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ABSTRACT

This Study Guide on Contemporary Problems, Number 8, is one of a series prepared by the American Association for the Advancement of Science representing part of the 1974-75 National Science Foundation Chautauqua-Type Short Courses for College Teachers Program. This is a test edition and contains evaluation forms to be completed by the users and to be used in possible revision. Thorough discussions of several mathematical models are presented along with computer programs in BASIC to solve the related problems. This is followed by self-study problems and solutions. The contents include: population models--deterministic and discrete, deterministic and continuous, and stochastic; a predator-prey model; an economic model; and modeling in probability. (MP)

TEST EDITION

Mathematical Modeling and Computing

by Jack K. Cohen
and
William S. Dorn

AAAS Study Guides on Contemporary Problems

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Since 1970 the American Association for the Advancement of Science has conducted the NSF Chautauqua-Type Short Courses for College Teachers Program with the support of the Education Directorate of the National Science Foundation. More than 7,000 college teachers of undergraduate students have participated in the courses which have dealt with either broad interdisciplinary problems of science or the applications of science and mathematics to college teaching. All of the courses are designed to make available the most current information. In view of the growing demand for course materials in published form and thereby available to larger numbers of college teachers and their students, the AAAS STUDY GUIDES ON CONTEMPORARY PROBLEMS, a part of the 1974-75 NSF Chautauqua-Type Short Courses for College Teachers Program, are being prepared to test the feasibility of publication. After testing and revision, the following titles will be available from AAAS in the late fall of 1975:

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MATHEMATICAL MODELING

and

COMPUTING

By

Jack K. Cohen

and

William S. Dorn

Study Guide No. 8

AAAS STUDY GUIDES ON CONTEMPORARY PROBLEMS

Printed by the American Association for the Advancement of Science, 1776 Massachusetts Avenue, NW, Washington, D.C. 20036.

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PREFACE FROM AAAS

TO STUDY GUIDE REVIEWERS:

The test editions of the initial set of eight Study Guides were prepared on relatively short notice by the course directors during the summer of 1974. To provide as much information as possible to the authors for use in revising this study guide for publication, we ask you as a participant in the NSF Chautauqua-Type Short Course, or a colleague or student of a participant, to test these materials (as if they had been published) and provide your reactions. Your efforts will contribute significantly to the quality of the revised Study Guide.

If this Study Guide has been successfully prepared, upon completing it, you will: (i) have an overall comprehension of the scope of the problem, (ii) understand the relationships between aspects of the problem and their implications for human welfare, and (iii) possess a reliable guide for studying one or more aspects of the problem in greater depth. We ask you to evaluate the study guide on the basis of how well each of these objectives are achieved. Of less importance but most welcome are your specific editorial suggestions, including punctuation, syntax, vocabulary, accuracy of references, effectiveness of illustrations, usefulness and organization of tabular materials, and other aspects of the draft that are related to its function. Three copies of an evaluation form follow this page and additional copies may be reproduced if needed. Each evaluator should return a completed form to: NSF Chautauqua-Type Short Course Program, 1776 Massachusetts Avenue, NW, Washington, D.C. 20036. Please type or print legibly. Feel free to include any additional comments you care to make. This evaluation is in addition to any evaluative requests made by the study guide authors; however, we do encourage you to cooperate with all requests from authors.

We hope that having used this guide will provide some remuneration to you and that if you were able to participate in the sessions of the Short Course, you gained satisfaction from that. Your efforts in evaluating this study guide are a worthwhile contribution to the improvement of undergraduate education and we express our appreciation to you. Apart from this, we can only offer to include your name among the evaluators in the revised edition.

We hereby gratefully acknowledge the services of Joan G. Creager, Consulting Editor, and Orin McCarley, Production Manager for this series.

Arthur H. Livermore
Deputy Director of Education
AAAS

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After completing this study guide, tear out one copy of this sheet, complete it, fold, and mail. No envelope is needed.
Circle the response that best matches your feeling about the study guide. Also, please make specific suggestions wherever possible.

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PREFACE FROM THE AUTHORS

The *Preface From The AAAS*, which immediately precedes this preface, contains a general evaluation form applicable to all study guides. That preface also asks you to complete and return the evaluation to the AAAS offices. The authors urge you to do so at your earliest convenience. The responses to those questions will be of immeasurable value in revising this study guide.

In addition the authors have prepared some evaluation forms designed specifically for this particular study guide on *Mathematical Modeling and Computing*. A rather broad evaluation form which covers the entire manuscript appears at the close of this preface. In addition there are chapter evaluation forms at the end of each chapter. We would appreciate your completing and returning as many of these forms as you feel are appropriate. They have purposely been kept brief in order not to make an unduly large demand on your time. If you wish to make any other comments or criticisms, be assured that they will be taken seriously when the rewriting process begins.

Please send all of your responses (except for the evaluation form in the preceding preface) directly to:

Professors Jack K. Cohen and William S. Dorn
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University of Denver
Denver, Colorado 80210

You may wish to remain anonymous, and the authors respect your right to do so. Our sincere thanks for your assistance and cooperation in this difficult task.

Jack K. Cohen
William S. Dorn

STUDY GUIDE EVALUATION

1. Did you participate in a short course in 1974-75? Yes No

2. Which one or two chapters were the most interesting? ____________________

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5. In general were the descriptions too detailed? Yes No

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   (a) Modeling Yes No

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CHAPTER I

POPULATION MODELS

DETERMINISTIC AND DISCRETE MODEL

1.1 Two Simple Models

Consider a single species of life in a closed environment. Assume that the population of this species can be measured at various points in time. For example, the population of the United States is counted every ten years through a census. We will call the time between countings a *period*.

Let N_k be the number of individuals alive at the *end* of the kth period for $k = 0, 1, 2, \ldots$. The beginning of the $(k + 1)$st period coincides with the end of the kth period so that N_k is the population at the beginning of the $(k + 1)$st period. We assume throughout Part I that the change in population, $N_{k+1} - N_k$, during the $(k + 1)$st period depends only upon N_k. We ask the reader to suspend judgment on the validity of this assumption until Part II of these notes, where it will be critically examined.

Assume now that the number of births (deaths) in any period is proportional to the population at the start of the period. The increase in population is the excess of births over deaths which may, of course, be negative. Suppose the increase in population is A% of the population. If A is in a decimal form then the increase in the $(k + 1)$st period is

$$AN_k$$

But this increase is also given by

$$N_{k+1} - N_k$$

so

(1.1)
$$N_{k+1} - N_k = AN_k$$

where for now we assume that

$$A > 0$$

Given an initial population N_0 we can calculate $N_1, N_2, \ldots$. A program in BASIC to do so is

```
100  PRINT "TYPE VALUE FOR A"
200  INPUT A
300  PRINT "TYPE INITIAL POPULATION"
400  INPUT N
500  PRINT "TYPE NO. OF FUTURE PREDICTIONS"
600  INPUT M
700  PRINT
800  PRINT "PERIOD","POPULATION"
900  FOR I = 1 TO M
1000  PRINT I,N
1100  LET N = (1 + A)*N
1200  NEXT I
1300  END
```

SELF-STUDY: PROBLEM #1.1

Run the above program with the values $A = .5$, $N_0 = 1000$. Choose values for A and N_0 and re-run the program. Select the additional values in a way which will enable you to make a conjecture about the nature of the solution to this model. Try to prove, by analytical means, that your conjecture is correct.

Solution to Self-Study: Problem # 1.1

The results of running the program with $A = .5$ and $N_0 = 1000$ are shown below.

```
TYPE VALUE FOR A
?.5
TYPE INITIAL POPULATION
?1000
TYPE NO. OF FUTURE PREDICTIONS
?25

PERIOD          POPULATION
 1               1000
 2               1500
 3               2250
 4               3375
 5               5062.5
 6               7593.75
 7               11390.63
 8               17085.94
 9               25628.91
 10              38443.36
 11              57665.04
 12              86497.56
 13              129746.3
 14              194619.5
 15              291929.3
 16              437893.9
 17              656840.8
 18              985261.3
 19              1477892.
 20              2216838.
 21              3325257.
 22              4987885
 23              7481828
 24              11222741
 25              16834112
```

The correct conjecture is that (1) for $A > 0$, the population grows in an unbounded (geometric) way. (2) For $A = 0$, the population does not change at all. (3) For $-1 < A < 0$, the population gradually becomes extinct. (4) For $A \leq -1$, the population immediately becomes extinct. Since N_k is intrinsically positive, we shall interpret the first *negative* value of N_k as indicating extinction.

The first three conjectures can be established by using induction to demonstrate that

$$N_k = N_0(1 + A)^k$$

(see also Section 1.2 of the appendix) and observing that, while

$$(1 + A)^k \rightarrow \begin{cases} +\infty & \text{for } A > 0 \\ 1 & \text{for } A = 0 \\ 0 & \text{for } -1 < A < 0 \end{cases}$$

The last conjecture, (4), may be verified by direct substitution into (1.1).

We now turn to a discussion of the appropriateness of this model, i.e., does it bear any resemblance to reality?

For the values $A = .5$ and $N_0 = 1000$, this model (1.1) predicts that the population will grow in an unbounded way. While this may be satisfactory in the short term, it is not acceptable as a long term solution since eventually the individuals in the population will occupy all of the available space. One possible solution to this dilemma is to choose $A \leq 0$. In this case the population either does not change or becomes extinct.

A 'better' solution is to discard the assumption that the change in population is some fixed percentage of the current population. As long as there is ample room to place newly born individuals, then the assumption we have made may be quite all right. When the population becomes large, however, then the individuals consume all of the food supply, pollute the environment and in other ways make it difficult to maintain life. The result of this overcrowding is to reduce the birth rate and increase the death rate. Both of these rate changes will decrease A. From this argument it follows, rather than being a constant, that A should depend upon the population itself. The simplest way to achieve such a dependence is to replace A by

$$A - BN_k$$

where $A > 0$ and $B > 0$. If we do so then A will represent the growth rate per person which would exist in the absence of over-population pressures, while the term BN_k crudely models the effect of such pressures. Equation (1.1) is replaced by

(1.2) $$N_{k+1} - N_k = (A - BN_k)\, N_k$$

or

$$N_{k+1} = (1 + A - BN_k)\, N_k \; .$$

A program which reads A, B and N_0, the initial population, and computes

$N_1, N_2, \ldots$ is

```
100  PRINT "TYPE VALUE FOR A"
200  INPUT A
300  PRINT "TYPE VALUE FOR B"
400  INPUT B
500  PRINT "TYPE INITIAL POPULATION"
600  INPUT N
700  PRINT "TYPE NO. OF PREDICTIONS"
800  INPUT M
900  PRINT
1000  PRINT "PERIOD","POPULATION"
1100  FOR I = 0 TO M
1200  PRINT I,N
1300  LET N = (1 + A - B*N)*N
1400  NEXT I
1900  END
```

The results of running this program for one case are:

```
TYPE VALUE FOR A
?.5
TYPE VALUE FOR B
?.0001
TYPE INITIAL POPULATION
?1000
TYPE NO. OF PREDICTIONS
?25

PERIOD          POPULATION
 0               1000
 1               1400
 2               1904
 3               2493.478
 4               3118.474
 5               3705.223
 6               4184.967
 7               4526.056
 8               4740.565
 9               4863.552
 10              4929.914
 11              4964.466
 12              4982.107
 13              4991.021
 14              4995.503
 15              4997.749
```

16	4998.874
17	4999.437
18	4999.718
19	4999.859
20	4999.93
21	4999.965
22	4999.982
23	4999.991
24	4999.996
25	4999.998

Notice that the population grows rapidly at first but then tapers off and seems to be approaching 5000. We will return to a discussion of the behavior of this solution in Section 1.3.

Self Study: Problem # 1.2

It turns out that our second model, (1.2), has a much wider variety of solution types than the first model, (1.1). In this problem, we ask you to explore there behaviors and make conjectures about the conditions on A, B and N_0 which produces these behaviors. To do this, use equation (1.2) to derive formulas for A and B given N_0, N_1, N_2. Then use the formulas which you have derived to compute A and B for the N_0, N_1, N_2 values given below. Next use these values in the program above to compute the population for 25 time periods. You may find it convenient to combine these steps by suitably modifying the given program.

Consider the cases in which N_0, N_1, N_2 are given by

	N_0	N_1	N_2
(a)	1000	1400	1900
(b)	1000	2400	5500
(c)	1000	4900	22100
(d)	1000	2900	7900
(e)	1000	2900	7800

	N_0	N_1	N_2
(f)	1000	3900	14000
(g)	1000	3900	14100

Noting that N_0 and B were essentially fixed in the above cases, vary A and re-run the program until you can make suitable conjectures about the behavior of the solution for N_0 and B fixed. Next systematically vary B to see if your conclusions are affected. Finally, vary N_0.

Soluiton to Self-Study: Problem # 1.2

The solutions for A and B are

$$A = -1 + \frac{N_1{}^3 - N_0{}^2\ N_2}{N_0\ N_1\ (N_1 - N_0)}$$

and

$$B = \frac{N_1{}^2 - N_0\ N_2}{N_0\ N_1\ (N_1 - N_0)}$$

For the specific values given for N_0, N_1, N_2, we have:

(a) $A = .507$, $B = 1.07 \times 10^{-4}$, solution approaches 4738.3 in a monotone manner.

(b) $A = 1.477$, $B = .77 \times 10^{-4}$, solution approaches 19181.8 in an oscillatory manner.

(c) $A = 4.000$, $B = 1.0 \times 10^{-4}$, solution becomes negative (extinction).

(d) $A = 1.993$, $B = .93 \times 10^{-4}$, solution oscillates to 21522.7.

(e) $A = 2.011$, $B = 1.11 \times 10^{-4}$, solution oscillates without convergence.

(f) $A = 3.007$, $B = 1.07 \times 10^{-4}$. solution becomes negative.

(g) $A = 2.998$, $B = .98 \times 10^{-4}$, solution oscillates without convergence.

The correct conjecture for $N_0 = 1000$ and $B \simeq .0001$ is that for $0 < A \leq 1$, we have monotone convergence (to A/B); for $1 < A < 2$, we have oscillatory convergence (to A/B); for $2 \leq A \leq 3$, we have finite oscillations; and for $A > 3$, the solution becomes negative (extinction).

These results for A hold in general, so long as

$$N_0 < \frac{1 + A}{B} .$$

If, however,

$$N_0 \geq \frac{1 + A}{B} .$$

then the population becomes extinct regardless of the value of A. A proof of these and related results may be found on page 74 of the Quant. J. Math. (Oxford), 1936, in an article by T. W. Chaundy and Eric Phillips. In the text, below, we give only a heuristic derivation to these results. However, our methods also apply to more difficult difference equations.

1.2 The United State Census

How good is this model, (1.2)? One way to test the model is to use data from an actual population. To this end we look at the United States census. We start in 1890 (the first census with 48 states). The census figures (in millions) for the 48 contiguous states are:

1890	62.948
1900	75.995
1910	91.972
1920	105.711
1930	122.775
1940	131.669
1950	150.697
1960	178.464
1970	199.208

where Alaska and Hawaii have been subtracted from the 1960 and 1970 figures. A program which asks for values of A and B, the initial census year and its population, and a final year to be predicted follows:

```
100 PRINT "TYPE VALUE FOR A"
200 INPUT A
300 PRINT "TYPE VALUE FOR B"
400 INPUT B
500 PRINT "TYPE YEAR OF INITIAL CENSUS"
600 INPUT Y
700 PRINT "TYPE POPULATION IN YEAR JUST TYPED"
800 INPUT N
900 PRINT "TYPE YEAR OF FINAL CENSUS TO BE PREDICTED"
1000 INPUT F
1100 PRINT
1200 PRINT "YEAR","POPULATION"
1400 PRINT Y,N
1500 IF Y>=F THEN 2000
1600 LET N=(1+A-B*N)*N
1700 LET Y=Y+10
1900 GO TO 1400
2000 END
```

We use values of

A = .2329121   B = .0006710713

and start with 1890 and a population of 62.948. The results through the year 2200 are:

```
TYPE VALUE FØR  A
.2329121
TYPE VALUE FØR B
?
6.710713E-4
TYPE YEAR ØF INITIAL CENSUS
?
1890
TYPE PØPULATIØN IN YEAR JUST TYPED
?
62.948
TYPE YEAR ØF FINAL CENSUS TØ BE PREDICTED
?
3000

YEAR          PØPULATIØN
 1890          62.948
 1900          74.95026
 1910          88.63732
 1920          104.0097
 1930          120.9752
 1940          139.3306
 1950          158.7549
 1960          178.8177
 1970          199.0085
 1980          218.7826
 1990          237.6184
 2000          255.0722
 2010          270.8205
 2020          284.679
 2030          296.5991
 2040          306.6458
 2050          314.9654
 2060          321.7522
 2070          327.2199
 2080          331.5798
 2090          335.0277
 2100          337.7363
 2110          339.8529
 2120          341.5
 2130          342.7776
 2140          343.7662
 2150          344.5295
 2160          345.118
 2170          345.5713
 2180          345.92
 2190          346.1881
 2200          346.3942
```

Notice that through 1970 the predictions are reasonably accurate. On the basis of this agreement with the actual census figures then we can, at least tentatively, accept the model as being representative of the United States population with the given values of A and B .* We then use the same equation to predict the future population of the United States. The figures from 2210 to 2400 are:

2210	346.5524
2220	346.674
2230	346.7673
2240	346.8389
2250	346.8939
2260	346.936
2270	346.9684
2280	346.9932
2290	347.0123
2300	347.0269
2310	347.0381
2320	347.0467
2330	347.0533
2340	347.0584
2350	347.0623
2360	347.0652
2370	347.0675
2380	347.0693
2390	347.0706
2400	347.0716

* The values of A and B were actually chosen in a way which produces good estimates of the population figures through 1970. For an arbitrary population it is not always possible to make such a judicious choice.

Notice from these last results that the population seems to be leveling off at about 347 million, and it reaches that value by 2290. There are some modest gains in population thereafter but one century later the population has only increased by another .06 million (about 1/50th of 1%).

Self-Study: Problem #1.3

The U.S. Census population of Colorado and Alabama (in thousands of people) were:

Year	Colorado	Alabama
1890	413	1513
1900	540	1829
1910	799	2138
1920	940	2348
1930	1036	2646
1940	1123	2833
1950	1325	3062
1960	1754	3267
1970	2207	3444

(a) Using $A = .3165$ and $B = 7.82 \times 10^{-5}$ and $N_0 = 413$, calculate the successive populations of Colorado? What is the equilibrium population? Comment on the appropriateness of the model in this case.

(b) Using $A = .3155$ and $B = 8.394 \times 10^{-5}$ and $N_0 = 1513$, calculate the successive populations of Alabama? What is the equilibrium population? Comment on the appropriateness of the model in this case.

Solution to Self-Study: Problem #1.3

(a)

Colorado	Calculated Population
1890	413.0
1900	530.4
1910	676.2
1920	854.5
1930	1067.9
1940	1316.7
1950	1597.8
1960	1903.9
1970	2223.0

Equilibrium population = 4047.3 thousand.

The model does not appear to be satisfactory. Colorado has had considerable immigration and apparently over population presumes have not seriously affected growth to date. Notice that in recent decades the actual population exceeds the calculated population. Of course, different values for A and B might improve matters, but the values used here were chosen in a way which makes them a good choice. In particular, the values of A and B are least square approximations.

(b)

Alabama	Calculated Population
1890	1513.0
1900	1798.2
1910	2094.1
1920	2386.7
1930	2661.6
1940	2906.7
1950	3114.5

Alabama	Calculated Population
1960	3282.9
1970	3414.0

Equilibrium population = 3758.6

The model appears quite good. The population of Alabama is relatively stable and overpopulation forces are starting to be felt.

Self Study: Problem #1.4

In 1920 Pearl and Reed (see Proceedings of National Academy of Sciences, Vol. 6, p. 275 and also Lotka, *Elements of Mathematical Biology*, Dover, 1956, pp. 66-69) used the following census data to predict the equilibrium population of the United States:

Year	Population (x 100,000)
1790	3.929
1800	5.308
1810	7.240
1820	9.638
1830	12.866
1840	17.069
1850	23.192
1860	31.443
1870	38.558
1880	50.156
1890	62.948
1900	75.995
1910	91.972

Using $A = .3641$ and $B = 2.209 \times 10^{-3}$ and $N_0 = 3.929$, estimate the U.S. population. What is the equilibrium population? Compare these results with those given in the text here. Explain the discrepancies.

Solution to Self-Study: Problem #1.4

Year	Population (x 100,000)
1790	3.929
1800	5.325
1810	7.202
1820	9.709
1830	13.036
1840	17.407
1850	23.076
1860	30.302
1870	39.306
1880	50.205
1890	62.917
1900	77.080
1910	92.020
1920	106.820
1930	120.507
1940	132.305

The equilibrium population is 164.8 .

These results are much lower than those in the text. Moreover, this equilibrium population was exceeded in fact before 1960. Hence the results in the text seem more appropriate.

A glance at the census data will show that Pearl and Reed did not include territories in their data. In 1790, for example, only 17 states were taken into account while the 1910 figure included 48 states. Hence the area whose population was counted changed from census to census thereby contaminating the data.

Self-Study: Problem #1.5

Lotka (see above p. 70) gives the following data for growth of a bacteria colony.

Age of Colony (in days)	Area Covered (in cm^2)
0	0.24
1	2.78
2	13.53
3	36.30
4	47.50
5	49.40

Using $A = 6.525$ and $B = 0.1609$ and $N_0 = 0.24$, calculate the size of the colony in square centimeters. What is the equilibrium size? Explain the results.

Solution to Self-Study: Problem #1.5

Age	Size
0	0.24
1	1.797
2	13.001
3	70.636
4	-271.269
5	-13881.390

The equilibrium size is 40.55. The difficulty lies in the rate at which the population reproduces. The observations were made in days but the reproductive span for bacteria is much shorter. Thus the discrete model is inappropriate (see Section 2.1).

Different values of A and B might improve the agreement between the calculated and actual values. See also Self-Study Problem #2.1 in Chapter II.

1.3 Equilibrium and Stability

The value which the population seems to be approaching is called the *equilibrium population*. We will discuss how to compute this equilibrium population in this section. It will turn out that if the population ever reaches its equilibrium value, it will remain there.

Another interesting question which arises in connection with equilibrium population is: Suppose a population is in equilibrium and some catastrophe (such as a flood) kills a significant portion of the population, will the population return to its equilibrium value? We could ask the same question with regard to a certain influx of people through say immigration. That is, if a population in equilibrium is increased by a sudden flood of immigration, will the population decrease to its equilibrium value? Populations which do return to their equilibrium value when subjected to a sudden, but reasonably small change, are called *stable*. Populations that, when disturbed, do not return to equilibrium are called *unstable*.

Now if the population is in equilibrium then the population is not changing, i.e.,

$$N_{k+1} = N_k$$

If we set

$$N_{k+1} = N_k = N_E$$

(E for "equilibrium") then from (1.2)

$$0 = (A - B\,N_E)\,N_E \tag{1.3}$$

From this it follows that either

$$N_E = 0 \tag{1.4}$$

or

(1.5) $$N_E = A/B$$

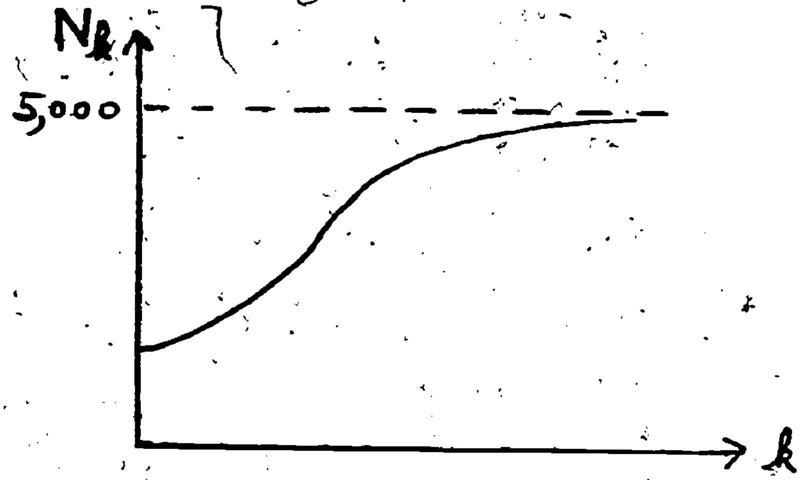
These are the two equilibrium populations. If N_0 equals either 0 or A/B then all succeeding populations also equal either 0 or A/B . For the United States census we used A = .2329121 and B = .0006710713 so A/B = 347.0750 ; which is quite close to the numbers produced by the computer program for the years 2300 and beyond.

To test the stability of the solutions we could try starting the population either above or below A/B or above 0 and see if the population seems to return to the equilibrium value. We should, of course, try different combinations of A and B since stability may depend upon the choice of A and B . The computer programs in the previous section are ideal for conducting such experiments of stability.

Suppose for example we let A = .5 and B = .0001 then N_E = 5000 . The results of this program have been shown on pages 6 and 7 . A sketch of the population growth as indicated by these results is shown in Figure 1.1. The sketch is an S-shaped curve and is typical of the behavior of populations in which there is a braking effect. The equilibrium, 5000, appears to be stable.

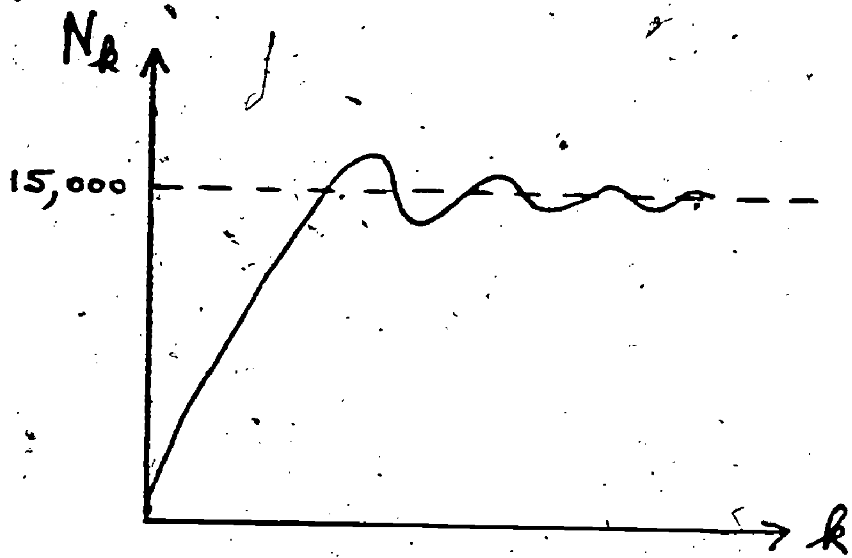
Will the population always follow on S-shaped curve if the equilibrium is stable? We try A = 1.5 and B = .0001 . The equilibrium population is 15000 . The numerical results are not shown, but a sketch of the results is given in Figure 1.2. This certainly is not an S-shaped curve, but equilibrium appears to be stable. Thus both A = .5 and A = 1.5 produced a stable equilibrium. Is the equilibrium always stable in this model?

