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ABSTRACT
This is unit fifteen of a fifteen-unit SMSG secondary school text for high school students. The text is devoted almost entirely to mathematical concepts which all citizens should know in order to function satisfactorily in our society. Chapter topics include analyzing geometric figures and measurement. (MP)

SCHOOL MATHEMATICS STUDY GROUP

UNIT NUMBER FIFTEEN

Chapter 27. Analyzing Geometric Figures.

Chapter 28. Measurement

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Chapter 27

ANALYZING GEOMETRIC FIGURES

27-1. Convex Regions

To analyze properties of geometric figures you will need to use some concepts of geometry which you have already studied in earlier chapters. Important among these concepts are the ones concerning parallelism, perpendicularity, and convexity. In this section we shall study further some ideas regarding convex regions.

The two quadrilaterals in the drawings below seem fundamentally different to us.

One of the reasons they seem so different is that the region bounded by the left-hand quadrilateral is convex while the region bounded by the right-hand quadrilateral is not convex.

Definition: A set of points S is called convex if for every two points P and Q of S, each point of the segment $\overline{PQ}$ is in set S.

Some simple examples of convex sets are a line, a segment, and a ray. In this section we shall use the above definition to distinguish between two different types of quadrilaterals.

We will follow the custom of using the phrase <u>convex quadrilateral</u> in place of the cumbersome expression "quadrilateral that forms the boundary of a convex region". Note that the boundary is not itself a convex set.

Now if you take any quadrilateral you will find that besides the four sides there are two other segments joining vertices of the polygon. These are the two <u>diagonals</u>.

1-a 1-b

Figure 1

In the case of a non-convex quadrilateral (see Figure 1-b) we note that one diagonal has the property that it does not pass through the interior of the quadrilateral. Furthermore since the other diagonal does lie in the convex region, the two diagonals do not intersect. In the more familiar case of the convex quadrilateral (see Figure 1-a) both diagonals lie in the convex region and they of course intersect at an interior point of the quadrilateral.

The distinction between the two situations can be described in another way that will be useful for our purposes. For a non-convex quadrilateral such as ABCD (see Figure 2-b) <u>one diagonal</u> $\overline{BD}$ has the property that the line $\overleftrightarrow{BD}$ containing it does <u>not split the quadrilateral</u>. All the rest of the figure lies in one half-plane bounded by the line $\overleftrightarrow{BD}$. The line $\overleftrightarrow{AC}$ containing the other diagonal $\overline{AC}$ does split the quadrilateral so that part lies in one half-plane and part lies in the other. In particular, the vertices B, D not on that diagonal are in opposite half-planes.

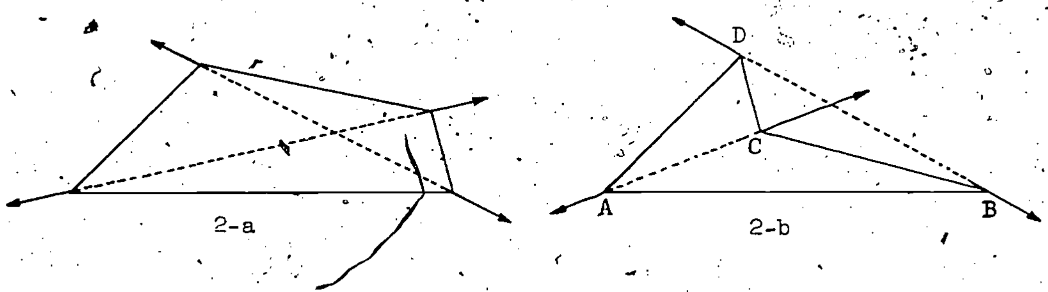


Figure 2

If <u>either</u> <u>diagonal</u> of <u>a</u> <u>convex</u> <u>quadrilateral</u> is given, then the line containing that diagonal <u>separates</u> the other two vertices, so that they lie in opposite half-planes bounded by that line.

We will take these properties of quadrilaterals for granted without attempting to prove them.

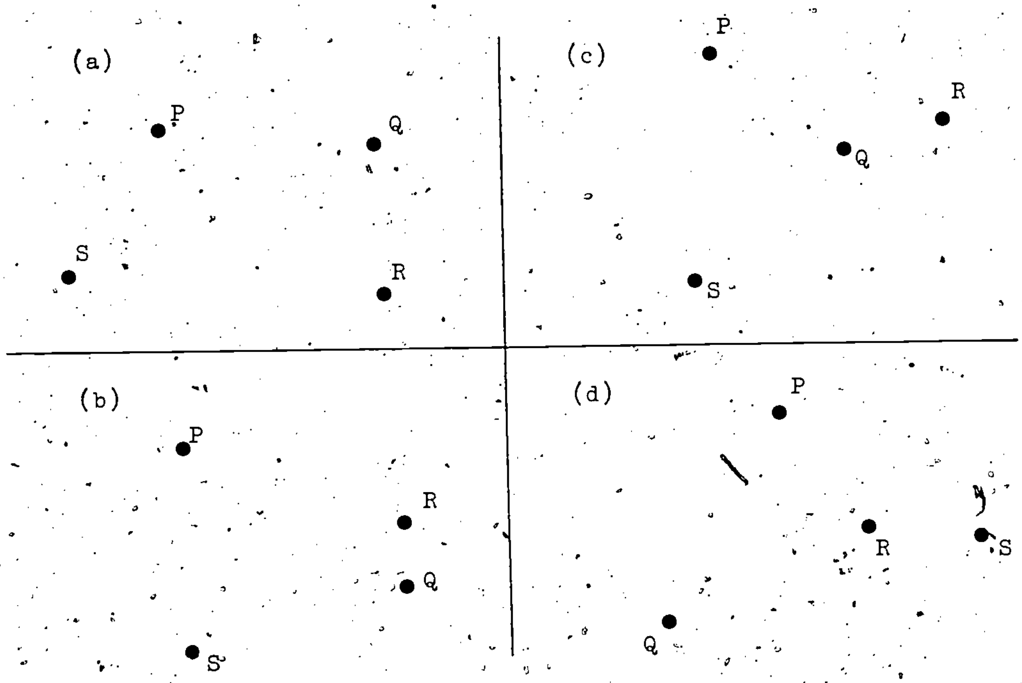
Check Your Reading

1. If for every two points A and B of a set S, the points of the segment $\overline{AB}$ are in set S, then what kind of a set do we call S?
2. What do we call the line segment that joins opposite vertices of a quadrilateral?
3. Do the diagonals of a non-convex quadrilateral intersect?
4. Is a convex quadrilateral a convex set?
5. Where do the diagonals of a convex quadrilateral intersect?
6. A line containing a diagonal of a convex quadrilateral separates the other two vertices in what particular way?

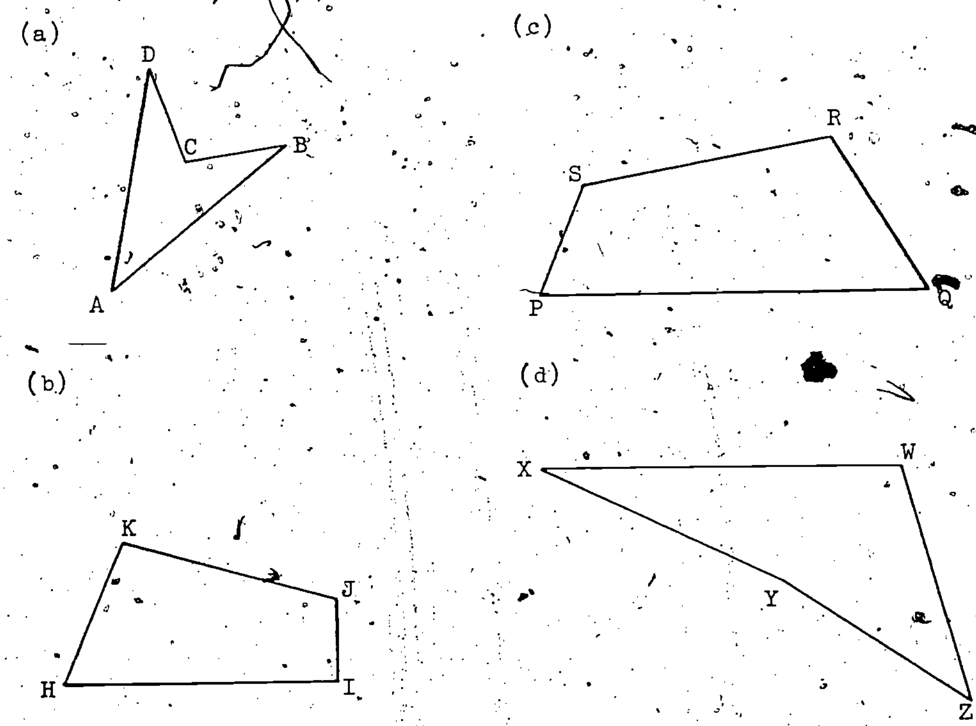
Exercises 27-1

(Class Discussion)

1. Copy the patterns of dots in the following figures and draw the figure PQRS. Join the points in order. That is, join P to Q, Q to R, R to S, and finally join S back to P. In each case tell whether the figure is a convex quadrilateral, a non-convex quadrilateral, or not a quadrilateral.



2. In each of the figures below, name the pairs of opposite vertices, the pairs of opposite sides, and the diagonals.



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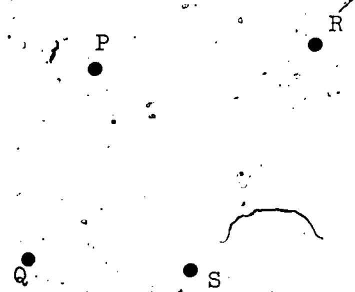
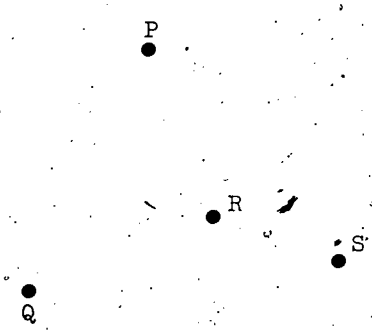
3. For each part of Exercise 2, name all lines through pairs of opposite vertices that separate the other two vertices.

4. In each of the following figues how many different quadrilaterals are there having the given points as vertices? Sketch them all. Name them (as PQRS, PRSQ, etc.).

(a)

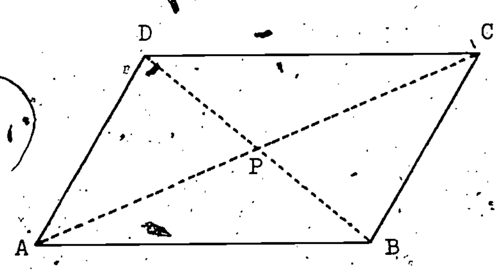


(b)



27-2. More About Parallelograms

Given that ABCD is a convex quadrilateral. How can you tell if it's a parallelogram? You recall the definition of a parallelogram: <u>A convex quadrilateral is a parallelogram if the opposite sides are parallel</u>. We also know quite a few things about a quadrilateral if it <u>is</u> a parallelogram. That is, given that ABCD is a <u>parallelogram</u>,



we know,

(1) $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$ (Opposite sides are parallel)

(2) $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$ (Opposite sides are congruent)

(3) $\angle A \cong \angle C$ and $\angle B \cong \angle D$ (Opposite angles are congruent)

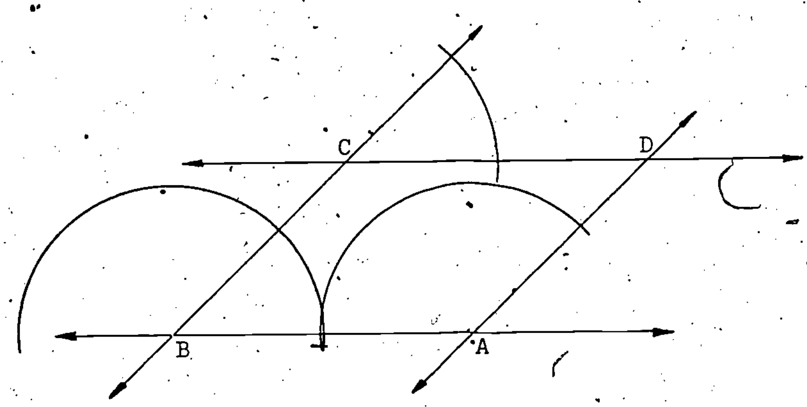
(4) $\triangle ABC \cong \triangle CDA$ and $\triangle BDC \cong \triangle DBA$ (Diagonals form congruent triangles)

(5) $\overline{AP} \cong \overline{PC}$ and $\overline{DP} \cong \overline{BP}$ (Diagonals bisect each other)

If we know that the opposite sides of a quadrilateral are parallel, then we can use the definition, stated above, to decide that the quadrilateral is a parallelogram. However, we will need some other ways to determine whether or not a given quadrilateral is a parallelogram.

Suppose that we have three non-collinear points. Can we locate a fourth point, in the plane of the three points, so that the four points are vertices of parallelogram ABCD?

We could, of course, draw $\overleftrightarrow{AB}$, and $\overleftrightarrow{BC}$, construct a line parallel to $\overleftrightarrow{AB}$ through C, and then construct a line parallel to $\overleftrightarrow{BC}$ through A.

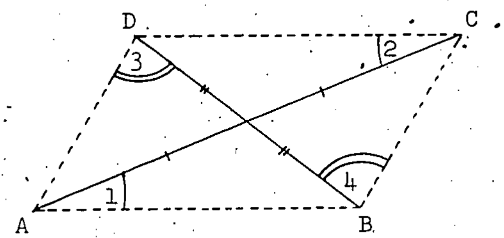


The intersection, D, of the line through C parallel to $\overleftrightarrow{AB}$ and the line through A parallel to $\overleftrightarrow{BC}$ is the fourth vertex of a parallelogram ABCD, since the opposite sides of the quadrilateral are parallel. (Remember that when we say that two segments like $\overline{BC}$ and $\overline{AD}$ are parallel, we mean that the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$, containing these segments, are parallel.) We know that the opposite sides are parallel since we constructed congruent corresponding angles.

Exercises 27-2a

(Class Discussion)

1. Suppose that $\overline{AC}$ and $\overline{BD}$ bisect each other at P.



(a) Why is $\triangle DPC \cong \triangle BPA$ and $\triangle DPA \cong \triangle BPC$?

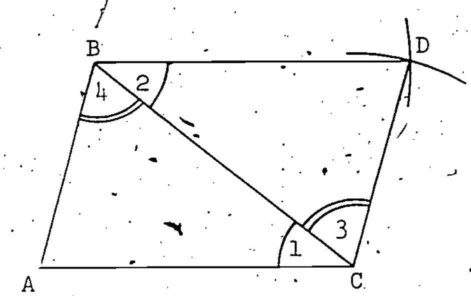
(b) Why is $\angle 1 \cong \angle 2$? Why is $\overline{DC} \parallel \overline{AB}$?

(c) Why is $\angle 3 \cong \angle 4$? Why is $\overline{AD} \parallel \overline{CB}$?

(d) What kind of quadrilateral is ABCD? Why?

(e) Suppose that you are given three non-collinear points R, S, and T. How would you use the above result to find a fourth vertex, V, of a parallelogram? How many possible positions are there for V?

2. Given three non-collinear points A, B, and C. With B as the center and a radius equal to AC, draw an arc of a circle, as shown in the diagram. With C as the center and a radius equal to AB, draw an arc of a circle intersecting the first arc in point D. Quadrilateral ABDC now has opposite pairs of congruent sides. $\overline{AB} \cong \overline{DC}$ and $\overline{BD} \cong \overline{CA}$ (by construction).



(a) Is $\triangle ABC \cong \triangle DCB$? Why?

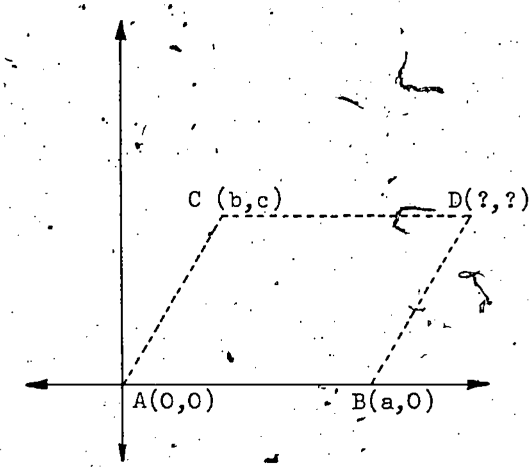
(b) Why is $\angle 1 \cong \angle 2$? Why is $\overline{BD} \parallel \overline{CA}$?

(c) Why is $\angle 3 \cong \angle 4$? Why is $\overline{AB} \parallel \overline{DC}$?

(d) Is ABDC a parallelogram? Why?

(e) Use the same procedure to find parallelogram ABCE and parallelogram AFBC.

3. If we have three non-collinear points, A, B, and C, then we know that they determine a plane. We can assign a coordinate system to the plane so that A is at the origin and $\overline{AB}$ lies along the positive horizontal axis. Let the coordinates of C be (b, c).



(a) What are the coordinates of D if we know that $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$? (That is, one pair of opposite sides of the quadrilateral are congruent and parallel.)

(b) What are the slopes of $\overline{AC}$ and $\overline{BD}$? Is $\overline{AC} \parallel \overline{BD}$? Why?

(c) Is the quadrilateral ABDC a parallelogram? Why?

4. Draw a quadrilateral which is <u>not</u> a parallelogram but

(a) which has 2 pairs of congruent sides;

(b) whose diagonals are perpendicular;

(c) which has one pair of parallel sides and one pair of congruent sides.

In the previous class exercises we have seen that a convex quadrilateral is a parallelogram if

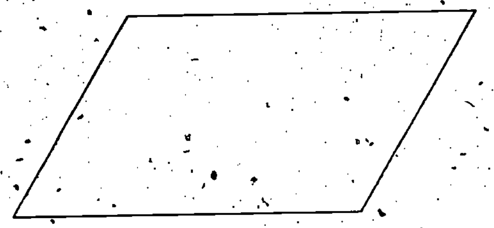
(1) the diagonals bisect each other, or

(2) the opposite sides are congruent, or

(3) one pair of opposite sides are congruent and parallel.

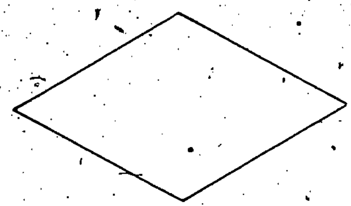
There are certain kinds of quadrilaterals which we encounter so frequently that it will be useful to name and define them here.

(1)



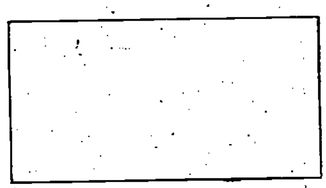
A quadrilateral is a <u>parallelogram</u> if the opposite sides are parallel.

(2)



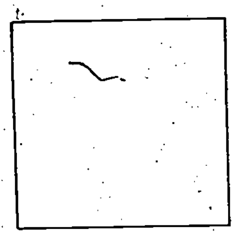
A quadrilateral is a <u>rhombus</u> if it is a parallelogram with all four sides congruent.

(3)



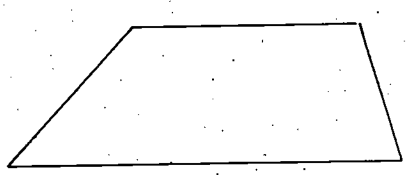
A quadrilateral is a <u>rectangle</u> if it is a parallelogram with all of its angles right angles.

(4)



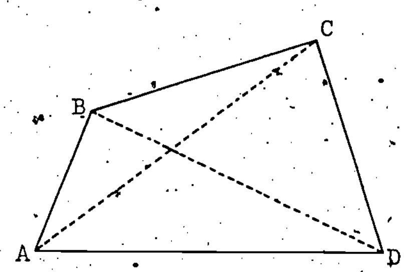
A quadrilateral is a <u>square</u> if it is a parallelogram with all four sides congruent, and if all of its angles are right angles.

(5)



A convex quadrilateral is a <u>trapezoid</u> if two, and only two, opposite sides are parallel.

(6) <u>Opposite</u> sides of a quadrilateral are two sides that do not intersect. (Such as, $\overline{BC}$ and $\overline{AD}$) Two of its <u>angles</u> are <u>opposite</u> if they do not contain a common side. ($\angle A$ and $\angle C$, for example) Two <u>sides</u> are called <u>consecutive</u> if they have a common endpoint. ($\overline{AD}$ and $\overline{DC}$, for example) Two <u>angles</u> are called <u>consecutive</u> if they contain a common side. ($\angle D$ and $\angle C$, for example) A <u>diagonal</u> is a segment joining two non-consecutive vertices. (for example, $\overline{AC}$)



<u>Exercises 27-2b</u>

In the following table, consider the five types of convex quadrilaterals listed across the top with respect to the sixteen statements. The statements are referred to by number and listed below the table. If the quadrilateral <u>ALWAYS</u> has the property at the left, fill in the table with an <u>A</u>; if the figure <u>SOMETIMES</u> has the property, use an <u>S</u>; and if it <u>NEVER</u> has that property, use an <u>N</u>. Draw a complete table and fill it in with the appropriate letters. (Problems 1-16)

	RECTANGLE	SQUARE	RHOMBUS	PARALLELOGRAM	TRAPEZOID
1.					
2.					
3.					
⋮					

STATEMENTS:

1. Both pairs of opposite angles are congruent.
2. Both pairs of opposite sides are congruent.
3. Each diagonal bisects two angles of the quadrilateral.

4. The diagonals bisect each other.

5. The diagonals are perpendicular.

6. Each pair of consecutive angles is supplementary.

7. Each pair of consecutive sides is congruent.

8. Each pair of consecutive angles is congruent.

9. The diagonals are congruent.

10. Both pairs of opposite sides are parallel.

11. Three of its angles are right angles.

12. Its diagonals are perpendicular and congruent.

13. Its diagonals are perpendicular bisectors of each other.

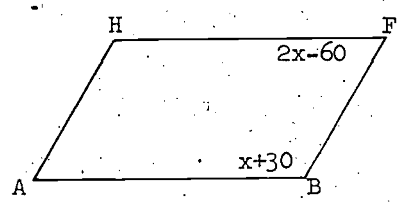
14. It is equilateral.

15. It is equiangular.

16. It is both equilateral and equiangular.

17. Prove: If, in a convex quadrilateral, one of the diagonals and the line joining the midpoints of a pair of opposite sides bisect each other, then the quadrilateral is a parallelogram.

