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TRANSFORMING A FAMILY OF CURVES INTO A FAMILY OF CURVES WITH A SINGLE SHAPE

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Abstract

The relatively hard problem of transforming a given set of curves to curves with the same shape can sometimes be reduced to the easier problem of rendering curves parallel. In this paper a group is associated with the given curves, and it is shown that the reduction from the hard problem to the easy problem is valid whenever the group is nonabelian. A comprehensive review of earlier work is included so that this paper can be read alone. Applications to psychological measurement are cited briefly.

Transforming a Family of Curves into a Family of Curves with a Single Shape

I. Introduction

A problem of considerable importance in psychological theories and numerous applications is this: Given the graphs of several continuous, strictly increasing functions, find (if possible) a transformation of the independent variable that carries the graphs into curves having the "same shape." Curves have the same shape when they differ in location and scale only, i.e., when any two can be superimposed by a horizontal translation and change of scale. Examples are discussed in my earlier papers, especially Levine [1972; 1973, appendix] and Levine and Saxe [1976].

It turns out that some of the most interesting and important aspects of the problem remain when a mathematically idealized version of the problem is considered. Furthermore, a recent successful application of related work [Levine, 1975; Levine and Saxe, 1976] indicates that the difficulties which have been removed in the idealized version are quite tractable in practice.

We consider collections of strictly increasing real functions. The familiar notion of "same shape" is formalized, along with the important auxiliary notion of "parallel" and of what it means to transform the x -axis by an increasing homeomorphism so as to make a family of curves the same shape or parallel. At this point it becomes possible to introduce group theory and obtain a fruitful algebraic reformulation of the original geometric problem. A procedure is specified for associating a group with a family of functions, and the original problem is translated into a problem about the group.

It was shown in an earlier paper [Levine, 1972, see also Theorem 2 of this paper] that <u>if</u> the associated group has certain special properties, then the problem of transforming given functions into functions with the same shape, can be reduced to another problem, namely transforming a related family of functions into parallel functions.

This result has important practical implications, for it is far simpler to transform to parallel functions than to transform to functions with the same shape. For parallel functions there is an extensive literature containing practical procedures of proven merit, while for the original problem, there is no such literature.

The reduction to parallel functions raises a mathematical question that is answered in this paper: How restrictive are the special assumptions made about the associated group in order to obtain the reduction? The answer, given in Theorem 4 below, is that the assumptions can generally be ignored; the reduction can be used whenever the associated group is nonabelian. The proof involves a theorem (Theorem 5) about groups of real functions which may be of independent interest.

II. Notation and Definitions

When the domain of function F is contained in the range of G, the composition of F and G, $x \to F[G(x)]$, will be denoted by FG. When F is 1 - 1 and onto its range, the inverse function G satisfying $x = GF(x)$ will be denoted by F^{-1}.

Frequent use will be made of the fact that composition is a group operation in the set of all increasing real homeomorphisms. All of the groups of functions considered in this paper are subgroups of the group of all increasing homeomorphisms.

The following notation conventions will be used. F, G, H are used to denote functions in a given set of functions $\{F\}$ or $\underset{\sim}{F}$. f, g, h are generally used to denote elements in a subgroup $\underset{\sim}{G}$ or $\underset{\sim}{H}$ of the group of all increasing homeomorphisms. The symbol e will be reserved for the identity homeomorphism $x \to x$.

Two increasing homeomorphisms f, g are <u>conjugate</u> if for some increasing homeomorphism u, $f = u^{-1}gu$. Two subgroups of homeomorphism $\underset{\sim}{G}, \underset{\sim}{H}$ are conjugate if for some increasing homeomorphism u, the mapping $g \to u^{-1}gu$ maps $\underset{\sim}{G}$ onto $\underset{\sim}{H}$.

Two subgroups that play an important role in this paper are

1. the group of all translations, where a <u>translation</u> is a mapping of form $x \to \ell(x) = x + b$;

2. the group of all (increasing) affine functions, where an affine function, is a mapping of form $x \to \ell(x) = ax + b$ where a is positive.

