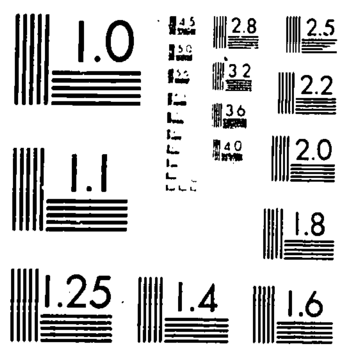

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

DOCUMENT RESUME

ED 168 834 SE 026 912

AUTHOR Lester, Frank K., Jr.

TITLE Mathematical Problem Solving Project Technical Report I: Documents Related to a Problem-Solving Model. Part B: Mathematical Problem Solving in the Elementary School - Some Educational and Psychological Considerations. Final Report.

INSTITUTION Indiana Univ., Bloomington. Mathematics Education Development Center.

SPONS AGENCY National Science Foundation, Washington, D.C.

PUB DATE May 77

GRANT NSF-PES-74-15045

NOTE 57p.; For related documents, see SE 026 911-934; Not available in hard copy due to marginal legibility of original document

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.

DESCRIPTORS *Educational Needs; Elementary Education; *Elementary School Mathematics; *Mathematics Education; Models; *Problem Solving; *Psychological Characteristics; *Research Needs

IDENTIFIERS *Mathematical Problem Solving Project

ABSTRACT

The philosophy and activities of the Mathematical Problem Solving Project (MPSP) are described. The main focus is on the research and development efforts, especially the attempt to develop a conceptual framework for problem solving through the creation of an appropriate model. An extensive discussion of models of problem solving is included. Four main topics are discussed: (1) critical issues and questions related to mathematical problem solving; (2) nature of MPSP; (3) thrust of the work of MPSP; and (4) plan for future research. An article by this author on a similar topic appeared in the November 1977 issue of "Arithmetic Teacher." (MP)

MATHEMATICAL PROBLEM SOLVING PROJECT

MPSP

A Project of the
MATHEMATICS EDUCATION DEVELOPMENT CENTER

Project Supported by
National Science Foundation Grant PES74-15045

MATHEMATICAL PROBLEM SOLVING PROJECT

POLICY BOARD

George Immerzeel

Donald R. Kerr, Jr.*

John F. LeBlanc, *project director*

George Springer, *co-principal investigator*

Maynard Thompson

David W. Wells*

* *acting co-directors, 1975-76*

ADVISORY BOARD

Robert P. Dilworth,
California Institute of Technology

James F. Gray,
St. Mary's University

John L. Kelley,
University of California at Berkeley

Jeremy Kilpatrick,
Teachers College, Columbia University

Eugene D. Nichols,
Florida State University

UNIVERSITY OF NORTHERN IOWA CENTER
Cedar Falls, Iowa

George Immerzeel,
center director
Joan E. Duea
Earl G. Ockenga
John E. Tarr
Jack D. Wilkinson

INDIANA UNIVERSITY CENTER
Bloomington, Indiana

John F. LeBlanc,
center director
Marilyn Hall Jacobson
Donald R. Kerr, Jr.**
Frank K. Lester, Jr.
George Springer
Arthur H. Stengel
Maynard Thompson
Norman L. Webb

***acting center director, 1975-76*

OAKLAND SCHOOLS CENTER
Pontiac, Michigan

David W. Wells,
center director
Stuart A. Choate
Douglas W. MacPherson

Project Graduate Students: Randall I. Charles, Fadia F. Harik, Tom S. Hudson, Barbara E. Moses, Gary Post, Linda A. Proudfit

FINAL REPORT
MATHEMATICAL PROBLEM SOLVING PROJECT
TECHNICAL REPORT I:
DOCUMENTS RELATED TO A PROBLEM-SOLVING MODEL
PART B: "Mathematical Problem Solving in the Elementary School: Some Educational and Psychological Considerations"

Report Prepared By

Frank K. Lester, Jr.

Prepared for the Research Workshop on Problem Solving in Mathematics Education
Center for the Study of Learning and Teaching Mathematics
The University of Georgia
May, 1975

MATHEMATICS EDUCATION DEVELOPMENT CENTER
Indiana University - Bloomington
May, 1977

Mathematical Problem Solving in the Elementary School: Some Educational and Psychological Considerations[1]

Introduction

One of the most important goals of elementary school mathematics is to develop in each child an ability to solve problems. In recent years more and more emphasis has been placed on problem solving in the elementary mathematics curriculum. A cursory look at the scope and sequence charts of the most popular textbook series points out this trend. In each of these series problem solving is identified as one of the key strands around which the mathematics program is built. At the same time there is concern among teachers, mathematicians, and mathematics educators that these programs are doing a poor job of developing problem solving ability in children. Points of view which are representative of the dissatisfaction with current programs are found in the reports of the Snowmass Conference on the K-12 Mathematics Curriculum and the Orono Conference on the National Middle School Mathematics Curriculum held during the summer of 1973. These reports called for extensive modification of current mathematics programs to include a more systematic approach to providing instruction in problem solving.

The current concern should raise a number of questions in the mind of anyone interested in the mathematics education of children.

---

[1]The author is indebted to Dr. Norman J. Webb and other members of the Mathematical Problem Solving Staff at Indiana University for their valuable suggestions. The views expressed in this paper do not constitute an official statement of policy regarding the goals of the Mathematical Problem Solving Project. The author accepts sole responsibility for all of the positions and views stated in this paper.

Exactly what is problem solving? Can students really be taught to be better problem solvers? If problem solving is so important and good problem solvers are not being developed, what steps should be taken to change present instructional practices? Certainly an answer to the first question must be obtained before the other questions and ones related to them are tackled. So, before proceeding any further a definition of problem solving should be provided.

Definition of a Problem

A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution.

Any one of a number of other definitions of a problem would be satisfactory for the purposes of this paper (e.g., Bourne, Ekstrand & Dominowski, 1971; Davis, 1966; Henderson & Pingry, 1953; Simon & Newell, 1972). Let it suffice to say that any reference to a problem or problem solving refers to a situation in which previous experiences, knowledge, and intuition must be coordinated in an effort to determine an outcome of that situation for which a procedure for determining the outcome is not known. Thus, finding the length of the hypotenuse of a right triangle given the lengths of the two legs probably does not involve problem solving for the student who understands the Pythagorean Theorem, but may be problem solving of a complex nature for the student who has not been exposed to the Pythagorean Theorem.

Since problem solving is viewed as such an important part of learning mathematics it seems natural to analyze carefully what is involved in the process so that effective instructional techniques can be developed. There is little or no argument on this point. Everyone agrees that serious attention must be given to instructional issues related to problem solving. However, beyond this point there is little, if any, unanimity of opinion concerning the process of problem solving.

Even the most successful problem solvers have difficulty in identifying why they are successful and even the best mathematics teachers are hard pressed to pinpoint what it is that causes their students to become good problem solvers. Unfortunately, in spite of the volumes that have been devoted to problem solving what is now universally accepted knowledge about problem solving can be boiled down to Georg Polya's words of advice to mathematics students: "Use your head." (Professor Polya's final statement in a presentation at the 1974 annual meeting of the American Mathematical Society.)

Out of frustration over an inability to deal successfully with the problem solving dilemma, mathematics educators have turned to psychology for guidance. The nature of problem solving and the measurement of problem solving ability have been the objects of considerable attention by psychologists (representative reviews of psychological research in problem solving have been written by Bourne & Dominowski, 1972; Davis, 1966; Green, 1966). Typically, psychological reports of problem solving research begin with a statement like: "Research in human problem solving has a well-earned reputation for being the most chaotic of all identifiable categories of human learning (Davis, 1966, p. 36)." Indeed, it has only been during the last twenty to twenty-five years that a major point of view or technique has developed which attempts to isolate the important variables which influence problem solving behavior.[2]

There appear to be a number of reasons for this condition. First, a variety of tasks has been used in problem solving research. The tasks found in the literature include such diverse problems as matchstick, Tower of Hanoi, and jigsaw puzzles, anagram problems, concept identification problems, arithmetic computation problems and standard mathematics textbook word problems. Also, problem solving research has been conducted by experimenters with quite different positions on the nature of problem solving. The traditional cognitive-Gestalt approach of such psychologists as Wertheimer, Maier, and Duncker is quite different from the associative learning theory approach characterized by the work of Maltzman (1955) and the Kendlers (Kendler & Kendler, 1962). More recently, especially within the past fifteen years, considerable effort has been devoted to the development of an information processing approach to the study of problem solving. The well-known work of Simon and Newell (1972) is representative of the information processing view of the problem solving process. Thus, although much exciting and potentially fruitful work is being conducted by psychologists, very few definitive answers to the questions concerning the nature of learning and instruction in mathematical problem solving are available at the present time. It is likely that these answers will result only from several years of intensive study that reflects a cooperative effort by mathematics educators, psychologists, and classroom teachers.