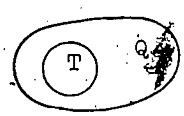
18. With the measures of the angles as given in parallelogram ABFH, give the degree measure of each angle.



$m\angle A =$ __?__ , $m\angle B =$ __?__ ,

$m\angle F =$ __?__ , $m\angle H =$ __?__ .

19. Let Q = the set of all quadrilaterals
T = the set of all trapezoids
P = the set of all parallelograms
R = the set of all rhombuses (rhombi)
C = the set of all rectangles
S = the set of all squares

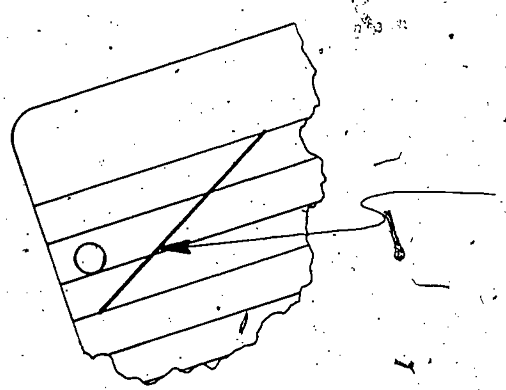


Since every trapezoid is a quadrilateral, we can state that set T is contained in set Q. Thus we write $T \subset Q$ and can represent the fact in a drawing like that shown above. Represent in an analogous way the relations between the indicated sets.

(a) Q, T, and P
(b) Q, P, R
(c) P, R, C, and S
(d) P, C, S
(e) Q, T, C, R

27-3. Families and Networks of Parallel Lines

The chemistry teacher gave Johnny Jones a piece of platinum wire to use in an experiment. Johnny had to share the wire equally with the other two boys in his lab group. Johnny didn't have a ruler handy so he divided the wire into three parts by a rather clever device. He took a piece of lined notebook paper and laid the wire on it as shown below.



We see that the endpoints of the wire are on two of the ruled lines and the wire is intersected internally by the other ruler lines. Johnny cut the wire at these two points. Does it seem to you that the three pieces of wire obtained would have the same length?

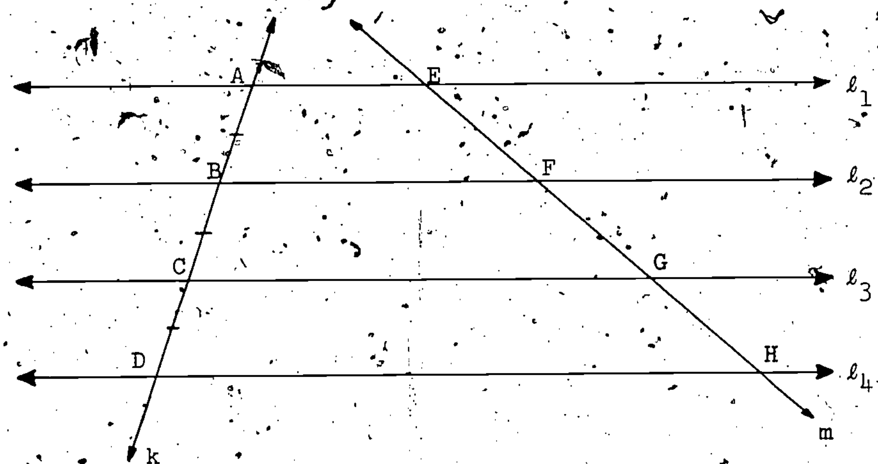
Johnny had the advantage of knowing some geometry. He reasoned as follows:

> "The four lines which touch the wire cut off three segments of equal length on the edge of the paper. Therefore these lines divide the wire into three pieces of equal length."

Johnny's reasoning suggests a theorem which we can possibly prove.

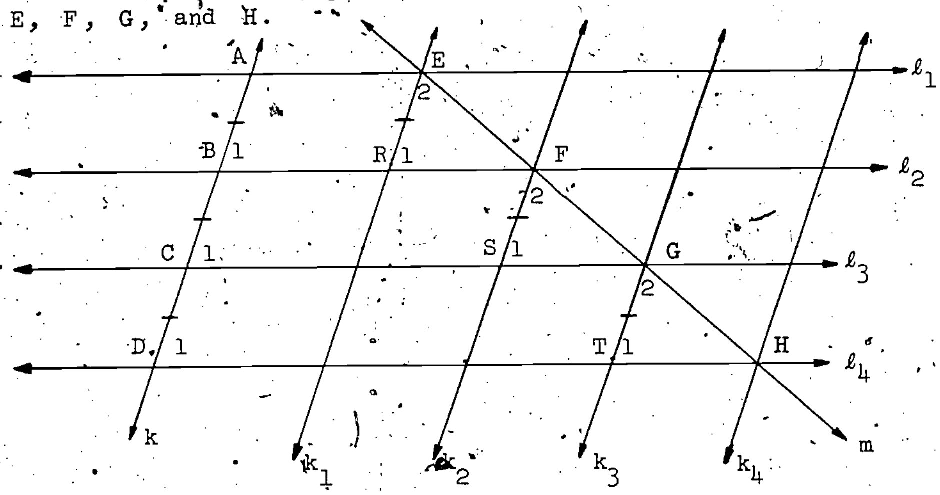
Theorem: If three or more parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.

Proof: Given that the segments cut off by ℓ_1, ℓ_2, ℓ_3, and ℓ_4 on k are congruent. We will show that the segments cut off on line m are congruent.



(We assume that the lines, k and m, are not parallel. If they were parallel, what could we tell about $\overline{EF}$, $\overline{FG}$, $\overline{GH}$? Why?)

Draw lines k_1, k_2, k_3, k_4 parallel to k through points E, F, G, and H.



All of the angles marked "1" are congruent. Why?

All of the angles marked "2" are congruent. Why?

$\overline{AB} \cong \overline{ER}$, $\overline{BC} \cong \overline{FS}$, and $\overline{CD} \cong \overline{GT}$. Why?

It was given that $\overline{AB} \cong \overline{BC} \cong \overline{CD}$. Can we conclude that $\overline{ER} \cong \overline{FS} \cong \overline{GT}$?

Therefore $\triangle ERF \cong \triangle FSG \cong \triangle GTH$ by ASA, and $\overline{EF} \cong \overline{FG} \cong \overline{GH}$ since they are corresponding parts of congruent triangles.

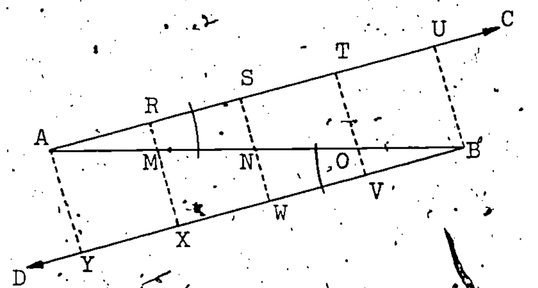
The theorem can be thus proved. It should be clear that a similar proof could be given no matter how many "ℓ" lines there are.

Now do you see why Johnny's method for dividing the platinum wire worked?

Exercises 27-3

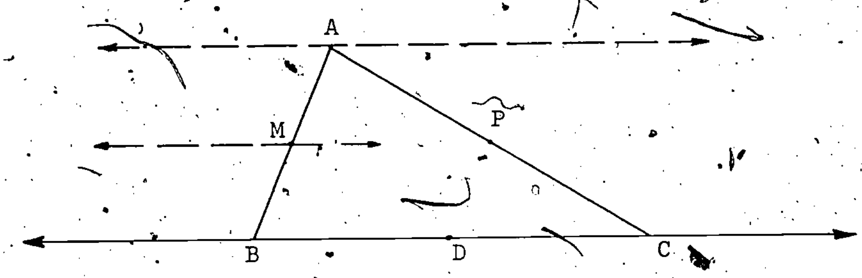
1. Given segment $\overline{AB}$ and any convenient ray, $\overrightarrow{AC}$

   (a) Construct $\overrightarrow{BD}$ parallel to $\overrightarrow{AC}$ and mark off the same number of congruent segments on each ray. (In this drawing we marked off four congruent segments.)



   (b) Draw $\overline{AY}$, $\overline{RX}$, $\overline{SW}$, $\overline{TV}$ and $\overline{UB}$. Show that $\overline{AM} \cong \overline{MN} \cong \overline{NO} \cong \overline{OB}$.

2. Given a triangle, $\triangle ABC$: Let M be the midpoint of $\overline{AB}$ and P be the midpoint of $\overline{AC}$.



   Draw lines through A and M parallel to $\overline{BC}$.

   (a) Since this family of lines divides $\overline{AB}$ into two congruent segments will the line through M pass through P, the midpoint of $\overline{AC}$? Why?

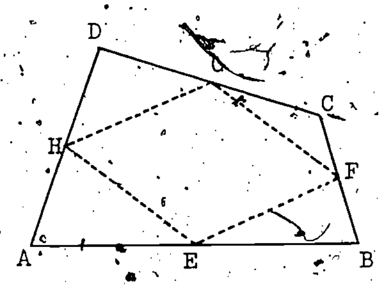
(b) Can we conclude from this that a line parallel to one side of a triangle and passing through the midpoint of a second side of the triangle will always pass through the midpoint of the third side of the triangle?

(c) Draw $\overline{PD} \parallel \overline{AB}$ where D is a point on $\overline{BC}$. What kind of quadrilateral is MPDB?

(d) Is $\overline{MP} \cong \overline{BD}$? Why?

(e) Is $MP = \frac{1}{2} BC$? Why?

3. Draw a convex quadrilateral like ABCD. E, F, G, and H are the midpoints of the respective sides.



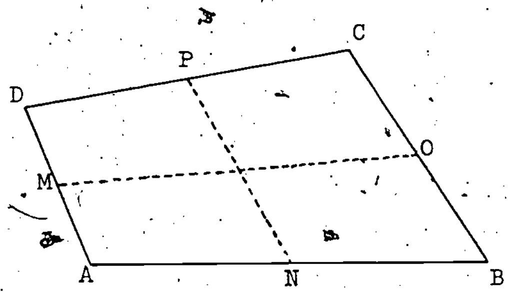
(a) Draw diagonal $\overline{BD}$. What can you say about $\overline{HE}$ and $\overline{BD}$; $\overline{GF}$ and $\overline{BD}$?

(b) Is $\overline{HE} \cong \overline{GF}$? Why?

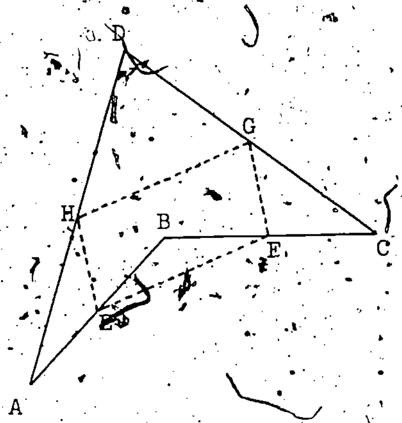
(c) Is HEFG a parallelogram? Why?

(d) Write a statement about the kind of figure you will always get if you join the midpoints of the adjacent sides of any convex quadrilateral.

4. Prove that in any convex quadrilateral the segments joining the midpoints of the opposite sides bisect each other. (Hint: Use Problem 3)



5. Draw a non-convex quadrilateral like ABCD. E, F, G, and H are midpoints of the respective sides.



(a) Draw diagonal $\overline{BD}$. What can you say about $\overline{HE}$, $\overline{BD}$, and $\overline{GF}$?

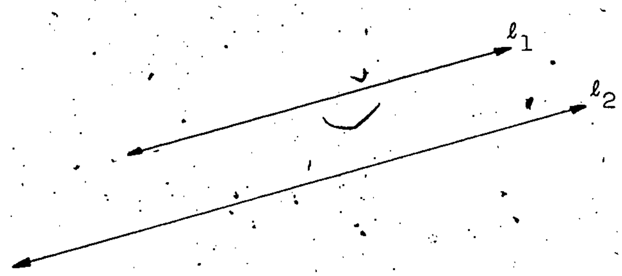
(b) Why is $\overline{HE} \cong \overline{GF}$?

(c) Why is HEFG a parallelogram?

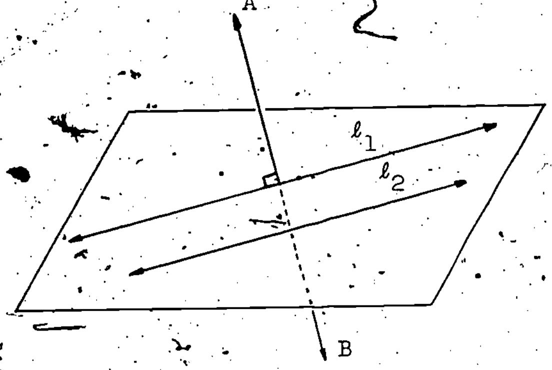
27-4. Combining Parallel and Perpendicular Relations in Space.

In this section we would like to have you think about the relationships among parallel and perpendicular lines and planes in space. You should use familiar objects around you, such as boxes, buildings, rooms, etc., to help you to visualize the different situations. Draw sketches when they appear to be needed to clarify the relationships. Be sure that you have looked at a situation in several different ways before you decide that a statement is ALWAYS, SOMETIMES, or NEVER true.

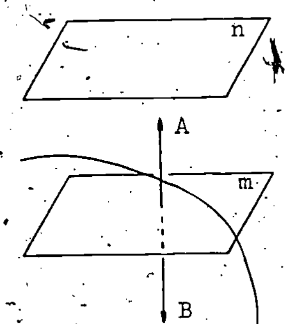
Is it possible to have a line perpendicular to just one of two parallel lines? For example, suppose that we are given two parallel lines, ℓ_1 and ℓ_2.



Is it possible to have a line, $\overleftrightarrow{AB}$, perpendicular to ℓ_1, but not perpendicular to ℓ_2? Look at the following drawing and state what must be true about $\overleftrightarrow{AB}$ so that $\overleftrightarrow{AB}$ will also be perpendicular to ℓ_2 (We know that ℓ_1 and ℓ_2 determine a plane.)

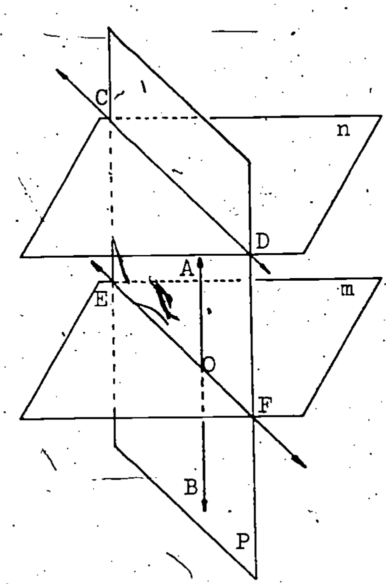


What can we conclude about a line perpendicular to one of two parallel planes? Plane m $\|$ plane n and $\overleftrightarrow{AB} \perp$ plane m. (Recall that two planes are parallel if their intersection is the empty set.) ____



If $\overleftrightarrow{AB}$ intersects plane m will $\overleftrightarrow{AB}$ be perpendicular to plane n?

Let's draw any plane, P, containing $\overleftrightarrow{AB}$. It will intersect plane m in $\overleftrightarrow{EF}$ and plane n in $\overleftrightarrow{CD}$. What relationship exists between $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$? Why? Since $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{EF}$ at O, can we conclude that $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$? Does this mean that we can conclude that $\overleftrightarrow{AB}$ is perpendicular to plane n? Can you draw another plane through $\overleftrightarrow{AB}$? How will this help to decide whether $\overleftrightarrow{AB} \perp$ n or not?



Exercises 27-4a

(Class Discussion)

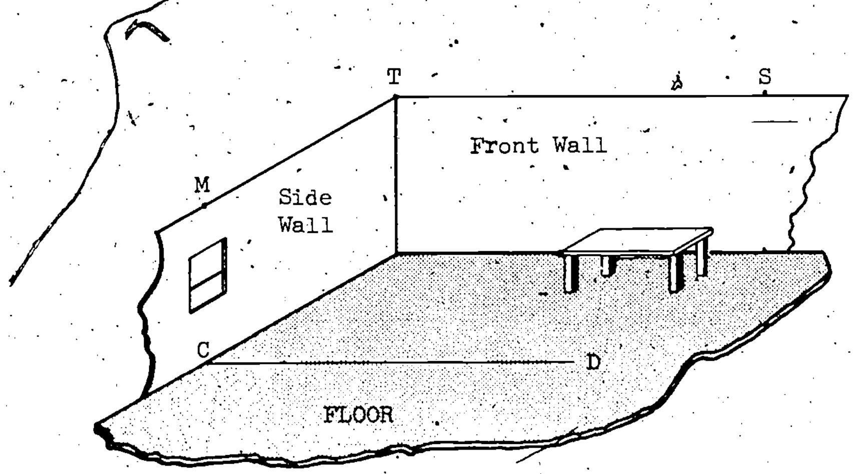
After thinking about the situations discussed previously, try to decide whether the following statements are ALWAYS, SOMETIMES, or NEVER true in space. Give some brief argument which supports each conclusion. Draw some sketches to help you decide. (Problems 1-4.)

1. If two lines are parallel, then a third line perpendicular to one of the two parallel lines is perpendicular to the other line.

2. If a third plane intersects two parallel planes, then the lines of intersection will be parallel.

3. If two planes are perpendicular to the same line, then the two planes are parallel.

4. If two planes are parallel and one plane is perpendicular to a line, then the other plane is perpendicular to the same line.



5. In the above picture of a corner of a room:

   (a) Identify a part of a plane that appears to be perpendicular to $\overleftrightarrow{CB}$.

   (b) Does the plane identified in part (a) appear to be perpendicular to any other line in the drawing?

   (c) Is there a line in the drawing which appears to be parallel to $\overleftrightarrow{CB}$? If so, name it.

(d) If the plane that contains the front wall is perpendicular to $\overleftrightarrow{CB}$, will the plane always be perpendicular to all lines parallel to $\overline{CB}$?

6. In the drawing of the corner of a room:

(a) Name a line in the front wall that is perpendicular to $\overleftrightarrow{TB}$.

(b) Name a line in the side wall that is perpendicular to $\overleftrightarrow{TB}$.

(c) Two lines perpendicular to the same line are (ALWAYS, SOMETIMES, or NEVER) parallel.

7. In the drawing of the corner of a room:

(a) Name two lines in the front wall that are perpendicular to the plane containing the side wall.

(b) Name a line in the floor that is perpendicular to the side wall.

(c) <u>All</u> of the lines perpendicular to a given plane (ALWAYS, SOMETIMES, or NEVER) lie in the same plane.

(d) <u>All</u> of the lines perpendicular to a given plane are (ALWAYS, SOMETIMES, or NEVER) parallel to each other.

8. In the drawing of the corner of a room:

(a) Does the plane containing the front wall appear to be perpendicular to $\overleftrightarrow{MT}$?

(b) Would you expect the plane containing the back wall also to be perpendicular to $\overleftrightarrow{MT}$?

(c) Two different planes, both perpendicular to the same line, are (ALWAYS, SOMETIMES, or NEVER) parallel.

9. In the drawing of the corner of a room:

(a) Is the plane containing the front wall perpendicular to the floor?

(b) Is the plane containing the side wall perpendicular to the floor?

(c) Two different planes, both perpendicular to the same plane are (ALWAYS, SOMETIMES, or NEVER) parallel.

10. In the drawing of the corner of a room:

(a) Would you normally expect the front and back wall to be parallel?

(b) Does the side wall appear to be perpendicular to the front wall?

(c) If the front and back wall are parallel and the side wall is perpendicular to the front wall, the back wall will (ALWAYS, SOMETIMES, or NEVER) be perpendicular to the side wall.

We can summarize these relationships in the following manner. You should be cautioned not to try to memorize these relationships but to establish them intuitively using convenient pictures of the situations as the need arises.

If / then	two planes are parallel	two lines are parallel
a line $\parallel$ to . . .	one plane is sometimes parallel to the other plane.	one line is always parallel to or identical with the other line.
a line $\perp$ to . . .	one plane is always perpendicular to the other plane.	one line is sometimes perpendicular to the other line.
a plane $\parallel$ to . . .	one plane is sometimes parallel to the other plane.	one line is sometimes parallel to the other line.
a plane $\perp$ to . . .	one plane is always perpendicular to the other plane.	one line is always perpendicular to the other line.

Exercises 27-4b

Fill in the following chart with phrases like the ones in the previous chart. Use a drawing like a room or a box to help visualize the relationships.

If	two planes are perpendicular	two lines are perpendicular
then		
1. a line $\|\|$ to . . .		
2. a line $\perp$ to . . .		
3. a plane $\|\|$ to . . .		
4. a plane $\perp$ to . . .		

We know that the relation of equality between real numbers possesses the following three properties.

For any real numbers a, b, and c

(1) $a = a$ (Reflexive property)

(2) If $a = b$, then $b = a$ (Symmetric property)

(3) If $a = b$, and $b = c$, then $a = c$ (Transitive property)

Consider the lines ℓ_1, ℓ_2, and ℓ_3, and the parallel relation. Is the parallel relation <u>reflexive?</u> That is,

$$\text{for any line } \ell_1, \text{ is } \ell_1 \parallel \ell_1?$$

In order to answer this question we have to decide whether a line is parallel to itself. It is desirable to have a number equal to itself, and a line segment congruent to itself. These relations are quite useful. Is the same true for the parallel relation? For our purposes, the answer is "no". We will assume that a line cannot be parallel to itself and hence the parallel relation for lines is <u>not reflexive</u>. We remind you that we could have decided otherwise, and proceeded just as well on that basis.

Is the parallel relation for lines symmetric? That is,

for any lines ℓ_1 and ℓ_2, if $\ell_1 \parallel \ell_2$, then is $\ell_2 \parallel \ell_3$.

We can appeal to our intuition and look at parallel lines in a room, or consider the definition of parallel lines. In either case you should conclude that the parallel relation for lines is symmetric.

Is the parallel relation for lines transitive? That is,

for any three distinct lines ℓ_1, ℓ_2, and ℓ_3 if $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$.

Look at the two parallel lines drawn in the plane of this page.

Can you hold a pencil, in space, so that it is parallel to ℓ_2, but not parallel to ℓ_1? We could, of course, prove formally that the parallel relation for three distinct lines is transitive, but it should be easy for you to see intuitively that this is the case.