Suppose $F: \mathbb{R} \to \mathbb{R}$ is a real function, and $u: \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Geometrically, transforming F by u means applying u to the x coordinate of every point in the graph of F and changing the graph from the curve $\{ \langle x, F(x) \rangle : x \text{ is real}\}$ to the curve $\{\langle u(x), F(x) \rangle\}$. While this definition is strongly motivated by a geometric approach in which we visualize the actual curves, it turns out that it is easier to prove theorems algebraically. Consequently, the <u>transformation of</u> F <u>by</u> u is defined to be the function Fu^{-1}. This definition expresses the geometric notion of transformation because the curve $\{\langle u(x), F(x) \rangle\}$ is the same as the curve $\{\langle x, Fu^{-1}(x) \rangle\}$.

Two real functions F and G are said to be <u>parallel</u> if there is a translation $\ell(x) = x + b$ that transforms one onto the other, so that $F\ell^{-1}(x) = G(x)$ for all real x. Equivalently, $F = G\ell$.

Geometrically this means that the shift along the x -axis $\langle x,y\rangle \to \langle x+b,y\rangle$ carries the graph of F onto the graph of G . This merely formalizes the idea that F and G differ only by a change of origin.

F and G have the same shape if some affine ℓ transforms F into G and $F\ell^{-1} = G$; i.e., $F = G\ell$. Geometrically, this means a shift along the x -axis followed by a dilation of the x -axis carries the graph of F onto the graph of G and this formalizes the notion that F and G differ only by a change of origin and scale.

It is easy to see that "having the same shape" and "being parallel" are each equivalence relationships, so that we may talk of a family of curves having the same shape or being parallel. Thus a <u>family or set of curves</u> $\{F\}$ <u>has the same shape</u> if every pair of curves in $\{F\}$ has the same shape.

Let $\{F\}$ be a (finite or infinite) set of strictly increasing real-valued functions of a real variable. A homeomorphism u is said to <u>transform</u> $\{F\}$ <u>into functions with the same shape if</u>

(a) the family $\{Fu^{-1}\}$ has the same shape.

This formalizes the idea that a <u>single</u> homeomorphism transforms all the curves into curves which differ only by change of origin and change of scale. It is easy to show that (a) is equivalent to

> (b) for every pair F and G in $\{F\}$, there is an affine transformation $\ell(x) = ax + b$ such that $F = Gu^{-1}\ell u$.

Analogously, u transforms $\{F\}$ into <u>parallel functions</u> if

> (a') the family $\{Fu^{-1}\}$ are parallel,

or equivalently if

> (b') for every pair F and G in F, there is a translation $\ell(x) = x + b$ such that $F = Gu^{-1}\ell u$.

A well-developed theory exists for functions transformable to parallel functions, and several different procedures are available to find u given $\{F\}$ or to approximate u from approximations of the functions of $\{F\}$. For a review of some of the mathematical literature on parallel functions and some recent results, see Levine [1970] and Levine [1975]. For a recent application to mental test data, see Levine and Saxe, 1976]. Additional areas of application are discussed in Levine [1970, 1972]. Data analysis procedures of proven merit (which were developed for other purposes but are clearly relevant to parallel functions) are discussed in Box and Cox [1964] and Kruskal [1965].

III: Review of Results on Relations between Families and Groups

Note that if the functions of $\underset{\sim}{F}$ can be transformed to the same shape, the range of any function of $\underset{\sim}{F}$ is equal to the range of any other. This is geometrically obvious and an immediate consequence of (b). Consequently we may, without loss of generality, restrict attention to families $\underset{\sim}{F}$ of strictly increasing, continuous functions mapping the reals onto a common range.

Let $\underset{\sim}{F}$ be a non-empty set of strictly increasing continuous functions mapping the reals onto a common range. The <u>associated group</u> $\underset{\sim}{G}(\underset{\sim}{F})$ is the group of real homeomorphisms generated by the set of functions

$$\{F^{-1}G: \ F,G \text{ is in } \underset{\sim}{F}\}$$

with group operation given by function composition.

Since $(F^{-1}G)^{-1}$ is $G^{-1}F$, $\underset{\sim}{G}(\underset{\sim}{F})$ is simply the set of functions of form

$$f = F_1^{-1}F_2F_3^{-1}F_4 \ \cdots \ F_{n-1}^{-1}F_n$$

where each F_i is $\underset{\sim}{F}$.