---

[2]Kilpatrick (1969) suggests that serious attention to problem solving by mathematics educators has developed primarily within the last ten or so years.

Overview of This Paper

The intent of this paper is to describe the philosophy and activities of the Mathematical Problem Solving Project (MPSP) at Indiana University. The paper will contain four main sections:

1. The Critical Issues and Questions Related to Mathematical Problem Solving
2. Nature of the MPSP
3. Thrust of the Work of MPSP at Indiana University
4. Plan for Future Research

The main focus of this paper is on the research and development efforts under way at Indiana University. Included in this effort is a serious attempt to develop a conceptual framework for mathematical problem solving. The development of such a framework will center on the creation of a model for mathematical problem solving. Since the creation of such a model is considered to be of utmost importance in developing a framework for future research and development efforts, an extensive discussion of models of problem solving is included.

It is hoped that the positions posed and the efforts described will stimulate valuable discussion concerning the key issues related to mathematical problem solving in the elementary schools.

Critical Issues and Questions Related to Mathematical Problem Solving

The opening sentence of this paper stated that the development of children's problem solving abilities is a major goal of elementary school mathematics. It is interesting that while few educators would disagree with this claim there is little evidence that a serious attempt is being made to attain this goal. No single factor can be identified as causing this state of affairs to exist. Instead the problem can be attributed to a number of causes. The following are among the most prominent:

1. Problem solving is the most complex of all intellectual activities; consequently it is the most difficult intellectual ability to develop.
2. Elementary school mathematics textbooks typically are deleterious rather than facilitative in developing problem solving skills and processes in children.
3. Elementary school teachers do not view problem solving as a key feature of their mathematics programs.

Before suggestions are presented for remedying the present situation it is appropriate to elaborate on causes 2 and 3.

It is the author's opinion that the overwhelming majority of the activities presented in elementary mathematics texts as problems are actually little more than exercises designed for practicing the use of a formula or algorithm. A second criticism is that textbooks do not include enough situations which involve real-world[3] applications of mathematics.

The third cause is the result of several factors. It is a fact that most elementary school teachers perceive mathematics to be a static and closed field of study. To them mathematics is more mechanics than ideas, and involves very little independent or original thought. Of course elementary teachers cannot be blamed for their perception of mathematics since it is based primarily on educational experiences which stressed memorization of rules, formulae, and facts. However, the view of mathematics which is held by elementary teachers is a part of a vicious cycle which has developed. Children are not learning to become good problem solvers because their mathematics textbooks do not provide appropriate opportunities for them to solve problems and because their teachers do not view problem solving as important. At the same time, teachers do not view problem solving as important because it was not given priority status when they studied mathematics. This condition cannot be rectified by attempting to convince preservice teachers of the importance of problem solving. At Indiana University preservice elementary school teachers are required to take nine semester hours of mathematics and three semester hours of methods of teaching mathematics. Even this uncommonly good situation does not allow sufficient time to

[3]The term "real-world" is difficult to define since a real-world or real-life problem for one person may not be a real-life problem for another. Although interest rate and grocery shopping problems are very <u>real</u> in the sense that such problems are encountered daily by adults, they are often not even problems for children because children are not interested in them.

overcome ten or more years of "bad" experiences with mathematics. Also, young teachers are prone to model their teaching behavior after the behavior of their supervising teachers. Consequently, if little or no provision is made for developing children's problem solving skills by a student teacher's supervising teacher, it is unlikely that the student teacher will consider problem solving as an important part of the mathematics program.

Remedies for the existing conditions cannot ignore the need to improve current teacher training programs, but improved teacher training is only a small part of the solution. Even if teachers can be trained to view mathematics as an area accessible through experimentation and independent thought, they will probably resort to using whatever written materials are available in the classroom and these materials are, for the most part, not conducive to enhancing the development of problem solving abilities. Thus, serious and extensive efforts must begin to develop exemplary instructional materials in mathematics which have problem solving as their main focus. The Mathematical Problem Solving Project (MPSP), which will be described in the next section, is attempting to satisfy the need for such problem solving materials by producing a series of modules devoted to the development of certain problem solving techniques and by collecting and categorizing problems suitable for use in the intermediate grades.

Attempts to develop instructional materials of any type must involve considerable reflection about the most important aspects of the topic being considered. In the course of developing modules which will teach children fundamental skills and processes of problem solving the following questions are among those which should be considered:

1. What kind and how much direction should be given in a module?
2. What instructional format is best suited to teaching children how to solve problems?

Of course, these are important questions but they are not specifically related to mathematical problem solving. Instead they are questions which are raised by writers of any sort of instructional materials. It is premature to attempt to answer these questions until answers to several more basic questions are found. Unfortunately, the knowledge that exists about how children solve problems and how problem solving should be taught is very limited. For example, no confident answers have been found for the most basic questions such as:

1. What prerequisite skills abilities, etc. must children have to solve particular kinds of problems?
2. What aspects of the problem solving process can be taught to intermediate grade children?
   a. Can children use various problem solving strategies effectively?
   b. Can children learn to coordinate the cognitive processes which are needed in solving complex problems?

Clearly the answers to these questions to a certain extent must be based upon the intuition and experience of the persons involved in writing the materials. However, it is equally as important that these questions be attacked by considering the theoretical and research base underlying the various views toward teaching problem solving. It would be most [illegible] another curriculum project [illegible]ch devotes all its ener[illegible] [illegible]velopment of materials to th[illegible] [illegible]sion of attempting to further the scientific knowledge regarding learning and instruction in mathematical problem solving.[4]

The issues raised thus far have been concerned primarily with the role of problem solving in the existing mathematics curriculum and the development of instructional materials. Before these issues can be dealt with in an appropriate way it is essential that several more fundamental issues and questions be considered. These issues include the four previously mentioned and are listed below with some discussion following.

Fundamental Issues Related to Mathematical Problem Solving

1. Can problem solving be taught?
2. If problem solving can be "taught,"[5] what types of experiences most enhance the development of this ability?
3. What are the specific characteristics of successful problem solvers?
4. What prerequisite skills, abilities, etc. and what level of cognitive development must a student have in order to solve a particular class of problems?
5. Educators and psychologists generally agree that there are several factors which influence problem difficulty. What are the primary

---

[4]This view is also held by Richard Shumway and is presented in a position paper prepared by him for the MPSP (1974).

[5]"Taught" is being used here in the sense that teaching can be viewed as facilitating the understanding of or knowledge about something. It does not imply necessarily direct intervention in the student's learning process.

determinants of mathematical problem difficulty for children in grades 4-6?

6. There [illegible] vation factors which influence children's ability and willingness to solve mathematical problems. For example:

   a. What types of problems are interesting to children in grades 4-6?

   b. To what extent does a child's cognitive and emotional style influence her/his willingness to solve problems?

7. What problem solving strategies can children (grades 4-6) learn to use effectively? More fundamentally, can problem solving strategies be taught which are generalizable to a class of problems?

8. Since problem solving is also important in non-mathematical areas, the question arises concerning the extent to which learning to solve various types of mathematical problems transfers to solving non-mathematical problems (the issue is just as important if modified to read ". . . transfers to solving other types of mathematical problems").

9. There are a number of issues related to the method of instruction. Among the most important are:

   a. Is the small group mode of instruction a better mode than either the large group mode or individual instruction in terms of teaching problem solving?

   b. What aspects of the problem solving process should influence the choice of method of instruction? For example, should the type of problem solving strategy appropriate for a problem affect the instructional mode used?

c. The specific role of the teacher in problem solving instruction is an open issue. Are there certain aspects of the problem solving process which specify a more directive role by the teacher than others?

d. How should problem solving instruction be organized and sequenced? For example, should specific skills (e.g., making tables) be developed before attention is directed toward teaching a particular strategy? To what extent should a Gagnéan hierarchy be followed in planning instruction in problem solving?

10. How do such characteristics of problems as difficulty, interest, setting, strategy, and mathematical content relate to one another?

11. Several models of the problem solving process have been suggested. Do any of these models adequately describe mathematical problem solving? Is there a need for developing a model for instruction in problem solving? An instructional model might be fundamentally different from a model of the solution process.

The MPSP at Indiana University has selected several of these issues and questions for study: namely, nos. 1, 5, 6(a), 7, and 11. Since these questions and issues have been given some careful thought, it is appropriate to discuss them briefly.

Question 1. Can problem solving be taught?

Clearly, this is the most important question of all. Kilpatrick's (1969) review of mathematical problem solving indicates that very little research has been done regarding the influence of instruction on problem solving ability. The answer to this question probably will not be determined until more is known about the nature of solving problems and the relationships among the many factors which influence mathematical problem solving.