Exercises 27-4c

1. Does the perpendicular relation for three arbitrary lines ℓ_1, ℓ_2, and ℓ_3 have the following properties in space?

(a) Reflexive? For any line ℓ_1, is $\ell_1 \perp \ell_1$?

(b) Symmetric? For any lines ℓ_1 and ℓ_2, if $\ell_1 \perp \ell_2$, is $\ell_2 \perp \ell_1$?

(c) Transitive? For any three lines ℓ_1, ℓ_2, and ℓ_3, if $\ell_1 \perp \ell_2$, and $\ell_2 \perp \ell_3$, is $\ell_1 \perp \ell_3$?

2. Does the parallel relation for three arbitrary planes, p_1, p_2, and p_3 have the following properties?

(a) Reflexive?

(b) Symmetric?

(c) Transitive?

3. Does the perpendicular relation for three arbitrary planes, p_1, p_2, and p_3 have the following properties?

(a) Reflexive?

(b) Symmetric?

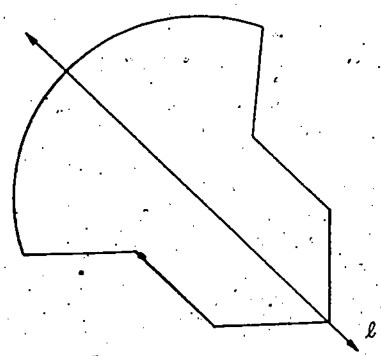
(c) Transitive?

4. Is the congruence relation for triangles reflexive? symmetric? transitive?

5. Are vertical lines sometimes, always, or never parallel?

6. Are horizontal lines sometimes, always, or never parallel?

27-5. Axes of Symmetry



What is the image of this figure for the reflection in ℓ?

Since the figure is invariant for the reflection in ℓ, you recall that line ℓ is called the axis of symmetry for the figure.

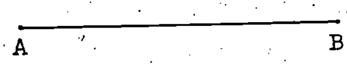
We say that the figure is symmetrical with respect to ℓ. Does this figure have any other axes of symmetry?

The concept of symmetry provides us with another very important tool for analyzing geometric figures.

Exercises 27-5a

(Class Discussion)

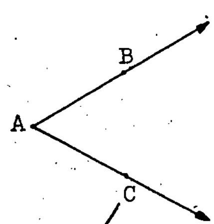
1. Formulate a definition of an axis of symmetry of a geometric figure. When is a geometric figure symmetrical with respect to a line?



2. (a) Show how you would find an axis of symmetry for the segment $\overline{AB}$.

   (b) How many axes of symmetry does a segment have? Explain.

   (c) How many axes of symmetry does a line have? Describe them.



3. (a) Show how you would find an axis of symmetry for $\angle BAC$. Justify your answer.

   (b) How many axes of symmetry does an angle have?

4. (a) Draw a figure consisting of two non-collinear segments $\overline{AB}$ and $\overline{AC}$ which have a common endpoint A.

   (b) If this figure has an axis of symmetry, what must be true about segments $\overline{AB}$ and $\overline{AC}$?

   (c) What must be true about the points B and C?

5. (a) Draw a triangle which has exactly one axis of symmetry.

   (b) What kind of triangle must it be? Explain.

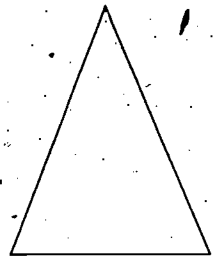
   (c) Describe the location of the axis of symmetry.

From the exercises above we see that two points have one and only one axis of symmetry, the perpendicular bisector of the line segment joining the two points. If A and A' are any two points in a plane, then we can refer to A' as the image of A in the axis of symmetry ℓ, and conversely.

<u>Exercises 27-5a</u>

Copy the following figures on your paper and draw all of the axes of symmetry that you can find in the plane of the paper. If there is none, state this fact. (Problems 1-21).

1.



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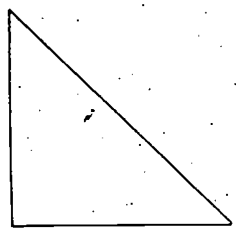
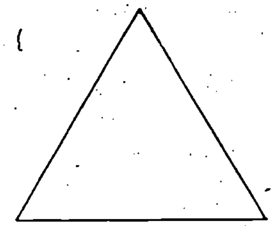
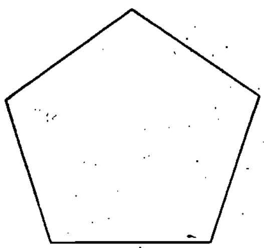


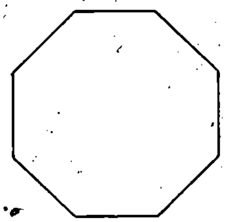
3.



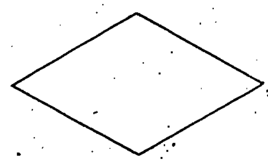
4.



5.



6.



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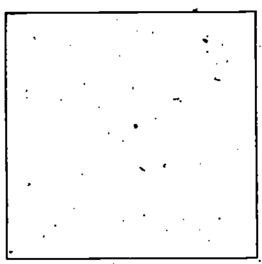

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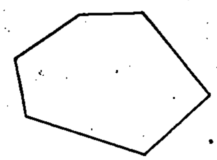

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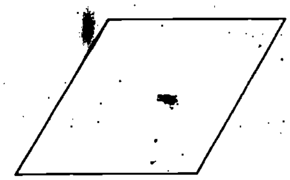

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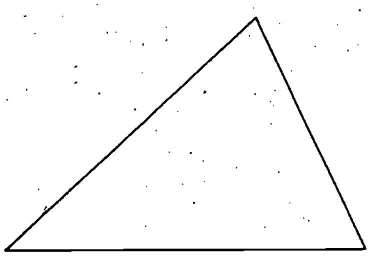

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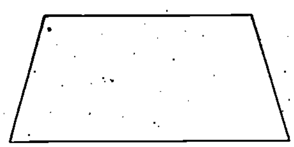
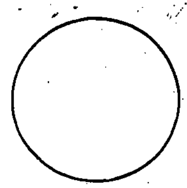
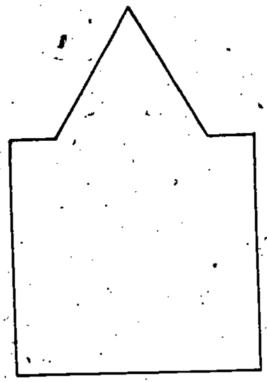


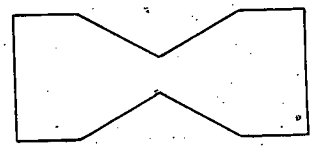
12.



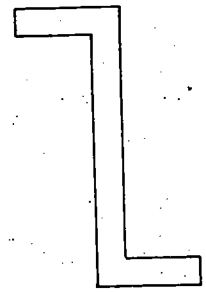
13.



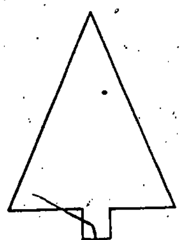
14.



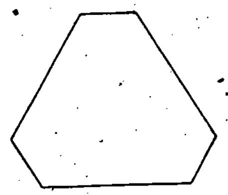
15.



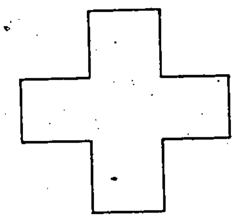
16.



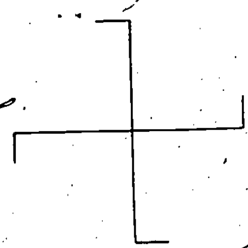
17.



18.



19.



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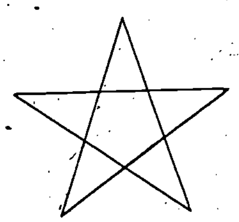
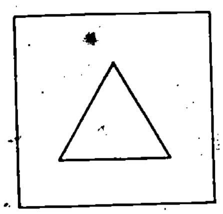


21.



22. Draw a triangle which has 2 axes of symmetry. If a triangle has 2 axes of symmetry then must it have 3 axes of symmetry? Explain.

23. Draw a quadrilateral which has exactly one axis of symmetry and which is

(a) kite-shaped.

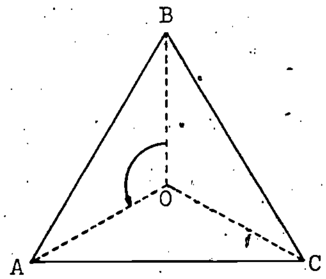
(b) a trapezoid.

24. Show that there are two kinds of quadrilaterals which have exactly two axes of symmetry.

25. What kind of quadrilateral has exactly four axes of symmetry? Draw an example and show the four axes.

27-6. Rotational Symmetry

In the equilateral triangle ABC shown below, the dotted lines intersecting at O show its three axes of symmetry.



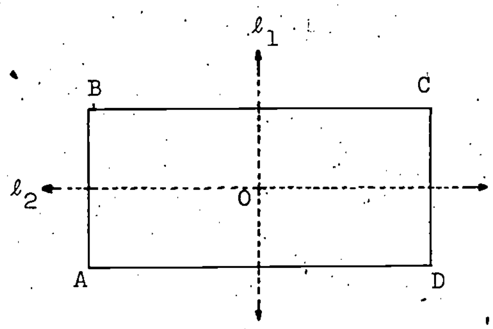
Consider the rotation with center at O defined by the curved arrow.

The image $\triangle A'B'C'$ falls exactly on $\triangle ABC$ such that A' is located at C. Where are B' and C' located?

Therefore this figure is invariant for this rotation about O.

<u>Definition</u>. A figure which is invariant for a rotation about O that is less than a full turn is said to have <u>rotational symmetry</u> about O.

Rectangle ABCD is shown below with its two perpendicular axes of symmetry ℓ_1 and ℓ_2 intersecting at O.



For a rotation of a half turn about O, describe how the image rectangle $A'B'C'D'$ falls on the original rectangle ABCD by telling where the points A', B', C', and D' are located.

Now consider the reflection in point O. Again the image rectangle $A'B'C'D'$ coincides with rectangle ABCD. The image B' is located at D because O is the midpoint of $\overline{BD}$. Likewise the image C' is located at A. Where are A' and D' located?

Notice that the image of rectangle ABCD for a rotation of a half turn about O is the same as the image of rectangle ABCD for a reflection in O. This was to be expected since we found in an earlier chapter that a reflection in a point O and a rotation of a half turn about the same point O result in the same rigid motion.

<u>Definition</u>. A figure which is invariant for a reflection in O, and therefore is invariant for a rotation of a half turn about O, is said to have <u>central symmetry</u>. Point O is called the <u>center of symmetry</u> for the figure.

Exercises 27-6a

(Class Discussion)

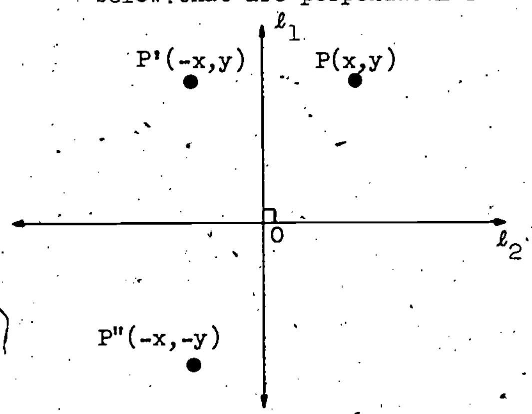
1. In the definition of rotational symmetry, why is it necessary to state that the rotation must be less than a full turn?

2. For the equilateral triangle ABC discussed at the beginning of this section, describe another rotation of less than a full turn for which the triangle is invariant.

3. Explain why an equilateral triangle has rotational symmetry but no central symmetry and therefore no center of symmetry.

4. Explain why a figure that has central symmetry also has rotational symmetry.

5. Suppose that a figure has two axes of symmetry ℓ_1 and ℓ_2 shown below that are perpendicular at point O.



We shall not draw the figure but we shall use ℓ_1 and ℓ_2 to form a coordinate system with origin at O and let P(x, y) be any point on the undrawn figure.

(a) Point P'(-x,y) must be a point on the figure. Why?

(b) Likewise P"(-x,-y) must be a point on the figure. Why?

(c) Show that the coordinates of the midpoint of $\overline{PP''}$ are (0,0) which is point O.

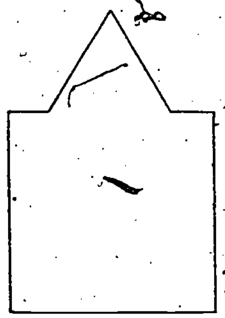
(d) Point P" is the image of P for a reflection in O. Why? Since P is any point on the figure, and the image of P for the reflection in O is also on the figure, then O is the center of symmetry for the figure.

In summary, if a figure has two axes of symmetry which are perpendicular at a point O, then O is the center of symmetry for the figure.

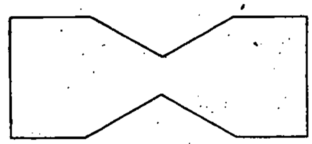
Exercises 27-6b

1. Which of the following figures have rotational symmetry? Copy the figures that do, and show with a curved arrow a rotation of less than a full turn for which the figure is invariant.

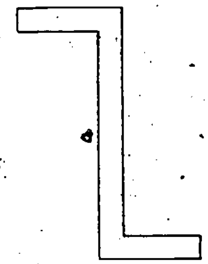
(a)



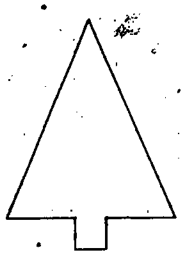
(b)



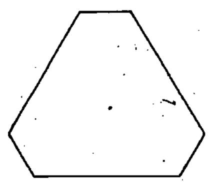
(c)



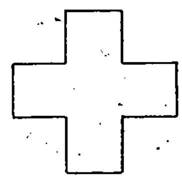
(d)



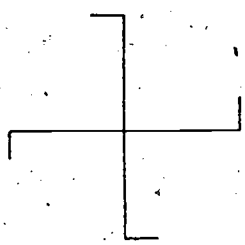
(e)



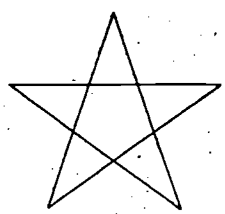
(f)



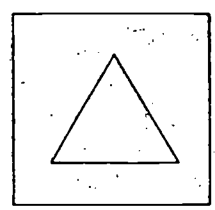
(g)



(h)

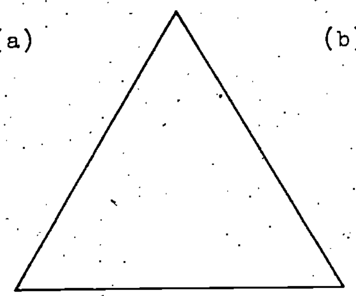


(i)

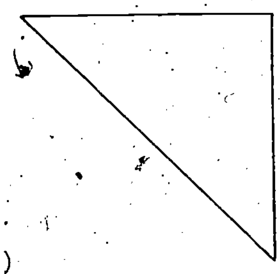


2. Which of the following figures have central symmetry? Copy the figures that have central symmetry and locate each center of symmetry if one exists.

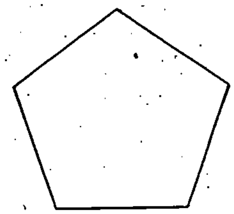
(a)



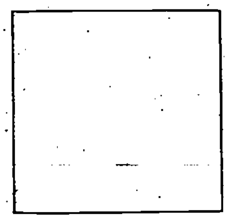
(b)



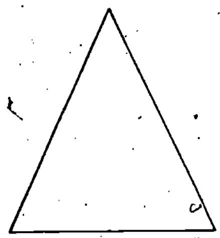
(c)



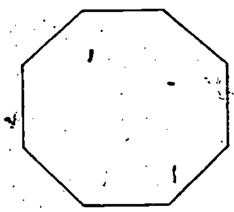
(d)



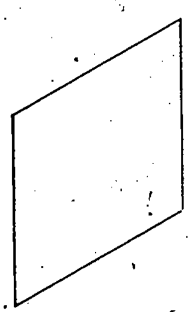
(e)



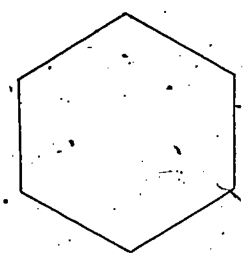
(f)



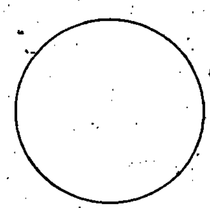
(g)



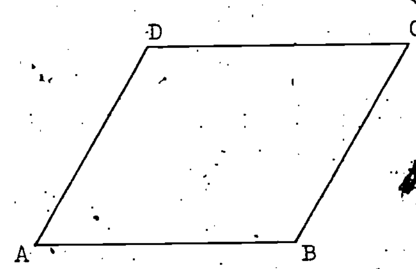
(h)



(i)



3. Given a parallelogram ABCD:



(a) Is the given figure symmetrical about a point? If so, locate that point.

(b) Is the figure symmetrical about a line? If so, locate that line.

(c) If a figure has central symmetry, must it have an axis of symmetry?

4. Draw a rhombus that is not a square and draw its two perpendicular axes of symmetry. The intersection of the axes should be the center of symmetry. Show that this is true.

5. In Exercise 2 which of the regular polygons have central symmetry? If so, how many sides do they have? (Remember that regular polygons are polygons with congruent sides and congruent angles.) Are there any regular polygons in Exercise 2 that do not have central symmetry but do have rotational symmetry? If so, how many sides do they have? Discuss.

27-7. The Symmetry of Figures in Space

Points P and P' are said to be symmetrical with respect to plane m if they

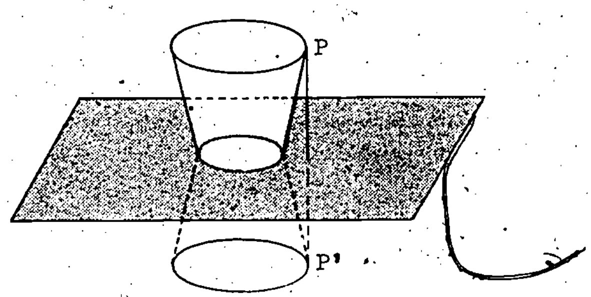
(1) lie on opposite sides of plane m;

(2) lie on the same perpendicular line to plane m; and

(3) are equidistant from plane m.

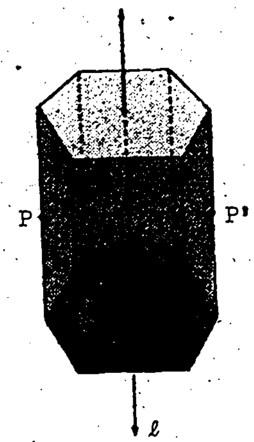
That is, plane m is the perpendicular bisector of $\overline{PP'}$.

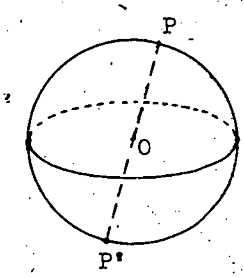
A transformation of space into itself which assigns to each point P the point P' symmetrical to P about some plane m is called a symmetry about the plane.



A figure is said to be symmetrical with respect to a plane m if it is transformed into itself by a reflection of space in this plane. In the above drawing, plane m is called the plane of symmetry of the figure.

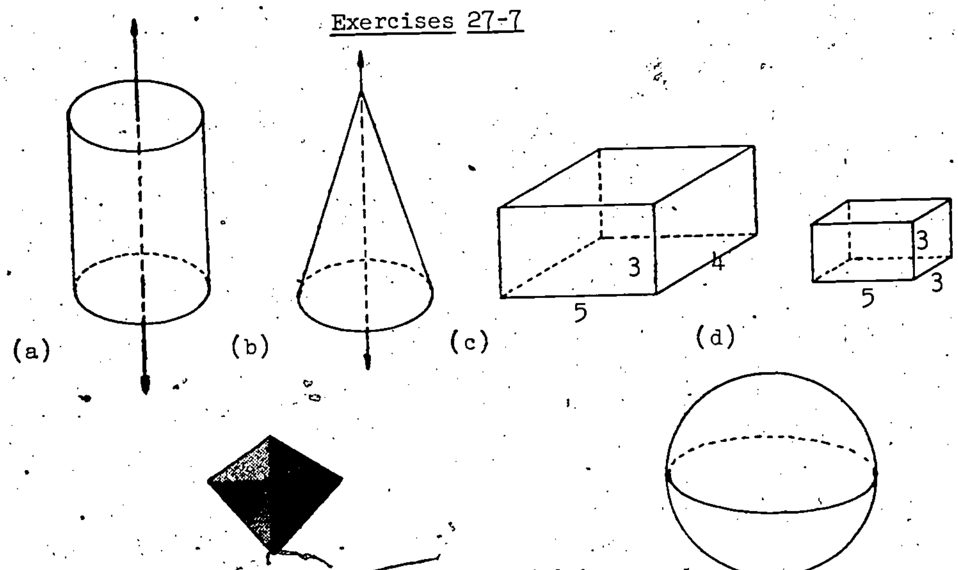
Figures in space can also be symmetric with respect to a line or a point.





Line ℓ and point O are called the <u>axis of symmetry</u> and <u>center of symmetry</u> of the figures respectively. If P and P' are symmetric points, then ℓ is the perpendicular bisector of $\overline{PP'}$, and O is the midpoint of $\overline{PP'}$.

<u>Exercises 27-7</u>

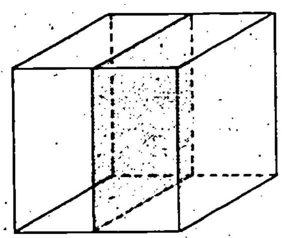


(e) Regular Tetrahedron (f) Sphere

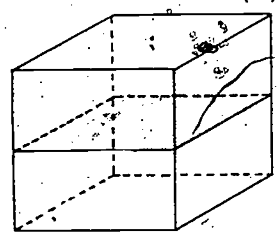
1. In each of the above figures, how many planes of symmetry are there?
2. In each of the above figures, how many axes of symmetry are there?
3. In each of the above figures, how many points are there that are centers of symmetry?

4. The figures below show a cube cut by a plane in several different ways. Indicate in which cases the cube is symmetric with respect to the plane.