Associated groups are often easier to study than families of functions, and the properties of families of central interest in this paper can easily be translated into properties of groups. For example,

Proposition 1: Let $\underset{\sim}{F}$ *be a set of increasing continuous functions and* u *a real homeomorphism. Then* u *transforms* $\underset{\sim}{F}$ *to functions with the same shape iff* u *transforms* $\underset{\sim}{G}(\underset{\sim}{F})$ *to functions with the same shape.*

Proof: If for each F, G in $\underset{\sim}{F}$ is some affine ℓ such that $Fu^{-1}\ell^{-1} = Gu^{-1}$, then $G^{-1}F = u^{-1}\ell u$. Thus the generators of $\underset{\sim}{G}(\underset{\sim}{F})$ have the form $u^{-1}\ell u$. Since composites and inverses of functions having this form also have this form, g_1, g_2 in $\underset{\sim}{G}(\underset{\sim}{F})$ implies that for some affine ℓ_1, ℓ_2,

$$g_1 = u^{-1}\ell_1 u \quad ;$$

$$g_2 = u^{-1}\ell_2 u \quad .$$

Thus $g_1u^{-1} = u^{-1}\ell_1 = u^{-1}\ell_2(\ell_1^{-1}\ell_2)^{-1}$ has the same shape as $g_2u^{-1} = u^{-1}\ell_2$ and u transforms $\underset{\sim}{G}(\underset{\sim}{F})$ to the same shape.

Conversely, let F, G be arbitrary elements of $\underset{\sim}{F}$. If u transforms $\underset{\sim}{G}(\underset{\sim}{F})$ to the same shape, then for each f, g in $\underset{\sim}{G}$ there is some affine ℓ such that $fu^{-1} = gu^{-1}\ell$. In particular, $f = e$ and $g = F^{-1}G$ are in $\underset{\sim}{G}$, so for some ℓ, $u^{-1} = F^{-1}Gu^{-1}\ell$, i.e., $Fu^{-1} = Gu^{-1}\ell$.

There is an analogue of Proposition 1 for parallel functions.

Proposition 2: *Let $\underset{\sim}{F}$ be a set of increasing continuous functions and u an increasing homeomorphism. Then u transforms $\underset{\sim}{F}$ to parallel functions iff u transforms $\underset{\sim}{G}(\underset{\sim}{F})$ to parallel functions.*

Proof: Same as proof for Proposition 1, except ℓ, ℓ_1, ℓ_2 are translations rather than affine.

The question of the existence of a transforming homeomorphism u can be concisely expressed in group theoretical terms as follows.

Proposition 3: *Let $\underset{\sim}{F}$ be a set of strictly increasing real functions with common range. There exists an increasing homeomorphism transforming $\underset{\sim}{F}$ to the same shape iff $\underset{\sim}{G}(\underset{\sim}{F})$ is conjugate to a subgroup* of the affine group. There exists an increasing homeomorphism transforming $\underset{\sim}{F}$ to parallel functions iff $\underset{\sim}{G}(\underset{\sim}{F})$ is conjugate to a subgroup of the translations.

Proof: $Fu^{-1} = Gu^{-1}\ell$ iff $G^{-1}F = u^{-1}\ell u$.

Associated groups can be used to study the uniqueness of homeomorphisms rendering functions parallel. In a sense made precise in Theorem 1 below, such homeomorphisms may be unique except for a change of origin and scale. This result is needed for reducing questions about functions with the same shape to questions about parallel functions. The proof uses the following well-known, easily proven facts about the additive group of real numbers.

1. If $\underset{\sim}{G},\underset{\sim}{H}$ are two subgroups of additive reals and ϕ : $\underset{\sim}{G} \rightarrow \underset{\sim}{H}$ is a strictly increasing homomorphism, then for some positive number a , $\phi(g) = ag$ for all g in $\underset{\sim}{G}$. More concisely, an additive, increasing function defined on a subgroup of the additive reals is linear.

2. A subgroup of the additive reals either has exactly one element, has a least positive element, or is dense in the reals.

In the proof and throughout the remainder of the paper, "cyclic group" includes infinite cyclic groups such as the integers.

<u>Theorem 1</u>: (<u>Uniqueness Theorem</u>) <u>If</u> u <u>and</u> v <u>both transform</u> $\underset{\sim}{F}$ <u>into parallel curves and the associated group</u> $\underset{\sim}{G} = \underset{\sim}{G}(\underset{\sim}{F})$ <u>is not the trivial group</u> $\{e\}$ <u>or cyclic, then for some affine transformation</u> ℓ, $v = \ell u$.

