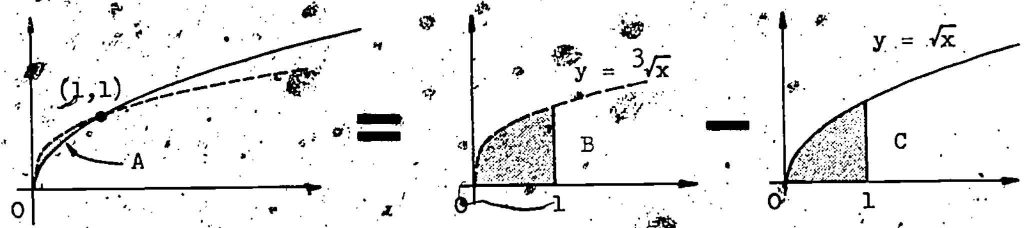


Figure 7-4b

Area of A = Area of B - Area of C

To find antiderivative of $x \to \sqrt[3]{x}$ and $x \to \sqrt{x}$, we first write $\sqrt[3]{x} = x^{1/3}$ and $\sqrt{x} = x^{1/2}$ and then recall the power formula

$$Dx^a = ax^{a-1}.$$

Differentiation amounts to multiplying by the exponent and reducing the exponent by 1. As was the case with our polynomial (Example 7-4c), antidifferentiation amounts to raising the exponent by 1 and dividing by the new exponent. Thus, we have

$$x \to \frac{3}{4}x^{4/3} \quad \text{and} \quad x \to \frac{2}{3}x^{3/2}$$

as respective antiderivatives of $x \to \sqrt[3]{x}$ and $x \to \sqrt{x}$. Therefore, our desired area is

$$\int_0^1 \sqrt[3]{x}\,dx - \int_0^1 \sqrt{x}\,dx = \frac{3}{4}x^{4/3}\Big|_0^1 - \frac{2}{3}x^{3/2}\Big|_0^1$$

$$= \frac{3}{4} - \frac{2}{3}$$

$$= \frac{1}{12}.$$

Example 7-4e. Evaluate $\int_{-\pi/2}^{\pi} |\sin x|\,dx.$

In Figure 7-4c we indicate (by shading) the region whose area is the integral we wish to evaluate.

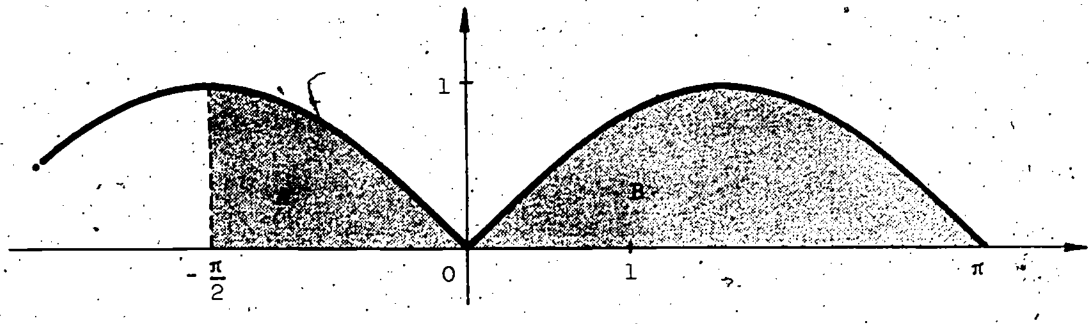


Figure 7-4c

We know that the area of region B is 2 (from Example 7-4b) and we should suspect that the total area of regions A and B is 3. We can confirm this suspicion and gain additional experience using antiderivatives. By definition of absolute value we have

$$x \to |\sin x| = \begin{Bmatrix} \sin x, & \text{for } \sin x \geq 0 \\ -\sin x, & \text{for } \sin x < 0 \end{Bmatrix}.$$

we express our integral as the sum of two integrals:

$$(4) \qquad \int_{-\pi/2}^{\pi} |\sin x|dx = \int_{-\pi/2}^{0} |\sin x|dx + \int_{0}^{\pi} |\sin x|dx$$
$$= \int_{-\pi/2}^{0} (-\sin x)dx + \int_{0}^{\pi} \sin x\, dx.$$

The antiderivatives of

$$x \to -\sin x \quad \text{and} \quad x \to \sin x$$

are, respectively,

$$x \to \cos x \quad \text{and} \quad x \to -\cos x.$$

Therefore, we have

$$\int_{-\pi/2}^{\pi} |\sin x|dx = \cos x \Big|_{-\pi/2}^{0} + (-\cos x) \Big|_{0}^{\pi}$$
$$= \cos 0 - \cos(-\tfrac{\pi}{2}) + (-\cos \pi) - (-\cos 0)$$
$$= 1 - 0 + (-(-1)) - (-1)$$
$$= 3.$$

**Example 7-4f.** Evaluate $\int_0^2 f(x)dx$ if

$$f(x) = \begin{cases} \sqrt{3x} & , \text{ for } 0 \le x \le 1 \\ (2x-1)^2, & \text{ for } 1 < x \le 2 \end{cases}.$$

The area of the shaded region in Figure 7-4d is given by the integral we wish to evaluate. Note the break in the graph of $f$ at $x = 1$. In order to be able to apply the Fundamental Theorem of Calculus, we first break our interval into subintervals over which the graph of $f$ has no gaps:

$$\int_0^2 f(x)dx = \int_0^1 \sqrt{3x}\,dx + \int_1^2 (2x-1)^2 dx.$$

Antiderivatives for $x \to \sqrt{3x}$ and $x \to (2x-1)^2$ are respectively

$$x \to \frac{2\sqrt{3}}{3} x^{3/2} \quad \text{and} \quad x \to (\tfrac{1}{2})(\tfrac{1}{3})(2x-1)^3.$$

(Check by differentiation and see Exercises 7-4, No. 5). We, therefore, have

$$\int_0^2 f(x)dx = \frac{2\sqrt{3}}{3} x^{3/2} \Big|_0^1 + \frac{(2x-1)^3}{6} \Big|_1^2$$

$$= \frac{2\sqrt{3}}{3}(1^{3/2} - 0^{3/2}) + \frac{1}{6}(3^3 - 1^3)$$

$$= \frac{2\sqrt{3} + 13}{3}.$$

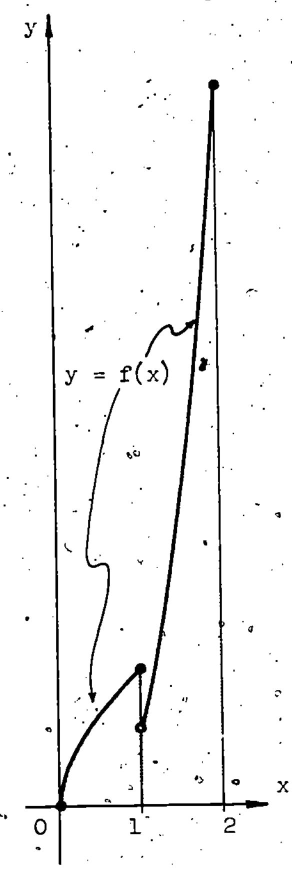

Figure 7-4d



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## Exercises 7-4

1. Find each of the following integrals.

(a) $\int_0^2 (x^2 + x + 3)dx$

(b) $\int_{-2}^0 (x^2 + x + 3)dx$

(c) $\int_{-2}^2 (x^2 + x + 3)dx$

(d) $\int_0^{\pi/3} \cos x \, dx$

(e) $\int_0^2 \sqrt{x^3} \, dx$

(f) $\int_{1/16}^1 (\sqrt{x} + \sqrt[4]{x})dx$

(g) $\int_{1/2}^1 \frac{1}{3x^2} dx$

(h) $\int_{-2}^{-1} (5x^{-6} + x^2)dx$

(i) $\int_1^2 \frac{1}{x} dx$

(j) $\int_1^\pi x^n \, dx$

(k) $\int_{-1}^2 e^x \, dx$

($\ell$) $\int_{-1}^2 (e^x + 1)dx$

(m) $\int_{-1}^2 (e^x + x)dx$

(n) $\int_1^2 (5x^4 + 3x^2 + 1)dx$

(o) $\int_{\pi/6}^{\pi/3} (\sin x + \cos x)dx$

(p) $\int_0^{4\pi/3} (e^x + \sin x)dx$

(q) $\int_3^3 (x^2 + 2x + 5)dx$

(r) $\int_{10}^{10} x^3 e^x \arctan (\sin^2 x)dx$

2. Sketch the regions bounded by the x-axis, the curve $y = f(x)$ and the vertical lines $x = a$ and $x = b$. Then find the areas

(a) $f : x \to x^3 + 2x + 1, \quad a = 1, \quad b = 3$

(b) $f : x \to e^x, \quad a = -1, \quad b = 1$

(c) $f : x \to e^x + x^2, \quad a = -1, \quad b = 1$

(d) $f : x \to \sin x + \cos x, \quad a = 0, \quad b = \frac{\pi}{2}$

(e) $f : x \to 2x^4 + \cos x, \quad a = -\frac{\pi}{2}, \quad b = \frac{\pi}{4}$

(f) $f : x \to x^{-10}, \quad a = -1, \quad b = -\frac{1}{2}$

(g) $f : x \to \sqrt[3]{x^2}, \quad a = -1, \quad b = 1$

3. Sketch the region bounded by the x-axis, $y = f(x)$ and the given vertical lines; then find its area.

(a) $f : x \to |x|$; vertical lines $x = -2$, $x = 4$
(Check your result by elementary geometry.)

(b) $f : x \to |4x^3|$; vertical lines at $x = -1$, $x = 3$

(c) $f : x \mapsto |\cos x|$; vertical lines at $x = -\frac{\pi}{3}$, $x = \frac{4\pi}{3}$

(d) $f : x \to |\frac{1}{2} - \sin x|$; vertical lines at $x = -\pi$, $x = 2\pi$

(e) $f : x \to |1 - \sqrt{x}|$; vertical lines at $x = 0$, $x = 4$

4. (a) Evaluate $(x^2 + 3\sqrt{x})\Big|_1^4$ and $(x^2 + 3\sqrt{x} + 50)\Big|_1^4$

(b) Suppose $F(x) = G(x) + \pi\log_e(\arctan 7)$ where $F(0) = 1$, $F(1) = -1$.
Find $G(x)\Big|_0^1$.

(c) What is $F(x)\Big|_a^b - G(x)\Big|_a^b$ if $F' = G'$?

5. (a) Find an antiderivative for each of the following functions.

(i) $f : x \to (x - 1)^3$

(ii) $F : x \to x^3 - 3x^2 + 3x - 1$

(iii) $g : x \to 8x^3 - 12x^2 + 6x - 1$

(iv) $G : x \to (2x - 1)^3$
[Hint: Try to put $G$ in the form $a(x - b)^n$.]

(b) Compare the functions $F$ with $f$ and $G$ with $g$. Compare the antiderivatives.

6. Find an antiderivative for each of the following functions

$$f : x \to 8(x + 1)^3$$

$$g : x \to (2x + 2)^3.$$

7. Find $\int_0^1 (3x + 4)^5\,dx$

(a) by first carrying out the indicated multiplication,

(b) by using the method found in Number 6.

8. Which of the following integrals are the same as $\int_a^b t^3\, dt$?

(a) $\int_a^b y^3\, dy$ (c) $\int_a^b A^3\, dA$

(b) $\int_a^b y^3\, dt$ (d) $\int_{a+1}^{b+1} (t - 1)^3\, dt$

9. Evaluate the following integrals using a line of symmetry appropriate to the problem. [e.g., $\int_{-3}^{3} x^2\, dx = 2\int_0^3 x^2\, dx = \frac{2}{3} x^3 \Big|_0^3 = 18.$]

(a) $\int_{-\pi/6}^{\pi/6} \cos x\, dx$

(b) $\int_{-2}^{2} (1 + 6x^2) dx$

(c) $\int_0^2 (x - 1)^2\, dx$

(d) $\int_0^{\pi} \sin x\, dx$

10. Find the area of the region bounded by the x-axis, the given curves $y = f(x)$, and the given vertical lines. (Sketch first.)

(a)
$$f : x \to \begin{cases} -x^3 , & x \le 0 \\ -x^2 + 2, & 0 < x \le \sqrt{2} \\ x , & \sqrt{2} < x \end{cases}$$
vertical lines at $x = -1$ and $x = 4$

(b)
$$f : x \to \begin{cases} |2x - 3| & \text{if } 0 \le x \le 3 \\ \frac{4}{3}\left(x - \frac{3}{2}\right)^2 & \text{if } x \le 0 \\ & \text{or } x \ge 3 \end{cases}$$
vertical lines at $x = -\frac{3}{2}$ and at $x = \frac{9}{2}$

In Problems 11-12 deduce part (b) from the solutions to part (a). (Sketch each first.)

11. (a) (i) Find $\int_0^1 (8 - x^2) dx$ (ii) $\int_0^1 x^2\, dx$

(b) Find the area of the region bounded above by $y = 8 - x^2$, below by $y = x^2$, to the left by the vertical line $x = -1$, and to the right by the vertical line $x = 1$.

12. (a) (i) Find $\int_0^2 (8 - x^2)dx$; (ii) $\int_0^2 x^2 \, dx$

(b) Find the area of the region bounded by $y = 8 - x^2$ and $y = x^2$.

13. (a) Find the solution of Number 11(b) directly without using part (a) of Number 11.

(b) Find the solution of Number 12(b) directly without using part (a) of Number 12.

14. Find the area bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$. (Sketch first.)

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## 7-5. Signed Area

Until now we have discussed the integral $\int_a^b f$ or $\int_a^b f(x)dx$ only in cases for which $a \leq b$ and the interval from $a$ to $b$ could be subdivided so that in each subinterval the function $f$ was nonnegative, always increasing (or always decreasing) and its graph had no gaps. We now extend our discussion to include situations for which $a > b$ or for which the graph of $f$ may contain portions below the x-axis, preserving, if possible, the result

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{if} \quad F' = f.$$

This can be accomplished by suitably interpreting $\int_a^b f(x)dx$ as <u>signed area</u>.

First consider the case for which $f$ is nonpositive on the interval $a \leq x \leq b$, and $F' = f$. In this case $-f$ is nonnegative and has antiderivative $-F$, so that

$$(1) \qquad \int_a^b -f(x)dx = -F(x)\Big|_a^b = -F(b) + F(a).$$

This can be interpreted as the area of the shaded region of Figure 7-5b. Note that this is the same as the area of the shaded region of Figure 7-5a.

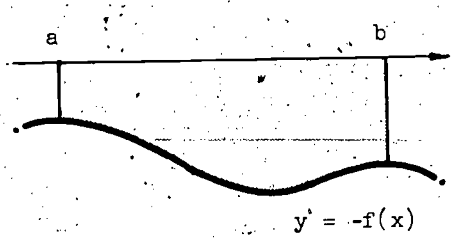

Figure 7-5a

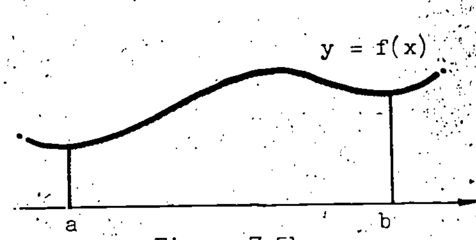

Figure 7-5b

If the Fundamental Theorem is to hold we should have

$$\int_a^b f(x)dx = F(b) - F(a).$$

Referring to (1), we see that this requires that

$$\int_a^b f(x)dx = -\int_a^b [-f(x)]dx;$$

that is $\int_a^b f(x)\,dx$ must be defined as the negative of the area of the shaded region of Figure 7-5a.

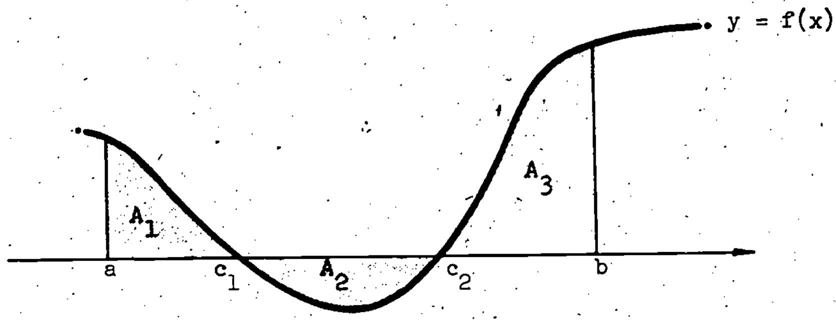


Figure 7-5c

Now suppose the graph of $f$ looks like that shown in Figure 7-5c and that $F$ is an antiderivative of $f$. We have

$$\text{area of } A_1 = \int_a^{c_1} f(x)\,dx = F(c_1) - F(a)$$

$$\text{area of } A_2 = \int_{c_1}^{c_2} -f(x)\,dx = F(c_1) - F(c_2)$$

$$\text{area of } A_3 = \int_{c_2}^{b} f(x)\,dx = F(b) - F(c_2).$$

Now note that

$$\begin{aligned} F(b) - F(a) &= F(b) - F(c_1) + F(c_1) - F(c_2) + F(c_2) - F(a) \\ &= [F(c_1) - F(a)] - [F(c_1) - F(c_2)] + [F(b) - F(c_2)] \\ &= (\text{area of } A_1) - (\text{area } \text{f } A_2) + (\text{area } A_3). \end{aligned}$$

In other words, if we wish

$$\int_a^b f \quad \text{to be} \quad F(b) - F(a)$$

then we must have

$$\int_a^b f = (\text{area of } A_1) - (\text{area of } A_2) + (\text{area of } A_3).$$

In summary, if $a \leq b$, $F' = f$ and if we define $\int_a^b f$ by

(2) $$\int_a^b f(x)dx = F(b) - F(a),$$

then $\int_a^b f$ will be the total area of the regions bounded by the graph of $f$ which lie above the interval minus the total area of the regions bounded by the graph of $f$ which lie below the interval. This is called the signed area determined by $f$ on the interval from $a$ to $b$.

It is also common practice to remove the restriction that $a \leq b$, by defining

$$\int_a^b f = -\int_b^a f \quad \text{if} \quad b < a.$$

The fundamental relation (2) will still hold, for if $b < a$ and $F' = f$ then

$$\int_a^b f = -\int_b^a f = -[F(a) - F(b)]$$
$$= F(b) - F(a).$$

The properties of the symbol $\int_a^b f$ discussed in Section 7-2 also hold for signed area:

(3) $$\int_a^b (f + g) = \int_a^b f + \int_a^b g;$$

(4) $$\int_a^b (\alpha f) = \alpha \int_a^b f, \quad \text{where } \alpha \text{ is any real number;}$$

(5) $$\int_a^b f = \int_a^c f + \int_c^b f, \quad \text{where } a, b, c \text{ are any real numbers.}$$

Notice, in fact, that (4) now holds without the restriction that $\alpha$ be non-negative and (5) doesn't require that $a \leq c \leq b$.

Of course, if $a \leq b$ and $f(x) \geq 0$ for $a \leq x \leq b$ then

$$\int_a^b f(x)dx \geq 0.$$

One consequence of this is the fact that

(6) $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ if $a \leq x \leq b$ and $f(x) \leq g(x)$.

For we then have $g(x) - f(x) \geq 0$, so that

$$\int_a^b (g(x) - f(x))dx \geq 0.$$

Adding $\int_a^b f(x)dx$ to both sides, we obtain (6).

Example 7-5a. Find $\int_{-\pi}^{\pi} \sin x \, dx$.

This integral can be interpreted as the signed area of the total shaded region shown in Figure 7-5d. Since the regions above and below the x-axis are

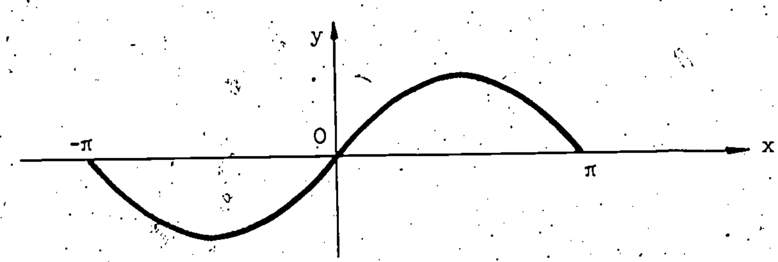


Figure 7-5d

$y = \sin x$

the same, we should expect that the signed area is 0. The defining relation (2) should corroborate our expectation. In this case

$$F : x \to -\cos x$$

is an antiderivative of $x \to \sin x$, so (2) gives

$$\int_{-\pi}^{\pi} \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi} = (-\cos \pi) - (-\cos(-\pi))$$

$$= (-(-1)) - (-(-1)) = 0.$$

<u>Example 7-5b</u>. Sketch the graph of $f : x \to 1 - x^2$ for $-2 \leq x \leq 3$.

Find $A = \int_{-2}^{-1} [-f(x)]dx$, $B = \int_{-1}^{1} f(x)dx$, and $C = \int_{1}^{3} [-f(x)]dx$.

Use the fundamental relation (2) to show that

$$\int_{-2}^{3} (1 - x^2)dx = -A + B - C.$$

The desired graph is shown in Figure 7-5e.

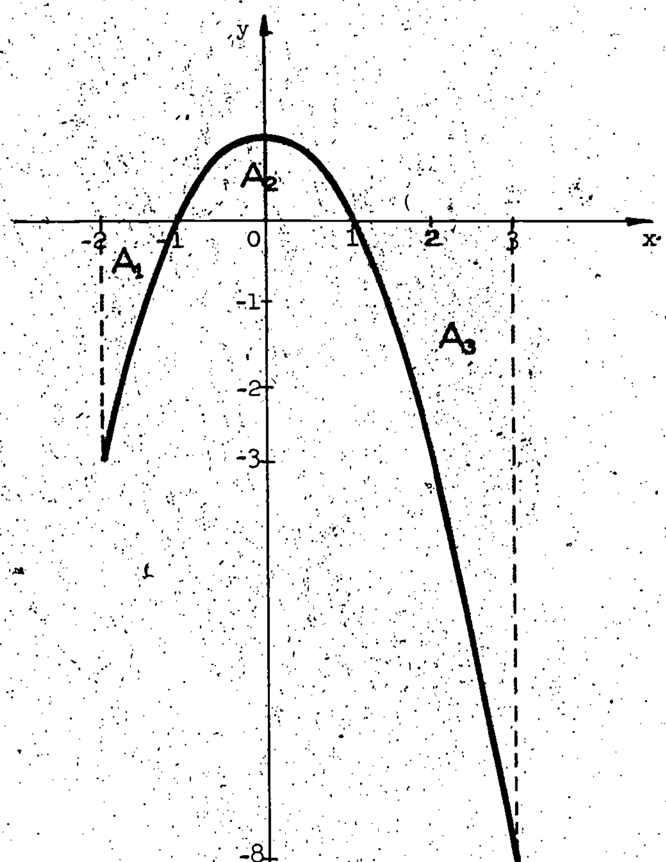


Figure 7-5e

$y = 1 - x^2$



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The function $F : x \to x - \frac{1}{3}x^3$ is an antiderivative for $f$ (as easily checked by showing that $F' = f$). We have

$$\int_{-2}^{-1} [-f(x)]dx = \int_{-2}^{-1} (x^2 - 1)dx = \frac{x^3}{3} - x \Big|_{-2}^{-1} = \frac{4}{3} = \text{area of } A_1;$$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (1 - x^2)dx = x - \frac{x^3}{3} \Big|_{-1}^{1} = \frac{4}{3} = \text{area of } A_2;$$

$$\int_{1}^{3} [-f(x)]dx = \int_{1}^{3} (x^2 - 1)dx = \frac{x^3}{3} - x \Big|_{1}^{3} = \frac{20}{3} = \text{area of } A_3;$$

The fundamental relation (2) gives

$$\int_{-2}^{3} f(x)dx = F(3) - F(-2) = x - \frac{x^3}{3} \Big|_{-2}^{3} = -\frac{20}{3},$$

which is the same as

$$-(\text{area of } A_1) + (\text{area of } A_2) - (\text{area of } A_3) = -\frac{4}{3} + \frac{4}{3} - \frac{20}{3} = -\frac{20}{3}.$$

<u>Example 7-5c</u>. Find $\int_{1}^{0} x^2\,dx$ and $\int_{0}^{1} x^2\,dx$.

We have

$$\int_{1}^{0} x^2\,dx = -\int_{0}^{1} x^2\,dx = -\frac{x^3}{3}\Big|_{0}^{1} = -\frac{1}{3}.$$

<u>Example 7-5d</u>. Find the area of the region enclosed by the graphs of the two functions

$$f : x \to x^2 - 6x + 7 \quad \text{and} \quad g : x \to -x^2 + 7x - 11.$$

(This is the same problem as Example 7-2c.) A sketch of the region whose area is sought is given in Figure 7-5f.

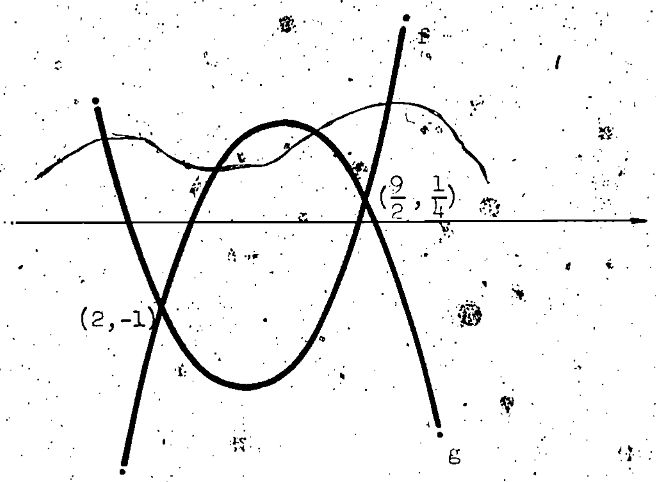


Figure 7-5f

We shall show that the desired area is given by

$$\int_{2}^{9/2} (g(x) - f(x))dx.$$

First we note that

$$(7) \qquad \int_{2}^{9/2} g(x)dx = -(\text{area of } A_1) + (\text{area of } A_2) + (\text{area of } A_3),$$

where $A_1$, $A_2$ and $A_3$ are the regions indicated in Figure 7-5g.

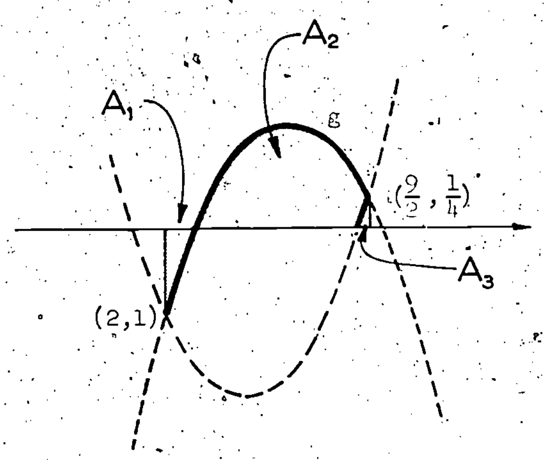


Figure 7-5g

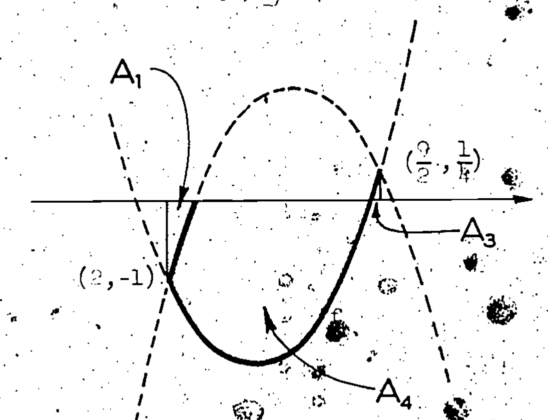


Figure 7-5h

Then we observe that

$$(8) \qquad \int_2^{9/2} f(x)\,dx = -(\text{area of } A_1) - (\text{area of } A_4) + (\text{area of } A_3),$$

where region $A_4$ is indicated in Figure 7-[illegible].

Subtracting (8) from (7), we obtain

$$\int_2^{9/2} g(x)\,dx - \int_2^{9/2} f(x)\,dx = (\text{area of } A_2) + (\text{area of } A_4),$$

which is the area we seek. Since

$$\int_2^{9/2} g(x)\,dx - \int_2^{9/2} f(x)\,dx = \int_2^{9/2} (g(x) - f(x))\,dx,$$

we establish that $\int_2^{9/2} (g(x) - f(x))\,dx$ determines the area of the region between the graphs of $g$ and $f$. A simple calculation now gives

$$\int_2^{9/2} (g(x) - f(x))\,dx = \int_2^{9/2} (-2x^2 + 13x - 18)\,dx$$

$$= -\frac{2}{3}x^3 + \frac{13}{2}x^2 - 18x \Big|_2^{9/2} = \frac{125}{24},$$

the same result as that of Example 7-2c.



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## Exercises 7-5

1. (a) Sketch the graph of the function

$$f : x \to x^2 - 1, \quad -1 < x < 2.$$

(b) Evaluate $\int_0^2 (x^2 - 1)dx$.

(c) Find the area of the region bounded by the x-axis and the graph of the function, $x \to x^2 - 1$, between the vertical lines at $x = 0$ and $x = 2$.

2. (a) Sketch the graph of the function

$$f : x \to x^2 - 1, \quad |x| \le 1.$$

(b) Evaluate $\int_{-1}^{1} x^3\, dx$.

(c) Find the area of the region between the graph of the function, $x \to x^3$, the x-axis, where $|x| \le 1$.

(d) Find $b$ $(b > 0)$, if $\int_0^b x^3\, dx = \frac{1}{2}\int_0^2 x^3\, dx$. Sketch.

3. (a) Evaluate $\int_{-1}^{1} x\, dx$.

(b) Evaluate $\int_{-1}^{1} |x|dx$.

(c) Sketch and then find the area bounded by the x-axis, $|x| = 1$ and $y = x$.

(d) Sketch and then find the area bounded by the x-axis, $|x| = 1$ and $y = |x|$.

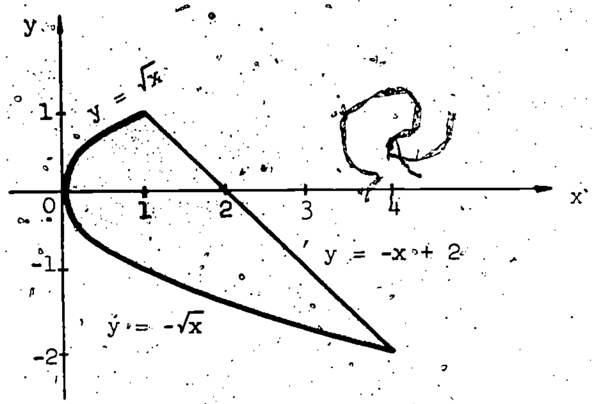
4. Sketch and then find the area of the region bounded by the coordinate axes and the curve

$$\sqrt{x} + \sqrt{y} = 1.$$

Can you identify the curve?

5. Sketch and then find the area of the region bounded by $x = 4$ and $2x = y^2$.

6. Sketch and then find the area of the region bounded by $y = x^3$, $y = -2x^2$ between the vertical lines $x = 0$ and $x = 1$.

7. 


Find the area of the region bounded by $y^2 = x$ and $x + y = 2$, indicated in the figure above.

(a) For the first method divide the required region into smaller regions which can be evaluated as follows:

$$A = \int_0^1 \sqrt{x}\,dx + \int_0^1 -(-\sqrt{x})\,dx + \int_1^2 (-x+2)\,dx + \left[\int_1^4 [-(-\sqrt{x})]\,dx - \int_2^4 [-(-x+2)]\,dx\right]$$

$$A = A_I + A_{II} + A_{III} + [\,A_{IV} - A_V\,]$$

Identify this smaller region with their respective integrals.

(b) Second, try dividing the required region into different smaller regions which are evaluated as follows:

$$A = \int_0^1 \sqrt{x}\,dx + \int_1^2 (-x+2)\,dx + \left[\int_0^4 [-(-\sqrt{x})]\,dx - \int_2^4 [-(-x+2)]\,dx\right]$$

$$A = A_X + A_Y + [\,A_Z - A_W\,]$$

Identify the smaller regions with their respective integrals.

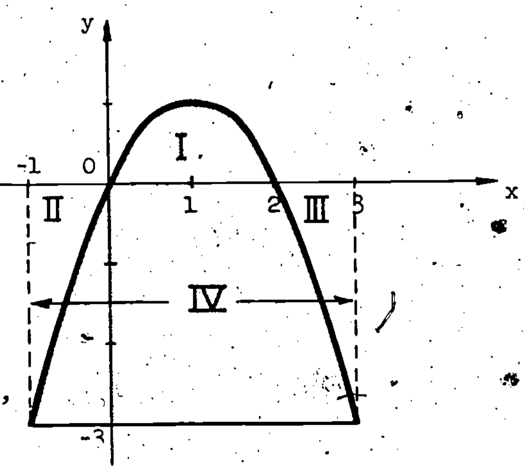
(c) Show that the expressions of area in part (a) and part (b) may be simplified to the following statement.

$$A = 2\int_0^1 \sqrt{x}\,dx + \int_1^4 [(-x+2) + \sqrt{x}]\,dx$$

Can you point out the relationship of this expression for the area and the figure representing the area? Could you have arrived at this expression without going through the smaller sub-regions of parts (a) and (b)?

(d) From the expression for the area in part (c) find the area of the region indicated in the figure.

8\. (a) Express an integral representing the area of each of the following regions: (DO NOT EVALUATE.)




(i) Region I: bounded by the x-axis and $y = 2x - x^2$.

(ii) Region II: bounded by $y = 0$, $x = -1$, and $y = 2x - x^2$.

(iii) Region III: bounded by $y = 0$, $x = 3$, and $y = 2x - x^2$.

(iv) Region IV: bounded by $y = 0$, $y = -3$, $x = -1$, and $x = 3$.

(b) Combine the integrals of part (a) and show that the area of the region bounded by $y = 2x - x^2$, and $y = -3$ can be expressed by the integral,

$$A = \int_{-1}^{3} (2x - x^2 + 3)\,dx.$$

(c) Find the area of the region described in part (b).

9\. (a) Find the area bounded only by the graphs of the functions

$$\begin{cases} f : x \to \cos x \\ f : x \to -\sin x \end{cases}$$

if $x$ is restricted to the closed interval $-\pi \leq x \leq \pi$. Sketch the curves in this interval.

(b) (i) Evaluate $\displaystyle\int_{-\pi/4}^{3\pi/4} \cos x\,dx.$

(ii) Evaluate $\displaystyle\int_{-\pi/4}^{3\pi/4} (-\sin x)\,dx$

(iii) Evaluate $\displaystyle\int_{-\pi/4}^{3\pi/4} (\cos x - \sin x)\,dx$

(iv) Interpret parts (i), (ii), and (iii) geometrically.

10. (a) Use a geometric argument to find

$$\int_{-a}^{a} f \quad \text{if } f \text{ is an } \underline{\text{odd}} \text{ function (i.e., } f(-x) = -f(x)).$$

(b) Show that $\int_{-a}^{a} f = 2\int_{0}^{a} f$ if $f$ is an even function (i.e., $f(-x) = f(x)$).

(c) Evaluate $\int_{-5}^{5} (x^3 - 3x)\sin x^2 \, dx$.

11. Show that if $F' = f$, $G' = g$, and $f(x) \leq g(x)$ for $a \leq x \leq b$ then

$$F(b) - F(a) \leq G(b) - G(a).$$

12. Verify (5). (Hint: $\int_{a}^{b} f = F(b) - F(a)$.)

13. Suppose $F(x) = \int_{x}^{1} f$ where $f : x \to e^x$.

(a) What is $F(1)$?

(b) Find an expression for $F(x)$.

(c) Use part (b) to find $F'(x)$.

(d) In general, suppose $G(x) = \int_{x}^{b} g$. Can you find $G'(x)$?

14. (a) Find the area bounded by the x-axis and the curve $y = x^2 - x^3$. Sketch.

(b) Find the area bounded by the y-axis and the curve $x = y^2 - y^3$. Sketch. (Hint: Note analogy to part (a).)

## 7-6. Integration Formulas

We have seen that the integral $\int_a^b f(x)dx$ can be evaluated, if we can find a function $F$ such that $F' = f$, for then we have

$$\int_a^b f(x)dx = F(b) - F(a).$$

In general we find antiderivatives by one or a combination of methods. A method may consist of recalling a differentiation formula, judicious guessing, or using tables of antiderivatives. In this section we review some of the basic formulas used previously, give some additional formulas and discuss the use of tables. Techniques for extending the scope of our formulas will be discussed in Chapter 9, where we also discuss methods for obtaining approximate values for integrals. Other integration methods are discussed in the appendices.

The common notation for an antiderivative of $f$ is

$$\int f(x)dx,$$

which is also called the <u>indefinite integral</u> of $f$. This symbol is quite similar to

$$\int_a^b f(x)dx,$$

the integral of $f$ from $a$ to $b$. The symbol

$$\int f(x)dx$$

<u>defines a function</u>, namely, a function whose derivative is $f$. The second symbol

$$\int_a^b f(x)dx$$

represents a <u>number</u>, which can be interpreted as the signed area determined by $f$ between $a$ and $b$.

