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ABSTRACT

A preliminary framework for research on effects of calculators on mathematics learning is presented. A research methodology is urged that places less emphasis on tight controls, exact replicability, and numbers amenable to statistical analysis. New curriculum needs to be developed that looks at the calculator as a mechanism to be explored in its own right and to be used to raise and reflect upon essentially philosophical questions. The extent to which the calculator coincides with youngsters' conceptions of the nature of mathematics and the extent to which that instrument is at odds with what they believe about mathematics needs to be explored. Other approaches to research on calculators include: (1) an analysis of the kinds of problems that relate to the internal anatomy of the calculator; (2) the effect of the calculator on society; and (3) investigating hardware usage. (MP)

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THE NEW CALCULATION IN EDUCATION: A RESEARCH AGENDA

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Disclaimer: I believe that I would not have been invited to develop this paper except for the fact that the conference organizers sought to raise issues for discussion. I believe that I have raised, but in no way exhausted, appropriate themes for concern, controversy and perhaps in the end consensus. Good luck to us. On one ground rule for this task I insisted: I would have no knowledge of who are to be the conference participants before delivering my paper to NIE. I did learn the name of only one participant because I discussed with Ed Esty possible use of this person as another consultant. Since he is attending the conference, I chose not to co-opt him in that way. At any rate you will be surprised to learn that I use him positively in the text. GRR

ROAD MAP

THE NEW CALCULATION IN EDUCATION: A RESEARCH AGENDA

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On Educational Research

This paper has been prepared under a contract with the National Institute of Education. The charge given by Ed Esty was straightforward: prepare "a preliminary framework for research [on] effects of calculators on mathematics learning." This seems easy. We have on the one hand activities---teaching and learning---with a history as old as civilization and an associated voluminous collection of more recent research. And on the other we have a new technology -- electronic computers, calculators and microprocessors -- a technology scarcely as old as many contemporary classroom teachers with some of its developments even younger than today's nursery school students. All we need to do, it seems, is superimpose the new technology on the old tradition, manipulate the dials and read off the easily identifiable pairings for research, the results of which will lead to educational improvement.

Lest you read that suggestion as sarcastic or facetious, let me assure you that is exactly what I believe should be done. That does not mean, of course, that I believe the task will be well carried out. Past history pre-

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*The following consultants made major contributions to this paper: Stephen I. Brown, SUNY Buffalo; Robert B. Davis, Myriam Steinback and Curtiss Mc Knight, University of Illinois; Betty Krist, West Seneca (New York) Senior High School; and Wallace Jewell, Edinboro State College. Final responsibility for the contents, however, rest with the author; in particular opinions and points of view do not necessarily represent or reflect N.I.E. policy.

dicts that it will not. Consider in this regard other recent technolo; education intersections -- or should we say traffic accidents: motion pictures, programmed instruction, television. In each cas: the opportunity was there to carry out a program like the one I have proposed; in each case the opportunity was lost and the technological advance largely wasted. In fact a case can be made that the research carried out, worse than merely failing to make these new technologies comfortable, actually contributed to their discomfort or demise. Because the lesson we should learn from this a· lies here, it is worth exploring why.

The major source of difficulty has been a narrow and uncompromising definition of research as an activity in which the most important characteristics are tight controls, exact replicability, and numbers amenable to statistical analysis. Assigned much less importance than methodology has been the asking of wise, penetrating and appropriate questions. It is almost as if we believe that the use of allegedly high-powered methodology ·r-come the superficiality and sometimes even the foolishness of the qu being asked. Researchers in the field have feared their own intelli and furthermore have not understood that judgment and taste are no relevant in their educational research questions than they are in rthwhile human endeavor.

It is a source of great irony to find out that investigators he-matical understanding should be so insensitive to the role played the asking of a question, for even a superficial study of mathematics if suggests how questions not only influence but may even misdirect courses of research. At any rate the inevitable result of this narrow and unintelligent approach to research has been that the domain of problems open to attack -- of questions that may be addressed -- is severely reduced in both significance and meaningfulness. Further, the answers forthcoming have been at best tangential to concerns of the educational enterprise. And of still greater concern is the fact that the nature of this research has led the researchers and through them even many school classroom teachers (our educational organization combines the role of researcher and teacher trainer) to view instruction and curriculum through these same distorting spectacles.

Consider in this regard just one example, an example which will be important to recall in the sequel when we address more specific problems of the new technology. In this example we modify the curriculum in some way -- choose your own from among the hundreds of recent but now frowned upon mathematics innovations -- and the time comes to justify that change. We now impose our strict empiricism* on the approach to justification: we test new against old in order to determine which is better. How is this comparison accomplished? We select test items -- for testing still means paper and pencil inventories and to researchers inventories easily machine scored -- that cover content common to the two approaches. "It would be unfair to the students if we did not. And who ever heard of testing students on content they never studied?" say the researchers. The result is, of course, predictable: seldom a significant difference in favor of the new curriculum. The old has almost everything going for it: the innate conservatism of teachers (new math taught by old math minds), the conservatism of the testing program and, most important, the conservatism of the statistical approach itself.

For contemporary researchers have misled us by discussions of Type I and Type II errors into missing the more important issue associated with the result "no significant difference". No significant difference arises under three con-

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*I will deal with definitions later, but I note here that I mean by empiricism experimentation, not its alternate definition in my dictionary: quackery.

ditions:

(1) There is no difference

(2) There is a difference but we demanded too much from our statistical treatment, the so called Type II error. Or

(3) The instrument is insensitive to the differences it seeks to identify.

Now the educational researchers focus us tightly on (1) and (2).* I submit here that over 99% of the time it is actually condition (3) that applies. We have not developed -- as often cannot develop -- instruments of the standard form that will identify the kinds of differences on which this approach focuses our attention. In fairness to those who have for so long addressed this problem, I note its extreme difficulty.** As Minnesota psychologist Jim ... an told me early on, teacher effects produce thousands of units of variance, curricular effects dozens: you're drowned in the surf. That does not, however, excuse the researchers for their having conveyed to the educational community the clear message:

NOTHING MAKES ANY DIFFERENCE

A different but equally significant message is clear to me: unless we alter our approach, we will kill the new computation with the same brand of kindness that has killed every significant curricular modification that has come down the pike in recent years.***

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* I sadly observe many of my own university colleagues closely associating no significant difference with no difference; I angrily observe them passing on this false view to classroom teachers.

** And I further note the overclaim of many curriculum modifiers. "We'll knock 'em dead," was too often heard, especially in the '60s.

*** Many readers will feel that "kill" is too strong here, "delay" the better word. Where, I ask them, are classroom TV, math films, CAI, programmed instruction, new math?

Having presented my case that we need change in our approach to research as we address the new technology, it now behooves me to indicate to what we should change. Let us seek our answer by going to the dictionary. Two I have in hand give the following definitions:

RESEARCH n. diligent and systematic inquiry or investigation into a subject in order to discover facts or principles. v.i. investigate carefully. (American College Dictionary, Random House, 1964)

RESEARCH n. careful, systematic, patient study and investigation in some field of knowledge, undertaken to establish facts or principles. v. i. study. (Webster's New World Dictionary of the American Language, College Edition, World, 1960)

There are, I suggest, many lessons to be learned from these two very similar definitions of research, but I stress here only one. They broaden our perspective on research remarkably. Research is not here restricted to experimentation; in fact it is appropriate to note that experimentation is not even mentioned.* I do not suggest by this that experimentation is not research, for indeed it is one well recognized form of systematic inquiry with ground rules firmly laid down in the physical sciences; rather I suggest that it is only one of a wide range of research techniques. Thus we should support experimentation when that research technique is appropriate but only when that approach is appropriate. And because the mathematics education research community and the educational establishment in general have so effectively

*This is not due merely to lack of space in a smaller dictionary. My Oxford English Dictionary (Oxford U. Press, 1971) in some 140 odd lines of definitions of research and related words does not once mention or imply experimentation.

barred other forms of research from acceptability, we must work aggressively to change this posture.

Some changes that are in order and that will be further identified by examples when we turn to our more specific analysis of the new computation are:

(1) The development of new curriculum is itself research. "Oh, him. He's just a textbook author," may well put the designated him in his place if indeed the work is boilerplate, but when the content developed is new or when the pedagogy is different or when the organization or approach is unique the matter is quite different. Then the "just" and quite probably the comment itself are inappropriate. I speak here, for example, of the work of Beberman, of Jacobs, of Braunfeld, of Usiskin, of Exner Kaufman and the Papys. A research product then is the text itself: UICSM Math, Mathematics: A Human Endeavor, Stretchers and Shrinkers, Usiskin's Transformation Geometry or the CSMP Elements of Mathematics and Elementary Mathematics. Another research product is a report on special characteristics of the program developed. Here Frederique Papy's Mathematics Play Therapy is a commendable example.

In each of these cases, we find that a deep understanding of the discipline enables the author to find interpretations of fundamental mathematical ideas that strip them of their formalism while still maintaining their essential qualities. It is not by "watering down" and oversimplifying the ideas that this translation is accomplished, but rather by seeing what is essential and what is peripheral to a mathematical or pedagogical construct.

(2) Analysis of textual material is research. With the exception of some of the groundbreaking texts of the kind mentioned above, how are most texts produced? Pick up any two at random that are intended for the same course and it will be clear that any differences are purely unintentional. Why? Because for the most part the author of a new text has other texts (rather than his own organizational scheme based upon an understanding either of the discipline or of how students learn) as a model; and because these other texts are as unintelligently eclectic as his own, we are sure to find that they lack coherence and integrity from many points of view. A useful piece of microresearch would be to "work through" part of such a textbook, reading the text carefully and writing out the exercise sets. A wide range of questions are immediately identified: What are the author's assumptions about students? What is required of the reader to learn from this text? Is there an intellectual line in the development or are the topics developed piecemeal? What would a serious and careful student take away from use of this text? What would a more typical student learn - a student who, for example, looks only at exercises, giving each one a ten to twenty second try? What is the author's definition of mathematics? To what philosophical school of mathematics does he belong? To what extent does the author attempt to involve students in development of ideas? How is course continuity handled by this text?

In this example I do not mean superficial analysis like the application of standardized readability tests; rather, I mean detailed and probing analysis and reporting with specific quotations from the text serving as examples. Such analyses would, I suggest, prove eye opening. The researcher would, I believe, gain some grudging respect for authors. And the analysis would convey both specific and general messages to commercial publishers. The specific messages would relate to the particular answers the researcher put forward, many of which would extrapolate beyond the specific text examined. The general message would be that mathematics education researchers are seriously concerned about <u>the</u> central tool for conveying mathematics in most classrooms.