<u>Proof</u>: For each g in $\underset{\sim}{G}$ there is some translation $\ell = \ell_g$ such that $g = u^{-1}\ell u = u^{-1}[u(\cdot) + b_g]$ and $ug(x) = u(x) + b_g$, for all real x.

In particular, for $x = u^{-1}(0)$ we have $ugu^{-1}(0) = b_g$. Thus $g \to \phi(g) = ugu^{-1}(0)$ defines a function from $\underset{\sim}{G}$ into $\mathbb{R}$. Since $gh = u^{-1}\ell_g uu^{-1}\ell_h u = u^{-1}\ell_{gh}u$, ϕ is a homomorphism into the additive group of real numbers. Similarly $g \to vgv^{-1}(0) = \psi(g)$ also defines a homomorphism. Since

$$\phi(g) > \phi(h) \text{ iff } g[u^{-1}(0)] > h[u^{-1}(0)]$$

and

$$\psi(g) > \psi(h) \text{ iff } g[v^{-1}(0)] > h[v^{-1}(0)] \quad ,$$

ϕ and ψ are in fact increasing isomorphisms and $\psi\phi^{-1}$ is an increasing isomorphism of one subgroup of the additive reals onto another. Consequently for some positive a , $\psi(g) = \psi\phi^{-1}[\phi(g)] = a\phi(g)$. Thus for each g in $\underset{\sim}{G}$,

$$\begin{aligned} v[g(0)] &= v(0) + \psi(g) \\ &= v(0) + a\phi(g) \\ &= v(0) + a[ug(0) - u(0)] \\ &= au[g(0)] + [v(0) - au(0)] \quad . \end{aligned}$$

Thus for $x = g(0)$ we have $v(x) = au(x) + b$. Since u and v are continuous, to verify this equation for all real x , it is sufficient to show the orbit of 0 , $\{g(0): g\epsilon\underset{\sim}{G}\}$, is dense in the reals. Since

G is not cyclic, its isomorphic image $\phi(G)$ cannot have a least positive element and thus is dense. Since u^{-1} is a homeomorphism and $g(0) = u^{-1}[u(0) + \phi(g)]$, the orbit is also dense in the reals and

$$v(x) = au(x) + [v(0) - au(0)]$$

for all real x.

A necessary condition for a set of curves with associated group G to be transformable to curves with the same shape can be obtained by studying the derived group of G, i.e., the subgroup* G' of G generated by the set of functions $\{ghg^{-1}h^{-1}: g \text{ and } h \text{ are in } G\}$. The proofs that follow use the characterization of the derived group for an arbitrary group G as the intersection of all normal subgroups X such that the quotient G/X is abelian.

Theorem 2 (Levine, 1972): If u transforms F into functions with the same shape, then u also transforms the derived subgroup of $G(F)$ into parallel functions.

<u>Proof</u>: If u transforms $\underset{\sim}{F}$ to functions with the same shape, then for g_1, g_2 in $\underset{\sim}{G}(\underset{\sim}{F})$ there are affine ℓ_1, ℓ_2 such that $g_i = u^{-1}\ell_i u$. If $\ell_i(x) = a_i x + b_i$ then $\ell_1\ell_2(x) = a_1 a_2(x) + a_1 b_2 + b_1$. Consequently the mapping ϕ given by

$$g = u^{-1}[au(\cdot) + b] \rightarrow a = ugu^{-1}(1) - ugu^{-1}(0) = \phi(g)$$

is a homomorphism of $\underset{\sim}{G}(\underset{\sim}{F})$ into the group of positive real numbers with multiplication as group operation. Since the multiplicative reals are abelian, $\underset{\sim}{G}(\underset{\sim}{F})/\phi^{-1}(1)$ is also abelian, and every element of the derived group is in the kernel of ϕ. Thus every function in the derived group has form $u^{-1}[u(\cdot) + b]$. Thus the derived group is conjugate to a subgroup of the translation, and u transforms the derived group to parallel functions.

If a non-elementary assumption is made about the associated group, this result can be strengthened by a simple application of the uniqueness theorem, as proven below and in an earlier paper. The goal of this paper is to dispose of this assumption.