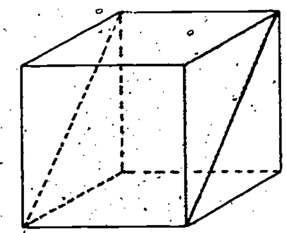
(a)



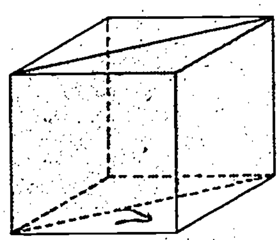
(b)



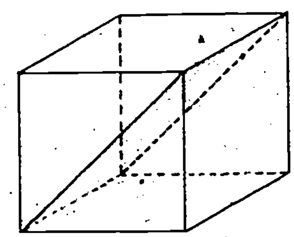
(c)



(d)



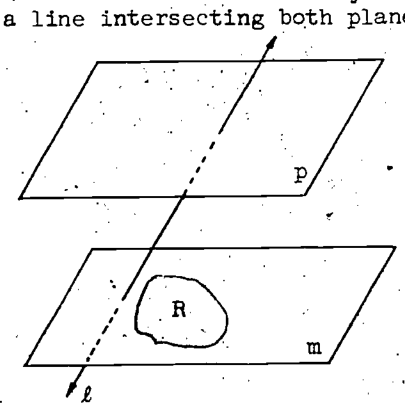
(e)



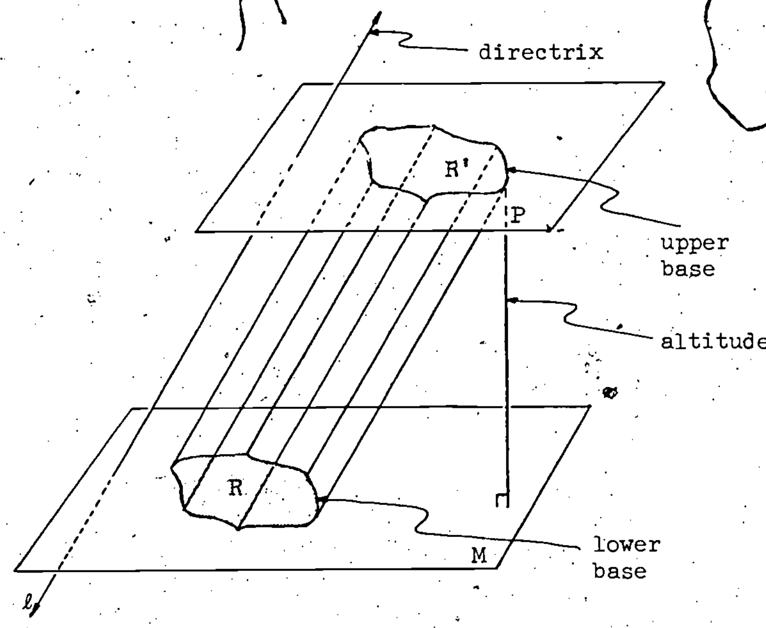
BRAINBUSTER: Find all of the axes of symmetry of a cube.
(<u>Hint</u>: There are 13.)

27-8. <u>Prisms and Cylinders</u>

Let m and p be two parallel planes, R a region in plane m, and ℓ a line intersecting both planes.



The set of all line segments $\overline{AA'}$, where A is in region R, A' is in plane p, and $\overline{AA'}$ is parallel to line ℓ is a solid region called a cylinder.

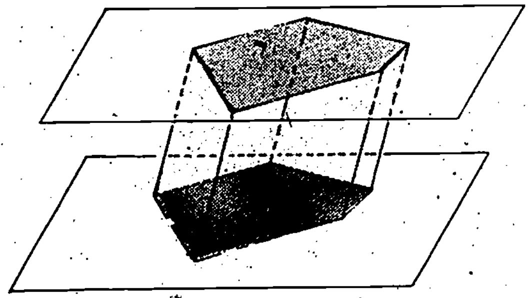


Line ℓ is called the <u>directrix</u> of the cylinder, R and R' are the <u>lower</u> and <u>upper</u> <u>bases</u> respectively, and the distance between plane p and plane m is the altitude.

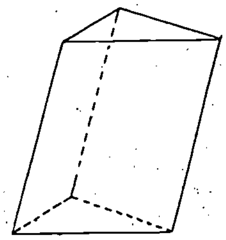
If ℓ is perpendicular to m then the cylinder is called a <u>right</u> <u>cylinder</u>, otherwise it is an <u>oblique</u> <u>cylinder</u>. We name certain common solids in the following way:

If region R is	then the solid is
(a) a polygonal region	a <u>prism</u>,
(b) a parallelogram region	a <u>parallelepiped</u>,
(c) a triangular region	a <u>triangular</u> <u>prism</u>,
(d) a square region	a <u>square</u> <u>prism</u>,
(e) a circular region	a <u>circular</u> <u>cylinder</u>.

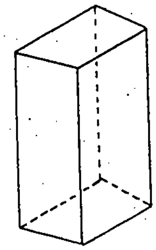
The following drawings are representations of some common solid figures.



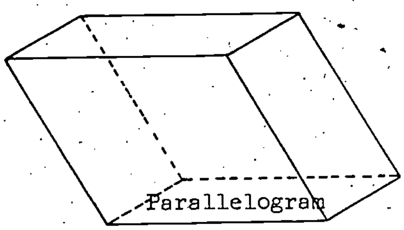
oblique prism



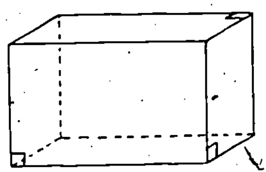
triangular prism



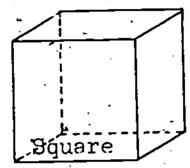
right prism



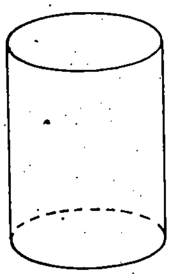
parallelepiped



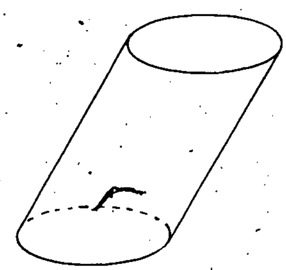
rectangular
parallelepiped



cube



right circular
cylinder



oblique circular
cylinder

Exercises 27-8a

(Class Discussion)

1. If we define a lateral edge of a prism as segment $\overline{AA'}$ where A is a vertex of the base of the prism, then are:

   (a) all lateral edges of the prism parallel? Why?

   (b) all lateral edges of the prism congruent?

2. If we define a lateral face of a prism as the union of all segments $\overline{PP'}$ for which P is a point in a given side of the base, then:

   (a) are all lateral faces parallelogram regions?

   (b) are all lateral faces congruent regions?

3. What kind of regions are the lateral faces of a right prism?

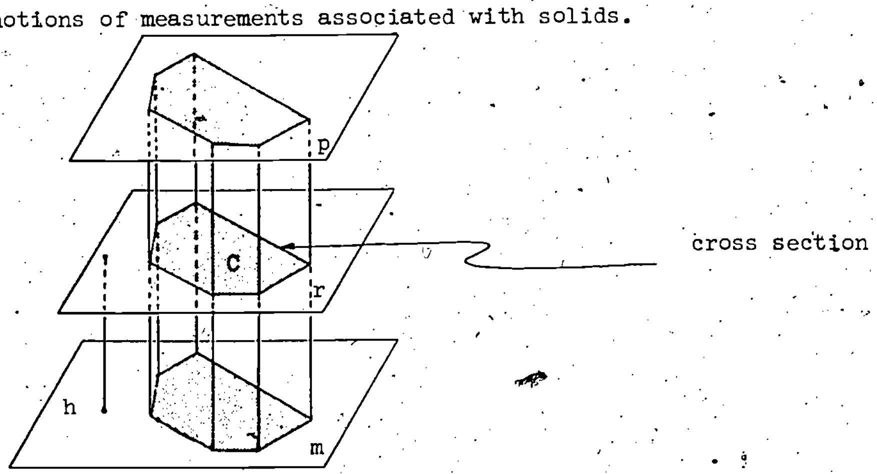
4. If we define the lateral surface of a prism as the union of its lateral faces, then:

   (a) find the lateral area of a rectangular parallelepiped of length 5, width 4, and height 3;

   (b) find the lateral area of a cube with edge 6;

   (c) find the total area of a cube with edge 6.

We find it quite useful to define a cross section of a prism. For example, such a definition will enable us to develop in a later chapter some notions of measurements associated with solids.

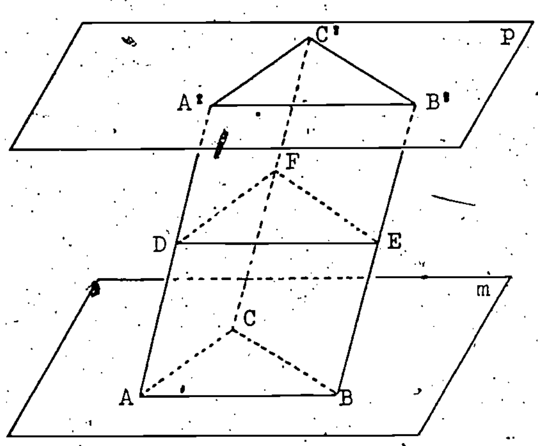


Definition. A cross section of a prism is the intersection of the prism and a plane parallel to the plane of the base.

In the drawing above, plane r is parallel to the plane of the base, plane m, and at a distance h above plane m. Region C is a cross section of the prism pictured above.

Exercises 27-8b

(Class Discussion)



1. Let the triangular region ABC be the base of a prism, and DEF a cross section. Is $\overline{AD} \parallel$ to $\overline{BE}$? Why?
2. Is $\overline{DE} \parallel \overline{AB}$? Why?
3. Quadrilateral ABED is a parallelogram. Why?
4. $\overline{AB} \cong \overline{DE}$. Why?
5. Quadrilaterals ADFC and BEFC are parallelograms. Why?
6. $\triangle ABC \cong \triangle DEF$. Why?
7. Are the upper and lower bases of a triangular prism congruent? Do they have the same area?
8. Do the cross sections of a given prism have the same area? Why?

We should be able to conclude from the previous exercises that all cross sections of a triangular prism are congruent to the base of the prism. It is possible to cut up the base of any polygonal prism into triangular regions, and hence any prism can be cut up into triangular prisms whose bases are the triangular regions.

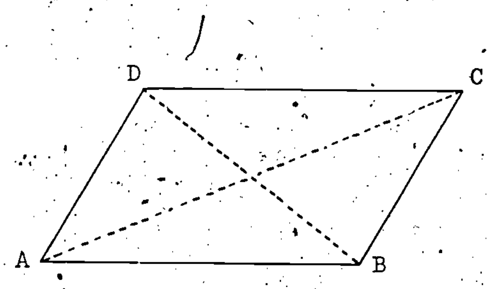
27-9. Summary

Section 27-1.

A region is called a convex region if for any two points of the region the line segment joining those two points lies entirely within the region. The diagonals of a convex quadrilateral intersect in the interior of the quadrilateral and the line that contains either diagonal separates the other two vertices so that they lie in opposite half-planes bounded by that line.

Section 27-2.

Quadrilateral ABCD is a parallelogram if

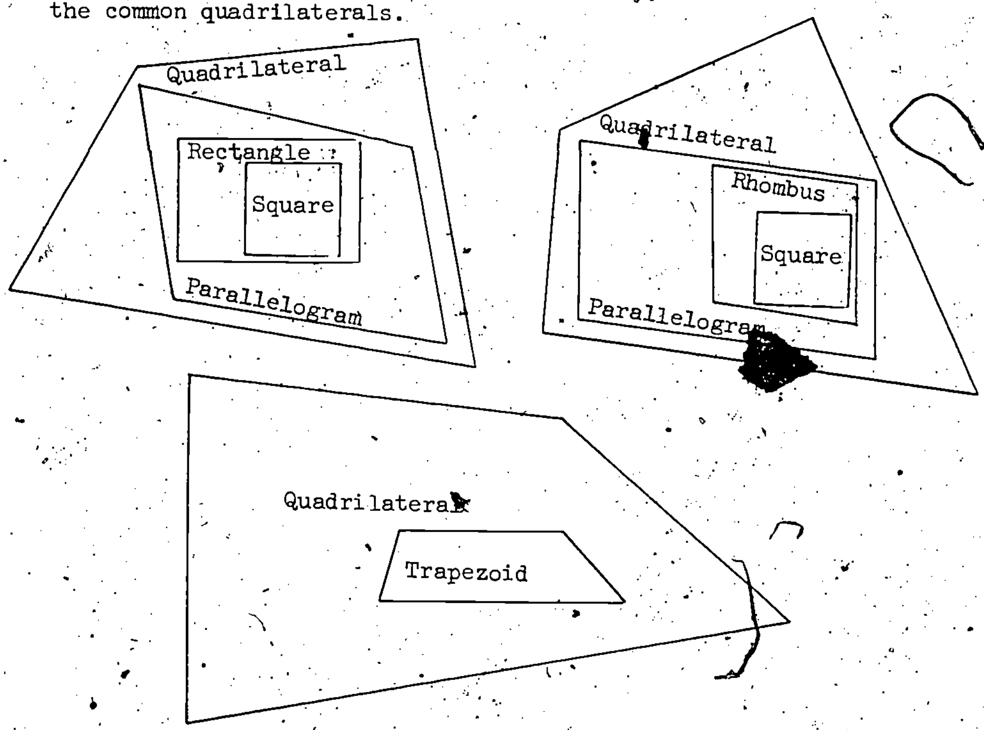


(1) $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$,

or (2) $\overline{AC}$ and $\overline{BD}$ bisect each other,

or (3) $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$.

We can illustrate in the following way the relationships between the common quadrilaterals.



In the previous drawings, if a figure is contained in the interior of another figure, then it will have the same properties as the enclosing figure plus some new characteristics.

Section 27-3.

If three or more parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on any transversal.

The above theorem enabled us to develop a construction method for dividing a segment into any number of congruent parts. Also we were able to show that a line through the midpoint of one side of a triangle and parallel to a second side passed through the midpoint of the third side.

Section 27-4.

Relationships between parallel and perpendicular lines and planes are numerous. It is not practical to try to memorize these relationships, but one can use physical models to help visualize them. However, you should not regard these models as final proofs and you should be cautious about "jumping to conclusions" on the basis of a single diagram.

The parallel and perpendicular relationships between three distinct lines or three distinct planes are not reflexive and the perpendicular relationships are not transitive. All of the relationships are symmetric.

Section 27-5.

If a geometric figure has an axis of symmetry, then this means that the figure is invariant for a reflection in that line.

An axis of symmetry of a figure is the perpendicular bisector of the line segments joining every point and its image for a reflection about that axis.

Section 27-6.

A figure which is invariant for a rotation that is less than a full turn about a point O is said to have rotational symmetry about O.

A figure which is invariant for a rotation of a half turn about a point O is said to have <u>central symmetry</u>. Central symmetry is just a special case of rotational symmetry.

Regular polygons with an even number of sides have central symmetry. Regular polygons with an odd number of sides have rotational symmetry, but not central symmetry.

<u>Section 27-7.</u>

Two points P and P' are said to be symmetrical with respect to plane m if plane m is the perpendicular bisector of $\overline{PP'}$.

A geometric figure, in space, is symmetrical with respect to a plane m if it is invariant in a reflection in plane m.

A figure in space can also be symmetric with respect to a line or a point.

<u>Section 27-8.</u>

A cylinder is a solid region formed by a region R in one plane and the set of <u>all</u> parallel line segments $\overline{AA'}$, where A is in R and A' is in a second plane parallel to the first plane. The line segments $\overline{AA'}$ are all parallel to a given line, ℓ, called the directrix of the cylinder.

Certain cylinders are given common names depending on the shape of the region R. If R is a convex polygonal region, then the solid is called a <u>prism</u>. If R is a circular region then the solid is called a circular cylinder.

A <u>cross section</u> of a prism is the intersection of the prism and a plane parallel to the plane of the base. The area of a cross section of a prism is the same as the area of the base.

Chapter 28

MEASUREMENT

28-1. Introduction. What Is Measurement?

Modern science dates from Galileo (1564-1642) who stated its program in the following words: "Measure that which is measurable and make measurable that which appears not to be." He followed his own advice by measuring the speed of a falling body and of motion down an inclined plane, and the speed of sound. He also invented a thermometer to measure how hot or cold an object is. Previously no one had thought of measuring temperature. Galileo's pupil, Torricelli, showed how to measure atmospheric pressure; and shortly afterward, Boyle measured the "spring of the air," (the pressure of an enclosed volume of air). Galileo also tried, but without success, to measure the speed of light. His successors had better luck.

Today we measure the wavelength of light of a given color, the charge on an electron, and the half-life of a radioactive element--to give a few examples.

It is clear that the idea of measurement is an important one. In the present chapter we shall take a good look at it. At first we shall be concerned with measurements of lengths, areas, and volumes, because they are simpler than others and because they are fundamental to other measurements. Later we shall consider different types of measurement.

Before getting down to business it will be useful to recall some of the facts about measurement with which you are familiar.

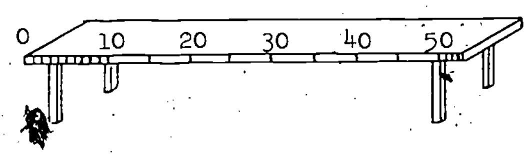
If you ask someone to measure the length of a table, what kind of an answer do you expect? Something like 5 feet, or 2 yards, or perhaps 54 inches. The answer, you notice, consists of two parts: a <u>number</u>, like 5, 2, or 54; and a <u>word</u>, like feet or yards, that tells you what unit is being used. If you received the answer "5" you would ask "5 what?"

Let us get our language straight. That which is to be measured is called a <u>quantity</u>. In the example, the quantity is length. The result is called a <u>measurement</u>. As we have said, when we report this result we use a number and a word. The number is called the <u>measure</u>. Thus 5 feet is a measurement. The measure is 5. The word "feet" gives the unit of measure.

What exactly is a unit? It is an example of the quantity to be measured to which we agree to give the number 1. If we wish to measure lengths of line segments, we choose a particular line segment and call its length 1. If we wish to measure areas of plane regions, we choose a particular region and label its area 1. To measure volumes we use a particular unit of volume. The unit that we choose is in each case a matter of convenience. For example, we may measure length in inches, feet, yards, or miles (English system), or in centimeters, meters, or kilometers (metric system). Scientists ordinarily use the metric system in their work. In everyday life English-speaking people are more likely to use the English system. Whichever system we use, we usually choose such a unit that in the measurements to be performed, the measures will be numbers that are neither too large nor too small. We do not measure the distance from New York to Chicago in inches, feet, or yards. You would not measure the length of your nose in yards or miles.

Of course you can make up your own units. But in that case you may have a little trouble in making other people understand what your measurements mean. It is a little like making up your own language.

Let us choose the segment shown as a unit of length. As accurately as we could draw it, this matches a standard inch. How do we use this unit to measure the length of an object like a table? We place copies of the unit end-to-end and see how many of these copies are required to reach from one end of the table to the other. We count units. Of course, we usually will not get an exact fit. We may discover, for example, that the length is between 53 and 54 inches. If this approximation is not good enough, we can subdivide our unit and use a quarter inches as a new unit (or $\frac{1}{8}$ inch or $\frac{1}{16}$ inch). All actual physical measurements are approximations. Approximations to what? To an ideal that we imagine. In the case of length we imagine that the table has a real length which our measurements approximate. In our imagination we replace the table by a model which has an exact length. Is there in reality such a length? This is the sort of question that we have



learned not to ask. We do know that if we think of the world in terms of models which correspond to sharp mathematical concepts, we manage to give a satisfactory account of our experience which is simple enough for human beings to understand. We also know that these models allow us to deal effectively with the physical world.

Check Your Reading

1. What is the difference between a measurement and a measure? Illustrate.

2. What name do we use for anything that can be measured?

3. Of what two parts does a measurement consist?

4. What do we mean by a unit?

5. Suppose that a quantity cannot be measured with satisfactory accuracy in terms of a given unit. What can we do to increase the accuracy of reporting the measurement?

Exercises 28-1

1. Name at least 5 quantities other than length, area, volume, pressure and temperature. State a unit of measure for each.

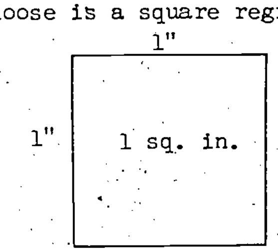
2. Name 5 units of length in the metric system. What are the relations among them?

3. Name 5 different units of length in the English system. Express each of them in terms of 1 foot.

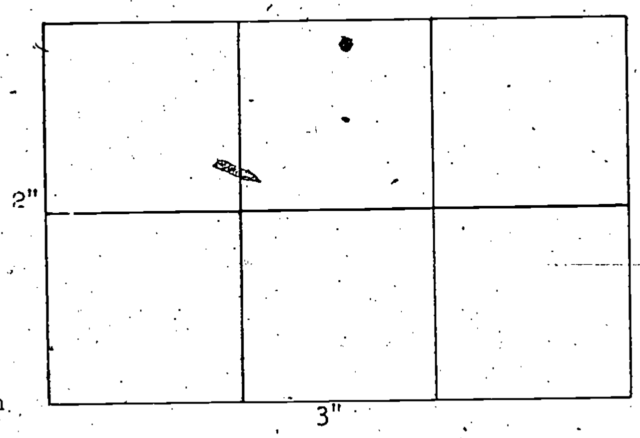
4. Measure the length of your schoolroom in paces. Compare your result with that of the other pupils in your class. Explain why it is important to establish a set of standard "weights and measures." Compare with the reasons for adopting a standard currency.

28-2. Measurement and Calculation

To measure area, a natural unit to choose is a square region with one unit of length on a side; examples of such regions are a square inch, a square foot, or a square meter.

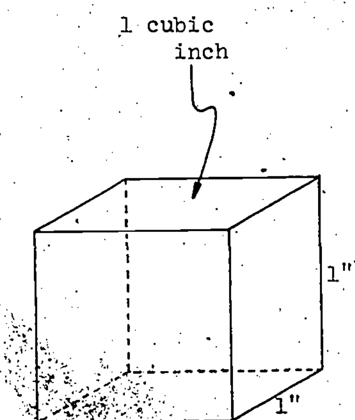


A rectangular region with sides 3" and 2" has the area 6 square inches because it can be covered exactly without overlapping by six square-inch units. We can think of these units as tiles.