For A = 4 and B = .0001 , the equilibrium population is 40000 . If we start with N_0 = 1000 the BASIC program produces the following results:



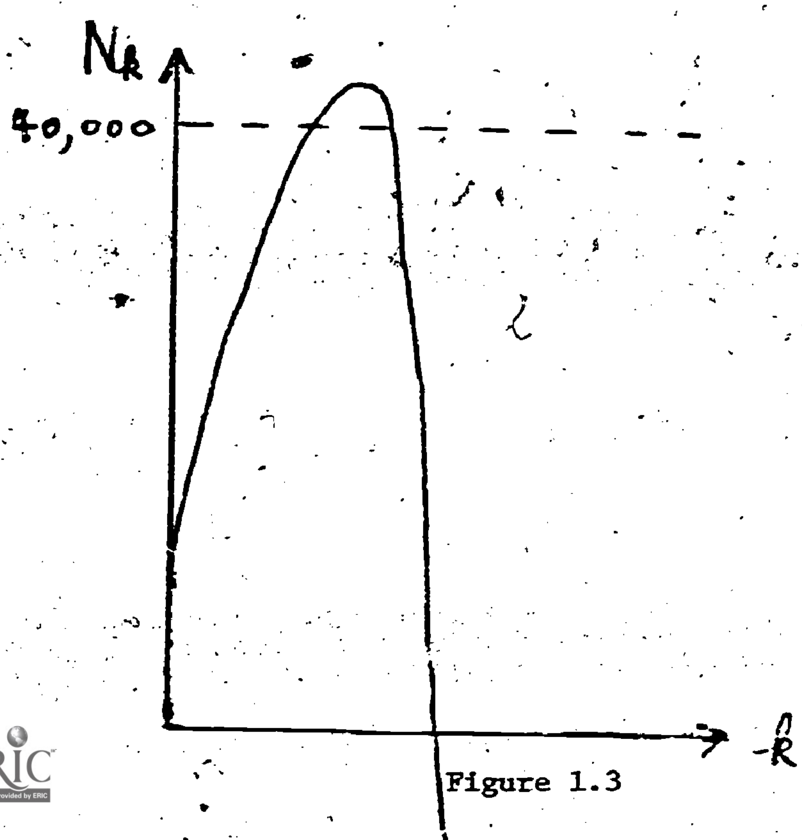
$A = .5$

Figure 1.1



$A = 1.5$

Figure 1.2



$A = 4.$

Figure 1.3

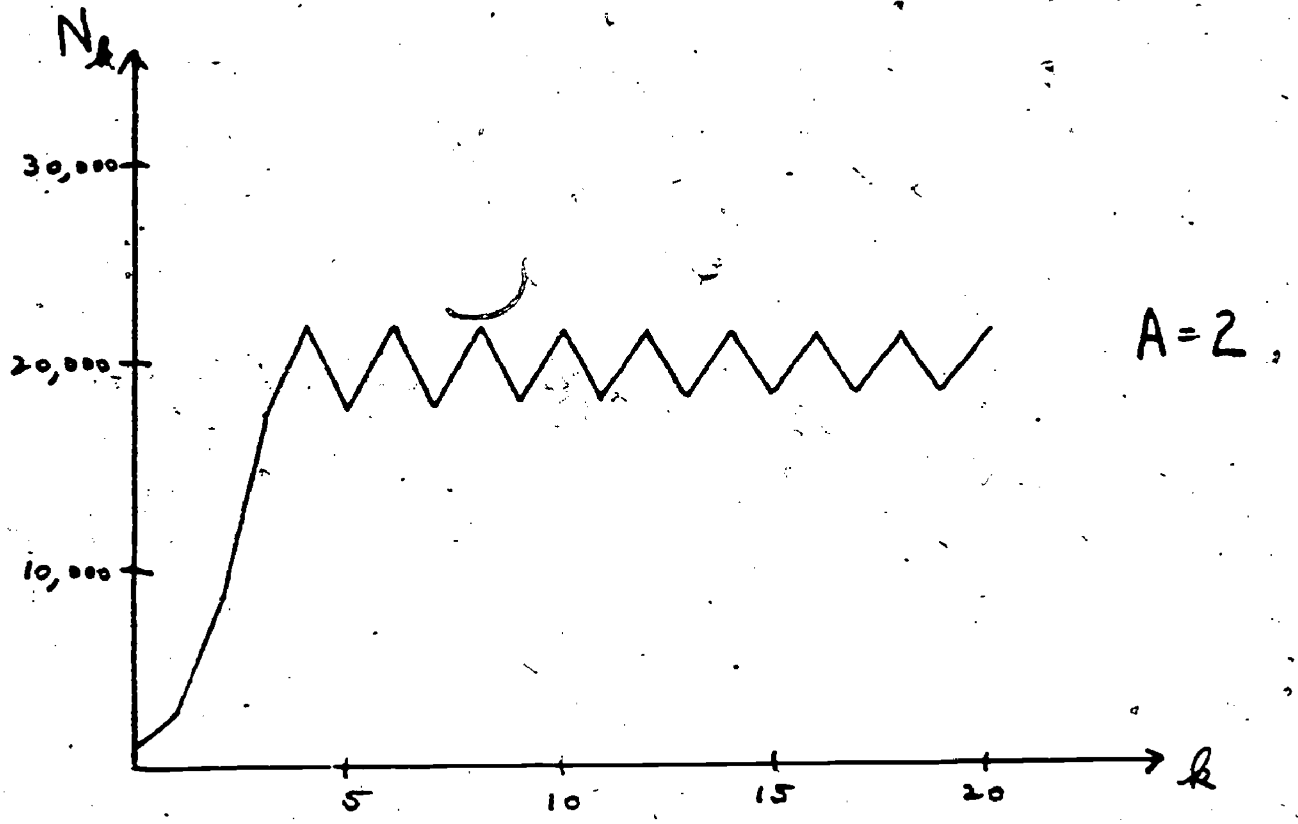


Figure 1.4

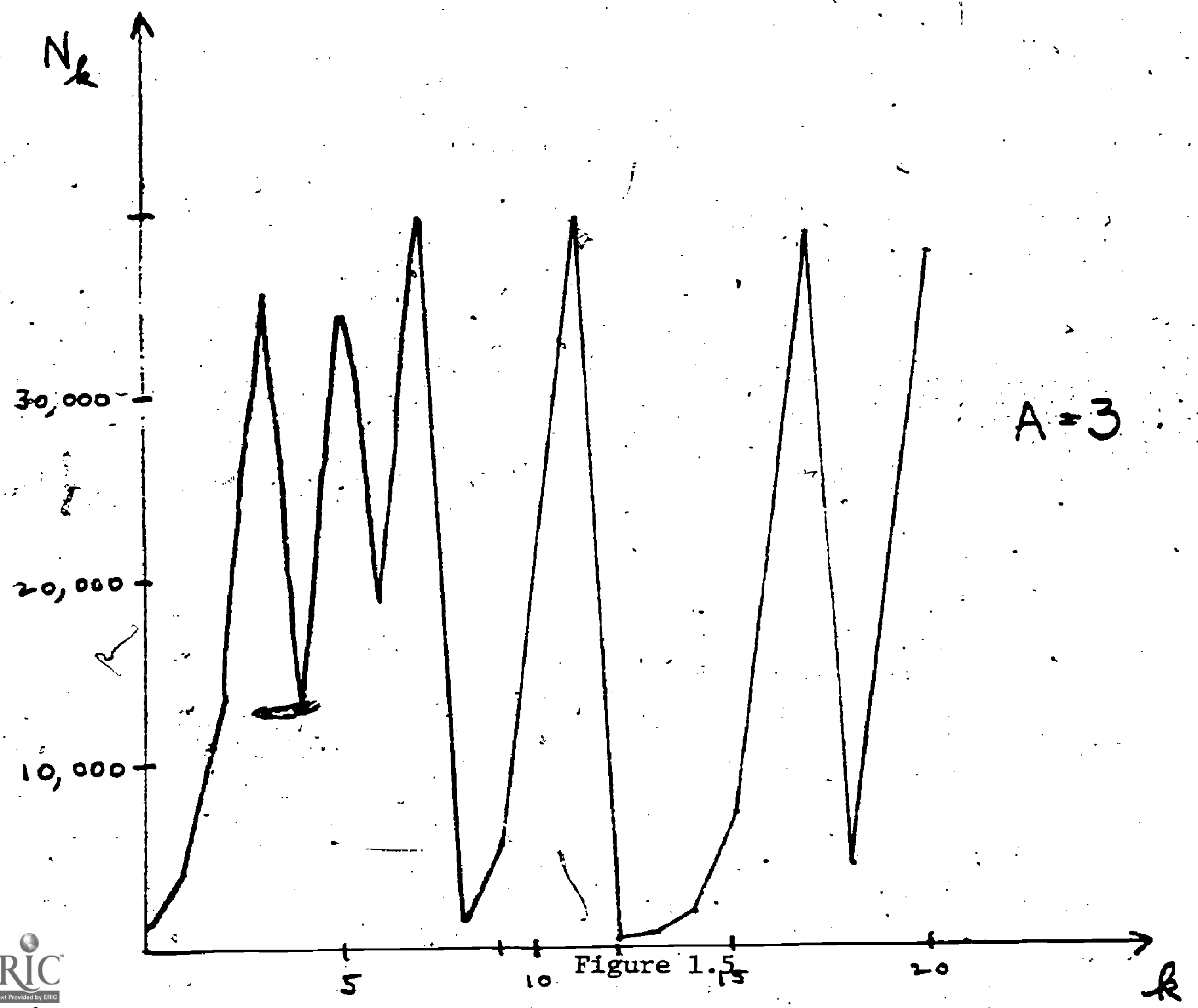


Figure 1.5

```
RUN
NØ1

TYPE VALUE FOR A
?4
TYPE VALUE FOR B
?.0001
TYPE INITIAL POPULATION
?1000
TYPE NO. OF PREDICTIONS
?8

PERIOD          POPULATION
 0               1000
 1               4900
 2               22099
 3               61658.4
 4              -71884.
 5              -876151.
 6              -8.11448E+07
 7              -6.58853E+11
 8              -4.34087E+19

DONE
```

A sketch of these results appears in Figure 1.3. Notice that at point 4 the population is negative, i.e., the species has become extinct. In one sense this again represents stability -- zero population. But the violent behavior is a clue that all is not well with our model.

Where is the source of the difficulty? The only parameter which we have changed is A. For A = .5 or 1.5, the equilibrium A/B was stable. For A = 4 it decidedly was not. A reasonable course of action is to try some values of A between 1.5 and 4. If we use A = 2 with B = .0001 we will produce Figure 1.4. A value of A = 3 yields Figure 1.5. Neither is stable.

We could, of course, continue to experiment with different values of A, but even with these results we can begin to make some 'educated guesses', and it would seem appropriate to delay any further experiments until we have analysed the results already obtained a little more thoroughly.

It appears that if $A \geq 2$, stability is lost. On the other hand, for $A \leq 1.5$, the equilibrium is stable. For values of A between 1.5 and 2, the question of stability is unanswered. But remember -- we have restricted ourselves to one value of B $(B = .0001)$ and have always started with $N_0 = 1000$.

The task of varying A, B and N_0 to determine which combinations of values produce stability appears hopeless due to the large number of cases which must be examined. For example, if we use five different values of A (as we did here) and also five different values of B and N_0 then there are 125 cases. The volume of data would become overwhelming and extremely difficult to analyse. Therefore, we look for some more profitable way of studying stability. However, our experiments have not been wasted. They have given us quite a few clues which we can use in our analysis. In particular, from our numerical results we expect to find stable equilibrium for the smaller values of A.

1.4 Determination of Stability

Suppose the population at some point is disturbed from its equilibrium value. Suppose further that the disturbance is "small".* Then

(1.6) $$N_k = N_E + n_k$$

where n_k is small. Similarly then let

(1.7) $$N_{k+1} = N_E + n_{k+1}$$

where n_{k+1} may or may not be small. Using (1.6) and (1.7) in (1.2)

$$N_E + n_{k+1} = (1 + A - B(N_E + n_k))(N_E + n_k)$$

* We will say shortly what we mean by "small".

or

$$N_E + n_{k+1} = (1 + A - BN_E)N_E + (1 + A - BN_E)n_k - BN_kN_E - Bn_k^2$$

Now N_E satisfies (1.3) so this last equation reduces to

$$n_{k+1} = (1 + A - BN_E)n_k - BN_En_k - Bn_k^2$$

or

(1.8) $$n_{k+1} = (1 + A - 2BN_E)n_k - Bn_k^2$$

Now since n_k is small, we will neglect terms in n_k^2 compared to n_k. In fact this is the definition of the word "small" as used here, i.e., that Bn_k^2 can be neglected compared to the terms in n_k. In any case we will ignore the term Bn_k^2 and write

(1.9) $$n_{k+1} = (1 + A - 2BN_E)n_k$$

This is a linear, first order difference equation for n_k. If

$$|1 + A - 2BN_E| < 1$$

then n_k approaches zero as k increases. In this case N_k approaches N_E as k increases and the equilibrium population is stable. If, on the other hand,

$$|1 + A - 2BN_E| \geq 1$$

then n_k does not decrease as k increases and the equilibrium solution is unstable.

We now examine the two equilibrium populations 0 and A/B for stability. First if

$$N_E = 0$$

then (1.9) becomes

$$n_{k+1} = (1 + A)n_k$$

and since $A > 0$ it follows that $|1 + A| > 1$ so the solution 0 is unstable. Next consider

$$N_E = A/B$$

then (1.9) becomes

$$n_{k+1} = (1 - A)n_k$$

If

(1.10) $$0 < A < 2$$

then

$$|1 - A| < 1$$

and the solution is stable. Otherwise it is unstable. The conclusion then is: *The equilibrium population 0 is never stable. The equilibrium population A/B is stable for $0 < A < 2$ and is unstable for $A \geq 2$.*

This certainly agrees with the numerical experiments which we performed at the close of the previous section. Just to be sure, however, we should try some other values of B and/or N_0. Suppose we return to $A = .5$ and $B = .0001$. This produced the S-shaped curve in Figure 1. Rather then start with $N_0 = 1000$, we use an initial population which exceeds N_E. We will try $N_0 = 15000$. The results are:

```
RUN
N01

TYPE VALUE FOR A
?.5
TYPE VALUE FOR B
?.0001
TYPE INITIAL POPULATION
?15000
TYPE NO. OF PREDICTIONS
?10

PERIOD        POPULATION
 0             15000
 1             0
 2             0
 3             0
 4             0
 5             0
 6             0
 7             0
 8             0
 9             0
 10            0

DONE
```

These results are somewhat surprising. The value of A is considerably less than 2, yet stability does not result. Apparently $0 < A < 2$ is not sufficient to guarantee stability. Is there a flaw in our analysis?

Recall that in the argument which led to $0 < A < 2$, we neglected terms in n_k^2 in (1.8). We justified neglecting these terms in n_k^2 on the basis that they were 'small' compared to the terms in n_k. We now examine what neglecting these terms implies about the validity of our stability condition.

Returning to (1.8), if we are to neglect the term $-Bn_k^2$, then it must be small compared to the terms in n_k, i.e.,

$$Bn_k^2 << |1 + A - 2BN_E| \cdot |n_k|$$

For $N_E = A/B$ this becomes

$$|n_k| << \frac{|1 - A|}{B}$$

Since we have concluded that

$$|1 - A| < 1$$

is necessary for stability, this condition may be rewritten

$$|n_k| << \frac{1}{B} \quad \text{for all } k .$$

From this and (1.6) we obtain

$$N_k = N_E + n_k << N_E + \frac{1}{B}$$

or

(1.11) $$N_k << \frac{1 + A}{B} \quad \text{for all } k .$$

For $A = .5$ and $B = .0001$, the right side of this last inequality is 5000 as is A/B so, $N_k << 10000$. Since this must hold for all k , it must hold for $k = 0$, i.e.,

$$N_0 << 10,000$$

Thus when we used $N_0 = 15,000$ we violated (1.11), i.e., we allowed n_0 to be so large that we could not safely neglect the term Bn_k^2 in (1.8).

The conditions (1.10) and (1.11) are necessary conditions for stability of the equilibrium value, A/B . But some unanswered questions still remain. Are they also sufficient to guarantee stability? How are we to verify (1.11), since it must hold for *all* k ?

The rather complex analysis of Chaundy and Phillips cited above shows that the stability condition (1.11) can be strengthened to

(1.12) $$0 < N_0 < \frac{1 + A}{B}$$

This together with (1.10) are both necessary and sufficient conditions for convergence to the equilibrium value A/B. Moreover, the convergence is monotone for $0 < A < 1$.

The analysis we have given here was confined to 'small' disturbances from equilibrium [recall (1.11)]. Such an analysis, therefore, applies only to *local* stability, i.e., not too far away from equilibrium. Typically it also produces only necessary conditions which may be overly restrictive. Nevertheless the analysis is straightforward and does at least produce conditions which, if satisfied, guarantees stability.

Chaundry and Phillips have used a more powerful analysis which avoids the question of 'smallness' of n_k. Hence they deal with *global* stability. Such an analysis usually is impossible to carry out. Even a brief glance at their paper will indicate the difficulty for even this relatively simple equation. Yet when a global analysis can be carried out, it produces stronger and more useful results.

Local stability is easily verified using the procedures described above, i.e., add a 'small' disturbance, neglect all non-linear terms, solve the resulting linear equation and, finally, investigate the 'size' of the neglected terms to determine what 'small' means. Global stability follows no such pattern and, indeed, cannot be carried out at all in most cases.

1.5 A More Refined Model

So far we have discussed a constant rate of population growth, i.e.

$$R = A$$

which lead to (1.1). We also have discussed a rate of growth which decreased as the population increased

$$R = A - BN$$

with $A > 0$ and $B > 0$. This lead to (1.2). If we plot these rates of growth as functions of the population we obtain the curves in Figures 1.6 and 1.7.

We observe that in Figure 1.7 the rate of growth R is negative for $N > A/B$. One interpretation of this negative growth rate is that over crowding puts a brake on the growth. Indeed, we could have motivated (1.2) by postulating that for N greater than *some* positive value N_E, the effect of over crowding should make the rate of growth negative. The simplest rate of growth which is positive for $N < N_E$ and negative for $N > N_E$ is the linear function

$$R = B(N_E - N)$$

so the model would be

$$N_{k+1} - N_k = BN_k(N_E - N_k) .$$

If we write

$$N_E = A/B$$

we obtain (1.2).

We can extend this model by assuming that there is some minimal population, N_e, below which the rate of growth is again negative because there are so few individuals that they cannot survive in their environment. If we retain the assumption that for $N > N_E$, overcrowding causes a negative rate of growth, then R must be negative for $N < N_\varepsilon$, positive for $N_e < N < N_E$ and negative for $N > N_E$. The simplest such function R is the quadratic

$$R = -C(N - N_E)(N - N_e) \tag{1.13}$$

where

$$0 < N_e < N_E$$

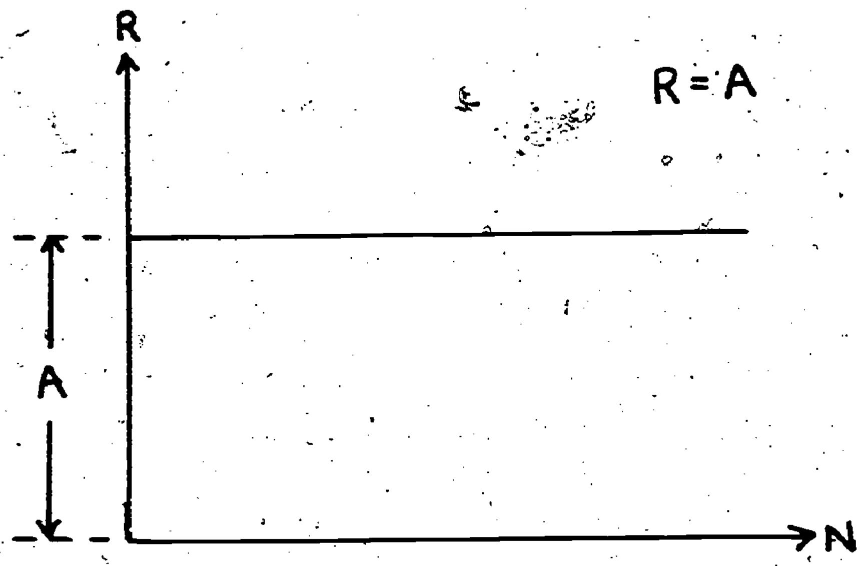


Figure 1.6

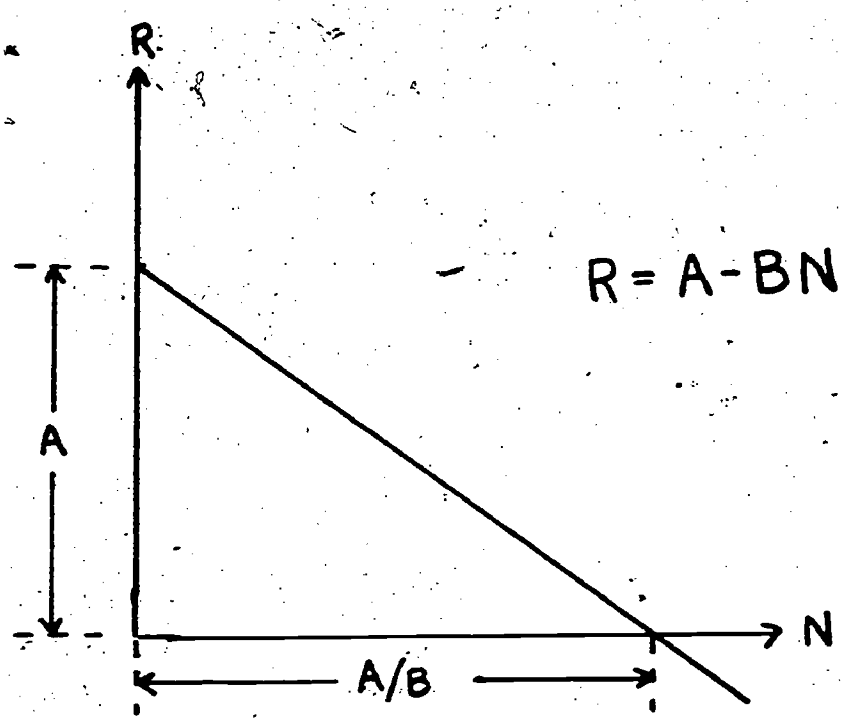


Figure 1.7

and $C > 0$. We depict this rate of growth in Figure 1.8. The e and E stand for "equilibrium" and, as we shall see, they are indeed equilibrium solutions. We also label the maximum value of R as R_M. The difference equation is

$$N_{k+1} = [1 - C(N_k - N_e)(N_k - N_E)]N_k , \quad 0 < N_e < N_E . \tag{1.14}$$

The equilibrium solutions are the values of N such that

$$N_{k+1} = N_k = N$$

or

$$N = (1 - C(N - N_e)\ (N - N_e))N$$

Thus either

$$N = 0$$

or

$$C(N - N_e)\ (N - N_E) = 0$$

which leads to the values N_e and N_E. There are, therefore three equilibrium populations:

$$0, N_e \text{ and } N_E$$

Of the three parameters C, N_e, N_E the latter two have direct biological interpretations. It is desirable to also replace C by a biologically meaningful quantity. Thus we introduce R_M the maximum rate of growth shown in Figure 1.8. If we set

$$N_M = \frac{1}{2}(N_e + N_E)$$

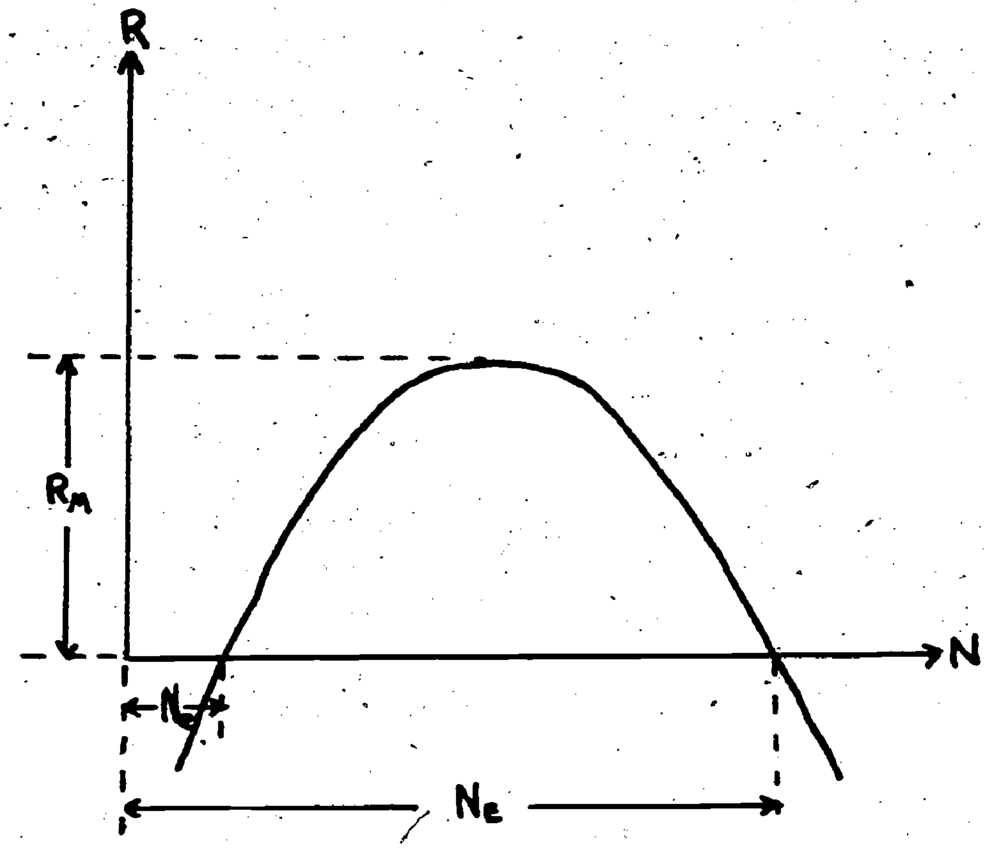


Figure 1.8

Thus we may write

$$\begin{aligned} R &= -C(N - N_e)(N - N_E) \\ &= -C(N - N_M + (N_M - N_e))(N - N_M + (N_M - N_E)) \\ &= -C\left(N - N_M + \frac{N_E - N_e}{2}\right)\left(N - N_M - \frac{N_E - N_e}{2}\right) \\ &= -C\left[\frac{1}{4}(N_E - N_e)^2 - (N - N_M)^2\right]. \end{aligned}$$

In this latter form, it is clear that the maximum value R_M of R occurs for $N = N_M$ and is given by

$$R_M = \frac{C}{4}(N_E - N_e)^2$$

Thus

(1.15)
$$C = \frac{4R_M}{(N_E - N_e)^2}$$

and our model may be written as

(1.16) $$N_{k+1} = [1 - 4R_M \frac{(N_k - N_e)(N_k - N_E)}{(N_E - N_e)^2}]\, N_k$$

Using an analysis quite similar* to the one in Section 1.4 we find that the three equilibrium values are stable under the following conditions:

Equilibrium Value	Stable When
0	$\frac{2R_M N_E N_e}{(N_E - N_e)^2} < 1$
N_E	$\frac{2R_M N_E}{N_E - N_e} < 1$
N_e	Never

While these do not appear to be very enlightening, let us consider the case when N_E is much larger than N_e (the largest equilibrium population is much larger than the smallest non-zero equilibrium population). Then write

$$\frac{2R_M N_E N_e}{(N_E - N_e)^2} = \frac{2R_M \frac{N_e}{N_E}}{(1 - \frac{N_e}{N_E})^2}$$

The denominator on the right is close to 1 so we replace the left hand member by the numerator on the right. The numerator then should be less than 1 for stability. Thus

$$R_M < \frac{1}{2}\frac{N_E}{N_e}$$

*In this case we let $N_k = N_E + n_k$ and neglect terms in n_k^2 and n_k^3.

Since N_E is much larger than N_e, the right side is very large. It follows then that under these conditions 0 is always stable.

More interesting is the equilibrium value N_E. Rewrite

$$\frac{2R_M N_E}{N_E - N_e} = \frac{2R_M}{1 - \frac{N_e}{N_E}} \doteq 2R_M$$

Again this should be less than 1 for stability or

$$R_M < 1/2$$

This says that if $N_E >> N_e$ then if the maximum rate of growth does not exceed 1/2, N_E is a stable equilibrium value. On the other hand, if $R_M \geq 1/2$ then N_E is unstable. Qualitatively this says that a high degree of correction in the system (steep parabola) renders N_E unstable, while a low degree of correction leaves N_E stable.

1.6 Some Numerical Experiments

We now write a program to test the model described in the previous section

$$N_{k+1} = [1 - 4 R_M \frac{(N_k - N_e)(N_k - N_E)}{(N_E - N_e)^2}] N_k .$$

The program is as follows:

```
100   PRINT "TYPE SMALLEST EQUILIBRIUM POPULATION"
200   INPUT N1
300   PRINT "TYPE LARGEST EQUILIBRIUM POPULATION"
400   INPUT N2
500   PRINT "TYPE MAXIMUM RATE OF GROWTH"
600   INPUT R
700   PRINT "TYPE INITIAL POPULATION"
800   INPUT N
900   PRINT "TYPE NO. OF FUTURE PREDICTIONS"
1000  INPUT M
1100  PRINT
1200  PRINT "PERIOD","POPULATION"
1300  FOR I=0 TO M
1400  PRINT I,N
1500  LET N=(1-4*R*(N-N1)*(N-N2)/(N2-N1)↑2)*N
1600  NEXT I
1700  END
```

We will use this program to verify the stability behavior predicted for N_E. Thus we will take $N_e << N_E$ and try values of R_M below and above $1/2$. For $R_M < 1/2$ the solution should be stable.

We try first $R_M = 1/8$. The results are:

```
TYPE SMALLEST EQUILIBRIUM POPULATION
#?
?
100
TYPE LARGEST EQUILIBRIUM POPULATION
?
10000
TYPE MAXIMUM RATE OF GROWTH
?
.125
TYPE INITIAL POPULATION
?
750
TYPE NO. OF FUTURE PREDICTIONS
?
40
```

PERIØD	PØPULATIØN
0	750
1	773.0047
2	797.4931
3	823.607
4	851.5064
5	881.3719
6	913.4084
7	947.8493
8	984.9609
9	1025.049
10	1068.464
11	1115.612
12	1166.966
13	1223.073
14	1284.576
15	1352.233
16	1426.937
17	1509.748
18	1601.934
19	1705.014
20	1820.818
21	1951.559
22	2099.923
23	2269.181
24	2463.31
25	2687.141
26	2946.497
27	3248.299
28	3600.543
29	4012.021
30	4491.472
31	5045.758
32	5676.478
33	6374.672
34	7114.44
35	7849.061
36	8516.474
37	9058.955
38	9448.579
39	9697.061
40	9840.886

A graph of these results starting at the 21st period is shown in Figure 1.9. Notice that $N_E = 10{,}000$ is stable and that $N_E \gg N_e$.