Question 5. What are the primary determinants of <u>mathematical</u> problem difficulty for children in grades 4-6?

Psychologists generally focus on four main areas for investigating problem difficulty: (i) type of problem task; (ii) method of presentation of the problem; (iii) familiarity of the problem solver with acceptable solution procedures (strategies, skills, etc.); (iv) problem size (e.g., a problem with several dimensions, both relevant or irrelevant, is more difficult than a problem having fewer dimensions). Each of these areas has direct relevance for elementary school mathematical problem solving. Clearly, not all types of problems are appropriate for children of this age. What is less clear is the best method of presenting particular classes of problems to children. Language factors, complexity of the problem statement, role of concrete and visual materials, child's prior experiences, and type of problem are among the several factors determining the most appropriate method of presentation. Much valuable information could be gained by posing problems to students in different forms and versions and under varying conditions.

That the student's familiarity with acceptable solution procedures is an important determinant of problem difficulty raises a number of questions which must be considered.

-- Which skills and strategies are most important for aiding problem solving in mathematics in grades 4-6?

-- Which skills and strategies should be taught first?

-- Which, if any, strategies do students use naturally?

-- Which skills and strategies can be taught efficiently and effectively? Can any be taught?

-- Should the skills (e.g., making a table) be developed before concentrating on teaching a strategy (e.g., pattern finding) or should they be developed as the strategy is taught?

-- Does teaching a particular strategy really improve problem solving ability in the sense that for any problem a student will be able to choose the most appropriate strategy to use?

More questions are being raised than answers in this paper. This reflects the author's earlier statement that there are few definitive answers to the questions about learning and instruction in mathematical problem solving. The questions posed in the preceding paragraph are no exceptions. However, despite the lack of answers based on firm research evidence, there is considerable agreement that strategies can and should be taught. This claim will be discussed when Question 7 is considered.

Issues related to problem size and problem complexity are a major focus of the research efforts of the MPSP. Since these efforts will be discussed in section IV of this paper no more will be said about problem size in this section.

The four determinants of problem difficulty that have been discussed are certainly not the only ones. Rather, they are the ones to which psychologists have devoted the most attention. Maier (1970) states that there are several other important factors which make a problem difficult. In determining a list of causes of difficulty, he begins with the assumption that there is no lack of knowledge on the student's part. Based upon this assumption he lists five potential causes of difficulty in addition to the four that have already been mentioned:

1. misleading incorrect solutions;
2. type of demands made upon idea getting processes versus idea-evaluation processes;
3. difficulty in locating subgoals that can be reached;
4. lack of motivation;
5. high degree of stress.

The factors which have been listed in the previous paragraphs illustrate the extreme complexity of problem solving. In addition these factors are ones that psychologists have determined primarily through highly controlled experimentation. In many of the "laboratory" studies there is no need to consider factors such as mathematical content, level of understanding of concepts, processes, and skills, and environmental influences since ability to perform the tasks used is not contingent upon these factors. Unfortunately, these factors are present in normal classroom instruction. Consequently, in addition to the determinants of problem difficulty which have already been mentioned, the teacher is confronted with the task of dealing with even more confounding factors in planning appropriate mathematical-problem solving activities.

Question 6 (a). What types of problems are interesting to children in grades 4-6?

This question cannot be answered without considerable knowledge of a student's background, experiences, cognitive ability, and psychological makeup. There is substantial evidence that learning is enhanced when instruction is meaningful and relevant to the student. It is reasonable to expect that this is also the case in learning to solve problems. There are no hard-and-fast rules for telling if a particular problem is interesting, but there are some general rules-of-thumb which can guide problem selection.

1. Be sure the problem statement (if written) is easy for the student to read.
2. Use personal words and terms in the statement of the problem. Try to make the student feel like he is a part of the problem.
3. Although "real-world" problems are often difficult to find, such problems have a high motivational value (Most of the "interesting" real-world problems are too sophisticated for the level of mathematical understanding which intermediate grade children have).
4. Encourage students to make up their own problems.
5. Do not place the student in a stressful situation. For example, insistence on getting a correct answer in a short period of time is a good way to kill enthusiasm for working a problem.

The MPSP is developing a problem bank for grades 4-6. One of the criteria for selecting a problem for inclusion in the bank is that it be interesting to children. Interest will be determined through extensive interviewing and observing children as they solve problems.

Question 7. What problem solving strategies can children learn to use effectively?

In papers prepared for the MPSP, Greenes (1974) and Seymour (1974) offered specific recommendations regarding skills and strategies which should be taught. Greenes not only listed several strategies which can be taught to children in grades 4-6 but also made suggestions for sequencing problem solving activities. The skills and strategies Greenes identifies include: estimate or guess, simplify, conduct an experiment, make a diagram, make a table, construct a graph, write an equation, search for a pattern, construct a flowchart, partition the decision space, and deductive logic.

Seymour considers such skills as "making a table" and "constructing a graph" as valuable aids to mathematical problem solving but would probably classify such skills as sub-strategies because they are really tools for applying a strategy. The strategies he considers appropriate for the intermediate grades include: analogy, pattern recognition, deduction, trial and error, organized listing, working backwards, combined strategies, and unusual strategies which are unique to a problem.

The belief of mathematics educators like Greenes, Seymour and Polya that strategies can be taught should be given serious consideration. Most of our knowledge about learning and instruction is based on the experiences of teachers who have thought long and hard about ways to help children learn. Although little research has been done on the effectiveness of teaching problem solving strategies, the fact that several master teachers are convinced of the feasibility of teaching children the use of certain strategies should encourage teachers who are planning to include problem solving as a part of their mathematics program.

Question 11. Do any of the models of the problem solving process adequately describe mathematical problem solving?

The primary purpose of a model is to describe the salient and essential characteristics of the process or phenomenon which is being modeled. Any model of the problem solving process should be evaluated on the basis of the extent to which it not only identifies the essential aspects of the process but also the extent to which stages and relationships among those stages are identified.

An investigation of this question has evoked considerable inquiry within the MPSP and it is a major theme of this paper. A discussion of models of mathematical problem solving is included in section IV of this paper.

The Nature of the Mathematical Problem Solving Project

The Mathematical Problem Solving Project (MPSP), which is co-sponsored by the National Council of Teachers of Mathematics and the Mathematics Education Development Center at Indiana University and funded by the National Science Foundation, is working toward the development of mathematical problem solving modules which can be inserted into existing curriculum of grades 4-6. Many types of problem situations will be included in these modules: real-world applications of mathematics (i.e., "real-world" as the student sees it), problems related to the mathematics studied in the standard curriculum, mathematical recreations, and problems involving various strategies such as guess and test and pattern finding. While the MPSP is primarily a development project the materials being developed will be based upon research into the teaching and learning of problem solving and will be pilot tested in a number of elementary schools.

The project is in operation at three different centers: the University of Northern Iowa, the Oakland Schools (Pontiac, Michigan), and Indiana University. While the project has identified the central goal as being the development of problem solving modules for use in grades 4-6, each center plays a distinct role.

Role of the University of Northern Iowa (U.N.I.)[6]:

The MPSP site at the University of Northern Iowa is directed by George Immerzeel. The primary role of the site is to develop a series

[6]This description summarizes the role of U.N.I. as reported by George Immerzeel and his associates.

of "skills"[7] booklets and associated problem solving experiences. Specifically, the center at U.N.I. is identifying the spectrum of required skills that are not part of the present curriculum, and writing materials that build this spectrum for particular problem solving strategies.

After considering an extensive list of required problem solving skills and classifying these skills into those that are simple (require a limited set of tactics) and complex (requiring a variety of tactics), seven were identified as appropriate for students in grades 4 through 6.

1. Using an equation
2. Using a table
3. Using resources (reading, formulas, dictionaries, encyclopedias)
4. Using a model (physical model, graph, picture, diagram)
5. Make a simpler problem
6. Guess and test
7. Compute to solve

Each of these skills is simple in that they involve a single principle tactic. They do not depend upon an interrelation among tactics as is the case in strategies such as pattern finding and goal stacking.[8]

[7] There is a semantics problem in trying to communicate ideas about problem solving. Terms like "skill," "strategy," "heuristic," and "technique" connote different things to different people. The word "skill," as used by the University of Northern Iowa staff, refers to generic problem solving techniques which are needed in order to use a particular strategy. Thus, "making a table" is a skill, whereas "pattern finding" is a strategy.

[8] See Simon and Newell (1972) and Wickelgren (1974) for a description of goal stacking.

A "skills booklet" will be written for each of the seven skills. These booklets will be designed to teach the subskills needed to use a particular skill. For example, for the Guess and Test Skills Booklet, approximately 100 problems were written and the skills necessary to solve the problems were identified. These skills were then incorporated into the booklet.