Analyzing text is just one example of a kind of analysis that can be conducted without focussing primarily upon the students as subjects of inquiry. But analysis of this sort is needed throughout mathematics education.* We need to examine our goals. We need to examine how we translate these goals into curricular materials. We need to examine the idea of mathematics conveyed to students by various approaches to instruction. We need to examine teacher training programs for what they convey to prospective teachers about mathematics. And more specifically to the point here: we need to examine technology-education interactions. All call for exactly that "careful, systematic, patient study" that I am translating here as analysis.

(3) <u>Observation of and interaction with individual students or teachers can be developed into research</u>. For too long researchers have stood outside the classroom, scarcely even peeping in. In the Sixties, in fact, it was considered <u>de rigueur</u> for researchers to stay away from students. They had to keep their hands clean and to maintain strict experimental neutrality. It is at least as important surely

* In fact it is exactly the lack of analysis that has made so much of what has constituted research in mathematics education so sterile. The average research report provides a dozen pages of tables and charts preceded and followed by hardly a paragraph of analysis.

to find out some of what is going on in the thinking of our students - singly or in small groups - as it is to find what happens when we apply gross achievement measures to groups of large n.

The message of Holt, Piaget, Erlwanger, Davis, and Ginsburg is clear here: It is not the act of observation or interaction that is research; it is the creative response of the insightful researcher to those acts that carries them to this level.

(4) Feasibility studies are research. By a feasibility study I mean an examination of whether something can be done, the associated question of whether it should be done temporarily held in abeyance. Can a nursery school student operate a particular kind of calculator? Can an identified group of students learn from a given set of curricular materials? Can a given teacher training regimen be implemented?* My reason for including this kind of study is my concern that we have often skipped this step and gone directly to comparisons. By separating feasibility in the way I propose here, we allow ourselves to address the question should independently: Yes, these students did succeed. Is what they did worthwhile in the general context of the instructional program? Can we now make the case that what they have done is more important than what they might have done alternatively with the same amount of time?

These are some of the different directions toward which research should take us today. They will better allow us to get to those "facts and principles" that our definition set as research goals and that our current research has been

* I do not mean to imply here that the answers are simply yes or no.

unable to approach. Central to what I have attempted to convey here is an approach to research captured in a comment by Max Perutz about James Watson, one of the discoverers of double helix structure of the DNA molecule: * "He never made the mistake of confusing hard work with hard thinking; he always refused to substitute the one for the other."

Because I believe it is to the point here, I close this section with a brief review. N. L. Gage has recently produced a book, The Scientific Basis of the Art of Teaching,** which is enjoying an enthusiastic reception from conservative educational researchers. They take it as vindication of their efforts since the book purports to show that research has given answers useful to education. But Gage's arguments do not stand careful inspection. He admits that "Most reviewers of research on teaching have concluded...,that past work has been essentially fruitless."*** He then proposes to add admittedly severely flawed studies together by means any elementary statistics student would reject, or alternatively to take counts - research by consensus.**** And finally he displays some results in the form of what he calls teacher-should statements. Here are those statements in Gage's own words:

> ---Teachers should have a system of rules that allows pupils to attend to their personal and procedural needs without having to check with the teacher.
>
> ---Teachers should move around the room a lot, monitoring pupils' seatwork and communicating to their pupils an awareness of their behavior,

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* Quoted by Horace Freeland Johnson in "Annals of Science: DNA", New Yorker, Nov. 27, 2978, p. 47.

** Teachers College Press, 1978.

*** Quotations are taken from excerpts in Phi Delta Kappan, (Nov. 1978): 229-235.

**** We are subject to that last method in mathematics education. See, for example, Harold L. Schoen, "Implications of Research for Instruction in Self-paced Mathematics Classrooms," NCTM 1977 Yearbook: Organizing for Mathematics Instruction, pp. 198 - 223.

while also attending to their academic needs.

---When pupils work independently, teachers should insure that the assignments are interesting and worthwhile yet still easy enough to be completed by each third-grader working without teacher direction.

---Teachers should keep to a minimum such activities as giving directions and organizing the class for instruction. Teachers can do this by writing the daily schedule on the board, insuring that pupils know where to go, what to do, etc.

---In selecting pupils to respond to questions, teachers should call on a child by name <u>before</u> asking the question, as a means of insuring that all pupils are given an equal number of opportunities to answer questions.

---With less academically oriented pupils, teachers should always aim at getting the child to give some kind of response to a question. Rephrasing, giving clues, or asking a new question can be useful techniques for bringing forth some answer from a previously silent pupil or one who says "I don't know" or answers incorrectly.

---During reading-group instruction, teachers should give a maximal amount of brief feedback and provide fast-paced activities of the "drill" type.

One way to sum up many of the implications of the research, as emphasized in these "teacher-should" statements, is to say that teachers should organize and manage their third-grade classes so as to optimize what David Berliner calls "academic learning time" - time during which pupils are actively and productively engaged in their academic learning tasks. And one way to do this is to avoid time-wasting activities, for example, waiting in line to have papers corrected or receive further instructions.

Are those the results of the millions of dollars that have gone into research on teaching -- and in particular into Gage's own NIE supported center? I suggest that that list includes imperatives all of which are trivial, some of which are open to serious question and some of which are just plain wrong. Most sitnificantly, these imperatives ignore differences of learning styles of pupils within a class and minimize the use of a teacher's intelligence to respond to subtle and frequent changes within a class environment. Furthermore, they completely by-pass a considration of goals that may require a challenge rather than an acceptance of values that (for example) identify the teacher as the only organizer and reinforcer in a classroom. I believe that far from making his case for experimentation, Gage underscores mine: We need to change our approach if we wish to make any kind of impact on educational practice.

So now let us turn to directions that might be explored that relate to the new computation. We first address the broad concern of curriculum development.

The Development of New Curriculum

What are some of the curriculum directions that have been explored so far for the hand-held calculator? Though there have been some frivolous, frothy kinds of suggestions - such as using the device to spell out words when held upside down - for the most part the instrument has been used as a tool for the purpose of focussing attention elsewhere. There have been some fine beginnings in curriculum which encourage the teacher to place attention on problem solving writ large without restricting students because of otherwise cumbersome calculations. Puzzles such as:

$$\begin{array}{r} 12345679 \\ \times\ 9 \\ \hline 111111111 \end{array}$$

$$\begin{array}{r} 12345679 \\ \times\ 18 \\ \hline 222222222 \end{array}$$

can be explored - provided the print-out has a large enough number of digits (or the student improvises appropriately) - without the accompanying frustration of calculating each answer.*

There are other curriculum suggestions which indicate how the calculator could be used as a tool to approach existing curriculum in slightly different ways.** For example, it would be possible to approximate irrational roots of a polynomial in ways that would have been tedious without such an instrument.

As valuable as these curriculum directions are, they all tend to focus on the calculator as a tool for doing other things in mathematics, much as one might use an idiot savant if he had one at instant call. There are new curriculum directions that have barely begun to be explored however, which make a very different kind of use of the calculator. We turn now to some of these possibilities.

Curriculum development is no easy business, especially if you construe the task in non-plagiaristic terms. There are all kinds of things that one must take into consideration - from the nature of a discipline, to the sophistication of youngsters, and even further to the effect that one would hope to make on society in general. If we have learned anything from curriculum reform of the past few decades it is that it is quite possible to oversimplify the task of curriculum development so as to exclude many important variables in an effort to reflect the nature of the discipline

With this caveat in place, we now look at the calculator not primarily as a best way of doing otherwise burdensome work, but as a mechanism

(1) to be explored in its own right - in the spirit of a black box

(2) to be used to raise and reflect upon essentially philosophical questions.

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* Many students would give up the investigation if the calculation itself was too cumbersome. On the other hand, others might find the problem fascinating enough so that they would be willing to put up with drudgery. Embedded here are some interesting research questions!

** See, for example, Wallace Jewell, The Calculator in Secondary School Mathematics: A Status Report," unpublished doctoral thesis, SUNY Buffalo, 1979.

In what follows, we do little more than provide the vaguest outline or sketch of potentiality. This section provides a new focus on the instrument and is written with the object in mind of encouraging the curriculum researcher with imagination to look at new terrain for the purpose of generating curriculum ideas.

No effort here is made to suggest appropriate sequencing or to indicate the teaching strategies that might be most successful. We are rather trying to suggest an avenue of curriculum development that requires considerable inspiration as well as sweat. In particular we respond here to one of the commonest complaints about texts: their T^3 approach -- tell, try and test. If we truly wish to improve classroom instruction, our research should lead to improved products which break this lock step. We talk about student involvement, why not seek to encourage this student participation in our texts?

It stands as a truism (but one that is honored more in the breach than the observation) that the practice and improvement of thinking is central to any educational enterprise. Let us take thinking seriously, and attempt to gain some possible new directions for curriculum research with the hand-held calculator, by examining calculators as they interact with thinking in three different ways.

(A) The Mind of the Beast

An interesting first approximation in the design of curriculum along the lines we suggested in (1) of the first section might be to take the concept of thinking and apply it to the calculator itself. What is the mind of this beast like? Though there are occasional first steps in thinking along these lines* there is no systematic curriculum that addresses the question.

Such a curriculum might begin not by teaching the youngster how to operate with (for example) order of operations so that he can use the calculator for other purposes, but might rather have him figure out what the order of operations ought to be. What does he hypothesize when he punches on an algebraic order calculator:

$$2 \times 2 + 3 \times 3$$

$$2 \div 3 + 4 \div 2$$

and gets answers that are unexpected? If the teacher has a programmable calculator, he might even preprogram bizarre re-organizations concerning parentheses and order of operations in order to encourage intelligent investigation of a calculator that has a "serious disease."

Similarly students could be directed to the unusual roles played by various special keys. I have four different calculators at hand that process

$$\boxed{3}\ \boxed{+}\ \boxed{4}\ \boxed{=}\ \boxed{=}\ \boxed{=}$$

to give these results:

(1) Rockwell 24 RD - II: 15

(2) Sharp EL-203: 7

(3) Monroe 326: 13

(4) HP-25: oops, no $\boxed{=}$ key at all!

The standard response to this disorder would be to restrict instruction to a single model - for convenience sake. While there are strong arguments for that decision there are equally strong arguments against. Students would

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* See, for example, Donovan R. Lichtenberg, "Minicalculators and Repeating Decimals," Mathematics Teacher 71, 6 (September 1978) and Donald Stover's manuscript in process on this same subject.

gain understanding of the ideosyncracies of processing and the strengths and weaknesses of each procedure.

The students might then be led to consider what would be an optimum organization for order of operation or a preferred use of the = key. The students might be encouraged to find interesting rather than oversimplified answers to the question of what is optimum? They would have to consider the different purposes for which these calculators are used, and might correctly conclude that for some purposes one scheme is optimum but for other purposes there are better schemes. Activities of this kind provide excellent transition from looking only at the mind of the calculator to finding out purposes for which this mind is to be used.