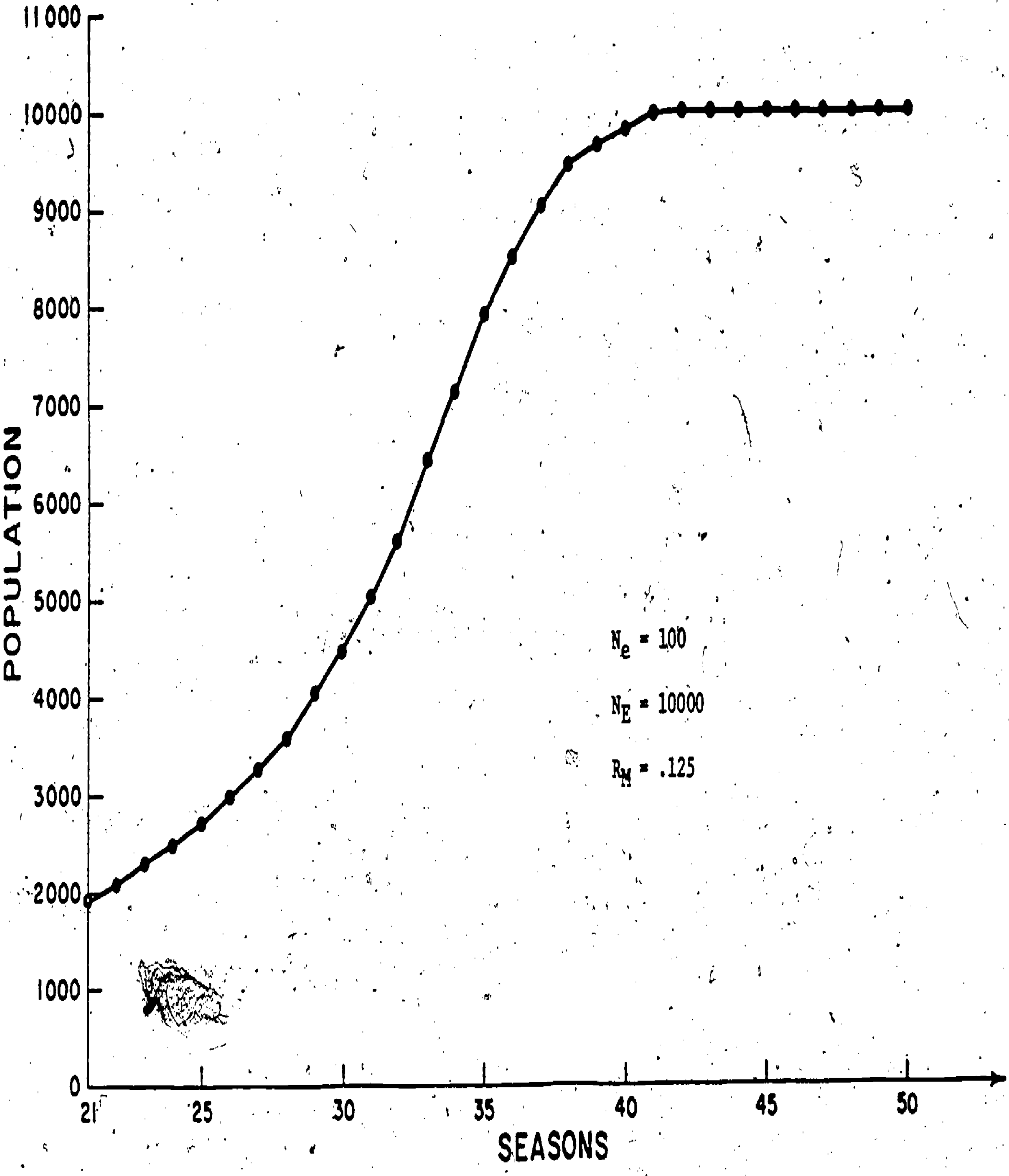

11000
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
POPULATION
21
25
30
35
40
45
50
SEASONS
N_e = 100
N_E = 10000
R_M = .125

FIGURE 1.9

Notice also that the population increases in each period gradually approaching N_E much as our earlier model with $R = A - BN$ did.

Next we try a larger value of R_M, i.e., $R_M = 3/8$. This is still less than 1/2 and thus N_E should be a stable solution. The results are:

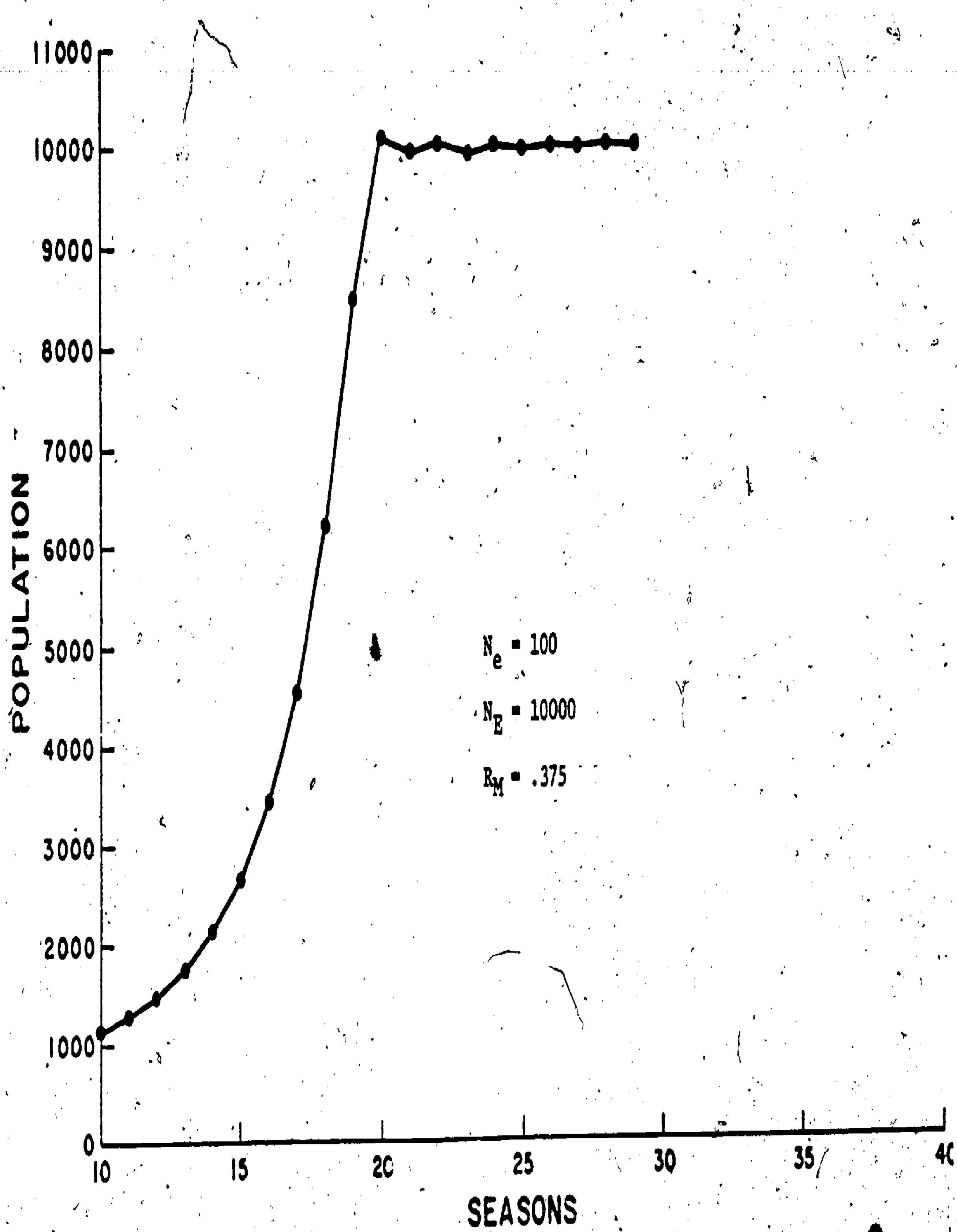
```
TYPE SMALLEST EQUILIBRIUM PØPULATIØN
#?
?
100
TYPE LARGEST EQUILIBRIUM PØPULATIØN
?
10000
TYPE MAXIMUM RATE ØF GRØWTH
?
.375
TYPE INITIAL PØPULATIØN
?
750
TYPE NØ. ØF FUTURE PREDICTIØNS
?
40

PERIØD          PØPULATIØN
 0               750
 1               819.014
 2               901.7585
 3               1002.431
 4               1127.002
 5               1284.178
 6               1487.026
 7               1755.75
 8               2122.549
 9               2640.113
 10              3395.496
 11              4526.555
 12              6205.032
 13              8405.226
 14              10109.03
 15              9940.19
 16              10029.73
 17              9984.418
 18              10007.95
 19              9995.884
 20              10002.12
```

21	9998.909
22	10000.56
23	9999.711
24	10000.15
25	9999.923
26	10000.04
27	9999.98
28	10000.01
29	9999.995
30	10000.
31	9999.999
32	10000.
33	10000.
34	10000.
35	10000.
36	10000.
37	10000
38	10000
39	10000
40	10000

A graph of these results is shown in Figure 1.10. In this case the solution oscillates about N_E and eventually reaches N_E.

Finally we try $R_M = 5/8$ which exceeds $1/2$. Thus we would expect N_E to be unstable. The numerical results are:

```
TYPE SMALLEST EQUILIBRIUM POPULATION
#?
?
100
TYPE LARGEST EQUILIBRIUM POPULATION
?
10000
TYPE MAXIMUM RATE OF GROWTH
?
.625
TYPE INITIAL POPULATION
?
750
TYPE NO. OF FUTURE PREDICTIONS
?
40
```



11000
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
POPULATION
N_e = 100
N_E = 10000
R_M = .375
10
15
20
25
30
35
40
SEASONS

FIGURE 1.10

PERIØD	PØPULATIØN
0	750
1	865.0233
2	1019.222
3	1233.843
4	1546.661
5	2029.119
6	2824.989
7	4233.868
8	6808.103
9	10526.4
10	9052.727
11	11011.03
12	7912.706
13	11204.1
14	7382.966
15	10972.34
16	8013.59
17	11226.8
18	7317.763
19	10931.43
20	8118.356
21	11242.71
22	7271.692
23	10900.97
24	8195.097
25	11249.31
26	7252.492
27	10887.9
28	8227.697
29	11250.8
30	7248.147
31	10884.91
32	8235.121
33	11251.03
34	7247.479
35	10884.45
36	8236.264
37	11251.06
38	7247.387
39	10884.39
40	8236.422

A graph of these results is shown in Figure 1.11. Notice that the solution oscillates about N_E with varying amplitude. Our analysis would seem to indicate that the oscillations should increase in amplitude. However, that analysis assumed that

$$N_k = N_E + n_k$$

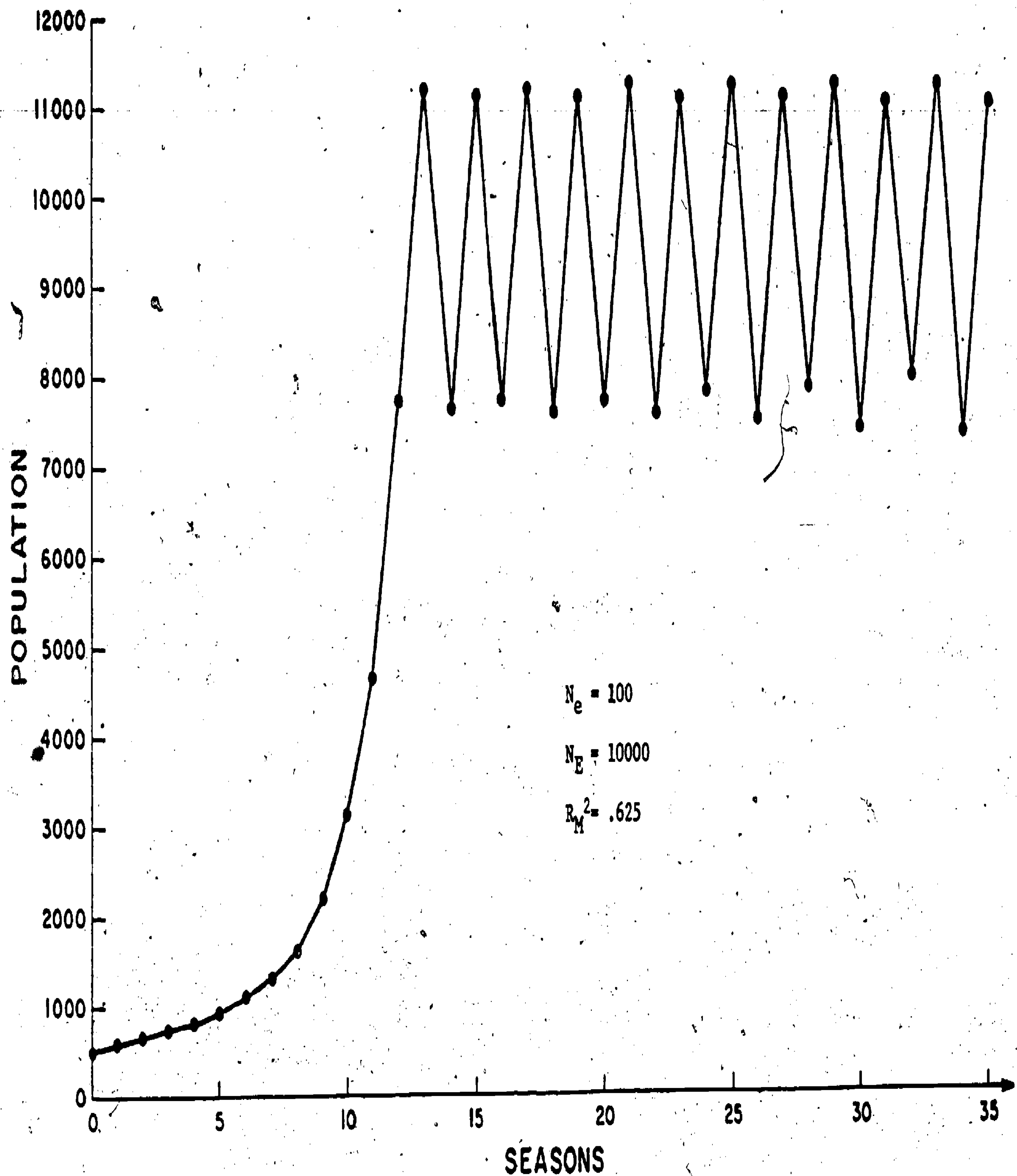


FIGURE 1.11

8-1.45

where n_k was small. Clearly in this case n_k is not small. Nevertheless once n_k does become small (N_k becomes close to N_E), the oscillations will grow. Therefore, the solutions never settles down to N_E.

* *

Interim Project #1

Examine the rate of growth curve shown in Figure 1.12. The equation of that curve is

$$R = A - C\,|N - B|$$

where $A > 0$, $B > 0$, $C > 0$ and $BC > A$. The equilibrium values are

$$N = 0$$

$$N_e = B - (A/C)$$

$$N_E = B + (A/C)$$

The stability of these equilibrium populations can be carried out both analytically and numerically quite analogously to the development already given here.

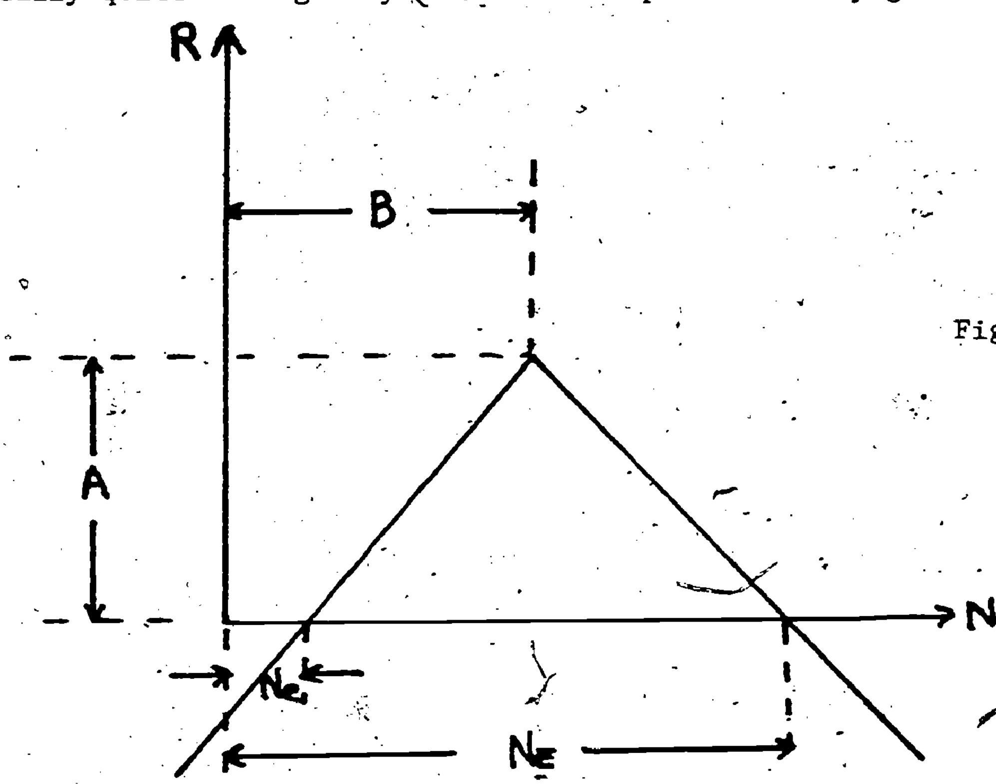


Figure 1.12

CHAPTER I

AUTHORS' EVALUATION

(Please circle one of the responses to each question)

1. Did you attend the short course in 1974-75? Yes No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ______________________

   ____

   ____

   can you suggest topics to be added? ______________________

   ____

   ____

   If (b), which topics should be abbreviated? ______________________

   ____

   ____

   which topics should be eliminated? ______________________

   ____

   ____

3. Could you read and understand the computer programs?

   (a) always (c) seldom

   (b) sometimes (d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy (b) Too difficult

6. Was the <u>number</u> of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attempt any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

____

____

____

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Difference equations in general	____	____	____	____
Models of population growth	____	____	____	____
Assumptions in the population models	____	____	____	____
Stability	____	____	____	____
Equilibrium	____	____	____	____
Feedback	____	____	____	____

CHAPTER II

POPULATION MODELS

DETERMINISTIC AND CONTINUOUS MODEL

2.1 Differential Equations

In Chapter I we made the assumption that the change in population, $N_{k+1} - N_k$, during the $(k + 1)$st time period was a function of N_k alone. There are circumstances where this assumption fits the biological situation quite well. This is the case, for example, when the breeding group as a whole has a fixed season for mating and the effects of the external environment are fairly constant from time period to time period. In such cases it is reasonable to select the unit of time to be the period between mating seasons or some other 'natural' period. However, we have not explicitly indicated the time period in the models examined in Part I. Assuming that the period between observations, $t_{k+1} - t_k$, is a constant, Δt, our general model,

(2.1) $$N_{k+1} - N_k = N_k R(N_k) ,$$

becomes

(2.2) $$N_{k+1} - N_k = N_k R(N_k) \Delta t .$$

This model, (2.2), is appropriate when there is a natural period which the observer regards as 'long'. In such cases, we speak of the 'nearly discrete' case. The reason for the adjective 'nearly' is that (2.2) is rarely, if ever, exactly true. Even some 17-year locusts appear after only 16 years!

We now wish to consider the opposite extreme; the 'nearly continuous' case. By this, we mean a situation in which growth takes place (almost) continuously. Again this idealization is never precisely true, but examples do exist where populations change over very short periods of time.

Suppose for example, we wish to study a caterpillar invasion. At the height of such an infestation, one can almost see the larvae come to life and begin to eat the leaves. From our human point of view, at least over a span of a few days, caterpillars are created continuously. This example, therefore, raises the following point. If, as indicated, we study the caterpillar population for the few days of the infestation, then the nearly continuous model is appropriate. However, if we study the same population making one observation a year (at the appropriate time), the 'nearly discrete' model applies. In these two cases (two day and yearly observation) we are studying two different growth processes (larvae-to-moths, and annual amounts of caterpillars) even though the same population is studied for both. In other words the observer's frequency of observation provides the scale against which we decide whether the process is *nearly* discrete or *nearly* continuous.

Chapter I was concerned with the nearly discrete case. We now turn our attention to the nearly continuous case. To study the nearly continuous case, we recast (2.2) in continuous notation as follows:

$$t = k\,\Delta t, \quad N_k = N(k\,\Delta t) = N(t) \;,$$

and find that

$$N(t + \Delta t) - N(t) = N(t)\,R(N(t))\,\Delta t \tag{2.3}$$

Since the process is taking place almost continuously, we commit a negligible error by letting Δt approach zero. Recall from the calculus that the derivative, $\frac{dN}{dt}$ is defined by

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t}$$

Thus on dividing (2.2) by Δt and letting Δt approach zero, we find that the appropriate model for the nearly continuous case is:

$$\frac{dN}{dt} = N\,R(N) \tag{2.4}$$

Before studying the nearly continuous model (2.4), we pause to examine the 'borderline' case. Commonly, we wish to study a situation in which growth can occur at *any* instant, but does not occur at *every* instant. In such situations either (or neither, if you are a pessimist) of the models might seem reasonable, and we wish to select the one which is better. The correct choice basically depends on the period between observations. Suppose this period is fixed for a moment. If the number of births less the number of deaths within successive time periods is 'small', we would expect that the continuous growth effect is negligible and that the difference equation model is satisfactory. Thus if

$$\frac{|N_{k+1} - N_k|}{N_k} = \frac{|N(t + \Delta t) - N(t)|}{N(t)} << 1$$

we would be justified in using the difference equation models of Chapter I. In this regard, let us return to the U.S. census data cited in Section 1.2. For the decade 1890–1900,

$$\frac{N_{k+1} - N_k}{N_k} \simeq \frac{76 - 63}{63} = .21$$

while for the decade 1960–1970, we have

$$\frac{N_{k+1} - N_k}{N_k} \approx \frac{199 - 178}{178} = .12$$

Thus the difference equation *should* give a reasonable fit. As we have seen in Section 1.2, it indeed *does* give a reasonable fit.

On the other hand, the relative growth between periods (.21 and .12 shown above) is not altogether negligible, and therefore it would seem that the differential equation could also be applied. That is to say, the U.S. census data is almost 'too close to call', and we probably should study both models since both seem to apply equally well and each represents a different type of approximation to reality.

In a case, such as the U.S. census data, where the continuous and discrete models both seem to apply, we would naturally expect the solutions to the models to be similar. This is easily seen to be true since (2.2) is the classic Euler approximation to the differential equation (2.3) (See e.g., pp. 366-367 of *Numerical Methods with FØRTRAN Case Studies* by Dorn and McCrackin, Wiley, 1972).

In part I, we studied the discrete model (2.2) for the following particular choices of R :

(2.4) $\quad R = A, \quad A > 0$

(2.5) $\quad R = B(N_E - N), \; N_E = A/B, \; A > 0, \; B > 0$

(2.6) $\quad R = -C(N - N_e)(N - N_E), \; C = 4R_M/(N_E - N_e)^2, \; C > 0, \; 0 < N_e < N_E$.

The reasoning behind the choices (2.4), (2.5) and (2.6) for $R(N)$ made in Part I is equally valid for the continuous model (2.3) and so we proceed to the study of these three choices.

Self-Study: Problem #2.1

If the data of Self-Study: Problem #1.3 [illegible] construed as arising from a 'borderline' case, decide whether one should use the difference model or the continuous model.

Solution to Self-Study: Problem #2.1

The successive values of $(N_{k+1}^{*} - N_e)/N_k$ are 10.58 , 3.867 , 1.683 , 0.309 and 0.040 . Since most of these are greater than 1 , the difference model does not appear to be satisfactory. This was borne out by Self-Study: Problem #1.5. The relative growth is far from negligible so the continuous model appears to be most appropriate.

2.2 Solution of the Differential Equations

The reader who is familiar with the elementary theory of first order differential equations will observe that equation (2.3) is of the type with separable variables. Thus, we formally write

$$\frac{dN}{N\,R(N)} = dt$$

and upon integrating on both sides, we obtain the formal solution

$$\int_{N_0}^{N(t)} \frac{dx}{x\,R(x)} = t \tag{2.7}$$

where $N_0 = N(0)$.

Although the integral on the left side of (2.7) can be evaluated explicitly for the choices of R (2.4), (2.5) and (2.6) for it is more instructive to examine the corresponding differential equation directly and obtain a geometric understanding of the nature of the solutions. To do this we need only the most rudimentary fact from the Differential Calculus, namely that the derivative of $N(t)$ at a point, t_0 , gives the slope of the graph of $N(t)$ at $t = t_0$. Thus

$\left.\frac{dN}{dt}\right|_{t=t_0} > 0$ implies the graph of N is rising at $t = t_0$

$\left.\frac{dN}{dt}\right|_{t=t_0} < 0$ implies the graph of N is falling at $t = t_0$.

$\left.\frac{dN}{dt}\right|_{t=t_0} = 0$ implies the graph of N is flat at $t = t_0$.

Consider first the case (2.4), the corresponding differential equation is

$$\frac{dN}{dt} = AN\ , \quad A > 0\ .$$

Suppose that at $t = 0$

$$N(0) = N_0 > 0 ,$$

Then the initial slope $\left(\frac{dN}{dt}\Big|_{t=0} \right)$ is AN_0 which is positive, so N rises. But now the slope is even greater so N rises even faster. Thus we see that N rises faster and faster as t increases and so in this case, we have the same type of behavior as occured in the discrete case, i.e., unbounded growth.

If we recall that the antiderivative of $\frac{1}{x}$ is $\ln x$, then we can verify our geometric conclusion by setting $R = A$ in (2.7) to obtain

$$t = \frac{1}{A} \int_{N_0}^{N(t)} \frac{dx}{x} = \frac{1}{A} \ln \frac{N(t)}{I} .$$

Thus

(2.8) $$N(t) = I e^{At} .$$

We next consider the choice (2.5) for R,

(2.9) $$\frac{dN}{dt} = (A - BN) N , \quad A > 0 , B > 0 , N_E = A/B .$$

Just as in the discrete case, we have the two equilibrium solutions

$$N = 0 \text{ and } N = N_E = A/B$$

That is, if any time t, $N = 0$, then $\frac{dN}{dt} = 0$ for all time. It follows that $N = 0$ for all time as well, and $N = 0$ is the solution to (2.9). Similarly that at any time t, $N = A/B$ then $N = \frac{A}{B}$ is the solution to (2.9) for all time. If initially $N_0 \neq 0$ and $N_0 \neq A/B$, then the solution can never attain these values. Suppose then

(2.10) $$0 < N_0 < N_E$$

we conclude at once that for all time

(2.11) $$0 < N(t) < N_E .$$

Moreover, since the right hand side of (2.9) is positive in view of (2.11), the solution is increasing. The only additional fact we need to establish in order to complete our geometric picture of the solution is that N approaches N_E as $t \to \infty$. To establish this, we appeal to a monotone convergence theorem:

If a function $N(t)$ is increasing and bounded from above, it has a limit.

If the limit were less than A/B the right hand side of (2.9) would be positive and hence N would continue to rise. Thus the limit is A/B. Typical graphs of N are shown in figures 2.1 and 2.2 for initial value satisfying (2.10).

Self-Study: Problem #2.2

By differentiating (2.9) and recalling the test for an inflection point, show that the graphs shown in figures 2.1 and 2.2 are essentially the only cases that can occur when (2.10) is satisfied.

Solution to Self-Study: Problem #2.2

$$\frac{d^2N}{dt^2} = A - 2BN$$

At a point of inflection

$$\frac{d^2N}{dt^2} = 0$$

so

$$N = A/2B = N_E/2$$

If $N_0 < N_E/2$ then one inflection point exists (Figure 2.1). If $N_0 > N_E$ then since $N(t) \geq N_0$, no inflection point exists (Figure 2.2).