Integration formulas are obtained by reversing the differentiation process, for

$$\int f(x)dx = F(x) \quad \text{means that} \quad DF(x) = f(x).$$

For example,

$$\int x^2\,dx = \frac{x^3}{3} \quad \text{since} \quad D\,\frac{x^3}{3} = x^2.$$

Of course, if $C$ is any constant, we have

$$D\left(\frac{x^3}{3} + C\right) = x^2;$$

more precisely we have

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

In fact, we know from the Constant Difference Theorem (Theorem 7-3b) that all antiderivatives of $x \to x^2$ have the form

$$x \to \frac{x^3}{3} + C, \quad \text{where } C \text{ is a constant.}$$

In some books this fact is stressed by writing

$$\int f(x)dx = F(x) + C,$$

where $C$ is a constant and $DF(x) = f(x)$. For convenience we follow the simple practice of ignoring this constant $C$ in our formulas, each integration formula giving only one function whose derivative is $f$, others obtained by adding constants to our antiderivatives.

<u>The Power Formula</u>

Recall that if $a$ is any real number then

$$Dx^a = ax^{a-1}.$$

If $a \neq 0$, we can write

$$D\left(\frac{1}{a}\,x^a\right) = x^{a-1},$$

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so that $x \to \frac{1}{a} x^a$ is a function whose derivative is $x \to x^{a-1}$. This tells us that

$$\int x^{a-1}\, dx = \frac{1}{a} x^a, \quad \text{if} \quad a \neq 0.$$

For convenience we replace $a$ by $p + 1$, where $p$ is any real number except $p \neq -1$, to obtain the formula

$$\int x^p\, dx = \frac{x^{p+1}}{p + 1}, \quad p \neq -1.$$

In other words, an antiderivative of a power function $x \to x^p$, $p \neq -1$, is obtained by raising the exponent by 1 and dividing by the new exponent.

Suppose $p = -1$, then our function is $x \to \frac{1}{x}$. In Section 6-6 we obtained the formula

$$D \log_e x = \frac{1}{x}, \quad x > 0.$$

This gives the integration formula

$$\int \frac{1}{x}\, dx = \log_e x, \quad x > 0.$$

<u>Circular and Exponential Functions</u>

From the formulas

$$D \sin x = \cos x; \quad D \cos x = -\sin x,$$

we obtain the integration formulas

$$\int \cos x\, dx = \sin x; \quad \int \sin x\, dx = -\cos x.$$

Since $De^x = e^x$, we have the formula

$$\int e^x\, dx = e^x.$$

It is a simple matter to extend these formulas to the case when $x$ is replaced by the linear expression $cx + d$. For example, we know that

$$D \sin (cx + d) = c \cos (cx + d)$$

so that

$$\int c \cos (cx + d) dx = \sin (cx + d).$$

If $c \neq 0$, we can write

$$\int \cos (cx + d) dx = \frac{1}{c} \sin (cx + d).$$

Analogous differentiation formulas were discussed in Volume One for polynomial, exponential and logarithmic functions. In Chapter 9 we shall discuss the formulas resulting from nonlinear substitutions. Here we state the general result for linear replacements:

$$\boxed{\begin{array}{ll} \text{if} & \int f(x)dx = F(x) \text{ and } c \neq 0, \\ \text{then} & \int f(cx + d)dx = \frac{1}{c} F(cx + d). \end{array}}$$

For easy reference we summarize current results in Table 7-6.

| Table 7-6<br>Some Integration Formulas |
|---|
| (1) $\int x^a \, dx = \frac{x^{a+1}}{a + 1}, \quad a \neq -1$ |
| (2) $\int \frac{1}{x} \, dx = \log_e x$ |
| (3) $\int \cos x \, dx = \sin x$ |
| (4) $\int \sin x \, dx = -\cos x$ |
| (5) $\int e^x \, dx = e^x$ |
| (6) $\int f(ax + d)dx = \frac{1}{c} F(cx + d)$ |



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Example 7-6a. Find $\int_1^{3/2} \frac{1}{x^2}\,dx$.

The power formula (1), with $a = -2$, gives

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} = -\frac{1}{x};$$

so that

$$\int_1^{3/2} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{3/2} = \left(-\frac{1}{\frac{3}{2}}\right) - \left(-\frac{1}{1}\right) = \frac{1}{3}.$$

Example 7-6b. Find $\int_2^4 \sqrt{x}\,dx$.

The power formula (1), with $a = \frac{1}{2}$, gives

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2x^{3/2}}{3};$$

so that

$$\int \sqrt{x}\,dx = \frac{2x^{3/2}}{3}\Big|_2^4 = \frac{2}{3}(4^{3/2} - 2^{3/2})$$

$$= \frac{16 - 4\sqrt{2}}{3}.$$

Example 7-6c. Find

$$\int_0^{\pi} (\sin x - 3\cos 2x)dx.$$

We have, from (4) and (3),

$$\int \sin x\,dx = -\cos x \quad \text{and} \quad \int \cos x\,dx = \sin x.$$

Replacing $x$ by $2x$ in the latter and using (6), we have

$$\int \cos 2x\,dx = \frac{1}{2}\sin 2x.$$

Therefore, we conclude

$$\int_0^{\pi} (\sin x - 3 \cos 2x)dx = \int_0^{\pi} \sin x \, dx - 3 \int_0^{\pi} \cos 2x \, dx$$

$$= -\cos x \Big|_0^{\pi} - \frac{3}{2} \sin 2x \Big|_0^{\pi}$$

$$= -[\cos \pi - \cos 0] - [\frac{3}{2} \sin 2\pi - \frac{3}{2} \sin 0]$$

$$= -[-1 - 1] - [\frac{3}{2} \cdot 0 - \frac{3}{2} \cdot 0] = 2.$$

Example 7-6d. Find $\int_{-10}^{-1} 2 \, e^x \, dx$.

We use (5) to obtain

$$\int_{-10}^{-1} 2 \, e^x \, dx = 2 \int_{-10}^{-1} e^x \, dx = 2 \, e^x \Big|_{-10}^{-1}$$

$$= 2e^{-1} - 2e^{-10}.$$

Example 7-6e. Find $\int_0^1 2^x \, dx$.

We first convert to base e:

$$2^x = e^{cx}, \quad \text{where} \quad c = \log_e 2.$$

Now we use (5) to obtain

$$\int e^x \, dx = e^x.$$

We replace $x$ by $cx$, so that (6) gives

$$\int e^{cx} \, dx = \frac{1}{c} e^{cx},$$

where $c = \log_e 2$. Converting to base 2, we have

$$\int 2^x \, dx = (\frac{1}{\log_e 2}) \, 2^x;$$

so that

$$\int_0^1 2^x\,dx = \left(\frac{1}{\log_e 2}\right)2^x \Big|_0^1 = \frac{2^1 - 2^0}{\log_e 2} = \frac{1}{\log_e 2}.$$

Example 7-6f. Find $\int_{-1}^{0} (x + 1)^3 dx$.

We can evaluate this integral in two ways. First we expand to obtain

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1,$$

so that

$$\int_{-1}^{0} (x + 1)^3 dx = \int_{-1}^{0} (x^3 + 3x^2 + 3x + 1)dx.$$

We apply the power formula (1) to each term to obtain

$$\int_{1}^{0} (x + 1)^3 dx = \left(\frac{x^4}{4} + x^3 + \frac{3x^2}{2} + x\right)\Big|_{-1}^{0} = [0] - [\tfrac{1}{4} - 1 + \tfrac{3}{2} - 1] = \tfrac{1}{4}.$$

Alternatively we can recognize that the power formula (1) gives

$$\int x^3\,dx = \frac{1}{4}x^4,$$

and the linear substitution formula (6) gives

$$\int (x + 1)^3 dx = \frac{1}{4}(x + 1)^4.$$

Therefore, we conclude that

$$\begin{aligned}\int_{-1}^{0} (x + 1)^3\,dx &= \frac{1}{4}(x + 1)^4 \Big|_{-1}^{0} \\ &= \frac{1}{4}(0 + 1)^4 - \frac{1}{4}(-1 + 1)^4 \\ &= \frac{1}{4}.\end{aligned}$$

The second method is certainly quicker.

Example 7-6g. Find $\int_0^1 \sin^2 \pi x \, dx$.

We have not yet obtained a differentiation formula which results in the square of the sine function. We use the fact that

$$\sin^2 \pi x = \frac{1 - \cos 2\pi x}{2}.$$

Thus, we have

$$\int_0^1 \sin^2 \pi x \, dx = \int_0^1 \left(\frac{1}{2} - \frac{\cos 2\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^1 1 \, dx - \frac{1}{2}\int_0^1 \cos 2\pi x \, dx.$$

To evaluate this second integral, we combine the cosine formula (3) with the linear substitution result (6) to obtain

$$\int \cos(2\pi x) dx = \frac{1}{2\pi} \sin(2\pi x).$$

We can write

$$\int_0^1 \cos 2\pi x \, dx = \frac{1}{2\pi} \sin 2\pi x \Big|_0^1$$

$$= \frac{1}{2\pi}(\sin 2\pi - \sin 0) = 0.$$

Since the second integral is 0, we conclude that

$$\int_0^1 \sin^2 \pi x \, dx = \frac{1}{2}\int_0^1 1 \, dx - 0 = \frac{1}{2} x \Big|_0^1 = \frac{1}{2}.$$

Example 7-6h. Show that the area of the shaded region of Figure 7-6a is twice that of the shaded region of Figure 7-6b.

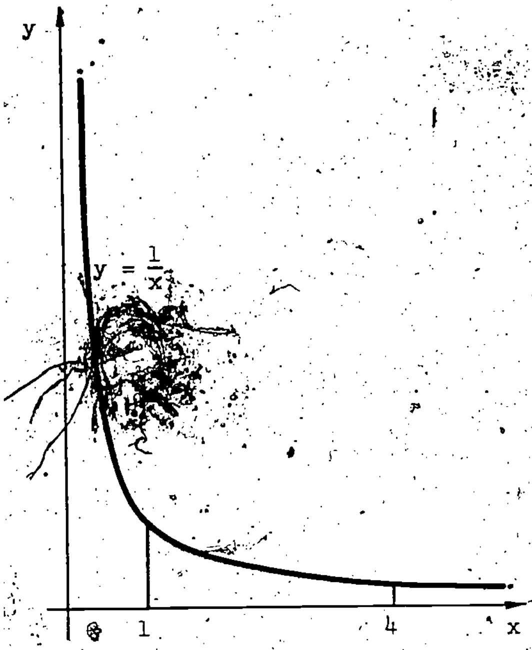



Figure 7-6a

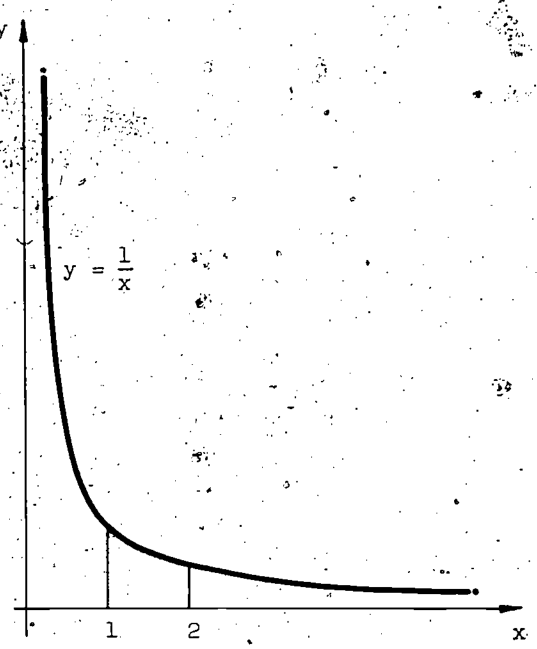



Figure 7-6b

Let $\alpha = \int_1^4 \frac{1}{x}\,dx$ and $\beta = \int_1^2 \frac{1}{x}\,dx$. We wish to show that $\frac{\alpha}{\beta} = 2$.

Formula (2) gives

$$\int \frac{1}{x}\,dx = \log_e x,$$

so that

$$\alpha = \int_1^4 \frac{1}{x}\,dx = \log_e x \Big|_1^4 = \log_e 4 - \log_e 1 = \log_e 4;$$

$$\beta = \int_1^2 \frac{1}{x}\,dx = \log_e x \Big|_1^2 = \log_e 2 - \log_e 1 = \log_e 2.$$

Thus, we conclude that

$$\frac{\alpha}{\beta} = \frac{\log_e 4}{\log_e 2} = \frac{\log_e 2^2}{\log_e 2} = \frac{2\,\cancel{\log_e 2}}{\cancel{\log_e 2}} = 2.$$

The Use of Tables

A longer table of integrals is given in a separate booklet (Table 7). As more differentiation methods are developed, we shall see how to construct these tables. The following examples make use of these tables.

Example 7-6i. Find $\int_0^1 xe^x\,dx$.

Formula 16 of the tables gives

$$\int xe^x\,dx = xe^x - e^x,$$

so that

$$\int_0^1 xe^x\,dx = (xe^x - e^x)\Big|_0^1$$

$$= (1e^1 - e^1) - (0e^0 - e^0)$$

$$= 1.$$

Example 7-6j. Find $\int_0^1 xe^{3x}\,dx$.

Formula 16 of the tables gives $\int xe^x\,dx = xe^x - e^x$. We replace $x$ by $3x$ and use (6) to obtain

$$\int 3xe^{3x}\,dx = \frac{1}{3}(3xe^{3x} - e^{3x});$$

so that

$$\int_0^1 x\,e^{3x}\,dx = \frac{1}{3}(3xe^{3x} - e^{3x})\Big|_0^1 = \frac{1}{3}(3e^3 - e^3) - \frac{1}{3}(0e^0 - e^0) = \frac{2}{3}e^3 + \frac{1}{3}.$$

Example 7-6k. Find $\int_0^1 \log_e (1 + x)dx$.

We use Formula 7 of the booklet tables: $\int \log_e x\,dx = x \log_e x - x$. Replace $x$ by $1 + x$ and use (6) from this chapter to obtain

$$\int_0^1 \log_e (1 + x)dx = [(x + 1)\log_e(x + 1) - (x + 1)]\Big|_0^1$$

$$= (2 \log_e 2 - 2) - (1 \log_e 1 - 1)$$

$$= 2 \log_e 2 - 1.$$

Example 7-6ℓ. Find $\int_{-\pi}^{\pi} \sin^4 x \, dx$.

Formula 28 of the booklet tables gives that

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

With $n = 4$, we have

$$\int \sin^4 x \, dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x \, dx.$$

To find this second integral we can use a trigonometric identity (as in Example 7-6g) or we can use Formula 28 again with $n = 2$ to obtain

$$\int \sin^2 x \, dx = \frac{-\sin x \cos x}{2} + \frac{1}{2} \int 1 \, dx$$

$$= \frac{-\sin x \cos x}{2} + \frac{1}{2} x.$$

Therefore, we have

$$\int_{-\pi}^{\pi} \sin^4 x \, dx = \left(\frac{-\sin^3 x \cos x}{4} - \frac{3 \sin x \cos x}{8} + \frac{3x}{8}\right) \Big|_{-\pi}^{\pi}.$$

Since $\sin \pi = \sin(-\pi) = 0$, this becomes

$$\frac{3x}{8} \Big|_{-\pi}^{\pi} = \frac{3}{8}(\pi - (-\pi)) = \frac{3\pi}{4}.$$

Example 7-6m. Find $\int_0^{10} e^{-x^2} dx$.

The tables give no formula for $\int e^{-x^2} dx$. There is a good reason for this: it is known that there is no elementary function whose derivative is $x \to e^{-x^2}$. Our integral, therefore, can't be found by using the Fundamental Theorem of Calculus and we must resort to some approximation method in order to estimate this integral. We shall have more to say about this in Section 9-4.

## Exercises 7-6

For problems 1-15 find the following indefinite integrals.

1. $\int (x^2 + 1)dx$

2. $\int (\frac{1}{x^2} + x + x^4)dx$

3. $\int 8\sqrt{x}\, dx$

4. $\int (x^2 - \sqrt{x})dx$

5. $\int (\frac{1 - x}{x})dx, \quad (x > 0)$ [Hint: Write as 2 fractions.]

6. $\int \sin 3x\, dx$

7. $\int \cos(2x - 5)dx$

8. $\int (-\sin 2x)dx$

9. $\int [-\cos(3x - 1)]dx$

10. $\int \frac{4}{3} \cos 3x\, dx$

11. $\int 2 \sin x \cos x\, dx$ [Hint: Use trigonometric identity.]

12. $\int (3 \sin 2x - 6 \cos 3x)dx$

13. $\int e^{2x}\, dx$

14. $\int e^{x/3}\, dx$

15. $\int (e^x + e^{-x})^2 dx$ [Hint: Remove parenthesis.]

For problems 16-25 find the following indefinite integrals, (using tables when necessary).

16. $\int x^2 e^x \, dx$

17. $\int x^3 e^x \, dx$

18. $\int x^4 e^x \, dx$

19. $\int x^2 \log_e x \, dx$

20. $\int x^3 \log_e x \, dx$

21. $\int x^4 \log_e x \, dx$

22. $\int x^2 \sin x \, dx$

23. $\int x^3 \sin x \, dx$

24. $\int e^{3x} \sin 4x \, dx$

25. $\int e^{x/2} \cos \frac{3x}{2} \, dx$

For problems 26-31, sketch a graph of the relevant region and find the value of the indicated integral.

26. $\int_0^{\pi} (x + \sin x)dx$

27. $\int_0^{2\pi} (x + \sin x)dx$

28. $\int_{-1}^{+1} \frac{e^x + e^{-x}}{2} \, dx$

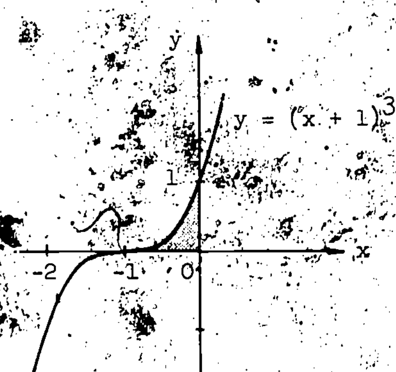
29. $\int_{-1}^{+1} \frac{e^x - e^{-x}}{2} \, dx$

30. $\int_0^{2\pi} x \sin x \, dx$

31. $\int_{1/e^2}^{e^2} \frac{\log_e x}{\sqrt{x}} \, dx$

For problems 32-33, the following instructions are to be followed (linear substitution: translation). In this section we were given an area represented by:

$$A = \int_{-1}^{0} (x + 1)^3 \, dx.$$




By replacement of $x + 1$ by $x$, (i.e., $x$ by $x - 1$), and by appropriate change of limits, we find an equivalent expression for the area. After the linear substitution we have

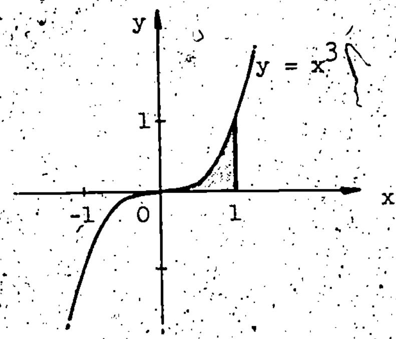
$$A_{L.S.} = \int_0^1 x^3 \, dx \qquad \text{and}$$

evaluating the two equivalent forms of the area:

$$A = \int_{-1}^{0} (x+1)^3 dx = \frac{1}{4}(x+1)^4 \Big|_{-1}^{0} = \frac{1}{4},$$

$$A_{L.S.} = \int_{0}^{1} x^3\, dx = \frac{1}{4} x^4 \Big|_{0}^{1} = \frac{1}{4},$$

we see that they are, indeed, the same.




In the following two problems, follow the format above: Sketch the area defined by the integral, make an appropriate linear substitution, sketch the equivalent area, and evaluate each.

32. $A = \int_{3}^{4} \frac{1}{(x-2)^2} dx$

33. $A = \int_{1}^{2} x(x-1)^3$

For problem 34-35, follow the instructions of problems 32 and 33, except in this case the linear substitution is a <u>scale change</u> instead of a translation. Draw two graphs as before.

34. $A = \int_{0}^{\pi/2} \sin 2x\, dx$

35. $A = \int_{1}^{4} \sqrt{3x}\, dx$

36. (a) Show that if $x < 0$ then

$$D \log_e (-x) = \frac{1}{x}.$$

(Hint: Sketch $f : x \to \log_e (x),\ x > 0$ and $g : x \to \log_e (-x),\ x < 0$.)

(b) Use part (a) to find

$$\int_{-3}^{-1} \frac{1}{x} dx$$

and sketch the area.



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37. (a) Can you apply the Fundamental Theorem to find $\int_0^1 \frac{1}{x}\,dx$? If so, do so. If not, state reasons.

(b) Show that

$$\lim_{n \to \infty} \int_{1/n}^{1} \frac{1}{x}\,dx = \infty$$

(c) Use part (b) to discuss what area, if any, you think should be assigned to the region bounded by $y = \frac{1}{x}$, $x \neq 0$, the $x$ and $y$ axes and the vertical line $x = 1$.

(d) What answer seems reasonable to you for $\int_{-1}^{1} \frac{1}{x}\,dx$? With what properties of area is your answer consistent? inconsistent?

Chapter 8

# DIFFERENTIATION THEORY AND TECHNIQUE

In volume one we showed that the derivative of a polynomial function was also a polynomial function (of one lower degree) and established the formulas

$$D(\sin x) = \cos x \qquad D(\cos x) = -\sin x$$

$$D(e^x) = e^x \qquad D(\log_e x) = \frac{1}{x}$$

$$D(x^a) = ax^{a-1}.$$

These are the basic differentiation formulas. Our primary purpose in this chapter is to obtain formulas for differentiating various algebraic combinations of these functions and to use these derivatives to discuss graphs and motion.

The first section of this chapter is primarily a review and extension of the terminology of derivatives, limits and approximation to general functions, as well as an introduction to the concept of continuity. Various geometric properties of graphs of continuous functions are stated in Section 8-2, making use of the relationship between differentiation and integration (the Fundamental Theorem of Calculus) to establish the connection between derivatives and the shape of the graph of a function. Derivatives of sums, multiples and products are discussed in Sections 8-3 and 8-4. Functions which are composites of simpler functions are discussed in Section 8-5 and the important "chain rule" for differentiating such functions is given in Section 8-6. Special cases of the chain rule, which enable us to differentiate powers, reciprocals and quotients are described in Sections 8-7 and 8-8. A general discussion of the "folding" process used in Chapters 5 and 6 to define and differentiate root and logarithmic functions is contained in Section 8-9. These results are applied, in particular, to the inverse trigonometric functions. The final section of this chapter gives a special technique for differentiating functions which are defined implicitly be relations.

## 8-1. Notation and Terminology

In this chapter we discuss the properties of various combinations of polynomial, circular, exponential and logarithm functions. Some common terminology and notation will be helpful. Some of what is said here is a review of previous discussion.

The notation

$$(1) \qquad \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

will be used as a shorthand for the phrase "the limit of $\frac{f(x + h) - f(x)}{h}$, as $h$ approaches 0."

The function $f$ is said to be <u>differentiable</u> at $x$ if the limit (1) exists; that is, there is a unique number which we can approximate as close as we please by the quotient

$$\frac{f(x + h) - f(x)}{h}$$

if $|h|$ is small enough. The derivative $f'$ is the function which is defined for all $x$ for which the limit (1) exists and whose value $f'(x)$ is given by

$$(2) \qquad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Our intuitive discussion makes use of elementary limit principles. A rigorous discussion of limits is left to the appendices.

If $f$ is differentiable at $x = a$, then the graph of $f$ has a nonvertical tangent at $(a,f(a))$, the slope of this tangent being $f'(a)$. The equation of this tangent line is

$$y = f(a) + f'(a)(x - a).$$

If $x$ is close to $a$, then the values $f(x)$ will closely approximate the values $f(a) + f'(a)(x - a)$ along the tangent; that is,

$$(3) \qquad f(x) \approx f(a) + f'(a)(x - a)$$

if $x$ is close to $a$.

The approximation (3) is a simple rewrite of the definition of $f'(a)$, for (2). We have

$$\frac{f(x + h) - f(x)}{h} \approx f'(x), \text{ if } |h| \text{ is small.}$$

If we multiply through by $h$ and add $f(x)$ to both sides we obtain

$$f(x + h) \approx f(x) + f'(x)h, \quad \text{for small } |h|.$$

In the preceding chapters it was shown that polynomial functions, the sine and cosine function and the exponential function are differentiable at each point. Explicit formulas for the derivatives of these functions were obtai[illegible] The logarithm function $x \to \log_e x$ is defined and differentiable only for positive $x$. Power functions $x \to x^a$ are differentiable for $x > 0$. In some cases, such as

$$x \to x^{3/2}$$

(where the exponent is not less than 1) a power function may also be differentiable at $x = 0$, while in other cases, such as

$$x \to x^{1/2}$$

the function will fail to be differentiable at $x = 0$. Functions such as $x \to x^{2/3}$ (where the exponent is rational with an odd denominator) will also be differentiable for negative values of $x$.

The square root function

$$x \to \sqrt{x}$$

fails to be differentiable at $x = 0$ because its graph has a vertical tangent line at $(0,0)$. A function will also fail to be differentiable at a point where its graph has a "corner." For example, consider the absolute value function

$$x \to |x|.$$

The absolute value $|x|$ is the larger of the two numbers $x$ and $-x$, so that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

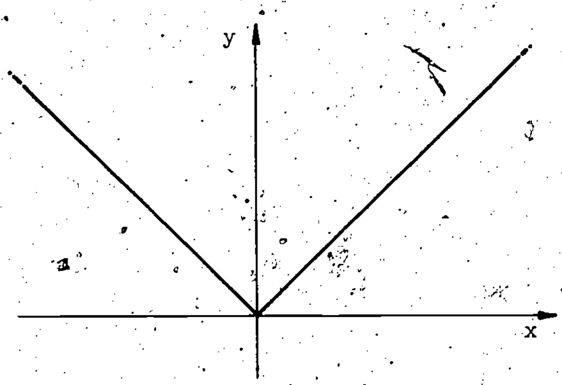


Figure 8-1a

$x \to |x|$

It is clear from Figure 8-1a that the absolute value function is differentiable for $x > 0$, where its derivative has the value $+1$; and for $x < 0$, where its derivative has the value $-1$. The difference quotient at $x = 0$ is

(4) $$\frac{|0 + h| - |0|}{h} = \frac{|h|}{h}.$$

The values of $\frac{|h|}{h}$ depend upon the sign of $h$. If $h > 0$, then

$$\frac{|h|}{h} = \frac{h}{h} = 1$$

while if $h < 0$, then

$$\frac{|h|}{h} = \frac{-h}{h} = -1.$$

In summary, the quotient (4) is $+1$ or $-1$ depending upon whether $h > 0$ or $h < 0$; hence, the difference quotient does not approximate a unique number for $|h|$ small.

The general principle is that the limit exists if and only if the left-hand and right-hand limits exist and are equal. Consequently, in order for the limit (1) to be defined, the number approximated by $\frac{f(x + h) - f(x)}{h}$ when $h$ is negative and near zero must be the same as the number approximated by this quotient when $h$ is positive and near zero. As the absolute value function illustrates, at a "corner" the left-hand and right-hand limits will differ, and the function will fail to be differentiable at the corner point.

It is also possible that a function oscillates so badly near a point that neither the left-hand nor the right-hand limits of (1) will exist.

In general, we shall use the notation $f'$ for the derivative of a function $f$. As in the previous chapters $D$ notation will be used in stating formulas. For example, $D(\sin x) = \cos x$ is a shorthand for the statement that the derivative of the sine function is the cosine function. The notation $\frac{dy}{dx}$ has also been introduced for the derivative. This notation is related to the concept of derivative as "rate of change." For example, if $y = x^2$, then

$$\Delta y = (x + \Delta x)^2 - x^2$$

represents the change $\Delta y$ in $y$ due to the change $\Delta x$. The <u>average rate of change</u> in $y$ per unit change in $x$ is then

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x.$$

As $\Delta x$ approaches $0$ we obtain the value of the derivative of $x \to x^2$ at the point $x$, in this case denoted by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x.$$

Let us now make explicit a concept which has been implicitly used in many of our previous discussions. A function $f$ is said to be <u>continuous</u> at $a$ if $f(a)$ is <u>defined</u> and

$$\lim_{x \to a} f(x) = f(a);$$

that is, the value $f(a)$ is approximated by the values $f(x)$ for all $x$ which are close enough to $a$. For example, use has been made of the fact that $h^2$ is small if $h$ is small. This amounts to saying that the function $x \to x^2$ is continuous at $0$.

In Chapter 6, use was made of the statements that

$$\log_e x \approx \log_e a, \text{ if } x \text{ is close to } a$$

and

$$e^x \approx e^a \text{ if } x \text{ is close to } a.$$

These two facts are summarized by saying that the two functions

$$x \to \log_e x \text{ and } x \to e^x$$

are continuous at each point for which they are defined.

If a function is continuous at each point where it is defined it is said to be a continuous function. It can be shown that polynomial functions and the sine, cosine, exponential, logarithm and power functions are all continuous functions. Such results are consequences of elementary properties of limits and are discussed in the appendices.

The concept of continuity is related to the concept of differentiability in the following sense:

(5) If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

This is a simple consequence of the tangent approximation idea (3),

$$f(x) \approx f(a) + f'(a)(x - a), \quad \text{if } x \text{ is close to } a.$$

As $x$ approaches $a$, the term $f'(a)(x - a)$ approaches $0$ so that $f(x)$ must approach $f(a)$. The converse of (5) is false. For example, the function

$$f : x \to |x|$$

is not differentiable at $0$ but is continuous at $0$;

$$f(x) = |x| \text{ will be close to } 0 = f(0)$$

if $x$ is close to $0$.

The concept of continuity arose out of the desire to formulate a condition which will guarantee that the graph has no "gaps." A relationship between continuity and the non-existence of gaps will be stated in the next section. For now we note that if the graph of $f$ has a "gap" at $a$, then $f$ will be discontinuous (that is, not continuous) at $a$. For example, consider the function $f$ whose values $f(x)$ are given by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The graph of $f$ is sketched in Figure 8-1b. Note the "jump" at the point $(0,f(0))$.

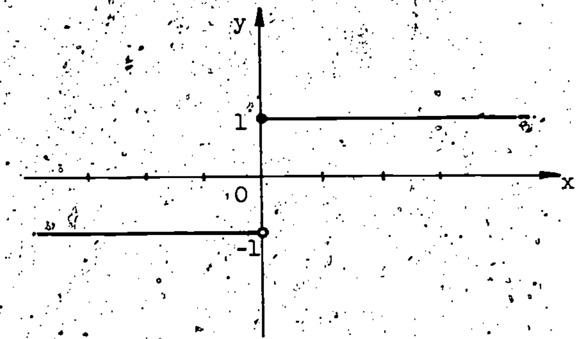


Figure 8-1b

$$x \to f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

As $x$ approaches 0 from the right, the values $f(x)$ are all equal to 1 so that the right-hand limit is +1. As $x$ approaches 0 from the left the values $f(x)$ are all -1 so that the left-hand limit is -1. Since the left-hand and right-hand limits differ we conclude that

$$\lim_{x \to 0} f(x) \text{ doesn't exist;}$$

that is, that $f$ is <u>discontinuous</u> at $x = 0$.

## Exercises 8-1

1. For each of the following find:

   (i) $\frac{\Delta y}{\Delta x}$, then find

   (ii) $\frac{dy}{dx}$ as $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

   (a) $y = x^3 - 3x + 3$

   (b) $y = \sqrt{x}, \quad x > 0$

   (c) $y = 3x^2 - \frac{1}{x}, \quad x \neq 0$

2. The notation $\frac{d^2y}{dx^2}$ is often used for the second derivative.

   (a) If $y = e^{cx}$ find $\frac{d^2y}{dx^2}$.

   (b) If $y = -\log_e x$ show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

   (c) Will the result of (b) hold for any $y$?

3. Suppose $f(x) \approx g(x) = 1 + 3(x - 2)$ in the sense that

$$\lim_{x \to 2} \frac{f(x) - g(x)}{x - 2} = 0.$$

   What is $f(2)$? $f'(2)$?

4. Find the value of the following by first specifying the function $f$ of which this is the derivative and then finding the derivative of $f$ by formula, (i.e., not by definition).

   (a) $\lim\limits_{\Delta x \to 0} \dfrac{\left[\frac{1}{(x + \Delta x)^2} + 3\right] - \left[\frac{1}{x^2} + 3\right]}{\Delta x}$

   (b) $\lim\limits_{\Delta x \to 0} \dfrac{\left(x + \Delta x + \frac{1}{x + \Delta x}\right)^2 - \left(x + \frac{1}{x}\right)^2}{\Delta x}$

   (c) $\lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)\sqrt{x + \Delta x} - x\sqrt{x}}{\Delta x}$

5. Find the value of the following: [Hint: In each of these, you are asked to find $f'(a)$; i.e., the value of the derivative at a particular point where $x$ equals some number, $a$.]

(a) $\lim_{h \to 0} \frac{(3 + h)^2 - (3)^2}{h}$

(b) $\lim_{\Delta x \to 0} \frac{(4 + \Delta x)^{3/2} - 4^{3/2}}{\Delta x}$

(c) $\lim_{h \to 0} \frac{1 - (1 + h)^6}{12h}$

(d) $\lim_{\Delta x \to 0} \frac{\pi \sin(\pi + h) - \pi \sin \pi}{h}$

(e) $\lim_{\Delta x \to 0} \frac{3 \cos [\pi - 2(\frac{\pi}{8} + \Delta x)] - 3 \cos(\pi - 2 \frac{\pi}{8})}{8\Delta x}$

6. Sketch a graph of each of the following and locate the points of discontinuity for each. ($f$ is discontinuous at $a$, if (1) $f(a)$ is not defined or (2) $\lim_{x \to 0} f(x) \neq f(a)$.)

(a) $f : x \to \frac{1}{x + 2}$ for $-1 < x < 3$

(b) $f : x \to \begin{cases} \frac{1}{x^2} & \text{for } 0 < |x| < 2 \\ 0 & \text{for } x = 0 \end{cases}$

(c) $f : x \to \begin{cases} \frac{x - 1}{x^2 - 1} & \text{for } |x| < \frac{3}{2},\ x \neq 1 \\ \frac{1}{2} & \text{for } x = 1 \end{cases}$

(d) $f : x \to \left\{ \frac{2x - 1}{|2x - 1|} \quad |x| < 2 \right.$

(e) $f : x \to \begin{cases} x - 4 & \text{for } -1 \leq x \leq 2 \\ x^2 - 6 & \text{for } -2 < x < -1, \text{ and } 2 \leq |x| < 3 \end{cases}$

7. We discovered in this section that the relationship between differentiability and continuity is a one-way affair:

> A function continuous at $x = a$ may or may not be differentiable there; a function differentiable at $x = a$ is continuous there.

Sketch each of the following and determine for what values of $x$ the function is not differentiable. Are there any points where the function is not differentiable and yet a tangent line exists?

(a) $f : x \to |\sin x|$ for $0 < x < 2\pi$

(b) $f : x \to x^{3/2}$ for $0 \le x \le 4$

(c) $f : x \to x^{2/3}$ for $0 \le x \le 8$

(d) $f : x \to x^{1/n}$ $n$, even $n \ge 2$ for $x \ge 0$

(e) $f : x \to x^{1/n}$ $n$, odd, $n \ge 3$ for all $x$

(f) $f : x \to x|x|$ for all $x$

(g) $f : x \to \dfrac{1}{x^2 - 6x + 10}$ for all $x$

(Hint: $f' : x \to \dfrac{2(3 - x)}{(x^2 - 6x + 10)^2}$)

8. (a) Use the inequality $|1 - \cos x| \le \frac{x^2}{2}$ show that $\lim\limits_{x \to 0} \cos x = 1$.

(b) Using the inequality $|\sin x| \le |x|$ show that $\lim\limits_{x \to 0} \sin x = 0$.

(c) Use the addition formulas and parts (a) and (b) to show that the sine function is everywhere continuous. [Hint: Look at $\lim\limits_{x \to a} \sin x = \lim\limits_{h \to 0} \sin (a + h)$ where $|x - a| = h$.]

9. Suppose $f : x \to \sin \frac{1}{x}$, $x > 0$.

(a) If $n$ is a positive integer find.

(i) $f(\frac{1}{n\pi})$, (ii) $f(\frac{2}{(4n + 1)\pi})$, (iii) $f(\frac{2}{(4n + 3)\pi})$

(b) What is the limit of each of the values of part (a) as $n \to \infty$?

(c) Is there any way to define $f(0)$ so that $\lim\limits_{x \to 0} f(x) = f(0)$?

8-2. Properties of Continuous Functions.

We have used several properties of functions in earlier discussions covering such topics as maxima and minima, intervals of increase and decrease, convexity and concavity, the use of anti-derivatives to calculate areas, and inverse functions. General theory was deliberately neglected while ideas were being treated separately for each type of function, using methods appropriate to the function under consideration. That there are underlying principles should now be obvious. Rather than proceed further in this ad hoc manner, we believe that now is the time to begin to discuss some general properties of functions, which are related to the concepts of differentiation, integration, and continuity.