In some cases it may not be necessary to program "sick" calculators, for like pollution they abound. Instead of sending defective calculators to the junk heap or to the dealer for repair, consider using them for educational purposes. Take a calculator, for example, that has a defective display. If you push the "8" button, the "print out" may be 8 or 9 depending upon the position on the display. Well, what exactly does it do incorrectly, and more importantly what can we do to compensate for or override the problem? Hand-held calculators are so inexpensive (a four banger purchased for $10 or less) that we have become quick to adopt the American way of life with them and to toss out defectives. Instead of thinking of defective calculators as disposable, suppose we imagine that they are very expensive and that as in the case of our ancestors, when we are stuck with lemons, we make lemonade.

So far, we have suggested that some curriculum be devised which encourages students to understand the mind of a healthy calculator, and which encourages them to understand the mind of artificially and naturally sick ones as well.

Each of these categories is worth further exploration with regard to subtle points of a calculator's well being.

For example, even healthy calculators have built- mitations. If a calculator displays only eight digits and we wish t out if a fraction is a repeating decimal of some period, then ther be some value in finding out how to "get around" the limitation of an eight digit display. We can see some significant material developed around the limitations of healthy calculators.

It should be noted that this kind of exercise also provides an excellent response to those who continue to view the calculator from the rot-the-mind perspective. Going beyond calculator or computer limitations demands serious thinking and deep insight into the structure of the concepts being probed. Consider in this regard, for example, the following exercises:*

(1) Give an exact sum for $\frac{23}{48} + \frac{11}{28}$

(2) Square exactly: 23451267

(3) Express 7^{120} in scientific notation with six digit accuracy

(4) Find the decimal repeatend for 1/17.

In addition to "getting around" limitations, there is considerable curriculum that could be devised which extends some of the thinking that was generated by exploring order of operations. That is, a psychoanalysis of sorts which informs us how the mind of the calculator behaves could take us into some mind-expanding exploration. For example, we know not just one but many algorithms for multiplying any two numbers. What is the algorithm that our calculator uses? Though we may not be able to find an answer for sure, we might be able to

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* If you have a calculator handy you may wish to compare your answers with mine: (1) 293/336; (2) 549,961,923,905,289; (3) 2.58086 E101; (4) 0.0588235294117647

eliminate possibilities, and in so doing will involve ourselves with significant mathematical and scientific thinking. For example, does the calculator use repeated addition? Does it use the standard multiplication algorithm? Does it multiply by the Russian peasant method? Does it multiply by using logarithms?

How could one begin to find out? One important variable that we have not mentioned involves the time needed for the calculator to perform an operation. How might curriculum be devised which encourages the youngster to focus on time elapsed to perform an operation? If calculation takes place too fast to discriminate, then what kinds of things can be done to slow down the general operation of the calculator?

There are, of course, many other ways in which we might "psychoanalyze" a healthy calculator. Some of this kind of exploration might provide a needed challenge to the accepted maxim that math is sequential and that we need to explore from the simple to the complex. For example, I have found an exciting way to introduce logarithms by use of the hand-held calculator. Instead of defining a logarithm and its properties as I had done in the past, I asked a group of eleventh grade students to gather data regarding the functioning of the $\boxed{\text{log}}$ button and to hypothesize how that function behaves based upon the data. Some very interesting hypothesizing took place and, of course, it was helpful for students to select special cases wisely in order to attempt to refute or verify their hunches. By the end of a 20-minute period of work independent of me, the students had generated many of the important properties of logarithms that I had intended to teach. In particular it was only a few minutes before one student suggested that logs were powers of ten. The students soon refined this and recorded

$$10^{\log N} = N.$$

How might this activity help to challenge the myth of sequentiality? For one thing, grade school students could also explore this button; instead of expecting a full blown analysis of the concept of logs at that level, however, we might instead encourage a partial understanding of what is involved. Are there any principles they can uncover regarding the log function? It is conceivable that in this particular case, because of their limited expertise and pre-conceptions they might see fundamental constructs that the older students overlook. For example, they might become intrigued by the monotone increasing quality of the function -- something the high school students never articulated. They might then be in a position to explore monotonicity with regard to many other functions. Activities of this sort might have a significant impact on re-structuring the curriculum so that the concept of "spiral curriculum" might gain in intellectual respectability.

(B) Second Reflections on Psychoanalysis

In psychoanalyzing the calculator we have essentially been asking "What does it do?" It is worth pointing out that in referring to either sick or healthy calculators that there is some ambiguity in the question just asked. In the case of investigating the log button we are essentially asking: What function (or relation) is generated by pairing the button pressed and number displayed? We are really trying to find out what the new "animal" looks like. We may try to graph the results or we may try to describe its properties, but we are, after all, trying to unravel the mystery of the pairings. In the case of investigating the manner in which (for example) multiplication is conducted, we assume that one already knows what the pairings will be - that for example, 2 and 7 paired under multiplication yield 14. Even if the specific answer is not known the meaning of the function is known. We are instead trying to hypothesize what its "strategy" must be for getting answers. We could, of course, go further and try to figure out what the circuitry must be that allows for the use of such strategies, but curriculum designed to address such questions would require considerably more sophistication on the part of the student than is implied by the two interpretations of the question "What does it do?" that we applied earlier. In addition it is interesting to note that this extreme mechanistic interpretation is not what is normally conveyed by a psychoanalytic metaphor. Much curriculum might be generated around the curriculum researcher's use of the two interpretations of "mind" that we have suggested, but we could take the metaphor of "mind" and "thinking" and go even further in developing curriculum. It is that possibility that we discuss in the next section.

(C) What it Means to Think

It is one thing to think and another thing to think about the nature of thought itself. It is controversial whether or not thinking is something that distinguishes man from beasts, but virtually everyone would grant that the latter are incapable of thinking about thought itself - a supremely human function.

An analysis of what is involved in thinking is essentially a philosophical task. What is the value of such issues for students and how might the calculator or computer add to such inquiry?

Let us digress for a moment and discuss a program that has been implemented in the elementary grades over the past few years. Matthew Lipman, the founder of the Institute for the Advancement of Philosophy for Children, has created a number of open ended non-didactic novels for children which encourage youngsters to reflect upon such topics as thinking, logic, and ethics. His fundamental concern is that the experience of school is disjointed and does not encourage students to reflect upon how the different experiences relate to each other. His novels involve the interaction of youngsters and adults in such a way that students are encouraged to figure out such things as:

-- What is thought about?

-- How do logic and creativity relate to each other?

-- What of value follows when I try to be illogical?

After learning that "If A then B" does not generally have the same truth value of "If B then A" for example, Lisa, one of the youngsters in Lipman's first novel, muses as follows:*

> The zebra had claws. The giraffes had long, furry tails. The elephants had high whiskers. A buffalo was trying to flatten himself on the ground, preparing to spring upon a green-eyed field mouse. The chimpanzees all had pointed ears and slanted eyes; and a grizzly bear kept licking his paw and then washing his face with it.

Realizing that if "all cats are animals," it does not follow that "all animals are cats," Lisa remarks,

> So all animals aren't cats...but in make believe they can be. I can imagine what I please, and when I do, Harry's rules won't apply.

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* Matthew Lipman, Harry Stottlemeyer's Discovery, Institute for the Advancement of Philosophy for Children, 1974, p. 12.

There are times in the novels at which such creative use of logic stands as a testimony to one's imagination and others at which such stretching leads back to strictly logical questions. (One such question here might be: "If A then B" does not usually have the same truth value as "If B then A." Aside from make-believe however, can we figure out when it does?)

Without going into further detail about the specifics of the program, it is worth citing some consequences of exposing youngsters to what may appear on the surface to be tangential to our tasks given our normal educational expectations and especially given our heavy emphasis on subject matter in the schools.

It turns out that even a modest exposure to philosophical thinking of this type has enormous consequences in terms of youngsters' performance on standardized test of logic, creativity, reading ability, even mathematics - and this with a population of so called culturally deprived youngsters. Wouldn't it be worth considering a modification of the mathematics experience which would accomplish some of these outcomes?

The calculator may very well provide the fodder for such an experience. In addition to using the calculator for the (standard) purpose of easing the burden of calculation and in addition to trying to understand its mind (the more radical use described here), we might try to devise some curriculum which would encourage the student to reflect upon the senses in which a calculator does think. What does it mean to think, and to what extent does the calculator do this? As Lipman points out in his teacher's manual for the novel Harry Stottlemeier's Discovery, thinking is an ambiguous word, and the youngsters in the novel frequently move from thinking meaning the content of thought (e.g., homework, cats, etc.) to thinking meaning the process of thought

(e.g., daydreaming, using modus ponens, etc.)*

After exposing youngsters to a curriculum which attempts to explore the mind of the calculator as described in (A) we might begin to confront them explicitly with the different interpretations of "What does it do" as described in (B). What we have pointed out in (B) is essentially two different interpretations of "mind" that are very much like the "process" vs "content" distinctions of thinking that we have just described. Compare "What is the log function?" (content) with "How does this beast do multiplication?" (process).

In addition to making these distinctions, the calculator would seem to be a natural device for encouraging further reflection on the nature of thought as suggested in (C):

- What does it do that is like what you do when you think?
- What does it do that is different from what you do when you think?
- What conceivable modifications could be made that would increase the potential of the calculator to think.

The existence of such functions as memory, programmability and random number generation for Monte Carlo applications may provide considerable challenge for youngsters who feel that the calculator or computer doesn't think and they do.

Anyone interested in writing curriculum along such lines would profit from looking not only at the work of Lipman but at books dealing with thinking, reasoning and the like.** An exploration of some of the literature dealing with Turing machines might be reasonable grist for this mill as well.

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*See pp. 1 - 4 of Matthew Lipman, et al., Instructional Manual to Accompany Harry Stottlemeier's Discovery, Institute for the Advancement of Philosophy for Children, 1975.

**See, for example Joseph Weizenbaum, Computer Power and Human Reason, W. H. Freeman, 1976; and Michael Scriven, Reasoning, McGraw-Hill, 1976.

A Concrete Tool and an Abstract Subject

There is a great need for the sensitive hypothesis generating kind of research that is typical of the work of Erlwanger, Davis and Ginsburg - work that is stimulated by attending to a very small number of subjects while keeping an open mind to the directions they suggest rather than using them to verify pre-conceived hunches.*

One important underlying question to be explored by these means is: To what extent does the calculator coincide with youngsters' conceptions of the nature of mathematics and to what extent is that instrument at odds with what they believe about mathematics?

The question is interesting because the concrete nature of the calculator does appear to encourage and reinforce student views of mathematics that may either be consistent with or at odds with what their teachers hope to convey. Let us provide an analogy to suggest why the question is appropriate and why the hypothesis generating type of research which we alluded to above might be an appropriate way of getting at the research question.