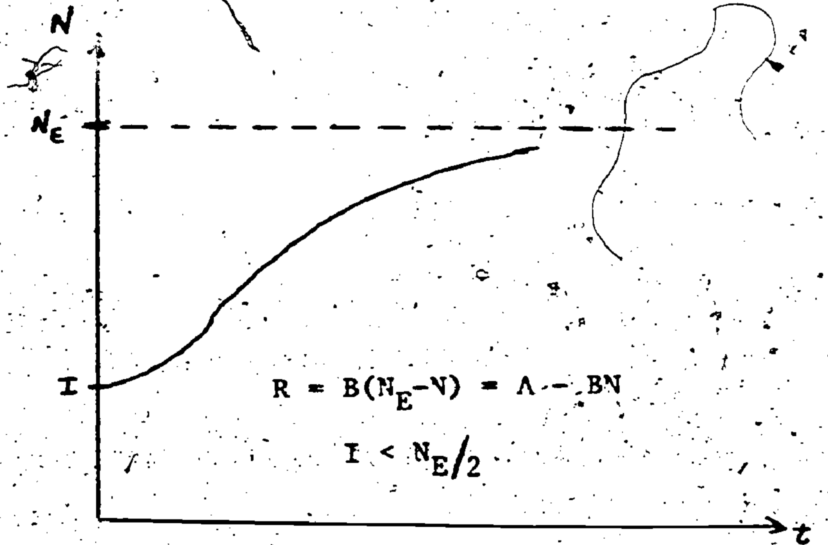


Figure 2.1

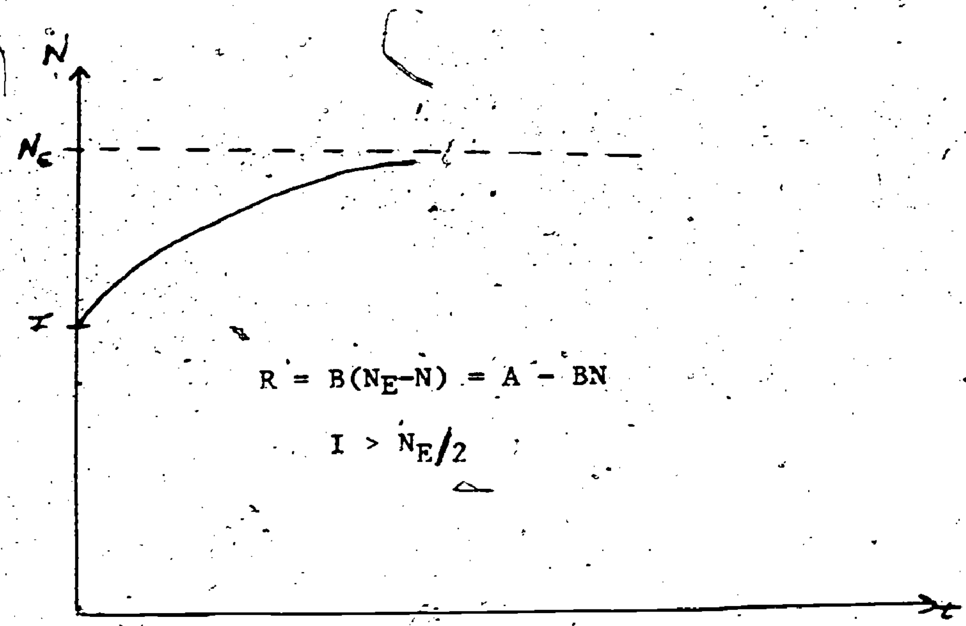


Figure 2.2

We now turn to the case in which the initial condition satisfies

$$N_0 > N_E \tag{2.12}$$

Hence by the above reasoning, we see that the solution

i) must remain above N_E

ii) is decreasing

iii) approaches N_E at $t \to \infty$.

Thus in all cases for which $N_0 \gtrless 0$, the solution monotonically approaches the equilibrium value A/B. That is, for the continuous model (2.9), $N_E = A/B$ is a globally stable equilibrium point.

Notice that in contrast to the discrete analogue, the value of A has no effect on the nature of the solution!

This fact gives rise to the question: Suppose we have a situation in which the discrete and continuous models both apply (borderline case), how can the results be so different? In fact, the two sets of results are not different. In the notation of this section, the conditions (1.10) and (1.11) for monotone convergence to A/B in the discrete case become

$$0 < A\,\Delta t < 1\,, \quad 0 < N_0 < \frac{1 + A\,\Delta t}{B\,\Delta t} = \frac{(1/\Delta t) + A}{B}$$

These conditions are obviously satisfied when the two models both apply.

What accounts for the existence of oscillatory and divergent behaviors (figures 1.2, 1.3, 1.4 and 1.5) in the discrete case and their total absence in the continuous case? The answer lies in the rate at which the population itself feeds back information about its own size into the growth process. When that information is fed back quickly (the continuous case), violent behavior does not occur. On the other hand, an appreciable delay in supplying information can be catastrophic. To better illustrate this idea of feedback we will use an analogy due to Richard Hamming.

Suppose you are taking a shower and the water is too cold. You open the hot water faucet. If additional hot water is added immediately in response to your turn, the water temperature gradually warms. This is the continuous case.

Suppose, however, that there is some delay in the additional hot water's arrival at the shower head. For example, if the hot water tank is empty, some time may be required to heat the water. Since the shower water remains cold, you continue to open the hot water faucet. The result is that when the hot water does arrive, it arrives in a rush and your shower becomes scalding hot. So you frantically turn the hot water faucet backwards (or turn on the cold water). Thus you produce violent oscillations in the shower's temperature --- for too cold to too hot to too cold and so on. The feedback is slow, i.e., the temperature of the shower responds slowly to the turning of the faucet. This is the discrete case, i.e., the response (hot water) comes only after some length of time. Clearly it is this slowness of response that causes the problem.

We return now to the population problem. A large population retards growth. But if the growth process itself is not aware of the large population, it continues unretarded and may produce a huge population, one in excess of the equilibrium value. This can happen in the discrete case since a large population in the middle of a period goes unnoticed. Indeed, information flows into the system only at period endings. Therefore oscillations, and violent ones, can occur. On the other hand, in the continuous case the information or population size flows continuously into the growth process thus preventing oscillations.

We close this portion of the discussion with one final remark. In the discrete case the greater the reaction to feedback, the greater the oscillations. In terms of the hot water problem, the hotter the water in the storage tank, the worse matters will become. In terms of the population model given by Figure 1.8, the stronger the reaction to over- or under-population, hence the more violent the oscillations. This of course, is born out by the instability when the peak of the curve (R_M) exceeds $\frac{1}{2}$. The oscillatory and divergent behaviors which are possible in the discrete model, can occur only in the 'nearly discrete' case.

Notice also that the above inequalities may be satisfied regardless of the value of A simply by choosing Δt sufficiently small.

<u>Self-Study: Problem #2.3</u>

Using the method of partial fractions, and equations (2.1) and (2.5), obtain a solution to (2.9) and verify our geometrical results.

Solution to Self-Study: Problem #2.3

$$N = \frac{N_E}{1 - \left(1 - \frac{N_E}{N_0}\right) e^{-BN_E t}} \tag{2.13}$$

Self-Study: Problem #2.4

For the choice (2.9), use geometrical reasoning to verify the graphs shown in Figures 2.3 and 2.4. Use the method of partial fractions to derive the implicit solution

$$\frac{N^{N_E} (N - N_E)^{N_e}}{N^{N_e} (N - N_e)^{N_E}} = \frac{I^{N_E} (N_0 - N_E)^{N_e}}{N_0^{N_e} (N_0 - N_e)^{N_E}} \, e^{-\frac{4N_E N_e R_M}{N_E - N_e} t} \tag{2.14}$$

Finally discuss the stability of the equilibrium solutions.

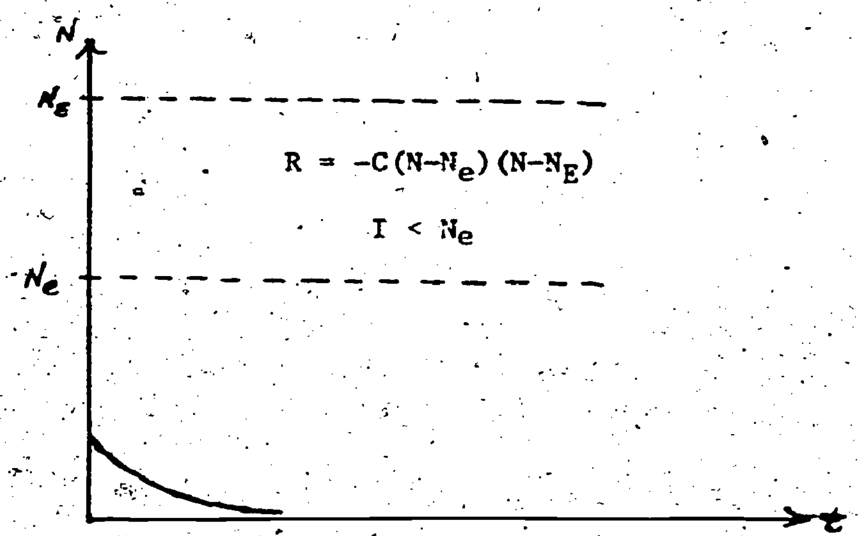


Figure 2.3

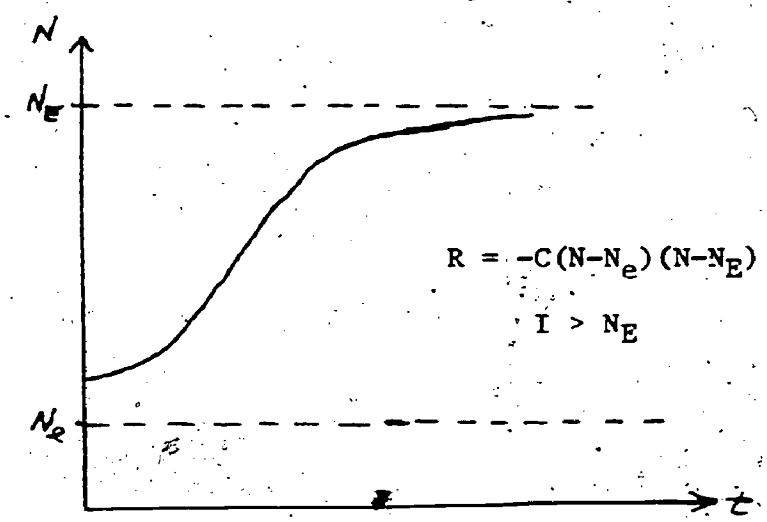


Figure 2.4

Solution to Self-Study: Problem #2.4

$N = 0$ is stable

$N = N_e$ is unstable

$N = N_E$ is stable.

These results are easy to obtain geometrically, but are not so easy to obtain from the explicit solution (2.14).

* *

Interim Project #2

Consider the following modification of (2.4). Suppose that due to medical advances the equilibrium increases time increases, e.g.

$$N_{eq} = N_E + kt$$

where $k > 0$. Now our model becomes

$$\frac{dN}{dt} = B(N_E + kt - N)N$$

or

$$\frac{dN}{dt} = B(N_E - N)\,N + k\,B\,Nt\,.$$

Use some numerical algorithm (e.g., the Runge-Kutta Method) starting with $N(0) = N_0$ to analyze the solution of this model.

* *

Interim Project #3

Reconsider the U.S. census problem discussed in part I. First use the solution (2.13) for the continuous analogue and the values

$$A\,\Delta t = .2329 \qquad\qquad B\,\Delta t = .00006711$$

to compute the predicted population for each decade from 1890 to 1970. Decide whether the discrete or continuous model gives a better fit (say in the sense that the maximum relative error is smaller). In Lotka's book, *Elements of Mathematical Biology*, Dover, 1956, the values

$$A = .03134 \qquad\qquad A/B = 197.273$$

are cited as giving the best fit for the continuous model for the years 1790-1910.

Compare the Lotka values for the parameters with the ones given above for the continuous model during 1890-1970. By examining the two sets of results for the continuous model, try to find values for the parameters, A and B, that are better than either Lotka's or ours for 1890-1970. Finally make comparisons between the discrete case and the continuous case.

* *

Interim Project #4

Using a decreasing linear function for R is only the crudest attempt to characterize the effect of population pressure. A more general form for R would be

$$R(N) = A - f(N) .$$

In the absence of population pressure, the growth law

$$R(N) = A$$

is reasonable and therefore we would expect that $f(0) = 0$. Since we want R to eventually become negative, we want R to be a decreasing function of N. Thus we would expect $f(N)$ to be an increasing function, which also satisfies $f(N_E) = A$ for a unique argument $N_E > 0$. Thus $f(N)$ might be BN^2 or BN^3 or $BN + CN^2$ or $B \cdot 2^N$ and so on. Investigate some functions which you feel have some biological justification. Remember that f should be increasing, $f(0) = 0$ and $f(N_E) = A$. Use either the discrete model (2.1) or the continuous model (2.4) or both.

* *

CHAPTER II

AUTHORS' EVALUATION

(Please circle one of the responses to each question).

1. Did you attend the short course in 1974-75? Yes No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ______________________

   ____

   ____

   can you suggest topics to be added? ______________________

   ____

   ____

   If (b), which topics should be abbreviated? ______________________

   ____

   ____

   which topics should be eliminated? ______________________

   ____

   ____

3. Could you read and understand the computer programs?

   (a) always (c) seldom

   (b) sometimes (d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy (b) Too difficult

6. Was the number of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attempt any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

____

____

____

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Relation between discrete and continuous models	____	____	____	____
Geometrical analysis of solution of differential equations	____	____	____	____
Equilibrium	____	____	____	____
Stability	____	____	____	____

CHAPTER III

POPULATION MODELS

STOCHASTIC MODELS

3.1 A Birth-Death Model

In Chapters I and II we assumed a birth and/or death rate which was proportional to the number of living individuals. Thus (1.1) and (2.1) both assumed that the difference between births and deaths in a time Δt was $AN\Delta t$ where N is the population, Δt is the time interval and A is the rate of change. We could have arrived at this result by assuming that each individual gives birth to $\lambda\Delta t$ new individuals in time Δt and that for each individual there are $\mu\Delta t$ deaths in the same time. Then the net change in the population *per individual* is $\lambda\Delta t - \mu\Delta t$. If there are N individuals the net change for the entire population is

$$(\lambda - \mu) N\Delta t$$

If we let $A = \lambda - \mu$ we arrive once more at our earlier model.

Using this as a guide we will now consider a similar but less deterministic process. In some sense this new model will also be more realistic. Rather than assume that each individual gives birth to $\lambda\Delta t$ individuals in a time Δt we will assume that "on the average" each individual will give birth to $\lambda\Delta t$ individuals in a time Δt. One way to achieve this end is to assume that each individual gives birth to precisely one other individual with a probability of $\lambda\Delta t$.*

Since the probability of a birth from any individual is $\lambda\Delta t$, the binomial law of probabilities shows that the probability of exactly k births in a population of

*More precisely $\lambda\Delta t + o(\Delta t)$. Those who wish a more rigorous derivation should similarly modify the probabilities derived below. The $o(\Delta t)$ terms will ultimately be removed, because we shall later divide by Δt and let $\Delta t \to 0$.

size N is $p(k) = \frac{N!}{k!\,(N-k)!}\,(\lambda\Delta t)^k\,(1-\lambda\Delta t)^{N-k}$.
Thus $p(1) = N\lambda\Delta t(1-\lambda\Delta t)^{N-1} \approx N\lambda\Delta t$ for Δt small. Similarly the probability of precisely two births is $p(2) = \frac{1}{2}N(N-1)(\lambda\Delta t)^2\,(1-\lambda\Delta t)^{N-2} \approx \frac{1}{2}N(N-1)(\lambda\Delta t)^2 \approx 0$ for Δt small. In a like manner we can show that the probability of 3, 4, 5, ... births is also zero. Thus we conclude that the probability of two or more births is ≈ 0.

Similarly we will assume that on the average each individual is responsibile for $\mu\Delta t$ deaths. This leads us to assume that the probability of one individual dying in time Δt is $\mu\Delta t$. In a population of size N then the probability of one death is $\approx N\mu\Delta t$. Once again we will assume that Δt is small enough so that two or more deaths or a combination of births and deaths occur with such a small probability in time Δt that these events may be neglected.

Of course, it is possible that neither a birth nor a death will occur in time Δt. Since this is the only possibility other than one birth or one death, the probability of no change in a population of size N is approximately

$$1 - N\lambda\Delta t - N\mu\Delta t = 1 - N(\lambda+\mu)\Delta t$$

Now let $p_N(t)$ be the probability that there are N individuals alive at time t. From our discussion N individuals where $N \geq 1$ can be alive at time t in only three ways:

(1) At time $t - \Delta t$ there were $N - 1$ individuals and a birth occurred.

(2) At time $t - \Delta t$ there were $N + 1$ individuals and a death occurred.

(3) At time $t - \Delta t$ there were N individuals and neither a birth nor a death occurred.

For $N = 0$ there cannot be $N - 1$ individuals alive at time $t - \Delta t$ so only the last two are possible in this special case.

Since these three events are mutually exclusive, the probability of any one of these occurring is the sum of the individual probabilities. The probability of N

individuals being alive at time t is, therefore, the sum of the probabilities that each of these three events occurs. That is

$$(3.1) \qquad p_N(t) = (N-1)\lambda\Delta t p_{N-1}(t-\Delta t) + (N+1)\mu\Delta t p_{N+1}(t-\Delta t) + \{1 - N(\lambda+\mu)\Delta t\}p_N(t-\Delta t)$$

for $N = 1, 2, 3, \ldots$ and

$$(3.2) \qquad p_0(t) = \mu\Delta t p_1(t-\Delta t) + p_0(t-\Delta t)$$

Subtracting $p_N(t - t)$ from both sides of (3.1) dividing by Δt and taking the limit as $\Delta t \to 0$

$$(3.3) \qquad \frac{dp_N}{dt} = \lambda(N-1)p_{N-1} + \mu(N+1)p_{N+1} - (\lambda+\mu)Np_N$$

for $N = 1, 2, 3, \ldots$. Carrying out this same process for (3.2) then

$$(3.4) \qquad \frac{dp_0}{dt} = \mu p_1$$

This is a system of differential-difference equations.

Now suppose at time $t = 0$ the population is I. Then

$$(3.5) \qquad p_I(0) = 1$$

and

$$(3.6) \qquad p_j(0) = 0 \quad \text{for } j \neq I$$

With these initial conditions the system of equations (3.3) and (3.4) may be solved.

In Section 3.3 we will determine the solution using generating functions and the method of characteristics to solve the resulting partial differential equation. Later in Section 3.4 we will use a computer program to simulate the process and thereby estimate some of the probabilities. In particular we will estimate the probability of extinction in time T, i.e., we will compute an approximation to $p_0(T)$.

We turn next to the computation of the expected value of the population at time t and to an estimation of the deviation of the population from this expected value.

Self Study: Problem #3.1

Carry out a derivation analogous to that of the subsection for our second model. That is, assume

$$Pr(\text{birth}) \simeq N(a_1 - b_1 N)$$

$$Pr(\text{death}) \simeq N(a_2 + n_2 N)$$

and assume that

$$Pr(\text{two or more births/deaths}) \simeq 0 .$$

Generalize your result to the case in which

$$Pr(\text{birth}) \simeq N \lambda_N , \qquad Pr(\text{death}) \simeq N \mu_N .$$

Solution to Self Study: Problem #3.1

$$\frac{dp_N}{dt} = (N - 1)\lambda_{N-1}\, p_{N-1}(t) - N[\lambda_N + \mu_N]\, p_N(t) + (N + 1)\mu_{N+1}\, p_{N+1}(t)\ .$$

3.2 Expected Value and Variance of the Population

Let $E(t)$ be the *expected value* of the population at time t. Since a population of size N occurs at time t with probability $p_N(t)$ and since any population of size $0, 1, 2, \ldots$ is possible, the expected value of the population is*

$$E(t) = \sum_{N=1}^{\infty} N p_N(t) \qquad (3.7)$$

Notice that the term $N = 0$ vanishes due to the factor N in each term of the sum. Differentiating both sides of (3.7) with respect to t

$$\frac{dE}{dt} = \sum_{N=1}^{\infty} N \frac{dp_N}{dt}$$

From (3.3) then

$$\frac{dE}{dt} = \lambda \sum_{N=1}^{\infty} N(N-1)p_{N-1} + \mu \sum_{N=1}^{\infty} N(N+1)p_{N+1} - (\lambda + \mu) \sum_{N=1}^{\infty} N^2 p_N \qquad (3.8)$$

But

$$\sum_{N=1}^{\infty} N(N-1)p_{N-1} = \sum_{K=0}^{\infty} (K+1)Kp_K$$

The first term in the sum on the right vanishes so

---

*The notion of expected value is analogous to the arithmetic mean of a frequency destribution. Suppose the values $x_1, x_2, \ldots x_n$ occur respectively $k_1, k_2, \ldots, k_n$ times. Then the relative frequencies are $k_1/K, k_2/K, \ldots, k_n/K$ where $K = \Sigma k_i$ and $\bar{x} = \Sigma x_i k_i/K$. The notion of expected value arises for theoretical destributions and the relative frequency k_i/K of x_i is replaced by the probability p_{x_i} of x_i, so that $E = \Sigma x_i p_{x_i}$. In equation (3.7) the x_i are the positive integers and the situation is made somewhat more complicated by the fact that the probabilities depend on time. Thus the expected value of the population can be interpreted as the mean value of the population.

(3.9) $$\sum_{N=1}^{\infty} N(N-1)p_{N-1} = \sum_{N=1}^{\infty} (N+1)Np_N$$

Similarly

$$\sum_{N=1}^{\infty} N(N+1)p_{N+1} = \sum_{K=2}^{\infty} (K-1)Kp_K$$

But when $K = 1$

$$(K-1)Kp_K = 0$$

so we may extend the sum on the right from $K = 1$ to ∞ so that

(3.10) $$\sum_{N=1}^{\infty} N(N+1)p_{N+1} = \sum_{N=1}^{\infty} (N-1)Np_N$$

Using (3.9) and (3.10) in (3.8)

$$\frac{dE}{dt} = \lambda \sum_{N=1}^{\infty} (N+1)Np_N + \mu \sum_{N=1}^{\infty} (N-1)Np_N - (\lambda + \mu) \sum_{N=1}^{\infty} N^2p_N$$

The coefficient of N^2 vanishes and we are left with

$$\frac{dE}{dt} = \lambda \sum_{N=1}^{\infty} Np_N - \mu \sum_{N=1}^{\infty} Np_N$$

But from (3.7)

$$\sum_{N=1}^{\infty} Np_N = E$$

so

(3.11) $$\frac{dE}{dt} = (\lambda - \mu)E$$

Moreover for $t = 0$ from (3.7)

$$E(0) = \sum_{N=1}^{\infty} Np_N(0)$$

But from (3.5) and (3.6) $p_N(0) = 0$ except when $N = I$ so

(3.12) $$E(0) = I$$

The solution of (3.11) and (3.12) is

(3.13) $$E(t) = Ie^{(\lambda - \mu)t}$$

Recalling that $A = \lambda - \mu$ this is identical with the deterministic solution (2.8).

We can conclude from (3.13) that if $\lambda > \mu$ (birth rate exceeds death rate) that the expected value of the population grows as time increases. On the other hand, if the death rate exceeds the birth rate $(\mu > \lambda)$, the expected value of the population decreases to zero as the time becomes large. Finally if $\lambda = \mu$, the expected value of the population is I for all time, i.e.,

(3.14) $$E = I \qquad \text{for } \lambda = \mu .$$

These results certainly agree with our intuition and, therefore, lend credence to our model.

Of course, the actual value of the population will vary from the expected value in any one case. It would be helpful to have some estimate regarding how much of a deviation from E should be expected. To that end we compute the *variance** of the population. The variance $V(t)$ at time t is

(3.15) $$V(t) = \sum_{N=1}^{\infty} N^2 p_N(t) - \{E(t)\}^2$$

We will derive a differential equation for $V(t)$ in much the same way as (3.11) was derived for $E(t)$.

*The variance of a sample can be defined as $\sum x_i^2 k_i/K - (\bar{x})^2$, where the k_i/K are the relative frequencies of the observed values x_i. If we again replace the relative frequencies by the probabilities p_{x_i}, and the mean by the expected value, we arrive at (3.15).

First we let

(3.16) $$H(t) = \sum_{N=1}^{\infty} N^2 p_N(t)$$

so

$$\frac{dH}{dt} = \sum_{N=1}^{\infty} N^2 \frac{dp_N}{dt}$$

Using (3.3)

(3.17) $$\frac{dH}{dt} = \lambda \sum_{N=1}^{\infty} N^2(N-1)p_{N-1} + \mu \sum_{N=1}^{\infty} N^2(N+1)p_{N+1} - (\lambda + \mu) \sum_{N=1}^{\infty} N^3 p_N$$

But

$$\sum_{N=1}^{\infty} N^2(N-1)p_{N-1} = \sum_{K=0}^{\infty} (K+1)^2 Kp_K = \sum_{N=1}^{\infty} (N+1)^2 Np_N$$

and

$$\sum_{N=1}^{\infty} N^2(N+1)p_{N+1} = \sum_{K=2}^{\infty} (K-1)^2 Kp_K = \sum_{N=1}^{\infty} (N-1)^2 Np_N$$

Thus (3.17) becomes

$$\frac{dH}{dt} = 2(\lambda - \mu) \sum_{N=1}^{\infty} N^2 p_N + (\lambda + \mu) \sum_{N=1}^{\infty} Np_N$$

From (3.7) and (3.16) then

(3.18) $$\frac{dH}{dt} = 2(\lambda - \mu) H + (\lambda + \mu)E$$

From (3.15) and (3.16) then

$$\frac{dV}{dt} = \frac{dH}{dt} - 2E \frac{dE}{dt}$$

Using (3.11) to replace dE/dt

$$\frac{dV}{dt} = \frac{dH}{dt} - 2(\lambda - \mu) E^2$$

From (3.18)

$$\frac{dV}{dt} = 2(\lambda - \mu) H + (\lambda + \mu) E - 2(\lambda - \mu) E^2$$

Using (3.15) and (3.16) to replace H

$$\frac{dV}{dt} = 2(\lambda - \mu) V + (\lambda + \mu) E \tag{3.19}$$

This is a differential equation for V, the variance. For an initial condition we note that at $t = 0$ the population is I with certainty so

$$V(0) = 0 .^{*} \tag{3.20}$$

The solution of this differential equation for $\lambda \neq \mu$ is

$$V = Ce^{2(\lambda - \mu)t} - \frac{\lambda + \mu}{\lambda - \mu} Ie^{(\lambda - \mu)t}$$

Using (3.20) to determine C

$$C = \frac{\lambda + \mu}{\lambda - \mu} I$$

so

$$V = \frac{\lambda + \mu}{\lambda - \mu} I e^{(\lambda - \mu)t} \{e^{(\lambda - \mu)t} - 1\}$$

For $\lambda = \mu$

$$E = I$$

*This follows from setting $t = 0$ in (3.15) and using (3.5), (3.6) and (3.12).

and from (3.19)

$$\frac{dV}{dt} = 2\lambda\ I$$

Integrating this differential equation with the initial condition (3.20)

$$V = 2\lambda\ It$$

In summary then

$$(3.22) \qquad V = \begin{cases} \dfrac{\lambda + \mu}{\lambda - \mu}\ Ie^{(\lambda - \mu)t}\ \{e^{(\lambda - \mu)t} - 1\} & \lambda \neq \mu \\ 2\lambda\ It & \lambda = \mu \end{cases}$$

We now examine the behavior of the expected value and variance for large time. For $\lambda > \mu$ (birth rate exceeds death rate) both E and V become large. Thus while the expected value of the population grows without bound, the variance also becomes infinitely large. We should, therefore, expect the deviations from the expected value to be large as time increases.

For $\mu > \lambda$ (death rate exceeds birth rate) the expected value and the variance approach zero as the time increases. For large times then the population should be close to zero. Later we shall see that the probability that the population is zero for large times is 1.

Finally for $\lambda = \mu$ (birth rate and death rate are equal) the expected value is constant for all time but the variance grows linearly without bound for increasing time. Thus as time becomes large the deviations from the constant expected value will become large. At large times then we should expect the population to deviate markedly from I. Indeed we should expect that in a significant number of cases, the population will differ from I by $\pm I$. If the deviation is $-I$ then the population is zero and the species becomes extinct. The interesting fact is that even when the birth rate and death rate are identical we should expect the species to become extinct a significant number of times for large time. Later we will see that the probability is 1 that the population will eventually become extinct when $\lambda = \mu$ (despite the fact that the expected value is I !).

3.3 Solution of the Stochastic Model

The solution for the expected value and variance of the population was obtained by elementary methods. These statistical parameters were useful in our study of the mathematical model. In order to obtain additional information we will need to analyse the differential-difference equations (3.3) and (3.4) in more detail. The analysis while relatively straightforward is more sophisticated than development in the previous section.

We turn then to the problem of finding the solution $p_N(t)$ to (3.3) and (3.4) given the initial conditions (3.5) and (3.6). To do so we introduce a *generating function*

$$f(x, t) = \sum_{N=0}^{\infty} p_N(t)\, x^N \tag{3.23}$$

We will use some of the techniques of the last section to obtain a partial differential equation which the function $f(x, t)$ must satisfy. We will then solve the partial differential equation and from the solution obtain $p_N(t)$.