We begin by listing two general theorems which hold for functions which are continuous at each point of a closed interval. The first asserts that the graph of such a function has no "gaps," while the second asserts that such a function has a maximum and a minimum in the interval. Proofs of these two theorems are in the appendices.

THEOREM 8-2a. The Intermediate Value Theorem

Suppose $f$ is continuous at each point of the interval $a \le x \le b$, and that $f(a) \ne f(b)$. If $d$ lies between $f(a)$ and $f(b)$, then there is at least one point $c$ between $a$ and $b$ such that

$$f(c) = d.$$

More simply, if the graph of a continuous function on an interval contains points on both sides of a horizontal line (the line given by $y = d$), then the graph must meet the line. (See Figure 8-2a.)

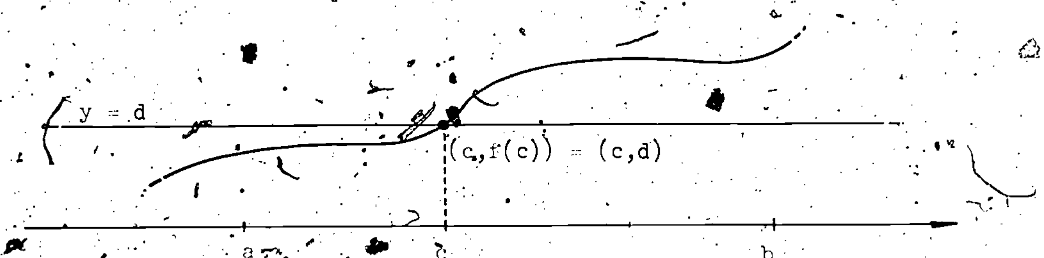


Figure 8-2a

A continuous function, f.

This result is, of course, a general form of the Location Theorem (Chapter 1), which asserts that a polynomial function has a zero between any two points $a$ and $b$ if $f(a)$ and $f(b)$ have opposite signs.

THEOREM 8-2b. Suppose $f$ is continuous at each point of the interval $a \le x \le b$. Then there are points $c$ and $d$ with $a \le c \le b$ and $a \le d \le b$ such that

$$f(d) \le f(x) \le f(c) \text{ for all } x,\ a \le x \le b.$$

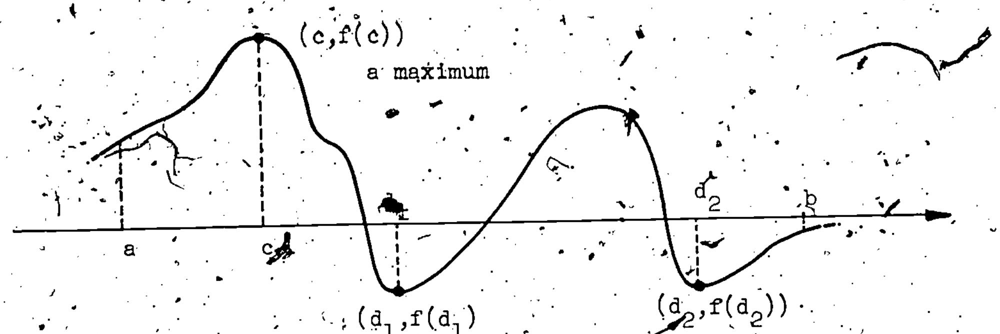


Figure 8-2b

Extreme values on an interval.

The value $f(d)$ is called the minimum while $f(c)$ is called the maximum of $f$ on the given interval. (See Figure 8-2b.) For polynomial functions this theorem was stated in Section 2-8. The hypothesis of continuity on the closed interval $a \le x \le b$ is essential in both of these theorems, as will be shown in the exercises. Note also that the theorem does not say that $c$ and $d$ are unique, a graph can have several points which are the highest (or lowest) points on an interval. (See Figure 8-2b.)

Theorem 8-2b is an existence theorem. It asserts the existence of highest and lowest points but gives no means for finding them. For polynomial functions (which are everywhere differentiable), we noted in Chapter 2 that maximum and minimum values can occur only at endpoints of the interval or at points where the derivative is 0. In general, we have the following result:

THEOREM 8-2c. Suppose $f$ is continuous at each point of the interval $a \le x \le b$ and that $f(c)$ is a maximum or minimum value for $f$ on this interval. Then it must be true that

$c = a$, $c = b$, $f'(c) = 0$, or $f$ is not differentiable at $c$.

In other words, extremal values can occur only at endpoints, zeros of the derivative, or places where the derivative doesn't exist (that is, points where the tangent line is vertical, or there is no tangent). The various graphs of Figure 8-2c illustrate each of the possibilities of Theorem 8-2c.

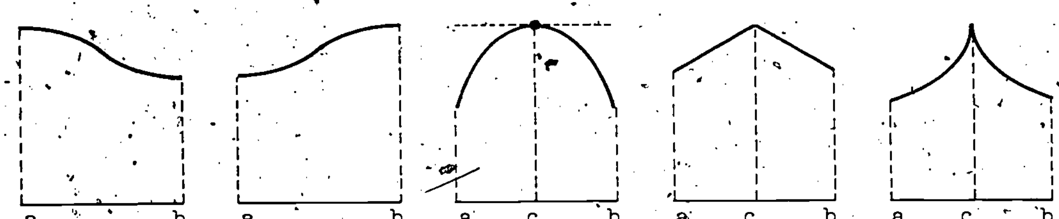


Figure 8-2c

Some possibilities for $f(c)$ = maximum.

Let us sketch a proof of Theorem 8-2c. Suppose $f(c)$ is a maximum and that none of the possible conclusions is true; that is, $c \ne a$, $c \ne b$, $f$ is differentiable at $c$ and $f'(c) > 0$ or $f'(c) < 0$. Consider the possibility that $f'(c) > 0$. Since

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h},$$

the values $\frac{f(c + h) - f(c)}{h}$ approximate the positive quantity $f'(c)$ and hence must themselves be positive if $|h|$ is small enough. Since $c < b$, we can find $h > 0$ so that $c + h < b$ and

$$\frac{f(c + h) - f(c)}{h} > 0;$$

multiply through by $h$ to obtain $f(c + h) - f(c) > 0$, so that $f(c + h) > f(c)$. This contradicts the assumption that $f(c)$ is the maximum of $f$ on the interval $a \le x \le b$. A similar argument (using $h < 0$ with $|h|$ sufficiently small) shows that the possibility $f'(c) < 0$ also leads to



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a contradiction. By this indirect argument we conclude that Theorem 8-2c must be true.

It will also be convenient to make use of the fact that if the derivative is positive on an interval the function is increasing. An instructive proof of this result can be given using the Fundamental Theorem of Calculus. (To do this vigorously it is first necessary to extend the Area Theorem and the Fundamental Theorem to continuous functions. This is done in the appendices.) At this point we assume that these theorems can be so extended, in particular, that $\int_a^b f$ can be defined for continuous functions $f$ so that if $f$ is nonnegative on the interval then $\int_a^b f$ is the area of the region bounded by the graph of $f$ over the interval; that is, if we use $\int_a^b f$ for this area then this is consistent with the area concepts of Chapter 7. Furthermore, it will be assumed that if $F' = f$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Replacing $f$ by $F'$, we can write

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Here we have the basic connection between the concept of area (or integral) and that of the slope of a tangent (or derivative). We summarize this discussion in the following theorem.

THEOREM 8-2d. The Fundamental Theorem of Calculus

If the derivative $F'$ exists and is continuous at each point of the interval $a \leq x \leq b$, then

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Let us now show how we can obtain results about increase-decrease and convexity-concavity from this version of the Fundamental Theorem. These results are established in the following two theorems.

THEOREM 8-2e. Suppose $f$ is differentiable at each point of the interval $a \leq x \leq b$ and that the derivative $f'$ is continuous and nonnegative at each point of the interval. Then $f$ is increasing* on the interval; that is;

(1) if $a \leq x_1 \leq x_2 \leq b$, then $f(x_1) \leq f(x_2)$.

If $f'$ is assumed to be nonpositive on the interval then $f$ is decreasing*; that is,

(2) if $a \leq x_1 \leq x_2 \leq b$, then $f(x_1) \geq f(x_2)$.

Proof. Suppose that $f'(x) \geq 0$, and $a \leq x_1 \leq x_2 \leq b$. The Fundamental Theorem tells us that

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)dx.$$

Since $f'$ is nonnegative, the area given by $\int_{x_1}^{x_2} f'$ must be nonnegative; that is,

$$f(x_2) - f(x_1) \geq 0.$$

This proves that (1) is true. The proof when the derivative is nonpositive is similar. (See Figure 8-2d)

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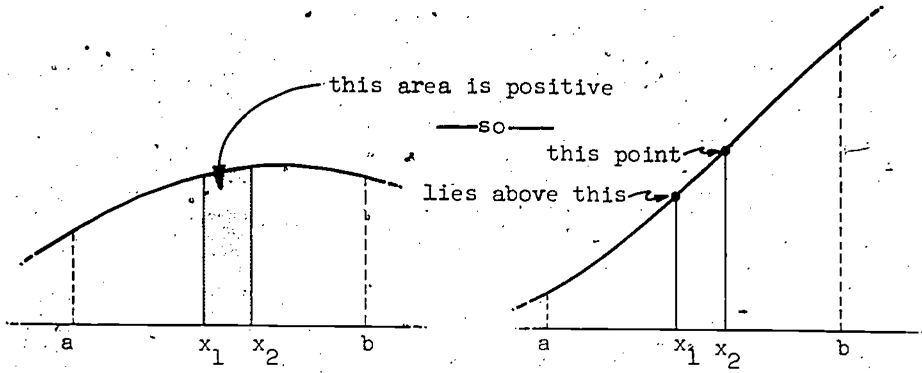
*This is slightly different terminology than that used previously, a function $f$ being increasing if

$$f(x_1) < f(x_2) \quad \text{whenever } a \leq x_1 < x_2 \leq b.$$

It is common to say that a function is strictly increasing if it satisfies this condition, and merely increasing if it satisfies (1). If it satisfies (2) then it is decreasing, with strictly decreasing used for the condition $f(x_2) > f(x_1)$ whenever $a \leq x_1 < x_2 \leq b$.




The graph of $f'$. The graph of $f$.

Figure 8-2d

A slightly different form of this result can also be proved, namely, that if

$$f'(x) > 0, \quad \text{for} \quad a < x < b$$

and $f'$ is continuous at each point of the interval $a \leq x \leq b$, then $f$ is strictly increasing on the interval. The proof is the same as the above and uses the fact that

$$\int_a^b f' > 0, \quad \text{if} \quad f' > 0 \quad \text{for} \quad a < x < b.$$

This fact just asserts that the area bounded by $f'$ will be positive if the graph of $f'$ lies above the x-axis in the interval. (The proof is given in the appendices.)

The techniques for finding intervals of increase and decrease and for locating maximum and minimum points discussed in Chapter 2 for polynomial functions can now be extended to more general functions. The basic method is to determine the intervals in which the derivative doesn't change sign. These methods will be applied in subsequent sections of this chapter as we learn to differentiate more functions.

The second derivative $f''$ is, of course, defined as the derivative of $f'$, that is, $f''(x)$ is defined for those $x$ in the domain of $f'$ for which the limit

$$\lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$$

exists. The value $f''(x)$ is then given by this limit.

A differentiable function $f$, is said to be convex* in the interval $a \leq x \leq b$ if

(3) $$f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1)$$

whenever $x_1$ and $x_2$ lie in the interval. This says that the graph in the interval doesn't go below its tangent at any point of the interval. (See Figure 8-2e. The inequality (3) asserts that B is not below C.)

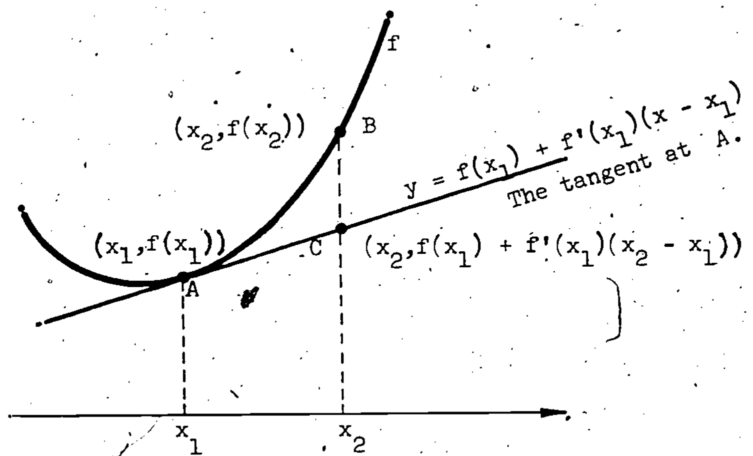



Figure 8-2e

A convex function.

The definition of concavity* is obtained by reversing the inequality in (3).

THEOREM 8-2f. Suppose $f$ is differentiable at each point of the interval $a \leq x \leq b$ and that its derivative $f'$ is differentiable at each point of this interval. If the second derivative $f''$ is continuous and non-negative at each point in the interval then $f$ is convex in the interval. If, instead, $f''$ is assumed to be nonpositive then $f$ is concave in the interval.

Proof. Assuming that $f''(x) \geq 0$, $a \leq x \leq b$, we have, by Theorem 8-2e, that the derivative $f'$ is increasing on the interval. Suppose $a \leq x_1 < x_2 \leq b$. The Fundamental Theorem then gives the result

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*This differs from the terminology used previously, which required that

$$f(x_2) > f(x_1) + f'(x_1)(x_2 - x_1)$$

This is usually referred to as strict convexity, with an analogous definition for strict concavity.

(4) $$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'.$$

The function $f'$ is increasing on the interval, $x_1 \le x \le x_2$, so that the minimum value of $f'$ on this interval must be $f'(x_1)$; that is,

$$f'(x_1) \le f'(x), \quad \text{for } x_1 \le x \le x_2.$$

It was noted in Section 7-2 that

$$\int_a^b g(x)dx \ge m(b - a) \quad \text{if} \quad g(x) \ge m \quad \text{for } a \le x \le b.$$

With $g = f'$, $m = f'(x_1)$, $b = x_2$, $a = x_1$, we have

$$\int_{x_1}^{x_2} f'(x)\, dx \ge f'(x_1)(x_2 - x_1).$$

If we combine this with (4) we obtain the desired inequality

$$f(x_2) \ge f(x_1) + f'(x_1)(x_2 - x_1).$$

The same result can be obtained if $x_2 < x_1$. Comparable arguments, with the signs reversed, establish concavity if $f''$ is nonpositive. (Figure 8-2f illustrates relationship between increase of the derivative and convexity.)

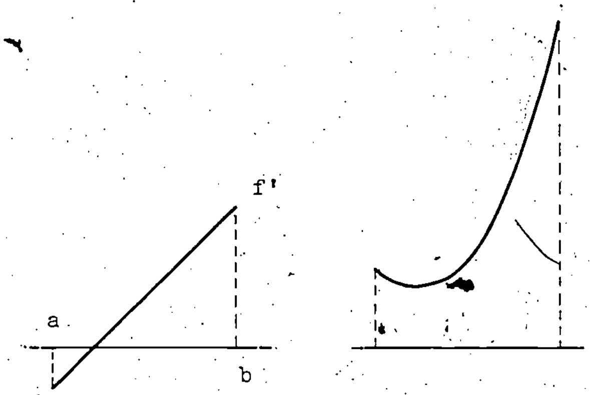


Figure 8-2f

These results about the relation between the sign of the derivatives and the shape of the graph of $f$ can also be derived without making use of the Fundamental Theorem, for which case assumptions that $f'$ or $f''$ be continuous can be dropped. A complete discussion of this is given in the appendices. Here we mention only the basic theorem used, the <u>Mean Value Theorem</u>.

THEOREM 8-2g. The Mean Value Theorem

Suppose $f$ is continuous at each point of the interval $a \leq x \leq b$ and differentiable at each point of the interval $a < x < b$. Then there is at least one number $c$, such that $a < c < b$ and

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

In other words, there is at least one point on the graph of $f$ between $a$ and $b$ where the slope of the tangent line is the same as the slope of the line connecting $(a,f(a))$ and $(b,f(b))$. (See Figure 8-2g, which illustrates a case where there are two points for which this is true.)

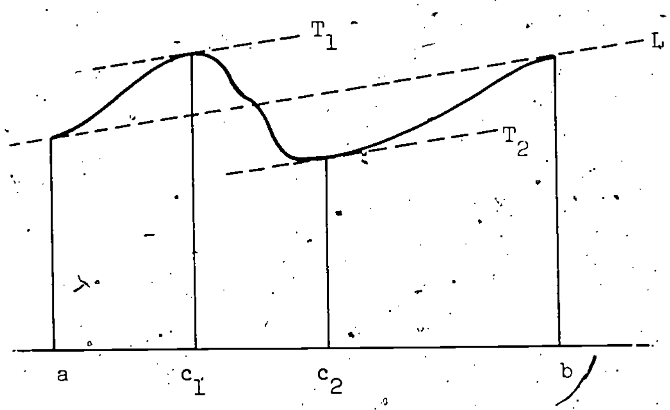


Figure 8-2g.

$L$, $T_1$, and $T_2$ are parallel.

## Exercises 8-2a

1. Suppose you drive from New York to Chicago, sometimes stopping and other times driving as fast as 70 miles per hour. Is there some time during the trip when your speed is 50 miles per hour? Give reasons.

2. Two cities are 200 miles apart. Starting from one you drive continually to the other in 4 hours, then stop.

   (a) Is there some place on the trip where your speedometer reads 50? Give reasons.

   (b) Is there some place on the trip where your acceleration was 0? Give reasons.

3. Suppose that you drive from Sacramento (elevation 200 feet) to Loggers Station Camp Ground (elevation 5480 feet). The map distance between the two points is exactly 100 miles. Was there some time during the trip when you were on a portion of road that had a slope of exactly 1%. Give your reason.

4. If the acceleration of a moving particle is always negative, what can you say about its velocity?

5. Suppose $f''$ is continuous in the interval $a \le x \le b$, and that $f'(c) = 0$ for some $c$ on the same interval. What can you say about $c$ if $f''$ is nonnegative on the interval? nonpositive?

6. Suppose $a < c < b$ can you deduce that $c$ is a maximum or minimum for $f$ on the interval if:

   (a) $f'(x) \le 0$, $a \le x < c$ and $f'(x) \ge 0$, $c < x \le b$.

   (b) $f'(x) \ge 0$, $a \le x < c$ and $f'(x) \le 0$, $c < x \le b$.

   (c) $f'(x) \le 0$, $a \le x < c$ and $f'(x) \le 0$, $c < x \le b$.

   (d) $f'(x) \ge 0$, $a \le x < b$ and $f'(x) \ge 0$, $c < x \le b$.

7. If $f'$ is positive on the interval $a \le x \le b$ how many zeros can $f$ have on the interval? At what point does $f$ have its maximum? its minimum?

8. Give an example of a function defined for $0 \le x \le 1$ such that $f(0) = 0$, $f(1) = 1$ and

   (a) there is no point $c$ in the interval, where $f(c) = \frac{1}{2}$.

   (b) $f$ is continuous and there are at least 10 points in the interval where $f$ has the value $\frac{1}{2}$.

9. Suppose $f$ is continuous for $a \le x \le b$, $f(a) \ne f(b)$ and $d$ lies between $f(a)$ and $f(b)$. Can there be two points $c$, between $a$ and $b$ such that $f(c) = d$? If not, why not? If so, give an example.

10. Give an example of a function $f$ defined for $0 \le x \le 1$ such that

(a) $f$ has a minimum but no maximum on the interval.

(b) $f$ has neither a maximum nor a minimum on the interval.

(c) Can your answers to (a) or (b) be continuous at each point of the interval.

11. Let $f : x \to x^2 + 1$ and $F : x \to \int_0^x f'$, $x \ge 0$.

(a) Find an expression for $F(x)$.

(b) What is the relation between $F(x)$ and $f(x) - f(0)$?

(c) Verify (without using Theorem 8-2e) that $f$ is an increasing function. Examine $f$ at $x$ and at $x + h$ for $h > 0$.

12. Give an example of a function $f$, continuous for $0 \le x \le 1$, whose maximum on the interval is at $c$, and

(a) $c = 0$, $f'(0) \ne 0$

(b) $c = 1$, $f'(1) \ne 0$

(c) $c = \frac{1}{2}$ and $f$ is differentiable at $c$.

(d) $c = \frac{1}{2}$ and the graph has a corner at $c$.

(e) $c = \frac{1}{2}$ and the graph has a vertical tangent at $c$.

13. Suppose $f$ is a polynomial function of degree two such that $f$, on the interval $0 \le x \le 1$, assumes its maximum value at two distinct points $c_1$ and $c_2$ of the interval.

(a) What must $c_1$ and $c_2$ be?

(b) Can there be a third point of the interval where $f$ has its maximum?

(c) Show that $f$ has a unique minimum on the interval.

14. Suppose $f$ is a polynomial function of degree three such that $f$ in the interval $0 \leq x \leq 1$, assumes its maximum value at two distinct points $c_1$ and $c_2$ on the interval.

Assume $f: x \rightarrow a_3x^3 + a_2x^2 + a_1x + a_0$, $a_3 > 0$.

(a) Sketch possible locations for $c_1$ and $c_2$.

(b) Can there be a third point of the interval where $f$ is a maximum?

(c) Give the possible cases for exactly one minimum when there are two maximums. Sketch your answer.

(d) Give the cases for exactly two minimums when there are two maximums. Sketch your answer.

15. Suppose $f$ is a polynomial function.

(a) If the degree of $f$ is 2, show that either $f$ is strictly convex or strictly concave.

(b) If the degree of $f$ is 3, then there are intervals on which $f$ is convex and there are intervals on which $f$ is concave.

(c) What can you say if $f$ has degree 4?

## Exercises 8-2b

1. The fact that the interval includes its endpoints in Theorem 8-2b cannot be weakened.

   (a) Show that the values of $f : x \to \frac{1}{x}$, $0 < x \leq 1$ are not even bounded above.

   (b) Show that $x \to \frac{1}{1 + x^2}$ has no maximum and no minimum on the interval $0 < x < 1$.

2. Show that if $f$ is continuous and strictly increasing for $0 \leq x \leq 1$, then given any $d$, $f(0) < d < f(1)$ there is a unique $c$, $0 < c < 1$ such that $f(c) = d$. Use Theorem 8-2a and indirect reasoning.

3. Show that if $f$ is continuous on an interval $a \leq x \leq b$, then the image of the interval is a closed interval. The image is the set of points $f(x)$ with $a \leq x \leq b$. (Hint: Use Theorems 8-2b to find the endpoints, then use the Intermediate Value Theorem).

4. Show that if $f'$ is continuous and positive at each point of an interval $a \leq x \leq b$, then

   (a) $f(a) \leq f(x) \leq f(b)$ for $a \leq x \leq b$.

   (b) there is a function $g$ such that
   $$g(f(x)) = x \quad \text{for} \quad a \leq x \leq b.$$
   (Hint: Use the remark following Theorem 8-2e and Exercise 8-2b, No. 2.)

5. Show that if $f'$ is continuous and $f$ strictly convex on an interval $a \leq x \leq b$ then $f$ has a unique minimum on the interval. Use Theorem 8-2c and an indirect argument. If $a < b$ can this minimum occur at $b$? at $a$?

6. (a) Show that if $f$ is a convex function then
   $$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$
   A sketch will help.

   (b) Show that
   $$2^{\alpha+\beta/2} \leq \frac{2^{\alpha} + 2^{\beta}}{2}.$$

7\. (a) Show that the following is a special case of the Mean Value Theorem: If $f$ is continuous for $a \leq x \leq b$, differentiable for $a < x < b$ and $f(a) = f(b) = 0$, then there is a number $c$ such that $f'(c) = 0$ and $a < c < b$.

This result is usually known as Rolle's Theorem. An algebraic trick will be used in the appendices to deduce the Mean Value Theorem from Rolle's Theorem.

(b) Deduce Rolle's Theorem from Theorems 8-2b, c.
(Hint: If $f$ is not constant then either its minimum or its maximum is not zero.)

8\. Deduce Theorem 8-2e from the Mean Value Theorem.
(Hint: If $f'$ is nonnegative on the interval show that $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is nonnegative.)

9\. Show that if $f'(x) = 0$, $a \leq x \leq b$, then $f$ is constant.
(Hint: Consider $\dfrac{f(x) - f(a)}{x - a}$ and use the Mean Value Theorem.)

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## 8-3. Sums and Multiples

The remaining sections of this chapter discuss methods for differentiating various combinations of known functions. In this section we examine sums and multiples of functions.

The sum of two functions has been previously encountered. For example, the graph of

$$f : t \to 3 \cos \pi t + 4 \sin \pi t$$

was obtained in Chapter 3 (p. 236)* by adding the corresponding ordinates (Figure 8-3a) of the two functions

$$u : t \to 3 \cos \pi t \quad \text{and} \quad v : t \to 4 \sin \pi t$$

at each value of $t$. Here we say that $f$ is the <u>sum</u> of the two functions $u$ and $v$ and write

$$f = u + v.$$

This means that for each $t$, the values $f(t)$, $u(t)$, and $v(t)$ are related by

$$f(t) = u(t) + v(t).$$

*In Chapter 3 we let $u$ and $v$ be the <u>values</u> of the functions $t \to u$ and $t \to v$. Here $u$ and $v$ are the <u>functions</u> $u : t \to u(t)$ and $v : t \to v(t)$ or $u : x \to u(x)$ and $v : x \to v(x)$.

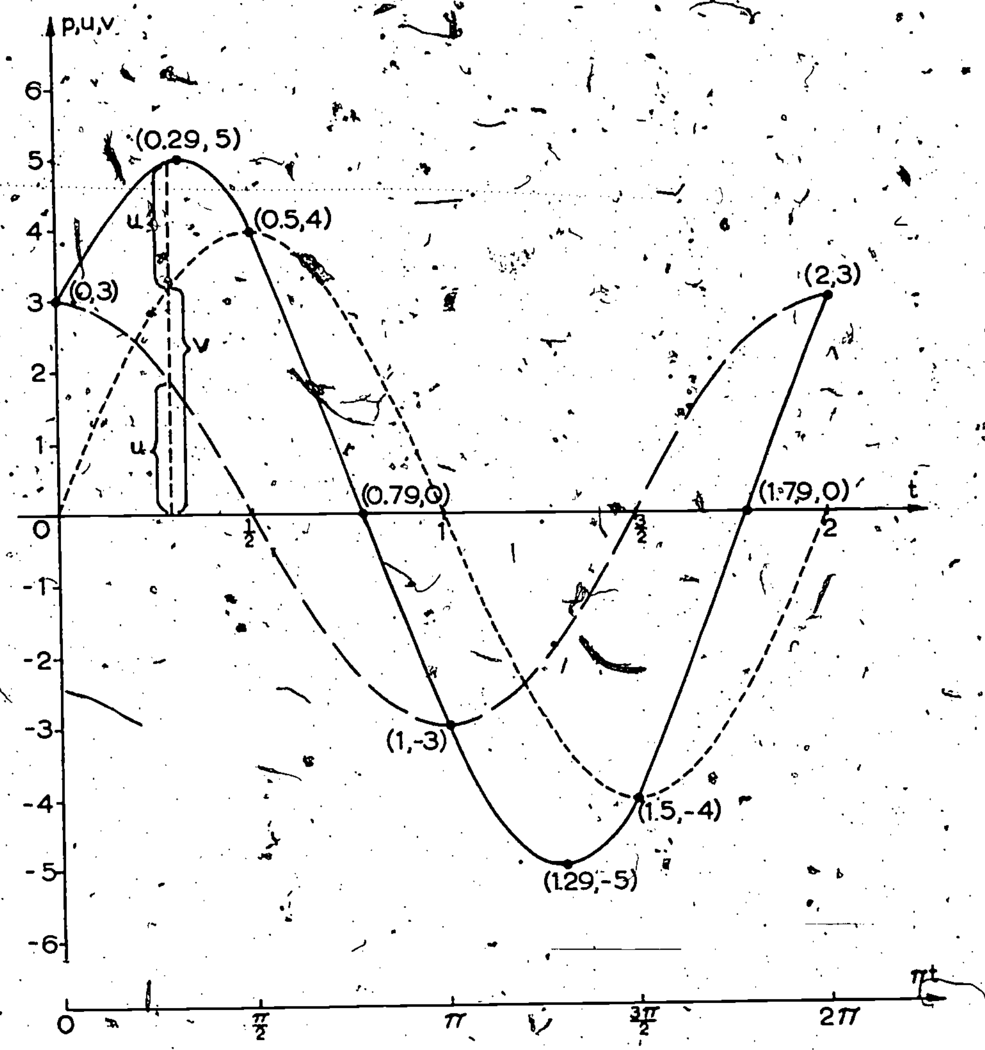


Figure 8-3a

The difference of two functions is defined analogously; for example,

$$f = u - v$$

if, for each $x$, the values $f(x)$, $u(x)$ and $v(x)$ are related by

$$f(x) = u(x) - v(x).$$

To be more concrete, if

$$f : x \rightarrow 2 \sin 3x - 3 \cos 3x$$

we can write $f = u - v$, where

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$$u : x \to 2 \sin 3x$$

$$v : x \to 3 \cos 3x.$$

The function $u : x \to 2 \sin 3x$ is a multiple of the function

$$g : x \to \sin 3x$$

in the sense that the values $u(x)$ and $g(x)$ are related by the equation $u(x) = 2g(x)$. The graph of $u$ is obtained from the graph of $g$ by multiplying the corresponding ordinate of the graph of $g$ by 2. (See Figure 8-3b.)

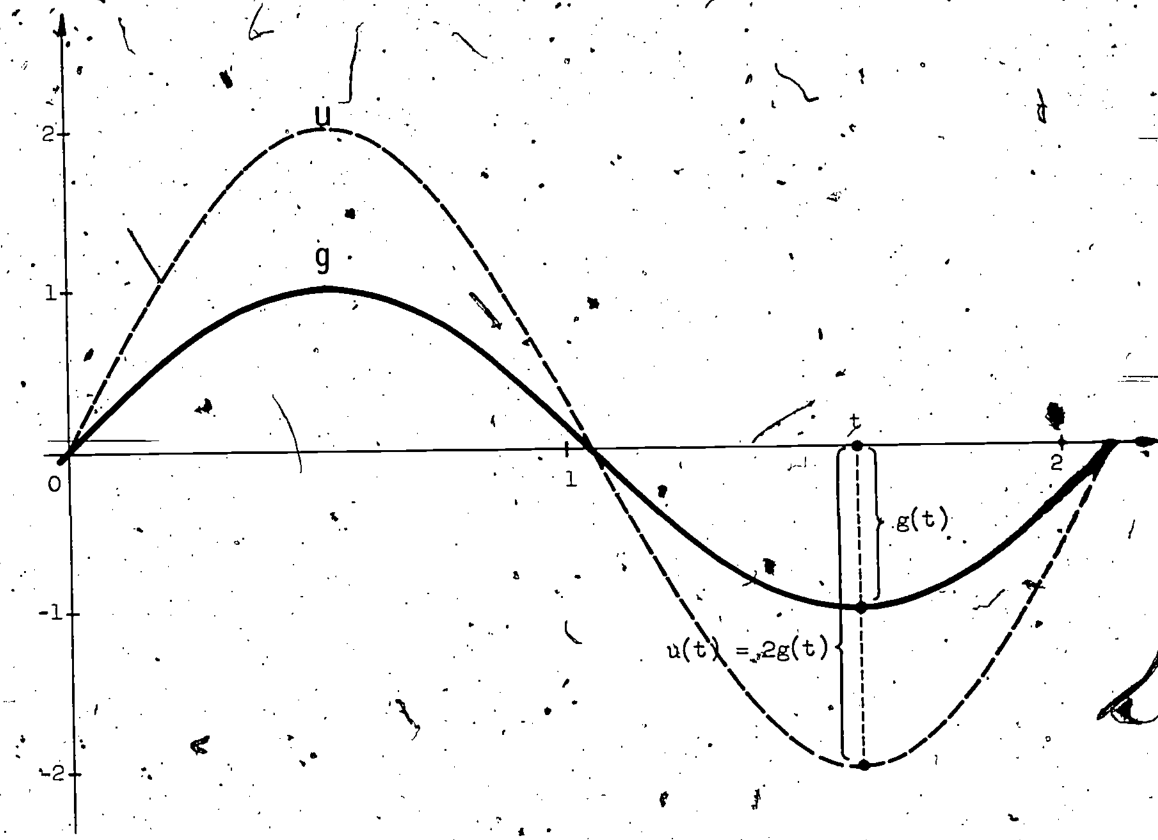


Figure 8-3b

$u = 2g$, where $u : x \to 2 \sin 3x$, $g : x \to \sin 3x$.

The basic rules for derivatives of sums and multiples are easily obtained and simply stated:

(1) $\boxed{\text{if } f = u + v, \text{ then } f' = u' + v';}$



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and, for any constant $a$,

(2) $$\boxed{\text{if } f = ag, \text{ then } f' = ag'.}$$

For example, if $f = u + v$, where

$$u : t \to 3 \cos \pi t \quad \text{and} \quad v : t \to 4 \sin \pi t,$$

then

$$f' = u' + v';$$

that is, for each $t$,

$$f'(t) = u'(t) + v'(t)$$
$$= -3\pi \sin \pi t + 4\pi \cos \pi t.$$

We also made use of (2). For example, that $u'(t) = -3\pi \sin \pi t$ makes use of the fact that $D(3 \sin \pi t) = 3D(\sin \pi t)$.

We can use the concept of approximation along the tangent line to the graph of a function to show that (1) and (2) hold. For example, suppose $f = u + v$, where $u$ and $v$ are each differentiable at $a$. We have

(3)
$$u(x) \approx u(a) + u'(a)(x - a),$$
$$v(x) \approx v(a) + v'(a)(x - a),$$

if $x$ is close to $a$. Adding, we have

(4) $$u(x) + v(x) \approx u(a) + v(a) + (u'(a) + v'(a))(x - a).$$

Now we use the assumption that $f = u + v$ to obtain

$$f(x) \approx f(a) + (u'(a) + v'(a))(x - a).$$

For $x \neq a$ we subtract $f(a)$ from both sides and divide by $x - a$ to get

$$\frac{f(x) - f(a)}{x - a} \approx u'(a) + v'(a).$$

We take the limit or $x$ approximates $a$ to obtain

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = u'(a) + v'(a).$$

We conclude that

$$f'(a) = u'(a) + v'(a).$$

We omit the easier intuitive argument which establishes that if $f = ag$ then $f' = ag'$.



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We can combine results (1) and (2) to differentiate $f = u - v$, for we can write

$$f = u + w, \text{ where } w = (-1)v,$$

so that

$$f' = u' + w' \quad \text{and} \quad w' = (-1)v' = -v'.$$

Thus, as we should expect:

(5) $$f' = u' - v', \text{ if } f = u - v.$$

Example 8-3a. Find the derivative of $f : x \to x - \sin x$ and discuss its graph in the interval $-2\pi \leq x \leq 2\pi$.

We can let $u : x \to x$ and $v : x \to \sin x$, so that $f = u - v$. Since, from (5), $f' = u' - v'$ and

(6) $$u' : x \to 1, \quad v' : x \to \cos x,$$

we have the result

$$f'(x) = 1 - \cos x.$$

For all $x$, $f'(x) \geq 0$, since $\cos x \leq 1$. This tells us that $f$ is an increasing function for all $x$. Furthermore, the graph of $f$ has a horizontal tangent at each of the points $(-2\pi, f(-2\pi))$, $(0, f(0))$ and $(2\pi, f(2\pi))$ since

$$f'(-2\pi) = f'(0) = f'(2\pi) = 0.$$

Let us differentiate again. Since

$$f' = u' - v'$$

we can apply (5) with $f$, $u$ and $v$ replaced by $f'$, $u'$ and $v'$ to obtain the result

$$f'' = u'' - v''.$$

Making use of (6) we have

$$u'' : x \to 0 \quad \text{and} \quad v'' : x \to -\sin x$$

so that

$$f'' : x \to \sin x.$$

The function $f''$ is nonnegative in the intervals

(7) $$-2\pi \leq x \leq -\pi, \quad \text{and} \quad 0 \leq x \leq \pi$$

and nonpositive in the intervals

(8) $-\pi \leq x \leq 0$ and $\pi \leq x \leq 2\pi$.

Thus, the graph of $f$ is convex in the intervals of (7) and concave in the intervals of (8).

The graph of $f$ (Figure 8-3c) is obtained by making use of this information and plotting a few points.