The skills booklet is written so that a student can use the booklet independent of teacher input and also so the teacher can use the booklet in a regular classroom setting. After completing each booklet the student is given an evaluation that not only determines the student's success in the skills but is a guide to group placement for the problem solving experiences designed for the skills.

The problem solving experiences consist of a set of cards for each skill. These cards represent five levels of difficulty and a variety of interests. Although a majority of the problems are supposed to have a "real world" setting, there are also examples from all aspects of the curriculum. From this set of problems each student should be able to find problems that not only fit her/his interests but also are at a level of difficulty where the student will be challenged but have a reasonable chance for success. Also included in the problem set are problems in which the use of the mini-calculator is appropriate. These problems are identified so the student knows the calculator is suggested for the problem. A separate skills booklet for the mini-calculator will be developed which can be used with any type of problem solving strategy.

As the skills booklets and problem sets are developed, they will be field tested with students in grades 4-6 in the Malcolm Price Laboratory School of the University of Northern Iowa.

Role of the Oakland Schools[9]

David Wells is the director of the Oakland Schools Center. This center is responsible for preparing teachers to field test and help develop materials. The teachers will use their classrooms to field test the materials developed at Oakland, U.N.I. and Indiana University. Thus, the Oakland Schools center operates the major field testing component of the project. Currently there are twelve teachers participating in the field testing. In addition, these twelve teachers participate actively in solving problems, discussing problem difficulty, identifying problem solving strategies, developing problems for use in modules, and contributing to the development of modules.

The participation of classroom teachers is an essential part of the project. It is also essential that these teachers teach in a school system which offers diverse socio-economic groupings of children. The Oakland Schools is ideally suited in this respect since it has approximately 260,000 students and 14,000 teachers and contains industrialized centers, suburban communities, and rural areas.

Role of Indiana University (I.U.)

The Mathematics Education Development Center, under the direction of John LeBlanc, is the third site involved in MPSP. The role of

[9]This description summarizes the role of the Oakland Schools as reported by Stuart Choate, Assistant Director of the Oakland Schools Center.

the I.U. center is twofold. First, it is involved in the development of one or more modules based on information gathered through work with individual and small groups of students. Second, the center has major responsibility for evaluating the materials developed at the other centers and for making suggestions for revision. At the same time the staff of the Mathematics Education Development Center is best qualified among the three centers to conduct developmental research into the questions which will arise inevitably as the modules and problems are being created. To date, research problems have been identified related to problem difficulty and complexity and techniques for observing and interviewing children as they attempt to solve problems. The thrust of the work of the I.U. center will be discussed in more detail in a later section.

The roles of the three centers have been described briefly but the interrelationships among the centers has not been specified. Interaction among the centers is determined on the basis of need for reaction to ideas being investigated and materials being developed. For example, it is expected that materials devised by one center will be reacted to by the other centers. In this respect there is a cyclic pattern of continual development, testing, and evaluation of materials which are produced (see figure 1). Also, all three centers will be involved in identifying researchable issues for close scrutiny by the I.U. center.

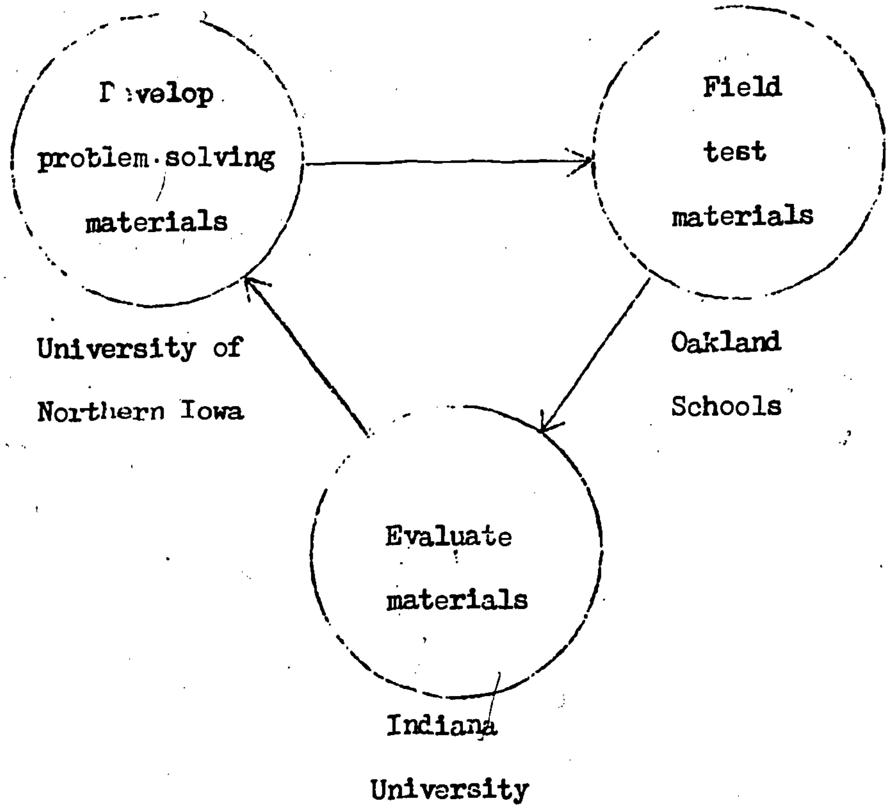

Figure 1: Interrelationship of Primary Roles of the MPSP Centers.

A final word should be said regarding the feasibility of a tri-site project. Such an organizational structure necessitates some confusion, inefficiency, and duplication of efforts that must be taken into account in assessing the project. However, despite these shortcomings the tri-site aspect is viewed as a strength rather than a weakness of the project. The collaboration of educators with different interests, experience and expertise has been proposed by several leading curriculum developers. Having three centers offers a broader base for disseminating the materials which will be developed and provides a wider range of expertise in the areas of teaching, materials development, evaluation, and research.[10]

---

[10]This view was articulated by James Gray who is the N.C.T.M. representative on the MPSP Advisory Board.

Focus of Efforts in the MPSP at Indiana University

This section is devoted to a description of the research and development work at Indiana University during 1974-75. Also the current status of the model of mathematical problem solving which is evolving will be given considerable attention. Although the development of a model has been given tertiary status during the past year, it seems appropriate to present it in this paper in order to ellicit the reader's reactions.

The work of the I.U. center during the past year focused primarily on intensive observation of students' problem solving behavior, the development of a problem bank, and the creation of a problem solving module.

Observation of Fifth Grade Students

In order to get a better feeling for what types of problems students find interesting and to investigate if students employ any discernible strategies as they solve problems, the decision was made to spend some time (approximately 6 weeks) observing fifth grade children as they attempted to solve problems without having any prior instruction. Fifth graders were used because it seemed reasonable to fix the age level of the children so that developmental factors related to age would not have to be dealt with.

Approximately eighty problems were found that were suitable for most fifth graders. The problems were selected on the basis of: relevance to fifth grade mathematics, potential interest for fifth graders, and "non-routineness" (i.e., problems that are not standard textbook "story problem"). Consideration also was given to selecting problems which could be solved in more than one way. Ten of the problems were selected for use in interviewing students.

Two classes of approximately sixty fifth grade students were interviewed as they attempted to solve some of the ten problems. The first class of students was interviewed individually and in groups of two, three, and four as they worked a set of four problems. When groups of students were interviewed, it proved too difficult to identify from audio recordings the processes used by individuals. Thus, all students in the second class were interviewed individually. The findings from the interviews were:

1. Very few of the students wrote anything down. Some drew a figure, but only after it was suggested by the interviewer.
2. Most students had difficulty retaining multiple conditions and considering two or more conditions at the same time.
3. Students often solved a problem that was not the stated problem. They misread the problem or misinterpreted the problem.
4. Students in general did not use strategies, although a few attempted to identify patterns for some problems.

The observation that many students were unable to coordinate multiple conditions in a problem (finding #2) deserves elaboration. One of the problems presented to students was the following:

> There are 5 cups on the table. John has 9 marbles, and he wants to put a different number of marbles under each cup. Can he do this? Explain.

There are three different conditions to coordinate: five cups, nine marbles, and a different number of marbles under each cup (Of course, "John" cannot perform this task). Some students ignored the third requirement and came up with 2, 2, 2, 2, 1 as their answer.

Other students ignored the condition of having nine marbles and arrived at 4, 3, 2, 1, 0 for an answer. It should be pointed out that although many students did not initially coordinate all of the conditions, they were able to do so after rereading the problem or being given a simple clue by the interviewer. It should be added that it is possible that students did not use all of the conditions because they would not have found a way to put the marbles under the cups otherwise. It is likely that they have been conditioned to find an "acceptable" answer at all costs. To them, getting an answer is the most important thing; getting an answer that makes sense is something else. This situation is probably not the fault of the students but the fault of a society which stresses immediate results and values quantity more than quality.