Multiplying (3.3) by x^N and summing from 1 to ∞

$$\sum_{N=1}^{\infty} \frac{dp_N}{dt} x^N = \lambda x^2 \sum_{N=1}^{\infty} (N-1)p_{N-1}\, x^{N-2} + \mu \sum_{N=1}^{\infty} (N+1)\, p_{N+1}\, x^N - (\lambda + \mu)x \sum_{N=1}^{\infty} Np_N\, x^{N-1}$$

Adding (3.4) to this last equation

$$\sum_{N=0}^{\infty} \frac{dp_N}{dt} x^N = \lambda x^2 \sum_{N=1}^{\infty} (N-1)p_{N-1}\, x^{N-2} + \mu \sum_{N=0}^{\infty} (N+1)p_{N+1}\, x^N - (\lambda+\mu)x \sum_{N=1}^{\infty} Np_N\, x^{N-1} \tag{3.24}$$

From (3.23) however

(3.25) $$\frac{\partial f}{\partial t} = \sum_{N=0}^{\infty} \frac{dp_N}{dt} x^N$$

and

(3.26) $$\frac{\partial f}{\partial x} = \sum_{N=1}^{\infty} Np_N x^{N-1}$$

By changing indices the latter equation may be expressed as

(3.27) $$\frac{\partial f}{\partial x} = \sum_{N=0}^{\infty} (N+1) p_{N+1} x^N$$

or

$$\frac{\partial f}{\partial x} = \sum_{N=2}^{\infty} (N-1) p_{N-1} x^{N-2}$$

When $N = 1$ then $N - 1 = 0$ so the last sum can be extended to $N = 1$, i.e.,

(3.28) $$\frac{\partial f}{\partial x} = \sum_{N=1}^{\infty} (N-1) p_{N-1} x^{N-2}$$

Using (3.25) through (3.28) in (3.24)

$$\frac{\partial f}{\partial t} = (\lambda x^2 + \mu - \{\lambda + \mu\} x) \frac{\partial f}{\partial x}$$

or

(3.29) $$\frac{\partial f}{\partial t} = \lambda(x - \frac{\mu}{\lambda}) (x - 1) \frac{\partial f}{\partial x}$$

This is the partial differential equation which $f(x, t)$ must satisfy. For $t = 0$ (3.23) becomes

$$f(x, 0) = \sum_{N=0}^{\infty} p_N(0) x^N$$

Using the initial conditions (3.5) and (3.6)

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Since the probability of a birth from any individual is $\lambda\Delta t$, the binomial law of probabilities shows that the probability of exactly k births in a population of

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*More precisely $\lambda\Delta t + o(\Delta t)$. Those who wish a more rigorous derivation should similarly modify the probabilities derived below. The $o(\Delta t)$ terms will ultimately be removed, because we shall later divide by Δt and let $\Delta t \to 0$.

8-3.1

occurred.

For $N = 0$ there cannot be $N - 1$ individuals alive at time $t - \Delta t$

last two are possible in this special case.

Since these three events are mutually exclusive, the probability

these occurring is the sum of the individual probabilities. The prob

...t so only the

...ty of any one of

...obability of N

$$f(x, \theta) = x^I \tag{3.30}$$

Self Study: Problem #3.2

Derive the corresponding partial differential equation for the first models of Self Study Problem #3.1.

Solution to Self Study: Problem #3.2

$$\frac{\partial f}{\partial t} = (x - 1)\ \{[(a_1 - b_1)\ x - (a_2 + b_2)]\ \frac{\partial f}{\partial x} - (b_1\ x + b_2)\ x\ \frac{\partial^2 f}{\partial x^2}\}\ .$$

The solution of (3.29) with initial condition (3.30) can be obtained by applying the "method of characteristics", which is the standard technique of solving first order partial differential equations. However, once again, important information can be obtained without solving the equation. Rather we simply examine the differential equation (3.29) itself. We shall, in fact, be able to derive the single most important fact about our stochastic model in this way — the long run probability of extinction, $\lim_{t\to\infty} p_0(t)$, which we denote by $p_0(\infty)$. First we note that from (3.23), $p_0(t) = f(o, t)$, thus

(3.31) $$p_0(\infty) = \lim_{\substack{t\to\infty \\ x\to 0}} f(x, t)$$

Next recall that in the model

$$\frac{dN}{dt} = BN(N - A/B)$$

of Chapter II, the values $N = 0$ and $N = A/B$ (the zeroes of the right hand side) played a crucial role. Now examine (3.29). The values $x = 1$ and $x = \mu/\lambda$, are clearly crucial ones. In fact, for these two values of x, we see that $\frac{\partial f}{\partial t} = 0$. That is f does not vary in time when x takes on either value so

$$f(1, t) = \text{constant}, \qquad f(\mu/\lambda, t) = \text{constant}.$$

We use the initial condition (3.30) to find these constants, thus

(3.32) $$f(1, t) = 1, \qquad f(\mu/\lambda, t) = (\mu/\lambda)^I$$

for all values of t.

Now if there is to be an equilibrium distribution as $t \to \infty$, then f must settle down and stop changing as $t \to \infty$. Thus we assume that

(3.33) $$\lim_{t\to\infty} \frac{\partial f}{\partial t} = 0.$$

Let $\lim_{t\to\infty} f(x, t) = F(x)$, then using (3.31) the quantity, $p_0(\infty)$, which we seek is given by

$$p_0(\infty) = F(0)$$

From (3.32) know that

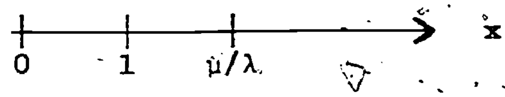
$$F(1) = \lim_{t\to\infty} f(1, t) = 1, \text{ and } F(\mu/\lambda) = \lim_{t\to\infty} f(\mu/\lambda, t) = (\mu/\lambda)^{I}$$

Moreover from (3.33) and the differential equation (3.29) we find

$$\lambda(x - \mu/\lambda)\ (x - 1)\ \frac{dF(x)}{dx} = 0 .$$

Hence *F(x) is a constant* except possibly at $x = 1$, $x = \mu/\lambda$ where it might jump.

<u>Case 1. $\mu \geq \lambda$</u>



Since $F(x) = \text{constant}$ for $0 \leq x < 1$, and since $F(1) = 1$, we have*

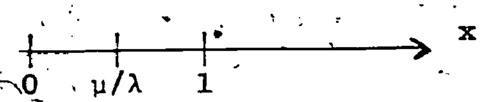
$$F(x) \equiv 1 , \qquad 0 \leq x \leq 1 .$$

Thus

$$(3.34) \qquad p_0(\infty) = F(0) = 1 , \qquad \mu \geq \lambda .$$

Extinction is certain in this case. This result is intuitively reasonable for $\mu > \lambda$, but is somewhat startling for $\mu = \lambda$ (in which case the expected value is I).

<u>Case 2. $\mu < \lambda$</u>



We have $F'(x) = 0$ for $0 \leq x < \mu/\lambda$, hence as above, $F(x) = \text{constant}$ for

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*We are assuming continuity from the left at $x = 1$. This can be established by noting that since the p_n's are probabilities, $0 \leq p_n$, and $\sum_{n=0}^{\infty} p_n = 1$ and using some standard results of analysis.

$0 \le x < \mu/\lambda$. Since* $F(\mu/\lambda) = (\mu/\lambda)^I$, we conclude, as in Case 1, that $F(0) = (\mu/\lambda)^I$, so

(3.35) $$p_0(\infty) = (\mu/\lambda)^I \quad \text{for} \quad \mu < \lambda .$$

Thus even when the birth rate is higher than the death rate, the probability of extinction, in the long run, is positive.

For those, who are equipped with the necessary prerequisites, we conclude this section by solving (3.29) by a special version of the method of characteristics.

Consider f as a function of a single parameter s and let

(3.36) $$\frac{dx}{ds} = -\lambda(x - \mu/\lambda)(x - 1)$$

(3.37) $$x(0) = \xi$$

and

(3.38) $$\frac{dt}{ds} = 1$$

(3.39) $$t(0) = 0$$

By the chain rule then

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial t}\frac{dt}{ds}$$

From (3.36), (3.39) and (3.29)

(3.40) $$\frac{df}{ds} = 0$$

(3.41) $$f(0) = \xi^I$$

*See previous footnote.

Now (3.40) is the partial differential equation (3.29) and the initial condition (3.41) is the initial condition (3.30). Therefore, if we can find a solution $x(s, \xi)$ to (3.36) and (3.37); then we can easily integrate (3.38) to replace s by t, and then integrate (3.40) to replace ξ by f. Thus having $x(t, f)$ we can invert this function to find $f(x, t)$. With this in mind we proceed to integrate (3.36).

The solution of (3.36) using partial fractions is

$$\frac{1}{\lambda - \mu} \log \left(\frac{x - 1}{\lambda x - \mu} \right) = -s + K_1$$

Using (3.37) to determine K_1

$$(\lambda - \mu)\, s = \log \left(\frac{\xi - 1}{\lambda \xi - \mu} \quad \frac{\lambda x - \mu}{x - 1} \right)$$

or

(3.42) $$\frac{(\xi - 1)\,(x - \mu)}{(\lambda \xi - \mu)\,(x - 1)} = e^{(\lambda - \mu)s}$$

From (3.38) and (3.39)

$$s = t$$

Using this in (3.42) and solving for ξ as a function of x and t

(3.43) $$\xi = \frac{x(\mu e^{(\lambda - \mu)t} - \lambda) - \mu(e^{(\lambda - \mu)t} - 1)}{\lambda x(e^{(\lambda - \mu)t} - 1) - (\lambda e^{(\lambda - \mu)t} - \mu)}$$

Now the solution of (3.40) is

$$f = K_2$$

But from (3.41)

$$K_2 = \xi^I$$

so

$$f = \xi^I$$

From (3.43) then

$$f(x,\ t) = \left[\frac{x(\mu e^{(\lambda - \mu)t} - \lambda) - \mu(e^{(\lambda - \mu)t} - 1)}{\lambda x(e^{(\lambda - \mu)t} - 1) - (\lambda e^{(\lambda - \mu)t} - \mu)}\right]^{I} \tag{3.44}$$

This is the solution of (3.29) with the initial condition (3.30) as may be verified by substitution.

From (3.23) then $p_N(t)$ is the coefficient of x^N in (3.44). In particular the probability of extinction (zero population) is $p_0(t)$ which is the coefficient of x^0 in (3.44). Thus

$$p\ (t) = \left[\frac{\mu(e^{(\lambda - \mu)t} - 1)}{\lambda e^{(\lambda - \mu)t} - \mu}\right]^{I} \tag{3.45}$$

We will consider three cases: $\lambda > \mu$; $\lambda < \mu$ and $\lambda = \mu$.

For $\lambda > \mu$ (birth rate exceeds death rate)

$$\lim_{t\to\infty} p_0(t) = \left(\frac{\mu}{\lambda}\right)^{I} \tag{3.46}$$

which is in agreement with (3.35).

For $\mu > \lambda$ (death rate exceeds birth rate), we again find

$$\lim_{t\to\infty} p_0(t) = 1 \tag{3.47}$$

which agrees with the previous result (3.34).

Finally for $\lambda = \mu$ (birth rate and death rate are equal) the right hand side of (3.45) is indeterminate. To evaluate $p_0(t)$ in this case we replace (3.36) by

$$\frac{dx}{ds} = -\lambda(x - 1)^2 \tag{3.48}$$

and repeat the steps above arriving at

$$f(x,\ t) = \left[1 + \frac{x - 1}{1 - (x - 1)t}\right]^{I} \tag{3.49}$$

It follows that

$$p_0(t) = \left(\frac{\mu t}{\mu t + 1}\right)^I \tag{3.50}$$

when $\lambda = \mu$. Finally then

$$\lim_{t \to \infty} p_0(t) = 1 \tag{3.51}$$

With this as background we now turn to a computer simulation of this birth-death process. We will compute an estimate of the probability of extinction. We will use (3.45) or (3.50) to compute the theoretical probability. We will also use (3.46), (3.47) or (3.51) to compute the limiting value of this theoretical probability. We will then compare the three results for $p_0(t)$: computer simulation, theoretical result and the limiting value as $t \to \infty$.

3.4 Monte Carlo Simulation

We now turn to a computer simulation of the stochastic birth-death process. We will use the simulation to estimate the probability of extinction for a given initial population, given time and for given probabilities of birth and death.

The simulation proceeds as follows. First a maximum time, M , is specified. If the population remains alive for this length of time, it will be said to have survived. If, on the other hand, the population is reduced to zero before a time of $\bar{M}$ has elapsed, the population becomes extinct. We will examine identical populations over and over again and count the number which survive and the number which become extinct. The ratio of the number of extinctions to the total number of populations examined is an estimate of the probability of extinction.

To this end we generate a random number* uniformly distributed between 0 and 1. If this random number, R , is between 0 and $\lambda N \Delta t$, then a birth is declared, and N is increased by one. If R is between $\lambda N \Delta t$ and $\lambda N \Delta t + \mu N \Delta t$, a death is declared, and N is decreased by 1 . If R exceeds $\lambda N \Delta t + \mu N \Delta t$, then neither a birth nor a death occurs, and N is unchanged. In any case the total time elapsed, X , is incremented by Δt .

At this point two checks are made. (1) If the total time exceeds the time for survival, M , a survival is recorded. The variable S records the number of survivals, so S is increased by one. (2) If the population, N , has reached zero, an extinction is recorded. The number of extinctions, E , is increased by one.

If neither of these checks are satisfied, we generate another random number, R , and proceed as before. Since each time we generate a random number we increase the time, X , eventually either $X > M$ or $N = 0$. Thus the process must stop with either survival or extinction.

This entire procedure is repeated a number of times, T , and the ratio of the number of extinctions recorded, E , to T is used as an estimate of the probability of extinction.

There is one technical problem yet to be considered. The probabilities of birth and death, λ and μ , are specified as is the time interval Δt . It is possible, therefore, that the population could become so large that $\lambda N \Delta t$ exceeds 1 . Thus R would always be less than $\lambda N \Delta t$. To prevent this from happeining, we allow Δt to vary. In particular we choose

$$\Delta t = \frac{1}{N}$$

where N is the population size. This allows us to use the following criteria:

*The computer generates a sequence of pseudo-random numbers, i.e., a sequence which "behaves" randomly in the sense that the sequence satisfies a set of statistical tests for randomness.

$$R < \lambda \qquad \text{implies a birth}$$

$$\lambda \leq R < \lambda + \mu \qquad \text{implies a death}$$

$$\lambda + \mu \leq R \qquad \text{implies no birth or death}$$

As the population becomes large, the time increments decrease but never reach zero.

A program to carry out this simulation and print the Monte Carlo estimate of $p_0(M)$ is given in Figure 3.1. In this program:

L = λ = probability of a birth in a time Δt.

U = μ = probability of a death in a time Δt.

M = time required until a survival is declared.

T = number of trials experiment is to be conducted (the larger T is, the better the estimate of $p_0(M)$).

E = number of extinctions.

S = number of survivals.

X = elapsed time at any given point in the calculation.

P = population at any given point in the calculation.

R = random number between 0 and 1.

K = index which indicates number of trial being conducted ($K \leq T$).

At the same time as the Monte Carlo simulation is being carried out, we can calculate $p_0(\tau)$ from (3.45) or (3.50) where $\tau = M$. To accomplish this we let

F = EXP(M*(L-U)

and

E1 = (U*(F - 1)/(L* F - U)) ↑ I

so

$$E1 = p_0(M)$$

If $\lambda = \mu$ then

$$E1 = (L^{*}M/(L^{*}M + 1)) \uparrow I$$

In either case the value of E1 is printed as the "THEORETICAL" value.

We also will calculate the probability $p_0(t)$ as $t \to \infty$. This is given by (3.46), (3.47) or (3.51). We let "LIMITING VALUE" be $\lim_{t \to \infty} p_0(t)$.

Then

$$\text{LIMITING VALUE} = \begin{cases} (U/L) \uparrow I & U < L \\ 1 & U \geq L \end{cases}$$

```
100  PRINT "TYPE STARTING RANDØM NØ."
200  INPUT Q
300  PRINT "TYPE BIRTH RATE"
400  INPUT L
500  PRINT "TYPE DEATH RATE"
600  INPUT U
700  IF L+U<=1 THEN 1000
800  PRINT "BIRTH RATE PLUS DEATH RATE EXCEEDS 1"
900  GØ TØ 300
1000  PRINT "TYPE INITIAL PØPULATIØN"
1100  INPUT I
1200  LET P=I
1300  PRINT "TYPE TIME REQUIRED FØR SURVIVAL"
1400  INPUT M
1500  LET X = 0
1600  PRINT "TYPE TØTAL NØ. ØF TRIALS"
1700  INPUT T
1800  IF L<=U THEN 2100
1900  LET E2=(U/L)↑I
2000  GØ TØ 2200
2100  LET E2=1
2200  PRINT
2300  PRINT "PRØBABILITY ØF EXTINCTIØN"
2400  PRINT "LIMITING VALUE",E2
2500  IF L=U THEN 2900
2600  LET F=EXP(M*(L-U))
2700  LET E1=(U*(F-1)/(L*F-U))↑I
2800  GØ TØ 3000
2900  LET E1=(L*M/(L*M+1))↑I
3000  PRINT "THEØRETICAL",E1
3100  LET E=0
3200  LET S=0
3300  FØR K=1 TØ T
3400  IF P=0 THEN 4500
3500  LET X = X +(1/P)
3600  LET R=RND(Q)
3700  IF R<L THEN 4000
3800  IF R<L+U THEN 4200
3900  GØ TØ 4300
4000  LET P=P+1
4100  GØ TØ 4300
4200  LET P=P-1
4300  IF X>=M THEN 4700
4400  GØ TØ 3400
4500  LET E=E+1
4600  GØ TØ 4800
4700  LET S=S+1
4800  LET P=I
4900  LET X = 0
5000  NEXT K
5100  PRINT "MØNTE CARLØ",E/(E+S)
5200  PRINT
5300  PRINT "EXTINCTIØNS",E,"SURVIVALS",S
5400  GØ TØ 100
5500  END
```

FIGURE 3.1

Suppose we start with $\lambda = \mu = 1/2$, an initial population of 20, and require survival for a time of 100 in order for survival to be declared. We will repeat the simulation 50 times and estimate the probability of extinction. The results are

```
TYPE STARTING RANDOM NO.
?#?
.451
TYPE BIRTH RATE
?.5
TYPE DEATH RATE
?.5
TYPE INITIAL POPULATION
?20
TYPE TIME REQUIRED FOR SURVIVAL
?100
TYPE TOTAL NO. OF TRIALS
?50

PROBABILITY OF EXTINCTION
LIMITING VALUE   1
THEORETICAL      .67297133307
MONTE CARLO      0.66

EXTINCTIONS      33              SURVIVALS      17
```

Notice that the Monte Carlo result is quite close to p_0 as calculated from (3.50) — see THEORETICAL. We now repeat the experiment for a longer time period. In particular we require 200 time intervals for survival. The results are

```
TYPE STARTING RANDOM NO.
?#?
.451
TYPE BIRTH RATE
?.5
TYPE DEATH RATE
?.5
TYPE INITIAL POPULATION
?20
TYPE TIME REQUIRED FOR SURVIVAL
?200
TYPE TOTAL NO. OF TRIALS
?50

PROBABILITY OF EXTINCTION
LIMITING VALUE   1
THEORETICAL      .81954447028
MONTE CARLO      0.8

EXTINCTIONS      40              SURVIVALS      10
```

Once again the Monte Carlo result is quite accurate. Notice also that the THEORETICAL value is closer to the LIMITING VALUE. Since the total time, M, is larger, we should expect threse two results, (3.50) and (3.51), to be closer together than they were in the first trial.

Next we try $\lambda < \mu$. In particular we let $\lambda = .4$ and $\mu = .6$. If we start with a population of 25 and require 100 time periods for survival, the results are:

```
TYPE STARTING RANDØM NØ.
?#?
.5
TYPE BIRTH RATE
?.4
TYPE DEATH RATE
?.6
TYPE INITIAL PØPULATIØN
?25
TYPE TIME REQUIRED FØR SURVIVAL
?100
TYPE TØTAL NØ. ØF TRIALS
?50

PRØBABILITY ØF EXTINCTIØN
LIMITING VALUE    1
THEØRETICAL       .99999998286
MØNTE CARLØ       1

EXTINCTIØNS       50              SURVIVALS        0
```

Somewhat surprisingly, all three probabilities are 1.

With these same birth and death rates, we start with a larger initial population (50) but require only 50 time periods for survival. The results are:

```
TYPE STARTING RANDØM NØ.
?#?
.5
TYPE BIRTH RATE
?.4
TYPE DEATH RATE
?.6
TYPE INITIAL PØPULATIØN
?50
TYPE TIME REQUIRED FØR SURVIVAL
?50
TYPE TØTAL NØ. ØF TRIALS
?50

PRØBABILITY ØF EXTINCTIØN
LIMITING VALUE   1
THEØRETICAL      .99924359218
MØNTE CARLØ      1

EXTINCTIØNS      50               SURVIVALS        0
```

Again the Monte Carlo results are quite good.

Finally we try $\lambda > \mu$. We let $\lambda = .501$ and $\mu = .499$. The results are

```
TYPE STARTING RANDØM NØ.
?#?
.5
TYPE BIRTH RATE
?.501
TYPE DEATH RATE
?.499
TYPE INITIAL PØPULATIØN
?20
TYPE TIME REQUIRED FØR SURVIVAL
?100
TYPE TØTAL NØ. ØF TRIALS
?50

PRØBABILITY ØF EXTINCTIØN
LIMITING VALUE   0.9231162479
THEØRETICAL      .64571382048
MØNTE CARLØ      0.64

EXTINCTIØNS      32               SURVIVALS        18
```

One note of caution. Monte Carlo simulations can consume an inordinate amount of computer time. They should be used only as a last resort. In general if there is any way other than Monte Carlo to solve a problem, use the other way.

In particular for this birth-death simulation, one should beware of increasing either the time required for survival on the total number of trials beyond those used in the above examples.

CHAPTER III

AUTHORS' EVALUATION

(Please circle one of the responses to each question)

1. Did you attend the short course in 1974-75? Yes No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ______

   ______

   ______

   can you suggest topics to be added? ______

   ______

   ______

   If (b), which topics should be abbreviated? ______

   ______

   ______

   which topics should be eliminated? ______

   ______

   ______

3. Could you read and understand the computer programs?

   (a) always (c) seldom

   (b) sometimes (d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy (b) Too difficult

6. Was the number of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attempt any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

___

___

___

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Stochastic model of birth-death process	___	___	___	___
Differential difference equation formulation	___	___	___	___
Method of characteristics for solving partial differential equations	___	___	___	___
Monte Carlo simulations	___	___	___	___

CHAPTER IV

A PREDATOR - PREY MODEL

4.1 A Simple Model

Consider two species one of which preys upon the other. For definiteness consider foxes as the predator which preys upon pheasants. In the absence of foxes the pheasants grow as described in Section 1.1 (see eq. (1.2)). That is

$$P_{k+1} = (1 + A - BP_k)P_k \tag{4.1}$$

where P_k is the pheasant population at the end of the kth period. However, if foxes are present they will retard the growth of the pheasants. We will assume that each fox alive at the start* of the kth period will consume some pheasants. The more pheasants there are the easier it is for a fox to find a pheasant on which to prey. Therefore, we assume that the number of pheasants consumed by any one fox is directly proportional to the number of pheasants alive at the start of the kth period.** Based on these assumptions the number of pheasants consumed by one fox is CP_k. The total number of pheasants consumed by F_k foxes is CP_kF_k where $C > 0$. The constant C depends upon the search capability of the foxes and the ground cover provided for the pheasants. If there is a large amount of foliage then the pheasants can hide more easily and fewer are killed by foxes. Thus good ground cover reduces C. Similarly large values of C correspond to poor ground cover.

*Foxes born during the kth period are too young to be predators.

**This assumes that foxes do not prey upon very young pheasants or upon eggs.

Equation (4.1) then becomes

(4.2) $$P_{k+1} = (1 + A - BP_k)P_k - CP_kF_k$$

We now turn our attention to the fox population. We will let F_k be the number of foxes alive at the start of the kth period. Suppose pheasants provide the sole food supply for the foxes. In the absence of pheasants then the fox population will vanish. We will assume that if the pheasant population becomes zero, the fox population will die out in one period. Thus $P_k = 0$ implies that the fox population at the end of the kth period vanishes, i.e., $F_{k+1} = 0$. We will also assume that the fox population at the end of a period is directly proportional to the population at the beginning of the period. These two assumptions lead to

(4.3) $$F_{k+1} = DP_kF_k$$

where $D > 0$ and represents the rate at which foxes convert their food supply into population growth. If the food conversion process is efficient, then D is large.

4.2 Equilibrium and Stability

Before carrying out some numerical experiments we turn to a discussion of equilibrium populations and the stability of these equilibrium populations.

Will the pheasant and fox populations reach equilibrium?

If we wish to institute predator control (kill foxes), when should we do so, and how many preadators should be eliminated.

If we wish to allow pheasants to be hunted, when should we do so, and how many pheasants should we allow to be killed?

What would be the effect of periodically adding more pheasants to the population?

We will attempt in what follows and in the exercises to answer these questions. The reader is once again urged to think of other similar questions which might be of interest to him or to ecologists and to use mathematics and/or computing to search for the answers. The following should provide a guide for such answer-seeking activities.

We first determine what the equilibrium populations are. If both pheasants and foxes are to reach equilibrium then for k sufficiently large both

$$P_{k+1} = P_k = P_*$$

and

$$F_{k+1} = F_k = F_*$$

where P_* and F_* are the equilibrium populations. Using these in (4.2) and (4.3)

(4.4) $$P_* = (1 + A - BP_*)P_* - C\,F_*P_*$$

(4.5) $$F_* = D\,P_*\,F_*$$

These are two algebraic equations in two unknowns P_* and F_*.

From (4.5) either

$$F_* = 0$$

or

$$P_* = 1/D$$

Consider first $F_* = 0$ then (4.4) becomes

$$P_* = (1 + A - BP_*)\,P_*$$

this is the same equilibrium equation we met in the single species model. Our earlier analysis then tells us there are two solutions: $P_* = 0$ or $P_* = A/B$. We have then found two equilibrium conditions. Either

$$F_* = 0 \qquad (4.6)$$
and
$$P_* = 0$$

or

$$F_* = 0 \qquad (4.7)$$
and
$$P_* = A/B$$

There is, of course, still the case where $P_* = 1/D$ to be considered. In this case (4.4) leads to

$$F_* = \frac{AD - B}{DC}$$

A third equilibrium condition then is

$$F_* = \frac{AD - B}{DC} \qquad (4.8)$$
and
$$P_* = 1/D$$

The first condition, where both populations vanish, is not of much interest, but we shall investigate the other two. For ease of reference we will define

$$P_E = A/B \qquad (4.9)$$

$$P_e = 1/D \qquad (4.10)$$

$$F_e = \frac{AD - B}{DC} \qquad (4.11)$$

The two equilibrium conditions then are (i) 0 foxes and P_E pheasants, and (ii) F_e foxes and P_e pheasants. The subscripts E and e derive from the word "equilibrium". As we shall see, the lower case e refers to a lower value of the pheasant population than does the upper case E .

To investigate the stability of these equilibrium conditions and, in more general terms, the behavior of the fox and pheasant populations, we will turn to a computer program and some numerical experiments. In order to do so, however, we would need to assign values of A, B, C and D. It is unlikely that we could obtain the values of all of these parameters even were we to ask an experienced ecologist. On the other hand, an ecologist might be able to estimate the values of P_E, P_e and F_e. For example, P_E is just the number of pheasants which the environment can support if no foxes are present. Similarly P_e and F_e are the numbers of pheasants and foxes which would be present in a completely balanced situation.

Could we compute A, B, C and D if we were given the values of P_E, P_e and F_e? The answer, unfortunately, is no. Recall that in the single species problem we could not compute both A and B from N_E. However, from N_E and A we could compute B. That is to say, we needed one of the original parameters in addition to the equilibrium population. The same is true here in the predator-prey problem. Given values of the equilibrium populations and the value of say A, we can compute all of the other parameters. Indeed, (4.9), (4.10) and (4.11) can be solved to obtain

$$B = \frac{A}{P_E} \tag{4.12}$$

$$C = \frac{A(P_E - P_e)}{P_E F_e} \tag{4.13}$$

$$D = \frac{1}{P_e} \tag{4.14}$$

We have assumed that none of P_E, P_e or F_e are zero. Moreover, since our original equations assumed that B, C and D are positive, it follows that $P_E > P_e$, i.e., the pheasant equilibrium in the absence of foxes is larger than the pheasant equilibrium population if foxes are present.