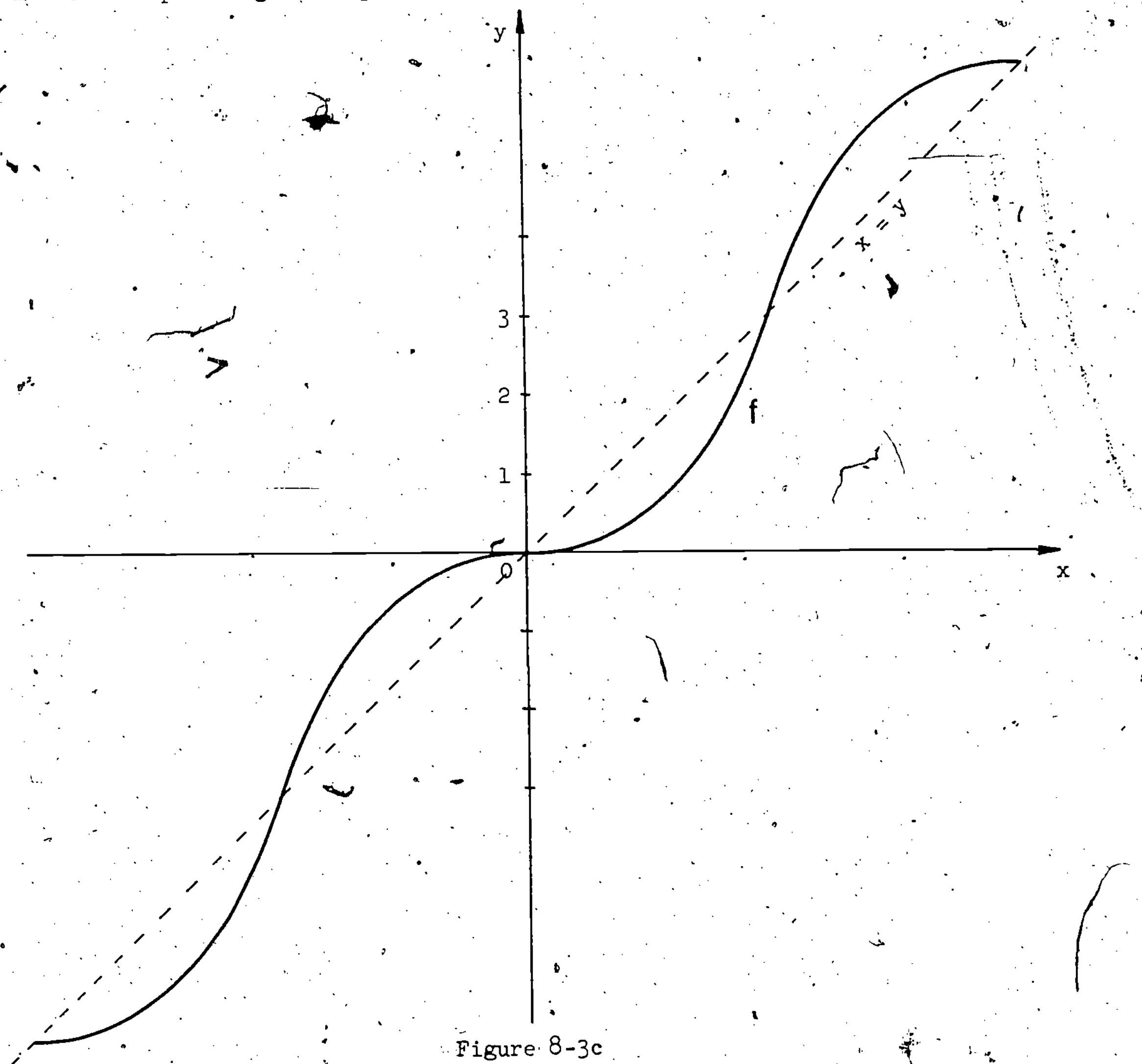



Figure 8-3c

$y = x - \sin x$

Example 8-3b. Suppose a particle moves along a horizontal line so that its distance from the origin at time $t > 0$ is given by $s = t + \frac{1}{t}$. Discuss the motion.

If $t$ is close to 0, then $s$ is nearly equal to and slightly larger than $\frac{1}{t}$, which is very large. If $t$ is very large, then $\frac{1}{t}$ is very small, so that $s$ is nearly equal to but slightly larger than $t$. Geometrically these observations mean that for $t > 0$ the graph of $s = t + \frac{1}{t}$ approaches the s-axis as $t$ approaches 0 and approaches the line given by $s = t$ as $t$ becomes large. In other words, the vertical line given by $t = 0$ is an asymptote for the graph of $s = t + \frac{1}{t}$ as $t$ approaches 0, while the line given by $s = t$ is an asymptote for the graph as $t$ grows large without bound through positive values (i.e., approaches $\infty$).

The derivative of $t \to t + \frac{1}{t}$ can be obtained using the sum formula (1). We have

$$D(t + \frac{1}{t}) = Dt + D(\frac{1}{t}) = Dt + D(t^{-1}).$$

Since $Dt = 1$ and $Dt^{-1} = -1t^{-2} = -\frac{1}{t^2}$, we conclude that

$$D(t + \frac{1}{t}) = 1 - \frac{1}{t^2}.$$

The value of the derivative $t \to s' = 1 - \frac{1}{t^2}$ is the <u>velocity</u> at time $t$. Since $s' < 0$ if $t < 1$ and $s' > 0$ if $t > 1$, the function $t \to t + \frac{1}{t}$ decreases in the interval $0 < t < 1$ and increases in the interval $t > 1$. When $t = 1$, the value of the derivative is 0 and 2 is the minimum value of $s$. This means that the particle moves toward the origin as $t$ increases from 0 to 1, is closest to the origin when $t = 1$ and then moves steadily away from the origin.

The second derivative is obtained by using the difference formula (5) and the power formula:

$$D\left(1 - \frac{1}{t^2}\right) = D1 - D(t^{-2}) = \frac{2}{t^3}.$$

Thus, the acceleration is always positive (since $t$ is positive), is very large when $t$ is close to 0, and approaches 0 as $t$ approaches $\infty$. The second derivative

$$t \to \frac{2}{t^3}$$

tells us that the graph of $s = t + \frac{1}{t}$ is convex.* The graph is given in Figure 8-3d.

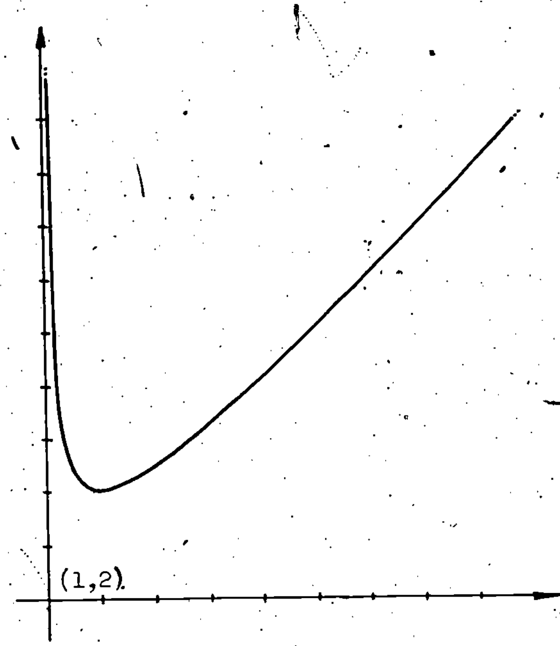



Figure 8-3d

$s = t + \frac{1}{t}$

*As we remarked earlier, in some texts the expression "concave upward" is used instead of "convex."

## Exercises 8-3

1. Find the derivatives of each of the following

(a) $y = x^{1/3} - 3x^{-2/5}$

(b) $y = x^2 + 2 \sin x$

(c) $y = (3x^2 + 1)(x^4 + 1)$

(d) $y = (1 - 2x)(\frac{1}{x^2} + \frac{1}{x})$

(e) $y = e^x + e^{2x} + \cos x$

(f) $y = \sqrt{x} - 3e^{-x}$

(g) $y = x + \log_e x^2 - 2 \log_e x$

(h) $y = x^e + e^x$

2. Sketch graphs of $f : x \to \sqrt{x} + \frac{1}{x}$, $u : x \to \sqrt{x}$ and $v : x \to \frac{1}{x}$ for $0 < x \leq 1$. What is the equation of the tangent line to each at the point where $x = \frac{1}{2}$? How are these tangent lines related?

3. (a) At what points on the graph of

$$y = \sin x - \sqrt{3} \cos x$$

is the tangent line horizontal.

(b) At what points on the graph of

$$y = 2^x - 2x$$

is the tangent line perpendicular to the line whose equation is $y = 3x + 2$?

(c) Suppose the tangent lines to the graphs of $y = 5f(x)$ and $y = 7f(x)$ are parallel and nonvertical at the point where $x = a$. Show that these tangent lines must be horizontal.

(d) Show that if $u$ and $v$ are differentiable at $x = a$ and the graphs of $f : x \to u(x) + 3v(x)$ and $g : x \to u(x) - 11v(x)$ have the same slope at the point where $x = a$ then $v$ has a horizontal tangent at $(a, v(a))$.

4. Show that if $a$ and $b$ are constants then

$$D(au + bv) = a\, Du + b\, Dv.$$

5. Analyze

(i) increase-decrease,

(ii) convexity-concavity and

(iii) asymptotes (if any)

for each of the following functions on the interval given. Sketch graphs.

(a) $f : x \to x - \cos x, \quad 0 \le x \le 2\pi$

(b) $f : x \to e^x - 2x, \quad 0 \le x \le 1$

(c) $f : t \to t^2 + \frac{3}{t}, \quad 0 < t$

(d) $f : x \to x^2 - \sqrt{2x}, \quad 0 \le x \le 2$

6. (a) Show that if $F(x) = \int_x^b f$, then $F'(x) = -f(x)$.

(b) Find $F'(x)$ if $F(x) = \int_x^0 e^{-t^2}\, dt$.

7. Show that the acceleration of a particle whose equation of motion is $s(t) = 2 \cos t + t^2$ is always nonnegative.

8. Suppose you know only that the rules of this section hold and that $Dx^n = nx^{n-1}$. Can you find the derivative of a polynomial?

9. Consider $g : x \to |x + 2| - |3 - x|$.

(a) Sketch the graph of $g$.

(b) Define $g(x)$ explicitly in terms of linear functions for all real $x$.

(c) For what values of $x$ is the derivative not defined?

10. (a) $1 + x + \frac{x^2}{2} \le e^x \le 1 + x + x^2, \quad 0 \le x \le 2$

(Hint; Put $f(x) = e^x - \left(1 + x + \frac{x^2}{2}\right)$ and find the minimum of $f$. Proceed in a similar manner for the right-hand side).

(b) Show that if $u(a) \le v(a)$ and $u'(x) \le v'(x)$ for $x \ge a$ then $u(x) \le v(x)$ for $x \ge a$. (Hint: Consider $f = v - u$.)

(c) Show that if $u(a) \le v(a)$, $u'(a) \le v'(a)$ and $u''(x) \le v''(x)$ for $x \ge a$ then $u(x) \le v(x)$ for $x \ge a$. (Hint: Use (b) twice. First show that $u'(x) \le v'(x)$ when $a \le x$.)

11. (a) Show that if $y = u$ and $y = v$ are solutions to the equation $y'' - 3y' + 6y = 0$, then so is $y = 3u + 8v$.

(b) Show that $y = e^x + e^{-x}$ and $y = e^x - e^{-x}$ are each solutions to the equation $y'' = y$. If $\alpha$ and $\beta$ are constants is

$$y = \alpha(e^x + e^{-x}) + \beta(e^x - e^{-x})$$

also a solution to $y'' = y$?

12. Suppose $u(x) = v(x) + ax + b$, where $a$ and $b$ are constants.

(a) What is $u'(x) - v'(x)$?

(b) Show that $u'' = v''$.

(c) Prove the following converse: If $u'' = v''$ then $u - v$ is a linear function. (Hint: Use the Constant Difference Theorem twice.)

13. Suppose $u$ and $v$ are continuous at $x = a$. Is $f = 2u - 3v$ also continuous at $x = a$?

14. Suppose $f = u + v$ and $f$ is differentiable and thus continuous at $x = a$. Must $u$ and $v$ also be differentiable and thus continuous at $x = a$? If so, why? If not, give an example.

## 8-4. Products

Each value of the function

$$f : x \to xe^x$$

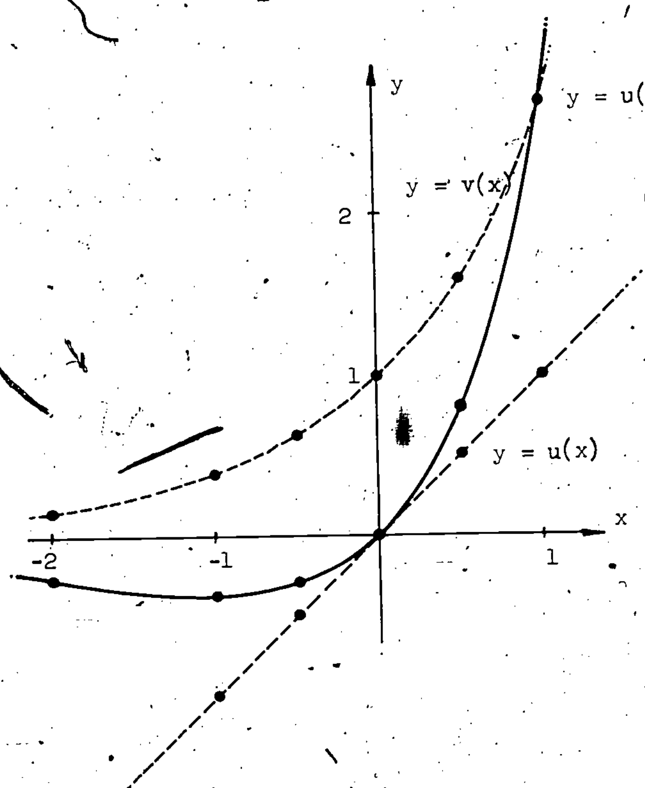
is just the product of the corresponding values of the two functions

$$u : x \to x \quad \text{and} \quad v : x \to e^x;$$

that is, for each $x$,

$$f(x) = u(x)v(x).$$

This relationship can be used to obtain the graph of $f$ from the graphs of $u$ and $v$, for the ordinate of a point on the graph of $f$ is the product of the corresponding ordinates of the graphs of $u$ and $v$. (See Figure 8-4a.)




| x | u(x) | v(x) | f(x) = u(x)v(x) |
|---|---|---|---|
| -2 | -2 | 0.14 | -0.28 |
| -1 | -1 | 0.38 | -0.38 |
| -0.5 | -0.5 | 0.61 | -0.30 |
| 0 | 0 | 1 | 0 |
| 0.5 | 0.5 | 1.6 | 0.8 |
| 1 | 1 | 2.7 | 2.7 |

Figure 8-4a

$y = xe^x$

In general, we say that the function $f$ is the <u>product</u> of the two functions $u$ and $v$ and write

$$f = uv$$

if for each $x$ the values $f(x)$, $u(x)$ and $v(x)$ are related by

(1) $$f(x) = u(x)v(x).$$

A formula for the derivative of $f = uv$ in terms of the derivatives of $u$ and $v$ can be obtained by using tangent line approximations. Suppose $u$ and $v$ are each differentiable at $x = a$ so that, if we take $x$ close to $a$, we have

$$u(x) \approx u(a) + u'(a)(x - a)$$

$$v(x) \approx v(a) + v'(a)(x - a).$$

For the product we get

$$u(x)v(x) \approx u(a)v(a) + [u(a)v'(a) + v(a)u'(a)](x - a) + u'(a)v'(a)(x - a)^2.$$

Since $f = uv$ we can rewrite this as

$$f(x) \approx f(a) + [u(a)v'(a) + v(a)u'(a)](x - a) + u'(a)v'(a)(x - a)^2$$

so that, for $x \neq a$

$$\frac{f(x) - f(a)}{x - a} \approx [u(a)v'(a) + v(a)u'(a)] + u'(a)v'(a)(x - a).$$

It follows that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = u(a)v'(a) + v(a)u'(a).$$

Thus, we obtain the <u>product rule</u>:

(2) $$f'(a) = u(a)v'(a) + v(a)u'(a).$$

This formula is sometimes written in the form

(3) $$(uv)' = uv' + vu'$$

or

(4) $$D(uv) = uDv + vDu,$$

or expressed in words:

(5)

| |
|---|
| The derivative of the product of two functions is the first times the derivative of the second plus the second times the derivative of the first. |



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For example, $f : x \to x \log_e x$ is the product of

$$u : x \to x \quad \text{and} \quad v : x \to \log_e x.$$

Since $u'(x) = 1$ and $v'(x) = \frac{1}{x}$, the product rule gives

$$f'(x) = x \cdot \frac{1}{x} + (\log_e x) \cdot 1 = 1 + \log_e x.$$

As another example we consider the function

$$f : x \to e^{3x} \sin 2x,$$

which is the product of

$$u : x \to e^{3x} \quad \text{and} \quad v : x \to \sin 2x.$$

The product rule gives

$$f'(x) = e^{3x} \cdot (2 \cos 2x) + (\sin 2x)(3e^{3x}).$$

Example 8-4a. Locate the intervals of increase and decrease, convexity and concavity for the graph of the function

$$f : x \to xe^{x}.$$

The function $f$ is the product of

$$u : x \to x \quad \text{and} \quad v : x \to e^{x},$$

so that

$$\begin{aligned} f'(x) &= u(x)v'(x) + v(x)u'(x) \\ &= x\, e^{x} + e^{x} \cdot 1 \\ &= (x + 1)e^{x}. \end{aligned}$$

This will be positive for $x > -1$ and negative for $x < -1$ so that the graph of $f$

> Falls until it reaches $(-1, f(-1)) = (-1, -\frac{1}{e})$ and rises after that point.

The function $f' : x \to (x + 1)e^{x}$ is the product of

$$u : x \to x + 1 \quad \text{and} \quad v : x \to e^{x}$$

so the product rule gives

$$f''(x) = u(x)v'(x) + v(x)u'(x)$$
$$= (x + 1)e^x + e^x \cdot 1$$
$$= (x + 2)e^x.$$

We conclude from this that the graph of $f$ is

concave for $x < -2$ and convex for $x > 2$.

An extension of our sketch (in Figure 8-4a) should reflect these conclusions. We should also note that as $x$ moves far to the left $f(x) = xe^x$ approaches 0; that is, the negative x-axis is an asymptote for the graph of $f$ as $x$ approaches $-\infty$.

<u>Example 8-4b</u>. Show that if $f : x \to e^{ax} \sin bx$, then $f''(x) - 2af'(x) + (a^2 + b^2)f(x) = 0$.

The product rule gives

$$f'(x) = e^{ax} D(\sin bx) + (\sin bx)\, D(e^{ax})$$
$$= e^{ax}(b \cos bx) + (\sin bx)(ae^{ax})$$
$$= e^{ax}[b \cos bx + a \sin bx].$$

Again we use the product rule to obtain

$$f''(x) = e^{ax}\, D[b \cos bx + a \sin bx] + [b \cos bx + a \sin bx]De^{ax}$$
$$= e^{ax}[-b^2 \sin bx + ab \cos bx] + [b \cos bx + a \sin bx]ae^{ax}$$
$$= e^{ax}[(a^2 - b^2)\sin bx + 2ab \cos bx].$$

Therefore,

$$f''(x) - 2af'(x) + (a^2 + b^2)f(x) = e^{ax}[(a^2 - b^2)\sin bx + 2ab \cos bx]$$
$$- 2ae^{ax}[b \cos bx + a \sin bx]$$
$$+ e^{ax}[(a^2 + b^2)\sin bx]$$
$$= e^{ax}[(a^2 - b^2 - 2a^2 + a^2 + b^2)\sin bx]$$
$$+ e^{ax}[(2ab - 2ab)\cos bx]$$
$$= 0.$$

Example 8-4c. Suppose $f$ is a polynomial function and that $a$ is a zero of $f$. Show that the multiplicity of $a$ is greater than 1 if and only if $a$ is a zero of $f'$.

If the multiplicity of $a$ exceeds 1 then $(x - a)^2$ is a factor of $f(x)$; that is

$$f(x) = (x - a)^2 q(x),$$

where $q$ is a polynomial function. Applying the product rule we have

$$f'(x) = (x - a)^2 q'(x) + q(x) \cdot 2(x - a)$$

so that indeed

$$f'(a) = 0.$$

If the multiplicity of $a$ is 1, then

$$f(x) = (x - a)g(x), \quad \text{where} \quad g(a) \neq 0.$$

The product rule gives

$$f'(x) = (x - a)g'(x) + 1 \cdot g(x)$$

so that

$$f'(a) = g(a) \neq 0.$$

In other words, if the multiplicity of $a$ is 1 then $a$ cannot also be a zero of $f'$.

## Exercises 8-4

1. Let $y_1 = a_1 + m_1(x - a)$ be the equation of the tangent line to the graph of $u : x \to x^2$ at $(a,a^2)$ and $y_2 = a_2 + m_2(x - a)$, the equation of the tangent line to the graph of $v : x \to x^3$ at $(a,a^3)$.

   (a) Find $a_1$, $m_1$, $a_2$, $m_2$.

   (b) Form the product of the expressions for $y_1$ and $y_2$, and omit the term involving $(x - a)^2$. The resulting expression is linear in $(x - a)$ and hence defines a line. Show that this line is the tangent line to $uv = f : x \to x^5$ at the point $(a,a^5)$.

2. Find the derivative of each of the following functions $f$, where $f(x)$ equals

   (a) $x(2x - 3)$

   (b) $(4x - 2)(4 - 2x)$

   (c) $(x^2 + x + 1)(x^2 - x + 1)$

   (d) $\sqrt{x}\,(ax + b)^3$

   (e) $\frac{1}{x} \cdot \sqrt{x}$

   (f) $\frac{1}{x} \cdot (5x + 2)$

   (g) $x\,e^x$

   (h) $x^{7/2}$, $x > 0$

   (i) $3x^4 - \frac{1}{\sqrt{x}}$

   (j) $3x^2(x^2 - 5)$

   (k) $\sqrt{x}\cos 2x$

   (ℓ) $e^{3x}\sin(x + 1)$

   (m) $x^2 \log_e x$

   (n) $(x - 1)^{1/2}\,e^{-x}$

   (o) $x\int_0^x e^{-t^2}\,dt$

   (p) $e^x\int_1^x \frac{\sin t}{t}\,dt$

   (q) $e^x \sin x$

   (r) $(\log_e x)(4x^2 + 2x)(\cos 2x)$

   (s) $2\sin x \cos x$

   (t) $x\,e^x \log_e(2x + 1)(\sin x)$

   (u) $x^2\,2^x$

   (v) $x \log_2(3x + 1)$

   (w) $x^e\,e^x$

3. Evaluate

(a) $D(3x^2 + 5x - 1)^2$

(b) $D(3 - 5x)^3$

(c) $D(3 - 5x)^4$

(d) $D(x(\sqrt{x} - 1)^2)$

(e) $D(x + \frac{1}{x})^2$

(f) $D\left(\frac{x^{3/2}}{3} - \frac{x^{1/2}}{2} + x^{-1/2}\right)$

(g) $D\left(4\sqrt{x^3} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

(h) $D(e^x \sin(1 - 2x))$

(i) $D(\sqrt{x} \log_e x)$

(j) $D(x^\pi \pi^x)$

(k) $D(x^2 \cos x)$

(ℓ) $D(\sin x \log_e x)$

(m) $D\left(\frac{\log_e x}{x}\right)$

4. (a) Suppose $f(x) = [u(x)]^2$. Show that $f'(x) = 2u(x)u'(x)$. (Hint: Use the Product Rule.)

(b) Show that $D[u(x)]^3 = 3[u(x)]^2u'(x)$.

(c) Show that $D[u(x)]^4 = 4[u(x)]^3u'(x)$.

(d) Make a conjecture about $D[u(x)]^n$.

5. Use the results of Number 4 to find $y'$ if

(a) $y = \sin^2 x$

(b) $y = \cos^3 (4x)$

(c) $y = (\log_e x)^2$

(d) $y = (e^x)^4$

(e) $y = (x^2 + 1)^2$

(f) $y = \sin^3 (2x - 1)$

(g) $y = \left(\int_1^x \sin t^2 \, dt\right)^4$

6. Combine the method of Number 4 with the Product Rule to find $\frac{dy}{dx}$ if

(a) $y = x^2(x^2 + 1)^2$

(b) $y = (x + 1)^3(x^2 - x + 1)$

(c) $y = (ax^2 + bx + c)(dx^2 + ex + f)$

(d) $y = (\cos^2 x)\sin 2x$

(e) $y = e^x \sin^2 (ax + b)$

(f) $y = \left(x\int_0^x e^{t^2}\,dt\right)^2$

(g) $y = x^3[\log_e(x+1)]^3$

7. For each of the following functions, find the intervals of increase (or decrease) and convexity (or concavity). Sketch graphs over the intervals indicated.

(a) $y = x\log_e x,\ 0 < x \le e$ (c) $y = \sin^3 x,\ 0 \le x \le 2\pi$

(b) $y = \frac{1}{x}\log_e x,\ 0 < x \le e^2$ (d) $y = x^2\log_e x,\ 0 < x \le 8$

8. Show that each of the following is an increasing function

(a) $x \to \sqrt{x}\,e^x,\ x > 0$

(b) $x \to \frac{e^x}{x},\ x \ge 1$

(c) $x \to \frac{e^x}{x^\alpha},\ x \ge \alpha > 0$

(d) $x \to x\sin x,\ 0 \le x \le \frac{\pi}{2}$

9. Show that if $f(x) = (x-a)^2 g(x)$ where $g$ is differentiable and $g(a) \ne 0$, then $f'(a) = 0$.

10. Show that if $a$ is a zero of the polynomial function $f$ of multiplicity greater than 2 then $f''(a) = 0$. If $f''(a) = 0$ must it be true that $a$ is a zero of $f$ of multiplicity greater than 2?

11. (a) Show that if $y = e^{ax}\cos bx$ then $y'' - 2ay' + (a^2+b^2)y = 0$.

(b) Show that if $y = x^2e^x + 2xe^x$ then $y''' - 3y'' + 3y' - y = 0$.

12. (a) Show that

$$(uv)'' = uv'' + 2u'v' + u''v.$$

(b) Use (a) to find the second derivative of

$$f : x \to x^2\cos x.$$

(c) What is $(uv)'''$?

(d) Does (c) lead you to a conjecture about the nth derivative of $uv$.

## 8-5. Composite Functions

The function $f : x \rightarrow \sqrt{x^2 + 1}$ is not a polynomial, circular, power, exponential or logarithm function; nor is it a sum or product of such functions. Our previous discussions and formulas do not cover even a simple a function of this type. The verbal description of $f$ can give a clue as to how to treat such a function. Verbally, the rule for $f$ is

(1) "the square root of the quantity $x$ squared plus one."

In other words, first calculate the quantity $x^2 + 1$, and then take the square root of the result. The operation defined by $f$ is composed of two simpler operations, finding $x^2 + 1$ and taking square roots. In this and the next two sections we discuss functions which are compositions of other functions.

The statement (1) can be translated into a symbolic form which will display the fact that $f : x \rightarrow \sqrt{x^2 + 1}$ is composed of the two operations, $x \rightarrow x^2 + 1$ and taking square roots. Let $g(x) = u = x^2 + 1$ and $h(u) = \sqrt{u}$, so that

$$f(x) = h(g(x)).$$

To evaluate $f(x)$ we first evaluate $g(x)$, then evaluate $h(g(x))$.

For example, if $x = 3$, then

$$u = g(3) = 3^2 + 1 = 10$$

and

$$f(3) = h(g(3)) = h(10) = \sqrt{10}.$$

In general, we say that a function $f$ is a <u>composition</u> of the two functions $h$ and $g$, if whenever $f(x)$ is defined, so are $g(x)$ and $h(g(x))$; and then

$$f(x) = h(g(x)).$$

The idea of composition has been previously used implicitly. For example, the function

$$f : x \rightarrow \sin (2x + 3)$$

is a composition of the functions $h : u \rightarrow \sin u$ and $g : x \rightarrow u = 2x + 3$; that is,

$$f(x) = h(g(x)).$$

Also, use has been made of the fact that the general exponential function

$f : x \to a^x$ is a composite function since we can write $a = e^{\alpha}$. If $h : u \to e^u$ and $g : x \to \alpha x = u$, then

$$f : x \to a^x = h(g(x)) = e^{\alpha x}.$$

Facility with composite functions depends upon ability to write complicated expressions as composites of simpler expressions. Some examples and practice exercises are provided to help you develop skill at doing this.

Example 8-5a. Express $x \to \sin \sqrt{x}$ as the composite of simpler functions.

Since $\sin \sqrt{x}$ is usually read "the sine of the square root of $x$," the function $x \to \sin \sqrt{x}$ is a composite of the sine and the square root functions. If we let $u = g(x) = \sqrt{x}$ and $h(u) = \sin u$, we then have

$$\sin \sqrt{x} = h(g(x)).$$

Example 8-5b. Express $x \to x^{2/3}$ as the composite of two simpler functions in two ways.

The expression $x^{2/3}$ can be read as

(2) "the cube root of the square of $x$"

or

(3) "the square of the cube root of $x$."

Put $g(x) = x^2 = u$ and $h(x) = \sqrt[3]{x} = v$. In symbolic form (2) becomes

(4) $$x^{2/3} = h(u) = h(g(x)),$$

while (3) becomes

(5) $$x^{2/3} = g(v) = g(h(x)).$$

In other words, in this case, it doesn't matter whether we square first and then take the cube root, or take the cube root and then square. It should, however, be noted that generally the order of composition is important. In the Example 8-5a we had

$$\sin \sqrt{x} = h(g(x)), \quad \text{where} \quad g(x) = \sqrt{x} = u \quad \text{and} \quad h(u) = \sin u.$$

Reversing the order of composition, we have

$$g(h(x)) = \sqrt{\sin x},$$

which is certainly not the same as $\sin \sqrt{x}$.

It should be observed that there are other ways of expressing $x \to x^{2/3}$ as a composite. For example,

$$x^{2/3} = f(g(x)), \tag{6}$$

where $g(x) = (x - 1)^{1/3}$ and $f(x) = (x^3 + 1)^{2/3}$, since

$$f(g(x)) = [(x - 1) + 1]^{2/3} = x^{2/3}.$$

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## Exercises 8-5

1. Express each of the following as a composite of two functions which are polynomials, exponentials, logarithms, power, sine or cosine functions.

   (a) $x \to \sqrt{1 - x^2}$

   (b) $x \to e^{x^2}$

   (c) $x \to \cos(x^3 - 3x)$

   (d) $x \to \dfrac{1}{1 + x^2}$

   (e) $x \to \log_e \sqrt{x^2 + 1}$

   (f) $x \to (2 - 3x^2)^{100}$

   (g) $x \to (2x^2 - 2x + 1)^{-1/2}$

   (h) $x \to \log_e (\sin x)^2$

   (i) $x \to e^{\cos^2 x}$

   (j) $x \to 3e^{2 \sin x}$

   (k) $x \to 2^{(x+1)^2}$

2. Express each of the following as the composition of three or more simpler functions.

   (a) $x \to \log_e |8x^2 + 5x + 2|$

   (b) $x \to \sqrt{1 + \cos x}$

   (c) $x \to \cos(\sin(\cos x))$

   (d) $x \to (x + 1)^{3/5}$

   (e) $x \to \sqrt{1 - (\log_e x)^2}$

   (f) $x \to \dfrac{1}{1 + e^{2x}}$

3. Express $x \to |x|$ as a composite of the function $x \to x^2$ and some other function.

4. (a) Show that the composite of two linear functions is linear.

   (b) Exhibit the composite of two quadratic functions. What is the degree of this composition?

   (c) Is the composite of two polynomial functions a polynomial function? If so, what is its degree?

5. (a) If $u : x \to x$ and $f : x \to u(u(x))$ what is $f(3)$?

   (b) Suppose $u : x \to \dfrac{1}{x}$. Find an expression for $f$, the function defined by $f(x) = u(u(x))$.

6. (a) Show that composition of power functions is a commutative operation, that is, if $u : x \to x^a$ and $v : x \to x^b$ then $u(v(x)) = v(u(x))$.

(b) Is the result of part (a) true for $u : x \to \cos x$ and $v : x \to \sin x$.

(c) Is the result of part (a) true for exponential functions $u : x \to a^x$ and $v : x \to b^x$? $(a, b > 1)$

(d) Is the result of part (a) true for $u : x \to e^x$ and $v : x \to \log_e x$?

7. Express the following as a composition of two functions

(a) $x \to \int_{-2}^{x^2} t^{2/3}\, dt$

(b) $x \to \int_{\sin x}^{1} e^t\, dt$

(c) $x \to \int_{0}^{x^2} e^{-t^2}\, dt$

8. What is the domain of the function

$$x \to \sqrt{1 - (\log_e x)^2}?$$

## 8-6. The Chain Rule

Suppose we can express $f$ as a composite of two functions $g$ and $h$ whose derivatives are known. The derivative of $f$ can then be expressed in terms of the derivatives of $g$ and $h$.

(1)
$$\boxed{\begin{array}{ll} \text{If} & f(x) = g(h(x)) \\ \text{then} & f'(x) = g'(h(x))h'(x). \end{array}}$$

This result is usually known as the chain rule. We have used the chain rule for particular functions in the case for $h$ a linear function. For example, suppose

$$f : x \to \sin(ax + b)$$

so that

$$f(x) = g(h(x))$$

where $g : u \to \sin u$ and $h : x \to ax + b = u$. Since $g' : u \to \cos u$ and $h' : x \to a$, the chain rule (1) gives

$$\begin{aligned} f'(x) &= g'(h(x))h'(x) \\ &= [\cos(ax + b)]a \\ &= a \cos(ax + b), \end{aligned}$$

which agrees with our previous result.

The general result for linear substitution is as follows. Suppose $f(x) = g(ax + b)$. Let $h(x) = ax + b$. The chain rule (1) gives

$$\begin{aligned} f'(x) &= g'(ax + b)h'(x) \\ &= ag'(ax + b) \end{aligned}$$

which shows that replacement of $x$ by $ax + b$ in a general function $g$ multiplies the derivative by $a$.

A special case of the chain rule was used in Section 6-7 to differentiate a power function. Suppose $f : x \to x^a$. We can write $f(x) = g(h(x))$, where $g : u \to e^u$ and $h : x \to a \log_e x = u$. The derivatives of $g$ and $h$ are given by

$$g' : u \to e^u \quad \text{and} \quad h' : x \to \frac{a}{x}.$$

The chain rule gives

$$f'(x) = g'(h(x))h'(x) = g'(a \log_e x) \cdot \frac{a}{x}$$

$$= e^{a \log_e x} \cdot \frac{a}{x}$$

$$= x^a \cdot \frac{a}{x}$$

$$= a\, x^{a-1}.$$

Let us now prove the chain rule by generalizing the tangent approximation arguments used in Section 6-7. Suppose that $f$ is related to $g$ and $h$ by composition

$$f(x) = g(h(x)).$$

If $h$ is differentiable at $a$ and $g$ is differentiable at $h(a)$, we can write

(2) $$h(x) \approx h(a) + h'(a)(x - a), \quad \text{for } x \text{ close to } a,$$

and

(3) $$g(u) \approx g(h(a)) + g'(h(a))(u - h(a)), \quad \text{for } u \text{ close to } h(a).$$

In particular, if $x$ is close to $a$ the second term of (2) is close to zero so that $h(x)$ is close to $h(a)$.

We can replace $u$ by $h(x)$ in (3) to obtain

$$g(h(x)) \approx g(h(a)) + g'(h(a))(h(x) - h(a)),$$

which will hold if $x$ is close to $a$ (so that $h(x) \approx h(a)$). We now use (2) again, this time to replace $h(x) - h(a)$ by $h'(a)(x - a)$. Thus, we have

(4) $$g(h(x)) \approx g(h(a)) + g'(h(a))h'(a)(x - a).$$

By assumption $f(x) = g(h(x))$ so we can rewrite (4) as

$$f(x) \approx f(a) + g'(h(a))h'(a)(x - a)$$

then subtract $f(a)$ and divide by $x - a$ to obtain

$$\frac{f(x) - f(a)}{x - a} \approx g'(h(a))h'(a).$$

Therefore,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = g'(h(a))h'(a),$$

which establishes the chain rule:

$$f'(a) = g'(h(a))h'(a).$$

The Leibniz notation $\frac{dy}{dx}$ for the derivative provides a convenient mnemonic device for the chain rule. Suppose $y = g(h(x))$; that is

$$y = g(u) \quad \text{where} \quad u = h(x).$$

We can then write $g'(u) = \frac{dy}{du}$, $h'(x) = \frac{du}{dx}$. The chain rule can then be expressed

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 8-6a. Find the derivative of $x \to \sqrt{x^2 + 1}$.

Put $g(x) = x^2 + 1 = u$ and $h(u) = \sqrt{u}$ so that

$$\sqrt{x^2 + 1} = h(g(x)).$$

Recall that $h'(u) = \frac{1}{2\sqrt{u}}$ and that $g'(x) = 2x$. The chain rule tells us that

$$\begin{aligned} D(\sqrt{x^2 + 1}) &= h'(g(x))g'(x) \\ &= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2 + 1}}. \end{aligned}$$

Example 8-6b. Find $D(e^{\sin x})$.

To express $x \to e^{\sin x}$ as a composite of functions with known derivatives, put

$$u = h(x) = \sin x, \quad g(u) = e^u$$

so that

$$e^{\sin x} = g(h(x))$$

and

$$h'(x) = \cos x, \quad g'(u) = e^u.$$

The chain rule gives

$$\begin{aligned} D(e^{\sin x}) &= g'(h(x)) \cdot h'(x) \\ &= e^{\sin x} \cdot \cos x. \end{aligned}$$

Example 8-6c. For $f : x \to (x^2 + x + 1)^{100}$, find $f'(-1)$.