Theorem 3: (Levine, 1972). If $\underset{\sim}{G}(\underset{\sim}{F})$ is nonabelian and its derived group $\underset{\sim}{G}'$ is not cyclic then

u transforms G' into parallel functions

if and only if

u transforms $\underset{\sim}{F}$ into functions with the same shape.

Proof: Theorem 2 gives the backward implication. To prove the forward implication let g be an arbitrary element of $\underset{\sim}{G}'$. Then

$$g(\cdot) = u^{-1}[u(\cdot) + b]$$

for some b. Let F,G be arbitrary functions $\underset{\sim}{F}$. Since G' is normal and $f = F^{-1}G$ is in $G(\underset{\sim}{F})$, fgf^{-1} is also in G'. Thus, for some d, $fgf^{-1}(\cdot) = u^{-1}[u(\cdot) + d]$; i.e., $g(\cdot) = (uf)^{-1}[uf(\cdot) + d]$, and $v = uf$ also transforms $\underset{\sim}{G}'$ into parallel curves. Thus, by Theorem 1, for some affine ℓ

$$v = uF^{-1}G = \ell u .$$

But this is equivalent to $Gu^{-1} = Fu^{-1}\ell$, and Fu^{-1} has the same shape as Gu^{-1}.

Before proceeding it may be worth emphasizing two facts about the significance of the correspondence between groups and families.

First of all, " u transforms $\mathbf{F}$ to functions of the same shape" implies that the functions in $\mathbf{G}(\mathbf{F})$ have the form $u^{-1}\ell u$. This would seem to suggest that working with the group necessitates consideration of geometric transformations of both abscissa and ordinate of form $\langle x,y\rangle \to \langle u(x),u(y)\rangle$ rather than the conceptually simpler transformations of the x-axis alone, $\langle x,y\rangle \to \langle u(x),y\rangle$. Fortunately, as Proposition 1 shows, that this conclusion is incorrect and transforming the associated group of a family can be handled in the same way as transforming the family.

A second consideration that might appear to discourage the use of the associated group, especially in applications, is that the group $\mathbf{G}(\mathbf{F})$ is generally infinite whereas the family $\mathbf{F}$ may have only finitely many curves. Fortunately, it is generally possible to select two curves f, g in $\underset{\sim}{G}(\underset{\sim}{F})$ that are equivalent to $\underset{\sim}{F}$, in the following sense:

> u transforms $\underset{\sim}{F}$ to parallel functions iff u transforms $\{f, g\}$ to parallel functions [Levine, 1970, Section V.4].

In view of Theorem 3 of this section and Theorem 4 of the next section this means we may be able to specify two functions f, g of the derived group such that u transforms $\underset{\sim}{F}$ to the same shape iff u transforms $\{f, g\}$ to parallel functions. For further discussion of the extent to which a group can be considered to be equivalent to a small number of its curves see Levine [1972, Section IV].

IV: Strengthening Theorem 3

The purpose of this section is to strengthen Theorem 3 by removing the reference to cyclic derived groups to obtain the following result.

Theorem 4: *If* $\underset{\sim}{G}(\underset{\sim}{F})$ *is nonabelian, then* u *transforms the derived group* $\underset{\sim}{G}(\underset{\sim}{F})'$ *into parallel functions if and only if* u *transforms* $\underset{\sim}{F}$ *into functions with the same shape.*

The restriction to nonabelian groups is acceptable because curves with an abelian group can either be transformed to parallel curves or are of a special type. A well-developed theory for parallel curves is available in the references given above, and the special other type of abelian group has been treated in detail in Levine [1972, section V].

Theorem 4 follows at once from Theorem 3 and the following fact about groups of real homeomorphisms.

Theorem 5: *If* $\underset{\sim}{G}$ *is a nonabelian group of increasing real homeomorphisms and its derived group* $\underset{\sim}{G}'$ *is conjugate to a subgroup of the translations, then* $\underset{\sim}{G}'$ *is not cyclic.*

The proof frequently uses the easily verified fact that conjugates have the same fixed point structure. More specifically, if two increasing homeomorphisms are conjugate, then one has a fixed point if and only if the other does. In particular, if f is conjugate to a translation and has fixed points, then f is the identity e.

We also use the well-known fact that every fully ordered, Archimedean group is abelian [Kurosh, 1965, p. 287].