Mini-Instruction of Fifth Grades

The results of the interviews suggested that although the students were unsuccessful for a variety of reasons, they did benefit from the question asked and the hints given by the interviewers. Thus, it seemed feasible to devise short sequences of instructional activities which would focus on helping children in the areas that appeared to cause them the most trouble.

A fifth grade class, different from those interviewed, was divided into four groups (3 groups of 8 Ss and 1 group of 7 Ss). The groups were approximately equal in ability based on the scores from a pretest on mathematical reasoning. Each group was given forty-five minutes of instruction on each of four consecutive days. The instruction varied among groups by what was stressed. The four different instructional stresses were based on the findings from the interviews. They were:

1. Using Strategies - This group worked on using "pattern finding" and "simplification" in solving problems.

2. Coordinating Conditions - This group considered the conditions of the problems and checked that the solution satisfied all of the conditions.

3. Understanding the Problem- This group was given ways to help understand what a problem is asking such as drawing a figure or distinguishing between relevant and irrelevant information.

4. Working Problems - This group was given no particular instruction. The students were given the problems and asked to work them. They were told if they had the solutions right or wrong and given hints when necessary.

Each group was given nearly the same set of problems over the four-day period. These problems were selected because they were appropriate for instruction in each group. At the end of the four-day instructional period a posttest of four problems was given to all the students to see if any change in their problem solving behavior had occurred. In addition, two students from each group were individually interviewed as they worked the posttest.

There was no attempt to compare the groups statistically in terms of problem solving performance. This was not an experimental study to determine which of four instructional techniques was the best, but rather an exploratory investigation of the feasibility of providing instruction in very specific aspects of the problem solving process. At this point the primary interest was to try out ideas in order to gain a narrower focus, not to conduct careful planned and controlled experiments to test well-formed hypotheses.

The results of the mini-instruction were inconclusive. Although the group which received instruction on using strategies seemed to benefit the most from the instruction, the teacher variable may well have been the factor that caused this to happen since each group had a different teacher. In general the extent of the influence of the small group instructional sessions is unclear. However, the insight gained into the behavior of fifth graders in small group problem solving situations was invaluable. Interviewing and observing students as they work on mathematical problems has continued to be a primary activity at the I.U. center.

Development of a Problem Bank and Problem Categorization Scheme

The second major thrust of the I.U. center has been toward the development of a large bank of problems of a wide variety of types. As the size of this bank has grown it has become necessary to determine a scheme for categorizing the problems so that retrieval of problems will be efficient. A substantial effort has been undertaken to devise a suitable categorization scheme. In pursuit of this scheme the purposes of having a problem bank had to be clarified. The purposes of the problem bank are:

1. to provide classroom teachers with a source of problems of various types;
2. to have available a wide range of problems with respect to structure and mathematical complexity, mathematical content, problem setting, strategies used in solving the prob-

lems, interest, etc. for use in development of problem solving materials.

One important use of the problem bank is as a source of problems exemplifying a particular strategy. For example, if a teacher wishes to illustrate the use of the "pattern finding" strategy, he/she can go to the problem bank and choose problems designated as "pattern finding" problems.

In order to categorize the problems in the bank four dimensions were identified: the setting of the problem, the complexity of the problem, strategies applicable for a problem, and the mathematical content of the problem. Initial attempts to sort out the components of each category resulted in the following outline for a categorization scheme.

I. The setting of problems
   A. Verbal setting
      1. simple statement
      2. statement in story form
      3. statement in game form
      4. statement in project form
   B. Auxiliary non-verbal setting (a verbal setting accompanied by non-verbal information or materials which are not essential to solving the problem)
      1. diagram/picture/graph
      2. concrete objects
      3. acting out the problem
      4. hand calculators and other "facilitative" devices
   C. Essential non-verbal setting (non-verbal information or materials essential to solving the problem)
      1. diagram/picture/graph
      2. concrete objects
      3. acting out the problem
      4. hand calculators and other "facilitative" devises.
II. Complexity of problems
   A. Complexity of the problem setting
      1. number of words
      2. number of conditions (numerical and non-numerical)
      3. type of connectives among conditions
      4. familiarity of setting
      5. amount of superfluous information
      6. number of clues provided (verbal and non-verbal)

B. Complexity of the solution process
   1. familiarity with the type of solution
   2. number of questions posed
   3. type of connectives among questions
   4. number of variables
   5. type of connectives among variables
   6. number of different operations required
   7. type of operations required
   8. number of steps required to reach solution

III. Problem solving strategies
   A. Pattern finding
   B. Systematization
   C. Visual perception
   D. Inference
   E. Trial-and-error
   F. Use and/or development of visual aids
   G. Use and/or development of simpler problems
   H. Recall and use of previous experiences.

IV. Mathematical content
   Since the problem bank will be used within the structure of the existing mathematics curriculum, the components of this category should be determined on the basis of topics included in various grade five mathematics textbooks.

Problems which exemplify the use of various strategies have not been difficult to find. Carole Greenes and Dale Seymour have provided the MPSP with large collections of excellent problems which illustrate particular strategies and which are appropriate for use in the intermediate grades. Complexity has proven to be the most challenging category to consider. Several weeks of intensive study resulted in a revision of the outline related to the complexity of problems. The revised outline is presented here without discussion. Work is now underway to determine if factors included in this outline are critical in the determination of problem complexity.

Revised Outline of Complexity of a Problem

I. Complexity of problem statement
   A. Vocabulary
      1. Word frequency
      2. Specialized use

B. Sentence factors (conceptualization of phrases)
   1. Number of Simple Sentences
   2. Average number of words per sentence
   3. Decodability of phrases
C. Amount of information
   1. Numerals and Symbols
   2. Necessary numerical and non-numerical data
   3. Questions asked
D. Interest factor
   1. Number of personal words
   2. Number of concrete non-mathematical words

II. Complexity of the focusing process
   A. Interrelationships of Conditions
      1. Number of bits of irrelevant data
      2. Types of connectives between conditions (an, or, if . . . then)
      3. Order of presentation of the givens and/or operations
      4. Logical structure of the problem
   B. Interrolationships of goals
      1. Leading questions
      2. Corollary questions
      3. Completely disjoint questions
      4. Related questions

III. Complexity of the Solution Process
   A. Unique vs. non-unique vs. no solution
   B. Mathematical content involved
   C. Types of strategies that oould be used effectively
   D. Minimum number of subgoals
   E. Types of goals

IV. Complexity of Evaluation
   A. Ease of checking solution
   B. Ease of generalizing solution

Module Development

The development of instructional materials on pattern finding was begun. Pattern finding was chosen as the focus of the module because the students had an accurate understanding of the word "pattern" and used it in conversation. Also, there is a wealth of problems which involve pattern finding in their solutions. Preliminary versions of parts of the module have been tested in fifth grade classrooms. No formal evaluation of the extent to which students learn to use a pattern finding strategy has been conducted. Instead, the testing has concentrated on readability of the materials, clarity of presentation, format used, and interest level.

Toward a Model of the Problem Solving Process

The attempt to develop a problem solving module on pattern finding and determine a scheme for categorizing mathematical problems necessitated a careful examination of the behaviors, both affective and cognitive, which are demonstrated as a student tries to solve a problem. This analysis involved an attempt to determine a model of the problem solving process which emphasizes the most important components of the process and provides an accurate description of how successful problem solvers think. A search of the literature on problem solving revealed that several attempts have been made to devise a model which describes problem solving. It was appropriate to study some of these models in order to create a model which approximates the process for solving mathematical problems.

Dewey's Model of Reflective Thinking

In his classic book, How We Think, Dewey proposes five phases of reflective thought (Dewey, 1933). While reflective thought is not synonomous with problem solving, it is clear that reflective thought is an essential part of problem solving.

The five phases are:

1. Suggestion: direct action upon a situation is inhibited thereby causing conscious awareness of being "in a hole" (p. 107);
2. Intellectualization: an intellectualization of the felt difficulty leading to a definition of the problem;
3. Hypothesizing: various hypotheses are identified to begin and guide observations in the collection of factual material;

4. Reasoning: each hypothesis is mentally elaborated upon through reasoning:
5. Testing the hypothesis by action: ". . . some kind of testing by overt action to give experimental corroboration, or verification, of the conjectural idea (pp. 113-4)."

Dewey is careful to point out that these phases do not necessarily follow one another in any set order. This analysis is valuable in identifying stages in reflective thinking and thus, in problem solving. However, it considers only the logical aspects of reflective thought but does not consider nonlogical "playfulness" or intuition. It has been suggested that Dewey's formal steps are more a statement of one type of scientific method than an accurate description of how people think (Getzels, 1964). As a result, this model of the process of solving problems may describe how students ought to think, but it does not describe how students usually do think when they are solving problems.