We could expand and then differentiate. Obviously, such a procedure would be quite lengthy. Instead we let $h(x) = x^2 + x + 1 = u$ and $g(u) = u^{100}$, so that

$$f(x) = g(h(x)).$$

We have $h'(x) = 2x + 1$, $g'(u) = 100u^{99}$, so that (by the chain rule)

$$f'(x) = 100(x^2 + x + 1)^{99} \cdot (2x + 1).$$

Thus $f'(-1) = -100$.

Example 8-6d. Use the chain rule to show that $D(\log_e (\cos x)) = -\tan x$, thus verifying integration formula 12 of the Table of Integrals:

$$\int \tan x \, dx = -\log_e (\cos x).$$

Put $h(x) = u = \cos x$, $g(u) = \log_e u$, so that

$$\log_e (\cos x) = g(h(x)),$$

and hence

$$\begin{aligned} D(\log_e (\cos x)) &= g'(h(x))h'(x) \\ &= \frac{1}{h(x)} \cdot (-\sin x) \\ &= -\frac{\sin x}{\cos x} \\ &= -\tan x. \end{aligned}$$

Example 8-6e. Find $\frac{dy}{dx}$ if $y = \frac{1}{1 + \sin (x^2)}$.

We let $u = x^2$ and $v = 1 + \sin u$, whence $y = \frac{1}{1 + \sin u} = \frac{1}{v}$. We obtain $\frac{du}{dx} = 2x$, $\frac{dv}{du} = \cos u$, and $\frac{dy}{dv} = -\frac{1}{v^2}$. We have $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$. Therefore,

$$\frac{dy}{dx} = \left(-\frac{1}{v^2}\right)(\cos u)(2x)$$

$$= \left(-\frac{1}{(1+\sin u)^2}\right)(\cos u)(2x)$$

$$= -\frac{2x\cos(x^2)}{(1+\sin(x^2))^2}.$$

<u>Example 8-6f</u>. Analyze the graph of $y = xe^{-x^2}$.

The product rule gives

$$y' = D\left(xe^{-x^2}\right) = xD\left(e^{-x^2}\right) + e^{-x^2}\,Dx$$

$$= xD\left(e^{-x^2}\right) + e^{-x^2}.$$

Applying the chain rule to $e^{-x^2}$, we get

(5) $$D\,e^{-x^2} = e^{-x^2}(-2x) = -2xe^{-x^2},$$

so that

(6) $$y' = -2x^2 e^{-x^2} + e^{-x^2}$$
$$= (-2x^2 + 1)e^{-x^2}.$$

The derivative $y'$ will have the same sign as

$$-2x^2 + 1 = -2\left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right).$$

(7) The graph falls until it reaches $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}e^{-1/2}\right)$, then rises to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{-1/2}\right)$, then falls.

To analyze convexity we find the second derivative. Apply the product rule to (6) to obtain

$$y'' = D\left[(-2x^2 + 1)e^{-x^2}\right]$$

$$= (-2x^2 + 1)D\left(e^{-x^2}\right) + e^{-x^2}\,D(-2x^2 + 1).$$

Now use (5) and the fact that $D(-2x^2 + 1) = -4x$ to obtain

$$y'' = (-2x^2 + 1)\left(-2xe^{-x^2}\right) + e^{-x^2}(-4x)$$
$$= (4x^3 - 6x)e^{-x^2}.$$

The second derivative $y''$ has the same sign as

$$4x^3 - 6x = 4x\left(x - \sqrt{\tfrac{3}{2}}\right)\left(x + \sqrt{\tfrac{3}{2}}\right).$$

(8) The graph is convex for $-\sqrt{\frac{3}{2}} < x < 0$ or $\sqrt{\frac{3}{2}} < x$, and concave for $x < -\sqrt{\frac{3}{2}}$ or $0 < x < \sqrt{\frac{3}{2}}$.

We can show that if $|x|$ is large then $xe^{-x^2} \approx 0$, so that the x-axis is an asymptote. We know that $|x|e^{-|x|} \approx 0$ if $|x|$ is large. Then noting that

$$-x^2 \leq -|x| \text{ if } |x| \geq 1, \text{ we have } e^{-x^2} \leq e^{-|x|},$$

since $x \to e^x$ is an increasing function. Therefore, we have:

$$|xe^{-x^2}| \leq |x|\, e^{-|x|} \approx 0, \text{ if } |x| \text{ is large.}$$

See Figure 8-6a for the graph of $y = xe^{-x^2}$.

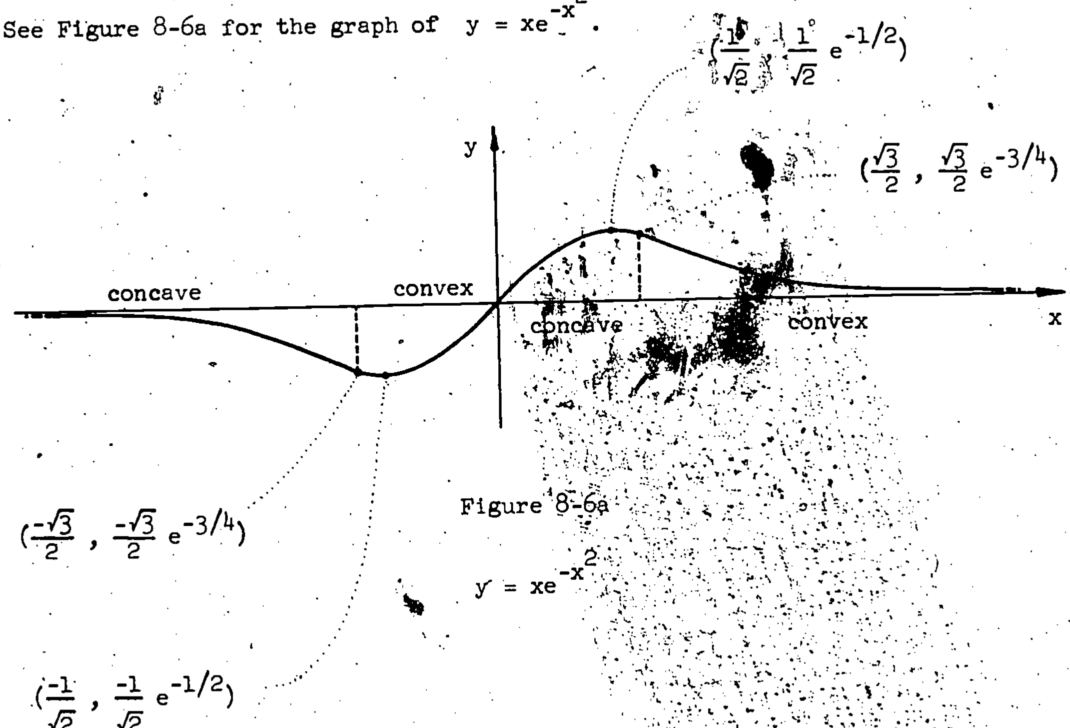


Figure 8-6a

$y = xe^{-x^2}$

## Exercises 8-6

1. Find the derivatives of each of the following by making an appropriate substitution:

(a) $x \to \sqrt{1 - x^2}$

(b) $x \to e^{x^2}$

(c) $x \to \cos(x^3 - 3x)$

(d) $x \to \dfrac{1}{1 + x^2}$

(e) $x \to \log_e \sqrt{x^2 + 1}$

(f) $x \to (2 - 3x^2)^{100}$

(g) $x \to (2x^2 - 2x + 1)^{-1/2}$

(h) $x \to \log_e (\sin x)^2$

(i) $x \to e^{\cos^2 x}$

(j) $x \to 3e^{2 \sin x}$

(k) $x \to 2^{(x+1)^2}$

2. Find the derivatives of each of the following functions by making one <u>or more</u> substitutions.

(a) $x \to \sqrt{1 + \cos x}$

(b) $x \to \sqrt{1 - (\log_e x)^2}$

(c) $x \to \dfrac{1}{1 + e^{2x}}$

(d) $x \to \cos(\sin(\cos x))$

3. Find the derivatives of each of the following functions by using the chain rule, along with the sum and product rules.

(a) $x \to (x^2 + 1)^{1/2} + (x^2 + 1)^{-1/2}$

(b) $x \to \dfrac{\sqrt{x^2 - a^2}}{\sqrt{x^2 + a^2}} = [x^2 - a^2]^{1/2}[x^2 + a^2]^{-1/2}$

(c) $x \to x(2x^2 + 2x + 1)^{-1/2}$

(d) $x \to x^2 \sqrt{\sin x}$

(e) $x \to \sin^2(e^x)$

(f) $x \to e^{x \sin x}$

(g) $x \to \log_e (\sqrt{x} \cos x)$

(h) $x \to e^{\log_e x + \cos x}$

(i) $x \to \sin x \cos x \log_e \sqrt{x}$

(j) $x \to \cos^2 (\log_e x) + \sin^2 (\log_e x)$

4. (a) Show that if $f(x) = \int_a^{g(x)} h(t)dt$ then $f'(x) = h(g(x))g'(x)$.

(b) Deduce from (a) that if $F(x) = \int_{x^2}^{b}$ then $F'(x) = -2x\, f(x^2)$.

(c) Verify (a) by evaluating $\int_{-\pi}^{x^2} \sin t\, dt$ and then calculating its derivative.

5. Find the derivatives of each of the following functions

(a) $x \to \int_{-2}^{x^2} t^{2/3}\, dt$

(b) $x \to \int_{\sin x}^{1} e^t\, dt$

(c) $x \to \int_0^{x^2} e^{-t^2}\, dt$

6. (a) Find the derivative of $f : x \to x^x$, $x > 0$. (Hint: Write $x^x = e^{x \log_e x}$.)

(b) What is the minimum value of $f$.

(c) Find the second derivative of $f$ and show that the graph of $-f$ is convex.

7. Determine intervals of increase-decrease and convexity-concavity. Then sketch a graph.

(a) $f : x \to \dfrac{x}{x^2 - 1} = [x(x^2 - 1)^{-1}]$

(b) $f : x \to e^{1/x}$

(c) $f : x \to \log_e \dfrac{1 + x^2}{1 - x^2}$, $-1 < x < 1$

8. Find the tangent line to the curve at the point indicated:

(a) $y = xe^{-x^2}$, $x = 0$

(b) $y = e^{-11x^2}$, $x = 1$

(c) $y = \sin(\pi - x^2)^{3/2}$, $x = \sqrt{\pi}$

(d) $y = \log_e (1 - x^2)$, $x = \frac{1}{2}$

(e) $y = e^{e^x}$, $x < 0$

(f) $y = (e^x)^\pi$, $x = e$

9. If $f(x) = (Ax + B)\sin x + (Cx + D)\cos x$, determine the value of constants $A, B, C, D$ such that for all $x$, $f'(x) = x \sin x$.

10. If $g(x) = (Ax^2 + Bx + C)\sin x + (Dx^2 + Ex + F)\cos x$, determine the value of constants $A, B, C, D, E, F$ such that for all $x$, $g'(x) = x^2 \cos x$.

The notation $\left.\frac{dy}{dx}\right|_{x=a}$ is sometimes used for the value of the derivative of $y$ at $x = a$. This notation is used in the following problems.

13. Let $y = \sin x$ and $x = t^2 + \frac{1}{t}$. Find $\left.\frac{dy}{dt}\right|_{t=1}$ and $\left.\frac{dy}{dx}\right|_{x=1}$.

14. Let $y = f(x)$ and $x = h(t)$. Express $\left.\frac{dy}{dt}\right|_{t=t_0}$ in terms of $t_0$.

15. Let $y = f(x)$, $x = h(t)$, $x_0 = h(t_0)$. Show that

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}}.$$

16. Find the following:

(a) $D \sin x\big|_{x=0} + D \sin x\big|_{x=\pi/4}$

(b) $D(x^2 + \sin a \sin x)\big|_{x=5\pi/3}$

(c) $\frac{d}{dx}(x^2 - a^2)\Big|_{x=a}$ $[\frac{d}{dx} = D]$

(d) $D(f(a)\sin x + f(x)\sin a + f(x)\sin x)\Big|_{x=a}$

17. Let $y = f(t)$, $w = g(t)$, $t = h(x)$, $z = \frac{y}{w}$.

(a) Using Leibnizian notation, find $\frac{dz}{dx}$ in terms of $\frac{dy}{dt}$, $\frac{dw}{dt}$, and $\frac{dt}{dx}$.

(b) Using (a) express $\frac{dz}{dx}\Big|_{x=x_0}$ in terms of $f'$, $g'$, and $h'$.

## 8-7. The General Power and Reciprocal Rules

A special case of the chain rule, known as the general power rule, occurs so frequently that it is worth discussing separately.

Suppose the values of the function $f$ can be expressed as

$$f(x) = (h(x))^a$$

where $a$ is a fixed real number and $h$ is a function. In other words,

$$f(x) = g(h(x)), \text{ where } h : x \to h(x) = u \text{ and } g : u \to u^a.$$

If $h$ is differentiable at $x$ and if $a(h(x))^{a-1}$ is defined (that is, $g$ is differentiable at $u$), then the chain rule gives

$$f'(x) = g'(h(x))h'(x).$$

Since $g' : u \to au^{a-1}$, we can write this as

(1) $$f'(x) = a(h(x))^{a-1}h'(x).$$

This is the general power rule. Using the $D$ notation it can be expressed as

(2) $$\boxed{Du^a = au^{a-1} Du.}$$

For example, suppose

$$f : x \to \sin^3 x$$

that is

$$f(x) = (h(x))^3, \text{ where } h : x \to \sin x.$$

The power formula (1) gives

$$f'(x) = 3(h(x))^2h'(x)$$
$$= 3 \sin^2 x \cos x.$$

As an example of the case when the exponent $a$ is not an integer, consider the function

$$f : x \to \sqrt{x^2 + 1} = (x^2 + 1)^{1/2}.$$

The power formula gives

$$f'(x) = D[(x^2 + 1)^{1/2}] = \frac{1}{2}(x^2 + 1)^{-1/2} D(x^2 + 1)$$

$$= \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x$$

$$= \frac{-x}{\sqrt{x^2 + 1}}.$$

As an example of the case when $a$ is a negative integer, consider the function

$$f : x \to \frac{1}{(\log_e x)^2} = (\log_e x)^{-2}.$$

The power formula then gives

$$f'(x) = D[(\log_e x)^{-2}] = -2(\log_e x)^{-1} D(\log_e x)$$

$$= -2(\log_e x)^{-1} \cdot \frac{1}{x}$$

$$= \frac{-2}{x \log_e x}.$$

The case when $a = -1$ is so important that it deserves special consideration. Suppose the values of the function $f$ can be expressed as

$$f(x) = \frac{1}{g(x)},$$

where $g$ is a function. We can then write

$$f(x) = (g(x))^{-1}$$

and apply the power formula to obtain

$$f'(x) = D[(g(x))^{-1}] = -(g(x))^{-2} D(g(x))$$

$$= -(g(x))^{-2} g'(x)$$

$$= \frac{-g'(x)}{(g(x))^2}.$$

This will hold, provided $g(x) \neq 0$ and $g$ is differentiable at $x$. In words, the derivative of the reciprocal of a function is the negative of the derivative of the function times the reciprocal of the square of the function. Using $D$ notation, we summarize:

$$\boxed{D\left(\frac{1}{g(x)}\right) = \frac{-D\,g(x)}{[g(x)]^2}}$$

We shall refer to this as the <u>reciprocal rule</u>.

For example, suppose

$$f : x \to \frac{1}{x^2 + 2}.$$

The reciprocal rule gives

$$f'(x) = D\left(\frac{1}{x^2 + 2}\right) = -\frac{D(x^2 + 2)}{(x^2 + 2)^2}$$

$$= \frac{-2x}{(x^2 + 2)^2}.$$

A differentiation formula for the secant function can be found using the reciprocal rule. The secant function is defined by

$$\sec: x \to \frac{1}{\cos x}.$$

The expression $\frac{1}{\cos x}$ is not defined if $\cos x = 0$, that is, if $x$ is an odd multiple of $\frac{\pi}{2}$. Thus the secant function is defined only for those values $x$ which are not odd multiples of $\frac{\pi}{2}$. The reciprocal rule gives the derivative

$$D(\sec x) = D\left(\frac{1}{\cos x}\right) = -\frac{D(\cos x)}{\cos^2 x}$$

$$= -\frac{(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}.$$

Since $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ this result is usually expressed as

(4) $$\boxed{D(\sec x) = \sec x \tan x.}$$

A corresponding formula for the cosecant function is given in the exercises.

## Exercises 8-7

1. Use the power formula to find the derivative of each of the following:

(a) $x \to \sqrt{\sin x}$

(b) $x \to (\log_e x)^{\pi}$

(c) $s \to (s^3 + 3s)^{25}$

(d) $t \to (e^t)^{-10}$

(e) $x \to \dfrac{1}{\sqrt[3]{(1 - x)^2}}$

(f) $t \to (1 + \frac{1}{t})^{4/3}$

(g) $v \to \cos^{10} 2v$

(h) $x \to (\int_0^x \sqrt{t^3 + 1}\, dt)^{1/2}$

2. Use the reciprocal rule to find $\frac{dy}{dx}$ if

(a) $y = \dfrac{1}{1 - x^2}$

(b) $y = \left(\dfrac{1}{1 - x^2}\right)^5$

(c) $y = \dfrac{1}{1 + e^{2x}}$

(d) $y = (1 + \log_e x)^{-1}$

(e) $y = \dfrac{1}{\sqrt{x + \frac{1}{x}}}$

(f) $y = (\sin x + \cos x)^{-1}$

3. Find an equation for the tangent line to each of the following curves at the indicated point.

(a) $y = \sin^{3/2} (2x), \quad x = \frac{\pi}{6}$

(b) $y = (\int_0^x e^{-t^2} dt)^2, \quad x = 0$

(c) $s = \sqrt{t + \frac{1}{t}}, \quad t = 1.$

4. For each of the following

(i) state where defined,
(ii) find the intervals of increase-decrease,
(iii) convexity-concavity,
(iv) asymptotes (if any), and
(v) sketch.

(a) $y = \dfrac{1}{1 + x^2}$

(b) $y = \sqrt{\sin x}$

5. Show that each of the following is an increasing function

(a) $x \to \frac{1}{1 - e^{x}}$ ; $x > 0$

(b) $x \to (x^3 + 3x)^{10}$, $x \geq 0$

6. Find expressions for the derivatives if

(a) $y = \sec x = \frac{1}{\cos x}$

(b) $y = \csc x = \frac{1}{\sin x}$

(c) $y = \tan x = \frac{\sin x}{\cos x} = (\sin x)(\cos x)^{-1}$

(d) $y = \cot x = \frac{\cos x}{\sin x}$

Use the results of (a), (b), (c) and (d) to obtain the following:

(e) $D(\tan 3x)$

(f) $D\sqrt{\tan 2x}$

(g) $D(\sec^2 x^2)$

(h) $D(\csc 3x)^{1/6}$

(i) $D[\sec(\csc x)]$

7. In what intervals is the secant function increasing? convex? Sketch its graph.

8. (a) Find $D(\sec x \csc x)$

(i) in terms of $\sec x$ and $\csc x$

(ii) in terms of $\tan x$ and $\cot x$

(iii) in terms of $\csc 2x$ and $\cot 2x$

(b) Find

(i) $D(\tan x \cot x)$

(ii) $D(\sin x \csc x)$

(iii) $D(\cos x \sec x)$

(c) Find

(i) $D(\sin x \cot x)$

(ii) $D(\cos x \tan x)$

9. Show that

(a) $D\left(\frac{\tan^{(k+1)} x}{k+1}\right) = \tan^k x \sec^2 x, \quad k \neq -1$

(b) $D(\frac{1}{k} \csc^k x) = -\csc^k x \cot x, \quad k \neq 0$

(c) $D(\cot^2 x) = D(\csc^2 x)$

10. (a) Use the product and reciprocal rules to show that $(\frac{u}{v})' = \frac{uv' - u'v}{v^2}$.

(b) $D\left(\frac{x^2 + 1}{3x^2 - x}\right)$

## 8-8. The Quotient Rule

By combining the product rule and the reciprocal rule we can obtain a rule for differentiating quotients of functions. Suppose the values of the function $f$ can be expressed as

$$f(x) = \frac{p(x)}{q(x)},$$

where $p$ and $q$ are functions (and, of course, $q(x) \neq 0$). It is then common to write $f = \frac{p}{q}$ and call $f$ the <u>quotient</u> of $p$ and $q$. Since we can write

$$f(x) = p(x) \cdot \frac{1}{q(x)},$$

the function $f$ is just the product of $p$ and the reciprocal of $q$. If $p$ and $q$ are differentiable at $x$ and $q(x) \neq 0$, then the product rule gives

$$\begin{aligned} f'(x) &= D(p(x) \cdot \frac{1}{q(x)}) \\ &= p(x)\ D(\frac{1}{q(x)}) + \frac{1}{q(x)}\ D\,p(x). \end{aligned}$$

The reciprocal rule gives

$$D(\frac{1}{q(x)}) = -\frac{D\,q(x)}{(q(x))^2} = \frac{-q'(x)}{(q(x))^2},$$

so that

$$\begin{aligned} f'(x) &= p(x)\left(\frac{-q'(x)}{(q(x))^2}\right) + \frac{1}{q(x)}\,p'(x) \\ &= \frac{-p(x)q'(x) + q(x)p'(x)}{(q(x))^2}. \end{aligned}$$

This is usually written in the form

(1) $$\boxed{f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}}$$

and is referred to as the <u>quotient rule</u>. With $D$ notation it can be written as

(2) $$\boxed{D(\frac{p(x)}{q(x)}) = \frac{q(x)\ Dp(x) - p(x)\ Dq(x)}{(q(x))^2}}$$

In words, the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all over the square of the denominator.

Example 8-8a. Use the quotient rule to find the derivative of the tangent function and discuss its graph in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The tangent function can be expressed as

$$\tan : x \to \frac{\sin x}{\cos x}.$$

This function is defined for those $x$ for which $\cos x \neq 0$; that is, the tangent function is defined only when $x$ is not an odd multiple of $\frac{\pi}{2}$. The quotient rule gives the derivative

$$\begin{aligned} D(\tan x) = D\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x\, D(\sin x) - \sin x\, D(\cos x)}{\cos^2 x} \\ &= \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}. \end{aligned}$$

Since $\sec x = \frac{1}{\cos x}$ this is usually expressed as

(3) $$\boxed{D(\tan x) = \sec^2 x.}$$

The function $x \to \cos x$ is not zero in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ so that

$$\sec^2 x > 0 \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Therefore, the tangent function is an increasing function in the interval. In fact the tangent function is strictly increasing on this interval.

Let us denote the second derivative of $y = \tan x$ by $y''$. We have

$$\begin{aligned} y'' = D(\sec^2 x) &= 2 \sec x\, D(\sec x) \\ &= 2 \sec x\, (\sec x \tan x) \\ &= 2 \sec^2 x \tan x, \end{aligned}$$

where we used the power rule and the fact that $D(\sec x) = \sec x \tan x$.

The second derivative $y'' = 2\sec^2 x \tan x$ will be negative for $-\frac{\pi}{2} < x < 0$ and positive for $0 < x < \frac{\pi}{2}$; that is, the graph of the tangent function is concave in the left interval and convex in the right interval.

As $x$ approaches $\frac{\pi}{2}$, $\cos x$ goes to zero while $\sin x$ approaches 1. Thus the line given by $x = \frac{\pi}{2}$ is an asymptote and $y = \tan x$ becomes large as $x$ approaches $\frac{\pi}{2}$ from the left; that is,

$$\lim_{x \to \frac{\pi}{2}} \tan x = +\infty .$$

Similar arguments show that

$$\lim_{x \to -\frac{\pi}{2}} \tan x = -\infty .$$

A graph of the tangent function in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is given in Figure 8-8a.

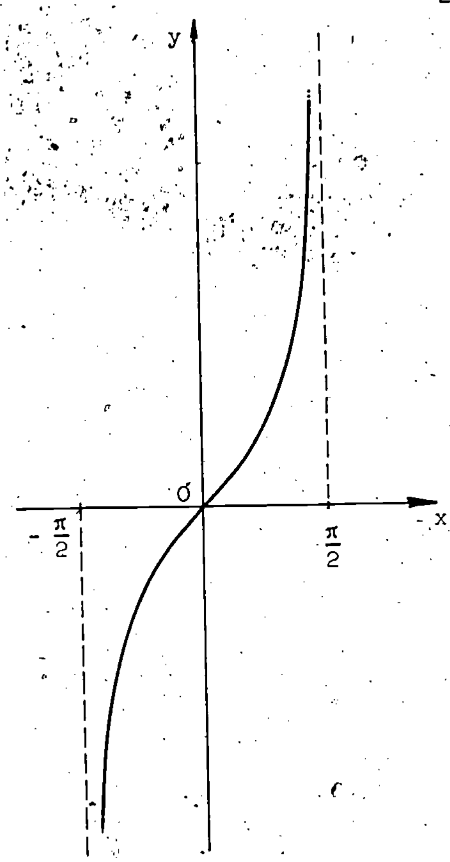


Figure 8-8a

$y = \tan x$

Rational functions, that is, quotients of polynomials, can be differentiated using the quotient rule. Such a function is discussed in the following example.

Example 8-8b. Discuss the graph of the function

$$f : x \to \frac{x^3 + x^2 - 1}{x^2 - 1}.$$

This function is not defined when $x = \pm 1$. As $x$ approaches $+1$ from the left the numerator approaches $1$, while the denominator is negative and near zero. Thus $|f(x)|$ becomes large and $f(x)$ negative as $x$ approaches $+1$ from the left; that is, $f(x)$ approaches $-\infty$. Similar arguments show that $f(x)$ approaches $+\infty$ as $x$ approaches $+1$ from the right.

Suppose $x$ approaches $-1$ from the left. The numerator approaches $-1$, while the denominator is positive and approaches $0$. Thus as $x$ approaches $-1$ from the left, $f(x)$ approaches $-\infty$.

To discuss the behavior when $|x|$ is large we rewrite the expression for $f(x)$ as

$$x\left(\frac{1 + \frac{1}{x} - \frac{1}{x^3}}{1 - \frac{1}{x^2}}\right)$$

If $|x|$ is large, the expression in the parenthesis is nearly $1$. Thus

$$\lim_{x \to \infty} f(x) = +\infty \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

Note that $f$ is continuous except when $x = \pm 1$. For example, if $a \neq \pm 1$ then as $x$ approaches $a$, the numerator approaches $a^3 + a^2 - 1$, while ~~the~~ denominator approaches $a^2 - 1$. Thus $f(x)$ approaches $\frac{a^3 + a^2 - 1}{a^2 - 1} = f(a)$. This is illustrative of the fact that a rational function is continuous except at the zeros of its denominator.

We now determine the intervals of increase and decrease. The quotient rule gives:

$$f'(x) = \frac{(x^2 - 1)D(x^3 + x^2 - 1) - (x^3 + x^2 - 1)D(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)(3x^2 + 2x) - (x^3 + x^2 - 1)(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 3x^2}{(x^2 - 1)^2} .$$

The derivative $f'$ is a rational function. (In fact, the derivative of a rational function is always a rational function.) In factored form, we have

$$f'(x) = \frac{x^2(x - \sqrt{3})(x + \sqrt{3})}{(x - 1)^2(x + 1)^2} ,$$

from which we see that the sign of $f'$ is determined by the sign of $(x - \sqrt{3})(x + \sqrt{3})$. We conclude:

> the graph of $f$ is rising when $x < -\sqrt{3}$ or $x > \sqrt{3}$ and is falling when $-\sqrt{3} < x < -1$, $-1 < x < 0$, $0 < x < 1$ or $1 < x < \sqrt{3}$.

An analysis of the sign of $f''$ to determine convexity-concavity is quite lengthy and will be omitted. The graph of $f$ is given in Figure 8-8b.

$$f(x) = \frac{x^3 + x^2 - 1}{x^2 - 1} = \frac{x^3}{x^2 - 1} + 1$$

$$f(-\sqrt{3}) = \frac{-3\sqrt{3}}{3 - 1} + 1 = \frac{-3\sqrt{3}}{2} + 1 \approx -1.6$$

$$f(\sqrt{3}) = \frac{3\sqrt{3}}{3 - 1} + 1 = \frac{3\sqrt{3}}{2} + 1 \approx 3.6$$

$$f(0) = 1$$

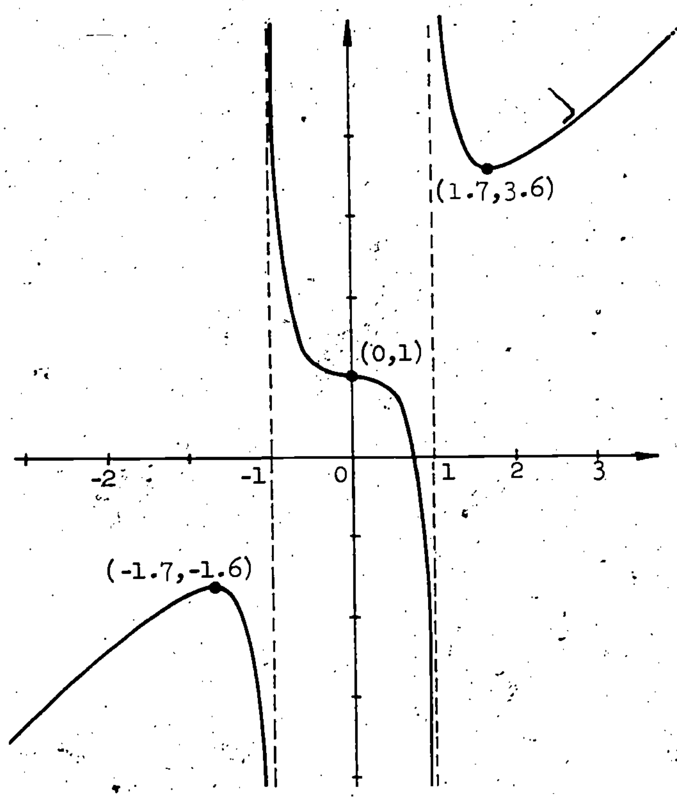


Figure 8-8b

$$y = \frac{x^3 + x^2 - 1}{x^2 - 1}$$

## Exercises 8-8

1. Evaluate:

(a) $D(\frac{x}{x - 1})$

(b) $D\left(\frac{x^2}{1 + x^2}\right)$

(c) $D(1 - \frac{1}{x})^{-1}$

(d) $D\left(\frac{3 + 2x^2}{2 - x^2}\right)$

(e) $D(\frac{1}{x} + \frac{1}{1 - x})$

(f) $D\left(\frac{\sqrt{x}}{1 + x^2}\right)$

(g) $D\left(\frac{1}{1 + \sqrt{x}}\right)$

(h) $D\left(\frac{x^2 - 1}{x^2 + 1}\right)$

(i) $D(\frac{\sin x}{1 + \tan x})$

(j) $D\left(\frac{e^x}{1 + x^2}\right)$

(k) $D\left(\frac{x \log_e x}{1 - 2x}\right)$

($\ell$) $D(\cos x \sec x)$

(m) $D\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

(n) $D[(1 + \frac{1}{x})(1 + \log_e x)]$

(o) $D\left(\frac{\log_e x^2}{\sqrt{x^2 + 1}}\right)$

2. Show that $D(\cot x) = -\csc^2 x$.

3. Discuss the graphs of each of the following, as in Example 8-8a, b. Sketch.

(a) $y = \frac{x + 2}{x^2 - 1}$

(b) $y = \frac{x - 1}{x + 1}$

(c) $y = \frac{e^{-2x}}{1 + x}$

4. Find

(a) $\int_0^{\pi/4} \sec^2 x \, dx$

(b) $\int_{-\pi/3}^{0} \sec x \tan x \, dx$

## 8-9. Inverse Functions

Let us review our discussions of Section 5-1 and 6-1 where we defined the square root function and found its derivative. The function

$$g : x \to x^2, \; x \geq 0$$

is a strictly increasing function and its graph meets each horizontal line given by $y = c$, $c \geq 0$. In other words

$$g(x_1) < g(x_2) \quad \text{if} \quad 0 \leq x_1 < x_2$$

and each nonnegative number $c$ is in the range of $g$; i.e., $c = g(d)$. The function

$$f : x \to \sqrt{x}$$

is defined for each nonnegative real number $c$ by

$$f(c) = d \quad \text{if} \quad g(d) = c;$$

that is $\sqrt{c}$ is the nonnegative real number $d$ such that $c = d^2$. This defines a function $f$, since for each $c \geq 0$ there is a unique $d \geq 0$ such that $c = d^2$. This follows from the fact that $g$ is strictly increasing.

The graph of $f$ is obtained by folding the graph of $g$ over the line given by $y = x$; that is

(1) $(c,d)$ lies on the graph of $f$ if and only if $(d,c)$ lies on the graph of $g$. (See Figure 8-9a.)

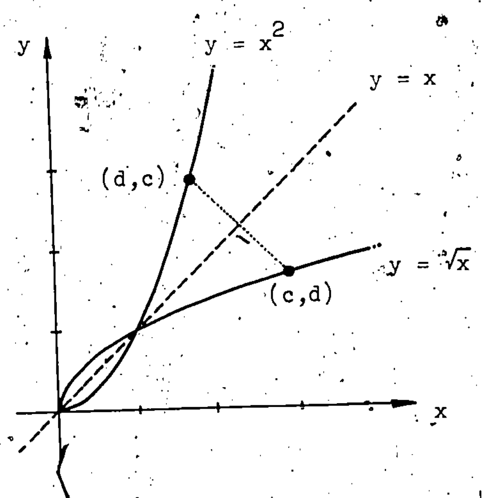


Figure 8-9a



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The tangent to the graph of $g$ at $(d,c)$ is given by the equation

$$y = g(d) + g'(d)(x - d) = d^2 + 2d(x - d).$$

If $c > 0$ then $d$ must also be positive and this line folds over the line given by $y = x$ into the line whose equation is

$$y = d + \frac{1}{2d}(x - d^2).$$

This is the tangent to the graph of $f$ at the point $(c,d)$. Replacing $d$ by $\sqrt{c}$, we see that the tangent to the graph of $f$ at $(c,d)$ has the equation

$$y = \sqrt{c} + \frac{1}{2\sqrt{c}}(x - c).$$

The coefficient of $x$ is the derivative of $f$ at $c$, so that

(2) $$f'(c) = \frac{1}{2\sqrt{c}}, \quad c > 0.$$

This same method was used to define

$$f : x \to \log_e x, \quad x > 0$$

in terms of the function $g : x \to e^x$ and to obtain the derivative formula

$$f' : x \to \frac{1}{x}.$$

In this section we discuss a general form of the folding process. Suppose the function $g$ is defined for those numbers $x$ in an interval $I$, which may be the entire real number line (as in the case $g : x \to e^x$), a ray (as in the case $g : x \to x^2, x \geq 0$), or a line segment. Suppose further that $g$ is continuous at each point of $I$ and that $g$ is strictly increasing; that is,

(3) $$g(x_1) < g(x_2) \quad \text{if } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2.$$

If we fold the graph of $g$ over the line given by $y = x$, then we obtain the graph of a function $f$. The function is called the <u>inverse</u> of $g$ and is defined by

$$f(c) = d \quad \text{if } g(d) = c;$$

that is, $f(c)$ is defined for those numbers $c$ in the <u>range</u> of $g$ (meaning that $c = g(d)$ for some $d$ in $I$). This defines a function since for a number $c$ in the domain of $f$ there is exactly one number $d$ in $I$ such that $g(d) = c$. This follows from the assumption (3) that $g$ is strictly increasing. That the domain of $f$ is an interval is a consequence of the assumption that

$g$ is continuous. In the appendices, it will be shown that the inverse $f$ is continuous at each point of its domain.

) The graphs of $f$ and $g$ are related by the condition

(4) $(c,d)$ lies on the graph of $f$ if and only if $(d,c)$ lies on the graph of $g$;

that is, the graph of the inverse $f$ can be obtained by folding the graph of $g$ over the line given by $y = x$.

The folding process used to find the derivative of the square root function also works in the general case. Suppose $f$ is the inverse of the continuous function $g$ and that $g(d) > 0$. The tangent to the graph of $g$ at $(d,c)$ has the equation

$$y = g(d) + g'(d)(x - d).$$

This folds over the line given by $y = x$ into the line whose equation is,

$$y = d + \frac{1}{g'(d)}(x - c),$$

the equation of the tangent line to the graph of the inverse $f$ at the point $(c,d)$. The value $f'(c)$ is the coefficient of $x$,

$$f'(c) = \frac{1}{g'(d)}, \text{ if } g'(d) > 0.$$

To obtain a formula for $f'(c)$ in terms of $c$, we replace $d$ by $f(c)$, to obtain the <u>inverse function rule</u>:

(5) $$f'(c) = \frac{1}{g'(f(c))}, \text{ if } g'(f(c)) > 0.$$

The geometrically intuitive folding process can be justified by rigorous arguments. In the appendices it is shown that limit concepts give the same results; that is,

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

is indeed equal to $\frac{1}{g'(f(c))}$.

Definitions and derivatives of the inverse circular functions can be obtained using this process.

The Arcsine Function.