Stephen Brown has been teaching our undergraduate methods course at the State University of New York at Buffalo for several years.** His students have all performed well in their previous mathematics courses, and their professors have all attempted to convey the nature of mathematics as an abstract discipline derived from axioms, definitions, theorems, and so forth. An appreciation for the roles of logic (and in particular contradiction) as a way

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* The Journal of Children's Mathematical Behavior consistently provides fine examples of such studies.

** For a more extended discussion of this work see G. Rising, "Minimal Content for a Mathematics Methods Course" in Designing Methods Courses for Secondary School Mathematics Teaching, ERIC, 1977, pp. 31-49.

of understanding mathematics is assumed. In this context Steve has been collecting a number of exercises like the following:

> In grade 9, we are told that x^2 is always greater than or equal to zero. We are given a number of axioms that describe the system in which such a conclusion is valid. Later in grade 11, we appear to tell students that if we now add the following axiom:
>
> There is an element x such that $x^2 = -1$,
>
> then we arrive at a system in which all the old axioms hold but in which we can now have complex numbers as well. But the difference between grade 9 and grade 11 appears to be that in grade 11 we append an axiom that contradicts the original system - something we said we could not do in grade 9. How can that be?

These good quality mathematics students have answered that question in some very peculiar ways. For example, Steve has received responses like:

You always find out as you get older that it is possible to do what you previously thought impossible.

Maybe there's a physical model to justify adding the new axiom.

By adding the new axiom, you're defining a new system. So there's no contradiction.

Now these are peculiar responses. None of them appear to respond to the apparent paradox in so extending systems. We are essentially finding out important things about these prospective teachers' world view of mathematics that is very much at odds with what they have been explicitly taught in their previous courses. In addition, such assertions are never made if you just ask these students straight-out what they think the nature of mathematics is or what they believe the role of logic or contradiction to be in mathematics.

The calculator appears to provide us with a concrete tool much like the specific examples Steve has collected to enable us to do research that might illuminate our students' conception of the nature of mathematics.

Why should this be so? The calculator is a concrete tool that has input-output functions. We do something to it and we get a response. It is as if we say:

What is $f(x_i)$?

The machine answers: $f(x_i)$ is y_i. We in a sense have propositional answers to every question we can ask - even if the answer is "Error".

There are several possible directions for the purposes of using this machine to explain conception of knowledge in a way analogous to what Brown has done with undergraduates and in an Erlwanger-Davis-Ginsburg spirit.

First of all, the calculator sometimes takes what we know to be two equivalent expressions and comes up with different answers. Consider for example the following:

$\frac{1}{2} \times 2 = 1$

$\frac{1}{4} \times 4 = 1$

$1/5 \times 5 = 1$

$1/8 \times 8 = 1$, but

$1/3 \times 3 = 0.9999999$

$1/6 \times 6 = 0.9999999$

$1/7 \times 7 = 0.9999999$

Now it is worth doing research in which we listen very carefully to what students at all levels say when they arrive at what appears to be such contradictions. Examples of this kind can be found at many levels of sophistication so that the apparent contradiction may not be easily dismissed in light of technical knowledge that the student may have. It will thus be necessary for him to speculate in a way that might illuminate his view of the nature of mathematics vis à vis the calculator.

In analyzing student responses it may be helpful to look at some of research on conception of knowledge that flows from Perry's work in which he uncovers nine stages of development - beginning with an absolute right/wrong conception of all knowledge through a stage of relativity ("it depends") to a stage of commitment.* That is, of course, only one handle, but it is one that we have found to be powerful in studying the development of conceptual knowledge among our students.

There are other research directions suggested by the concreteness of the calculator beyond the one in which different answers emerge for equivalent questions.

Consider what happens for example when students or authors ask "rich" and interesting mathematics questions - questions like:

"What are the prime numbers in the set of fractions?"

Frequently these questions are ambiguous, and allow for many different interpretations or they may be vague allowing for no interpretations or they may have assumptions embedded which make them into only apparent questions as in:

"How can you prove the parallel postulate from the other postulates of Euclidean geometry?"

The calculator never says,

"It depends..."

"What do you mean?"

"There are several ways of interpreting this question or phenomenon."

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* See William Perry, <u>Forms of Intellectual Development in the College Years</u>, Holt, Rinehart and Winston, 1970.

—though the overload display comes close to performing this function on occasion.

It is worth the extent to which considerable use of calculators affects the inclination of youngsters to see the potential ambiguity and vagueness in the study of mathematics. Does such use tend to suppress the asking of such questions? Does it tend to de-value these kinds of questions when they are raised?

Does the use of a calculator appear to encourage a view of mathematics as open, exploratory, mind expanding? Does it provide solace for those who view it as right/wrong, closed and filled primarily with answers? Or is the calculator neutral with regard to these differing conceptions? Here we would expect to find differences as a function of age but we do not predict wholesale increases in sophistication - as the analogy with our students in methods of teaching mathematics courses suggests.

There is of course more to explore in this area, and though we are not suggesting specific machinery, the questions appear to us to be critical ones. Some people for example, make considerable use of imagery, metaphor and other literary devices in an effort to understand and to solve mathematical problems. Though very little of this kind of thinking is explicitly encouraged in the standard curriculum, there are some people who engage in that type of thinking naturally.* How does the presence and use of an instrument with no imagination affect the inclination to do and view mathematics in such a way?

* See W. W. Sawyer, Mathematician's Delight, Penguin, 1943, and Stephen I. Brown, "Mathematics and Human Liberation," Occasional Papers of the Learning and Instruction Research Group, SUNY Buffalo, 1978.

We do not have in our hip pocket the instruments or the methodology to pursue these questions, but we have already suggested the work of Perry as a point of departure. In addition it is conceivable that some of the exploration in synectics will yield research insight.* In addition, collaboration with new kinds of people (for mathematics educators) like poets or philosophers might yield some of the needed research tools.

As we have argued in the first section of this paper, however, we have for too long been hamstrung by conservative methodology. We need the courage to explore profound questions despite the limitations of available tools of analysis.

Errors

In this section we look more closely at some of the kinds of problems that relate to the internal anatomy of this beast. We will see that there are man problems to address, but we will also see that researchers themselves have a great deal of homework to do here. Too often we researchers start and end at about the mathematics and computer science knowledge and sophistication level of the students with whom we are concerned. In one sense then this section not only raises research problems but also exposes some areas where prerequisite researcher knowledge is important.

At the same time that the new computation gives us new calculating power it creates new problems and raises new curricular concerns. Until now educators

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* See, for example, William J. Gordon, Synectics: The Development of Creative Capacity, Harper, 1961.

have glibly identified these problems at the generic level with recommendations like the one that students should be taught approximation skills or rounding techniques.* But closer examination reveals that what appears simple on the surface is increasingly complex the farther you venture into deeper waters.

Consider in this regard several exercises that seem quite straightforward until we examine them closely.** First we evaluate $100 \times \frac{2}{3}$ in three different ways on the same calculator:***

(a) $\underbrace{\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \ldots + \frac{2}{3}}_{100 \text{ terms}}$ gives 66.666609

(b) (2/3) x 100 gives 66.666660

(c) (2 x 100)/3 gives 66.666666

Now (b) and (c) represent straightforward results, but (a) is more complicated. Clearly these computations are on a calculator which truncates (or rounds down) at eight digits or seven in the case of decimals between zero and one. It is the latter case which in fact leads to the error in (b): 2/3 is expressed as 0.6666666.

But why (a)? In this case the calculator is not only accumulating the rounding error of the initial approximation to 2/3, but in fact compounds this error as it truncates within the computation. Thus

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* We do not argue against approximation skills here, but only the facile unthoughtful recommendation.

** In this development I draw on D. E. Bailey, "The effect on the solution of a problem of errors in the calculation," Mathematical Gazette 62, 421 (October 1978):157 - 164.

*** For these computations my Rockwell 24RD-II and a Sharp EL-203 give the same results.

$$\underbrace{\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \dots + \frac{2}{3}}_{15 \text{ terms}} \quad \text{gives } 9.9999990$$

while

$$\underbrace{\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \dots + \frac{2}{3}}_{16 \text{ terms}} \quad \text{gives } 10.666665$$

This additional and different error is the result of the addition

$$9.9999990 + .6666666 = 10.6666656$$

which the calculator must further truncate. Thus we have both entering truncations and processing truncations which accumulate in calculations. And these problems are not solved by either of the two standard machine responses: (a) additional accuracy carried internally,* and (b) standard rounding procedures applied. For example, my HP-97 gives 66.66666695 for the result of calculating by method (a) of the previous page.

Such problems are further complicated quite unexpectedly by more sophisticated types of errors. Bailey** gives the result of

$$\underbrace{\frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \dots + \frac{4}{5}}_{1000 \text{ terms}} \quad \text{as } 799.864$$

on his (unspecified) computer. He used the straightforward BASIC program:

```
S = 0
Y = 415
FOR K = 1 to 1000
S = S + Y
NEXT K
```

---

* My Monroe 326, for example, displays nine decimal digits but calculates internally with 13 digits rounded in the 13th place!

** *Loc. cit.*, pp. 159-160.

Why this startling result? Bailey's computer calculates in hexadecimal arithmetic in which the decimal 0.8 converts to the repeating hexadecimal 0.cccccc, thus producing a storing error of 0.000000c. (c here represents the digit in hexadecimal corresponding to twelve in decimal: thus in hexadecimal representations there are c inches in a foot.) This storing error is compounded in the computation.

It becomes clear that the internal anatomy of this beast is indeed a subject worth serious study. The following represent kinds of questions that cry out for research:

- What are the internal processing characteristics (both generic and specific) of various calculators and computers?
- How do these processing characteristics affect computation?
- What are the implications for teaching? More specifically,
- Should we communicate these processing characteristics to students? If yes,
- How?
- When? and
- What are the teacher training implications of these questions and answers?

Mathematics and Computer Science

Here we are straddling the boundary between mathematics and computer science, a boundary that should be more clearly delineated, another reasonable research task. We have today in too many mathematics classes far overstepped this border. Students are learning computer languages and programming tricks and essentially non-mathematical procedures like alphabetizing, all in mathematics classrooms. Surely this is wrong. What we need then is a serious consideration of just how far into computer science we can go before we find ourselves beyond neutral territory.

Included in this area are a wide range of activities for some of which decisions are quite difficult. A quite reasonable research task would be to derive from a set of sample activities characteristics of mathematical, computer-math and strict computer tasks together with samples of each for comparison. These empirically developed guidelines could then be applied to a larger number of those activities that fill the pages of today's mathematics education journals.*

So that this kind of technique would not be translated into too strict and mindless a procedure, I offer here an example picked essentially at random from one of the many sources for teachers:**

> Add any two numerals; for example $3 + 4 = 7$. Add the second digit to the sum to get a third numeral, that is, $4 + 7 = 11$. Add the new sum to the previous sum to get a fourth numeral $7 + 11 = 18$. Continue this procedure as in the column shown below. Do this until you have ten numerals. Add up the ten numerals, and the sum will always equal the product

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* Would not such a list and in particular the summary of mathematics activities better serve the mathematics education community than the seemingly endless government supported summaries of so-called research studies?