<u>Proof</u>: Suppose $\underset{\sim}{G}$ is nonabelian with cyclic derived group $\underset{\sim}{G}'$ generated by some function $p \neq e$ so that

$$G' = \{p^n : \ n \text{ is an integer}\} .$$

A contradiction will be obtained by showing $\underset{\sim}{G}' = \{e\}$, i.e., $\underset{\sim}{G}$ is abelian.

As conjugates of translations, the elements of $\underset{\sim}{G}'$ are continuous functions without fixed points. Consequently, if g is in $\underset{\sim}{G}'$ and $g \neq e$, then either

$$x < g(x) \quad \text{for all real } x$$

or

$$g(x) < x \quad \text{for all real } x \quad .$$

In the former case g will be called <u>positive</u>, in the latter, <u>negative</u>.

Since the functions of $\underset{\sim}{G}'$ other than e are either positive or negative, it is natural to introduce an order relation in $\underset{\sim}{G}'$. Let $\leq$ be defined by

$$f \leq g \text{ iff} \begin{cases} f = g \\ f(x) < g(x) \quad \text{for all real } x \end{cases} ,$$

i.e., $f \leq g$ iff $f = g$ or $f^{-1}g$ is positive.

It is routine to verify that $(\underset{\sim}{G}', \leq)$ is a fully ordered group. That is, in terminology needed later,

i. $\leq$ is a partial order relation in the set of elements of $\underset{\sim}{G}'$,

ii. $\leq$ is a connected relation, i.e., for all $f,g \in \underset{\sim}{G}'$

$$f \leq g \quad \text{or} \quad g \leq f$$

iii. $(\underset{\sim}{G}',\leq)$ is an ordered group, i.e., for all $f,g,h \in \underset{\sim}{G}'$

$$f \leq g \text{ implies } hf \leq hg \text{ and } fh \leq gh \quad .$$

The generator p of $\underset{\sim}{G}'$ can be assumed, without loss of generality, to be positive. For p and p^{-1} generate the same group and, using iii, $p \leq e$ implies $p^{-1}p = e \leq p^{-1}e = p^{-1}$.

The remainder of the proof is divided into short, numbered, separately proven parts.

1. <u>The positive generator</u> p <u>is the least positive element of</u> $\underset{\sim}{G}'$.

<u>Proof</u>: $e \leq p$ implies by iii $p^{-n} \leq \ldots \leq p^{-1} \leq e \leq p \leq \ldots p^{n}$ for for each positive integer n . Each g in cyclic $\underset{\sim}{G}'$ is of form $g = p^{m}$ for some integer m . If g is positive then $1 \leq m$ and $p \leq g$.

2. <u>If a real homeomorphism</u> f <u>has no fixed points, then for each real</u> x_o, <u>the set</u> $\{f^n(x_o): n \text{ is an integer}\}$ <u>is unbounded above and below. In particular, for each</u> x_o, $\{p^n(x_o)\}$ <u>is unbounded above and below.</u>

<u>Proof</u>: If $f(x_o) > x_o$, then $f(x) > x$ for all real x. Consequently, $f^n(x_o) = f[f^{n-1}(x_o)] > f^{n-1}(x_o)$ and $\{f^n(x_o)\}$ is an increasing sequence. If the sequence were bounded, it would converge. But then by continuity, $\lim_n f^n(x_o) = f[\lim_n f^n(x_o)]$ would be a fixed point of f.

The same argument proves that the sequence is unbounded below.

If $f(x_o) < x_o$ then $f^{-1}(x_o) > x_o$ and the set $\{f^n(x_o): n \text{ is an integer}\} = \{(f^{-1})^n(x_o): n \text{ is an integer}\}$ is unbounded above and below.

3. p <u>is in the center of</u> $\underset{\sim}{G}$.

<u>Proof</u>: Let g be any element of $\underset{\sim}{G}$. For some n , $gpg^{-1}p^{-1}$ equals p^{n-1} . To show p and g commute it clearly suffices to show n equals one. If n exceeds one, then p equals $g^{-1}p^ng = (g^{-1}pg)^n$. Since $g^{-1}pg$ is also in $\underset{\sim}{G}'$ this contradicts the choice of p as least positive element. n is not zero since p is not e . If n is negative, gpg^{-1} equals p^{-m} for $m = -n > 0$. But $p(x) > x$ for all x implies $gpg^{-1}(x) > x$, $p^{-m}(x) > x$, $(p^{-1})^m(x) > x$, $p^{-1}(x) > x$ and $p(x) < x$ for all x . Consequently n is 1 and p commutes with every element of $\underset{\sim}{G}$.