Johnson's Model of Problem Solving

Whereas Dewey's model reflects a logical analysis of problem solving, Johnson (1955) has provided an analysis which is oriented to the psychological processes related to problem solving. Johnson's model is of particular interest because it provides a framework in which ". . . to interpret measures of problem difficulty such as solution time (Bourne et al, 1971, p. 56)." Three stages are included in his model:

1. Preparation and orientation: the student gets an idea of what the problem involves.

2. Production: the consideration of alternative approaches to a solution and the subsequent generation of possible solutions.

3. Judgment: the determination of the adequacy of a solution and the validity of the approach used to arrive at the solution.

In addition to providing information about problem difficulty this model offers a dimension that is not present in Dewey's model--it leads to speculation about the effects of instruction. In Johnson's model pre-production activity by the problem solver is just as important as the production stage. Unfortunately, little is known about the preparation stage because researchers have preferred to investigate problem situations which are well-defined for the student. Thus, the preparation stage plays a less important role. The suggestions for future research efforts which appears in the final section of this paper include a plea for studies which focus on the preparation stage of problem solving by examining problems for which the student is not fully prepared.

Polya's Model of Problem Solving

Georg Polya's extensive writings have been a source of much valuable information regarding the problem of teaching problem solving in mathematics (Polya, 1957; 1962). Unlike Dewey and Johnson, Polya's concern lies primarily with mathematical problem solving. To him, teaching problem solving involves considerable experience in solving problems and serious study of the solution process. The teacher who wants to enhance her/his students' ability to solve problems must direct their attention to certain key questions and suggestions which correspond to the mental operations used to solve problems. In order to group these questions in a convenient manner Polya suggests four phases in the solution process.

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

Since Polya's four phases are familiar to most mathematics educators interested in mathematical problem solving, no discussion of his model will be presented here. It should be pointed out that instead of being a description of how successful problem solvers think, his model is a proposal for teaching students how to solve problems. While this model may be valuable as a guide in organizing instruction in problem solving, it is too gross to be of much help in identifying potential areas of difficulty for students or clearly specifying the mental processes involved in successful problem solving.

Webb's Model of Problem Solving

After reviewing the existing literature on mathematical problem solving, Webb (1974) devised a model which is purported to be a synthesis of the various models described in the literature. This model contains three main stages in solving a problem: preparation, production, and evaluation.

1. Preparation: includes defining and understanding the problem; understanding what is unknown, what is given, and what the goals are.
2. Production: includes the search for a path to attain the

goals; recall of principles, facts, and rules from memory; generation of new concepts and rules to be used in solving the problem; development of hypotheses and alternative plans that may lead to one or more goal.

3. Evaluation: includes checking subgoals and the final solution; checking the validity of procedures used during preparation and production.

Webb states that his model ". . . is not a hierarchical model in that preparation always comes before production which always must precede evaluation. This is more a cyclic model (Webb, 1974, p. 4)." This model has proved to be useful to the staff at the Indiana University center of the MPSP as a rudimentary model from which a more detailed and refined model can be developed.

Some Other Models of Problem Solving

In addition to the models proposed by Dewey, Johnson, Polya, and Webb several other thoughtful models have been developed. Representative of these models are those of Klausmeir and Goodwin (1966) and Wallas (1929). Without discussion these models are presented below.

Klausmeir and Goodwin's Model

1. Setting a goal
2. Appraising the situation
3. Trying to attain the goal
4. Confirming or rejecting a solution
5. Reading the goal

Wallas' Model

1. Preparation
2. Incubation (a mulling over period)
3. Illumination (the conception of a solution)
4. Verification

A Working Model of Problem Solving for the MPSP at Indiana University

The primary limitation of each of the models that have been discussed is that they are either prescriptive (viz. Dewey and Polya) or only grossly descriptive (viz. Johnson, Klausmeir and Goodwin, Wallas, and Webb). The prescriptive models suggest techniques to help the student be a better problem solver. The descriptive models may be more valuable in the sense that they identify phases the student goes through during problem solving. The goal of the MPSP is to devise a more detailed and refined descriptive model.

The search for such a model has led to an investigation of information processing approaches to problem solving research. With the possible exception of gestalt psychology, information processing theory seems to be the only psychological theory which has problem solving as a central focus. A primary thrust of information processing theory is to develop a description of specific types of problems that is precise enough to enable an explanation of problem solving behavior in terms of basic cognitive processes. The most complete description of information processing theory has been presented by Newell and Simon (1972). Wickelgren (1974) has attempted to develop an operationalized theory of problem solving by combining elements of information processing theory and the ideas of master teachers like Georg Polya.

The work of Newell and Simon, and Wickelgren has led the author to the model for solving mathematics problems which is described in the paragraphs which follow. This model is, of course, not as refined as it should be nor does it necessarily generalize to all types of successful mathematical problem solving behavior. However, it does pinpoint some critical components of problem solving behavior which are missing in the other models. Six distinct, but not necessarily disjoint, stages are included in this model.

1. Problem Awareness
2. Problem Comprehension
3. Goal Analysis
4. Plan Development
5. Plan Implementation
6. Procedures and Solution Evaluation

It should be emphasized that these stages are not necessarily sequential. In fact it only rarely happens that these stages do occur sequentially and distinctly from each other.

In keeping with an information processing approach to building a model it would be desirable to devise a flow chart that would describe the student's cognitive processes as progress is made from Problem Awareness through Plan and Solution Evaluation. However, since the stages are not hierarchically ordered or even distinct, for most problems it is not possible to devise a completely accurate diagram of the flow of progress during problem solving. Figure 3 is a rough description of the way in which the stages of the model are related.

Stage 1: Problem Awareness

A situation is posed for the student. Before this situation becomes a problem for the student he/she must realize that a difficulty exists. A difficulty must exist in the sense that the student must recognize that the situation cannot be resolved readily. This recognition often follows from initial failure to attain a goal. This view of what constitutes a problem is consistent with Bourne's description of a problem situation as one in which initial attempts fail to accomplish some goal (Bourne et al, 1971). A second component of the awareness stage is the student's willingness to try to solve the problem. If the student either does not recognize a difficulty or is not willing to proceed in trying to solve the problem, it is meaningless to proceed (see Figure 2).

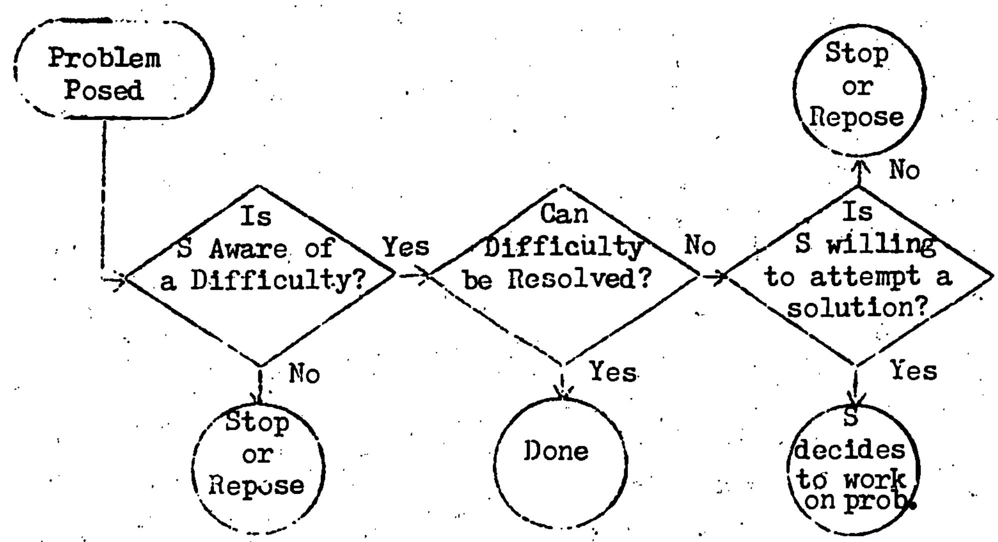

Figure 2: Schematic representation of problem awareness.

Stage 2: Problem Comprehension

Once the student is aware of the problem situation and declares a willingness to eliminate it as a problem, the task of making sense out of the problem begins. This stage involves at least two sub-stages:

translation and internalization. Translation involves interpretation of the information the problem provides into terms which have meaning for the student. Internalization requires that the problem solver sort out the relevant information and determine how this information interrelates. Most importantly, this stage results in the formation of some sort of internal representation of the problem within the problem solver. This representation may not be accurate at first (or it may never be accurate, hence the student fails to solve the problem), but it furnishes the student with a means of establishing goals or priorities for working on the problem. It is here that the non-sequential nature of the model shows up for the first time. The accuracy of the problem solver's internal representation may increase as progress is made toward a solution. Thus, the degree of problem comprehension will be a factor in several stages of the solution process.