** This one from Bernard R. Yvon and Davis A. Downing, Math Explorations with the Simple Calculator, J. Weston Walch, 1978, pp. 30-31.

of the seventh numeral and 11, or 47 x 11 = 517

3
4
7
11
18
29
47 x 11
76
123
<u>199</u>
517

Challenge your friends to explore why. They'll admire the speed with which you can add without a calculator, while they use calculators. Once they reach the seventh numeral, you can probably beat them to the answer.

POSSIBLE ANALYSIS: Mathematics. Here the calculator is used strictly as a motivating device. The possibilities for student discovery and for analysis leading to exploration of Fibonacci numbers and their characteristics are excellent. Another value is the clear communication to students of the mind over machine quality of their understanding of underlying structure. Note, however, that these values revolve around the search for Why? and not on the trick by itself. Without this question the activity is neither mathematics nor computer science.

*

Errors of the Second Kind

In our discussion of computational errors we addressed ourselves to machine limitations that generate essentially systematic errors. Another most important source of error is to be found in the act of measurement which (aside only from trivial examples*) always involves approximation. Now many of us studied at one time or another in physics classes some of the rules for

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*The only example that comes to mind is the length of the standard meter bar in Paris, which we would have to agree was indeed <u>exactly</u> one meter in length at the time it was the standard.

computing with these approximations. If you can recall these rules, you are a much better student than I am. (I have just searched for them in three old texts without success.)

The calculator makes these rules take on importance, for this machine churns out eight or ten digit numbers in cases where it should be clear that most of the digits make no sense. For example, suppose John drives a stake at A, paces off 127 paces from A north to B and drives a stake at B. He then returns to A and paces off 127 paces east from there to C. He now seats himself at C with his calculator to compute the distance to B. Recalling his geometry he knows that the distance is $127\sqrt{2}$ paces which he quickly calculates to be 179.60511 paces, a number of paces that should only appeal to that silly breed known as sport statistician. Most of us would surely agree that it is about 180 paces from B to C; most of our students would agree that it is 179.60511 paces.

Research tasks again leap out at us.

- What rounding rules should we teach?
- How?
- When?
- With what kinds of materials and activities?
- How can we give our students a "feel" for accuracy of measurement?

Now for any of you who believe that <u>any</u> of those questions have simple and straightforward answers, I say hold on. The main reason I do not remember those statics rules for computation with measures is that they are generally unacceptable to me as mathematics. Consider, for example, one of the rules I do recall:

— In multiplication the product has the same number of significant digits as the lesser number of significant digits of the factors.

Let's try a multiplication:

$$84 \times 77 = 6468$$

And applying my poorly stated rule we should round this to 6500. Seems reasonable. But 6500 means something between 6450 and 6550 (excluding 6550 if you will). What is the real range of possibilities?

The smallest: $83.5 \times 76.5 = 6387.75$

The largest: $84.5 \times 77.5 = 6548.75$*

Something is clearly wrong here. We have many values in our real range of calculations that fall outside of our expressed range. And in this (rather extreme) case it does not even help to round to one significant digit because 6000 --that is, values between 5500 and 6500 -- still do not collect all of the possible products.

The problem here is that the "Laws" for rounding are really (a) "rule of thumb" reasonable approximations, and (b) partly statistically based. To see how the latter applies here an experiment (by the reader) is in order. Form the Cartesian product of the sets

$$\{76.5, 76.6, 76.7, \ldots, 77.5\} \times \{83.5, 83.6, 83.7, \ldots, 84.5\},$$

to produce a 121 element set:

$$\{(76.5, 83.5), (76.5, 83.6), (76.5, 83.7), \ldots, (77.5, 84.5)\}.$$

Now replace each element in this second set by the product of its coordinates:

$$\{6387.75, 6395.4, 6403.05, \ldots, 6548.75\}$$

* I do not wish to lose my point in an argument about .5 rounding up, so those who wish (wrongly) to use $84.4 \times 77.4 = 6532.56$ here should do so. This will also reduce the number of calculations in the following exercise by eleven without changing the results.

And plot the resulting points along a number line. If you do this carefully you will note that far from being spread evenly along the line, the points "pile up" in mid-range -- not in anything like a normal distribution by the way!

Now how do we deal with this kind of situation. Do we attack it head on and drive every kid with any mathematical bent away from the subject? Do we drop the whole thing and leave it to the physicists? These are serious questions that require equally serious attention. So I cringe every time I read one of those calls for action that says so glibly: We must teach rounding. We must teach estimation. We must teach you name it -- with no sense of the difficulty of responding to those oh so simple "musts." But having said that, I must (!) add that it is exactly responding to those so difficult problems that curricular research is about.

A Final Error Type: Vulgar Fractions vs. Calculators

One of the best self-characterizing recommendations that has been put forward lately is the one to drop common fractions from the curriculum. Envision with me if you will the textbooks of this brave new world: Whenever a time comes to portion out candy or pieces of pie there are always two or four or five or ten children to share, never three or seven. Or perhaps (later) three or seven can share. To share a pie among three, you first cut ten slices, distributing three to each. Next you cut the remaining piece into ten sections. (Seven is so complicated that you would probably again cut ten pieces, give four one slice each and three two slices each.) I place this recommendation in the same category as the one of my neighbor, now a vice president of a department store in Birmingham, Alabama, who suggested that we pass out calculators in grade one and forget math except for a few science types for the rest of the school program.

But this issue of common fractions is an important one <u>for mathematics</u> and I should not treat it too lightly. The situation should be explored and explicated. Let me give a parallel example which clarifies the issue for me but which may well only complicate things for many others. Consider a plane lattice, that is, the points in the coordinate plane with integer coefficients. (This corresponds to a geoboard or pin-board.)

Exercise (1) Locate the vertices of a square on this lattice.

Exercise (2) Locate the vertices of an equilateral triangle on this lattice.

Exercise (1) is trivial, exercise (2) nontrivial and not even possible. So we relax the conditions in a remarkable way. Allow all rational points in a new lattice. Even under these conditions exercise (2) is not possible.

Now who cares that we need merely one nonrational point -- (0,0), (1,0) and ($\frac{1}{2}$, $\sqrt{3}/2$) will do nicely -- to solve this problem. We can come so close on either of the simpler lattices that no one could detect the difference. In fact we can go further: there is no need for real numbers in the real world! Measurement deals in rationals and could in fact be reduced to dealing only in integers. We can get on perfectly well in our real world lives without real numbers. But mathematics, on the other hand, would die!

The situation is parallel with vulgar fractions. We can get on adequately in the real world of measurement with decimals, especially with metrication. But where does this lead us. When we divide one by three, in grade two do we get .3 and later .33 and then as adults .333? Or do we offer $.\dot{3}$ as an answer? (That last "solution" leads us into representing one divided by 17 as $.\dot{0}58823529411764\dot{7}$.) The simple fact of the matter is that the rationals are not adequately represented by decimals; they are completely represented by common fractions.*

In fact it is a contribution that calculators are making bringing this matter to a head. In this regard I recall my own experience in junior high school being led to believe that decimals were somehow better than fractions. "More accurate" sticks in my mind. I, like practically everyone else in the world, accepted the greater accuracy of 3.14 as a approximation when compared to $3\frac{1}{7}$. It is a simple exercise which I leave to the reader to prove the contrary.

The matter falls in a still broader context when related to metrication. Clearly metrication and our denary numeration system complement each other beautifully. But is this the <u>best</u> answer? It would be a good <u>mathematical</u> experience for our students (and more of us researchers) to meet the world of six fingers and toes, the duodecimal world. In that world of twelve digits - say 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, and B in order - we would have, for example:

$$\frac{1}{2} = .6 \quad \frac{1}{3} = .4 \quad \frac{1}{4} = .3 \quad \frac{1}{6} = .2 \quad \frac{1}{10} = .1 \quad \frac{1}{8} = .16 \quad \frac{1}{9} = .14^{**}$$

---

* Other matters arise here as well such as, for example, two different representations for many decimals. Thus $1 = .\dot{9}$. This problem does not arise as readily when dealing with common fractions.

** The point should not be missed here. Common duodecimal fractions up to twelfths (1/10ths) have finite representations except only for 1/5, 1/7, 1/A (denary tenth), and 1/11. Compare denary in which 1/3, 1/6, 1/7, 1/9, 1/11 and 1/12 have non-finite representations and 1/8 requires three digits.

1 foot = .4 yards   1 inch = .1 foot and

100 has two-thirds again as many factors as its counterpart in denary

I make no case here for a change to a duodecimal system. Like Teddy Roosevelt's spelling reform proposal which rated a famous New York Times one word editorial, "Thru," history has bypassed the Duodecimal Society. Still it would be a nice piece of curricular development to place metrication in this more general context. By developing this setting we might provide (old style research question?) some students with a better understanding of numeration and a deeper sense of what mathematical representation is about.

Decimals and Scientific Notation

It would be wrong to leave the topic of decimals without raising some other questions about their role in the new computation. We should probably be making a careful point-in-time record of student understanding of decimals at, say, the end of eighth grade. For the nature of that understanding will surely change over the next decade or two as these already ubiquitous four bangers become even more prevalent. I suspect that, as is so often the case, we will find improvement of understanding in certain areas - probably those most closely tied to the computational processes - and decrements in understanding in others - perhaps, for example, in areas in which fractions and decimals interact.

In some early work with tenth graders I called their attention to the reciprocal key and invited them to key:

$\boxed{4}$ $\boxed{1/x}$

and then to tell me what factor they would need to multiply the displayed result by in order to obtain one. Many hands went up but I called upon one lad who sat looking spellbound at his calculator display. "I just don't know," he said.

"Let's look at this in a different way," I suggested. "Everyone key in the following," and I turned to the chalkboard and wrote:

$\boxed{1}$ $\boxed{\text{ENTER}}$ $\boxed{4}$ $\boxed{\div}$*

"Now what should I use as my multiplying factor to get back to a display of 1." More hands waving, but my target sat again entranced by that display

0.25

After some additional thought, he ventured an answer: "Twenty-five." But when I turned to the board again and wrote

$$1 \div 7 \times ? = 1$$

he gave the correct response without hesitation.

It seems to me that this episode suggests several things, some trivial, some not so trivial:

(1) Clearly the display form can be misleading. If my example does not speak to you personally, consider the display in scientific notation:

2.50 -01

Now what factor do you multiply by to get a display of one?

(2) We need to identify what are the gaps in understanding of our average students <u>which are curriculum and instruction related</u> and seek to revise curriculum and instruction to respond to those inadequacies.

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* This class was using HP-25 RPN calculators. The students were already familiar with key order.

(3) We need to design and test curricular units that address the new problems that the calculators themselves create, as suggested by my example in (1).