We now construct a computer program which takes P_E, P_e, F_e and A as input. (A is the unrestricted pheasant growth rate). The program computes B, C and D then asks for starting pheasant and fox populations and for the number of periods to be predicted. The program is shown in Figure 4.1. It should be self explanatory except perhaps for the statements numbered 2400 and 2600. Each of those checks to see if one of the two species has dropped below 1. If so that population is set equal to zero. Numerically, of course, we could obtain negative values for the populations. In the single species case this was of little concern since we merely assumed, after the fact, that negative values meant zero populations. In this two species case, however, if the fox population became negative it would affect the pheasant population in an unnatural way. Therefore, it is crucial that we prevent negative populations. Finally we note that it may seem more natural to set a population equal to zero if its value drops below 2 or even some larger number. Our choice of 1 is somewhat arbitrary and could be changed without affecting the general behavior of the populations.

```
0100 PRINT "TYPE PHEASANT EQUILIBRIUM PØPULATIØN IN ABSENCE ØF FØXES"
0200 INPUT P1
0300 PRINT "TYPE PHEASANT EQUILIBRIUM PØPULATIØN WITH FØXES PRESENT"
0400 INPUT P2
0500 PRINT "TYPE FØX EQUILIBRIUM PØPULATIØN"
0600 INPUT F2
0700 PRINT "TYPE PHEASANT UNRESTRICTED GRØWTH RATE"
0800 INPUT A
0900 LET B=A/P1
1000 LET C=A*(P1-P2)/(P1*F2)
1100 LET D=1/P2
1200 PRINT "TYPE INITIAL PHEASANT PØPULATIØN"
1300 INPUT P
1400 PRINT "TYPE INITIAL FØX PØPULATIØN"
1500 INPUT F
1600 PRINT "TYPE NØ. ØF PERIØDS TØ BE PREDICTED"
1700 INPUT N
1800 PRINT
1900 PRINT "PERIØD","PHEASANTS","FØXES"
2000 FØR I=0 TØ N
2100 PRINT I,P,F
2200 LET P3=(1+A-B*P)*P-C*F*P
2300 F3=D*P*F
2400 IF P3>1 THEN 2600
2500 LET P3=0
2600 IF F3>1 THEN 2800
2700 LET F3=0
2800 LET P=P3
2900 LET F=F3
3000 NEXT I
3100 END
```

Figure 4.1

We now use this program and examine some specific examples. One case is exhibited in Figure 4.2. The pheasant and fox populations are started relatively close to the equilibrium values. In particular the initial number of pheasants (9,000) is somewhat less than the equilibrium value (10,000), and the initial number of foxes (1,100) is somewhat larger than the equilibrium value (1,000). In each case the difference between the initial population and the equilibrium population is 10%.

Notice that for all practical purposes each 20 periods the populations return to the same values. In order to study these results more carefully we draw a graph of the populations. We could, of course, plot each of the populations of the two species as functions of time (number of periods). However, we will find it more useful to plot the number of foxes as a function of the number of pheasants. The graph is shown in Figure 4.3. The initial point is indicated by the large circle. The arrows indicate the order in which the points appear as time progresses. The equilibrium solution $F_E = 1,000$ and $P_E = 10,000$ is indicated by the large X .

From the cyclical nature of this behavior it appears that the equilibrium P_e and F_e is unstable. Indeed this is the case.

Let us now try another example as shown in Figure 4.4. Notice that in this case even though the foxes start at twice the equilibrium value, the pheasant population is insufficient to sustain the foxes and the latter become extinct rather quickly. Thereafter, the pheasant population continues to thrive and should eventually reach 10,000.

We could continue such numerical experiments, but we shall not do so here. Rather we turn our attention to a discussion of hunting seasons and predator control.

Self-Study: Problem #4.1

Assume that the growth of the pheasant population is *not* affected by over-crowding. Simplify the model discussed above accordingly and discuss the equilibrium and stability question.

```
TYPE PHEASANT EQUILIBRIUM POPULATION IN ABSENCE OF FOXES
?20000
TYPE PHEASANT EQUILIBRIUM POPULATION WITH FOXES PRESENT
?10000
TYPE FOX EQUILIBRIUM POPULATION
?1000
TYPE PHEASANT UNRESTRICTED GROWTH RATE
?.2
TYPE INITIAL PHEASANT POPULATION
?9000
TYPE INITIAL FOX POPULATION
?1100
TYPE NO. OF PERIODS TO BE PREDICTED
?40
```

PERIOD	PHEASANTS	FOXES
0	9000	1100
1	9000.	990.
2	9099.	891.
3	9280.17	810.722
4	9522.62	752.363
5	9803.9	716.447
6	10101.1	702.397
7	10391.5	709.5
8	10652.7	737.278
9	10863.	785.4
10	11002.4	853.184
11	11053.7	938.708
12	11004.9	1037.62
13	10853.	1141.89
14	10606.4	1239.29
15	10288.3	1314.44
16	9935.12	1352.33
17	9591.52	1343.55
18	9301.18	1288.67
19	9097.68	1198.62
20	8999.08	1090.46
21	9007.74	981.318
22	9113.95	883.945
23	9300.47	805.623
24	9546.31	749.268
25	9828.98	715.274
26	10125.7	703.042
27	10413.6	711.876
28	10670.6	741.32
29	10875.1	791.032
30	11007.2	860.252
31	11050.1	946.892
32	10992.8	1046.33
33	10832.7	1150.2
34	10579.8	1245.98
35	10258.2	1318.22
36	9905.29	1352.26
37	9565.74	1339.45
38	9282.57	1281.28
39	9088.07	1189.36
40	8998.85	1080.9

Figure 4.2

8-4.9

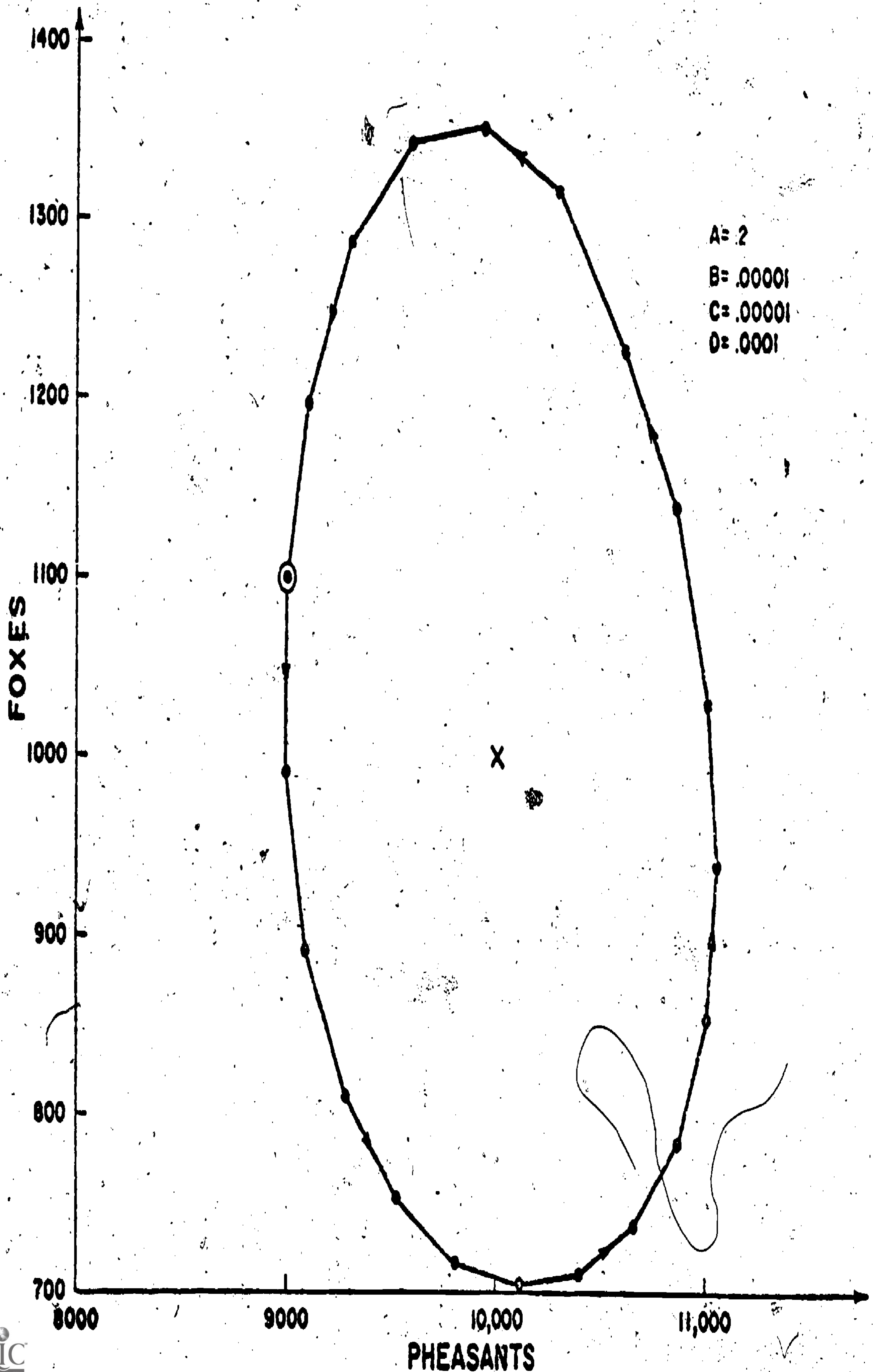


Figure 4.3

```
TYPE PHEASANT EQUILIBRIUM POPULATION IN ABSENCE OF FOXES
?10000
TYPE PHEASANT EQUILIBRIUM POPULATION WITH FOXES PRESENT
?5000
TYPE FOX EQUILIBRIUM POPULATION
?1000
TYPE PHEASANT UNRESTRICTED GROWTH RATE
?.1
TYPE INITIAL PHEASANT POPULATION
?1000
TYPE INITIAL FOX POPULATION
?2000
TYPE NO. OF PERIODS TO BE PREDICTED
?20
```

PERIOD	PHEASANTS	FOXES
0	1000	2000
1	990	400.
2	1059.4	79.2
3	1149.92	16.7809
4	1250.72	3.85933
5	1359.91	0
6	1477.41	0
7	1603.32	0
8	1737.95	0
9	1881.54	0
10	2034.29	0
11	2196.34	0
12	2367.73	0
13	2548.44	0
14	2738.34	0
15	2937.19	0
16	3144.64	0
17	3360.21	0
18	3583.32	0
19	3813.25	0
20	4049.17	0

Figure 4.4

Solution to Self-Study: Problem #4.1

If crowding is neglected then $B = 0$. Then the model becomes

$$P_{k+1} - P_k = AP_k - C P_k F_k \qquad A > 0, C > 0$$

$$F_{k+1} = D P_k F_k \qquad D > 0$$

Letting P_* and F_* be the equilibrium populations

$$P_* (A - C F_*) = 0$$

$$F_* (1 - D P_*) = 0 ,$$

The equilibria are

$$F_* = 0, P_* = 0$$

$$F_* = \frac{A}{C}, \quad P_* = \frac{1}{D}$$

The stability of these equilibria can be investigated either by means of a computer program similar to that of Figure 4.1 or by techniques similar to those employed in Part I of these notes. The changes required in Figure 4.1 are

```
 100
 200
 900   LET B = 0
1000   LET C = A/F2
```

The equilibrium $F_* = 0$, $P_* = 0$ is easily seen to be unstable. For if we consider the starting values $F_0 = 0$, P_0 arbitrary, we find $F_k = 0$, $P_k = (1 + A)^k P_0$. (See e.g. Section 1.1 of the appendix).
Thus no matter how small P_0 is, as long as P_0 and A are positive, $P_0 \to \infty$

To investigate $F_* = A/C$, $P_* = 1/D$, we set

$$F_k = f_k + A/C , \quad P_k = p_k + 1/D$$

and drop terms in $f_k p_k$. On eliminating f_k from the resulting equations, we find

$$p_{k+2} - 2 p_{k+1} + (1 + A) p_k = 0$$

$$f_k = \frac{D}{C} (p_k - p_{k+1})$$

The second order difference equation may be solved using the techniques described in Section 2.2 of the appendix. The solution for p_k is

$$p_k = \alpha(1 + A)^{k/2} \cos(k\theta + \beta)$$

where

$$\theta = \pi - \cos^{-1} \frac{\sqrt{1 + A}}{1 + A}$$

and where α and β are constants determined by the initial values p_0 and f_0. The solution oscillates and the amplitude increases if $A > 0$. Therefore the equilibrium is unstable.

Self-Study: Problem # 4:2

Consider weakening the assumption that in the absence of pheasants at period k, the foxes are extinct at period $k + 1$ to the assumption that in the absence of pheasants, the foxes obey a simple (negative) growth law. Find the equilibrium populations using this new assumption and modify the program in Figure 4.1 in order to study the stability of the equilibria. Check your results by noting that the model discussed in the text is a special case of this one.

Solution to Self-Study: Problem #4.2

In the absence of pheasants

$$F_{k+1} - F_k = - G F_k , \quad 0 < G \leq 1$$

Thus, the full model is

$$P_{k+1} - P_k = (A - B P_k)P_k - C P_k F_k$$

$$F_{k+1} - F_k = - G F_k + D P_k F_k .$$

Note that $G = 1$ gives the model discussed in the text. The equilibrium populations are

(a) $F = 0$ and $P = 0$

of

(b) $F_E = 0$ and $P_E = A/B$

or

(c) $F_e = \dfrac{AD - BG}{CD}$ and $P_e = \dfrac{G}{D}$

Thus

$$B = A/P_E$$

$$C = \frac{A(P_E - P_e)}{P_E F_e}$$

$$D = \frac{G}{P_e}$$

Notice that the first two of these are identical with (4.12) and (4.13) and that the last reduces to (4.14) for $G = 1$.

The only changes required in the BASIC program in Figure 4.1 are

```
 820  PRINT "TYPE FOX UNRESTRICTED DEATH RATE"
 830  INPUT  G
1100  LET D = G/P2
2300  LET F3 = (1-G)* F + D*P*F
```

4.2 Predator Control

We continue our study of pheasants and foxes, and assume the two populations are governed by (4.3) and (4.4). We will consider the particular example shown graphically in Figure 4.3 although our discussion is applicable to more general cases as well.

Suppose that a decision has been made to institute predator control. Such a decision might result from one or several of the following policies:

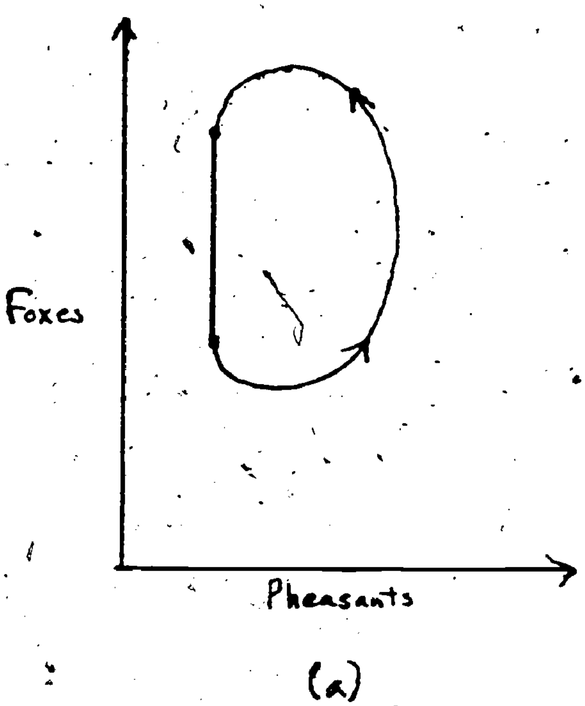
(1) The pheasant population should be kept above some minimum value, say 10,000.

(2) The fox population should be kept below some maximum value, say 1,200.

(3) It is desirable to allow foxes to be hunted and killed.

As has been our habit, we ask the reader to try to think of other policies which might lead to a decision to institute predator control. Moreover, we encourage the reader to think of more fundamental problems which might in turn give rise to the above policies. By way of example, it may be that if the fox population exceeds 1,200; the animals are so numerous that they create a nuisance to the human inhabitants and thus policy (2) is instituted.

Be that as it may, we now try to decide upon a sensible way of implementing predator control without upsetting the ecological balance. Our first inclination would be to kill foxes when the fox population was relatively high. Thus we look towards the upper part of the graph in Figure 4.3. As a start, suppose we decided to kill foxes at the end of period 18 when there are approximately 9,300 pheasants and 1,290 foxes. We will assume that the control season (time during which foxes are killed) is quite short compared with the time periods in our computations. Thus when we kill foxes, the pheasant population remains unchanged. This means that on our graph (Figure 4.3) we would travel downward on a vertical line from period 18. If we then abandon predator control the two populations would continue around the oval as before. The result would be the pattern indicated in Figure 4.5(a). This assumes, of course, that the predator control is reinstituted each time we reach 9,300 pheasants and 1,290 foxes.



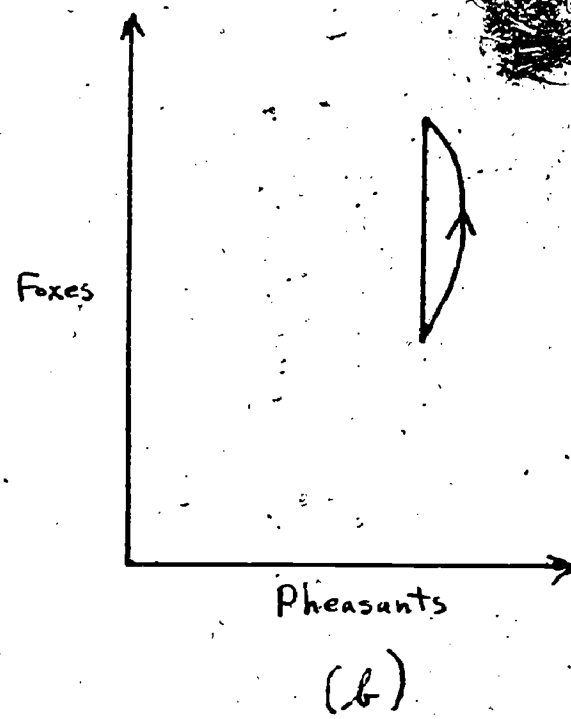


Figure 4.5

Rather than use control at period 18, suppose we kill foxes at the close of period 14 where the pheasant population is 10,600 and the fox population is 1,240. Once again we travel vertically downward and then resume the oval. The result is shown in Figure 4.5(b).

Let us now examine these two alternatives and make some observations. The first alternative has relatively little effect overall. The pheasant and fox populations both vary between approximately the same limits. Therefore if our goal was to satisfy policy (3) above, this might be a reasonable choice. The second choice which leads to Figure 4.5(b), on the other hand, has considerable effect. While the fox population still varies between widely separated values, the pheasant population changes very little. There is yet another observation to be made. In Figure 4.5(a) the time between hunting seasons is about 15 periods since we will drop from period 18 to period 3 conditions. In Figure 4.5(b), however, the time between hunting seasons is only 6 periods (period 14 to period 8).

What conclusions can we draw from these observations. The second course of action, the one in Figure 4.5(b), is better adapted to policy (1), i.e., keeps the pheasant population high. Neither solution does much for policy (2) since in both cases the fox population grows quite large. Even so, the second alternative does keep the maximum fox population a little lower than does the first choice. Finally if an objective is to have frequent hunting seasons on foxes then the second choice is clearly preferable. On the other hand, if predator control is an expense, and we wish to use it as infrequently as possible, then the first choice is the better one.

On balance then the second choice as shown in Figure 4.5(b) seems a wise one, and we shall choose it. We will concentrate on policy (2) of keeping the fox population below 1,200. Before devising a detailed strategy of predator control, however, we reexamine the assumptions we have made and discuss the consequences of any variations in these assumptions.

We assumed that the predator control season was quite small compared to the time periods in our calculations. This led to vertical lines in the graph. Suppose this is not so. As control is started in either of the cases in Figure 4.5 the pheasant population is decreasing. Therefore, the populations would follow a line, perhaps with slight curvature, down but slightly to the left. One possible pattern is shown in Figure 4.6(a). It corresponds to predator control at period 14, i.e., the second course of action.

A second assumption was that our control was so accurate that we killed precisely the number of foxes necessary to bring us back to our oval curve. This degree of precision is quite unlikely, and we are more likely to overkill or underkill. In either case, provided we do not miss our target number by too much, we simply end on another similar shaped curve. The case of two successive overkills are shown in Figure 4.6(b) where the pattern moves from left to right. Of course, upon observing

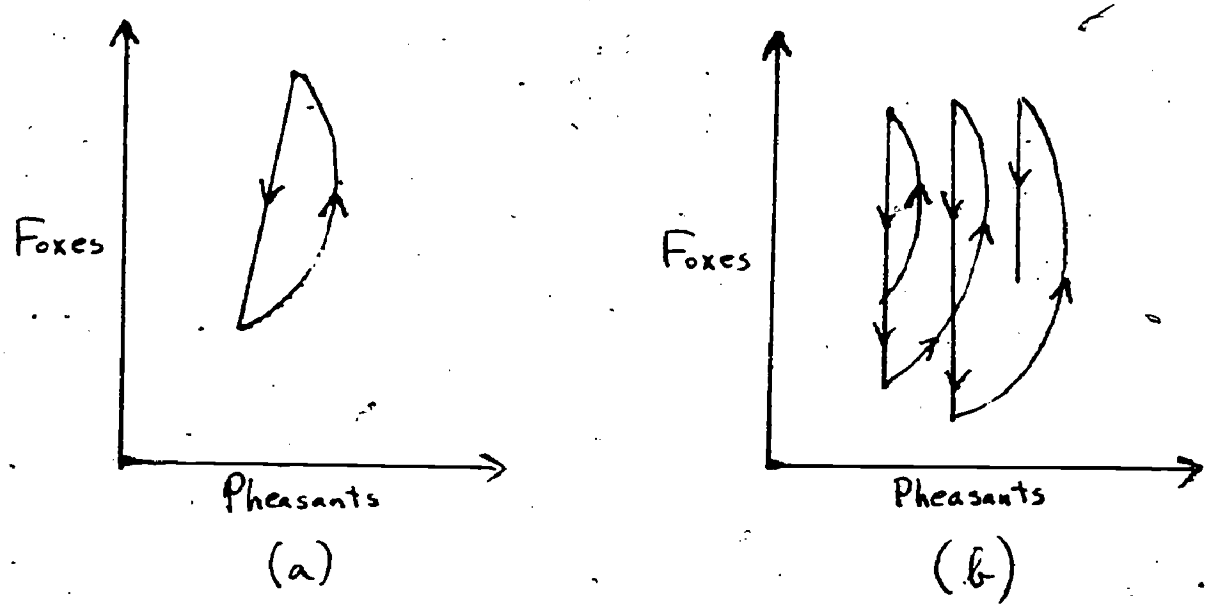


Figure 4.6

such a pattern, it is likely that an adjustment to produce an underkill would be made for one or two hunting seasons.

We now return to the details of our predator control and to the program in Figure 4.1. We will change the program so that if the fox population exceeds 20% of the equilibrium fox population*, the program will (a) tell us what the two populations would be in the absence of control, (b) allow us to determine a number of foxes to be killed, and (c) continue the computations with the same numbers of pheasants but a reduced number of foxes.

To accomplish this we add the program steps in Figure 4.7 to the BASIC program in Figure 4.1.

```
2800   IF F3 < 1.2 * F2 THEN 2900
2810   PRINT "PREDATOR CONTROL WITH NO CONTROL THERE WILL BE"
2820   PRINT F3; "FOXES AND"' P3; "PHEASANTS"
2830   PRINT "HOW MANY FOXES SHOULD BE KILLED?"
2840   INPUT  X
2850   LET F3 = F3 - X
2950   LET P = P3
```

Figure 4.7

*In this case this means if the fox population exceeds 1,200.

The first statement checks to see if $F_{k+1} < 1.2\ F_e$. If so we still bypass control even though the pheasant population is on the decline. Statement 2820 to 2850 should be self explanatory. Statement 2950 is necessary since we deleted the previous LET P = P3 by our new statement 2800.

An example of running this revised program is shown in Figure 4.8. Some explanation of this figure is in order. The first predator control is at period 14. Since there are 10606.4 pheasants we look back in time for a season with approximately the same number of pheasants. If we could find a period with precisely that number of pheasants, we would kill a number of foxes which would return us to that state. The best we can do, however, is season 8 where there are 10652.7 pheasants. The fox population there is $F_8 = 737.278$. To reduce the fox population to this latter figure we would need to kill 502.012 foxes. However, since the pheasant population at period 14 is slightly less than P_8 , we choose to kill more than 502 foxes. In fact we kill 520.

The second time that control is exerted is six periods later where $P_{20} = 10707.3$. This pheasant population is about midway between P_{14} and P_{15} so we kill a number of foxes which will leave the fox population about midway between F_{14} and F_{15} . This leads to a kill of 480 foxes.

At period 26 we kill 530 foxes and arrive at populations slightly less than period 8 so that we are slightly inside the original oval. At period 32 we kill 465 foxes to achieve populations slightly larger than period 8 so we are slightly outside of the original oval. The remainder of the computer printout and the strategy used to achieve it should be self evident.

Our strategy has been simple enough. At each point where the fox population exceeds 1,200 we look back in history for a state in which the pheasant population is close to the present pheasant population. Upon finding such a state we make note of the fox population of that past state. We then kill enough foxes to bring the present fox population down to the lower value of the past state. Of course, we are not able to find the precise pheasant population in any past season. Having chosen the nearest one, we make some compensating adjustment in the fox kill.

```
TYPE PHEASANT EQUILIBRIUM POPULATION IN ABSENCE OF FOXES
?20000
TYPE PHEASANT EQUILIBRIUM POPULATION WITH FOXES PRESENT
?10000
TYPE FOX EQUILIBRIUM POPULATION
?1000
TYPE PHEASANT UNRESTRICTED GROWTH RATE
?.2
TYPE INITIAL PHEASANT POPULATION
?9000
TYPE INITIAL FOX POPULATION
?1100
TYPE NO. OF PERIODS TO BE PREDICTED
?35

PERIOD          PHEASANTS       FOXES
 0               9000            1100
 1               9000.           990.
 2               9099.           891.
 3               9280.17         810.722
 4               9522.62         752.363
 5               9803.9          716.447
 6               10101.1         702.397
 7               10391.5         709.5
 8               10652.7         737.278
 9               10863.          785.4
 10              11002.4         853.184
 11              11053.7         938.708
 12              11004.9         1037.62
 13              10853.          1141.89
```

Figure 4.8

(Part 1)

```
PREDATOR CONTROL. WITH NO CONTROL THERE WILL BE
 1239.29     FOXES AND 10606.4    PHEASANTS
HOW MANY FOXES SHOULD BE KILLED?
?520
 14                10606.4          719.287
 15                10839.8          762.904
 16                11005.8          826.973
 17                11085.5          910.149
 18                11064.8          1008.95
 19                10937.1          1116.38
PREDATOR CONTROL. WITH NO CONTROL THERE WILL BE
 1220.99     FOXES AND 10707.3    PHEASANTS
HOW MANY FOXES SHOULD BE KILLED?
?480
 20                10707.3          740.993
 21                10908.9          793.404
 22                11035.1          865.516
 23                11069.3          955.108
 24                11000.6          1057.24
 25                10827.6          1163.03
PREDATOR CONTROL. WITH NO CONTROL THERE WILL BE
 1259.28     FOXES AND 10561.5    PHEASANTS
HOW MANY FOXES SHOULD BE KILLED?
?530
 26                10561.5          729.279
 27                10788.1          770.225
 28                10950.9          830.926
 29                11032.           909.943
 30                11017.5          1003.85
 31                10901.1          1105.98
PREDATOR CONTROL. WITH NO CONTROL THERE WILL BE
 1205.65     FOXES AND 10687.4    PHEASANTS
HOW MANY FOXES SHOULD BE KILLED?
?465
```

Figure 4.8

(Part 2)

Self-Study: Problem #4.3

Devise other strategies for predator control and modify the BASIC program in Figure 4.1 to implement that control. In particular, when the fox population is more than 20% above equilibrium reduce the fox population to equilibrium.

Solution to Self-Study: Problem #4.3

The required modifications to Figure 4.1 are

```
2800   IF F3 < 1.2 * F2 THEN 2900
2810   PRINT "PREDATOR CONTROL"
2820   LET  X = F3 - F2
2830   PRINT "KILL"; X ; "FOXES"
2840   LET  F3 = F2
2950   LET P = P3
```

Self-Study: Problem #4.4

Referring back to Self-Study Problem #4.2 write down a pair of first-order differential equations which represent the continuous model for the pheasant-fox problem.

Solution to Self-Study: Problem #4.4

$$\frac{dP}{dt} = (A - BP)P - CFP.$$

$$\frac{dF}{dt} = - GF + DPF$$

Notice that $G = 1$ no longer implies the model used in the text since this G is Δt times the G in Self-Study Problem # 4.2.