Stage 3: Goal Analysis

It seems that the problem solver may jump back and forth from this stage to another. For some problems it is appropriate to establish subgoals, for others subgoals are not needed. It is often true that the identification and subsequent attainment of a subgoal aids both problem comprehension and procedure development.

Goal analysis can be viewed as an attempt to reformulate the problem so that familiar strategies and techniques can be used. It may also involve an identification of the component parts of a problem. It is a process which moves from the goal itself backwards in order to separate the different components of the problem. Thus, goal analysis actually includes more than a simple specification of given information, specification of the relationships among the information, and specification of the operations which may be needed (see Resnick and Glaser, 1975, for a more detailed discussion of goal analysis).

Stage 4: Plan Development

It is during this stage that the problem solver gives conscious attention to devising a plan of attack. Developing a plan involves much more than identifying potential strategies (e.g., pattern finding and solving a simpler related problem). It also includes ordering subgoals and specifying the operations which may be used. It is perhaps this stage more than any other that causes difficulty for students. It is common to hear mathematics students proclaim after watching their teacher work a problem: "How did he ever think of that? I never would have thought of that trick." The main sources of difficulty in learning how to formulate a plan of attack emanate from the fact that students are prone to give up if a task cannot be done easily. Of course, if problems can be done too easily, they are not really problems. A good problem causes initial failure which too often results in a refusal to continue. This state of affairs is not the fault of students but rather the fault of teachers who do not recognize that initial failure is a necessary condition for problem solving (Shumway, 1974). It may also be true that students are unable to devise good plans because they have few plans at their disposal. There is preliminary evidence from work done at the Indiana University center of MPSP that many children in grades 4-6 proceed primarily in a trial-and-error fashion until they either find a "solution" that satisfies them or give up. Equipping students of this age with a few well-chosen strategies may facilitate their ability to plan.

Another source of difficulty for students at this stage is in ordering subgoals and specifying the operations to be used. For many students the hardest part of problem solving lies with knowing what to do first and organizing their ideas. Consequently, in addition to teaching students strategies, attention must be given to helping them organize their thinking and planning.

Stage 5: Plan Implementation

At this stage the problem solver tries out a plan which has been devised. The possibility that executive errors may arise confounds the situation at this stage. The student who correctly decides to make a table and look for a pattern may fail to see the pattern due to a simple computation error. Errors of this type probably cannot be eliminated but they can be reduced if instruction on implementing a plan also considers the importance of evaluating the plan while it is being tried. Thus, while stages 5 and 6 are distinct, they are not disjoint. The main dangers of stage 5 are that the problem solver may forget the plan, become confused as the plan is carried out, or be unable to fit together the various parts of the plan. Fitting together the parts of a plan can be a very difficult task in itself. This difficulty may arise from the fact that the best sequencing of steps in the plan or the best ordering of subgoals may not be clear to the problem solver. For some problems the sequencing of subgoals does not matter, while for others it is essential that sub-goals be achieved in a particular order. The reader is referred to Chapter 6 of Wickelgren's book How To Solve Problems for an in-depth analysis of techniques for defining subgoals and using them to solve problems (Wickelgren, 1974).

Stage 6: Procedures and Solution Evaluation

Successful problem solving usually is the result of systematic evaluation of the appropriateness of the decisions made during problem solving and thoughtful examination of the results obtained. The role of evaluation in problem solving goes far beyond simply checking the answer to be sure that it makes sense. Instead it is an ongoing process that begins as soon as the problem solver begins goal analysis and continues long after a solution has been found. Procedure and solution evaluation may be viewed as a process of seeking answers to certain questions as the problem solver works on a problem. Representative of the questions which should be asked at each stage are the following:

A. Problem Comprehension: The problem solver evaluates how well he/she understands what the problem is.

-- What are the relevant and irrelevant data involved in the problem?

-- Do I (problem solver) understand the relationships among the information given?

-- Do I understand the meaning of all the terms that are involved?

B. Coal Analysis: The problem solver categorizes the information into classes like givens, operations, variables, etc., and attempts to identify the structure of the problem.

-- Are there any subgoals which may help me achieve the goal?

-- Can these subgoals be ordered?

-- Is my ordering of subgoals correct?

-- Have I correctly identified the conditions operating in the problem?

C. Plan Development: The problem solver searches for a method of proceeding.

-- Is there more than one way to do this problem?

-- Is there a best way?

-- Have I ever solved a problem like this one before?

-- Will the plan lead to the goal or a subgoal?

D. Plan Implementation: The problem solver tries out a plan.

-- Am I using this strategy correctly?

-- Is the ordering of the steps in my plan appropriate or could I have used a different ordering?

E. Solution Evaluation: The problem solver analyzes the results.

-- Is my solution generalizable?

-- Does my solution satisfy all the conditions of the problem?

-- What have I learned that will help me solve other problems?

The diagram which follows attempts to illustrate the interrelationships that exist among the stages in the model. It also suggests how a student might proceed in solving a problem.

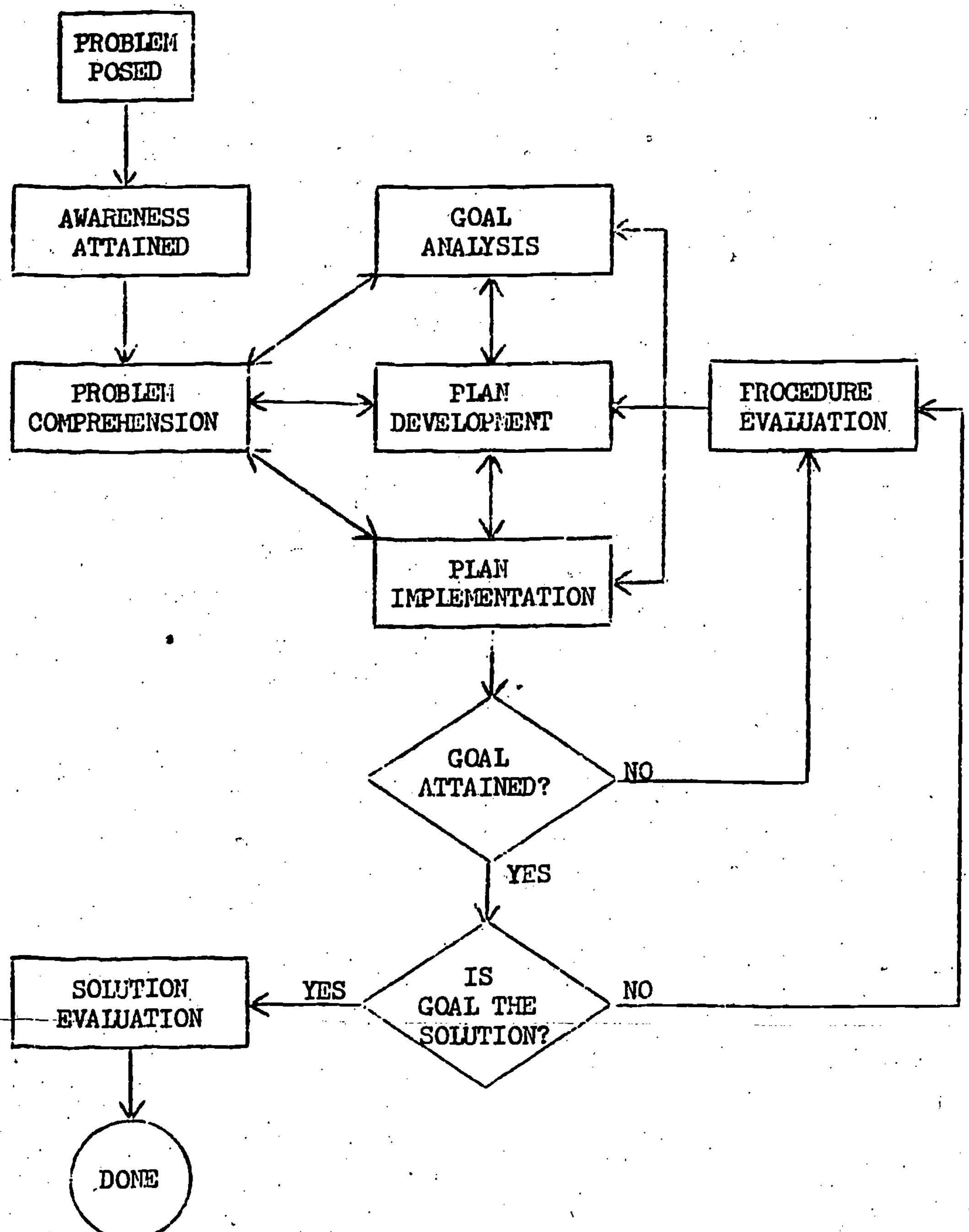

Figure 3: Schematic representation of a model of mathematical problem solving.