(4) We need to observe and record students responding to set-up situations like that of my example in order to catalog not only what are the gaps but also how widely distributed they seem to be. It is one thing, for example, to respond to a difficulty of one student who perhaps was absent during key instruction on a topic like decimals; quite another to respond to a widespread deficiency among students.

(5) It is not enough to seek answers to (2) and (4) by reviewing texts to see if this or that "was covered." That is quite reasonably a first step. But the most interesting situations will be those for which a seemingly adequate textbook presentation was given and a seemingly reasonable instructional regimen mounted <u>without success</u>. Sensitive probing of those kinds of situations has the potential of identifying more serious learning problems.

I used an example of scientific notation to make a point in (1) above, because for me scientific notation plays very much the confounding role that decimals did for my student. That despite the fact that I have used scientific notation a good deal and have, I believe, a good handle on most of the meanings it conveys. For example, I am comfortable with the relation between logs and scientific notation. Yet, despite this <u>conceptual</u> understanding of scientific notation, I too find it most difficult to identify

2.50 -01

with one fourth.

Now we all have agreed that converting units back and forth between SI metric and English interfaces with metric instruction; however, it seems to me that we have a quite different situation, one in which we do not seek to supplant the old but rather to append the new to the old. In this latter case we need then to find the best ways of relating the two. Thus instruction in scientific notation and instruction in decimals have parallel concerns: incorporation and interrelations. How is this accomplished? We should set out immediately to find out.

The Social Setting

One approach to research on calculators and computers is to think of them in terms of their effect on society as a whole. We are after all literally overwhelmed by the ubiquitous microprocessor chip. It is in our televisions, our clocks and watches, our airplanes, our toys, our ovens, and our cars as well as in the space vehicles for which they were originally designed. Only incidentally they are also in some 80 million calculators already sold in the United States.

The societal changes that have been occasioned by this remarkably rapid move into the age of computation may well be comparable to those of earlier ages: the stone age and the industrial revolution for example. And not a few of those changes are humorous. We're all aware of the take over of clerical responsibilities by the modern cash register. This machine often calculates and even returns change, thus avoiding human errors. But sometimes the human clerk can still be clever. The other day I offered a five dollar bill to a drug store clerk for a $4.31 purchase. As I fumbled out another six cents to make my return come out in larger coins, the clerk rang up my $5.00 and out tumbled the change into a tray. With a bright smile she took my six cents from my hand and added it directly to the tray. I was so charmed by her alert thinking that I walked off with my collection of coins already wearing a hole in my pocket.

We are rightly concerned today about applications of mathematics to the real world. This is a topic that deserves as much research attention as do calculators. Do not the two come together here in an interesting way? It would be an extremely beneficial experience for our students, our teachers, and our researchers to make school students research assistants who observe calculator and computer users in order to identify what those users do with these tools. Watch and talk to store clerks, bank tellers, nurses, lab technicians, draftsmen, salesmen, scientists, and surveyors. How often do they use a calculator? What are the types of problems they process? What are some of the techniques they use? I have suggested this as a student activity for several reasons:

- It would be a good experience for them to see mathematics - even low level computation - in use.
- Their watching and talking to others in this way might well educe a different kind of motivation for self improvement.
- Collections of such observations would give us a much better feel for the nature of use of these tools. The multiplier effect would be operating here with a vengeance.
- There could be a bonus in this for the schools. Much citizen resentment of the educational establishment derives from a sense that the schools do not respond to societal needs. Here would be one clear response to that concern.

Whether or not this kind of task is done by students, it should be done. And here I digress briefly to comment on our training of doctoral students. Nate Gottfried, then a psychologist at the University of Minnesota, complained to me about such training: "Few students," he said, "have the opportunity to build testing equipment, interview students, run smaller experiments. These prerequisite tasks are skipped and we turn out half-trained [he used a different word] scientists." I suggest that this situation is getting worse as doctoral programs proliferate. What has contributed significantly to the situation described in the first part of this paper has been the narrow focus of doctoral programs on superficial machine statistics and the applications of behavioral psychology. Too few of our students are encouraged to address the philosophical problems that are embedded in teaching and learning and too few of our students are trained to interact with subjects in a clinical environment. We at SUNY Buffalo have begun to address these concerns in a doctoral practicum. One of the kinds of experiences that would fit that program perfectly would be carrying out this kind of intensive analysis of calculator use outside the classroom.

The support of doctoral students is, I should note here, a problem with which all of us should be concerned. If NIE could provide support for even a few dozen of these students to undertake such tasks, the benefits would be excellent for mathematics education in terms of both useful data and an improved experiential base.

At any rate one thrust of study could be in this natural history direction. One aspect of particular concern is parental attitude toward calculators and computers, specifically and mathematics in general, for it is this attitude that strongly influences school students. We should be especially interested in identifying these attitudes and then in developing the means for manipulating them. We may well have discharged the rot-the-mind fallacy within the education community but we are probably talking only to ourselves. In many homes this anti-calculator attitude is probably having an important negative effect on our students. If this hunch is correct, we have a serious problem to confront.

Quite another focus of this kind of research could be on that singular artifact of modern technological society, the computer freak. Joseph Weizenbaum has given us a portrait in broad brush strokes of this individual who redoubles his effort as he loses track of his goals. Like the idiot savant, the computer freak~~, because he is so much a contemporary phenomenon,~~ is a worthy subject of investigation. What is his expertise? What are his methods? What are the components of the intense interest that draws him into this tight man-machine commensalism? (I find the comparison with the pilot fish at the shark's mouth an attractive one.) How does this machine fervor compare with cult behavior? Is it age related?

We must not forget, of course, the sociology of the classroom. We need a better handle on teacher attitudes in order to think about ways to confront those attitudes. Consider in this regard a kind of thinking that is widespread among classroom teachers:

EXERCISE: Find the three rational values for x in the equation:

$$x^3 - 0.9x^2 - 4.09x + 4.641 = 0$$

A POSSIBLE PROGRAMMABLE CALCULATOR SOLUTION:

Rewrite the equation (for simpler programming):

$$[(x - 0.9)\ x - 4.09]\ x + 4.641 = 0$$

This is an equation of the form $f(x) = 0$. Program f.

An HP-19 program:

LABEL A	.	9	1
R/S	9	-	+
ENTER	-	x	PRINT X
ENTER	x	4	GSB A
ENTER	4	.	
SPACE		6	
PRINT X	0	4	

Now test various values for solutions using the idea that

$f(x_i) = 0$ implies x_i is a root.

"But that's not mathematics," several teachers have argued. "That's trial and error."* We need to come to grips with this kind of wrong and wrong-headed perception of what mathematics is so that we can improve our teacher (and student) education programs. And the first step in doing this is to identify what are ideosyncratic and what are more general perceptions of classroom teachers. We need then to develop and apply thoughtful responses to these deep seated attitudes about this subject and particularly about calculators.

There is a specific aspect of instruction with calculators that applies directly to this concern. We have found in our work with calculators in the classroom that the teacher is called upon to broaden his range of teaching techniques. It is well recognized that calculators encourage a free-wheeling math laboratory atmosphere for many activities, but what is not so well recog-

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* We do not argue here that this is high quality math, only that such critics fail to realize that they are refusing to accept for calculator solution the <u>same</u> trial-and-error solution technique they regularly apply but with synthetic division!

nized is the need for some tightly regimented lock-step activities when a class is learning calculator technique. If, for example, you wish to make a point about what happens after you depress keys a, b, and c, you must assure yourself that your students have pressed exactly those keys and no others.* How will teachers respond to such additional - and often as in my case unexpected - strains?

What methods do we use to find these answers? We can get help here from colleagues in other fields: anthropologists** and social psychologists in particular. This kind of research is accomplished by one researcher asking penetrating but essentially neutral questions of one subject at a time. It is accomplished by unobtrusive and long term observation. It is accomplished by cycles of the following kind: observation ⟶ reflection ⟶ hypothesis generation ⟶ observation to test and refine hypotheses ⟶ further reflection and discussions with colleagues to place refined hypotheses in a larger conceptual framework.

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* One of my tenth grade students made so many errors for a time that I found myself constantly exchanging his calculator thinking he was working with defectives. I finally determined that he was consistently depressing two keys at once. To respond I had him keying with one finger held in vertical (piano) position. While this was effective, it made me feel lika a teacher out of a previous century.

** Anthropology professor Fred Gearing at SUNY Buffalo has developed some techniques for microanalysis of interactional behavior in the school classroom that should provide some useful assistance here. The power of the anthropologists' approach to monitoring behavior in the classroom is their insistence on neutrality of data gathering. As Fred himself says, "You must choose. You either buy in on a particular psychological school or you're forced to wing it." The first approach distorts the observation. Today I (Rising) believe that mathematics education observers, the few that we do have, almost without exception enter classrooms wearing thick glasses of one of three types: behaviorist, Piagetian or artificial intelligence. I do not oppose their approach because it is at least theory based, but I believe that we have much to gain from a neutral approach that offers the possibility of a paradigm shift.

The Calculator as Machine

Having considered the calculator as everything but what it is - beast, human being, instrument of social change - we turn now to the calculator and the computer as sophisticated technological marvels to be sure but still just pieces of electronic equipment.

And here we meet at the outset a remarkable fact. While our mathematics education experimentalists have been out applying their statistical weapons to the classroom where they have least possibility of providing useful information, my engineering colleagues at SUNY Buffalo have stolen the march on them in investigating one problem to which those techniques can make a contribution. These engineers* have explored in very interesting and suggestive ways the question of which calculator processing language - algebraic or reverse Polish (Lukasiewicz) -- is better. This is an important question which unfortunately is being answered for us by early and ill considered recommendations (for algebraic) by NCTM and now by our Second International Mathematics Study team. The evidence provided by the engineers gives strong support for reverse Polish (RPN) with the college student population. In particular, speed and accuracy differences favor RPN, the full power of RPN calculators is more often exploited, and retraining from algebraic to RPN is so simple as to be accomplished in ten to fifteen minutes. Their proposed extension of this work with college students to students from various elementary and secondary school levels could prove interesting.

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* See S. J. Agate and C. G. Drury, "Electronic Calculators: Which Notation is Better?" Applied Ergonomics, in press; and D. M. Kasprzyk, C. G. Drury and W. F. Bialas, "Human Behavior and Performance in Calculator Use," currently an in-house publication of the Operations Research-Human Factors Group of the Department of Industrial Engineering, SUNY Buffalo.

But their heady goal is still more interesting. They seek optimum calculators for students of various ages and will experiment with not only different operating orders but key availability, size, spacing and location, display size and form, and appropriate peripherals like printers and plotters. The state of the art is such that their recommendations should prove valuable to manufacturers and indirectly of even more value to students and teachers.