If we restrict the sine function to an interval in which it is strictly increasing then the methods we have been using can be applied to obtain an inverse function. It is conventional to use the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The function $g : x \to \sin x$ is strictly increasing on this interval. Its inverse function $f$ is usually called the arcsine (or inverse sine) function, and denoted by arcsin. The range of $g$ is the interval $-1 \le x \le 1$ so that

$$f : x \to \arcsin x$$

is defined for $-1 \le x \le 1$. Its value at $c$, $\arcsin c$, is that real number $d$, such that

$$\sin d = c \quad \text{and} \quad -\frac{\pi}{2} \le d \le \frac{\pi}{2} .$$

In other words,

(6) $\qquad f(c) = d$ if and only if $|d| \le \frac{\pi}{2}$ and $\sin d = c$.

For example,

$$\sin 0 = 0; \ \sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}; \ \sin\frac{\pi}{2} = 1$$

so that

$$\arcsin 0 = 0; \ \arcsin(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}; \ \arcsin 1 = \frac{\pi}{2} .$$

The graph of $f : x \to \arcsin x$ can be obtained by folding the graph of $g : x \to \sin x$ over the line given by $y = x$, as shown in Figure 8-9b.

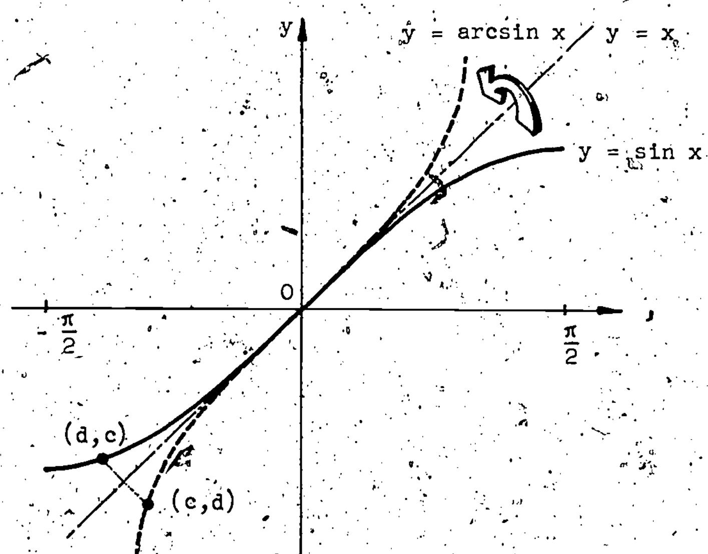


Figure 8-9b



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Using the inverse function rule (5), we can express the derivative of the arcsine function $f$ in terms of the sine function $g$. We have

$$f'(c) = \frac{1}{g'(f(c))} \quad \text{if } g'(f(c)) > 0.$$

In this case $g' : x \to \cos x$, so that

$$g'(f(c)) = \cos(\arcsin c)$$

and we have

$$f'(c) = \frac{1}{\cos(\arcsin c)}, \quad \text{if} \quad \cos(\arcsin c) > 0.$$

Referring to Figure 8-9b we see that

$$\cos(\arcsin c) = \sqrt{1 - c^2}$$

and hence we have

$$f'(c) = \frac{1}{\sqrt{1 - c^2}} \quad \text{if} \quad |c| < 1;$$

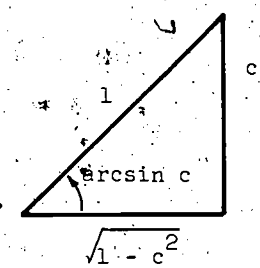


Figure 8-9b

that is,

(7) $$\boxed{D(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}, \quad \text{if } |x| < 1.}$$

The graph of the arcsin function has a vertical tangent at $x = \pm 1$. This seems reasonable as we recall the fact that the sine function has a horizontal tangent at $x = \pm \frac{\pi}{2}$.

The integration formula corresponding to (7) is

(8) $$\boxed{\int \frac{1}{\sqrt{1 - x^2}}\, dx = \arcsin x, \quad |x| < 1.}$$

Thus for $|a| < \frac{\pi}{2}$ and $|b| < \frac{\pi}{2}$ the Fundamental Theorem gives

$$\arcsin b - \arcsin a = \int_a^b \frac{1}{\sqrt{1 - x^2}}\, dx.$$

Replacing $b$ by $t$, $a$ by $0$, and using the fact that $\arcsin 0 = 0$, we have

(9) $$\arcsin t = \int_0^t \frac{1}{\sqrt{1-x^2}}\,dx; \quad |t| < 1.$$

The Arctangent Function

The function $g$ defined by

$$g(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

is strictly increasing and continuous. Furthermore, the range of $g$ is the entire real line; that is, if $c$ is any real number, then there is a number $d$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that $g(d) = c$. The inverse function $f$, known as the arctangent function, is accordingly defined for all real numbers $c$ as follows:

(10) $f(c) = \arctan c$ is real number $d$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that $\tan d = c$.

Graph of $y = \arctan x$ and $y = \tan x$ are sketched in Figure 8-9c.

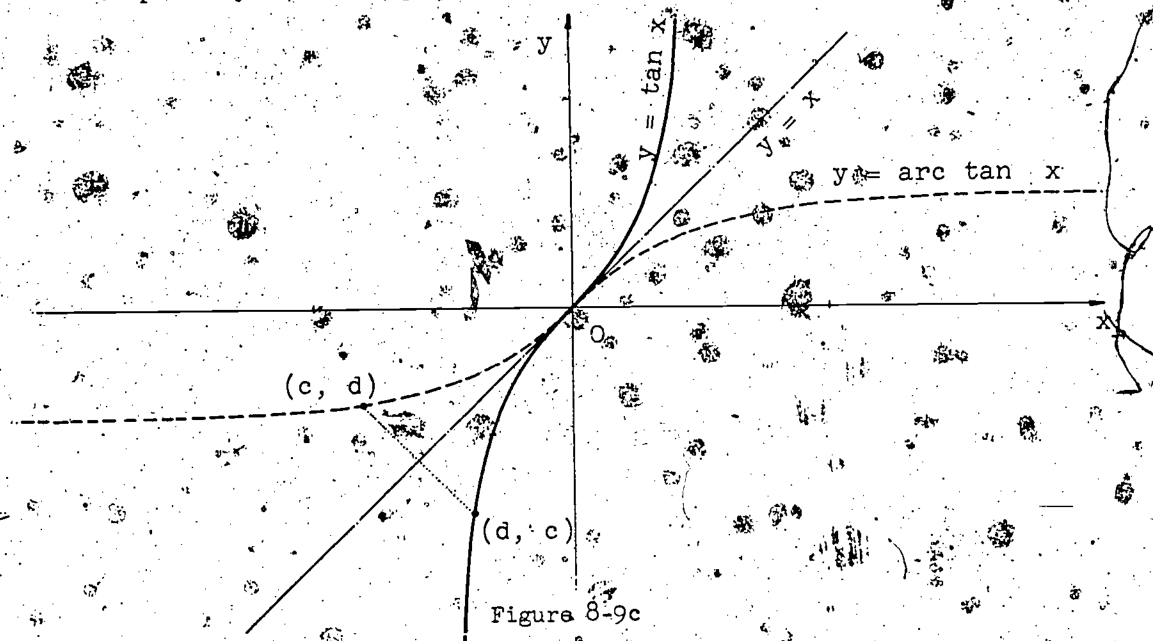


Figure 8-9c

The inverse function formula (5) gives

$$f'(c) = \frac{1}{g'(f(c))} = \frac{1}{\sec^2(\arctan c)}$$

since $D \tan x = \sec^2 x$. Referring to Figure 8-9d, we see that



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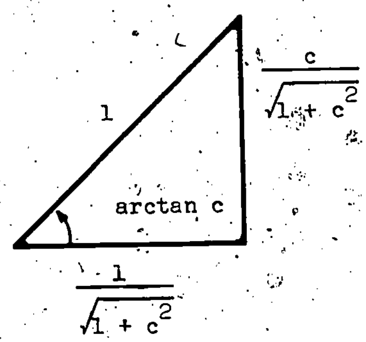


Figure 8-9d

$$\sec^2(\arctan c) = 1 + c^2$$

and hence

$$f'(c) = \frac{1}{1 + c^2}.$$

This fraction is always positive. In summary, we have

(11) $$\boxed{D(\arctan x) = \frac{1}{1 + x^2};}$$

and the corresponding integral form

(12) $$\boxed{\int \frac{1}{1 + x^2}\,dx = \arctan x.}$$

## Exercises 8-9

1. Determine the domain and range and draw the graph of the function

   (a) $f : x \to \arcsin(\sin x)$

   (b) $f : x \to \sin(\arcsin x)$

   (c) $f : x \to \arcsin(\cos x)$

   (d) $f : x \to \cos(\arcsin x)$

   (e) $f : x \to \arctan(\tan x)$

2. Derive the formula

$$D \arccos x = \frac{-1}{\sqrt{1 - x^2}} .$$

3. Derive each of the following formulas.

   (a) $D \operatorname{arccot} x = -\frac{1}{1 + x^2}$

   (b) $D \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}}$

   (c) $D \operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2 - 1}}$

4. Evaluate:

   (a) $D(\arcsin x + \arccos x)$

   (b) $D(x^2 \arcsin x)$

   (c) $D \frac{x^2}{\arctan x}$

   (d) $D(\arcsin x)^3$

   (e) $D\left(\frac{1}{1 + \arcsin x}\right)$

   (f) $D\left(\frac{1 - \arctan x}{1 + \arctan x}\right)$

5. Find $\lim\limits_{h \to 0} \frac{\arcsin h}{h}$. (Hint: What is the definition of the derivative of $f(x) = \arcsin x$ at $x = 0$?)

6. Find $\frac{dy}{dx}$ if

   (a) $y = \arcsin x^2$

   (b) $y = \arctan(3x + 2)$

   (c) $y = e^{\arcsin x}$

   (d) $y = e^{2x} \arcsin\left(\frac{1}{x}\right)$

7. Evaluate

(a) $\int_0^1 \frac{1}{1 + x^2}\,dx$

(b) $\int_{-\pi/4}^{\pi/6} \frac{1}{\sqrt{1 - t^2}}\,dt$

8. Find $F'(x)$ if $F(x)$ is given by

(a) $\int_0^x \frac{2}{1 + t^2}\,dt$

(b) $\int_0^{x^3} \frac{3}{\sqrt{1 - t^2}}\,dt$

(c) $\int_0^{\sin x} \frac{1}{1 + t^2}\,dt$

9. What is $\lim_{n \to \infty} \int_0^n \frac{1}{1 + t^2}\,dt$?

10. Show that each of the following functions $g$ has an inverse $f$ and find the derivative of $f$.

(a) $g : x \to \frac{1 - x}{1 + x}$, $x > -1$

(b) $g : x \to x|x|$ (a sketch is helpful.)

11. Show that if $f$ is the inverse of $g$ then $f(g(x)) = x$ for all $x$ in the domain of $g$. Assuming that $f$ and $g$ are differentiable apply the chain rule to obtain a formula for the derivative of $f$. Is this the same as the rule (5)?

12. Suppose $f_1$ and $f_2$ are the respective inverses of $g_1$ and $g_2$. Let $g$ be the function defined by $g(x) = g_1(g_2(x))$.

(a) Find an expression for the inverse of $g$.

(b) Use this method to find the inverse $f$ of $x \to (3x + 2)^2$, $x \geq -\frac{2}{3}$.

(c) What is the derivative of the function $f$ of part (b)?

13. Suppose $f$ is the inverse of $g$. Put $y = g(x)$, $x = f(y)$. Show that

$$\left.\frac{dx}{dy}\right|_{y=a} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=f(a)}}$$

(The symbol $\left.\frac{ds}{dt}\right|_{t=\alpha}$ means the value of the derivative of $s$, considered as a function of $t$ at the point where $t = \alpha$). This is the basis for the mnemonic expression of the diverse rule: $\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$.

14. The notation of Number 13 gives a method for finding derivatives. For example if $y = \arcsin x$, then $x = \sin y$ so $\frac{dx}{dy} = \cos y$ and hence

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}}.$$

Use this method to find the derivative of

(a) $y = \arctan x$ (c) $y = \sqrt{x}$

(b) $y = \log_e x$ (d) $y = x^\pi$

8-10. Implicitly Defined Functions

A function which is described in terms of rational operations on, and compositions and inverses of, known functions is said to be defined explicitly. No matter how complicated the description, if it is explicitly defined in terms of differentiable functions we know how to differentiate the function. You should, if pressed, be able to differentiate the explicit concoction.

$$(1) \qquad y = \arctan\left[-1 + \sqrt{1 + \sqrt[3]{\frac{\sin^2 x}{x^2} + 1}}\right]^{1/2}$$

It often happens that a function is defined indirectly or implicitly. Thus the conditions

$$(2) \qquad (\tan^4 y + 2\tan^2 y)^3 - \frac{\sin^2 x}{x^2} = 1, \quad 0 < y < \frac{\pi}{2}$$

determine $y$ as a function of $x$.

Sometimes we can find explicit expressions representing functions defined implicitly. This is the case for (2), which has the explicit solution (1). We put the implicit relationship (2) in the form

$$\tan^4 y + 2\tan^2 y - \sqrt[3]{\frac{\sin^2 x}{x^2} + 1} = 0$$

and recognize that this is a quadratic equation for $\tan^2 y$. Solving, we obtain

$$\tan^2 y = -1 + \sqrt{1 + \sqrt[3]{\frac{\sin^2 x}{x^2} + 1}}$$

where the positive square root has been taken since $\tan^2 y$ is positive. Taking the square root and then the arctangent of both sides gives (1) since $0 < y < \frac{\pi}{2}$.

In other cases there is either no equivalent explicit definition of a function defined implicitly or it is very difficult to obtain one. An example is provided by the relation

$$(3) \qquad x^2 \arctan z + z = \sin x.$$

This equation determines a unique value $z$ for every number $x$; that is, it defines a function $x \to z$ but we are unable to obtain an explicit expression for $z$.

It is easy to see that (3) defines $z$ as a function of $x$. For any given number $x$, $f : z \to x^2 \arctan z + z$ is a continuous function and has arbitrarily large values, both positive and negative. Hence (by the Intermediate Value Theorem), there is some value $z$ for which (3) is satisfied; since $f$ is an increasing function, that value must be unique, and the function $x \to z$ is defined implicitly by relation (3).

For the function defined by (1) we know that we can differentiate $y$ but the execution of the differentiation would be a punishment. A much more convenient way to find the derivative is to start from (2). Applying the chain rule and other techniques of differentiation, we obtain

$$3(\tan^4 y + 2 \tan^2 y)^2(4 \tan^3 y + 4 \tan y)\sec^2 y \cdot \frac{dy}{dx}$$

$$- \frac{x^2 \cdot 2 \sin x \cos x - 2x \sin^2 x}{x^4} = 0,$$

which is easily solved for $\frac{dy}{dx}$.

It is true that the formula obtained in this way will itself be somewhat implicit, since it will express $\frac{dy}{dx}$ in terms of both $x$ and $y$, unlike the one we could have obtained by differentiating (1) directly, where only $x$ would have appeared on the right side. We can still get a formula involving $x$ alone if we want it, by using (1) to eliminate $y$, but it is clearly more convenient to write $y$ instead of the complicated expression it represents. For most purposes, we do not need the completely explicit formula for the derivative. If we wish to find the value $\frac{dy}{dx}$ for a specified value of $x$, for instance, we can first compute the corresponding value $y$ (explicitly from (1) in this case, but by numerical approximation in most practical problems), and then compute $\frac{dy}{dx}$ from the shorter formula.

From (3) we obtain no explicit formula for $z$ in the first place. But we can still obtain a formula for $\frac{dz}{dx}$ by implicit differentiation. Thus, if $z$ is a differentiable function of $x$, we may apply the rules of differentiation and obtain

$$2x \arctan z + x^2 \frac{1}{1 + z^2} \frac{dz}{dx} + \frac{dz}{dx} = \cos x$$

or

(4) $$\frac{dz}{dx} = \frac{\cos x - 2x \arctan z}{\frac{x^2}{1 + z^2} + 1}.$$

If we wish to evaluate this for a specific $x$, we will first have to find $z$ from (3), probably by some approximate numerical technique.

We emphasize that we have not shown that (4) holds, merely that if $\frac{dz}{dx}$ exists it must have the value given by (4). There is in fact a theorem which applies under rather general conditions (which covers the present case and most of those that arise in practice) that if an equation defining a function implicitly can be formally differentiated and the result solved for the derivative of the function, then the derivative of the function exists and has the value found. To prove, or even precisely state, this theorem would take us too far afield; hereafter we shall use implicit differentiation freely to solve problems, without each time reiterating the warning that the derivative has not been proved to exist.

That we cannot solve for the derivative at every point even though the function is well defined is illustrated by the example

(5) $$u^5 + x^2 u = x,$$

which defines $u$ unambiguously for each $x$. Implicit differentiation yields

$$(5u^4 + x^2) \frac{du}{dx} + 2xu = 1,$$

which can be solved for $\frac{du}{dx}$ everywhere except where $5u^4 + x^2$ vanishes. Since from (5) we have $u = 0$ when $x = 0$, we cannot solve for $\frac{du}{dx}$ at $x = 0$. In fact, $u$ is not differentiable at $x = 0$.

Even if a function is differentiable at a given point the method may fail. For instance, consider the implicit definition

(6) $$v^5 + v^3 = x^3.$$

As before, at $x = 0$ we have $v = 0$ and no solution for $\frac{dv}{dx}$ from the implicitly differentiated result

$$(5v^4 + 3v^2) \frac{dv}{dx} = 3x^2.$$

In this case, however, there is a derivative at $x = 0$, and we can find it by writing (6) in the equivalent form

$$v(v^2 + 1)^{1/3} = x$$

and then differentiating:

$$\left[(v^2 + 1)^{1/3} + \frac{2}{3} v^2 (v^2 + 1)^{-2/3}\right] \frac{dv}{dx} = 1.$$

This gives $\frac{dv}{dx} = 1$ at $x = 0$.

## Exercises 8-10

1. For positive $x$, if $y = x^r$, where $r$ is a rational number, say $r = \frac{p}{q}$ (p, q integers), then $y^q = x^p$. Assuming the existence of the derivative $Dy$, derive the formula $Dy = rx^{r-1}$ using implicit differentiation and the differentiation formula $Dx^n = nx^{n-1}$, for integral $n$.

2. For each of the following, find $y'$ without solving for $y$ as a function of $x$.

   (a) $5x^2 + y^2 = 12$

   (b) $2x^2 - y^2 + x - 4 = 0$

   (c) $y^2 - 3x^2 + 6y = 12$

   (d) $x^3 + y^3 - 2xy = 0$

3. For each of the following use implicit differentiation to find $Dy$.

   (a) $x^2 = \frac{y - x}{y + x}$

   (b) $x^2y + xy^2 = x^3$

   (c) $x^m y^n = 10$ (m, n integers)

   (d) $\sqrt{xy} + x = y^{-1}$

4. Each of the following defines $x$ as a function of $y$. Use implicit differentiation to find $\frac{dx}{dy}$.

   (a) $x\sqrt{y} + y\sqrt{x} = a\sqrt{a}$

   (b) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$

   (c) $(x + y)^{1/2} + (x - y)^{1/2} = 4$

   (d) $3x^2 + x^2y^2 = y^4 + 5$

   (e) $4x^2 + 3xy - 7y^2 = 0$

5. For each equation, find the slope of the curve represented, at the stated point.

   (a) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$ at the point $(-2,1)$

   (b) $x^3 + y^2x^2 + y^3 - 1 = 0$ at the point $(1,-1)$

   (c) $x^2 - x\sqrt{xy} - 6y^2 = 2$ at the point $(4,1)$

   (d) $x \cos y = 3x^2 - 5$ at the point $(\sqrt{2}, \frac{\pi}{4})$

6. For each equation, find the slope of the curve represented at the point or points where $x = y$. Give a geometric explanation for these results.

(a) $x^3 - 3axy + y^3 = 0$

(b) $x^m + y^m = 2$

(c) $x^2 + y^2 = 2axy$

7. Find $y'$ by implicit differentiation.

(a) $a \sin y + b \cos x = 0$

(b) $x \cos y + y \sin x = 0$

(c) $\sin xy = \sin x + \sin y$

(d) $\csc(x + y) = y$

(e) $x \tan y - y \tan x = 1$

(f) $y \sin x = x \tan y$

(g) $xy + \sin y = 5$

8. If $0 < x < a$, then the equation $x^{1/2} + y^{1/2} = a^{1/2}$ defines $y$ as a function of $x$. Assuming the existence of the derivative, show without solving for $y$ that $f'(x)$ is always negative.

## Appendix 3

## MATHEMATICAL INDUCTION

### A3-1. The Principle of Mathematical Induction

The ability to form general hypotheses in the light of a limited number of facts is one of the most important signs of creativeness in a mathematician. Equally important is the ability to prove these guesses. The best way to show how to guess at a general principle from limited observations is to give examples.

Example A3-1a. Consider the sums of consecutive odd integers:

$$1 = 1$$

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

$$1 + 3 + 5 + 7 + 9 = 25.$$

Notice that in each case the sum is the square of the number of terms.

Conjecture: The sum of the first $n$ odd positive integers is $n^2$. (This is true. Can you show it?)

Example A3-1b. Consider the following inequalities:

$$1 < 100, \quad 2 < 100, \quad 3 < 100, \quad 4 < 100, \quad 5 < 100, \quad \text{etc.}$$

Conjecture: All positive integers are less than 100. (False, of course.)

Example A3-1c. Consider the number of complex zeros, including the repetitions, for polynomials of various degrees.

Zero degree: $a_0$, no zeros $(a_0 \neq 0)$.

First degree: $a_1x + a_0$, one zero at $x = \frac{-a_0}{a_1}$.

Second degree: $a_2x^2 + a_1x + a_0$, two zeros at

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}.$$

Conjecture: Every polynomial of degree $n$ has exactly $n$ complex zeros when repetitions are counted. (True.)

Example A3-1d. Observe the operations necessary to compute the roots from the coefficients in Example A3-1c.

Conjecture: The zeros of a polynomial of degree $n$ can be given in terms of the coefficients by a formula which involves only addition, subtraction, multiplication, division, and the extraction of roots. (False.)

Example A3-1e. Take any even number except 2, and try to express it as the sum of as few primes as possible.

$$4 = 2 + 2,\ 6 = 3 + 3,\ 8 = 3 + 5,\ 10 = 5 + 5,$$
$$12 = 5 + 7,\ 14 = 7 + 7,\ \text{etc.}$$

Conjecture: Every even number but 2 can be expressed as the sum of two primes. (As yet, no one has been able to prove or disprove this conjecture.)

Common to all these examples is the fact that we are trying to assert something about all the members of a sequence of things: the sequence of odd integers, the sequence of positive integers, the sequence of degrees of polynomials, the sequence of even numbers greater than 2. The sequential character of the problems naturally leads to the idea of sequential proof. If we know something is true for the first few members of the sequence, can we use that result to prove its truth for the next member of the sequence? Having done that, can we now carry the proof on to one more member? Can we repeat the process indefinitely?

Let us try the idea of sequential proof on Example A3-1a. Suppose we know that for the first $k$ odd integers $1, 3, 5, \ldots, 2k - 1$,

(1) $$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

can we prove that upon adding the next higher odd number $(2k + 1)$ we obtain the next higher square? From (1) we have at once by adding $2k + 1$ on both sides,

$$[1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2.$$

It is clear that if the conjecture of Example A3-1a is true at any stage then it is true at the next stage. Since it is true for the first stage, it must be true for the second stage, therefore true for the third stage, hence the fourth, the fifth, and so on forever.

Example A3-1f. In many good toy shops there is a puzzle which consists of three pegs and a set of graduated discs as depicted in Figure A3-1a. The problem posed is to transfer the pile of discs from one peg to another under the following rules:

1. Only one disc at a time may be transferred from one peg to another.
2. No disc may ever be placed over a smaller disc.

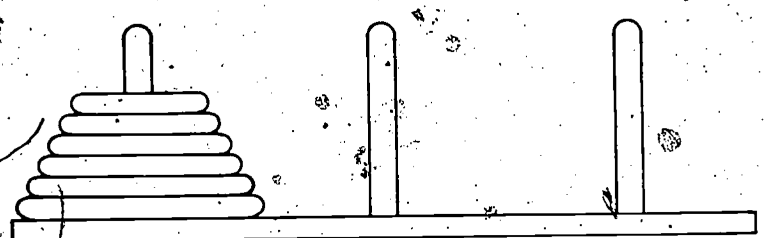

Figure A3-1a

Two questions arise naturally: Is it possible to execute the task under the stated restrictions? If it is possible, how many moves does it take to complete the transfer of the discs? If it were not for the idea of sequential proof, one might have difficulty in attacking these questions.

As it is, we observe that there is no problem in transferring one disc. If we have to transfer two discs, we transfer one, leaving a peg free for the second disc; we then transfer the second disc and cover with the first.

If we have to transfer three discs, we transfer the top two, as above. This leaves a peg for the third disc to which it is then moved, and the first two discs are then transferred to cover the third disc.

The pattern has now emerged. If we know how to transfer $k$ discs, we can transfer $k + 1$, in the following way. First, we transfer $k$ discs leaving the $(k + 1)$-th disc free to move to a new peg; we move the $(k + 1)$-th disc and then transfer the $k$ discs again to cover it. We see then that it is possible to move any number of graduated discs from one peg to another without violating the rules (1) and (2), since knowing how to move one disc, we have a rule which tells us how to transfer two, and then how to transfer three, and so on.

To determine the smallest number of moves it takes to transfer a pile of discs, we observe that no disc can be moved unless all the discs above it have been transferred, leaving a free peg to which to move it. Let us designate by $m_k$ the minimum number of moves needed to transfer $k$ discs. To move the $(k + 1)$-th disc, we first need $m_k$ moves to transfer the discs above it to another peg. After that we can transfer the $(k + 1)$-th disc to the free peg. To move the $(k + 2)$-th disc (or to conclude the game if the $(k + 1)$-th disc is last) we must now cover the $(k + 1)$-th disc with the preceding $k$ discs; this transfer of the $k$ discs cannot be accomplished in less than $m_k$ moves. We see then that the minimum number of moves for $k + 1$ discs is

$$m_{k+1} = 2m_k + 1.$$

This is a recursive expression for the minimum number of moves, that is, if the minimum is known for a certain number of discs, we can calculate the minimum for one more disc. In this way, we have defined the minimum number of sequential moves: by adding one disc we increase the necessary number of moves to one more than twice the preceding number. It takes one move to move one disc, therefore it takes three moves to move two discs, and so on.

Let us make a little table (Table A3-1a).

Table A3-1a

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $m_k$ | 1 | 3 | 7 | 15 | 31 | 63 | 127 |

$k$ = number of discs

$m_k$ = minimum number of moves

Upon adding a disc we roughly double the number of moves. This leads us to compare the number of moves with the powers of two: 1, 2, 4, 8, 16, 32, 64, 128, ...; and we guess that $m_k = 2^k - 1$. If this is true for some value $k$, we can easily see that it must be true for the next, for we have

$$\begin{aligned} m_{k+1} &= 2m_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1, \end{aligned}$$

and this is the value of $2^n - 1$, for $n = k + 1$. We know that the formula for $m_k$ is valid when $k = 1$, but now we can prove in sequence that it is true for 2, 3, 4, and so on.

According to persistent rumor, there is a puzzle of this kind in a most holy monastery hidden deep in the Himalayas. The puzzle consists of 64 discs of pure beaten gold and the pegs are diamond needles. The story relates that the game of transferring the discs has been played night and day by the monks since the beginning of the world, and has yet to be concluded. It also has been said that when the 64 discs are completely transferred, the world will come to an end. The physicists say the earth is about four billion years old, give or take a billion or two. Assuming that the monks move one disc every second and play in the minimum number of moves, is there any cause for panic? (Cf. Ball, W. W., Mathematical Recreations. New York: Macmillan Co., 1947; p. 303 ff.)

The principle of sequential proof, stated explicitly, is this. (First Principle of Mathematical Induction): Let $A_1, A_2, A_3, \ldots,$ be a sequence of assertions, and let $H$ be the hypothesis that all of these are true. The hypothesis $H$ will be accepted as proved if

1. There is a general proof to show that if any assertion $A_k$ is true, then the next assertion $A_{k+1}$ is true;

2. There is a special proof to show that $A_1$ is true.

If there are only a finite number of assertions in the sequence, say ten, then we need only carry out the chain of ten proofs explicitly to have a complete proof. If the assertions continue in sequence endlessly, as in Example 1, then we cannot possibly verify directly every link in the chain of proof. It is just for this reason--in effect that we can handle an infinite chain of proof without specifically examining every link--that the concept of sequential proof becomes so valuable. It is, in fact, at the heart of the logical development of mathematics.

Through an unfortunate association of concepts this method of sequential proof has been named "mathematical induction." Induction, in its common English sense, is the guessing of general propositions from a number of observed facts. This is the way one arrives at assertions to prove: "Mathematical induction" is actually a method of deduction or proof and not a procedure of guessing, although to use it we ordinarily must have some guess to test. This usage has been in the language for a long time, and we would gain nothing by changing it now. Let us keep it then, and remember that mathematical usage is special and often does not resemble in any respect the usage of common English.

In Example A3-1a, above, the assertion $A_n$ is

$$1 + 3 + 5 + \dots + (2n - 1) = n^2.$$

We proved first, that if $A_k$ is true (that is, if the sum of the first $k$ odd numbers is $k^2$) then $A_{k+1}$ is true, so that the sum of the first $k + 1$ odd numbers is $(k + 1)^2$. Second, we observed that $A_1$ is true: $1 = 1^2$. These two steps complete the proof.

Mathematical induction is a method of proving a hypothesis about a list or sequence of assertions. Unfortunately it doesn't tell us how to make the hypothesis in the first place. In the example just considered, it was easy to guess from a few specific instances that the sum of the first $n$ odd numbers is $n^2$, but the next problem (Example A3-1g) may not be so obvious.

<u>Example A3-1g</u>. Consider the sum of the squares of the first $n$ positive integers,

$$1^2 + 2^2 + 3^2 + \dots + n^2.$$

We find that when $n = 1$, the sum is $1$; when $n = 2$, the sum is $5$; when

$n = 3$, the sum is 14; and so on. Let us make a table of the first few values (Table A3-1b).

Table A3-1b

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| sum | 1 | 5 | 14 | 30 | 55 | 91 | 140 | 204 |

Though some mathematicians might be immediately able to see a formula that will give us the sum, must of us would have to admit that the situation is obscure. We must look around for some trick to help us discover the pattern which is surely there; what we do will therefore be a personal, individual matter. It is a mistake to think that only one approach is possible.

Sometimes experience is a useful guide. Do we know the solutions to any similar problems? Well, we have here the sum of a sequence and Example A3a also dealt with the sum of a sequence: the sum of the first n odd numbers is $n^2$. Consider the sum of the first n integers themselves (not their squares)--what is

$$1 + 2 + 3 + \dots + n?$$

This seems to be a related problem, and we can solve it with ease. The terms form an arithmetic progression in which the first term is 1 and the common difference is also 1; the sum, by the usual formula, is therefore

$$\frac{n}{2}(n + 1) = \frac{1}{2}n^2 + \frac{1}{2}n.$$

So we have

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n^2 + \frac{1}{2}n.$$

Is there any pattern here which might help with our present problem?

These two formulas have one common feature: both are quadratic polynomials in n. Might not the formula we want here also be a polynomial? It seems unlikely that a quadratic polynomial could do the job in this more complicated problem, but how about one of higher degree? Let's try a cubic: assume that there is a formula,

$$1^2 + 2^2 + \dots + n^2 = an^3 + bn^2 + cn + d.$$

where $a$, $b$, $c$, and $d$ are numbers yet to be determined. Substituting $n = 1, 2, 3,$ and $4$ successively in this formula, we get

$$1^2 = a + b + c + d$$

$$1^2 + 2^2 = 8a + 4b + 2c + d$$

$$1^2 + 2^2 + 3^2 = 27a + 9b + 3c + d$$

$$1^2 + 2^2 + 3^2 + 4^2 = 64a + 16b + 4c + d.$$

Solving, we find

$$a = \frac{1}{3}, \quad b = \frac{1}{2}, \quad c = \frac{1}{6}, \quad d = 0.$$

We therefore conjecture that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

$$= \frac{1}{6} n(n + 1)(2n + 1).$$

This then is our assertion $A_n$; now let us prove it.

We have $A_k$:

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6} k(k + 1)(2k + 1).$$

Add $(k + 1)^2$ to both sides, factor, and simplify:

$$1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = \frac{1}{6} k(k + 1)(2k + 1) + (k + 1)^2$$

$$= (k + 1)[\frac{1}{6} k(2k + 1) + (k + 1)]$$

$$= \frac{1}{6} (k + 1)(k + 2)(2k + 3),$$

and this last equation is just $A_{k+1}$, which is therefore true if $A_k$ is true. Moreover, $A_1$, which states

$$1^2 = \frac{1}{6} (1)(2)(3),$$

is true; and $A_n$ is therefore true for each positive integer $n$.

There is another formulation of the principle of mathematical induction which is extremely useful. This form involves the assumption in the sequential step that every assertion up to a certain point is true, rather than just the one assertion immediately preceding. Specifically, we have the following (Second Principle of Mathematical Induction): Again let $A_1$, $A_2$, $A_3$, ..., be a sequence of assertions, and let $H$ be the hypothesis that all of these are true. The hypothesis $H$ will be accepted as proved if

1. There is a general proof to show that if every preceding assertion $A_1$, $A_2$, ..., $A_k$, is true, then the next assertion $A_{k+1}$ is true.
2. There is a special proof to show that $A_1$ is true.

It is not hard to show that either one of the two principles of mathematical induction can be derived from the other. The demonstration of this is left as an exercise.

The value of this second principle of mathematical induction is that it permits the treatment of many problems which would be quite difficult to handle directly on the basis of the first principle. Such problems usually present a more complicated appearance than the kind which yield directly to an attack by the first principle.

Example A3-1h. Every nonempty set $S$ of natural numbers (whether finite or infinite) contains a least element.

Proof. The induction is based on the fact that $S$ contains some natural number. The assertion $A_k$ is that if $k$ is in $S$, then $S$ contains a least element.

Initial Step: The assertion $A_1$ is that if $S$ contains 1, then it contains a least number. This is certainly true, since 1 is the smallest natural number and so is smaller than any other member of $S$.

Sequential Step: We assume $A_n$ is true for all natural numbers up to and including $k$. Now let $S$ be a set containing $k + 1$. There are two possibilities:

1. $S$ contains a natural number $p$ less than $k + 1$. In that case $p$ is less than or equal to $k$. It follows that $S$ contains a least element.
2. $S$ contains no natural number less than $k + 1$. In that case $k + 1$ is least.

This example is valuable because it is a third principle of mathematical induction equivalent to the other two, although not an obvious one to be sure. An amusing example of a "proof" by this principle is given by Beckenbach in the American Mathematical Monthly, Vol. 52; 1945.

THEOREM. Every natural number is interesting.

Argument. Consider the set $S$ of all uninteresting natural numbers. This set contains a least element. What an interesting number, the smallest in the set of uninteresting numbers! So $S$ contains an interesting number after all. (Contradiction.)

The trouble with this "proof" of course is that we have no definition of "interesting"; one man's interest is another man's boredom.

One of the most important uses of mathematical induction is in definition by recursion, that is, in defining a sequence of things as follows: a definition is given for the initial object of the sequence, and a rule is supplied so that if any term is known the rule provides a definition for the succeeding one.

For example, we could have defined $a^n$ $(a \neq 0)$ recursively in the following way:

Initial Step: $a^0 = 1$

Sequential Step: $a^{k+1} = a \cdot a^k$ $(k = 0, 1, 2, 3, \ldots)$

Here is another useful definition by recursion: Let $n!$ denote the product of the first $n$ positive integers. We can define $n!$ recursively as follows:

Initial Step: $1! = 1$

Sequential Step: $(k + 1)! = (k + 1)(k!)$ $(k = 1, 2, 3, \ldots)$

Such definitions are convenient in proofs by mathematical induction. Here is an example which involves the two definitions we have just given.

Example A3-1i. For all positive integral values $n$, $2^{n-1} \leq n!$ The proof by mathematical induction is direct. We have the following steps.

Initial Step: $2^0 = 1 \leq 1! = 1$

Sequential Step: Assuming that the assertion is true at the k-th step, we seek to prove it for the $(k + 1)$-th step. By definition, we have

$$(k + 1)! = (k + 1)(k!).$$

From the hypothesis, $k! \geq 2^{k-1}$, and consequently,

$$(k + 1)! = (k + 1)(k!) \geq (k + 1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k$$

since $k \geq 1$ ($k$ is a positive integer). We conclude that $(k + 1)! \geq 2^k$. The proof is complete.

Before we conclude these remarks on mathematical induction, a word of caution. For a complete proof by mathematical induction it is important to show the truth of both the initial step and the sequential step of the induction principle being used. There are many examples of mathematical induction gone haywire because one of these steps fails. Here are two examples.

Example A3-1j.

Assertion: All natural numbers are even.

Argument: For the proof we utilize the second principle of mathematical induction and take for $A_k$ the assertion that all natural numbers less than or equal to $k$ are even. Now consider the natural number $k + 1$. Let $i$ be any natural number with $i \leq k$. The number $j$ such that $i + j = k + 1$ can easily be shown to be a natural number with $j \leq k$. But if $i \leq k$ and $j \leq k$, both $i$ and $j$ are even; and hence $k + 1 = i + j$, the sum of two even numbers, and must itself be even!