4. <u>If</u> $f \in \underset{\sim}{G}$ <u>has a fixed point, then</u> f <u>is in the center of</u> $\underset{\sim}{G}$.

<u>Proof</u>: Say $f(x_o) = x_o$. Then $fp^n(x_o) = p^nf(x_o) = p^n(x_o)$. Consequently the fixed point set of f is unbounded above and below. Let g be any element of $\underset{\sim}{G}$. Since $\underset{\sim}{G}' = \{p^n\}$, for some m

$fg = p^m gf$ and g equals $p^m f^{-1} gf$. Replacing g by $p^m f^{-1} gf$ in the product $(p^m f^{-1}) g(f)$ n times and simplifying gives $g = p^{nm} f^{-n} g f^n$ for all n. In particular $g(x_0) = p^{nm} f^{-n} g(x_0)$. Let x_1 and x_2 be fixed points of f such that $x_1 \leqq g(x_0) \leqq x_2$. Then for all n

$$x_1 \leqq f^{-n} g(x_0) \leqq x_2 \quad \text{and}$$

$$p^{nm}(x_1) \leqq p^{nm} f^{-n} g(x_0) = g(x_0) \leqq p^{nm}(x_2) \quad .$$

Consequently m is zero and fg equals gf.

5. <u>Let</u> $\underset{\sim}{H}$ <u>denote</u> $\{f \in \underset{\sim}{G}:$ f has a fixed point$\}$. <u>Then</u> $\underset{\sim}{H}$ <u>is a normal subgroup of</u> $\underset{\sim}{G}$.

<u>Proof:</u> Clearly $e \in \underset{\sim}{H}$ and $\underset{\sim}{H}$ is closed under formation of inverses and inner automorphisms. It remains only to show $f \in \underset{\sim}{H}$ and $g \in \underset{\sim}{H}$ implies $fg \in \underset{\sim}{H}$. If fg has no fixed points then $(fg)^n(x)$ diverges to $+\infty$ or $-\infty$ according to whether fg is positive or negative. Let x_0 be a fixed point of g and let x_1, x_2 be fixed points of f such that $x_1 \leqq x_0 \leqq x_2$. Then $(fg)^n(x_0) = f^n g^n(x_0) = f^n(x_0) \in [f^n(x_1), f^n(x_2)] = [x_1, x_2]$. Since this implies the sequence $\{(fg)^n(x_0)\}$ is bounded, fg is in $\underset{\sim}{H}$.

6. Let $\bar{f}$ denote the typical element of $\underset{\sim}{G}/\underset{\sim}{H}$, $\bar{f} = \{fg: g \in \underset{\sim}{H}\}$.

Let $\mathcal{G}f$ denote the graph of f , $\{ \langle x, f(x) \rangle : x \text{ is real}\}$, and

$\mathcal{G}\bar{f} = \bigcup_{g \in \bar{f}} \mathcal{G}g$. Then each $\mathcal{G}\bar{f}$ is a pathwise connected subset of the plane.

Proof: Let (x_1, y_1) and (x_2, y_2) be points of $\mathcal{G}\bar{f}$. Then there are f_1, f_2 in $\bar{f}$ such that $f_i(x_i) = y_i$. Since $f_1^{-1} f_2$ is in $\underset{\sim}{H}$, it has a fixed point x_o satisfying $f_1(x_o) = f_2(x_o)$. There is a path connecting (x_1, y_1) and (x_2, y_2) (i.e., a continuous function $\phi: [0,1] \to \mathbb{R} \times \mathbb{R}$ such that $\phi(0) = (x_1, y_1)$ and $\phi(1) = (x_2, y_2)$) since there obviously are paths connecting (x_1, y_1) with (x_o, y_o) and (x_o, y_o) with (x_2, y_2) .