An example of the kind of hardware question that is important because of its curricular implications is: Should a scientific calculator have a factorial $\boxed{n!}$ key? This question is important to manufacturers because it happens that this key uses up a significant amount of the finite program space on the chip. Thus the question may mean a necessary trade-off: if this function is introduced something else must go. Now if we wish to incorporate more combinatorics and probability in our program, we would probably be most willing to give up something else, say $\boxed{\text{cos}}$ or $\boxed{\text{tan}}$, recognizing that we can easily reconstitute either function by means of the simple identities:

$$\cos x = \sin (90 - x), \text{ or}$$

$$\tan x = \frac{\sin x}{\cos x}.^{*}$$

So here we have a kind of question that is important to educators and that requires the same level of serious consideration that all curricular modification

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* I offer this as only a theoretical example. The process for calculating a trig function does indeed itself take much program space, but that is for the <u>first</u> trig function. The calculator almost surely generates only one, say sine, by this means and uses exactly the kind of identity we suggest here to convert. Calculator companies are not at all enthusiastic about providing information about the internal processing characteristics of their equipment. When asked by Wallace Jewell why one model of a particular manufacturer gave "error" for 0^n, for a natural number, and a more recent model gave 0, a company representative would only respond, "Forty dollars." (The answer while not forthcoming is not inappropriate since powers are certainly computed with logs; thus the exception requires extra program - and extra expense.)

does: When you add here, you have to subtract somewhere else.

While I give my engineering colleagues strong support for their search, I must enter a disclaimer. There is a danger in a search for the optimum that must always be kept in mind. The very nature of an optimum demands a choice from what is available. We do not want this search to preempt the development of new directions. The Papert turtle is a useful example of the kind of divergence from the narrow path of development that is to be encouraged. Similarly history shouts at us - I think it is the Committee of Ten calling - to beware of single answers. It is a great temptation to set readiness standards for calculators and computers. Thus we have the quite natural progression:

- Four bangers
- Scientific calculators
- Programmables
- Microprocessors
- Full computer power

But not a fixed progression. I can see great value in development of microprocessor activities for lower grades without delaying to meet some schedule of prerequisites. A statement by Lee Shulman, made in quite another context, is very much to the point here:* "Research can serve teachers...by clarifying the complexities of nature, thereby making possible better grounded judgments, not by replacing judgments with ironclad rules." Thus we should be providing alternatives, noting carefully the dangers as well as the advantages of those

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* Lee S. Shulman, "Investigations of Mathematics Teaching: A Perspective and Critique," Michigan State University Institute for Research on Teaching, 1978.

alternatives, for informed choice. Pluralism, not uniformity, should generally be the end result.

Before leaving the subject of machine design I note that again we can serve ourselves well by establishing lines of communication in new directions outside of education. The engineers can contribute and are doing so, yes, but so too can, for example, the linguists. We are concerned here with machine <u>languages</u> and we should not lose sight of our common concern merely because the word language has a non-standard meaning. Such intrauniversity communication and sharing of ideas has exciting possibilities especially when we focus on very young children first learning language as well as mathematical processing.

I should mention here the work of Lauren Resnick and her colleagues at Carnegie Mellon. Some of their work explores the same blind alley as does so much of the artificial intelligence research. Here I join Weizenbaum, Chomsky, and Dreyfus* in being highly critical of the thoughtlessness of their approach. An example of this was (I use past tense advisedly because I believe that the approach is now rejected) the proposal by David Klahr to search for a computer MOLIM - model of a learner in mathematics - against which we would test curriculum! Still I believe that there are possibilities here once the search for the android is set aside. The human-machine parallel - within its limitations - has possibilities at two levels: (1) The machine can replicate some of the less complex (and less experience and genetic based) algorithmic processes of the human brain, and (2) The machine provides a much more useful metaphor for more significant thinking about mental information processing. These directions are,

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* A useful reference here is H. L. Dreyfus, <u>What Computers Can't Do: A Critique of Artificial Reason</u>, Harper & Row, 1972.

I believe, well worth following up and I am assured that the Carnegie group is moving in at least one of those directions.

I have already directed attention to curricular possibilities leading from the computer as metaphor. Here the possibilities lie in psychological theory building. For example it would be most interesting to teach BASIC to primary school children and to observe the metalevel effect on their own information processing. Could we get them to better organize their own thinking by focusing their attention on making machines "think," or is this another of those areas where we are like the centipede who, when asked by the ant how he was able to control all those legs at once, was effectively crippled. Either direction would be an interesting, yea exciting, result.

The Machine in the Classroom

We must think deeply about how to use this readily available technology in education, but we must also draw those thoughts together and communicate them in usable form to classroom teachers. And this involves some very practical activities which, I claim (perhaps plead is a better verb here), should constitute appropriate research activity. Surely the precedent for compilers as researchers is well established. As a case in point, the other paper prepared for this conference is more acceptable as standard research in the mathematics education community than is this paper.* Carefully organized compilations of calculator activities fall in this category. QED.

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* Compare Journal of Research in Mathematics Education: Research compilations are major features, papers about research are subsumed under a department, "A Forum for Researchers."

As one of my students said some years ago after I had painfully displayed the reasons for <u>pons asinorum</u>: perhaps. Well, research or no in some strictured use of that word: we need the intelligent compilations. Such compilations could be <u>the</u> most important direct contribution to classroom instruction that NIE could make. Some compilations might well address themselves to particular courses or specific broad topics such as seventh grade mathematics, trigonometry, graphing polynomials, exponents and logarithms, probability. Others might be directed to activities that cut across subject boundaries or vertically through the mathematics program: problem solving, iteration techniques, algorithms. Only within such compilations should differentiation by equipment be a concern; it is a different activity - albeit a reasonable one - to develop curriculum specific to a calculator type.

An excellent start was made by David C. Johnson working with classroom teachers in the Minneapolis - St. Paul area while he was still at the University of Minnesota.* Dave superimposes a content x purpose matrix scheme on his examples that is too refined for my taste, but his purpose categories are suggestive as at least a first approximation: calculations, patterns, exploration, consumer applications, societal applications, new or renewed content. The value of this paper, however, derives from the examples provided for the categories, examples that display intelligent eclecticism, an aspect of what David Hawkins has denoted the wisdom of the practitioner. Here the compilers are doing more than collecting examples: they are selecting, interpreting,

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* "Calculators in School Mathematics - A Classification for Curriculum Development," in manuscript. This seminal paper was rejected by at least one major journal.

and adding their own where the available art is thin.

The point that should not be lost in this* is that calculators or computers do not stand on their own. Vincent Glennan once made an astute observation in this regard: "Teachers' closets," he said, "are full of yesterday's world saving devices." I can give evidence that this is already true of calculators. Ask your local school principal if his teachers have calculators for instruction; he'll take you to a remote storage cabinet where a set of calculators are buried under a collection of the other detritus of schooling. In order to insure any use, to say nothing of good use, teachers must be provided with support. The best and most effective support is a program built into the student text for the course, but short of that some form of printed material with explanations and exercise sets is necessary. Since the logistics of commercial textbook publishing argues strongly against ties with calculators or in fact any supplementary devices,** it is necessary to choose the second option.

I indicate here some of the kinds of things that might be included in the calculation compilations. No attempt is made to exhaust possibilities or even to suggest a range; rather my examples are chosen to identify a few of the kinds of points that need to be raised. The examples are not fleshed out in the way they should be in a useful compilation.

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*But has been by federal agencies which have supported activities of the form: put some calculators in a classroom and see what happens, without any curricular preparation or teacher training. The outcome of such foolishness is predictable.

**I cannot argue with publishers here. Their attitude is that any such tie will lose more sales than it will gain. Until we change the dynamics of this situation they must continue to be wary of "peripherals."

- Sometimes calculator solutions provide surprising insights into the structure of a given situation. Suppose, for example, someone has programmed a quadratic function of the form f: $x \rightarrow ax^2 + bx + c$ with $0 \leq a, b, c \leq 99$ in a calculator or computer. Without looking at the program, what <u>single</u> input for x will identify via the output all of the parameters a, b and c? (Answer: 100. Why?)

- Too often we offer many digit number problems as a (usually false) indication that we are dealing with the real world. Solving an equation like

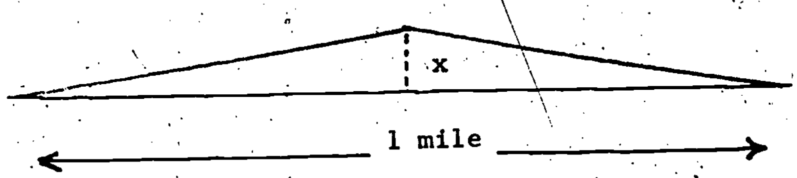
$$3.25x = 87.75$$

provides no benefits to students that go beyond solving

$$3x = 15.$$

But there are some nice problems in which the numbers do get large and rather difficult to manage without a calculator. Here is one:

> Railroad men leave expansion gaps between the ends of track rails to allow for the effects of temperature changes. Suppose a new crew boss forgets and a mile of track is laid without these gaps. The temperature rises and the track, fixed at its ends, expands one inch in length. Assume that the track bows to form an isosceles triangle over the full track length. What would be the altitude of this triangle? Does the answer



> justify the concern over expansion? (Answer: 14.8 feet!)*

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*This, like so many other calculator problems, offers a nice opportunity for non-calculator preliminary processing. Here, for example, we have (in inches):

- We need to show our students not only the power of calculators and computers but also their limitations. A nice problem that shows the kind of limitation we should convey is summing a finite number of terms of the divergent series of reciprocals of natural numbers.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

It is a simple matter to program a computer or programmable calculator to determine how many terms (n) are required to make the sum of this series $\geq N$ for given rational N. Thus we have for a few integral values of N:

N	n
1	1
2	4
3	11
7	616

And now the calculator that I used, an HP-25 with an eleven step program, begins to take a significant amount of processing time. When N = 7, for example, it took over six minutes to process.

Let us go further, but focusing now on the time for processing:

N	time
8	17 minutes
9	44 minutes
10	2 hours
13	40 hours (the highest value I calculated)

$$x^2 = (\underline{12\cdot5280\cdot\tfrac{1}{2}} + \tfrac{1}{2})^2 - (\underline{12\cdot5280\cdot\tfrac{1}{2}})^2$$

the right member of which is of the form

$$(a + b)^2 - a^2 \quad (= 2ab + b^2)$$

Thus we have

$$x^2 = 12\cdot5280 + \tfrac{1}{4}.$$

At this rate of increase in time (which appears to converge on e), to calculate n for N = 20 would take 5 years! The problem has become as impractical for the calculator as carrying out the process for, say, N = 10 (adding the reciprocals of the first 12367 natural numbers) is for paper and pencil calculation. While a computer may process faster, it too will reach a point of diminishing return in terms of time -- and in that case cost. Thus the new calculation has its limits as well.*

- Theorem: $n^2 + n + 17$ represents a prime for all n. Use a program that tests numbers for factors to check this formula for n = 1, 2, 3, ..., 15. Is the theorem true so far? Now test 16. (Can you show a way for testing 16 without your calculator?) What does your result suggest about attempts to prove a statement by showing that it is true for many specific cases?