Find the hole in this argument.

Example A3-1k.

Assertion: All girls are the same.*

Argument: Given girls designated by $a$ and $b$, let $a = b$ mean that $a$ and $b$ are the same. Consider any set $S_1$ containing just one girl. Clearly, if $a$ and $b$ denote girls in $S_1$, then $a = b$. Now suppose it is true for any set of $k$ girls that they are all the same. Let $S_{k+1}$ be a set containing $k + 1$ girls $g_1, g_2, \ldots, g_k, g_{k+1}$. By hypothesis the $k$ girls, $g_1, g_2, \ldots, g_k$, are all the same, but by the same argument so are the $k$ girls $g_2, g_3, \ldots, g_k, g_{k+1}$. It follows that $g_1 = g_2 = \cdots = g_k = g_{k+1}$. We conclude that all girls of a set containing any positive integral number of them are the same. Since there is only a positive integral number of girls in the whole world, the assertion is proved.

Find the flaw in this argument.

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*We are not trying to express an overly blasé attitude about girls. The original of this example (attributed to the famous logician Tarski) had it that all positive integers are the same; however, isn't it more interesting to write about girls?

## Exercises A3-1

1. Prove by mathematical induction that $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$.

2. By mathematical induction prove the familiar result, giving the sum of an arithmetic progression to $n$ terms:

$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) = \frac{n}{2}[2a + (n - 1)d].$$

3. By mathematical induction prove the familiar result, giving the sum of a geometric progression to $n$ terms:

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

Prove the following four statements by mathematical induction.

4. $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n)$.

5. $2n \leq 2^n$.

6. If $p > -1$, then, for every positive integer $n$, $(1 + p)^n \geq 1 + np$.

7. $1 + 2 \cdot 2 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = 1 + (n - 1)2^n$

Prove the following by the second principle of mathematical induction.

8. For all natural numbers $n$, the number $n + 1$ either is a prime or can be factored into primes.

9. For each natural number $n$ greater than one, let $U_n$ be a real number with the property that for at least one pair of natural numbers $p$, $q$ with $p + q = n$, $U_n = U_p + U_q$.
When $n = 1$, we define $U_1 = a$ where $a$ is some given real number.
Prove that $U_n = na$ for all $n$.

10. Attempt to prove 8 and 9 from the first principle to see what difficulties arise.

In the next three problems, first discover a formula for the sum, and then prove by mathematical induction that you are correct.

11. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)}$

12. $1^3 + 2^3 + 3^3 + \ldots + n^3$. (Hint: Compare the sums you get here with Examples A3-1a and A3-1g in the text, or, alternatively, assume that the required result is a polynomial of degree 4.)

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1)$. (Hint: Compare this with Example A3-1g in the text.)

14. Prove for all positive integers $n$,

$$(1 + \frac{3}{1})(1 + \frac{5}{4})(1 + \frac{7}{9}) \ldots (1 + \frac{2n + 1}{n^2}) = (n + 1)^2.$$

15. Prove that $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2^n}) = \dfrac{1 - x^{2^{n+1}}}{1 - x}$.

16. Prove that $n(n^2 + 5)$ is divisible by 6 for all integral $n$.

17. Any infinite straight line separates the plane into two parts; two intersecting straight lines separate the plane into four parts; and three non-concurrent lines, of which no two are parallel, separate the plane into seven parts. Determine the number of parts into which the plane is separated by $n$ straight lines of which no three meet in a single common point and no two are parallel; then prove your result. Can you obtain a more general result when parallelism is permitted? If concurrence is permitted? If both are permitted?

18. Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{p_n}{q_n}, \ldots$$

where each fraction is obtained from the preceding by the rule

$$p_n = p_{n-1} + 2q_{n-1}$$

$$q_n = p_{n-1} + q_{n-1}.$$

Show that for $n$ sufficiently large, the difference between $\frac{p_n}{q_n}$ and $\sqrt{2}$ can be made as small as desired. Show also that the approximation to $\sqrt{2}$ is improved at each successive stage of the sequence and that the error alternates in sign. Prove also that $p_n$ and $q_n$ are relatively prime, that is, the fraction $\frac{p_n}{q_n}$ is in lowest terms.

19. Let $p$ be any polynomial of degree $m$. Let $q(n)$ denote the sum

(1) $$q(n) = p(1) + p(2) + p(3) + \dots + p(n).$$

Prove that there is a polynomial $q$ of degree $m + 1$ satisfying (1).

20. Let the function $f(n)$ be defined recursively as follows:

Initial Step: $f(1) = 3$

Sequential Step: $f(n + 1) = 3^{f(n)}$

In particular, we have $f(3) = 3^{3^3} = 3^{27}$, etc.

Similarly, $g(n)$ is defined by

Initial Step: $g(1) = 9$

Sequential Step: $g(n + 1) = 9^{g(n)}$

Find the minimum value $m$ for each $n$ such that $f(m) \geq g(n)$.

21. Prove for all natural numbers $n$, that $\dfrac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$ is an integer. (Hint: Try to express $x^n - y^n$ in terms of $x^{n-1} - y^{n-1}$, $x^{n-2} - y^{n-2}$, etc.)

## A3-2. Sums and Sum Notation

### (i) Sum Notation

In the preceding section we made frequent use of extended sums in which the terms exhibit a repetitive structure. For example, consider the sum

(1) $$1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + \dots + n(2n - 1).$$

We adopt a concise notation which indicates the repetition instead of spelling it out. In this notation the sum (1) is written

$$\sum_{k=1}^{n} k(2k - 1).$$

This symbol means, "the sum of all terms of the form $k(2k - 1)$ where $k$ takes on the integer values from $1$ to $n$ inclusive." The Greek capital "$\Sigma$" (sigma) corresponds to the Roman "S" and is intended to suggest the word "sum."

The notation can be used more generally to express the sum of any quantities $\phi_k$ where $k$ takes on consecutive integral values; we may begin with any integer $m$ and end with any integer $n$ where $n \geq m$. Thus

$$\sum_{k=m}^{n} \phi_k = \phi_m + \phi_{m+1} + \phi_{m+2} + \dots + \phi_n.$$

(Note the trivial special case, $n = m$, a "sum" of one term: $\sum_{k=m}^{n} \phi_k = \phi_m$.)

Example A3-2a. If each of the regions $R_k$ in (1) is a rectangle with height $h_k$ and width $w_k$, the sum of the areas may we written

$$w_1h_1 + w_2h_2 + w_3h_3 + \dots + w_nh_n = \sum_{k=1}^{n} w_kh_k.$$

Here are other typical examples:

$$\sum_{k=0}^{3} \frac{k}{1 + k^2} = \frac{0}{1 + 0} + \frac{1}{1 + 1} + \frac{2}{1 + 4} + \frac{3}{1 + 9}$$

$$= 0 + \frac{1}{2} + \frac{2}{5} + \frac{3}{10}$$

$$= \frac{6}{5},$$

$$\sum_{j=2}^{5} (j + 3) = 5 + 6 + 7 + 8 = 26.$$

A linear combination of $n$ functions:

$$\sum_{j=1}^{n} a_j f_j(x) = a_1 f_1(x) + a_2 f_2(x) + \ldots + a_n f_n(x).$$

A polynomial of degree no greater than $m$:

$$\sum_{i=0}^{m} c_i x^i = c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m.$$

Example A3-2b. A simple but important sum is $\sum_{j=1}^{n} c$, where $c$ is a constant, that is, a quantity independent of the index $j$ of summation. The quantity $\sum_{j=1}^{n} c$ is the sum of $n$ terms each of which is $c$; it therefore has the value $nc$.

In any summation the values of the terms and the total are not affected by the choice of the index letter; thus

$$\sum_{k=m}^{n} \phi_k = \sum_{j=m}^{n} \phi_j.$$

We are free to choose the index letter and its initial value to suit our own convenience.

Example A3-2c.

(a) $$\sum_{j=0}^{2} a_j = a_0 + a_1 + a_2 = \sum_{p=1}^{3} a_{p-1} = \sum_{n=0}^{2} a_{2-n}$$

(b) $$\sum_{i=0}^{n} a_i^{n-i} = a_0^{n} + a_1^{n-1} + \ldots + a_n^{0} = \sum_{j=0}^{n} (a_{n-j})^j$$

Summation is a linear process; the proof is left as the first excercise below.

## Exercises A3-2a

1. Prove

$$\sum_{k=1}^{n}(\alpha f_k + \beta g_k) = \alpha \sum_{k=1}^{n} f_k + \beta \sum_{k=1}^{n} g_k .$$

2. Write each of the following sums in expanded form and evaluate:

(a) $\sum_{k=1}^{5} 2k$

(b) $\sum_{j=5}^{10} j^2$

(c) $\sum_{r=-1}^{3} (r^2 + r - 12)$

(d) $\sum_{m=2}^{5} m(m - 1)(m - 2)$

(e) $\sum_{i=0}^{10} 2^i$

(f) $\sum_{r=0}^{4} \frac{4!}{r!(4 - r)!}$

3. Which of the following statements are true and which are false? Justify your conclusions.

(a) $\sum_{j=3}^{10} 4 = 7 \cdot 4 = 28$

(b) $\sum_{j=m}^{n} 4 = 4((n - m) + 1)$

(c) $\sum_{k=1}^{10} k^2 = 10 \sum_{k=1}^{9} k^2$

(d) $\sum_{k=1}^{1000} k^2 = 5 + \sum_{k=3}^{1000} k^2$

(e) $\sum_{k=1}^{n} k^3 = n^3 + \sum_{j=2}^{n} (j - 1)^3$

(f) $\sum_{m=1}^{10} k^2 = \left(\sum_{m=1}^{10} k\right)^2$

(g) $\sum_{m=1}^{10} k^3 = \left(\sum_{m=1}^{10} k\right)^2$

(h) $\sum_{i=0}^{n} i(i - 1)(n - i) = \sum_{i=2}^{n-1} i(i - 1)(n - i)$

(i) $\sum_{k=0}^{m} f(a_{m-k}) = \sum_{k=0}^{m} f(a_k)$

(j) $n \sum_{k=0}^{n} A_k - \sum_{k=0}^{n} k A_k = \sum_{k=0}^{n} k A_{n-k}$

(k) $\sum_{k=0}^{m} k^2(A_k - A_{m-k}) = m^2 \sum_{k=0}^{m} A_{m-k} - 2m \sum_{k=0}^{m} k A_{m-k}$

4. Evaluate $\sum_{k=1}^{n} f(\frac{k}{n})\frac{(b - a)}{n}$ if $f(x) = x^2$, $a = 0$, $b = 1$, and

(a) $n = 2$

(b) $n = 4$

(c) $n = 8$

5. Subdivide the interval $[0,1]$ into $n$ equal parts. In each subinterval obtain upper and lower bounds for $x^2$. Using sigma notation use these upper and lower bounds to obtain expressions for upper and lower estimates of the area under the curve $g = x^2$ on $[0,1]$. If you can evaluate these sums without reading elsewhere, do so.

6. (a) Write out the sum of the first 7 terms of an arithmetic progression with first term $a$ and common difference $d$. Express the same sum in sigma notation.

(b) In sigma notation, write the expression for the sum of the first $n$ terms of a geometric progression with first term $a$ and common ratio $r$.

7. (a) Consider a function $f$ defined by

$$f(n) = \sum_{r=1}^{n} \left\{(r - 1)(r - 2)(r - 3)(r - 4)(r - 5) + r\right\}.$$

Find $f(n)$ for $n = 1, 2, \ldots, 5$.

(b) Give an example of a function $g$ (similar to that in (a)) such that

$g(n) = 1$ $n = 1, 2, \ldots, 10^6$,

$g(10^6 + 1) \neq 0$.

8. Write each of the following sums in expanded form and evaluate.

(a) $\sum_{n=1}^{4}\left\{\sum_{r=1}^{3} r(n - r)\right\}$

(b) $\sum_{n=1}^{N}\left\{\sum_{r=1}^{R} (rn - 1)\right\}$

9. The double sum $\sum_{i=0}^{m}\sum_{j=0}^{n} F(i,j)$ is a shorthand notation for

$$\sum_{i=0}^{m}\left\{F(i,0) + F(i,1) + \dots + F(i,n)\right\}$$

or

$$\begin{aligned} &F(0,0) + F(0,1) + \dots + F(0,n)\\ &+ F(1,0) + F(1,1) + \dots + F(1,n)\\ &\quad\vdots\\ &+ F(m,0) + F(m,1) + \dots + F(m,n). \end{aligned}$$

In particular $\sum_{i=1}^{2}\sum_{j=1}^{3} i \cdot j = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 2 \cdot 2$
$+ 2 \cdot 3 = 18.$ Evaluate:

(a) $\sum_{i=1}^{m}\sum_{j=1}^{n} i \cdot j$

(b) $\sum_{i=1}^{m}\sum_{j=1}^{n} (i + j)$

(c) $\sum_{i=1}^{m}\sum_{j=1}^{n} \max\{i,j\}$

(d) $\sum_{i=1}^{m}\sum_{j=1}^{n} \min\{i,j\}$

10. (a) Show that $\frac{1}{k(k - 1)} = \frac{1}{k - 1} - \frac{1}{k}$, $k \neq 0, 1$.

(b) Evaluate $\sum_{k=2}^{1000} \frac{1}{k(k - 1)}$

11. If $S(n) = \sum_{i=1}^{n} f(i)$, determine $f(m)$ in terms of the sum function $S$.

12. Determine $f(m)$ in the following summation formulae:

(a) $1 = \sum_{i=1}^{n} f(i)$

(b) $n = \sum_{i=1}^{n} f(i)$

(c) $n^2 = \sum_{i=1}^{n} f(i)$

(d) $an^2 + bn + c = \sum_{i=1}^{n} f(i)$

(e) $\cos nx = \sum_{i=1}^{n} f(i)$

(f) $\sin (an + b)$

(g) $n! = \sum_{i=1}^{n} f(i)$

13. Binomial Theorem: We define $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ where $r$, $n$ are integers such that $0 \le r \le n$. Also $0! = 1$ and $\binom{n}{r} = 0$ if $r > n$. Show that

(a) $\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{1} = \binom{n}{n-1} = n$

(b) $\binom{n}{r} = \binom{n}{n-r}$

$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

(c) Establish the Binomial Theorem

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + nx^{n-1}y + \dots + nxy^{n-1} + y^n,$$

$n = 0, 1, 2, \dots,$ by mathematical induction.

14. Using the Binomial Theorem, give the expansions for the following:

(a) $(x + y)^3$

(b) $(x - y)^3$

(c) $(2x - 3y)^3$

(d) $(x - 2y)^5$

15. Evaluate the following sums.

(a) $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$

(b) $\binom{n}{0} - \binom{n}{1} + \dots + (-1)^n \binom{n}{n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$

16. Sum $\sum_{r=0}^{n} r\binom{n}{r}$ by first showing $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$ and using 15(a).

17. If $P_n(x)$ denotes a polynomial of degree $n$ such that $P_n(x) = 2^x$ for $x = 0, 1, 2, \ldots, n$ find $P_n(n + 1)$.

(ii) Summation

Exercises A3-1, No. 10 illustrates a particularly useful summation technique, i.e., representation as a telescoping sum. It was possible to write

$$\sum_{k=2}^{1000} \frac{1}{k(k-1)} = \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \cdots + \frac{1}{1000 \cdot 999}$$

in the form

$$\sum_{k=2}^{1000} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{998} - \frac{1}{999}\right) + \left(\frac{1}{999} - \frac{1}{1000}\right).$$

Each quantity subtracted in one parenthesis is added back in the next, so that the first two terms telescope from a sum of four numbers to a sum of two numbers, the first three terms telescope from a sum of six numbers to a sum of two numbers, etc. Finally, the entire summation telescopes (or collapses) into a sum of two numbers--the first number in the first term and the second number in the last term. Symbolically, a telescoping sum has the form

$$(1) \qquad \sum_{k=m}^{n} \left\{f(k) - f(k-1)\right\} = f(n) - f(m-1).$$

In the above example, we have $m = 2$, $n = 1000$, and $f(k) = -\frac{1}{k}$ so that the sum telescopes to $f(1000) - f(1) = -\frac{1}{1000} + 1 = \frac{999}{1000}$.

We now use (1) to establish a short dictionary of summation formulae by considering different functions $f(k)$. Also, we let $m = 1$ without loss of generality. Let $f(k) = k$, then

$$(2) \qquad \sum_{k=1}^{n} \left\{k - (k-1)\right\} = \sum_{k=1}^{n} 1 = n.$$

This result is nothing new. Now let $f(k) = k^2$, then.

$$\sum_{k=1}^{n}\left\{k^2 - (k - 1)^2\right\} = \sum_{k=1}^{n}(2k - 1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = n^2$$

or, equivalently,

(3) $$\sum_{k=1}^{n}k = \frac{1}{2}n(n + 1).$$

By linearly combining (2) and (3), we obtain the sum of a general arithmetic progression.

$$\sum_{k=1}^{n}(ak + b) = a\left\{\frac{n(n + 1)}{2}\right\} + bn.$$

To obtain the sum $\sum_{k=1}^{n}k^2$, we let $f(k) = k^3$. Then,

$$\sum_{k=1}^{n}\left\{k^3 - (k - 1)^3\right\} = \sum_{k=1}^{n}(3k^2 + 3k + 1) = n^3,$$

$$3\sum_{k=1}^{n}k^2 - 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 = n^3.$$

Using (2) and (3), we obtain

$$\sum_{k=1}^{n}k^2 = \frac{1}{3}\left\{n^3 - \frac{3n(n + 1)}{2} - n\right\} = \frac{n(n + 1)(2n + 1)}{6}.$$

We now can establish a sequential method of obtaining sums of the form $\sum_{k=1}^{n}P(k)$ whose terms are values $P(k)$ of a polynomial function. Because a polynomial is a linear combination of powers, and summation is a linear process it is sufficient to give a sequential method for $\sum_{k=1}^{n}k^r$, $r$ a nonnegative integer.

Choosing $f(k) = k^{r+1}$ in summation, formula (1) gives us,

$$\sum_{k=1}^{n}\left\{k^{r+1} - (k - 1)^{r+1}\right\} = n^{r+1}.$$

Using the Binomial Theorem, we obtain

(4) $$k^{r+1} - (k - 1)^{r+1} = (r + 1)k^r + P(k)$$

where $P(k)$ is a polynomial of degree $r - 1$. Thus, the sum $\sum_{k=1}^{n} k^r$ can be expressed in terms of sums of lower degree. Since we already have the sum for $r = 0$, 1, and 2, we can repeat the method sequentially to obtain the sum for any $r$ (compare with Exercises A3-1, No. 19).

We can enlarge our summation table by choosing other functional forms $f(k)$, e.g., $\sin(ak + b)$. By (1),

$$(5)\qquad \sum_{k=1}^{n} \left\{\sin(ak + b) - \sin\left(a(k - 1) + b\right)\right\} = \sin(an + b) - \sin b .$$

Using the identity

$$\sin A - \sin B = 2 \sin \frac{A - B}{2} \cos \frac{A + B}{2} ,$$

in Equation (5), we obtain

$$(6)\qquad \sum_{k=1}^{n} \cos(ak + b - \frac{a}{2}) = \cos(b + \frac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}} .$$

If $b = \frac{a}{2}$, (6) reduces to

$$(7)\qquad \sum_{k=1}^{n} \cos ak = \cos\left((a + 1)\frac{n}{2}\right) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}}$$

If $b = \frac{a}{2} + \frac{\pi}{2}$, (6) reduces to

$$(8)\qquad \sum_{k=1}^{n} \sin ak = \sin(b + \frac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}} .$$

By choosing other functions $f(k)$, we can enlarge our list of summation formulae. We leave this for exercises.

## Exercises A3-2b

1. Write the following sums in telescoping form, i.e., in the form $\sum_{k=1}^{n} \left\{u(k) - u(k-1)\right\}$, and evaluate.

(a) $\sum_{k=1}^{n} k(k+1)$

(b) $\sum_{k=1}^{n} k(2k-1)$

(c) $\sum_{k=1}^{n} 2k(2k+1)$

(d) $\sum_{k=1}^{n} k(k+1)(k+2)$

(e) $\sum_{k=1}^{n} k^3$

(f) $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$

(g) $\sum_{k=1}^{n} k \cdot k!$

(h) $\sum_{k=1}^{n} r^k$

2. Using $\sum_{k=1}^{n} \left\{u(k) - u(k-1)\right\} = u(n) - u(0)$, establish a short dictionary of summation formulae by considering the following functions $u$:

(a) $(a + kd)\left(a + (k+1)d\right) \ldots \left(a + (k+p)d\right)$

(b) The reciprocal of (a).

(c) $r^k$

(d) $kr^k$

(e) $k^2r^k$

(f) $k!$

(g) $(k!)^2$

(h) $\arctan k$

(i) $k \sin k$

3. Simplify:

$$\frac{\sin x + \sin 3x + \ldots + \sin\left((2n-1)x\right)}{\cos x + \cos 3x + \ldots + \cos\left((2n-1)x\right)}$$

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4. Another method for summing $\Sigma P(k)$ ($P$, a polynomial) can be obtained by using a special case of problem 2a, i.e.,

$$\sum_{k=1}^{n}\left\{(k+1)(k)(k-1)\ \ldots\ (k-r+1) - (k)(k-1)(k-2)\ \ldots\ (k-r)\right\}$$

$$= (n+1)(n)(n-1)\ \ldots\ (n-r+1),$$

$$\text{or} \quad \sum_{k=1}^{n} k(k-1)\ \ldots\ (k-r+1) = \frac{(n+1)(n)(n-1)\ \ldots\ (n-r+1)}{r+1}.$$

First, we show how to represent any polynomial $P(k)$ of $r^{th}$ degree in the form

$$\text{(i)} \quad P(k) = a_0 + a_1 k + \frac{a_2 k(k-1)}{2!} + \ldots + \frac{a_r k(k-1)\ \ldots\ (k-r+1)}{r!}$$

If $k = 0$, then $a_0 = P(0)$; if $k = 1$, then $a_1 = P(1) - P(0)$; if $k = 2$, than $a_2 = P(2) - 2P(1) + P(0)$. In general, it can be shown that

$$\text{(ii)} \quad a_m = P(m) - \binom{m}{1}P(m-1) + \binom{m}{2}P(m-2) - \ldots + (-1)^m P(0),$$

$$m = 0, 1, \ldots, r.$$

Since both sides of (i) are polynomials of degree $r$ and (i) is satisfied for $m = 0, 1, \ldots, r$, it must be an identity.

Now sum $\displaystyle\sum_{k=1}^{n} P(k)$.

5. Using Prob. 4, find the following sums:

(a) $\displaystyle\sum_{k=1}^{n} k^2$

(b) $\displaystyle\sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2$

(c) $\displaystyle\sum_{k=1}^{n} k^4$

6. (a) Establish Equation (ii) of Number 4.

(b) Show that $a_m$ is zero for $m > r$.

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## Appendix 4

## FURTHER TECHNIQUES OF INTEGRATION

### A4-1. Substitutions of Circular Functions

Although it is not always possible to integrate a given function in terms of elementary functions, there are important broad classes of explicitly integrable functions. All powers and hence, clearly, all polynomials are explicitly integrable. It is not so clear but it is true that all rational functions are explicitly integrable (see Section A4-3). It follows that all integrals which can be transformed by substitution into integrals of rational functions are explicitly integrable. In this section we shall show that an integral of any rational combination of $x$ and $\sqrt{Q(x)}$, where

$$Q(x) = Ax^2 + Bx + C,$$

can be transformed into an integral of a rational combination of circular functions, and further that an integral of a rational combination of circular functions can be transformed into an integral of a rational function.

We should consider the substitution of a circular function whenever an integrand is a combination of $x$ and one of the expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, $(a > 0)$ suggestive of the Pythagorean expression for one of the sides of a right triangle in terms of the other two.

Example A4-1a. Consider

$$I = \int_0^{a/2} \frac{dx}{\sqrt{a^2 - x^2}}.$$

We utilize the substitution

$$x = a \sin\theta, \quad \sqrt{a^2 - x^2} = a \cos\theta \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right),$$
$$dx = a \cos\theta \, d\theta.$$

(See Figure A4-1a.) Observing that for $x = \frac{a}{2}$, $\theta = \frac{\pi}{6}$, we obtain by the substitution rule,

$$I = \int_0^{\pi/6} \frac{a \cos\theta}{a \cos\theta}\, d\theta = \int_0^{\pi/6} d\theta = \frac{\pi}{6}.$$

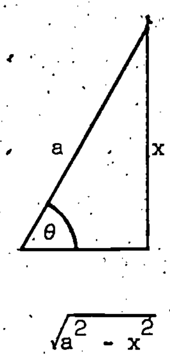


Figure A4-1a

Example A4-1b. For the integral

$$I = \int \frac{dx}{(x^2 + a^2)^{3/2}}$$

we employ the substitution (see Figure A4-1b)

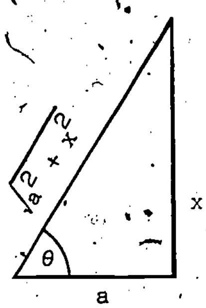


Figure A4-1b

$$x = a \tan \theta \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right),$$

$$dx = \frac{a}{\cos^2 \theta}\, d\theta$$

$$\sqrt{a^2 + x^2} = \frac{a}{\cos \theta}.$$

Thus we obtain

$$I = \int \frac{\cos^3 \theta}{a^3} \; \frac{a}{\cos^2 \theta}\, d\theta = \frac{1}{a^2} \int \cos \theta \, d\theta$$

$$= \frac{\sin \theta}{a^2} + C = \frac{x}{a^2 \sqrt{a^2 + x^2}}$$

Example A4-1c The integration

$$I = \int \frac{1}{x^2 \sqrt{x^2 - a^2}}\, dx$$

is performed with the aid of the substitution (see Figure A4-1c)*

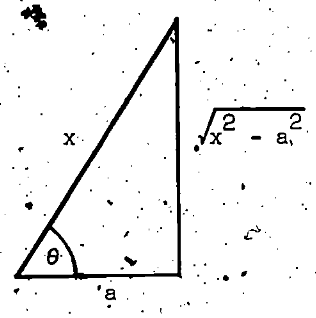


Figure A4-1c

$$x = \frac{a}{\cos\theta},$$

$$dx = \frac{a\sin\theta}{\cos^2\theta}\,d\theta,$$

$$\sqrt{x^2 - a^2} = a\tan\theta.$$

We have

$$I = \int\left(\frac{\cos^2\theta}{a^2}\right)\left(\frac{1}{a\tan\theta}\right)\left(\frac{a\sin\theta}{\cos^2\theta}\right)d\theta$$

$$= \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{\sin\theta}{a^2} + C = \frac{\sqrt{x^2 - a^2}}{a^2 x}.$$

Example A4-1d. Consider the integral

$$I = \int\frac{1}{\sqrt{x^2 - a^2}}\,dx.$$

Using the substitution of Example A4-1c we obtain

$$I = \int\frac{1}{a\tan\theta}\left(\frac{a\sin\theta}{\cos^2\theta}\right)d\theta = \int\frac{1}{\cos\theta}\,d\theta.$$

To complete the job algebraic trickery is needed (the objective of the manipulations will be clearer after Section A4-3 on decompositions into partial fractions). We have

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{\cos^2\theta} = \frac{\cos\theta}{1 - \sin^2\theta} = \frac{\cos\theta}{2}\left[\frac{1}{1 - \sin\theta} + \frac{1}{1 + \sin\theta}\right].$$

With this much as a hint we leave the integration as an exercise.

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*Here take $0 < \theta < \frac{\pi}{2}$ for $x > 0$, and $\frac{\pi}{2} < \theta < \pi$ for $x < 0$.

THEOREM A4-1a. An integral of any rational combination of $x$ and $\sqrt{Q(x)}$ where

(1) $$Q(x) = Ax^2 + Bx + C, \qquad (A \neq 0)$$

can be transformed by a substitution $x = f(\theta)$, where $f$ is a circular function, into an integral of a rational combination of $\sin \theta$ and $\cos \theta$.

Proof. We are concerned with integrals of the form

(2) $$I = \int \phi\left(x, \sqrt{Q(x)}\right) dx$$

where $\phi$ is a rational expression and $Q(x)$ is given by (1). For the proof we first make a preliminary linear transformation to replace $Q(x)$ by one of the standard forms of Examples A4-1a, b, c.

We "complete the square" to obtain

(3) $$Q(x) = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right].$$

We set $a = \sqrt{\left|\frac{C}{A} - \frac{B^2}{4A^2}\right|}$, $b = \frac{B}{2A}$, $c = \sqrt{|A|}$, and $x = u - b$ in (3), and separate the problem into three cases.

Case (i).

If $A < 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} < 0$ we have

$$\sqrt{Q(x)} = c\sqrt{a^2 - u^2}.$$

Since $dx = du$, the substitution $x = u - b$ yields

(4) $$I = \int \phi(u - b, c\sqrt{a^2 - u^2})du.$$

Now, employing the substitution $u = a \sin \theta$ of Example A4-1a, we transform the integral into the form

(5) $$I = a\int \phi(a \sin \theta - b, c\, a \cos \theta) \cos \theta \, d\theta, \quad \theta = \arcsin \frac{x + b}{a}.$$

Since $\phi$ involves only rational operations, we have established the theorem in this case.

Case (ii).

If $A > 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} < 0$, the substitution

$$x + b = u = a \tan \theta,$$

as in Example A4-1b, confirms the theorem for this case.

Case (iii).

If $A > 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} > 0$, the substitution

$$x + b = u = \frac{a}{\cos \theta},$$

as in Examples A4-1c, yields the desired result.

The integral (2) can be also transformed into an integral of a rational combination of $\sinh t$ and $\cosh t$ by an appropriate transformation $x = f(t)$ where $f$ is a hyperbolic function. The proof is left as an exercise.

THEOREM A4-1b. An integral of a rational combination of $\sin x$ and $\cos x$ can be transformed into an integral of a rational function by a suitable substitution.

Proof. We consider integrals of the form

(8) $$\int \psi(\sin x, \cos x)dx$$

where $\psi$ is a rational expression. We observe that $\sin x$ and $\cos x$ are rational expressions in $t = \tan \frac{x}{2}$; namely,

(9) $$\sin x = \frac{2t}{1 + t^2}, \quad \cos x = \frac{1 - t^2}{1 + t^2}.$$

Furthermore,

(10) $$dx = d(2 \arctan t) = \frac{2}{1 + t^2} dt.$$

Consequently we may transform the integral (8) into the integral of a rational function by employing the substitution

(11) $$x = 2 \arctan t;$$

thus, entering (9) and (10) in (8) we obtain the integral in the form

$$(12)\qquad \int \psi\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,dt .$$

Theorems 10-3a and 10-3b do not necessarily point the way to the simplest method of integration for a function of one of the types considered here; they simply indicate a line of approach which is sure to work but may lead to enormous complication. Often some special device leads to the solution far more simply and directly.

## Exercises A4-1

1. Integrate the following functions, the numbers $a$ and $b$ being positive.

(a) $\dfrac{\sqrt{a^2 - x^2}}{x^2}$

(b) $\dfrac{\sqrt{1 + x^2}}{x^4}$

(c) $x^2\sqrt{a^2 - x^2}$

(d) $\dfrac{1}{x^2\sqrt{x^2 - a^2}}$

(e) $\dfrac{x}{(x^2 + a^2)\sqrt{x^2 - b^2}}$

(f) $\dfrac{1}{(x^2 + a^2)\sqrt{a^2x^2 + 1}}$

(g) $\dfrac{x + 2}{\sqrt{m^2 \pm x^2}}$

(h) $x^3\sqrt{(4 - x^2)^5}$

(i) $\dfrac{1}{\sqrt{a^2x - x^2}}$

(j) $\dfrac{x^2 + ax + b}{x^2 + 1}$

(k) $\sqrt{a^2x + x^2}$

2. Let $R(x,y)$ denote a rational function in $x$ and $y$. Reduce the following integrals to integrals of rational functions.

(a) $\displaystyle\int R(x, \sqrt{ax + b})\,dx, \quad a \neq 0.$

(b) $\displaystyle\int R\left(x, \sqrt[n]{\frac{ax + b}{cx + d}}\right)dx,$ $n$ an integer, $ad - bc \neq 0.$

3. Using the result of Number 2, integrate $\dfrac{x}{\sqrt{ax + b} + \sqrt{(ax + b)^3}}$.

4. Reduce to rational form $\displaystyle\int \frac{dx}{\sqrt{\dfrac{1 - x}{1 + x}} + \sqrt[4]{\dfrac{1 - x}{1 + x}}}$.

5. Express as elementary functions

   (a) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$

   (b) $\displaystyle\int \frac{dx}{1 + \sin x}$

   (c) $\displaystyle\int \frac{dx}{1 - \cos 2x}$

   (d) $\displaystyle\int \frac{dx}{x \sqrt[4]{1 + x^4}}$

   (e) $\displaystyle\int \frac{dx}{\sqrt[4]{1 + x^4}}$

6. (a) The integral $\displaystyle\int \frac{P(x)}{\sqrt{ax^2 + 2bx + c}}\,dx$, where $P(x)$ is a polynomial of degree $n$ and $a \neq 0$ can be reduced to a rational trigonometric form as described in the text. It can be also reduced to the integration of $\dfrac{1}{\sqrt{ax^2 + 2bx + c}}$; namely for some polynomial $Q$ of degree $(n - 1)$ and constant $k$.

$$\frac{P(x)}{\sqrt{ax^2 + 2bx + c}} = D\left(Q(x)\sqrt{ax^2 + 2bx + c}\right) + \frac{k}{\sqrt{ax^2 + 2bx + c}}.$$

   Show how to find $Q$ and $k$.

   (b) Using (a), integrate $\dfrac{t^5 - t^3 + t}{\sqrt{1 - t^2}}$.

   (c) Calculate the integral of (b) by using trigonometric substitutions, and compare the merits of the two methods.

7. Integrate

(a) $\frac{1}{\sin x}$

(b) $\frac{1}{\cos x}$ (by a method other than that of Example A4-1d).

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## A4-2: Integration by Parts

(i) The basic formula. The method of integration by parts is used to integrate certain kinds of products. The method corresponds to the formula for the derivative of a product.

THEOREM A4-2a. If $f$ and $g$ have continuous derivatives over a common interval containing $a$ and $b$ then.

$$(1)\qquad \int_a^b f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x)g(x)dx.$$

The theorem follows directly from the product rule ((4) of Section 8-4) and the Fundamental Theorem of Calculus.

In Leibnizian notation, for $u = f(x)$, $du = f'(x)dx$ and $v = g(x)$, $dv = g'(x)dx$ we obtain for the definite integral corresponding to (1),

$$(2)\qquad \int u\,dv = uv - \int v\,du.$$

Integration by means of (2) is called integration by parts.

Example A4-2a. To integrate $x \to \log_e x$ observe that $\log_e x$ has an especially simple derivative and set $u = \log_e x$ and $dv = 1 \cdot dx$. For $v$, then, we take $v = x$. Consequently, from (2)

$$\int \log_e x\,dx = x\log_e x - \int \frac{x}{x}dx$$
$$= x\log_e x - x$$

the formula we have already obtained.

In application, (2) is used as above for the integral of a product where the product of the integral of one factor and the derivative of the other is formally integrable.

The Leibnizian notation in (2) was introduced as a shorthand for the explicit formula. But the notation suggests that we might interpret $u$ as a function of $v$, and $v$ as the inverse function of $u$. This idea yields an illuminating geometrical interpretation of integration by parts. Suppose that $u = f(x)$ and $v = g(x)$ where $f$ and $g$ have inverses. Then we can write $u = \phi(v)$ and $v = \psi(u)$ where $\phi$ and $\psi$ are inverses. (The proof is left to Exercises A4-2, No. 2). Set $u_0 = f(a)$, $u_1 = f(b)$ and $v_0 = g(a)$, $v_1 = g(b)$. We have $u_i = \phi(v_i)$ and, inversely, $v_i = \psi(u_i)$ for $i = 1, 2$. Now suppose $\phi$ and $\psi$ are increasing and nonnegative. Then, from the familiar interpretation of integral as area (see Figure A4-2a) we immediately have

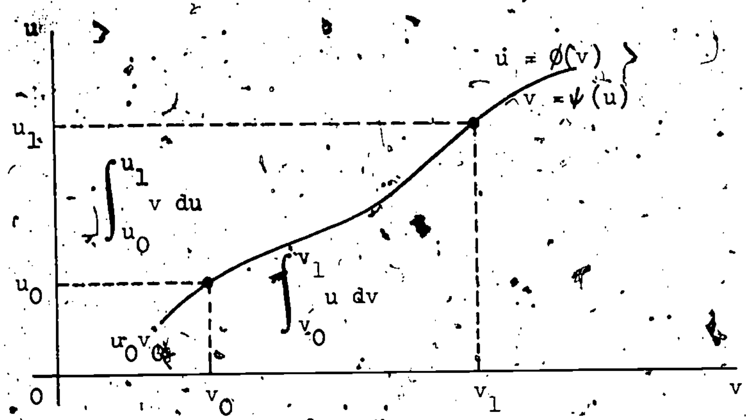


Figure A4-2a

$$u_1v_1 = \int_{v_0}^{v_1} u\,dv + \int_{u_0}^{u_1} v\,du + u_0v_0,$$ from which we at once obtain

$$\int_{v_0}^{v_1} u\,dv = [u_1v_1 - u_0u_0] - \int_{u_0}^{u_1} v\,du.$$

From the Substitution Rule we immediately recognize this equation as a form of (1). A like geometrical argument gives the same result when $\phi$ and $\psi$ are decreasing.