CHAPTER IV

AUTHORS' EVALUATION

(Please circle one of the responses to each question)

1. Did you attend the short course in 1974-75? Yes No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ______________________

   ____

   ____

   can you suggest topics to be added? ______________________

   ____

   ____

   If (b), which topics should be abbreviated? ______________________

   ____

   ____

   which topics should be eliminated? ______________________

   ____

   ____

3. Could you read and understand the computer programs?

   (a) always (c) seldom

   (b) sometimes (d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy (b) Too difficult

6. Was the <u>number</u> of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attempt any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

____

____

____

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Cyclical Nature of population				
Predator Control Methods				
Effects of Over (Under) kill				
Multiple Species Models in General				

CHAPTER V

AN ECONOMIC MODEL

5.1 A Simple Model

Assume that the total national income, T, can be separated into three parts: consumer expenditures, C; private investment, I; and government expenditures, G. Thus

$$T = C + I + G$$

All of these quantities vary with time. Usually the values of each of these are known only at specific times --- the end of a year or the end of a quarter of a year. We will assume then that each of these four quantities is measured and known at fixed points in time. Let T_n, C_n, I_n and G_n be the values of the total income, consumer expenditures, private investment and government expenditures for the nth period where $n = 0, 1, 2, \ldots$. Then

(5.1) $$T_n = C_n + I_n + G_n$$

Next we assume that consumers' buying habits are affected favorably by the total national income. However, consumers only know the value of the national income for the periods prior to the current one. We will assume that consumers have a short memory so their buying habits in the nth period are only affected by the total national income in the $(n - 1)$st period. Moreover, we will assume that C_n is some percentage (perhaps greater than 100%) of T_{n-1}, i.e.,

(5.2) $$C_n = AT_{n-1} \qquad n = 1, 2, 3, \ldots$$

The constant of proportionality is called the *marginal propensity to consume*. We assume that

$$A > 0$$

Next we assume that an increase in consumer spending will increase investment of private capital. We suppose that investment is proportional to a *change* in consumer spending so that

(5.3) $$I_n = B(C_n - C_{n-1}) \qquad n = 1, 2, 3, \ldots$$

We assume that

$$B > 0$$

Finally we assume that government spending is constant

(5.4) $$G_n = 1 \qquad n = 0, 1, 2, \ldots$$

Using (5.2), (5.3) and (5.4) in (5.1) we arrive at

(5.5) $$T_n = A(1 + B)T_{n-1} - ABT_{n-2} + 1 \qquad n = 2, 3, 4, \ldots$$

which can be rewritten

(5.6) $$T_{n+2} - A(1 + B)T_{n+1} + ABT_n = 1 \qquad n = 0, 1, 2, \ldots$$

Given T_0 and T_1 we can calculate, successively, T_2, T_3, T_4 and so on.

5.2 Numerical Solution

We will write a BASIC program which takes as input A, B, T_0 and T_1 together with the final period to be predicted. If M is this final period then the program computes and prints $T_2, T_3, \ldots, T_M$.

The program is

```
100 PRINT "TYPE VALUE FOR A"
200 INPUT A
300 PRINT "TYPE VALUE FOR B"
400 INPUT B
500 PRINT "TYPE NATIONAL INCOME IN ZEROTH PERIOD"
600 INPUT TO
700 PRINT "TYPE NATIONAL INCOME IN FIRST PERIOD"
800 INPUT T1
900 PRINT "TYPE FINAL PERIOD TO BE PREDICTED"
1000 INPUT M
1100 PRINT
1200 PRINT "PERIOD","NATIONAL INCOME"
1300 FOR K = 2 TO M
1400 LET T2 = A*(1+B)*T1 - A*B*TO + 1
1500 PRINT K,T2
1600 LET TO = T1
1700 LET T1 = T2
1800 NEXT K
1900 END
```

This program was run several times each time with $T_0 = 2$ and $T_1 = 3$. (See computer output on the following pages.).

The first case ($A = .5$ and $B = 1$) shows an economy which oscillates about 2 and eventually settles down to 2. Do you think changing T_0 and/or T_1 would effect the long run behavior of this economy?

The second case ($A = .8$ and $B = 2$) also oscillates (about 5 not 2) but the oscillations become quite large. Indeed T_{12} is negative! From an economist's point of view the national economy has collapsed at period 12 and the solution thereafter is meaningless. We return to this case in the next section.

The third solution ($A = .5$ and $B = -1$) produces a national income which steadily decreases to the value 2. Do you think a change in T_0 and/or T_1 would change this long run behavior? Try values for T_0 and T_1 where both are larger than 2 and where both are smaller than 2.

```
TYPE VALUE FOR A
?.5
TYPE VALUE FOR B
?1
TYPE NATIONAL INCOME IN ZEROTH PERIOD
?2
TYPE NATIONAL INCOME IN FIRST PERIOD
?3
TYPE FINAL PERIOD TO BE PREDICTED
?40

PERIOD          NATIONAL INCOME
 2               3
 3               2.5
 4               2
 5               1.75
 6               1.75
 7               1.875
 8               2
 9               2.0625
 10              2.0625
 11              2.03125
 12              2
 13              1.984375
 14              1.984375
 15              1.992188
 16              2
 17              2.003906
 18              2.003906
 19              2.001953
 20              2
 21              1.999023
 22              1.999023
 23              1.999512
 24              2
 25              2.000244
 26              2.000244
 27              2.000122
 28              2
 29              1.999939
 30              1.999939
 31              1.999969
 32              2
 33              2.000015
 34              2.000015
 35              2.000008
 36              2
 37              1.999996
 38              1.999996
 39              1.999998
 40              2
```

```
TYPE VALUE FOR A
?.8
TYPE VALUE FOR B
?2
TYPE NATIONAL INCOME IN ZEROTH PERIOD
?2
TYPE NATIONAL INCOME IN FIRST PERIOD
?3
TYPE FINAL PERIOD TO BE PREDICTED
?20

PERIOD            NATIONAL INCOME
 2                 5
 3                 8.2
 4                 12.68
 5                 18.312
 6                 24.6608
 7                 30.88672
 8                 35.67085
 9                 37.19128
 10                33.18572
 11                21.13968
 12               -1.36198
 13               -36.0921
 14               -83.44197
 15               -141.5134
 16               -205.1249
 17               -264.8784
 18               -306.5084
 19               -310.8146
 20               -254.5416
```

```
TYPE VALUE FOR A
?.5
TYPE VALUE FOR B
?.1
TYPE NATIONAL INCOME IN ZEROTH PERIOD
?2
TYPE NATIONAL INCOME IN FIRST PERIOD
?3
TYPE FINAL PERIOD TO BE PREDICTED
?20

PERIOD          NATIONAL INCOME
 2               2.55
 3               2.2525
 4               2.111375
 5               2.048631
 [illegible]     2.021178
 7               2.009817
 8               2.00401
 9               2.001745
 10              2.000759
 11              2.00033
 12              2.000144
 13              2.000063
 14              2.000027
 15              2.000012
 16              2.000005
 17              2.000002
 18              2.000001
 19              2.
 20              2.
```

```
TYPE VALUE FOR A
?.5
TYPE VALUE FOR B
?6
TYPE NATIONAL INCOME IN ZEROTH PERIOD
?2
TYPE NATIONAL INCOME IN FIRST PERIOD
?3
TYPE FINAL PERIOD TO BE PREDICTED
?20

PERIOD            NATIONAL INCOME
 2                 5.5
 3                 11.25
 4                 23.875
 5                 50.8125
 6                 107.2188
 7                 223.8281
 8                 462.7422
 9                 949.1133
 10                1934.67
 11                3925.005
 12                7934.507
 13                15996.76
 14                32186.14
 15                64668.21
 16                129760.3
 17                260175.5
 18                521334.2
 19                1044144.
 20                2090583.
```

The final case shown above (A = .5 and B = 6) produces a national income which grows without bound. Economists call this an expanding economy. Again try different initial conditions, T_0 and T_1, to see if the behavior changes.

Self-Study: Problem #5.1

The examples have exhibited the following behaviors: oscillating with decreasing amplitude, oscillating with increasing amplitude, exponential decay and exponential growth. For the following values of A and B determine the behavior of the economy

(a) $A = .5$, $B = .5$, $T_0 = 2$, $T_1 = 3$

(b) $A = .5$, $B = 2$, $T_0 = 2$, $T_1 = 3$

(c) $A = .5$, $B = 4$, $T_0 = 2$, $T_1 = 3$

(d) $A = 0.75$, $B = 6$, $T_0 = 2$, $T_1 = 3$

(e) $A = .5$, $B = 4$, $T_0 = 2$, $T_1 = 2$

(f) $A = .5$, $B = 6$, $T_0 = 2$, $T_1 = 2$

(g) $A = .75$, $B = 6$, $T_0 = 4$, $T_1 = 4$

(h) $A = 1$, $B = .5$, $T_0 = 2$, $T_1 = 4$

(i) $A = 1$, $B = .5$, $T_0 = 2$, $T_1 = 5$

Solution to Self-Study: Problem #5.1

(a) Decaying oscillations $T_{10} = 2.00239$

(b) Oscillations with 'nearly' constant amplitude

$T_6 = 0.594$

$T_{11} = 3.505$

$T_{15} = 0.506$

$T_{20} = 3.436$

$T_{24} = 0.492$

(c) Increasing oscillations $T_{11} = -52.827$

(d) Exponential growth $T_{10} = 596,859$

(e) Constant $T_k = 2$

(f) Constant $T_k = 2$

(g) Constant $T_k = 4$

(h) Linear growth $T_{10} = 22$

(i) 'Almost' linear growth $T_{10} = 23.998$

5.3 Government Pump Priming

In the second case of the previous section (A = .8 and B = 2) the national income became negative at period 12. Suppose we decide to prevent a negative income by increasing government spending to avoid such a collapse of the economy. Some natural questions which arise are: Will such pump priming for a few periods put the economy back on schedule? If not, does this policy lead to ever increasing government spending?

To answer these questions we modify the program as follows:

```
100 PRINT "TYPE VALUE FOR A"
200 INPUT A
300 PRINT "TYPE VALUE FOR B"
400 INPUT B
500 PRINT "TYPE NATIONAL INCOME IN ZEROTH PERIOD"
600 INPUT T0
700 PRINT "TYPE NATIONAL INCOME IN FIRST PERIOD"
800 INPUT T1
900 PRINT "TYPE FINAL PERIOD TO BE PREDICTED"
1000 INPUT M
1100 PRINT
1200 PRINT "PERIOD","TOTAL INCOME","GOVT SPENDING"
1250 LET G = 1
1300 FOR K = 2 TO M
1400 LET T2 = A*(1+B)*T1 - A*B*T0 + G
1410 IF T2>=0 THEN 1500
1420 LET G = 1 - T2
1440 LET T2 = 0
1500 PRINT K,T2,G
1550 LET G = 1
1600 LET T0 = T1
1700 LET T1 = T2
1800 NEXT K
1900 END
```

If $T_n < 0$ we increase grovernment spending so that $T_n = 0$, i.e.,

$$G_n = A(1 + B)T_{n-1} - ABT_{n-2} \tag{5.7}$$

To find G_n then we compute the right side of (5.7). If this is less than 1 we set $G_n = 1$ and compute T_n from (5.5). If not, we let G_n be the value defined by (5.7) and let $T_n = 0$. The program actually computes

$$T_n = A(1 + B)T_{n-1} - ABT_{n-2} + 1$$

If this is less than zero, then the program sets

$$G_n = 1 - T_n$$

and then sets

$$T_n = 0$$

The number of the period, the value of T_n and the value of G_n are printed. The results of running this program are shown on the following page. Notice that every 14 periods, government spending must increase above 1 for two consecutive periods. Notice also that this pump priming does not increase with time. A total additional government investment of 52.47875 is required in each 14 years (plus of course the normal spending of 14).

* *

Interim Project #5.1

See if you can reduce the total government expenditures by making government spending negative for several periods in between the periods where government spending is abnormally high. Try different strategies to see how well you can do.

* *

```
TYPE VALUE FOR A
?.8
TYPE VALUE FOR B
?2
TYPE NATIONAL INCOME IN ZEROTH PERIOD
?2
TYPE NATIONAL INCOME IN FIRST PERIOD
?3
TYPE FINAL PERIOD TO BE PREDICTED
?45

PERIOD          TOTAL INCOME    GOVT SPENDING
 2               5               1
 3               8.2             1
 4               12.68           1
 5               18.312          1
 6               24.6608         1
 7               30.88672        1
 8               35.67085        1
 9               37.19128        1
 10              33.18572        1
 11              21.13968        1
 12              0               2.36192
 13              0               33.82349
 14              1               1
 15              3.4             1
 16              7.56            1
 17              13.704          1
 18              21.7936         1
 19              31.37824        1
 20              41.43802        1
 21              50.24605        1
 22              55.2897         1
 23              53.3016         1
 24              40.46032        1
 25              12.82221        1
 26              0               33.96322
 27              0               20.51553
 28              1               1
 29              3.4             1
 30              7.56            1
 31              13.704          1
 32              21.7936         1
 33              31.37824        1
 34              41.43802        1
 35              50.24605        1
 36              55.2897         1
 37              53.3016         1
 38              40.46032        1
 39              12.82221        1
 40              0               33.96322
 41              0               20.51553
 42              1               1
 43              3.4             1
 44              7.56            1
 45              13.704          1
```

5.4 Analytical Solution

From Part II of the appendix on "Difference Equations" we can find a solution of (5.6). As we have seen, we may have solutions which oscillate with increasing or decreasing amplitudes, and we may have solutions which decay or grow exponentially. Actually we may also obtain constant solutions or solutions which increase or decrease linearly.

To determine the nature of the solutions we examine the homogeneous equation

$$T_{n+2} - A(1 + B)T_{n+1} + ABT_n = 0 \tag{5.8}$$

Recall that in this equation $A > 0$ and $B > 0$. The characteristic equation of (5.8) is

$$x^2 - A(1 + B)x + AB = 0 \tag{5.9}$$

If the discriminant of this equation is negative then the general solution oscillates and the particular solution will be a constant. For oscillation then

$$A^2 (1 + B)^2 - 4AB < 0 \tag{5.10}$$

Suppose A is fixed. Then the discriminant is a function of B, i.e.,

$$f(B) = A^2 B^2 + 2A(A - 2) B + A^2 \tag{5.11}$$

For $B = 0$, $f(B) = A^2 > 0$. Similarly for B large, the term $A^2 B^2$ dominates in (5.11) so $f(B) > 0$. Consequently since $B > 0$ it must be that $f(B) < 0$ only *between* the roots of $f(B)$. These roots are

$$B = \frac{2 - A \pm 2\sqrt{1 - A}}{A} \tag{5.2}$$

If $A > 1$ then both of these roots are imaginary so the discriminant of the characteristic equation is never negative. If $A = 1$ then the two roots given in

(5.12) are real and equal. Indeed in this case $B = 1$. Thus we arrive at one conclusion:

If $A \geq 1$ then the solution of the difference equation (5.6) does not oscillate.

The only possibility for oscillating solutions is:

$$0 < A < 1 \tag{5.13}$$

The solution oscillates if B is between the roots (5.12), i.e., if B satisfies

$$\frac{2 - A - 2\sqrt{1 - A}}{A} < B < \frac{2 - A + 2\sqrt{1 - A}}{A} \tag{5.14}$$

Notice that both bounds on B are positive.

We now have a second conclusion: *The solution of (5.6) oscillates if and only if A and B satisfy both (5.13) and (5.14).*

The complete solution is

$$T_n = C_1 r^n \cos(n\,\theta + C_2) + \frac{1}{1-A}$$

where

$$r = \sqrt{AB}$$

$$\theta = \cos^{-1}\left(\frac{-A(1 + B)}{2\sqrt{AB}}\right)$$

The amplitude of the oscillation is given by $(\sqrt{AB})^n$. Now $A > 0$ and $B > 0$ hence $AB > 0$. Thus we can conclude that if:

(a) $AB > 1$, the amplitude of the oscillations increase as n increases.

(b) $AB = 1$, the amplitude of the oscillations is constant.

(c) $AB < 1$, the amplitude of the oscillations decreases as n increases.

Of course if T_0 and T_1 are such that $C_1 = 0$ then the solution is a constant.

EXAMPLES

Example 1: Recall that for $A = .5$ and $B = 1$ the solution oscillated with decreasing amplitude. Clearly (5.13) is satisfied. The bounds on B are

$$.18 \doteq 3 - 2\sqrt{2} < B < 3 + 2\sqrt{2} \doteq 5.82$$

So $B = 1$ satisfies (5.14). Thus the solution should oscillate. Since $AB = 1/2$ the amplitude should decrease as n increases.

Example 2: Recall that for $A = .8$ and $B = 2$ the solution oscillated with increasing amplitude. Again (5.13) is satisfied. The bounds (5.14) are

$$.38 < B < 2.62$$

Since B lies between these bounds, the solution oscillates. Then since $AB = 1.6 > 1$, the amplitude of the oscillations should increase with increasing n as it does. Notice that the long run behavior of the solution is independent of the initial values, T_0 and T_1.

Now A may satisfy (5.13) and the solution may not oscillate. If either

$$B > \frac{2 - A + 2\sqrt{1-A}}{A} \tag{5.15}$$

or

$$B < \frac{2 - A - 2\sqrt{1-A}}{A} \tag{5.16}$$

then $f(B)$ given in (5.11) is positive and the roots of the characteristic equation (5.9) are positive. The largest root of (5.9) is

$$x = \frac{A(1 + B) + \sqrt{A^2(1 + B)^2 - 4AB}}{2} \tag{5.17}$$

Suppose (5.15) holds then

$$B > \frac{2 - A + 2\sqrt{1-A}}{A} > \frac{2 - A}{A}$$

so

$$\frac{A(1 + B)}{2} > 1$$

Thus

$$\frac{A(1 + B) + \sqrt{A^2 (1 + B)^2 - 4AB}}{2} > 1$$

Therefore, at least one root of (5.9) exceeds 1 and the solution grows exponentially.

On the other hand, suppose (5.16) holds.

Then

$$B < \frac{2 - A - 2\sqrt{1-A}}{A} < \frac{2 - A}{A}$$

so

(5.18) $$2 - A(1 + B) > 0$$

Now if (5.17) is to be less than 1 then

(5.19) $$2 - A(1 + B) > \sqrt{A^2 (1 + B)^2 - 4AB}$$

Of course, (5.18) does not guarantee that (5.19) is valid, but (5.18) is certainly necessary if (5.19) is to hold. The only other condition required to assure (5.19) is

$$\{2 - A(1 + B)\}^2 > A^2(1 + B)^2 - 4AB$$

which reduces to

$$A < 1$$

Therefore, both roots of (5.9) are less than 1 if (5.16) holds. In this case the solution decays exponentially to $1/(1 - A)$.

We now have another conclusion: *The solution of (5.6) decays exponentially to $1/(1 - A)$ if A satisfies (5.13) and B satisfies (5.16). The solution of (5.6) grows exponentially if A satisfies (5.13) and B satisfies (5.15).*

EXAMPLE

For $A = .5$ and $B = 6$ the solution grows exponentially. A satisfies (5.13) and

$$\frac{2 - A + 2\sqrt{1 - A}}{A} = 5.82$$

so B satisfies (5.15). Thus we should expect the solution to grow exponentially as it does.

Finally we look at

$$B = \frac{2 - A + 2\sqrt{1 - A}}{A} \tag{5.20}$$

and

$$B = \frac{2 - A - 2\sqrt{1 - A}}{A} \tag{5.21}$$

In either case the discriminant of (5.9) is zero, and there are two real, equal roots both equal to

$$\frac{A(1 + B)}{2}$$

The complete solution of (5.8) then is

$$T_u = (C_1 + C_2 n)\left(\frac{A(1 + B)}{2}\right)^n + \frac{1}{1 - A} \tag{5.22}$$

If (5.20) holds then

$$B > \frac{2 - A}{A}$$

or

$$\frac{A(1 + B)}{2} > 1$$

Therefore, the solutions given by (5.22) grows exponentially. On the other hand, hand, if (5.21) holds

$$B < \frac{2 - A}{A}$$

and

$$\frac{A(1 + B)}{2} < 1$$

Thus the solution in (5.22) decays exponentially to $1/(1 - A)$.

This leads to our final conclusion: *The solution of (5.6) decays exponentially to $1/(1 - A)$ if A satisfies (5.13) and B satisfies (5.21). The solution grows exponentially if A satisfies (5.13) and B satisfies (5.20).*

All of these conclusions may be summarized as follows:

If

$$0 < A < 1$$

then if

$$B \leq \frac{2 - A - 2\sqrt{1 - A}}{A}$$

The solution decays exponentially to $1/(1 - A)$. If however,

$$\frac{2 - A - 2\sqrt{1 - A}}{A} < B < \frac{2 - A + 2\sqrt{1 - A}}{A}$$

the solution oscillates. If

$$AB < 1$$

the amplitude of the oscillations decays; if

$$AB = 1$$

the amplitude of the oscillations is constant; if

$$AB > 1$$

the amplitude of the oscillations increases.

Finally if

$$B \geq \frac{2 - A + 2\sqrt{1 - A}}{A}$$

the solution grows exponentially without bound.

Self-Study:, Problem #5.2

Show that if $A > 1$ then the solution of (5.6) increases without bound.

Solution to Self-Study: Problem #5.2:

If $A > 1$ the $0 > 4(1 - A)$ adding $A^2(1 + B)^2 - 4AB$ to both sides of this inequality

$$A^2 (1 + B)^2 - 4AB > [2 - A(1 + B)]^2$$

or

$$\sqrt{A^2(1 + B)^2 - 4AB} > 2 - A(1 + B)$$

or

$$\frac{A(1 + B) + \sqrt{A^2 (1 + B)^2 - 4AB}}{2} > 1$$

But the left-side of this inequality is a root of the characteristic equation (5.9). Hence the solution increases (or decreases without bound depending upon the sign of the coefficient of this term. See also Case I in Section 2.2 of the Appendix.

CHAPTER V

AUTHORS' EVALUATION

(Please circle one of the responses to each question)

1. Did you attend the short course in 1974-75? Yes No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ______________________

   ____

   ____

   can you suggest topics to be added? ______________________

   ____

   ____

   If (b), which topics should be abbreviated? ______________________

   ____

   ____

   which topics should be eliminated? ______________________

   ____

   ____

3. Could you read and understand the computer programs?

   (a) always (c) seldom

   (b) sometimes (d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy (b) Too difficult

6. Was the number of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attemtp any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

____

____

____

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Components of Total Income	____	____	____	____
Types of Economics, e.g., expanding	____	____	____	____
Pump Priming	____	____	____	____
Analysis of Second Order Difference Equations	____	____	____	____
Economic Models in General	____	____	____	____

CHAPTER VI

MODELING IN PROBABILITY

In this chapter, our view point changes somewhat since the models we treat here are also treated at an elementary level in a number of finite mathematics texts. Thus we shall give only brief introductions which are intended to bring the modeling aspects to the fore.

6.1 Probability Models

Probabalistic models occur in many fields. In this section we shall not attempt to survey the applications of probability theory, but instead we shall try to explain the nature of a probability (or as it is sometimes called a stochastic) model.

The basic notion in probability theory is that of a *random experiment*. A random experiment is one for which the experimental outcomes vary significantly (in the opinion of the modeler) from one time to another. It is customary to assume that the set of all possible experimental outcomes (the *sample space*) is well defined and that the relative frequency (*probability*) of each collection of outcomes (*events*) is known. For simplicity we shall assume in the general discussion that the experiment has only a finite number of outcomes, although we shall later present two models which involve infinitely many outcomes.

The sample space and the probabilities associated with the experiment must be specified by the modeler - they are *not* supplied by probability theory itself. For example, in the experiment of flipping a coin, most modelers would assume that the sample space consisted of the two outcomes, "head" and "tail". Even in this simple situation other outcomes are conceivable, for example, "coin lands on edge", "coin rolls away and gets lost". Most modelers would decide that the last two outcomes were so rare that they should not be considered; but in any case, an important facet of any probability model is the careful specification of the sample space.

The assignment of probabilities to the events is usually made in one or a combination of the following ways:

1). A priori method - For example, symmetry considerations or other prior theoretical assumptions may lead to an assignment of probabilities to the events. In rolling a die, the assumption of a "fair die" leads to the assignment of probability 1/6 to each of the six outcomes. In Chapter III certain theoretical assumptions led to the assignment of the probability, $\lambda N\Delta t + o(\Delta t)$, of a birth in a population of size N.

2). A posteriori method - We observe many repetitions of the random experiment and use these observations to estimate the probability of each outcome. For example, we could roll a die several thousand times; record the number of times each event occurs; divide this number by the total number of rolls; and then use the last result as an estimate of the probability of the event.

3). Subjective method - We judge or we call in an expert to judge the probabilities. This is often the method employed in decision theory problems. For example, the sales manager of a large corporation may be called upon to judge the relative probabilities of having sales of $100,000, $150,000 or $200,000 next year for some product line.

Regardless of the way in which the probabilities are assigned, the assignment must satisfy the axioms of probability theory which are:

$$
\text{(6.1)}\qquad
\begin{aligned}
&1)\quad P[E] \geq 0 \quad \text{for every event } E\,,\\
&2)\quad P[S] = 1 \quad \text{for the certain event } S\,,\\
&3)\quad P[E \cup F] = P[E] + P[F] \quad \text{if } E \cap F = \phi\,.^{*}
\end{aligned}
$$

---

* This axiom must be strengthened if the sample space is not finite, see e.g., Parzen, p. 21.

We can now make the basic definition of this section. A *probability model** for a random experiment is a set of assumptions which lead to (1) a well defined sample space and (2) an assignment of probabilities to the events of that sample space. Naturally a given probability model may apply to many different random experiments, and we shall now briefly discuss four commonly occuring models and some applications of each.

1) Discrete Uniform Model

This forbidding title is merely a fancy way of refering to the "equally likely" model with which even non-probabalists are familiar. Concrete examples are

a) Coin flipping: P[head] = 1/2 , P[tail] = 1/2

b) Die tossing: P[one] = 1/6 , P[two] = 1/6, ... P[six] = 1/6 .

In general if

Axiom 1: Sample space has N elements, $s_1, s_2, \dots, s_n$

and

Axiom 2: $P[s_i] = P[s_j]$ for all i and j

are satisfied, then we speak of the "equally likely" or "discrete uniform" probability model. In this model it is easy to prove the

Theorem:

(6.2) $$P[s_i] = 1/N \quad \text{for all } i .$$

Proof:

By axiom 1 of the model and probability axiom 2, we have

$$\sum_{i=1}^{n} P[s_i] = 1 .$$

By axiom 2 of the model, $N \cdot P[s_i] = 1$ for any i, which leads to (6.2). This completes the proof.

*This usage differs slightly from that of some texts. See Adams, p. 118, for example.

It is now clear that the theorem and probability axiom 3 lead to a unique specification for the probability of any event in our sample space. Thus our two axioms do provide a probability model.

2) Binomial Model

Axiom 1. The sample space consists of a sequence of n trials.

Axiom 2. Each trial has only 2 outcomes, denoted by s (success) and f (failure).

Axiom 3. $P[s] = p$ for each trial, independently of all other trials.

Examples:

a) Repeated coin tosses with s being the event "head".

b) Getting 3, 4, 5, 6 (a "success") vs. getting 1 or 2 (a "failure") in repeated tosses of a die.

c) Getting a defect (success?!) as we examine a random sample of size n of one day's manufacturing output (sampling with replacement).

Although it is not difficult to show that the above axioms do specify the probability of any event in the sample space and hence are a probability model, we confine outselves to stating the most interesting result of this model. Let K denote the number of successes in n trial then we have the*

Theorem:

$$P[K = k] = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{6.3}$$

One also can show that the mean number of successes is $\mu_K = np$ and that the variance is $\sigma_K^2 = np(1 - p)$.

3) Poisson Model

This model concerns an experiment in which certain types of events occur repeatedly. The outcomes of the experiment are taken to be the number of these events which occur in a fixed portion of time or space. For definiteness, we shall

*See Parzen, P. 52 ff for the proof.

assume that we are interested in the number of events which occur in the time interval $[0, t]$. The sample points are the non-negative integers. The Poisson model applies when there exists a positive constant λ such that

(6.4)
1) $P[\text{exactly one event will occur in } \Delta t] = \lambda\Delta t + o(\Delta t)$ as $\Delta t \to 0$.
2) $P[2 \text{ or more events occur in } \Delta t] = o(\Delta t)$ as $\Delta t \to 0$.
3) The number of events occuring in non-overlapping sub-intervals of time are independent of each other.

Notice that the Poisson model is the one used in Chapter 3 of the number of births (and deaths). In fact that model really was concerned with the competition between these two Poisson processes.

If we let K be the number of events occuring in $[0, t]$, then we can prove the

<u>Theorem</u>:

(6.5) $$P[K = k] = e^{-\mu}\frac{\mu^k}{k!} ; \quad \mu = \lambda t , \quad k = 0, 1, 2, \ldots .$$

The probabilities expressed in the theorem are known as the Poisson law of probabilities with parameter μ . In our model μ has the value λt .

To establish the theorem, we can proceed as in Chapter 3. Let t now vary continuously and let $K(t)$ = number of events that have occurred by time t and

$$P_k(t) = P[K(t) = k] , \quad k = 0, 1, 2, \ldots .$$

Hence, as in Chapter 3, for $k > 0$ *

$$P_k(t + \Delta t) = P_{k-1}\lambda\Delta t + P_k(1 - \lambda\Delta t) + o(\Delta t)$$

*The expert will observe that assumption (3) has been employed.