How the Model May be Used

The most valuable aspect of this model is that is provides a conceptual framework for identifying the factors which most influence success in problem solving. This framework can be useful to the teacher who is trying to organize appropriate problem solving experiences for students by highlighting various potential sources of difficulty for problem solvers. It also emphasizes that teachers cannot be content to teach students how to solve problems by simply showing a few "tricks of the trade." Of course, the model does not describe problem solving for all types of problems and, in this sense at least, it is incomplete. But, it does supply a partial explication of a theory of problem solving which, although not fully conceptualized, is being created. The development of a theory of problem solving will give direction and add focus to any research efforts. Such a theory is needed critically within mathematics education at the present time. Many of the research efforts in mathematical problem solving which have been conducted were well-conceived and carefully done but the results of these efforts have had little impact on instructional practice. This is partially due to the diversity of types of research and the conflicting results which have been obtained. It is also due to the fact that none of the results seem to be generalizable to all types of mathematical problems. It may be that no single theory, and hence no single model, can accurately depict problem solving for all types of problems and all types of problem solvers. Even with the posibility of such a state of affairs it is worthwhile to continue the search for a suitable model since such a search will provide valuable information about the nature of the problem solving process.

Plans for Future Research

Although the MPSP is primarily a development project, an investigation of a few research questions will be included as a part of the efforts during 1975-76. Much of the work done at I.U. during the past year can be classified as exploratory. Emphasis was placed on intensive observation of students, the collection of problems, the creation of a problem solving module, and the design of a suitable model for mathematical problem solving. While none of these endeavors can be considered research in the usual sense, all of the work at I.U. was conducted with a research spirit. That is, every effort was made to approach each issue in an open-minded and objective manner and to apply the scientific method of inquiry. Perhaps the most valuable result of the work at the I.U. center was the identification of three areas within the problem solving process which cause difficulty for fifth graders. Two of these difficulties are related to problem comprehension, while the third is related to plan development and implementation.

1. Students often misread or misinterpreted problems.
2. Students had difficulty retaining and coordinating multiple conditions in a problem.
3. Students do not appear to use any strategies during problem solving.

Further investigation of the first difficulty suggested that students often perceive a simplified version of a stated problem. The students then proceed to solve the problem as they perceive it. In a few cases, the students were not even aware that a problem existed. In other cases students had trouble understanding phrases in problems (e.g., "a checker in every row and in every column" and "every sixth night"). Clearly, students cannot solve problems they don't fully understand. It is important, then, to pay special attention to the factors which influence problem comprehension. More specifically, it is important to determine the primary determinants of reading difficulty since most mathematical problems are presented in a written form.

Several measures of comprehension of written passages have been developed by reading specialists. However, there is reason to believe that these measures may not be appropriate for written mathematical passages since mathematical English appears to be much different from ordinary English. Kane (1968) suggests that there are at least four differences between mathematical English and ordinary English:

1. Redundancies of letters, word, and syntax are different;
2. Names of mathematical objects usually have a single denotation;
3. Adjectives are more important in mathematical English than in ordinary English;
4. The grammar and syntax of mathematical English are less flexible than in ordinary English.

If mathematical English is significantly different from ordinary English, it is essential that the nature of these differences be determined. Two members of the MPSP staff at I.U., Norman Webb and Barbara Moses, have designed a study which aims at identifying a reliable and accurate measure of comprehension of written mathematics problems. Their study will investigate the following questions:

1. Is the cloze procedure[11] a reliable measure of comprehension for individual mathematical problems?
2. What is the relationship of certain stimulus measures of mathematical problem statements to the mean cloze score percentage?
3. What stimulus measures are the best predictors of mean cloze score percentage?

Stimulus measures will include such variables as the number of: one syllable words per 100, nouns per 100 words, personal words per 100 words, words with specialized mathematical meanings, symbols per 100 words, connectives per 100 words, average sentence length, number of sentences per 100 words, and number of clauses per 100 words.

Webb and Moses expect that one or two stimulus measures will be found that can be used to predict the difficulty of comprehending a mathematical problem. They also expect the cloze procedure to prove to be an adequate measure of readability for mathematical problems. If such expectations are supported the task of classifying problems according to complexity will be greatly reduced.

The fact that many of the fifth graders were unable to coordinate and retain the conditions given in a problem has led to the design of a study to investigate particular issues related to this fact. Another MPSP staff member, Fadia Harik, has decided to explore the influence the number of conditions in a problem has on success in solving problems. In addition she will investigate the effect certain types of teacher

[11]The cloze procedure is a popular technique for measuring readability of long passages. The procedure involves deleting every nth word or symbol of a passage and replacing them with blanks. The student must fill in the blanks. The score is determined by the number of responses matching the deleted material. A high score indicates high readability.

clues has on problem solving success. This aspect of her study arose from the observation that although fifth graders do not initially coordinate multiple conditions simultaneously, they are able to do so in some problems if the teacher provides clues or asks the students to reread the problem.

Research studies like those of Webb and Moses, and Harik have been carefully conceived, organized, and planned. Their questions have risen from a concern for developing a sensible theory of mathematical problem solving. It is only by conducting research based on a sound conceptual framework that any significant progress will be made toward developing instructional materials which will enhance children's ability to solve mathematical problems.

REFERENCES

1. Bourne, L. E. & Dominowski, R. L. Thinking. Annual Review of Psychology, 1972, 105-130.

2. Bourne, L. E., Ekstrand, B. R., & Dominowski, R. L. The Psychology of Thinking. Englewood Cliffs, New Jersey: Prentice-Hall, 1973.

3. Davis, C. A Current Status of research and theory in human problem solving. Psychological Bulletin, 1966, 66, (1), 36-54.

4. Dewey, J. How We Think. Boston: D. C. Heath & Co., 1933.

5. Getzels, J. W. Creative thinking, problem-solving, and instruction. In H. G. Richey (Ed.) Theories of Learning and Instruction, Part I. Chicago: The National Society for the Study of Education, 1964.

6. Green, B. F. Introduction: Current trends in problem-solving. In B. Kleinmuntz (Ed.), Problem Solving: Research, Methods and Theory. New York: John Wiley, 1966.

7. Greenes, C. untitled paper prepared for the Mathematical Problem Solving Project, 1974 (available in mimeograph form only).

8. Henderson, K. B. & Pingry, R. E. Problem-solving in mathematics. In The Learning of Mathematics Its Theory and Practice, Twenty-first Yearbook. Washington, D.C.: The National Council of Teachers of Mathematics, 1953, 228-270.

9. Johnson, D. M. The Psychology of Thought and Judgment. New York: Harper & Row, 1955.

10. Kane, R. B. The readability of mathematics books. Journal of Research in Science Teaching, 1968, 5, 196-298.

11. Kendler, H. H. and Kendler, T. S. Vertical and horizontal processes in problem solving. Psychological Review, 1962, 69, 1-16.

12. Kilpatrick, J. Problem solving and creative behavior in mathematics. In J. W. Wilson and L. R. Carry (Eds.) Reviews of Recent Research in Mathematics Education. Studies in Mathematics Series, vol. 19. Stanford, Calif.: School Mathematics Study Group, 1969, 153-187.

13. Klausmier, H. J. & Goodwin, W. Learning and Human Abilities: Educational Psychology. 2nd ed. New York: Harper & Row, 1966.

14. Maier, N. R. F. Problem Solving and Creativity in Individuals and Groups. Belmont, Calif.: Brooks-Cole, 1970.

15. Maltzman, I. Thinking: From a behavioristic point of view. Psychological Review, 1955, 66, 367-386.

16. Newell, A. & Simon, H. A. Human Problem Solving. Englewood Cliffs, N. J.: Prentice-Hall, 1972.

17. Polya, G. How to Solve It. 2nd ed. Garden City, N. Y.: Doubleday, 1957.

18. Polya, G. Mathematical Discovery. vol. 1. New York: John Wiley, 1962.

19. Resnick, L. B. & Glaser, R. Problem solving and intelligence. In L. B. Resnick (ed.) Nature of Intelligence (in press).

20. Seymour, D. Considerations for a mathematics problem solving curriculum for the intermediate grades. Paper prepared for the Mathematical Problem Solving Project, 1974 (available in mimeograph form only).

21. Shumway, R. J. untitled paper prepared for the Mathematical Problem Solving Project, 1974 (available in mimeograph form only).

22. Wallas, G. The Art of Thought. New York: Harcourt, Brace, 1926.

23. Webb, N. A review of the literature related to problem-solving tasks and problem-solving strategies used by students in grades 4, 5 and 6. Progress report prepared for the Advisory Board of the Mathematical Problem Solving Project, 1974 (available in mimeograph form only).

24. Wickelgren, W. A. How to Solve Problems. San Francisco: W. H. Freeman, 1974.