In general, this interpretation of integration by parts gives the formal integral of any function which has a formally integrable inverse.

Example A4-2b. Consider

$$\int x^n \arcsin x\,dx, \qquad (n \text{ integral, } n \neq -1).$$

Since the arcsin has a simple algebraic derivative we set $u = \arcsin x$, $dv = x^n dx$ and take $v = \frac{x^{n+1}}{n+1}$. For the domain $0 < x \leq \frac{\pi}{2}$ we have $u = \arcsin \sqrt[n+1]{(n+1)v}$ and $v = \frac{1}{n+1} \sin^{n+1} u$. From Theorem A4-1b we know that $\int v\,du$ can be transformed into the integral of a rational function. As we shall see (Section A4-3) rational functions are always formally integrable. It follows that $\sin^{n+1} u$ is formally integrable with respect to $u$ and hence that $x^n \arcsin x$ is formally integrable with respect to $x$. Reduction to the integral of a rational function is not necessarily the most efficient way to carry out these integrations, but integration by parts can be used more effectively in other ways to execute the integrations.

The idea of Example A4-2b, for $u = f(x)$ $dv = x^n\,dx$, establishes the formal integrability of $x^n f(x)$ where $f$ is any inverse circular function, and, in view of Example A4-2a, if $f(x) = \log x$.

Example A4-2c. Consider

$$\int x^r \log x\,dx, \qquad (r \text{ real}).$$

Since $\log x$ has a simple derivative, we set $u = \log x$, $dv = x^r dx$. If $r \neq -1$ we take $v = \frac{x^{r+1}}{r+1}$ to obtain

$$\int x^r \log x\,dx = \frac{x^{r+1}}{r+1} \log x - \frac{1}{r+1} \int x^r\,dx$$

$$= \frac{x^{r+1}}{r+1} \log x - \frac{x^{r+1}}{(r+1)^2}.$$

If $r = -1$, we may take $v = \log x$ to obtain

$$\int \frac{\log x}{x}\,dx = (\log x)^2 - \int \frac{\log x}{x}\,dx,$$

which yields

$$\int \frac{\log x}{x}\,dx = \frac{(\log x)^2}{2} + C,$$

- a result which is obtained more directly from the substitution $\log x = t$.

The method of Example A4-2c, for $u = f(x)$ and $dv = x^n dx$, exhibits the formal integrability of any function of the form $x^n f(x)$, when $n \neq -1$, where $f'(x)$ is any rational combination of $x$ and $\sqrt{Q(x)}$ and $Q(x)$ is a quadratic polynomial. Integration by parts expresses the given integral in terms of the integral of $\frac{x^{n+1}}{n+1} f'(x)$ which may be transformed into the integral of a rational function by Theorem A4-1a. From the assumed integrability of rational functions, the result follows. It follows as a slight generalization that $P(x)f(x)$ is formally integrable for any polynomial function $P$. From this argument we observe again that if $f$ is a logarithmic or inverse circular function, then $x^n f(x)$ is formally integrable. In addition, for $h(x) = \phi(x, \sqrt{Q(x)})$, a rational combination of $x$ and $\sqrt{Q(x)}$, the expressions $x^n \log h(x)$ and $x^n \arctan h(x)$ and are all formally integrable since the derivatives of log and arctan and are rational functions.

Example A4-2d. Consider the integral

$$\int x\, e^x\, dx$$

We integrate by parts. Set $u = x$ $dv = e^x dx$ and $v = e^x$. Then by (2)

$$\begin{aligned}\int x\, e^x\, dx &= xe^x - \int e^x\, dx\\ &= xe^x - e^x.\end{aligned}$$

Integration by parts may be used to produce a simplification rather than a final complete integration as in Example A4-2c when $r = -1$.

Example A4-2e. Consider

$$I = \int e^{bx} \sin ax \, dx$$

For $u = \sin ax$, $dv = e^{bx}\,dx$, $v = \frac{e^{bx}}{b}$, we obtain

$$I = \frac{1}{b} e^{bx} \sin ax - \frac{a}{b} \int e^{bx} \cos ax \, dx$$

$$= \frac{1}{b} e^{bx} \sin ax - \frac{a}{b} J,$$

where

$$J = \int e^{bx} \cos ax \, dx$$

presents the same difficulties of formal integration as $I$. However, by the same technique, we can express $J$ in terms of $I$ and hopefully may obtain an equation which can be solved for $I$. Now take $u = \cos ax$ and $v = \frac{e^{bx}}{b}$ in (2) to obtain

$$J = \frac{1}{b} e^{bx} \cos ax + \frac{a}{b} \int e^{bx} \sin ax \, dx$$

$$= \frac{1}{b} e^{bx} \cos ax + \frac{a}{b} I.$$

Entering the expression for $J$ above in the expression for $I$ and solving for $I$, we obtain

$$I = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax).$$

(ii) Recurrence relations. The idea here is to express an integral of the general form $\int f_n(x)\,dx$ in terms of $\int f_{n-k}(x)\,dx$.

Example A4-2f. Consider

$$I_n = \int x^r(1 - x)^n\, dx \qquad , (n \geq 0, r \neq -1).$$

Set $u = (1 - x)^n$, $dv = x^r dx$, $v = \frac{x^{r+1}}{r + 1}$. Then

$$I_n = \frac{x^{r+1}(1 - x)^n}{r + 1} + \frac{n}{r + 1}\int x^{r+1}(1 - x)^{n-1} dx$$

where, for $n = 0$, the result yields, correctly, $I_n = \frac{x^{r+1}}{r + 1}$. Now, observe that

$$x^{r+1}(1 - x)^{n-1} = -x^r[(1 - x)^n - (1 - x)^{n-1}];$$

whence,

$$I_n = \frac{x^{r+1}(1 - x)^n}{r + 1} + \frac{n}{r + 1}[I_{n-1} - I_n].$$

This equation may then be solved for $I_n$ in terms of $I_{n-1}$:

$$I_n = \frac{x^{r+1}(1 - x)^n}{n + r + 1} + \frac{n}{n + r + 1} I_{n-1}$$

or

$$\int x^r(1 - x)^n\, dx = \frac{x^{r+1}(1 - x)^n}{n + r + 1} + \frac{n}{n + r + 1}\int x^r(1 - x)^{n-1} dx.$$

Now this formula may be applied recursively to express $I_{n-1}$ in terms of $I_{n-2}$, $I_{n-2}$ in terms of $I_{n-3}$, etc., to yield

$$I_n = \frac{x^{r+1}}{n + r + 1}\left[(1 - x)^n + \frac{n(1 - x)^{n-1}}{n + r} + \frac{n(n - 1)(1 - x)^{n-2}}{(n + r)(n + r - 1)} + \dots + \frac{n(n - 1)\ \dots\ 1}{(n + r)(n + r - 1)\ \dots\ (r + 1)}\right]$$

Sometimes it is necessary to prepare for integration by parts by some preliminary rearrangement, as we show in the following useful example.

<u>Example</u> <u>A4-2g</u>. Consider

$$I_n = \int \cos^n x \, dx$$

We write $\cos^n x = \cos^{n-1} x \cos x$, set $u = \cos^{n-1} x$, $dv = \cos x \, dx$, $v = \sin x$, to obtain

$$I_n = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \sin^2 x \, dx$$

$$= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x (1 - \cos^2 x) dx.$$

Thus,

$$I_n = \cos^{n-1} x \sin x + (n - 1)[I_{n-2} - I_n].$$

Solving for $I_n$, we have

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n - 1}{n} I_{n-2}.$$

Since the subscript is lowered by 2 at each step we observe for $n$ even that the recursive reduction of the integral terminates at $n = 0$ with $I_0 = \int dx = x$, and for $u$ odd, at $n = 1$ with $I_1 = \int \cos x \, dx = \sin x$.

Often the principle use of a recurrence relation is not to obtain the formal integral in terms of elementary functions (which may not be possible) but to obtain the original integral in terms of a simpler integral.

<u>Example</u> <u>A4-2h</u>. Consider

$$I_n = \int x^n e^{-x^2} dx.$$

From $u = x^{n-1}$, $dv = x e^{-x^2} dx$, $v = -\frac{1}{2} e^{-x^2}$, we obtain

$$I_n = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{(n - 1)}{2} \int x^{n-2} e^{-x^2} dx$$

or

$$I_n = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} I_{n-2}.$$

If $n$ is odd, the recurrence relation gives $I_n$ in terms of elementary functions and $I_1$, but $I_1 = -\frac{1}{2} e^{-x^2}$ is elementary and $I_n$ is formally integrable in terms of elementary functions. If $n$ is even, then the integration of $I_n$ is reduced to the integration of

$$I_0 = \int e^{-x^2}\, dx.$$

This integral is not elementary. However, it is well known and much used. In terms of the error function erf (the area under the normal probability curve) given by

$$\operatorname{erf} x = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\, dt$$

we have

$$I_0 = \sqrt{\pi}\, \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

The common tables of the error function enable us to work with it numerically just as conveniently as the circular functions.

## Exercises A4-2

1. Integrate the following.

(a) $x \sin 3x$

(b) $x \cdot 5^x$

(c) $x^3 e^{-2x}$

(d) $\sqrt{x} \log ax$

(e) $\log^2 bx$

(f) $\log^3 x$

(g) $\arccos 7x$

(h) $\arctan \sqrt[3]{x}$

(i) $x \arctan x$

(j) $\dfrac{\arccos x/m}{\sqrt{x+m}}$

(k) $x \sin^2 x$

(ℓ) $x^2 \sin x$

(m) $x^2 \arcsin ax$

(n) $\cos^3 2x$

(o) $\sin^5 x$

(p) $\sin (\log ax)$

(q) $x \tan^2 x$

(r) $(\arcsin x)^2$

(s) $\sin ax \cos bx$

2. Support the geometrical interpretation of integration by parts by showing for $u = f(x)$ and $v = g(x)$ where $f$ and $g$ have inverses, that $u = \phi(v)$ and $v = \psi(u)$ where $\phi$ and $\psi$ are inverse functions.

3. Verify as alleged after Example A4-2b that the method of the example does demonstrate the reducibility of $\int x^n f(x)dx$ to the integral of a rational function if $f$ is any inverse circular function, or if $f$ is the logarithmic function.

4. Establish recurrence relations for each of the following (in each case $m$ and $n$ are positive integers).

(a) $\int \sin^n x \, dx$

(b) $\int x^m \log^n x \, dx$

(c) $\int \sin^m x \cos^n x \, dx$

(d) $\int x^n \arctan x \, dx$

(e) $\int x^n e^{ax} \, dx$

(f) $\int x^n \arcsin x \, dx$

(g) $\int \frac{1}{\sin^n x} \, dx$

(h) $\int \frac{e^x}{x^n} \, dx$

(i) $\int x^n \cos x \, dx$

(Note the difference between $n$ odd and $n$ even).

## A4-3. Integration of Rational Functions

The problems of formal integration in the preceding sections of this appendix were often recast in the form of the problem of integrating a rational function. For a rational function there always exists a formal integral in terms of elementary functions. The formal integral is obtained by reducing the rational function to a sum of a polynomial function and functions defined by the elementary forms

(1) $$\frac{r}{(x - c)^n},$$

(2) $$\frac{px + q}{[(x - a)^2 + b^2]^n}, \qquad (b > 0).$$

It can be proved that such a reduction is possible, either from the Fundamental Theorem of Algebra which requires the theory of functions of a complex variable, or directly by new algebraic techniques. In either case a complete proof would take us outside the frame of this text.

The reduction of a rational function into the sum of a polynomial and terms of the form (1) and (2) is called a decomposition into partial fractions. We give one simple example.

Example A4-3a. A common case is given by the rational expression

(3) $$\frac{1}{(x - a)(x - b)} = \frac{1}{b - a}\left(\frac{1}{x - b} - \frac{1}{x - a}\right), \qquad a \neq b.$$

From the decomposition (3) we immediately obtain the integral

$$\int \frac{1}{(x - a)(x - b)} = \frac{1}{b - a}(\log(x - b) - \log(x - a))$$
$$= \frac{1}{b - a} \log \left(\frac{x - b}{x - a}\right).$$

Let $R$ be any rational function. By long division it is always possible to put $R(x)$ in the form

$$R(x) = S(x) + \frac{P(x)}{Q(x)}$$

where $S$, $P$, $Q$ are polynomials and the degree of $P$ is less than that of $Q$. Since the polynomial $S$ is immediately integrable, we may omit it from consideration. It follows from the Fundamental Theorem of Algebra (Appendix 2) that every polynomial $Q(x)$ with real coefficients has a unique factorization of the form

$$(4)\quad Q(x) = A(x-c_1)^{n_1}(x-c_2)^{n_2}\cdots[(x-a_1)^2+b_1^2]^{m_1}[(x-a_2)^2+b_2^2]^{m_2}\cdots$$

where the $c_k$ are the distinct real roots of $Q$, and $a_k \pm ib_k$, the distinct imaginary roots $(b_k > 0)$.

Now suppose that $R(x) = \frac{P(x)}{Q(x)}$, where the degree of $P$ is less than that of $Q$, and that $P$ and $Q$ have no common factors. Then we assert that $R(x)$ is the sum of expressions of two standard forms: for each real root $C$, an expression of the form

$$(5)\qquad \frac{r_1}{x-c} + \frac{r_2}{(x-c)^2} + \cdots + \frac{r_n}{(x-c)^n} \qquad (r_n \neq 0)$$

where $n$ is the multiplicity of $c$: for each pair of conjugate imaginary roots $a \pm ib$ an expression of the form

$$(6)\qquad \frac{p_1x+q_1}{(x-a)^2+b^2} + \frac{p_2x+q_2}{[(x-a)^2+b^2]^2} + \cdots + \frac{p_mx+q_m}{[(x-a)^2+b^2]^m}, \qquad (p_m^2+q_m^2 \neq 0)$$

where $m$ is their common multiplicity. We merely use this format as a guide without proof. In each particular case it can be verified directly that the decomposition obtained is correct. Once we have obtained and verified the correctness of the partial fraction decomposition we have reduced the integration problem to that of integrating the simple form (1) and (2).

Before we embark on the problem of integration let us see what is involved in the algebraic problem of obtaining the partial fraction decomposition. The first problem is to obtain the roots of the polynomial $Q(x)$. In general the roots of a polynomial cannot be obtained from the coefficients by a formula involving only rational operations and rational powers. There are such formulas for the roots of polynomials of third and fourth degree, but these formulas are generally useless. For example, the formula for the roots of a polynomial of third degree may involve complex quantities even when all three roots are real. For computational purposes it would be sufficient to estimate the roots numerically, but it is usually easier to estimate the integral directly (see Chapter 9). Nonetheless, the method of decomposition is valuable because often the factorization of $Q(x)$ is given by the conditions of the problem and often the factorization is easily obtained.

Next, we turn our attention to the problem of obtaining the partial fraction decomposition once the denominator is given in factored form.

First we consider the problem of obtaining the partial fraction decomposition of

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - c_1)(x - c_2) \cdots (x - c_n)}$$

where the roots of $Q$ are all real and simple (of multiplicity 1) and the degree of $P$ is less than that of $Q$. From the foregoing, there exist constants $A_k$, $(k = 1, 2, \ldots, n)$ such that

(7) $$\frac{P(x)}{Q(x)} = \frac{A_1}{x - c_1} + \frac{A_2}{x - c_2} + \cdots + \frac{A_n}{x - c_n}.$$

For $x \neq c_1$ we obtain on multiplication by $(x - c_1)$

$$A_1 = \frac{P(x)(x - c_1)}{Q(x)} - S(x)(x - c_1) = T(x)$$

where $S(x)$ is the sum of all the partial fractions but the first. In a neighborhood of $x = c_1$ this equation states that the expression $T(x)$ defines the constant function $T : x \to A_1$. Therefore

$$A_1 = \lim_{x \to c_1} \frac{P(x)(x - c_1)}{Q(x)}$$
$$= \lim_{x \to c_1} \frac{P(x)}{(x - c_2)(x - c_3) \cdots (c_1 - c_n)}$$

whence,

(8) $$A_1 = \frac{P(c_1)}{(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)}.$$

This last expression can be written tidily if we observe that since $Q(c_1) = 0$,

$$\lim_{x \to c_1} \frac{Q(x)}{(x - c_1)} = \lim_{x \to c_1} \frac{Q(x) - Q(c_1)}{x - c_1} = Q'(c_1).$$

Thus $A_1 = \dfrac{P(c_1)}{Q'(c_1)}$. Since $c_1$ is simply a symbol for any one of the roots, it does not matter which for the purpose of this discussion, we have in general

(9) $$A_k = \frac{P(c_k)}{Q'(c_k)}.$$

Example A4-3b. We obtain the partial fraction decomposition of

$$\frac{x^2 + x - 1}{(x + 1)x(x - 1)}$$

Here $P(x) = x^2 + x - 1$, $Q(x) = x^3 - x$; $Q'(x) = 3x^2 - 1$. The denominator has simple zeros at $-1$, $0$, and $1$. From

$$\frac{P(-1)}{Q'(-1)} = \frac{-1}{2}, \quad \frac{P(0)}{Q'(0)} = \frac{-1}{-1}, \quad \frac{P(1)}{Q'(1)} = \frac{1}{2},$$

we have

$$\frac{P(x)}{Q(x)} = -\frac{1}{2(x + 1)} + \frac{1}{x} + \frac{1}{2(x - 1)},$$

which is easily verified to be correct.

There are general techniques for the case of multiple real roots or imaginary roots, but in such cases it is often easier to determine the decomposition by the method of equated coefficients.*

Example A4-3c. From

$$\frac{x^3 - 1}{x(x^2 + 1)^2} = \frac{r}{x} + \frac{p_1x + q_1}{x^2 + 1} + \frac{p_2x + q_2}{(x^2 + 1)^2}$$

we obtain on multiplying both sides by $x(x^2 + 1)^2$

$$\begin{aligned} x^3 - 1 &= r(x^4 + 2x^2 + 1) + p_1(x^4 + x^2) + q_1(x^3 + x) + p_2x^2 + q_2x \\ &= (r + p_1)x^4 + q_1x^3 + (2r + p_1 + p_2)x^2 + (q_1 + q_2)x + r, \end{aligned}$$

provided $x \neq 0$. Now the coefficients of like powers on the right and left must be equal (Exercises A4-3, No.3). Thus we obtain the equations

$$\begin{aligned} r + p_1 &= 0 \\ q_1 &= 1 \\ 2r + p_1 + p_2 &= 0 \\ q_1 + q_2 &= 0 \\ r &= -1, \end{aligned}$$

from which $r = -1$, $p_1 = 1$, $q_1 = 1$, $q_2 = -1$, $p_2 = 1$. This yields

*Also called the method of undetermined coefficients.

$$\frac{x^3 - 1}{x(x^2 + 1)^2} = -\frac{1}{x} + \frac{x + 1}{x^2 + 1} + \frac{x - 1}{(x^2 + 1)^2}$$

which is easily verified to be correct.

Given the partial fraction décomposition of a rational function we complete the work of formal integration by showing how to integrate the standard forms (1) and (2). For (1) the integrals are already found. If $n > 1$, we have

(10a) $$\int \frac{r}{(x - c)^n} dx = -\frac{r}{(n - 1)(x - c)^{n-1}} + C$$

and if $n = 1$, then

(10b) $$\int \frac{r}{x - 1} dx = r \log |x - 1| + C.$$

For (2) we introduce the substitution

$$(x - a) = b \tan u \qquad (-\tfrac{\pi}{2} \le u \le \tfrac{\pi}{2}),$$

where we assume $b > 0$ (compare Example A4-1b). Using $dx = \frac{b}{\cos^2 u} du$ we obtain

$$\int \frac{px + q}{[(x - a)^2 + b^2]^n} dx = \int \frac{p \tan u + pa + q}{b^{2n}[1 + \tan^2 u]^n} \frac{b}{\cos^2 u} du$$

$$= \frac{p}{b^{2n-1}} \int \cos^{2n-3} u \sin u \, du + \frac{pa + q}{b^{2n-1}} \int \cos^{2n-2} u \, du.$$

Of the last two integrals, the first is immediately formally integrable and the second is given by the recurrence relation of Example A4-2g. We leave as an exercise the problem of completing the integration and representing the formal integral in terms of $x$. The resulting integral is a sum of terms of the following types,

(11a) $$\frac{Ax + B}{[(x - a)^2 + b^2]^k}$$

where $k$ is a positive integer, $k < n$,

(11b) $$A \log [(x - a)^2 + b^2],$$

(11c) $$A \arctan \frac{x - a}{b}.$$

Finally, we observe that if we know the factorization of $Q(x)$ we know the form of the integral of $\frac{P(x)}{Q(x)}$ from (10) and (11). Therefore it is sufficient to differentiate this form and determine the constants by the method of equated coefficients.

Example A4-3d. Consider

$$\int \frac{x + 1}{x^2(x^2 + 4)} \, dx ,$$

The integral must be of the form

$$a \log x + \frac{b}{x} + \alpha \log (x^2 + 4) + \beta \arctan \frac{x}{2} + C .$$

The derivative of this expression is

$$\frac{a}{x} - \frac{b}{x^2} + \frac{2\alpha x}{x^2 + 4} + \frac{2\beta}{x^2 + 4} = \frac{(a + 2\alpha)x^3 + (2\beta - b)x^2 + 4ax - 4b}{x^2(x^2 + 4)}$$

Since the numerator of this expression should be $x + 1$ we have on equating coefficients

$$a + 2\alpha = 0, \quad 2\beta - b = 0, \quad 4a = 1, \quad -4b = 1,$$

whence

$$a = \frac{1}{4}, \; b = -\frac{1}{4}, \; \alpha = -\frac{1}{8}, \; \beta = -\frac{1}{8} .$$

It is easy to verify that this yields the correct integral.

## Exercises A4-3

1. Integrate the following

(a) $\dfrac{x + 2}{x^2 + 3x + 1}$

(b) $\dfrac{x^3}{x^2 + 3x - 10}$

(c) $\dfrac{x^3}{x^2 + 2ax + b^2}$ $(b > |a|)$

(d) $\dfrac{x^2 + \alpha x + \beta}{(x - a)(x - b)}$

(Consider the cases $a \neq b$ and $a = b$)

(e) $\dfrac{x^2}{(x - a)(x - b)(x - c)}$ $(a \neq b \neq c)$

(f) $\dfrac{x^3 + 1}{x^3 - 1}$

(g) $\dfrac{1}{x^3 + a^2}$

(h) $\dfrac{(x + 2)^2}{x(x - 1)^2}$

(i) $\frac{1}{x^4 - 1}$

(j) $\frac{x^2}{x^4 - 1}$

(k) $\frac{1}{x^6 + x^4}$

(ℓ) $\frac{x^4}{x^4 + 1}$

(m) $\frac{1}{x^6 - 1}$

2. Prove from Equation (3) that if

$$Q(x) = (x - a_1)(x - a_2) \dots (x - a_n),$$ where

$a_1 < a_2 < \dots < a_n$, then $\frac{1}{Q(x)}$ has a decomposition into partial fractions of the form

$$\frac{1}{Q(x)} = \frac{r_1}{x - a_1} + \frac{r_2}{x - a_2} + \dots + \frac{r_n}{x - a_n} .$$

3. Prove if

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$

for all but finitely many numbers $x$, that the coefficients of like powers on the right and left are equal; i.e., $a_k = b_k$ for $k = 0, 1, \dots, n$.

4. Verify that $\int \frac{px + q}{[(x - a)^2 + b^2]} dx$ can be expressed as the sum of terms of the forms (11a, b, c).

## A4-4. Definite Integrals

In Chapter 9 and earlier sections of this appendix we addressed ourselves primarily to the problem of finding the indefinite integral of a given function. In principle, this solves the problem of evaluating any definite integral of the function. In practice, it is often desirable or necessary to evaluate a definite integral, not by formal integration, but by some other method altogether. It may be impossible to obtain an explicit representation of the indefinite integral in terms of elementary functions, yet some special symmetry may yield the value of a given definite integral effortlessly. Even if the formal expression for the indefinite integral is obtainable, the use of a symmetry condition may be a worthwhile shortcut. Often the idea of integral remains appropriate when the Riemann integral, as strictly defined, does not exist because the range or domain of the integrand may be unbounded. In these cases, we have to extend the definition of integral in a meaningful way. All these problems are treated in this section.

(i) Symmetry. Watch for symmetries; the observation that a symmetry exists often provides a direct solution to a problem or an important simplification. We have already pointed out one useful symmetry in Section 6-4.

If $f$ is an odd function and integrable on $[-a,a]$, then

(1) $$\int_{-a}^{a} f(x)dx = 0.$$

Example A4-4a. Consider

$$I = \int_{-\pi}^{\pi} x\, e^{x^2} \sin^4 x\, dx.$$

It is hopeless to find the indefinite integral, and it is not needed, since $I = 0$.

If $f$ is an integrable even function on $[-a,a]$, then

(2) $$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Example A4-4b. Consider

$$I = \int_{-x}^{x} (a_0 + a_1 t + a_2 t^2 + \dots + a_{2n} t^{2n}) dt.$$

The odd powers contribute zero and for the even powers we obtain

$$I = 2\int_{0}^{x} (a_0 + a_2 t^2 + \dots + a_{2n} t^{2n}) dx$$
$$= 2\left(a_0 x + \frac{a_2 x^3}{3} + \dots + \frac{a_{2n} x^{2n+1}}{2n + 1}\right).$$

Often an integral which exhibits no obvious symmetry can be transformed into a symmetric integral. This is specific for each case and no general rule for discovering such symmetries can be given.

Example A4-4c. Consider

$$I = \int_{-1}^{5} \sqrt[3]{x - 2}\, dx$$

Since the graph $y = \sqrt[3]{x - 2}$ has a center of symmetry at $x = 2$, we set $u = x - 2$ and find

$$I = \int_{-3}^{3} \sqrt[3]{u}\, du = 0.$$

Another important symmetry of a function is periodicity.

> If the function $f$ is integrable and periodic with period $p$, then the integrals of $f$ over intervals of length $p$ are all the same; i.e.,
>
> (3) $$\int_{a}^{a+p} f(x)dx = \int_{b}^{b+p} f(x)dx$$
>
> for all $a$ and $b$.

The statement is geometrically obvious. The graph $y = f(x)$ over any interval of length $p$ represents the complete graph in the sense that the picture of the function from $a$ to $p$ is identical to the picture from $a + kp$ to $a + (k + 1)p$ where $k$ is an integer. The entire graph can be thought of as a sequence of identical pictures of width $p$, laid end-to-end (Figure A4-4a). If a frame of width $p$ is laid over the graph (the

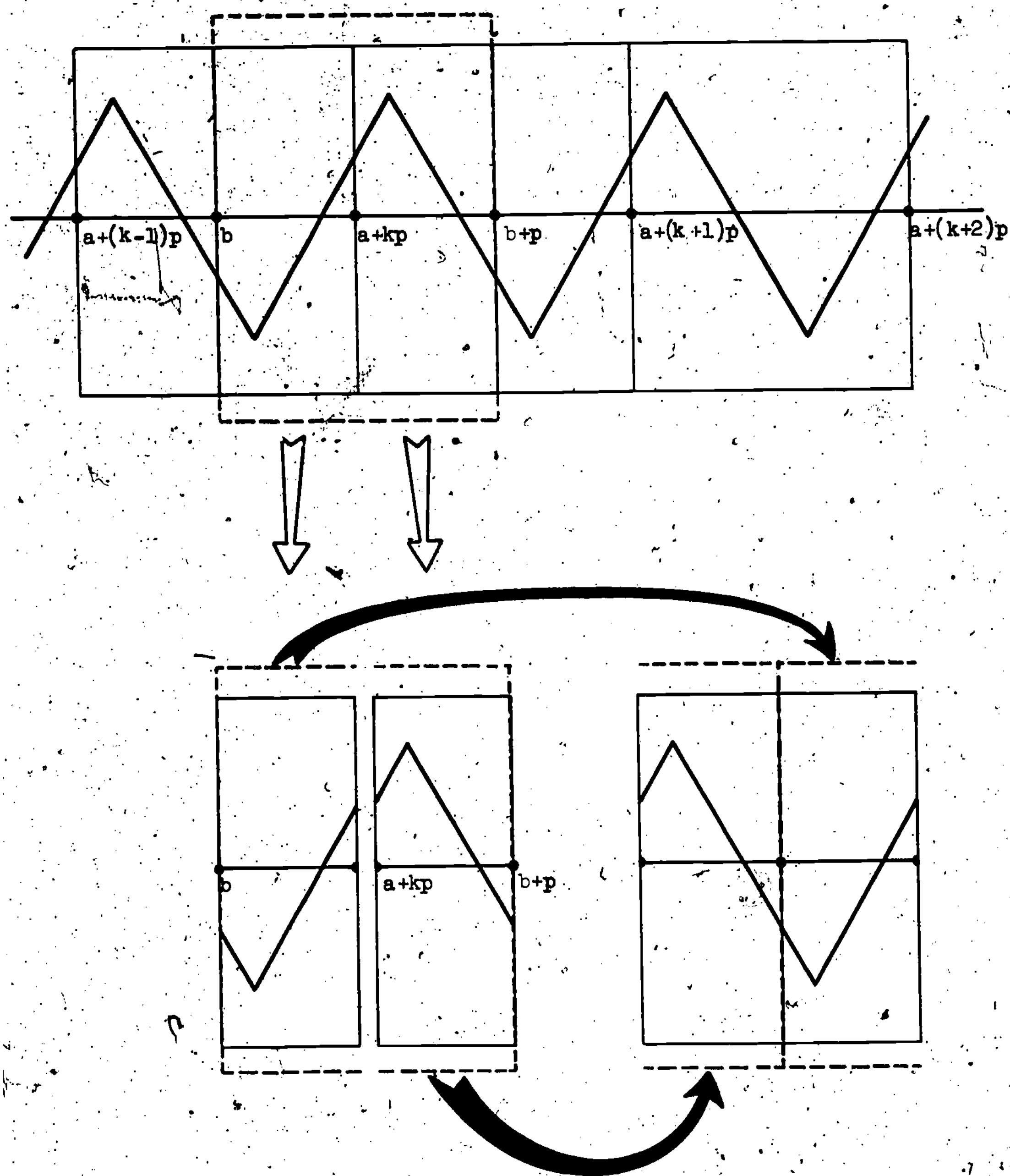


Figure A4-4a

interval $[b, b + p]$ in the figure) then the part of the total graph within the frame may be cut along a line $a + kp$ and reassembled to form the original picture by interchanging the two piece formed by the cut. This geometrical discussion is exactly paraphrased by the analytical proof. The proof is left to Exercises A4-4, Number 12.

Example A4-4d. Consider

$$I = \int_0^{n+1/4} (a_0 + a_1 \cos 2\pi x + \ldots + a_k \cos 2k\pi x)dx.$$

Since the integrand is periodic with period 1,

$$I = n\int_0^1 \sum_{\nu=0}^{k} a_\nu \cos 2\nu\pi x\, dx + \int_0^{1/4} \sum_{\nu=0}^{k} a_\nu \cos 2\nu\pi x\, dx.$$

For $\nu > 0$,

$$\int_0^1 \cos 2\nu\pi x\, dx = \frac{\sin 2\nu\pi x}{2\nu\pi}\Big|_0^1 = 0$$

and

$$\int_0^{1/4} \cos 2\nu\pi x\, dx = \frac{\sin(\frac{\nu\pi}{2})}{2\nu\pi}.$$

Consequently,

$$I = (n + \tfrac{1}{4})\, a_0 + \frac{a_1}{2\pi} - \frac{a_3}{6\pi} + \frac{a_5}{10\pi} - \ldots.$$

(ii) Special reductions. The general form of a recurrence relation for a definite integral is

$$\int_a^b f_n(x)dx = g_n(x)\Big|_a^b + c_n\int_a^b f_{n-1}(x)dx.$$

Quite often specific problems lead to integrals for which the "boundary" term

$$g_n(x)\Big|_a^b = g_n(b) - g_n(a),$$

is zero for $n > 0$, say. If so, we immediately have

$$\int_a^b f_n(x) = c_n c_{n-1} \cdots c_1 \int_a^b f_0(x).$$

Thus in Example A4-2f, we could conclude at once from

$$\int x^m(1 - x)^n\, dx = \frac{x^{m+1}(1 - x)^n}{m + n + 1} + \frac{n}{n + m + 1}\int x^m(1 - x)^{n-1}\, dx$$

that

$$\int_0^1 x^m(1 - x)^n\, dx = \frac{n(n - 1) \cdots 1}{(n + m + 1)(n + m) \cdots (m + 2)}\int_0^1 x^m\, dx$$

$$= \frac{n(n - 1) \cdots 1}{(n + m + 1)(n + m) \cdots (m + 1)}.$$

Thus we obtain an important connection with the binomial coefficients:

$$\int_0^1 x^m(1 - x)^n\, dx = \left[(n + m + 1)\binom{n+m}{m}\right]^{-1}.$$

Example A4-4e. A case of special interest is

$$I_\nu = \int_0^{\pi/2} \cos^n x\, dx.$$

From the result of Example A4-2g, we have

$$I_\nu = \frac{\cos^{\nu-1} x \sin x}{\nu}\Bigg|_0^{\pi/2} + \frac{\nu - 1}{\nu} I_{\nu-2}.$$

For $\nu > 1$, this yields simply

(4) $$I_\nu = \frac{\nu - 1}{\nu} I_{\nu-2}.$$

For $\nu$ even, $\nu = 2n$, we obtain

(5a) $$I_{2n} = \frac{(2n - 1)(2n - 3) \cdots 1}{2n(2n - 2) \cdots 2}\frac{\pi}{2}.$$

For $\nu$ odd, $\nu = 2n + 1$, we obtain

(5b) $$I_{2n+1} = \frac{2n(2n - 2) \cdots 2}{(2n + 1)(2n - 1) \cdots 3}.$$

From (5a) and (5b) there can be obtained a graceful representation of $\frac{\pi}{2}$ known as Wallis's Product.* Observe that

$$\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n-1)(2n+1)} \frac{I_{2n}}{I_{2n+1}} .$$

Now, since $0 \leq \cos x \leq 1$ on $[0, \frac{\pi}{2}]$ we have $\cos^{\nu+1} x \leq \cos^{\nu} x$ for all $\nu$ so that $I_{\nu+1} \leq I_{\nu}$. It follows that $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$, and since $I_{2n-1} = \frac{2n+1}{2n} I_{2n+1}$, that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq 1 + \frac{1}{2n} .$$

Taking limits we obtain $\lim\limits_{n \to \infty} \frac{I_{2n}}{I_{2n+1}} = 1$, whence

$$\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots$$

where by this infinite product, we mean simply

$$\lim_{n \to \infty} \left[ \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n-1)(2n+1)} \right]$$

$$= \lim_{n \to \infty} \frac{1}{2n+1} \left[ \frac{2^{2n} (n!)^2}{(2n)!} \right]^2$$

The verification that the two expressions in these limits are equal is left as an exercise.

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* John Wallis (1616 - 1703), English.

## Exercises A4-4

Evaluate the following definite integrals:

1. $\displaystyle\int_{-99}^{99} \frac{\sin^{99} \frac{x}{99}}{x^2 + (99)^2}\, dx$

2. $\displaystyle\int_0^1 x^3 e^{-3x^2}\, dx$

3. $\displaystyle\int_1^e \log^3 x\, dx$

4. $\displaystyle\int_0^{\pi/2} \sin^m x\, dx$, ($m$, a positive integer)

5. $\displaystyle\int_0^{\pi/2} \sin^m x \cos^m x\, dx$, ($m$, a positive integer)

6. $\displaystyle\int_0^{\pi/2} \frac{dx}{a + b\cos x}$, $a > b \geq 0$

7. $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^3 x\, dx$

8. $\displaystyle\int_1^2 \frac{dx}{x + x^5}$

9. $\displaystyle\int_0^b \sqrt{b^2 - x^2}\, dx$

10. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^5 \theta + 1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, d\theta$, $a > 0$, $b > 0$

11. Compare $\displaystyle\int_0^{-a} f(x)dx$ with $\displaystyle\int_{-a}^0 f(x)dx$ when $f$ is even, or odd to derive the results (1) and (2) of the text by a method other than the one you employed for Exercises 6-4, Number 4.

12. Prove if $f$ is integrable and periodic of period $p$, then for all $a$ and $b$

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx.$$

13. Prove that if $n \geq 2$ then

$$.500 < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} < .524.$$

14. Prove that $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x}\, dx = \pi^2.$

15. Show $\displaystyle\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n - 1)(2n + 1)} = \frac{1}{2n + 1}\left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2$

16. Determine the value exact to two decimal places of

$$\int_{1}^{e^{36.1}} \frac{\sin(\pi \log x)}{x}\, dx.$$

17. Evaluate

$$\int_{-\pi/4}^{\pi/4} \frac{t + \frac{\pi}{4}}{2 - \cos 2t}\, dt.$$

(Hint: Express the integrand as the sum of a symmetric part and an integrable part.)

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