7. Let $\leq$ on $\underset{\sim}{G}/\underset{\sim}{H}$ be defined by

$$\bar{f} \leq \bar{g} \text{ iff } \begin{cases} \bar{f} = \bar{g} \text{ or} \\ \bar{f} \neq \bar{g} \text{ for all } fh_1 \in \bar{f}, \; gh_2 \in \bar{g} \text{ and real } x, \\ \qquad fh_1(x) < gh_2(x) \quad . \end{cases}$$

Then $\underset{\sim}{G}/\underset{\sim}{H}$ if fully ordered by $\leq$.

Proof: Clearly $\leq$ is a partial ordering of $\underset{\sim}{G}/\underset{\sim}{H}$. It remains only to show $\leq$ is connected. Let f,g be arbitrary elements of $\underset{\sim}{G}$. If $f^{-1}g$ has a fixed point, then $\bar{f} = \bar{g}$ and $\bar{f} \leq \bar{g}$. If $f^{-1}g$ doesn't have a fixed point, then either for all x, $f(x) < g(x)$ or for all x, $g(x) < f(x)$. For definiteness we assume $g(x) < f(x)$. To show $\leq$ is connected it is clearly sufficient to show $f_1 \in \bar{f}$ and x is real imply $g(x) < f_1(x)$. Suppose for some $f_1 \in \bar{f}$ and real x, $f_1(x) \leq g(x)$. Since $f_1^{-1}g$ is not in $\underset{\sim}{H}$, $f_1(x) \neq g(x)$ and $f_1(x) < g(x) < f(x)$. Since $\mathcal{G}\bar{f}$ is pathwise connected there is a path ϕ beginning at $\langle x, f_1(x)\rangle$ and terminating at $\langle x, f(x)\rangle$. More explicitly, there are continuous, real valued functions u,v defined on $[0,1]$ such that $\phi(t) = \langle u(t), v(t)\rangle \in \mathcal{G}\bar{f}$ and $\phi(0) = \langle x, f_1(x)\rangle$, $\phi(1) = \langle x, f(x)\rangle$. Since the continuous function $g[u(t)] - v(t)$ changes sign as t ranges from zero to one, for some t_o, $g[u(t_o)] - v(t_o)$ is zero. Thus, $\langle u(t_o), g[u(t_o)]\rangle$ is in $\mathcal{G}\bar{f}$, contrary to the hypothesis $\bar{g} \neq \bar{f}$. Thus, whether $f^{-1}g$ has a fixed point or not, $\bar{f} \leq \bar{g}$ or $\bar{g} \leq \bar{f}$, and $\leq$ is connected relation on $\underset{\sim}{G}/\underset{\sim}{H}$

8. $\underset{\sim}{G}/\underset{\sim}{H}$ <u>is an ordered group</u>.

<u>Proof</u>: If $\bar{f} \leq \bar{g}$ and $k \in \underset{\sim}{G}$, then $\bar{f} = \bar{g}$ and $\overline{kf} = \overline{kg}$, or for all x, $f(x) < g(x)$ and $kf(x) < kg(x)$. In the latter case $\overline{kf} \leq \overline{kg}$, since $\leq$ is a connected relation. Since $\overline{kf} = \bar{k}\bar{f}$ and $\overline{kg} = \bar{k}\bar{g}$, $\bar{k}\bar{f} \leq \bar{k}\bar{g}$. Thus $\bar{f} \leq \bar{g}$ implies $\bar{f}\bar{k} \leq \bar{g}\bar{k}$. Similarly $\bar{f} \leq \bar{g}$ implies $\bar{f}\bar{k} \leq \bar{g}\bar{k}$.

9. $\underset{\sim}{G}/\underset{\sim}{H}$ <u>is abelian</u>.

<u>Proof</u>: Since the iterates of functions without fixed points are unbounded, $\underset{\sim}{G}/\underset{\sim}{H}$ is Archimedean. Since every fully ordered Archimedean group is abelian, $\underset{\sim}{G}/\underset{\sim}{H}$ is abelian.

This leads to a contradiction that proves the theorem. For G' is a subgroup of every normal subgroup defining of an abelian quotient. Thus $\underset{\sim}{G}' \subset \underset{\sim}{H}$, and every function in $\underset{\sim}{G}'$ has fixed points. Since $\underset{\sim}{G}'$ is conjugate to a subgroup of the translations and e is the only translation with fixed points, $\underset{\sim}{G}' = \{e\}$ and $\underset{\sim}{G}$, contrary to hypothesis, is abelian.

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