It would be tiresome to find the area of a rectangle with the dimensions 13" by 27" by counting squares. We can of course replace counting by a simple multiplication giving 13×27 sq. in. That is, we calculate the area. As you know, if a and b are rational numbers, the area of any rectangle with sides a and b units is simply $a \times b$ square units. We shall assume that this area is correct even if a and b are irrational numbers.

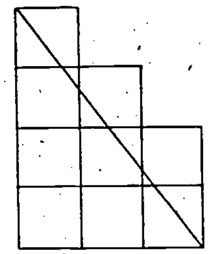
In the case of volume, a natural unit is a cube with one unit of length on a side; examples of units of volume are one cubic inch, one cubic foot, or one cubic meter. In simple cases, the volume of a box with dimensions a, b, and c length units can be found by counting unit cubes. Usually it is much easier to calculate the volume, V cubic units, by using the formula:



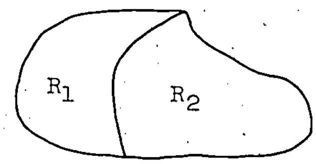
$$V = abc.$$

This result is easily seen to be valid if a, b, and c are any three positive rational numbers. It is true even if one or more of the dimensions are irrational. We actually multiply irrational numbers in such a way that this is true.

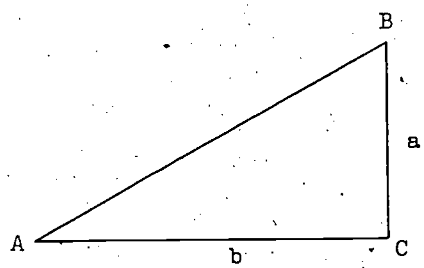
What can we do if the plane figure is not a rectangle or if the solid figure is not a box? What, for example, can we do about a right triangular region? We cannot tile it with unit squares.



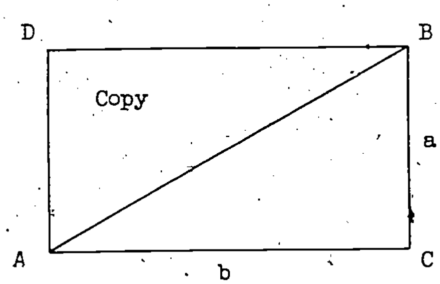
Let us ask ourselves what properties area ought to have if it is to agree with our idea of it. Let us assume that a unit has been chosen. It seems clear that we should make the following requirements: (1) Two congruent regions shall have the same area. If, for example, we make an exact copy of a triangular region, the area should not change. (2) If we join a region R_1 to a region R_2 so that there is no overlap, then the area of the combined region shall be the sum of the areas of R_1 and R_2.



Let us apply these two properties of area to the problem of finding the area of a right triangular region.



Whatever the area of $\triangle$ ABC may be, an exact duplicate of $\triangle$ ABC will have the same area. Now if we place the copy of $\triangle$ ABC in the proper position, we obtain a rectangle whose area measure is known to be $b \times a$. Then area (ABC) + area (ABD) = $b \times a$ and 2 area (ABC) = $b \times a$. We conclude therefore that the triangular region must have the area measure $\frac{1}{2}(b \times a)$.



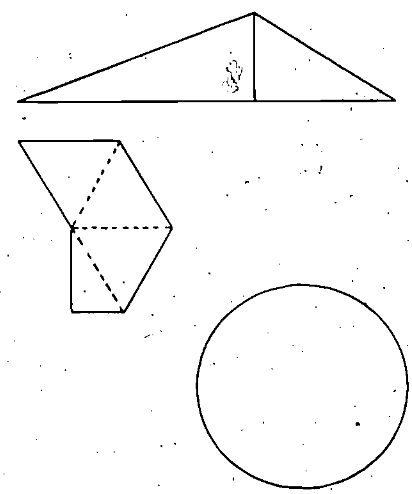
This result is commonly stated as follows: The area of a right triangle is $\frac{1}{2}$ the base times the altitude. This is not strictly correct.

In the first place, "right triangle" here really means "right triangular region". Secondly "base" means "length of base". Finally, the statement should include a reference to units. The precise statement might read:

> If the base of a right triangle has a length of $\underline{b}$ units and the altitude has a length of $\underline{a}$ units, then the area of the triangular region is A square units where
>
> $$A = \frac{1}{2}(b \times a).$$

Since it complicates speech to talk in this way, we shall often use the simpler form of expression as a kind of mathematical slang. When it is necessary you should be able to translate the slang into proper language.

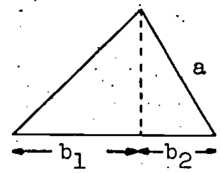
Since any triangular region is the union of two right triangular regions, its area is easy to find. From this point on it is easy to find the area of any polygonal region by cutting it into triangular regions. However, if the boundary of the plane region is curved (in whole or in part), the problem of finding the area is more difficult. The most important problem of this type is the case of a circular region.

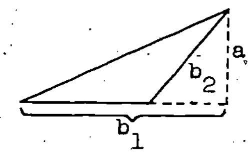


For solids the situation is somewhat similar. We assume that congruent solid figures have the same volume and that the volumes of nonoverlapping regions are added to find the volume of combined regions. If the solids are bounded by polygonal plane faces, it is possible to subdivide them into pyramids; and the volume of a pyramid ($\frac{1}{3}$ base times altitude) can be found by elementary methods. However, if the boundary is even partly curved, the volume problem is more difficult. Examples of solids with curved boundaries are cones, cylinders, and spherical balls. It turns out that the idea of average, studied in Chapter 25, comes to our rescue and enables us to find quite easily many areas and volumes which used to seem difficult. We shall soon turn to these matters. However, it will be helpful to discuss first the effect of changing our unit of measure. We shall do this in the next section.

<u>Exercises 28-2</u>

1. Given that the area of a right triangular region is $\frac{1}{2}$ (base $\times$ altitude), prove that in <u>any</u> triangle the area is $\frac{1}{2}$ (base $\times$ altitude). (Consider the two cases shown.)

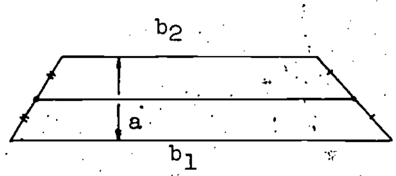




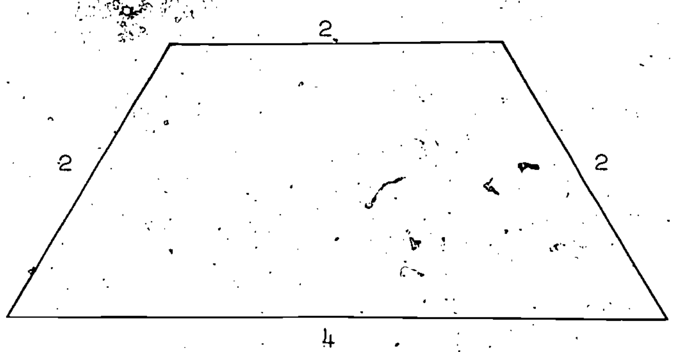
2. From the result of Exercise 1 show that for any parallelogram, the area is base $\times$ altitude.

3. (a) Show that the area of any trapezoid is $\frac{1}{2}$ the altitude times the sum of the bases.

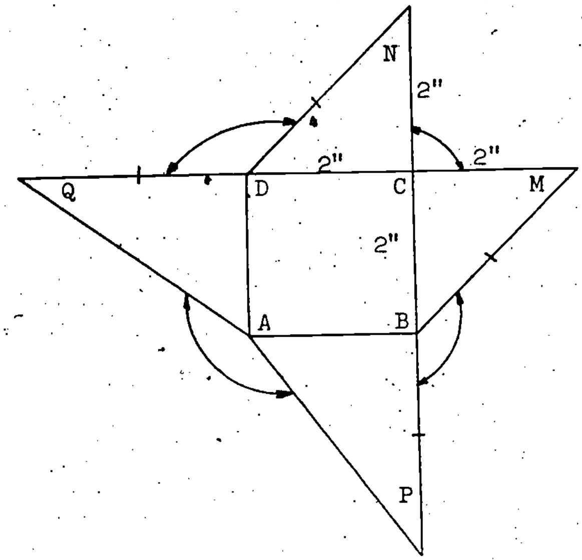
   (b) Show that we can replace this result by the altitude times the length of the segment which joins the midpoints of the non-parallel sides.



4. Find the area of the isosceles trapezoidal region shown.



5.



Construct a pyramid of stiff paper or thin cardboard by following steps (a) through (e).

(a) Draw square ABCD with each side of length 2 inches.

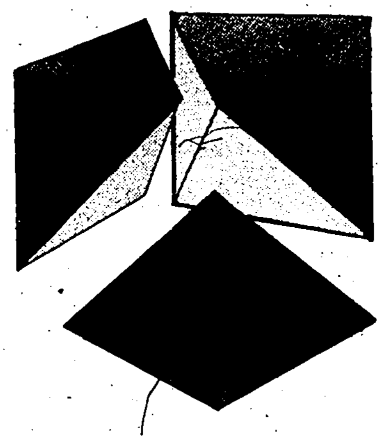
(b) Extend $\overrightarrow{DC}$ to M so that $CM = DC$.
Extend $\overrightarrow{BC}$ to N so that $CN = BC$.
Draw $\overline{MB}$ and $\overline{ND}$.

(c) Extend $\overrightarrow{CB}$ to P so that $BP = BM$.
Extend $\overrightarrow{CD}$ to Q so that $DQ = DN$.
Draw $\overline{AP}$ and $\overline{AQ}$.

(d) Cut out polygon APBMCNDQ.
Score and fold on segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.

(e) Scotch tape $\overline{CM}$ to $\overline{CN}$, $\overline{DN}$ to $\overline{DQ}$, $\overline{BM}$ to $\overline{BP}$, $\overline{AQ}$ to $\overline{AP}$.

6. By combining three pyramids like the one constructed in Exercise 5, show that a cube of side s may be cut into three congruent pyramids. (Hint: In each pyramid the altitude passes through one corner of the base as shown.) Thus show that

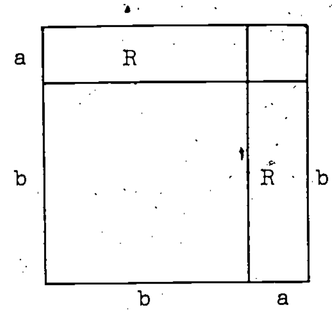


$$V = \frac{1}{3} a^3$$

without assuming the known result of any pyramid is

$$\frac{1}{3} \text{ (area of base) } \times \text{ altitude.}$$

7. Given that the area of a square is the square of its side and that congruent figures have the same area. Use the following figure to prove that the area of a rectangle must be base $\times$ altitude $(b \times a)$. Hint: Rewrite $(a + b)^2$ as the sum of three terms.

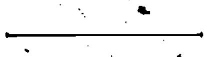


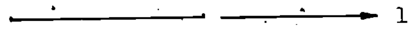
Associate the terms connected by plus signs with areas on the figure.

28-3. Measurement Functions

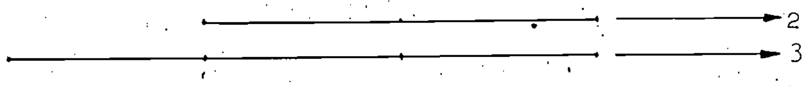
It will add to our understanding if we think of measurements as functions. For each measurement function, the inputs are objects which have the property that we wish to measure (a quantity like length, area, volume, weight). The outputs are non-negative real numbers.

For example, consider the function f that measures the length of line segments in inches f : segment $\rightarrow$ length measure in inches.

The real number 1 is the output when the input is the unit segment, . We show this as follows:

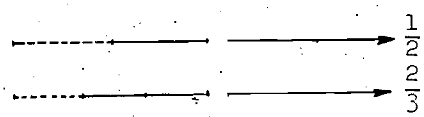


Putting segments together end-to-end, we have

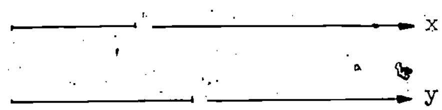


and so on.

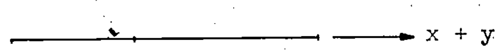
By subdividing the unit segment we have such results as



In general "addition" of segments corresponds to addition of the real number outputs (measures). Thus, if



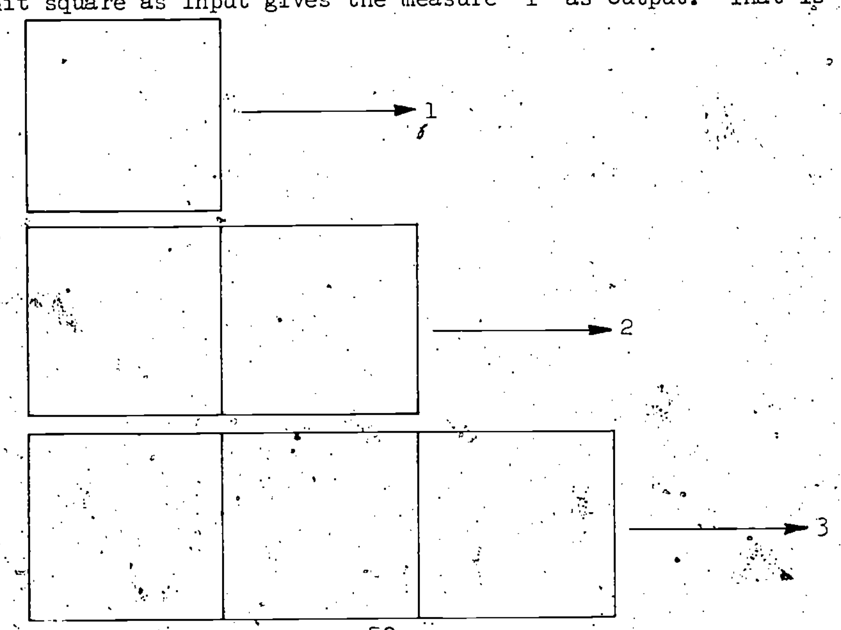
then



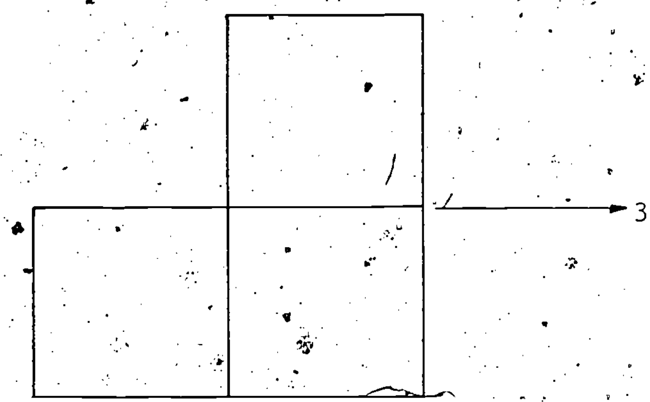
Similarly for areas we can introduce the function g.

$$g : \text{plane region} \rightarrow \text{area measure in square inches.}$$

Then the unit square as input gives the measure 1 as output. That is



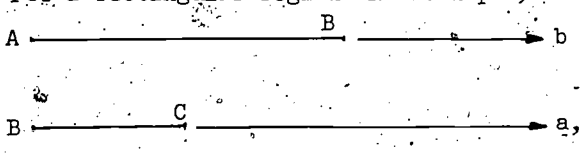
Also



and so on.

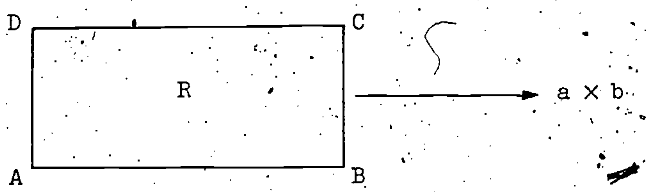
For a rectangular region R as input, we have:

If

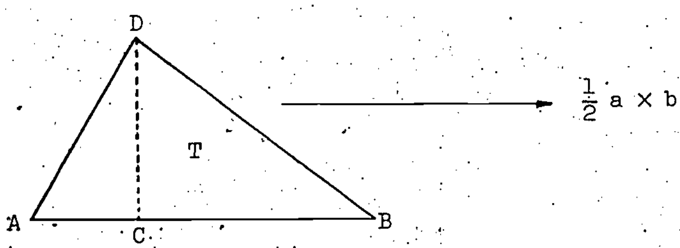


and

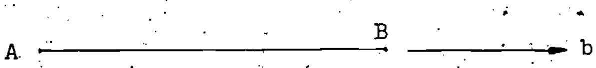
then



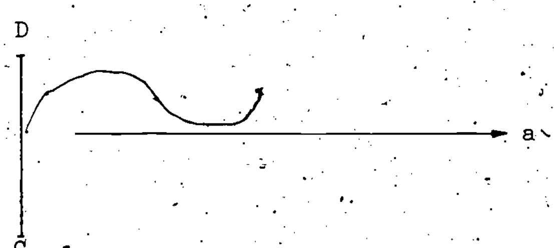
For a triangular region T



if

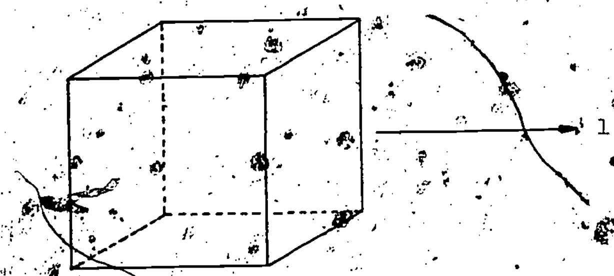


and

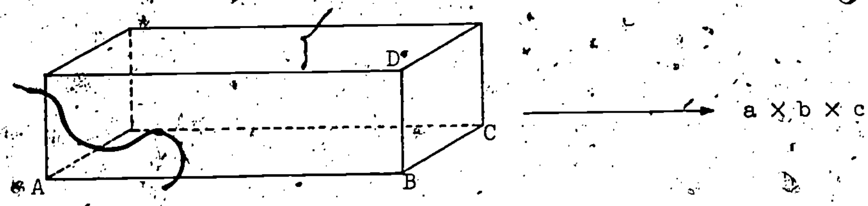


Moving into space we have the function h

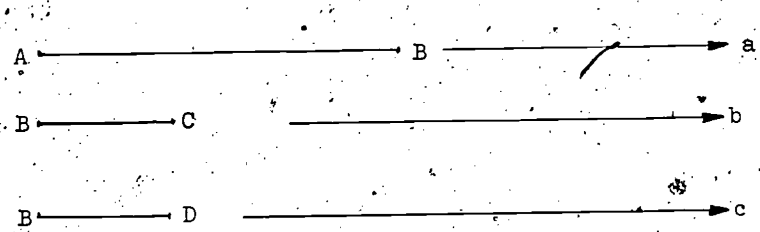
$$h : \text{space region} \to \text{volume measure in cubic inches.}$$



Then



if



<u>Exercises 28-3</u>

Draw arrow diagrams like those in the text to illustrate how the function f assigns an output (measure) for

1. The perimeter of triangle T.
2. The perimeter of rectangle R.
3. A diagonal of rectangle R.
4. The perimeter of a trapezoid.
5. The diagonal of the unit square.
6. The diagonal of the unit cube.

Draw arrow diagrams like those in the text to illustrate how the function g assigns an output (measure) for

7. The area of a trapezoidal region.

8. The surface area of a unit cube.

9. The surface area of a box with dimensions a, b, and c.

10. The surface area of the triangular prism obtained by cutting a solid cube by a plane through two opposite edges.

Draw arrow diagrams like those in the text to illustrate how the function h assigns an output (measure) for

11. The volume of a cube whose side measures 2 inches.

12. The volume of a pyramid with square base one inch on a side and an altitude of 1 inch. (Use for the volume, $\frac{1}{3}$ area of base times altitude.)

13. The volume of a pyramid whose altitude measures 2 inches and whose rectangular base measures 3 inches by 4 inches.

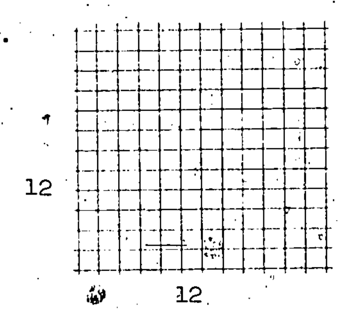
28-4. Changing Units and Some Consequences

Suppose that we measure the height of a door and find it to be 6 feet. What is its height in inches? Since a unit of 1 foot can be subdivided into 12 units of 1 inch, the answer is

$$6 \times 12 \text{ inches} = 72 \text{ inches}.$$

The measure in feet is multiplied by 12 to obtain the measure in inches.

What is the effect on area measures? That is, when we change from square feet to square inches what happens to the measure? Since there are $12 \times 12 = 12^2 = 144$ square inches in a square foot, all area measures are multiplied by 144. Thus, for example,



$$10 \text{ square feet} = 10 \times 144 \text{ square inches}.$$

We can write this

$$10 \text{ feet}^2 = 10 \times 144 \text{ inches}^2 .$$

Moving on to volumes we easily see that a cubic foot consists of $12 \times 12 \times 12 = 12^3$ cubic inches. Therefore

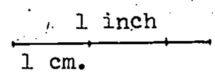
$$n \text{ cubic feet} = 1728\, n \text{ cubic inches},$$

whatever number n may be. More briefly

$$n \text{ ft.}^3 = 1728\, n \text{ in.}^3 .$$

Suppose that we wish to change length units from inches to centimeters. It has been agreed that

$$1 \text{ inch} = 2.54 \text{ centimeters}$$



exactly. Then

$$1 \text{ in.}^2 = (2.54)^2 \text{ cm.}^2 = 6.4516 \text{ cm.}^2$$

and

$$1 \text{ in.}^3 = (2.54)^3 \text{ cm.}^3 \approx 16.387 \text{ cm.}^3$$

where we have approximated the last result to three decimal places.

Hence

$$a \text{ in} = 2.54\, a \text{ cm},$$

$$b \text{ in}^2 \approx 6.45\, b \text{ cm}^2,$$

and

$$c \text{ in}^3 \approx 16.39\, c \text{ cm}^3$$

where approximations to 6.4516 and 16.387 have been chosen for convenience.