This kind of exercise was often assigned before the availability of calculators. The common response: Students didn't complete it because there is too much computation involved. And often a table of primes was not even provided to make the problem reasonable.

It is our observation that students who have learned to develop programs, like the factor test used here, love to use those programs to solve other problems. Thus in that setting the exercise conveys much better the failure of scientific induction to develop or prove theorems.

- Iteration now becomes a central theme of mathematics.** Book 0 Chapter 16 of the CSMP Elements of Matnematics program is a useful

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* It is interesting to note that the great Indian mathematician Ramanujian determined (without explanation and surely without calculator) that for n = 1000, N would be about $7\frac{1}{2}$. His result is remarkably close, 7.485+, with n = 1015 the exact value for N = $7\frac{1}{2}$.

** The other day I mentioned this to an audience of in-service teachers only to have one ask what iterations meant. This gives, I velieve, an accurate indication of the past investment in this useful technique. Exception: the divide and average square root algorithm, one of the poorer applications of this technique.

source of iteration problems. One that I made up for a talk several years ago provides a nice bonus.

Solve for x: $x = 2^{-x^2}$

A SOLUTION BY ITERATION: Let (1) $y = x^2$ and (2) $x = 2^{-y}$. This system represents the original equation. Starting with $x = 0$ (arbitrarily) in equation (1) find y. Substitute this in the second equation to produce a second x value. Continue this trading back and forth until the answers converge. Thus:

x	y
0	
	0
1	
	1
0.5	
	0.25
0.84^+	
	.
.	.
.	.
.	
	.501
.707	

After some 20 steps the numbers have converged to three digits (that is those digits are no longer changing). This is very simple with a programmable, not much more difficult with a scientific calculator.

The bonus here is that the x-value is suggestive and does indeed turn out to be a good lead to the correct real solution $\sqrt{2}/2$. Justification for this procedure is communicated by reference to the graph of the two equations. Until this technique gains the wide range of uses and is placed in its appropriate setting of limitations as well as values, examples such as this should be used to alert students to the method.

- One of the most significant roles the calculator can play is that of problem definer. We should recognize that one of the principal sources of difficulty for students in problem solving derives from the students' lack of clear understanding of what is being asked of them. In many cases the calculator can provide the necessary bridge. The CSMP Elementary Mathematics curriculum makes creative use of the calculator in this way. For example, they deactivate (by covering with paper tape) the following four-banger keys: 0, 1, 2, 3, 4, 7, and the decimal point. Now they pose the following generic problem:

> Each calculator key pressed costs one penny. Try to obtain the following displays for a cost of ten cents each or less:
>
> (a) 47 (d) 0.5
>
> (b) 65 (e) -0.5
>
> (c) 16 (f) 324

The provision of wide ranging compilations would provide what Max Bell* has so accurately called (and called for): an information base for work in this field. Their value as resources for teachers and researchers, especially those concerned with curriculum modification, would be tremendous.

Miscellaneous Recommendations

This section is not meant to serve as a summary for those who have been unwilling to read what has gone before. Rather it plays the role of a catch-all for recommended courses of action that did not fit the discussions of earlier sections.

- When New Math was introduced, the most effective teacher training

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*See his "Needed R&D on Hand-Held Calculators," Educational Researcher May 1977, pp. 7-13.

was found to be study of the content in exactly the form to be presented to school students. We face a similar situation now except that we have researchers confronting the new calculation We need to concern ourselves with the dynamics of this situation in which there are very few "experts," the continuing transitions in availability of hardware precluding our assigning this status.* We should think in terms of support for expertise development, in-service training calculator research activities.

This means two things: (1) It is too early to review propose research by asking a question which may be to the point at some later time: What does he know about calculators? (2) Activities that force researchers to work with calculators in educational settings should be generated.

As an example of what I mean here, I offer the current interesting explorations of Keith Harburn at the University of Toronto. Keith is exploring what different kinds of information, thinking, and attitudes are conveyed by computer and more standard processing of the same problem. For example, consider solving this old Math Olympiad (1960/1) problem:

> Find all three digit numbers which equal eleven times the sum of the cubes of their digits.

This problem has a delightful algebraic solution which calls upon and provides insights into divisibility, the quadratic formula, and

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*Of interest here, I suspect that exactly those colleagues whom I consider to be exceptions to the rule, Ruth Hoffman, Don Stover, and Max Bell for example, would be the first to agree with my point.

solution by cases.* The programmable calculator solution, on the other hand, seems to be more mecha ical and sterile; yet it focuses very tightly on the basic structure of the problem. There is, by the way, not so much time saved by a calculator solution when you count in programming time. Does the answer mean more one way or the other? The answer means something slightly different to each solver, but it seems to carry no more weight by having been found algebraically. (We make at least as many mistakes in algebraic processing as we do in computer processing.)

These and more subtle observations on this and other problems open to parallel attack can identify for us some information about computer processing that should be useful to both practitioners and other researchers. Keith's work in this area is meanwhile contributing to his growing confidence with calculators.

- Curriculum modification in both the small and the large should be supported in parallel, the former contributing to the latter. This suggests the extreme importance of communication and in particular the increasing value of Marilyn Suydam's compiling activity at ERIC. Support for increasing the communication of this information should be considered. There are already several good journals which overlap with researchers' concerns in this area but it may be that at least a newsletter should be considered, quite possibly as an extension of activities already underway at ERIC. We need better information

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* See Samuel L. Greitzer, ed., International Mathematical Olympiad, 1959-1977, MAA 1978, pp. 27-29.

about: what specific people are doing in both capsule and full account form, what publications and in particular what specific articles add to the growing knowledge in this field: I'm thinking here of at least brief squibs like those Phillip Peak paragraphs in old Mathematics Teachers.

I also urge that we explore the possibilities of utilizing a right-to-copy system so that experimental materials can be copied inexpensively and locally from an available master. This responds to the single most costly publication problem: maintaining inventory.

- Here as elsewhere in education we need best effort studies. What could we do for youngsters if we took the lid off and provided them curricular materials, equipment and setting, and instruction that approach optimum. I blame the research community for destruction of the best setting for this kind of research, the campus school, and I believe we owe education something for this ill advised and wanton act. The few possibilities left - George Immerzeel in the Price Laboratory School for example - should be utilized and other similar settings developed. The politics of such an effort within a metropolitan school system are horrendous, but it should be attempted and the time, as it happens, is right for such an attempt. School closings, population shifts and even the move to special interest schools as a response to integration pressures all speak to the opportunity available.
- Some very specific studies - quite possibly under contracts - should be mounted to develop answers to the standard public and education

community concerns, for example, the rot-the-mind belief of the public and the worry of many teachers and mathematicians about black boxing. These real concerns deserve thoughtful responses, responses that include acceptance of the concern, suggestions of steps taken in response, and reassurance that the concerns are overridden by positive returns.

A case in point: A high school teacher told me that he would never use a scientific calculator in his mathematics classroom until his students could calculate by paper and pencil the functions of the various keys. Now I find this a reasoned argument. It parallels the belief that elementary school students should never touch a calculator until after they have learned paper and pencil algorithm processing. But in the secondary school at least we're being two faced here. Should we not let our students use the trig and log keys because they cannot generate those values when we have always provided them with tables without explanation? While the argument is reasoned, it is inappropriate when carried to extremes or applied pedagogically. It is often at least as good to address how to do something first, why afterward.

Earlier in this paper I mentioned the learning that went on when my students set out to determine what the log key did. This is a good example, I believe, of what Lee Shulman has called "a strategic research site," a key moment to study in great detail because of the discontinuities in thinking that are bound to arise.*

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* *Loc. cit.*, pp. 16-17.

We should seek to create such situations and to record them carefully* in order to prize out details of the act of learning. I give strong support to this kind of study no matter how artificial are the settings developed: - learning to play a game like Mastermind, for example. The calculator in its most blatant new toy character provides many opportunities for generation of such set-ups.

- Just as we mount small and large studies of the curriculum, we should focus attention on both old and new topics. I have already mentioned iteration as a topic that now should have a place in the curriculum. The Monte Carlo method is another. What we must be concerned with here is our conservative-liberal separation in curricular reform. Some of us want to impose discontinuous and complete change on the school curriculum; some of us want no change at all - back to basics. Both extremes, the nihilists and the reactionaries, are clearly wrong, but there are intermediate positions that have stronger defenses. I suspect that there would be general agreement among thoughtful mathematics educators that there is too much application of Pope's dictum, "Whatever is is right." What this implies then is that we must continually support examination of our basic goals in mathematics instruction and how those goals apply to the specifics of our content.

One important thing that such reexamination requires and that should not be forgotten is the need for exemplars of what could be

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*Fred Gearing suggests devoting a year to study of a two or three minute video tape.

done that is different; otherwise we have no available alternatives to what is. What has happened in the past in the absence of specific worked out curriculum alternatives has been: (1) Ill-thought out and often ill conceived recommendations - like we should teach probability in the grades - have been promulgated, and (2) other possibilities have been avoided because we have prejudged students: they could never do that.

I conclude this opus - and I apologize for it having grown to that - with a recommendation that I should probably type in 24 point bold face caps. I believe that leaders in funding agencies like NIE and in national organizations such as NCTM do realize the need for new paradigms, for new thrusts, for new directions, and for newcomers to research. I also believe that individuals and groups are trying to move in this direction. Frankly I take the assignment of one to undertake the development of this working paper as a positive (but quite possibly another unsuccessful) act in evidence of this. The trouble is, however, that movement is a matter of critical mass. (History provides thousands of examples of this: Semmelweiss comes immediately to mind.) Without it we are today at a standstill.

But there is one way to move - really to prevent total focus on more of the same - and that option should be tried. We need to establish some review teams (for grants and for journals) which are made up <u>completely</u> of non-conformists, conservative researchers excluded just as completely as the others were excluded in the recent past. To these teams should go the non-standard proposals for evaluation. Only then will some of the truly creative activities that are now regularly being turned down gain support.

Many readers will not understand why this is and I attempt an explanation here. Support is a go-no go decision for federal agencies. The preponderance of the applications are conservative - more of the same. Review teams of three or four people examine proposals and it usually takes no more than one reader to kill a proposal. Thus non-conformists scattered through these committees kill a few old style projects - just as a group of mathematics education leaders killed all of the National Science Foundation centers several years ago - but enough others get through to continue the bulk of this activity. Meanwhile <u>every</u> new thrust is being rejected: too small an n, a quasi-experiment, the director is not known to me, no control group, statistics not carefully described, not feasible with today's teachers, differences cannot be measured; the list, no item of which addresses quality of the endeavor, is endless. The <u>only</u> way around this is to segregate until the critical mass situation is reversed.