We may wish to go in the opposite direction, that is, change from the metric measures to the corresponding English ones. Since

$$1 \text{ inch} = 2.54 \text{ centimeters},$$

$$\frac{1}{2.54} \text{ in} = 1 \text{ cm},$$

and

$$1 \text{ cm} \approx .3937 \text{ in}.$$

Then

$$1 \text{ cm}^2 = \left(\frac{1}{2.54}\right)^2 \text{in}^2 \approx (.3937)^2 \text{ in}^2 \approx .155 \text{ in}^2$$

and

$$1 \text{ cm}^3 = \left(\frac{1}{2.54}\right)^3 \text{in}^3 \approx (.3937)^3 \text{ in}^3 \approx .061 \text{ in}^3 .$$

Exercises 28-4a

(Class Discussion)

1. How many feet are there in a yard?

   A certain cube measures 4 yd on an edge.

   (a) What is the measure of an edge in feet?

   (b) What is the surface area of this cube in sq yds? in sq ft?

   (c) What is the volume of the cube in cu yds? in cu ft?

   (d) Compare by division the area measure in ft^2 with the area measure in yd^2; the volume measure in ft^3 with the volume measure in yd^3.

2. In any measurement, if you change from a larger unit to a smaller unit, do the measures get larger or do they get smaller?

Let us suppose that we make a measurement using a particular length unit U. If we change the unit U to a new unit u so that $U = ku$, what effect does this have on the measures of length, area, and volume? Our previous examples suggest that:

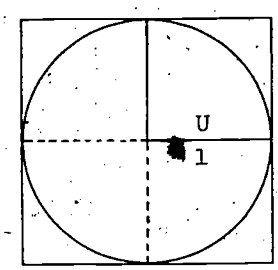
(1) All length measures are multiplied by k.

(2) All area measures are multiplied by k^2.

(3) All volume measures are multiplied by k^3.

We have seen three examples of this principle: Feet to inches with $k = 12$; inches to centimeters with $k = 2.54$; centimeters to inches with $k \approx .3937$. We shall assume that the principle holds for all positive numbers k and for all geometrical figures.

Let us apply this principle to the measurement of the area of a circular region. First choose a unit of length U equal to the radius of the circle. If the area of the circle is measured using a square U on a side as a unit, the result is a certain number less than 4 and greater than 3. The name of this number is π, pronounced pī.



In a later section we shall turn to the problem of calculating the number π. For the moment, let us imagine that this problem has been solved. What is the area of the circular region if we choose such a unit of length that the radius measures r units? The answer is simple. We take $k = r$ in the statement of our principle and conclude that the new area measure is $A = \pi r^2$.

By this simple argument we can reduce our problem to the special case where the radius is 1.

In a similar way, if we want the volume of a ball of radius r, we can consider the special case where the radius is chosen as the unit of length. If we can solve this special case, we can multiply the answer by r^3 to get the result when the radius is r length units.

It is clear that it is sufficient to find a length, an area, or a volume for a single choice (U) of the unit of length. If this has been done, we can use the fundamental principle of length, area and volume to find the measures that correspond to any other choice of the unit of length U. It is merely necessary to find the value of k so that $U = ku$.

Exercises 28-4b

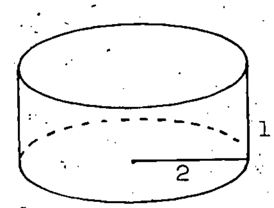
1. How many times as much material does it take to make a solid hemisphere of radius one yard than one with radius one inch?

2. The surface area of a sphere of radius 1 unit has the measure 4π. What is the measure of surface area if the radius measures r?

3. If a person is given 4 square feet on which to stand and if the population of the world is 3 billion, how many square miles would it take to allow the whole human race to stand all in one group? About how many miles would there be on the side of a square of this area?

4. Light travels in space about 186,000 miles a second. A light-year is the distance that light travels in a year. About how many miles are there in a light year? Take a year to be $365\frac{1}{4}$ days and round off the number of seconds in a year to the nearest 100,000.

5. When a certain unit of length is used, a cylinder is found to have the following measures: height 1, radius of base 2,



area of base 4π, and volume 4π.

(a) If the unit of length is changed so that the height measures 5 units, show that according to our principle the radius now measures 10, the areas of base 100π, and the volume 500π.

(b) Show that in each case the volume equals area of base $\times$ height.

6. When a certain unit of length is used, a cylinder has the following measures: height 1, radius of base R, area of base πR^2, and volume πR^2. If the unit of length is changed so that the height measures h units, show that the area of the base is still π times the square of the radius, and the volume is the area of the base times the height.

28-5. Areas and Averages

Suppose that the graph of a function f rises as x increases from 0 to 1 and that the y-intercept is non-negative. We wish to assign a number A to measure the area of the region below the graph and above the interval [0,1]. How shall we do this?

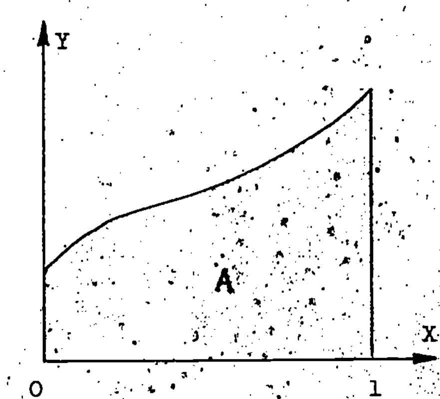


Figure 1

We might begin by locating the points $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots, \frac{9}{10}$ on the X axis, and erecting perpendiculars which extend up to the graph. Let $y_1, y_2, \ldots, y_9$ be the lengths of these perpendiculars, that is, the values of the function at $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}$. We call the ordinate at 1 y_{10} since $1 = \frac{10}{10}$.

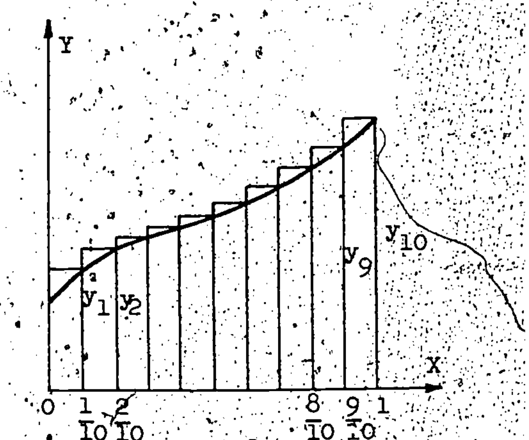


Figure 2

Now draw horizontal line segments as in Figure 2 so that ten rectangular regions are formed with heights $y_1, y_2, \cdots, y_{10}$. Each rectangle has the width $\frac{1}{10}$.

Surely we shall wish the required area to be less than the sum of these rectangular areas. Then

$$A < \frac{1}{10} y_1 + \frac{1}{10} y_2 + \cdots + \frac{1}{10} y_{10}$$

or

(1) $$A < \frac{y_1 + y_2 + \cdots + y_{10}}{10}.$$

Notice that the right side of the inequality is the <u>average</u> of the heights of the rectangles. We shall call this an <u>upper</u> average of the function f on the interval $[0,1]$.

To obtain a <u>lower</u> average we draw horizontal line segments as in Figure 3 so as to form rectangles of width $\frac{1}{10}$ and heights $y_0, y_1, \cdots, y_9$. (Of course, y_0 is the ordinate at $x = 0$.) The total areas of these rectangles measures

$$\frac{1}{10} y_0 + \frac{1}{10} y_1 + \cdots + \frac{1}{10} y_9.$$

We shall require that A be greater. That is

(2) $$\frac{y_0 + y_1 + \cdots + y_9}{10} < A.$$

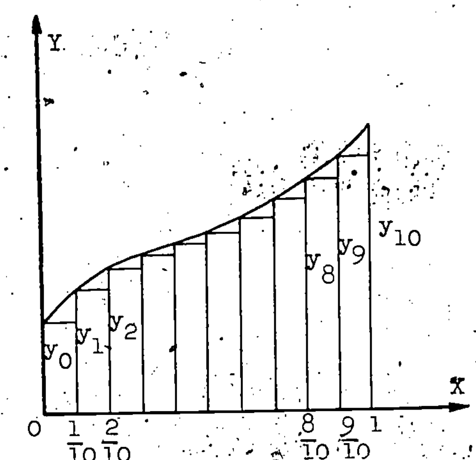


Figure 3

On the left we have the averages of the heights of the inner rectangles. We shall call this a <u>lower</u> average of the function f on $[0,1]$.

From (1) and (2) we know that

(3) $$\frac{y_0 + y_1 + \cdots + y_9}{10} < A < \frac{y_1 + y_2 + \cdots + y_{10}}{10}.$$

Of course, this double inequality does not tell us <u>exactly</u> how large A is. It gives an <u>approximation</u>. How accurate is this approximation? The accuracy is measured by the difference between the upper average and the lower average.

If we subtract $\frac{y_0 + y_1 + \dots + y_9}{10}$ from $\frac{y_1 + y_2 + \dots + y_{10}}{10}$,

what do we get? To do the subtraction we write these averages as follows:

$$\left(\frac{1}{10} y_1 + \frac{1}{10} y_2 + \dots + \frac{1}{10} y_9 + \frac{1}{10} y_{10}\right) \quad \text{(Upper average)}$$

$$-\left(\frac{1}{10} y_0 + \frac{1}{10} y_1 + \frac{1}{10} y_2 + \dots + \frac{1}{10} y_9\right) \quad \text{(Lower average)}$$

$$= \frac{1}{10} y_{10} - \frac{1}{10} y_0.$$

The difference is $\frac{y_{10} - y_0}{10}$. It is the difference between the approximations pictured in Figure 2 and Figure 3. This difference may also be pictured as the area of the shaded rectangle in Figure 4. It is the area of the rectangle with height $y_{10} - y_0$ and width $\frac{1}{10}$.

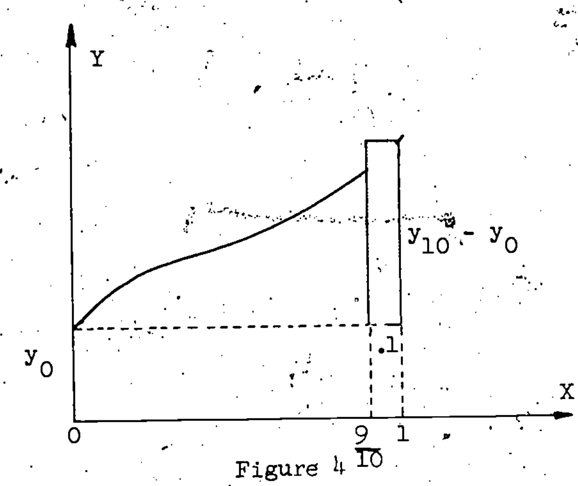


Figure 4

Now what we did was to obtain upper and lower approximations by subdividing the unit interval [0,1] into 10 congruent parts. We could have chosen 100 congruent intervals or 1000 or 1,000,000! To discuss all of these possibilities at once, let us assume that [0,1] is subdivided into n congruent intervals, each of length $\frac{1}{n}$. The upper average is now $\frac{y_1 + y_2 + \dots + y_n}{n}$ and the lower average is $\frac{y_0 + y_1 + \dots + y_{n-1}}{n}$ and so the inequality (3) is replaced by

(4) $$\frac{y_0 + \dots + y_{n-1}}{n} < A < \frac{y_1 + \dots + y_n}{n}.$$

The difference between the two averages is $d = \frac{y_n - y_0}{n}$. The numerator $y_n - y_0$ is simply RQ. This does not depend on the number n of subdivisions. But we can choose n large enough so that the difference d is as small as we please.

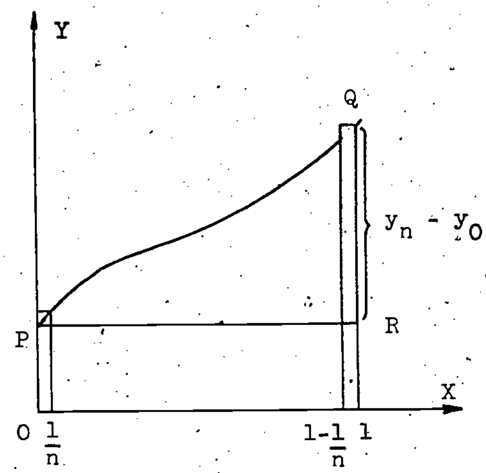


Figure 5

We shall prove that there can be only one number A which satisfies (4) for all positive integers n. That is, there can be only one number which is entitled to measure the area. For suppose, on the contrary, that there were two possible numbers, a smaller one, say p, and a larger one $p + s$ (s for "something"), each of which could be a value of A. Then p must satisfy (4) and so must $p + s$. Then their difference, $(p + s) - p = s$, is positive and cannot exceed the difference d between the two averages.

That is, we would have

$$s < \frac{RQ}{n}$$

for all positive integers n. But $s < \frac{RQ}{n}$ is true only if $n < \frac{RQ}{s}$. It is false if $n > \frac{RQ}{s}$. Surely n can be chosen this large.

A number A which is between all lower averages $\frac{y_0 + \dots + y_{n-1}}{n}$ and corresponding upper averages $\frac{y_1 + \dots + y_n}{n}$ will be called the <u>average of y</u> over the interval $[0,1]$ and will be denoted by $\overline{y}$. As we have seen, there cannot be more than one such number $\overline{y} = A$.

A geometrical interpretation will make it reasonable to write $A = \overline{y}$. If we draw a horizontal line at the height $\overline{y}$ (see Figure 6), a rectangle will be formed of base 1. If this rectangle has the same area as that under the graph, $1 \times \overline{y} = \overline{y} = A$. We may say therefore

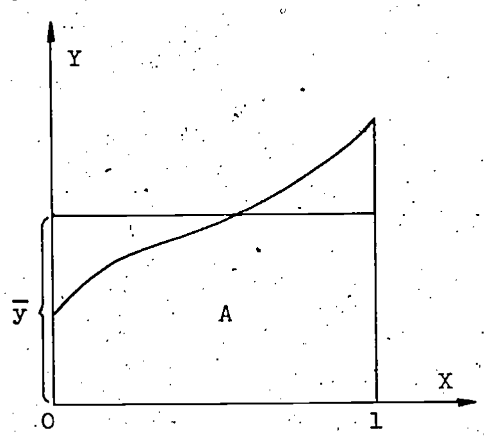


Figure 6

that $\bar{y}$ represents that constant height which gives an area equal to that below the graph and above $[0,1]$.

Exercises 28-5

1. Let the function f be given by

$$f : x \to x$$

so that the graph is that of $y = x$. Find the upper and lower averages of $y = x$, for $n = 10$, on the interval $[0,1]$. That is, put in the proper values of the y's in the inequality (3).

$$\frac{y_0 + y_1 + \dots + y_9}{10} < A < \frac{y_1 + y_2 + \dots + y_{10}}{10}$$

You should find that $.45 < A < .55$. Hint: $y_1 = .1$, $y_2 = .2$, ...

2. Let f be given by

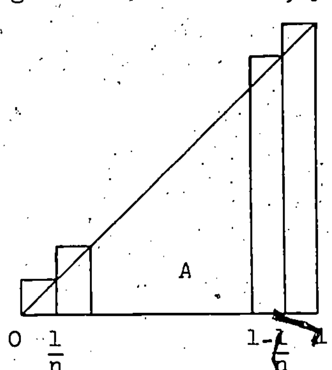
$$f : x \to x^2$$

so that the graph is that of $y = x^2$. Find the upper and lower averages of $y = x^2$, for $n = 10$, on the interval $[0,1]$, using $y_1 = (.1)^2 = .01$, $y_2 = (.2)^2 = .04$, and so on. You should find that

$$.285 < A < .385.$$

28-6. Some Special Cases

The question arises whether it is possible in practice to find the upper and lower averages discussed in the last section. Can we actually find the precise value of A? The answer is that in some important special cases this can be done very easily. If you go on in mathematics, you will find that many other cases can be treated effectively. We shall be concerned with only two very simple ones.



Example 1. To find the area below the graph of $y = x$ and above the interval $[0,1]$.

Of course we know the answer. For the triangle, $A = \frac{1}{2}$. Nevertheless it will be useful to see how this result appears as an average. Since $y = x$, $\bar{y} = \bar{x}$, and $A = \bar{y}$ becomes $A = \bar{x}$. We shall show that $\bar{x} = \frac{1}{2}$.

For n subdivisions, the upper average is

$$\frac{y_1 + y_2 + \dots + y_n}{n}.$$

But since $y = x$ and $x_n = 1$, we have

$$\frac{x_1 + x_2 + \dots + 1}{n}.$$

The x values are equally spaced. Therefore the average is simply

$$\frac{1 + x_1}{2} = \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

Finally, then, the upper average is

$$\frac{1}{2} + \frac{1}{2n}.$$

The lower average is

$$\frac{y_0 + y_1 + \dots + y_{n-1}}{n} = \frac{0 + x_1 + \dots + x_{n-1}}{n}.$$

Again the x's are equally spaced so the average is

$$\frac{0 + x_{n-1}}{2} = \frac{0 + \left(1 - \frac{1}{n}\right)}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{2n}.$$

Therefore

(1) $$\frac{1}{2} - \frac{1}{2n} < A < \frac{1}{2} + \frac{1}{2n}.$$

What is the difference between the upper and lower averages?

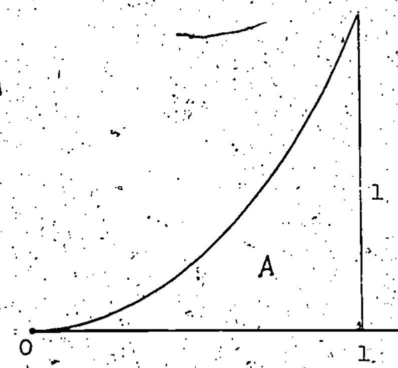
The double inequality (1) must be true for all positive integers n. There is only one possible value of A. What is this value? Certainly $A = \frac{1}{2}$. In fact

$$\frac{1}{2} - \frac{1}{2n} < \frac{1}{2} < \frac{1}{2} + \frac{1}{2n}$$

for all positive integers n.

As we expected, $A = \bar{x} = \frac{1}{2}$.

Example 2. The next case is more interesting because the result is new. We wish to obtain the area below the parabola $y = x^2$ and above the unit interval $[0,1]$. This area is measured by $A = \overline{x^2}$. Since



$y_1 = (\frac{1}{n})^2 = \frac{1}{n^2}$ and

$y_2 = \frac{4}{n^2}, \ldots, y_n = \frac{n^2}{n^2} = 1,$

the upper average

$$\frac{y_1 + y_2 + \ldots + y_n}{n}$$

is

$$\frac{\frac{1}{n^2} + \frac{4}{n^2} + \frac{9}{n^2} + \ldots + \frac{n^2}{n^2}}{n}$$

This is

$$\frac{1}{n^2}\left(\frac{1 + 2^2 + \ldots + n^2}{n}\right)$$

When we studied averages we learned that the average of the first n squares is

$$\frac{(n + 1)(2n + 1)}{6} = \frac{2n^2 + 3n + 1}{6} = \frac{n^2}{3} + \frac{n}{2} + \frac{1}{6}.$$

The upper average of x^2 is $\frac{1}{n^2}$ times this, that is,

$$\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

The lower average is $\frac{1}{n}$ less than the upper average. It is therefore

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

So

(2) $$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} < \overline{x^2} < \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

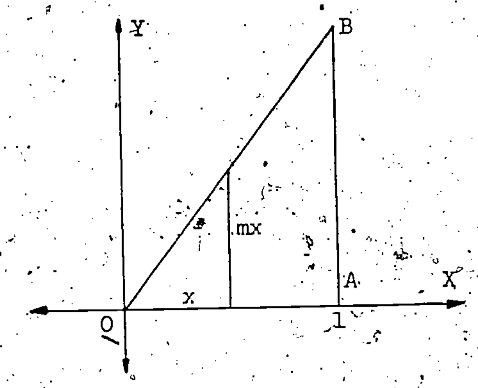
There can be only one solution of this double inequality. Of course that solution is $\overline{x^2} = \frac{1}{3}$.

Exercises 28-6

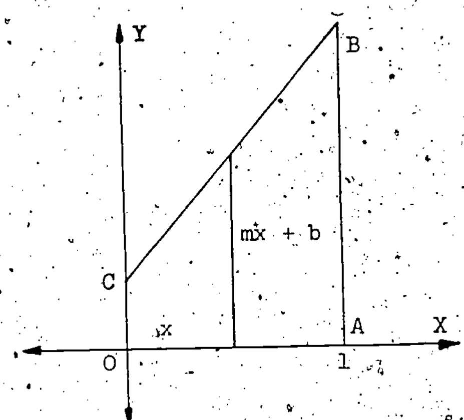
1. In Example 1, what are the upper and lower averages for 100 congruent subdivisions of $[0,1]$? for 1000 congruent subdivisions? Use inequality (1) with $n = 10$ to check your answer for Exercise 1 in Exercises 28-5.

2. In Example 2, what are the upper and lower averages for 100 congruent subdivisions of $[0,1]$? for 1000 congruent subdivisions? (Write the results in decimal form to 5 decimal places.)

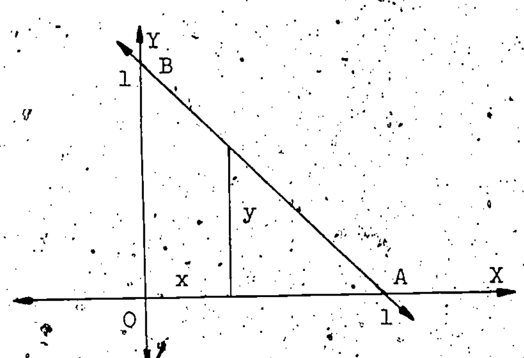
3. Find $\overline{y} = \overline{mx}$ and show that this gives the correct area of the triangular region OAB.



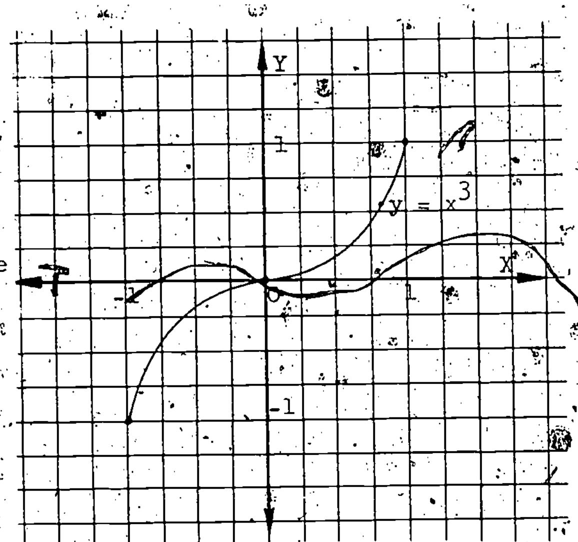
4. Find $\overline{y} = \overline{mx + b}$ and show that this gives the correct area of the trapezoidal region OABC.



5. Find y in terms of x, then find $\overline{y}$, and compare the result with the area of triangular region AOB. Why does this result agree with that in Example 1 of this section?



6. We wish to find the area below the graph of $y = x^3$ and above $[0,1]$.



(a) Show that the upper average for n equal subdivisions is

$$\frac{1}{n^3}\left(\frac{1^3 + 2^3 + \ldots + n^3}{n}\right) = \frac{1}{n^3}\left(\frac{n(n+1)^2}{4}\right).$$

(Hint: In Exercise 2 of Exercises 25-7b you showed that $\overline{i^3} = \frac{n(n+1)^2}{4}$.)

(b) Show that the upper average is equivalent to $\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}$.

(c) Show that the lower average is $\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}$.

(d) What is the required area?

28-7. The Area of a Circle

In Figure 7 we have drawn a quarter of a circular region of radius 1. Let A measure the area of this region. Then $4A$ measures the area of the complete circular region. Since the radius is 1, $4A$ is called π. How can we find the value of π?

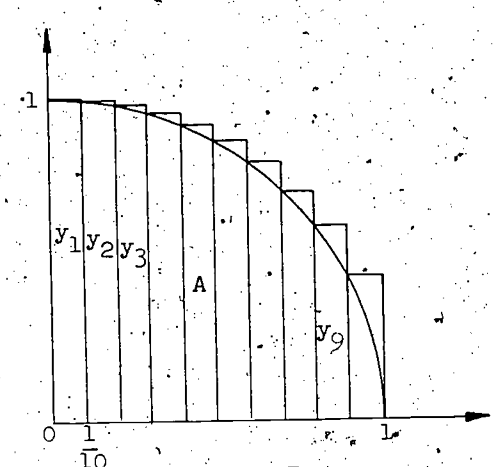


Figure 7

We can think of A as the average value of y over the interval $[0,1]$, where $x^2 + y^2 = 1$ (why? See Figure 8).

Then

$$y^2 = 1 - x^2$$

and

$$y = \sqrt{1 - x^2}.$$

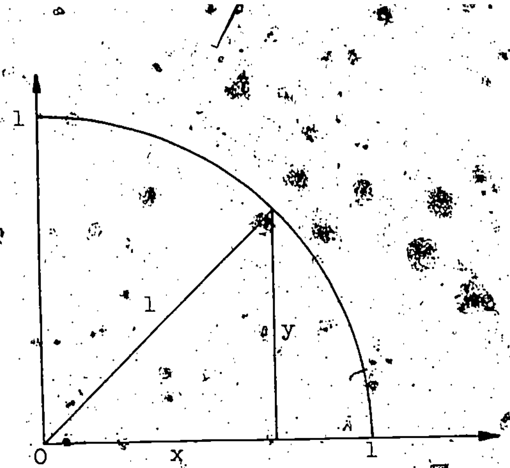


Figure 8

Unfortunately we have no simple way to average $\sqrt{1 - x^2}$ like the methods used to find $\overline{x}$ and $\overline{x^2}$. We must be content with <u>approximations</u> using upper averages and lower averages.

In Figure 7, we have cut the interval $[0,1]$ into 10 subintervals each of length $\frac{1}{10}$, and we have drawn ten rectangles which enclose the quarter circle. From the figure we see that

$$A < \frac{1 + y_1 + \dots + y_9}{10}.$$

Similarly from Figure 9 we see that

$$\frac{y_1 + y_2 + \dots + y_9 + 0}{10} < A.$$

(We have included 0, the height of the 10th rectangle, so as to have 10 numbers to average.)

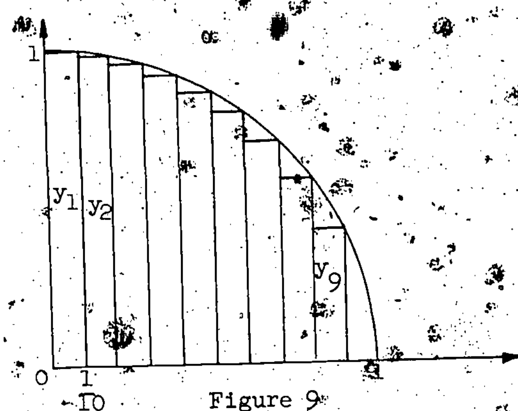


Figure 9

We put these two results together, and write

$$\frac{y_1 + y_2 + \dots + y_9 + 0}{10} < A < \frac{1 + y_1 + \dots + y_9}{10}.$$

The difference between the upper and lower averages is simply $\frac{1}{10}$.

To use the double inequality we must calculate the y-values. For example,

$$y_1 = \sqrt{1 - x_1^2} = \sqrt{1 - (\tfrac{1}{10})^2} = \sqrt{1 - \tfrac{1}{100}} = \sqrt{\tfrac{99}{100}} = \frac{\sqrt{99}}{10}.$$

From the table of square roots in Chapter 21, $\sqrt{99} \approx 9.95$. Therefore $y_1 \approx .995$. Following this pattern, we have calculated each of the y's to three decimal places.

$y_1 = .995$ $y_6 = .800$

$y_2 = .980$ $y_7 = .714$

$y_3 = .954$ $y_8 = .600$

$y_4 = .916$ $y_9 = .436$

$y_5 = .866$

If we add and divide by 10 we obtain the lower average .726. The upper average is .826 (.1 greater).

We can improve the estimate for A by averaging .726 and .826. The result is .776. Let us see what this new average means geometrically.

Exercises 28-7a

(Class Discussion)

We are averaging

$$\frac{1 + y_1 + \ldots + y_8 + y_9}{10}$$

and

$$\frac{y_1 + y_2 + \ldots + y_9 + 0}{10}$$

This new average can be written

$$(1) \quad \frac{1}{10}\left(\frac{1 + y_1}{2}\right) + \frac{1}{10}\left(\frac{y_1 + y_2}{2}\right) + \ldots + \frac{1}{10}\left(\frac{y_8 + y_9}{2}\right) + \frac{1}{10}\left(\frac{y_9 + 0}{2}\right).$$

1. What region has the area measure $\frac{1}{10}\left(\frac{1 + y_1}{2}\right)$? (See Figure 10.)

2. What region has the area measure $\frac{1}{10}\left(\frac{y_1 + y_2}{2}\right)$?

3. What does $\frac{1}{10}\left(\frac{y_8 + y_9}{2}\right)$ represent?

4. How can you interpret $\frac{1}{10}\left(\frac{y_9 + 0}{2}\right)$?

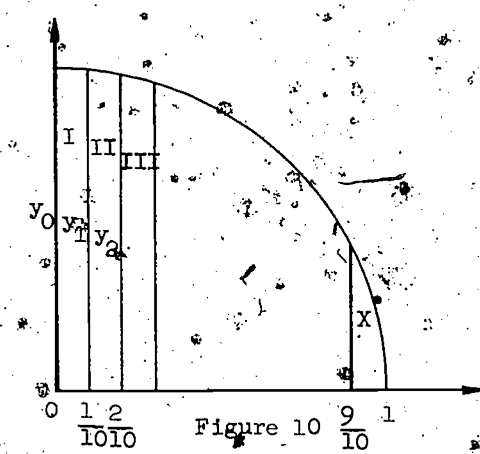


Figure 10

5. What total area does the required average (1) measure?

6. Is this area smaller or larger than the area of the quarter circle?

If you have answered these questions correctly you will conclude that

$$.776 < A.$$

You will soon discover in the exercises that

$$A < .793.$$

Therefore

$$.776 < A < .793,$$

and so

$$3.104 < 4A < 3.172;$$

that is,

$$3.104 < \pi < 3.172.$$

We can get a closer estimate to π by working with a larger number of subdivisions of the unit interval, say 20 or even 100. With patience or a machine you would discover that

$$3.1415 < \pi < 3.1416.$$

It is possible to obtain much greater accuracy than this. In 1959, as a training problem, π was calculated to 10,000 decimals on an IBM 704 computer. Time: 1 hour, 40 minutes. It has been said, "The mysterious and wonderful π is reduced to a gargle that helps computing machines clear their throats."*

Incidentally, it can be proved that π is irrational. What does this mean?

Exercises 28-7b

1. Calculate y_3, y_5, y_7 and y_9 for Figure 9 and verify the results given in the text. Hint: Follow the method used in finding y_1.

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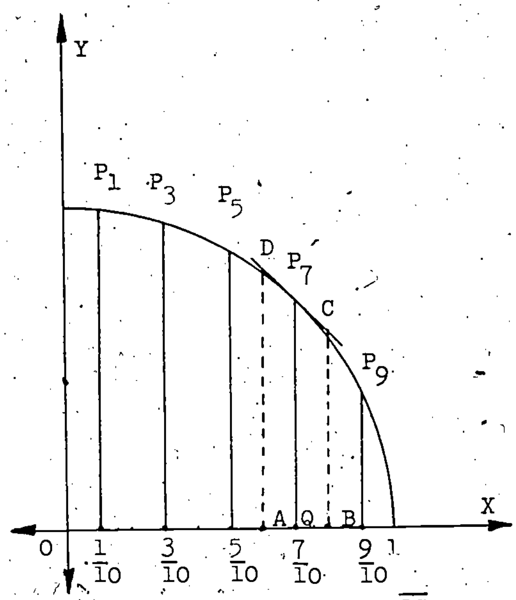
* Philip J. Davis, "The Lore of Large Numbers," New Mathematics Library, p. 74.

2. In the figure, let P_1 be the point on the circle above $\frac{1}{10}$, P_3 the point above $\frac{3}{10}$, and so on to P_9 above $\frac{9}{10}$.



(a) Copy this figure and draw tangents to the circle at each of the points P_1, P_3, P_5, P_7, and P_9. Cut off segments of these tangents which lie above the intervals [0,.2], [.2,.4], [.4,.6], [.6,.8] and [.8,1.0]. (For example, $\overline{DC}$ is tangent to the circle at P_7 and $\overline{DC}$ lies above [.6,.8].)

(b) Find the measures of the five trapezoidal areas. (<u>Hint</u>: $\overline{QP_7}$ joins the midpoints of the nonparallel sides of trapezoid ABCD.)

(c) Show that the sum of the measures of the five areas $\approx .793$ and that the total area is greater than the area of the quarter circle.

28-8. <u>Volumes</u>

The figure shows a pyramid with altitude 1, with a square base of side 1, and the vertex directly above one corner of the base.

You remember that three copies of this pyramid can be put together to form a unit cube. It follows that for the pyramid, $V = \frac{1}{3}$.

Let us look at this result with fresh eyes. If we slice the pyramid by a plane parallel to the base and, at the distance x from the vertex,

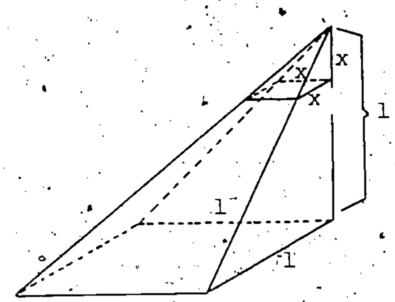
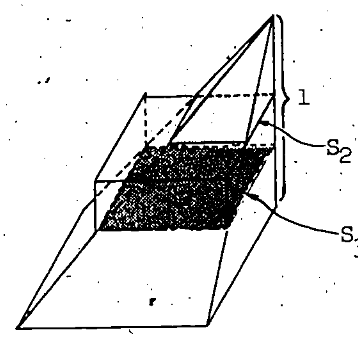


Figure 11

we obtain a cross section which is a square of side x. The area of this cross section therefore measures $S = x^2$.

We shall show that the <u>volume has a measure equal to the measure of the average cross section area.</u> To simplify our language we shall hereafter say that S is the cross-sectional area and V is the volume where the correct verb in each case is "measures."



The proof is like the one that was used when we talked about the area under a graph. We divide the altitude 1 into n equal parts, each of length $\frac{1}{n}$. Then we take plane sections parallel to the base through each of the points of division. In the figure the second and the third sections from the top are shown shaded. Let $S_1, S_2, S_3, \ldots, S_{n-1}$ be the areas of the successive cross sections and S_n the area of the base. We can now enclose the pyramid within n flat boxes each of thickness $\frac{1}{n}$. We have drawn one of these boxes with area of base S_3 and volume $\frac{1}{n}(S_3)$. The total volume of all of the outside boxes is

$$\frac{1}{n} S_1 + \frac{1}{n} S_2 + \ldots + \frac{1}{n} S_n$$

or

$$\frac{S_1 + S_2 + \ldots + S_n}{n}.$$

This is an <u>upper</u> average since it is greater than the required volume V.

Similarly, we can construct flat boxes of thickness $\frac{1}{n}$ and faces of area $S_0 = 0, S_1, \ldots, S_{n-1}$. Each of these boxes is inside the pyramid. The total volume is $\frac{1}{n} S_0 + \frac{1}{n} S_1 + \ldots + \frac{1}{n} S_{n-1} = \frac{S_0 + S_1 + \ldots + S_{n-1}}{n}$.

This is a <u>lower</u> average since it is less than V. Therefore

$$\frac{S_0 + S_1 + \ldots + S_{n-1}}{n} < V < \frac{S_1 + S_2 + \ldots + S_n}{n}$$

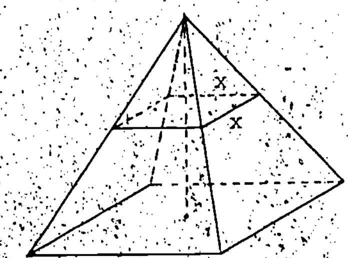
V must be a number which makes this double inequality true for every positive integer n. As before, we can show that there can be only one number V for which this is true. Therefore it is natural to call V the average of S over the x-interval $[0,1]$ and to write $V = \bar{S}$.

Since $S = x^2$,

$$V = \overline{S} = \overline{x^2}.$$

As you know, $\overline{x^2} = \frac{1}{3}$, which agrees with our previous result. (See Section 28-6.)

If the vertex of the pyramid is directly above the center of the base (or anywhere else), the cross section at the distance x from the vertex is still a square with side x. Therefore $S = x^2$ and once again

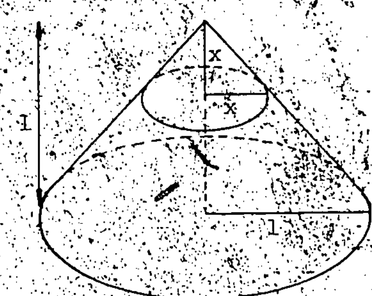


$$V = \overline{x^2} = \frac{1}{3}.$$

If the length is measured in such units that the altitude and each side of the square base is a, the volume measure is multiplied by a^3 so that $V = \frac{a^3}{3}$.

The result $V = \overline{S}$ was proved for certain pyramids of altitude 1. The argument is equally valid for any solid of altitude 1, if the cross-sectional area increases with the distance from its base. We apply this more general result in the following examples.

Example 1. Take a cone with radius of circular base equal to the altitude. Choose the unit of length so that this altitude is 1. What is the volume of the cone?



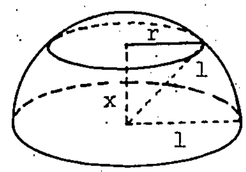
The cross section at distance x from the vertex is a circular region of radius x and therefore of area πx^2. Since

$$S = \pi x^2$$

$$\overline{S} = \overline{\pi x^2} = \pi \frac{1}{3} = \frac{\pi}{3}.$$

This is the volume measured in such units that a unit cube has an edge equal to the altitude. If a different choice of length unit is made so that the altitude is a, $V = \frac{\pi}{3} a^3$.

Example 2. The next example is a famous one: to find the volume of a solid hemisphere. As before we choose the length unit so that the altitude is 1. This means that the radius is 1.



Let us slice the solid at the height x above the base. The result is a circular region. If r is the radius of the circle, by the Pythagorean property, $x^2 + r^2 = 1$ so that $r^2 = 1 - x^2$. Then the area of the cross section is

$$S = \pi r^2 = \pi(1 - x^2) = \pi - \pi x^2.$$

The volume

$$V = \overline{S} = \overline{\pi - \pi x^2}.$$

Hence

$$V = \pi - \pi \frac{1}{3} = \frac{2}{3} \pi.$$

If the unit of length is such that the radius of the hemisphere is a, the volume measure is multiplied by a^3, and therefore

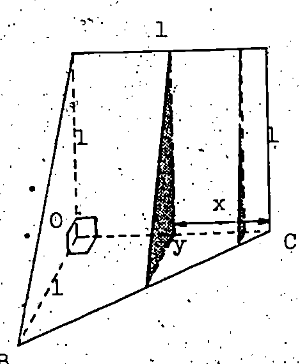
$$V = \frac{2}{3} \pi a^3.$$

For a whole sphere this result must be doubled so that

$$V = \frac{4}{3} \pi a^3.$$

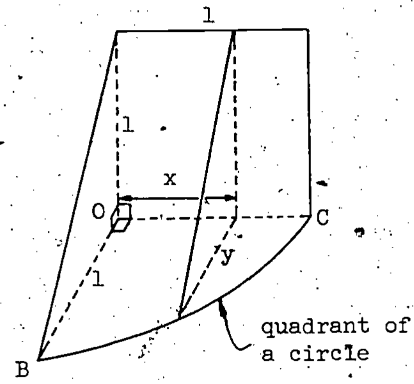
Exercises 28-8

1. In the figure every section perpendicular to $\overline{OC}$ is a right triangular region as shown. What is the volume of the solid? If units are chosen so that the lines marked 1 have measure a, what is the volume?

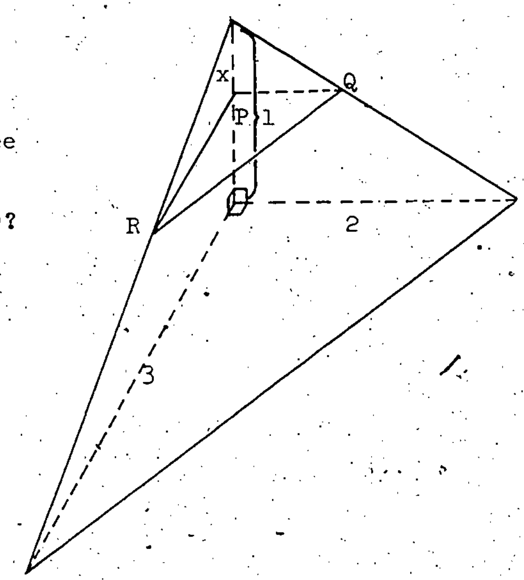


2. In this figure the triangular region OCB of Exercise 1 is replaced by a quarter of a circular region. Find the volume.

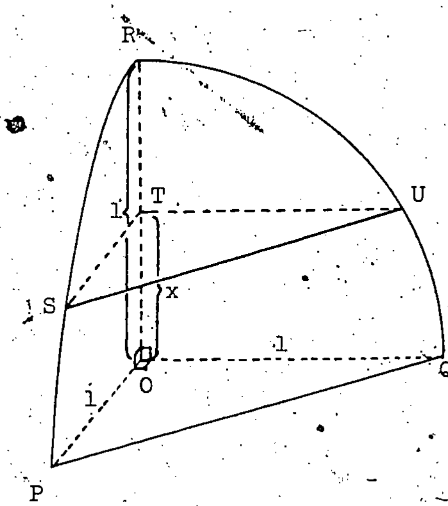
Hint: You must find $\bar{y}$. How can you do this?



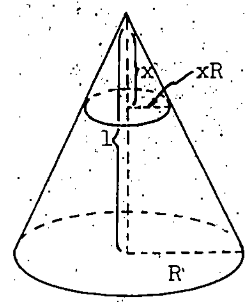
3. In the figure what is PQ? PR? What then is the cross-sectional area S? Find the volume. Does the result agree with the rule volume is $\frac{1}{3}$ (area of base times altitude)?



4. Two quadrants of circles lie in vertical planes at right angles to each other. A ruler is slid along them, remaining parallel to $\overline{PQ}$ so as to sweep out a surface. Find the volume enclosed by this surface, the vertical planes OPR and OQR and the base OPQ. Hint: Take a section at the distance x above the base. Find TU and ST. Remark: The figure may be thought of as forming one quarter of a certain type of tent.

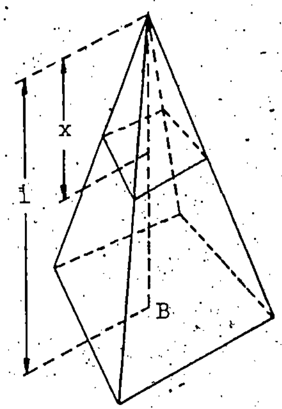


5. Given any right circular cone. Choose the unit of length so that the altitude is 1. Let R be the radius of the base measured with this unit.



Show that the section at distance x from the vertex has radius xR and therefore the cross-sectional area $S = \pi R^2 x^2$. Find $V = \overline{S}$ and show that the volume is $\frac{1}{3}$ (area of the base times the altitude). Prove that this is still true if the unit of length is changed so that altitude is h.

6. In terms of a certain unit of length the height of a pyramid is 1 and the area of its base is B.



Show that at distance x from the vertex $S = Bx^2$ (this is true if the base is any polygonal region) and the volume is $\frac{1}{3}$ (area of the base times the altitude). Prove that the same statement is correct if the altitude is h units.

7. Show that the volume of any right circular cylinder is the area of the base times the altitude. <u>Hint</u>: First choose a unit for which the altitude equals 1. What is S?

28-9. Some Other Measurements

We have studied measurements of length, area, and volume. As you know, there are many other quantities that we can measure: weight, pressure, temperature, speed, elapsed time, voltage, electric current--to name a few.

In the case of the measurement of length, area, and volume

(1) it is possible to choose a unit that can be copied or duplicated;

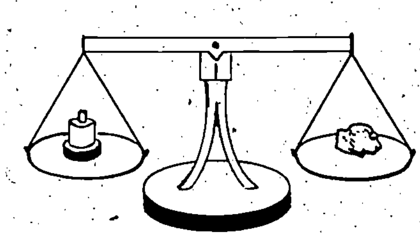
(2) these units can be put together in a way that corresponds to counting;

(3) it is possible to divide the given unit into a sufficiently large number of interchangeable sub-units to reach any desired accuracy.

Measurements with these characteristics will be called direct or fundamental measurements.

Let us consider the measurement of weight with these ideas in mind. There are of course many familiar units of weight, for example the pound weight and the kilogram.

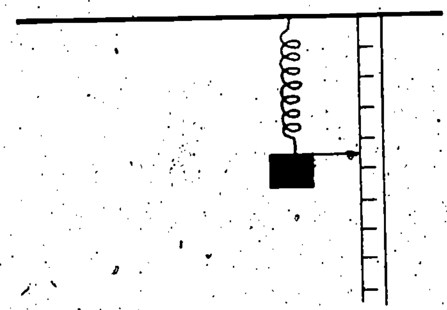
(1) Can a weight unit be copied or duplicated? Yes, of course. We can construct as many interchangeable pound weights as we please. To test whether two pieces of metal (whose weights are supposed to be units) are equal in weight it is sufficient to put them in opposite pans of an equal-arm balance to see whether the bar is horizontal.



(2) To weigh an object we can count the number of unit weights in one pan that balance the object in the other pan.

(3) In general we must use sub-units to bring about such a balance. But we can easily make sub-units. For example we can use ounce weights, 16 of which balance a pound weight.

The measurement described is a direct measurement because it determines the weight of the object by using a certain number of units of weight. But there are other ways of determining the weight. For example, we can use a spring balance. That is, we can find out how much the object stretches a spring. A pointer attached to the spring moves along a scale. This scale may be marked in pounds, but we are apparently measuring the weight by a stretch, that is, a distance. This is an <u>indirect</u> measurement of weight. How can we say that it measures weight at all? Because of the discovery that a spring behaves in a certain way. Robert Hooke (1635-1703) found that the stretch is proportional to the weight (if we do not overload the spring). For example, if we double the weight we double the stretch, if we triple the weight we triple the stretch, and so on. This property of springs has been thoroughly tested. We can be sure that if, for example, the spring balance reads 3.5 lbs., the stretch is 3.5 times the stretch for 1 lb. Therefore <u>indirectly</u> we are actually measuring the desired weight in terms of a unit <u>weight</u>.



Many measurements commonly made are of this indirect sort. We follow Galileo in measuring temperature changes by changes in the length of a column of liquid. Galileo used oil; we use mercury.

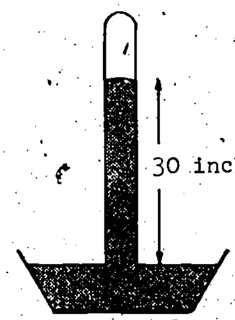
There are some quantities which <u>cannot</u> be measured directly. Density (weight per unit volume) is a good example. The usual unit is a gram per cubic centimeter (gm/cm^3). The very fact that we use a unit like gm/cm^3 shows that we think of the measurement of density as the result of measurements of weight in grams and volume in cubic centimeters. In fact, we divide the weight measure by the volume measure to obtain the density measure.

Water has the density $1\ gm/cm^3$ because $1\ cm^3$ weighs 1 gm., $2\ cm^3$ weighs 2 gm., $50\ cm^3$ weighs 50 gm. Indeed $x\ cm^3$ weighs x gm. Density is a property of the substance (water), not of the amount of it.

The density of mercury is about $13.6\ gm/cm^3$, but we cannot put together $13.6\ cm^3$ of water to make $1\ cm^3$ of mercury weighing 13.6 gm! We cannot combine density units directly the way we combine length units. In the case of fundamental or direct measurements, units can be put together by counting and a unit can be subdivided into as many sub-units as one pleases. This is not possible with indirect measurements. Nevertheless, in the example of density, a scale of densities can be constructed which is like a scale of length. We can say that a substance has a density of x units if 1 cubic centimeter of it weighs x gms. (or if c cubic centimeters weighs cx grams), and x can be any positive real number, integer or not.

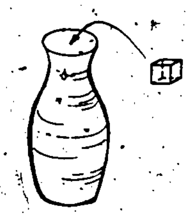
Speed is another quantity that cannot be measured directly. It must be calculated. We cannot combine a unit speed of 1 ft/sec with a copy of it to obtain a speed of 2 ft/sec. Nor can we split a speed of 1 ft/sec into two speeds of $\frac{1}{2}$ ft/sec each. As the name of the unit (ft/sec) suggests, we find the measure of speed as the result of a measurement of distance in feet and time in seconds. In fact, we divide the distance measure by the time measure to obtain the measure of speed.* If an object moves xt feet in t seconds, its speed is x ft/sec. We can therefore show speed measures on a number line. That is, we can represent speed on a scale.

It might be asked: When we read a speedometer on a car don't we measure the speed directly? The answer is "no." This is like the case of weight and the spring. It is based upon the way in which the reading of the speedometer is related to the speed. Actually the needle responds to an electric current whose amount is proportional to the speed of turning of the front wheels. The measurement depends upon the behavior of an electric generator.



Another example of an indirect measurement is the common practice of measuring the atmospheric pressure by the height of a column of mercury that it will support. The figure shows this form of barometer invented by Galileo's pupil, Torricelli. The space above the mercury is called the Torricellian vacuum. Pressure is defined to be force per unit area. It is therefore, not a length. Nevertheless, the length is a measure of pressure since the two are proportional. It may be shown that 30 inches of mercury corresponds to 15 lb/in^2, approximately 60 inches of mercury to 30 lb/in^2, and so on.

Going back to area and volume we see that when we calculate the areas and volumes of various figures we are really finding them in terms of length measurements and a calculation. We could measure the volume of an irregularly shaped jar by filling a unit cube with water and seeing how



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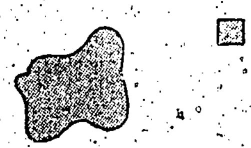
*For simplicity, we assume that the speed remains constant.

many such units of volume are needed to fill the jar. We could measure the area of an irregular piece of tin of uniform thickness by comparing its weight with the weight of a unit square cut from the same material. Why don't we <u>always</u> do something like this? What is the point of calculating areas and volumes?



One answer is this: This is a good way to measure the volume of a given object. If, for example, someone asked you for the volume of a certain electric light bulb, the simplest thing to do would be to dip the bulb in a tank full of water and measure the volume of the water than runs out. However, if we wish to <u>design</u> an object of a certain shape to fit some space, we would not like to construct a lot of samples in the hope of getting a fit. Think of designing an artificial satellite for example. It is expensive enough without resorting to trial and error.

Are there limitations to measurement? Are there things which cannot be measured even approximately, either directly or indirectly, because there is no possible unit? Most of us would agree that there are. We can say that John is the most valuable player on the baseball team and Jean the most beautiful girl in the school. It might be that no everyone would agree. But even if they did agree would we say that we could <u>measure</u> the value of the player or the beauty of the girl? What unit could be chosen? Sometimes we can <u>compare</u> without being able to measure. In some cases we cannot even compare. Who was greater, Newton or Beethoven? There are different kinds of greatness.

<u>Check Your Reading</u>

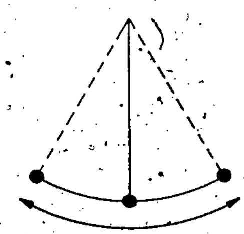
1. What three things do all direct measurements have in common?
2. Give two examples of direct measurements; two of indirect measurements.
3. Why don't we always measure a volume directly? (Why calculate it?)
4. If two quantities can be measured they can be compared. If they can be compared, can they always be measured? Illustrate.

Exercises 28-9

1. In antiquity, time was often measured by a water clock. A certain amount of water ran out of the bottom of a tank during the time interval to be measured. In using this method it is assumed that the rate of flow of the water is uniform. What precaution must be taken to make this a reasonable assumption? Does a water clock measure time directly or indirectly? How would you choose an appropriate unit of time?

2. Is the time from one sunset to the next sunset a good unit for measuring time? Are the units equal? How can you test whether they are or are not?

3. Suppose that you measure time by using as a unit the duration of a complete swing (over and back) of a pendulum of a certain length. (The unit depends on the length of the pendulum.) If it is desired to measure time in terms of a sub-unit that is $\frac{1}{4}$ of the original one, how could this be done experimentally?



4. Suppose that you measure time by counting your pulse. What objection can you make to this method?

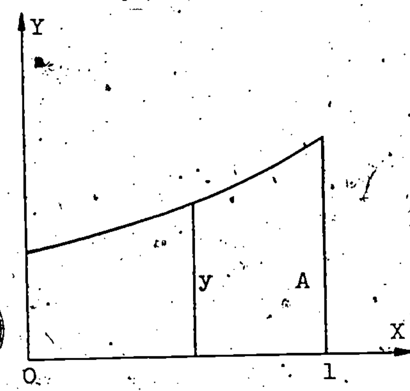
5. Suppose that you travel in a car a distance of 50 miles in 1 hour. If you say that your speed is 50 miles per hour ($50\ \frac{mi}{hr}$), does this mean that you were actually traveling at this speed for the entire hour? If this is not true, how can you make a new measurement to show that it is not true? The $50\ \frac{mi}{hr}$ gives of course the <u>average speed</u> over the hour. Describe a set of measurements that would give better and better approximations to the speed at a given instant, say at the time exactly $\frac{1}{2}$ hour after you start.

6. One method to measure weight is to move a slider along a bar until a balance is found with the object to be weighed. Is this a direct or an indirect measurement? Explain the principle on which it is based.

7. Name some things that you believe cannot be measured and justify your belief.

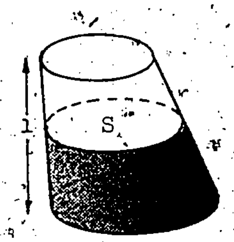
8. Can intelligence be measured? Are there perhaps different kinds of intelligence? Can creativity be measured?

28-10. Measurements and Averages



For the area below a graph whose ordinate y increases (or decreases) on the unit interval $[0,1]$, we have seen that $A = \overline{y}$, the average value of y over the interval.

Similarly for the volume of a solid whose cross-sectional area increases (or decreases) over the unit interval $[0,1]$, $V = \overline{S}$, the average of S over the interval.



Let the unit of length be changed so that $[0,1]$ is replaced by $[0,a]$. Then $A = \overline{y}$ must be replaced by $A = a\overline{y}$ and $V = \overline{S}$ by $V = a\overline{S}$, where the averages of y and S are now taken over the interval $[0,a]$.

This is easy to see. If we start with $A = \overline{y}$ and the interval $[0,1]$, then when we change the length unit as described above, A is multiplied by a^2 and $\overline{y}$ by a. Therefore to maintain equality we must have an extra multiplier a on the right.

Similarly if $V = \overline{S}$ on $[0,1]$ and if we change the unit of length so that 1 is replaced by a new units, the measure of volume is multiplied by a^3 while the measure of the average cross-sectional area is multiplied by a^2. Hence $V = a\overline{S}$ in terms of the new length unit.

To use $A = a\overline{y}$ and $V = a\overline{S}$ we need the averages of y and S over the interval $[0,a]$. The most important averages are

(1) $$\overline{x} \text{ on } [0,a] = \frac{a}{2},$$

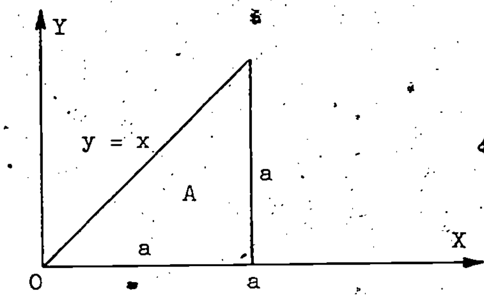
(2) $$\overline{x^2} \text{ on } [0,a] = \frac{a^2}{3}.$$

These results are easy to see geometrically.

In the figure $A = \frac{a^2}{2}$. But $A = a\overline{y} = a\overline{x}$. Hence

$$\frac{a^2}{2} = a\overline{x}$$

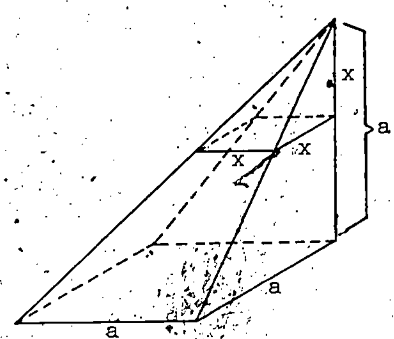
and therefore $\overline{x} = \frac{a}{2}$.



Similarly for the square pyramid shown,

$$V = \frac{1}{3} a^3 = \overline{x^2}\, a.$$

Hence $\overline{x^2} = \frac{1}{3} a^2$.



These results for $\overline{x}$ and $\overline{x^2}$ apply to other measurements than those of area and volume. We give two important examples.

Example 1. Everyone knows that if a body is dropped from rest, the speed (or velocity) increases with time. Galileo guessed that the velocity is proportional to the time x, that is, that

$$v = cx.$$

where c is constant.

Galileo asked himself this question: If it is true that

$$v = cx,$$

how far will the body fall in t seconds? In the case of a motion with uniform (constant) speed, the distance covered is the speed multiplied by the time. What should we do if the speed is not uniform? In such a case it is natural to assume that the distance covered is the average speed multiplied by the time, where the average is to be taken over the interval from $x = 0$ to $x = t$.

The average of cx over $[0,t]$ is $\overline{cx} = c\overline{x} = c \cdot \frac{t}{2}$. Then

$$d = \overline{cx}\, t = \frac{ct}{2} \cdot t$$

or

$$d = \frac{ct^2}{2}.$$

If we measure distance in feet and time in seconds, we find that in fact

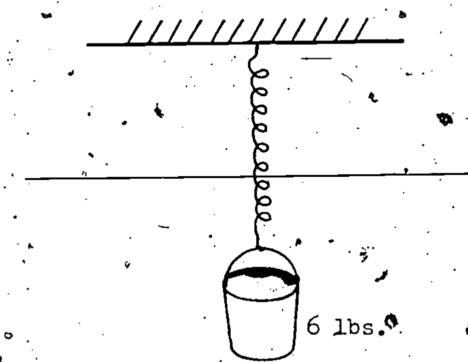
$$d \approx 16t^2.$$

(Actually 16 can be replaced by the better approximation 16.1.) This means that Galileo's guess works with $c = 32$, and the speed x seconds after release is

$$32x \text{ ft/sec.}$$

The next example concerns the idea of work. If a weight of 10 lbs is lifted a distance of 3 ft we say that an amount of work is done equal to 10×3 foot-pounds (ft-lbs).

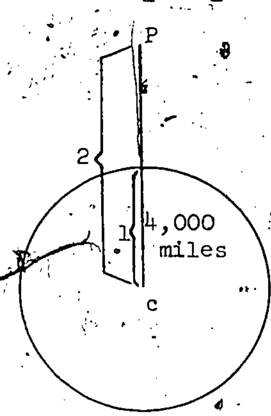
Example 2. A certain spring is stretched 1 inch ($= \frac{1}{12}$ foot) by a 1 pound weight. Then, as you know, W pounds will stretch the spring $W \cdot \frac{1}{12}$ feet.



Six pounds of sand are poured gradually into a bucket attached to the spring. As the weight increases the spring steadily lengthens until it is finally stretched $\frac{1}{2}$ foot. How much work is done? When the spring is stretched x feet, the corresponding number of pounds weight is $W = 12x$.

The work here is defined to be the average weight times the distance, $(\overline{W}\,\frac{1}{2})$, where the weight must be averaged over the distance interval $[0, \frac{1}{2}]$. Since $W = 12x$, $\overline{W} = 12\overline{x} = 12(\frac{1}{4}) = 3$ and the work is $(3)(\frac{1}{2}) = \frac{3}{2}$ measured in foot-pounds.

Example 3. Let us assume that the earth is a sphere with radius 4000 miles. A rocket is to be shot straight up to a point 4000 miles above the earth's surface. We shall take the radius of the earth to be a new unit of length. Then the rocket is to reach a point P that is two of these units from the center of the earth C.



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If the rocket weighs 10 tons, how much work (or energy) is required to raise the rocket to P? The work is the average weight times the distance. According to Newton's inverse square law, the weight changes with the distance x from the earth's center so that its measure in tons is

$$W = \frac{10}{x^2}.$$

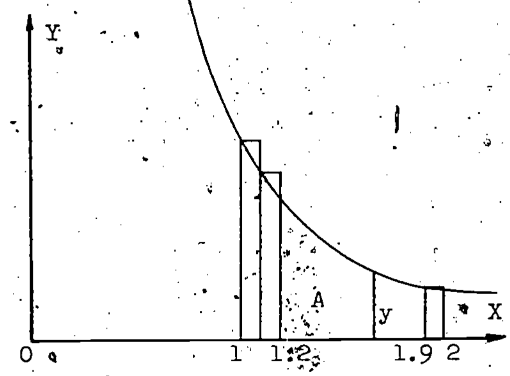
Notice that at the earth's surface where $x = 1$, W is 10 as it should be.

The required work is therefore

$$\overline{W} \cdot 1 = \overline{\left(\frac{10}{x^2}\right)} \cdot 1 = 10 \cdot \overline{\left(\frac{1}{x^2}\right)},$$

where the average of $\frac{1}{x^2}$ is to be taken over the interval [1,2].

Finding the average of $\frac{1}{x^2}$ over [1,2] is the same as finding $A \doteq \overline{y}$ for the graph of $y = \frac{1}{x^2}$ on the interval [1,2].



If we divide [1,2] into 10 congruent parts we get for the upper average

$$\frac{1 + \frac{1}{(1.1)^2} + \frac{1}{(1.2)^2} + \dots + \frac{1}{(1.9)^2}}{10}.$$

The numerator is obtained by adding

$$\begin{array}{r} 1.000 \\ .826 \\ .694 \\ .592 \\ .510 \\ .444 \\ .391 \\ .346 \\ .309 \\ .277 \\ \hline 5.389 \end{array}$$

Therefore the upper average is approximately .5389. The lower average is about .4639. The average of these two averages corresponds to using trapezoids instead of rectangles. The result .5014 is slightly too large to represent $A = \overline{y}$. As you might guess, the exact value is $.500 = \frac{1}{2}$.

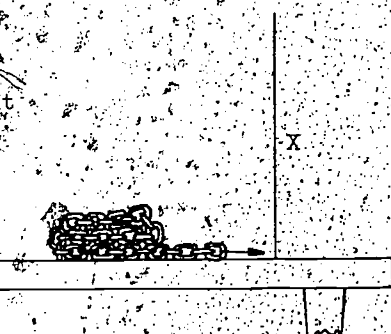
Returning to the rocket, the energy required measures

$$10(\tfrac{1}{2}) = 5$$

when weight is measured in tons and distance is measured in earth radii. To obtain the answer in ft-lbs we multiply by $2000 \times 4000 \times 5280$. The factor 2000 converts tons to pounds. The 4000 converts earth radii to miles and of course 5280 changes miles to feet. The answer is 21.12 billion ft-lbs.

Exercises 28-10

1. A chain 10 ft. long weighs 2 lbs/ft. It is resting on a table. You seize one end and lift the chain free from the table. Find the work required. Hint: When x feet of chain are above the table, what weight are you supporting?



2. In Example 3 of the text, check several of the terms in the numerator of the upper average. Verify that the lower average is .4639.

3. It can be shown that the average of $\frac{1}{x^2}$ over the interval $[1,a]$ is $\frac{1}{a}$. (Note that this agrees with our result in Example 3 of the text.)

   (a) If we assume that this result is correct, find the energy required to lift a rocket from $x = 1$ (the earth's surface) to $x = 1.5$ if the rocket weighs T tons when $x = 1$. (Use earth radii and tons as units.)

   (b) Show that the energy required to raise the rocket from $x = 1$ to $x = a$ is $(1 - \frac{1}{a})T$.

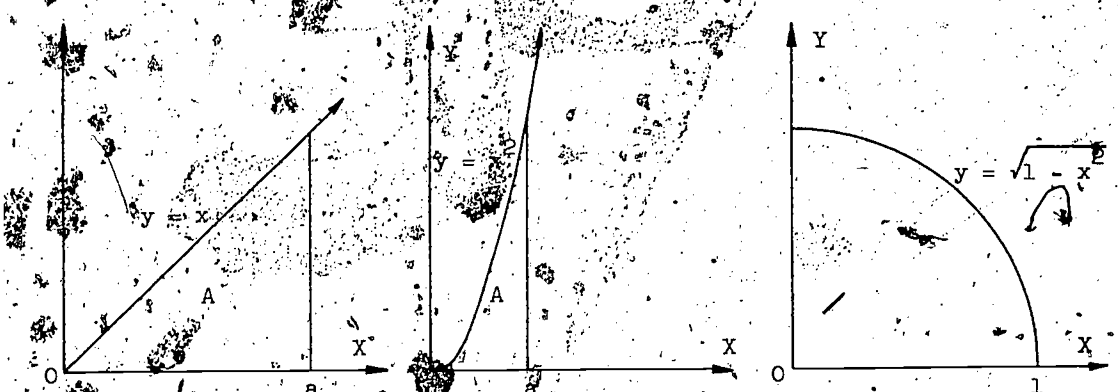
28.11. Summary

A measurement is a function from objects with a given property (length, area, weight, and so on) to non-negative real numbers. The real numbers are said to measure the property.

For some quantities it is possible to choose units which can be (1) copied, (2) put together in a way that corresponds to counting, (3) divided into interchangeable sub-units (as many as one pleases). Such quantities can be measured directly. Other quantities can be measured indirectly, that is, by a computation. The measure of a quantity depends upon the unit chosen.

Let U and u be two units of length. Then if $U = ku$, when we change from the units U, U^2 and U^3 for length, area, and volume to u, u^2, and u^3, all measures of length are multiplied by k, all measures of area by k^2, and all measures of volumes by k^3.

The area below the graph of a function that increases or decreases on an interval $[0,a]$ is measured by $A = a \cdot \bar{y}$, where $\bar{y}$ is the average value of the function over the interval $[0,a]$. If $y = x$, then $\bar{y} = \frac{a}{2}$ and $A = \frac{a^2}{2}$. If $y = x^2$, then $\bar{y} = \frac{a^2}{3}$ and $A = \frac{a^3}{3}$. If on the interval $[0,1]$, $y = \sqrt{1 - x^2}$, then $\bar{y} = \frac{\pi}{4}$ where $\pi \approx 3.1416$.



The volume of a solid bounded by two parallel planes at the distance a apart is measured by $V = a\bar{S}$, where S measures the cross-sectional area at the distance x from one of the planes and the average $(\bar{S})$ of S is taken over the interval $[0,a]$. For a sphere whose radius measures a units, $V = \frac{4\pi a^3}{3}$. For cones or pyramids, $V = \frac{1}{3}(Ba)$ where a measures the altitude and B measures the area of the base.