That is, to within negligible error, we have k events at $t + \Delta t$ when either there were $k - 1$ at time t and one more occured during $(t, t + \Delta t)$ or there were k at time t and none occured during $(t, t + \Delta t)$. Dividing by Δt and letting $\Delta t \to 0$, we then have

(6.6) $$\frac{dP_k}{dt} = \lambda(P_{k-1} - P_k), \quad k > 0 .$$

Similarly for $k = 0$, we obtain

(6.7) $$\frac{dP_0}{dt} = -\lambda P_0$$

Now

(6.8) $$P_k(0) = P[k \text{ events at } t = 0] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

With these initial conditions we can recursively solve for the P_k's. For example, for P_0 from (6.7)

$$\frac{dP_0}{dt} = -\lambda P_0 , \quad P_0(0) = 1 .$$

and the solution is

$$P_0 = e^{-\lambda t}$$

Then for P_1 from (6.6)

$$\frac{dP_1}{dt} = -\lambda P_1 + \lambda e^{-\lambda t} , \quad P_1(0) = 0$$

so

$$P_1 = \lambda t \; e^{-\lambda t}$$

Hence an induction argument will lead to the theorem.

Self-Study Problem #6 1:

Establish the above theorem using the generating function approach described in Chapter 3 .

Solution to Self-Study Problem #6.1:

Let $P(x, t) = \sum_{k=0}^{\infty} P_k(t) X^k$. Multiply (6.6) by X^k and sum over k . Then add (6.7) to obtain

$$\sum_{k=0}^{\infty} \frac{dP_k}{dt} X^k = \lambda \sum_{k=1}^{\infty} P_{k-1} X^k - \lambda \sum_{k=0}^{\infty} P_k X^k$$

$$= \lambda \sum_{k=0}^{\infty} P_k X^{k+1} - \lambda \sum_{k=0}^{\infty} P_k X^k .$$

Hence

$$\frac{dP}{dt} = \lambda x P - \lambda P = \lambda(x - 1) P .$$

The initial condition for P is

$$P(x, 0) = \sum_{k=0}^{\infty} P_k(0) X^k = 1$$

Thus

$$P = e^{\lambda(x-1)t} = e^{-\lambda t} e^{\lambda tx} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k X^k}{k!}$$

so

$$P_k(t) = e^{-\lambda t} \frac{(\lambda k)}{k!}$$

for any fixed t , as the theorem asserts.

The Poisson model has found wide application, some of the more well-known examples are:

1) Number of fatalities by horse-kick per annum in the Prussian Calvary,

2) Number of alpha particle emissions per minute from a radioactive material,

3) Number of certain blood components in a cc of blood plasma,

4) Number of vacancies per year on the U.S. Supreme Court ($\lambda \approx .5$).

Parzen gives many "textbook" examples for instance "Suppose one is observing the times at which automobiles arrive at a toll booth. Suppose the mean rate λ of the arrival of automobiles is given by $\lambda = 1.5$ autos/minute. Find the probability that k automobile arrive in a two minute period."

Here the assumptions (6.4) are at least plausible. Setting $\lambda = 1.5$, $t = 2$, we find from (6.5) that

$$P_k = e^{-3} \cdot \frac{3^k}{k!}, \quad k = 0, 1, 2, \ldots .$$

The binomial law and the Poisson law have a similarity in that they can be interpreted as giving the probability of k successes. The difference is that the binomial law deals with the number of successes in a fixed number of *trials*, while the Poisson law involves the number of successes in a fixed *time or space domain*.

4) <u>Exponential Model</u>

Our final model concerns an experiment in which the outcome is a waiting time. Specifically, suppose we are observing a sequence of events occuring in time in accordance with the Poisson Model. The experiment consists of measuring the time that elapses before the first event occurs. The outcome can be any positive time t, so our sample space contains an uncountably infinite set of sample points. In such cases the probability of any one value occurring is zero (what intuitive probability would you assign to waiting exactly 35.2 seconds?), but it is possible to assign positive probabilities to time intervals. This we now do, using our knowledge of the Poisson Model. Let

$$F(t) = P[\text{time of first event is} \leq t]$$

Then

$$1 - F(t) = P[\text{time of first event is} > t]$$

$$= P[\text{number of events in } (0, t) \text{ is } 0]$$

$$= P_0(t)$$

By the Poisson law, $P_0(t) = e^{-\lambda t}$, hence

$$F(t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

Again pursuing the Parzen example:

"Find the probability that a toll collector at a toll booth at which cars arrive with mean rate $\lambda = 1.5$ auto/minute will have to wait

a) less than three minutes,

b) between 3 and 10 minutes,

c) exactly 3 minutes,

for the first arrival." The solution is immediate:

a) $P[\text{time of first event is} \leq 3] = F(3) = 1 - e^{-4.5} = .98889$

b) $P[3 \leq t \leq 10] = F(10) - F(3) = e^{-3} - e^{-10} = .04974$

c) $P[t = 3] = \lim_{\varepsilon \to 0} P[3 \leq t \leq 3 + \varepsilon]$

$$= \lim_{\varepsilon \to 0} [F(3 + \varepsilon) - F(3)]$$

$$= 0$$

In conclusion we point out that all of the probability laws are models for various situations. The modeling viewpoints towards a given probability law is to ask

1) What are the underlying axioms for the law?

2) What are some of the real world situations in which these axioms are (approximately) satisfied?

Interterm Project:

Work out an alternate Monte Carlo simulation from that presented in Chapter 3. This time base the simulation on drawing 2 random numbers. The first number will be used to derive the *time* when the next event occurs, while the second random number will be used to decide whether that event is a birth or death. To find the time T of the next event, we use the exponential law:

$$1 - e^{-(\lambda + \mu)T} = R_1$$

where λ, μ are the mean birth and death rates and R_1 is a random number between 0 and 1.

Thus, the next event is assumed to occur after a lapse of time

$$T = - \frac{\log_e (1 - R_1)}{\lambda + \mu}$$

Given that an event occurs, the probability that it is a birth is

$$L = \frac{\lambda}{\lambda + \mu}$$

Thus we draw a second random number R_2 from the unit interval and if $R_2 < L$ declare a birth and otherwise declare a death.

By replacing λ and μ by λ_n and μ_n, (n = size of population) we can simulate more general birth and death processes. In particular, the analogue of the deterministic $A - BN$ case is $\lambda = \lambda_n = a_1 - b_1 n$, $\mu = \mu_n = a_2 + b_2 n$ (see Pielou).

6.2 Conditional Probability and Bayes Theorem.

Suppose you are an oil prospector. You would like to find a gusher, but will be content with a more modest oil deposit. From past experience you know that in this particular geographical area drilling wells at random will produce a gusher 10% of the time and a moderate supply 20% of the time. The other 70% of the borings are dry. The three events

$$A_1 = \{\text{gusher or large oil supply}\}$$

$$A_2 = \{\text{moderate oil supply}\}$$

$$A_3 = \{\text{no oil}\}$$

are a complete, mutually exclusive set. Moreover

$$P(A_1) = 0.1 \qquad P(A_2) = 0.2 \qquad P(A_3) = 0.7$$

Now suppose that cost of drilling wells is too high to take the risk when the chances of finding oil are so slim as this, so you decide to conduct seismic tests to increase the chances of finding oil. To determine the effectiveness of these tests, you conduct tests over existing wells and over some dry borings. When conducted where a gusher is known to be present, the seismic test produces a positive result 80% of the time, i.e.,

$$P(X \mid A_1) = 0.8$$

where X is the event "a positive seismic test result". Where modest wells exist the test is positive 60% of the time, and over dry borings it is positive 30% of the time. Thus

$$P(X \mid A_2) = 0.6 \qquad P(X \mid A_3) = 0.3$$

But what we want to know is: If a seismic test is positive, is there oil present? And how much? In terms of the events described above we would like to calculate $P(A_1 \mid X)$ and $P(A_2 \mid X)$. The first is the probability of a gusher and the second is the probability of a moderate oil supply.

To compute these probabilities we turn to Bayes' Theorem. We assume that the following are true:

1) A complete set of mutually exclusive alternatives can be found for the experiment, i.e., these are events $A_1, A_2, \ldots, A_m$ such that

$$\sum_{i=1}^{m} P(A_i) = 1$$

and

$$P(A_i \cap A_j) = 0 \qquad i, j = 1, 2, \ldots, m \quad i \neq j$$

2) The probabilities of each event in the complete set of mutually exclusive alternatives is known, i.e.,

$$P(A_i) \qquad i = 1, 2, \ldots, m$$

are given.

3) For a particular event X, the conditional probability for the event given that any one of the A_i occur is known, i.e.,

$$P(X \mid A_i) \qquad i = 1, 2, \ldots, m$$

are given.

These three assumptions are satisfied in the oil drilling example where $m = 3$. Notice that $2m$ probabilities must be given in general.

We use this information to invert the conditional probabilities and compute

$$P(A_i \mid X) \qquad x = 1, 2, \ldots, m$$

from:

Bayes Theorem:

(6.9) $$P(A_i \mid X) = \frac{P(X \mid A_i) \cdot P(A_i)}{\sum_{j=1}^{m} P(X \mid A_j) \cdot P(A_j)}$$

The proof may be found in most finite mathematics texts.

We now use this theorem to solve the oil drilling problem posed above. Since $m = 3$, (6.9) becomes

$$P(A_i \mid X) = \frac{P(X \mid A_i) \cdot P(A_i)}{P(X \mid A_1) \cdot P(A_1) + P(X \mid A_2) \cdot P(A_2) + P(X \mid A_3) \cdot P(A_3)} .$$

For $i = 1$ (gusher)

$$P(A_1 \mid X) = \frac{0.8 \times 0.1}{0.8 \times 0.1 + 0.6 \times 0.2 + 0.3 \times 0.7}$$

$$= 0.195$$

Similarly

$$P(A_2 \mid X) = 0.293$$

$$P(A_3 \mid X) = 0.512$$

Therefore the probability of finding oil – gusher or moderate – is 0.488 .

A BASIC program to evaluate (6.9) is:

```
LIST

100   PRINT "NØ. ØF MUTUALLY EXCLUSIVE EVENTS IS",
200   INPUT M
300   PRINT
400   REM  **  DATA: PRØBABILITIES, P(A(J)), ØF SET ØF
500   REM      MUTUALLY EXCLUSIVE EVENTS                 **
600   FØR J=1 TØ M
700   PRINT "PRØBABILITY ØF EVENT";J;" IS",
800   INPUT A(J)
900   NEXT J
1000  PRINT
1100  REM  ** DATA: CØNDITIØNAL PRØBABILITIES,P(X GIVEN A(J))  **
1200  FØR J=1 TØ M
1300  PRINT "PRØBABILITY ØF EVENT X GIVEN THAT EVENT";J;"ØCCURS IS";
1400  INPUT P(J)
1500  NEXT J
1600  PRINT
1700  REM  **  CØMPUTE DENØMINATØR ØF EQ. (6.9)   **
1800  LET D=0
1900  FØR J=1 TØ M
2000  LET D=D+P(J)*A(J)
2100  NEXT J
2200  REM  **  DENØMINATØR ØF EQ. (6.9) IS P(X)  **
2300  PRINT "PRØBABILITY THAT EVENT X ØCCURS IS";D
2400  PRINT
2500  REM  **  CØMPUTE BAYES PRØBABILITY FØR A(J) FRØM (6.9)   **
2600  FØR J=1 TØ M
2700  LET B(J)=P(J)*A(J)/D
2800  PRINT "PRØBABILITY ØF EVENT ";J;" GIVEN X IS";B(J)
2900  NEXT J
3000  END
#
```

Notice that in this program

$$A(J) = P(A_j) \qquad j = 1, 2, \ldots, m$$

$$P(J) = P(X \mid A_j) \qquad j = 1, 2, \ldots, m$$

and both of these are input to the program. The output is

$$B(J) = P(A_j \mid X) \qquad j = 1, 2, \ldots, m$$

In addition the program computes and prints

$$P(X) = \sum_{j=1}^{m} P(X \mid A_j) * P(A_j)$$

which is no more than the denominator of (6.9).

If the program is run for the oil drilling example the results are:

```
NO. OF MUTUALLY EXCLUSIVE EVENTS IS              ?3

PROBABILITY OF EVENT 1  IS    ?0.1
PROBABILITY OF EVENT 2  IS    ?0.2
PROBABILITY OF EVENT 3  IS    ?0.7

PROBABILITY OF EVENT X GIVEN THAT EVENT 1 OCCURS IS?0.8
PROBABILITY OF EVENT X GIVEN THAT EVENT 2 OCCURS IS?0.6
PROBABILITY OF EVENT X GIVEN THAT EVENT 3 OCCURS IS?0.3

PROBABILITY THAT EVENT X OCCURS IS 0.41

PROBABILITY OF EVENT  1  GIVEN X IS .19512195122
PROBABILITY OF EVENT  2  GIVEN X IS .29268292683
PROBABILITY OF EVENT  3  GIVEN X IS .51219512195
```

Self-Study Problem # 6.2: (Kemeny, Snell and Thompson).

If a person has tuberculosis, its early detection is important in order to save the patient's life. Chest x-rays are one method of detecting tuberculosis when it is present. If a patient is healthy, the x-ray will indicate tuberculosis is present 1% of the time. On the other hand, if a patient does have tuberculosis, the x-ray will fail to detect that fact 10% of the time. If tuberculosis is present in 5 out of every 10,000 persons, what is the probability that an x-ray indicates no tuberculosis when in fact the patient does have the desease?

Solution to Self-Study Problem #6.2:

The mutually exclusive events are "tuberculosis" and "no tuberculosis". Since we want to compute

P("tuberculosis" | "healthy x-ray")

we need to know

P("healthy x-ray" | "tuberculosis")

and

P("healthy x-ray" | "no tuberculosis")

The computer output is

```
NØ. ØF MUTUALLY EXCLUSIVE EVENTS IS
 2
PRØBABILITY ØF EVENT 1  IS   ?.0005
PRØBABILITY ØF EVENT 2  IS   ?.9995

PRØBABILITY ØF EVENT X GIVEN THAT EVENT 1 ØCCURS IS?0.1
PRØBABILITY ØF EVENT X GIVEN THAT EVENT 2 ØCCURS IS?0.99

PRØBABILITY THAT EVENT X ØCCURS IS 0.989555

PRØBABILITY ØF EVENT  1  GIVEN X IS 5.05277625-5
PRØBABILITY ØF EVENT  2  GIVEN X IS .99994947224
```

Since event 1 is "tuberculosis",

P("tuberculosis" | "healthy x-ray") = 0.0000505

Notice that tuberculosis is a rare disease, and the failure to detect it carries a stiff penalty. Therefore, if we are to err it should be on the side of a healthy patient with a unhealthy x-ray.

Self-Study Problem #6.3

It is essential that flaws in the equipment of a spacecraft be detected if the craft and its occupants are to survive an orbiting mission. Elaborate electronic sensing equipment is installed to detect such flaws even though they rarely occur. Suppose the failure rate of a critical part in a missile is 1/10 of 1%. Suppose also that if the part fails that 2% of the time the electronic sensing equipment does not detect this fact. Finally suppose that if the part is functioning correctly (not failing) the sensing equipment will say that it does fail 5% of the time. If the sensing equipment does not detect a failure, what is the probability that nonetheless a failure exists?

- -

Solution to Self-Study Problem #6.3

The mutually exclusive events are

A_1 = {part fails}

A_2 = {part does not fail}

and

$P(A_1) = .001 \qquad P(A_2) = .999$

The event X is "no failure detected" and we wish to compute $P(A_1 \mid X)$.

Therefore we need to know

$P(X \mid A_1)$ = P("no failure detected" | "part fails") = 0.02

$P(X \mid A_2)$ - P("no failure detected" | "part does not fail") = 0.95

The computer output is

```
NO. OF MUTUALLY EXCLUSIVE EVENTS IS          ?2

PROBABILITY OF EVENT 1  IS    ?0.001
PROBABILITY OF EVENT 2  IS    ?0.999

PROBABILITY OF EVENT X GIVEN THAT EVENT 1 OCCURS IS?0.02
PROBABILITY OF EVENT X GIVEN THAT EVENT 2 OCCURS IS?0.95

PROBABILITY THAT EVENT X OCCURS IS 0.94907

PROBABILITY OF EVENT  1   GIVEN X IS 2.10732612-5
PROBABILITY OF EVENT  2   GIVEN X IS .99997892673
```

$P(A_1 \mid X)$ = P("part fails" | "no failure detected") = 0.000021.

Notice that no failure is detected (event X) about 95% of the time.

6.3 Decision Theory Models

The theory of making decisions draws upon many fields including ethical theory, the theory of utility, game theory, probability, and classical and Bayesian statistics. Certain branches of decision theory have become commonly used in the behavioral and management sciences in recent years, particularly in the growing field known as operations research.

In general we have imperfect knowledge of the factors from the outside world. Nevertheless, we wish to make the 'best' decision even in the face of these knowledge gaps. In this section then we will develop several decision theory models each one taking into account and using as much knowledge of the world as there is available to us. As a pedagogical device, we shall develop a flow chart for decision making as we discuss the various decision theory models. The flow chart will be developed piecemeal with each piece being appropriate to some state of our knowledge. At the close of the section you will be asked to integrate the pieces into a coherent whole.

As in the earlier sections, we shall discuss finite sets in the general discussion, although some of our examples will involve infinite sets.

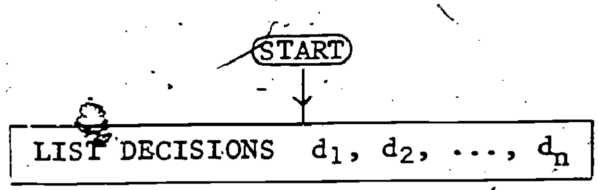


Figure 6.1

As indicated in figure 6.1, the first step in a decision theory problem is to define the set $D = \{d_1, \ldots, d_n\}$ of possible decisions. In some cases, each decision leads to a uniquely determined outcome. In this situation, we need "only" decide which outcome we prefer and make the appropriate decision. Of course, deciding which outcome we prefer may be either trivial or close to impossible.

<u>Example 1</u>: (Feibes)

"You may invest in one of A, B, C. Each investment costs \$500. The yearly returns on A, B, C are 3%, 5% and 6% respectively. Which investment will you choose?

<u>Analysis</u>: The possible decisions are d_1 = invest in A , d_2 = invest in B , and d_3 = invest in C . The outcome of d_1 is a profit of (.03)(\$500) = \$15 and similarly the outcomes of d_2 and d_3 are profits of \$25 and \$30 respectively. Here the decision is presumably trivial. Choose C .

<u>Example 2</u>:

Your father is 75 years old and needs an operation. Without the operation, he will die within 6 months. If he survives the operation (50% chance), he will probably live for at least 2 years, but will be an invalid. Do you recommend the operation to him?

<u>Analysis</u>: The possible decisions are d_1 = recommend the operation, d_2 = advise against the operation. The outcomes are as given in the example. Here the decision is far from trivial as it involves one's personal ethics and emotions.

The rating of outcomes is one of fundamental problems of ethical theory. There is a related mathematical theory known as utility theory (see Chapter II of Luce & Raiffa for an introduction to utility theory). In any case, as we shall see below, many decision theory models require that one assign a utility (equivalently a "payoff" or a "loss") to various outcomes. Although we shall not emphasize this point below, the possible difficulty in assigning such payoffs should be kept in mind whenever utilities are required (see e.g., example 2 above).

Formally we have the following axioms:

1) A set $D = \{d_1, \ldots, d_n\}$ of decisions is specified.

2) A set $D = \{O_1, O_2, \ldots, O_p\}$ $p \leq n$, of outcomes are specified.

3) For any given decision, d_i , there is a unique outcome, O_j .

The mechanism for making the decision is to rate the outcomes using ethical theory, utility theory or the like, and to select the decision which gives the most favorable outcome. This branch of decision theory is known as "decision under certainty". We may now further develop our flowchart as shown in Figure 6.2.

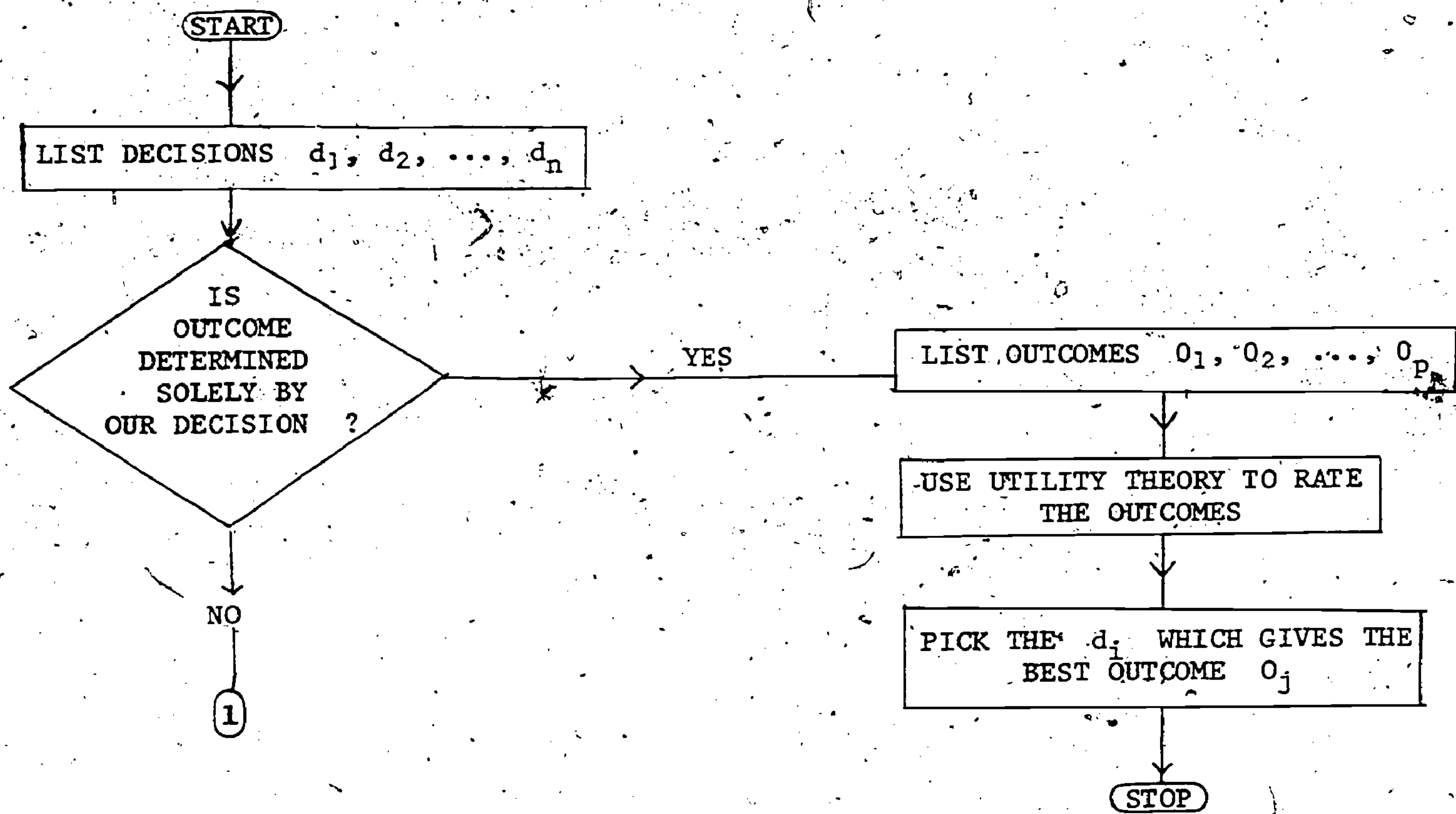


Figure 6.2

In our next decision model, we assume that the outcome of a decision is influenced by a rational opponent who has a set $S = \{s_1, \ldots, s_m\}$ of decisions or strategies available to him. We assume that to each decision d_i and opponent strategy s_j there is a definite outcome O_{ij}. This situation and it generalization to more than two rational agents is the subject of *game theory*. The classic expository text on game theory is that of Luce & Raiffa, but elementary treatments may be found in most texts on finite mathematics.

We will assume that numerical utilities for each opponent are known for each of the outcomes O_{ij}. The "game" is usually displayed by means of a payoff (utility) matrix whose entries are ordered pairs representing the utilities to each player.

For example:

decisions	opponent strategies s_1	s_2
d_1	(3, 2)	(0, 10)
d_2	(10, 0)	(4, 4)

Here the pair (0, 10) means that if we make decision 1 and our opponent employs strategy 2 then our payoff is 0 and our opponent's payoff is 10 . Similarly for the other entries in the matrix.

Example 3: (Adams)

"Two duopolists, the Row and Column Corporations, are competing for shares of a million dollar market. The following payoff matrix describes the profit (in hundred-thousand dollars) of the Row Corporation for various choices of strategy of the two corporations. The Column Corporation's profit for the given choices of strategy is the difference between one million dollars and the Row Corporation's profit.

		Column Corporation C_1	C_2	C_3
	R_1	5	4	4
Row Corporation	R_2	3	3	5
	R_3	6	4	3

Rows R_1, R_2, R_3, correspond to the Row Corporation selling 10,000 units, 12,000 units and 14,000 units, respectively of its product. Columns C_1, C_2, and C_3 correspond to the Column Corporation selling 10,000 , 12,000 , and 14,000 units respectively.

Analysis: The full utility matrix is

	C_1	C_2	C_3
R_1	(5, 5)	(4, 6)	(4, 6)
R_2	(3, 7)	(3, 7)	(5, 5)
R_3	(6, 4)	(4, 6)	(3, 7)

The Row Corporation reasons as follows. If we decide R_1, then no matter what the Column Corporation decides our profit is at worst 4. Similarly the worst we can do with R_2 is 3 and with R_3 is 3. Since the Column Corporation wishes to maximize its profit, we dare not decide R_2 if the Column Corporation is likely to decide C_1 or C_2, and we dare not decide R_3 if the Column Corporation is likely to decide C_3.

To decide what the Column Corporation is "likely to do" we need only reason from their point of view. If they decide C_1 the worst they can do is 4, similarly the worst outcome for C_2 is 6 and the worst outcome for C_3 is 5. A conservative strategy for the Column Corporation then is C_2 since no matter what the Row Corporation does, the Column Corporation will realize \$600,000.

Thus the Row Corporation's conservative strategy is R_1, and the Column Corporation's conservative is C_2 which leads to the utilities (4, 6). Now suppose the Row Corporation does not play R_1. Then if the Column Corporation continues its conservative strategy C_2, the payoffs will be either 3 or 4. But with R_1, the Row Corporation has a payoff of 4 already. Thus the Row Corporation has no incentive to alter its strategy from the conservative R_1. Similarly the Column Corporation has no incentive to alter its strategy from the conservative C_2, since if the Row Corporation continues R_1 the payoffs are 5 or 6 as opposed to 6.

The "rational behavior" for the Row Corporation is to decide R_1, i.e., sell 10,000 units and for the Column Corporation it is to decide C_2, i.e., sell 12,000 units.

In general, the conservative strategy for the "row player" is to compute his minimum payoff for each row and choose the row with the largest minimum payoff (the "maximum" strategy). Similarly the conservative strategy for the "column player" is to make a similar computation for the columns. If these considerations lead to a payoff for the "row player" which cannot be improved even if he *knows that* the "column player" will be conservative and to a payoff for the column player which cannot be improved even if he *knows that* the row player will be conservative, then the "maximum" strategy is taken to be optimal.

Example 4: (Adams)

"Determine optimal strategies for the row and column players for the game defined by the payoff matrix"

	C_1	C_2
R_1	(5, -5)	(1, -1)
R_2	(2, -2)	(3, -3)

Analysis: Here if the row player chooses R_1, his worst outcome is 1 and if he chooses R_2, his worst outcome is 2. Thus his maximum choice is R_2. Similarly, the maximum choice for the column player is C_2. However, if the column player *knew* that the row player would choose R_2, he could do better by changing his choice to C_1.

Continuing the analysis in this way, we soon conclude that there is no rational choice between R_1 and R_2. The way out of this dilemma is to adopt a "mixed strategy". That is, choose R_1 with a probability p and R_2 with a probability $1 - p$. It is only in decisions against a rational opponent that one would employ mixed strategies, since only in this case is one concerned with one's decision being discovered and exploited. It is a classic result of game theory, that in "zero-sum" games (such as that in the last table above), an *optimal* mixed strategy can be found. We refer the reader to the sources mentioned above for the theory of mixed strategy as well as for the the theory of competition and cooperation among two or more rational agencies. The axioms for the game theory model of decision making are

1) A set $D = \{d_1, d_2, \ldots, d_n\}$ of decisions is specified.
2) A set $S = \{s_1, s_2, \ldots, s_m\}$ of opponent strategies are specified.
3) The outcomes are determined by the pair (d_i, s_j).
4) For each outcome (d_i, s_j) each player has a specified utility and all of these utilities are known to each player.

Once again we remind the reader that the various sets above need not be finite (see Maki-Thompson, P. 60 ff, for an example). We can now extend our flow chart as shown in Figure 6.3.

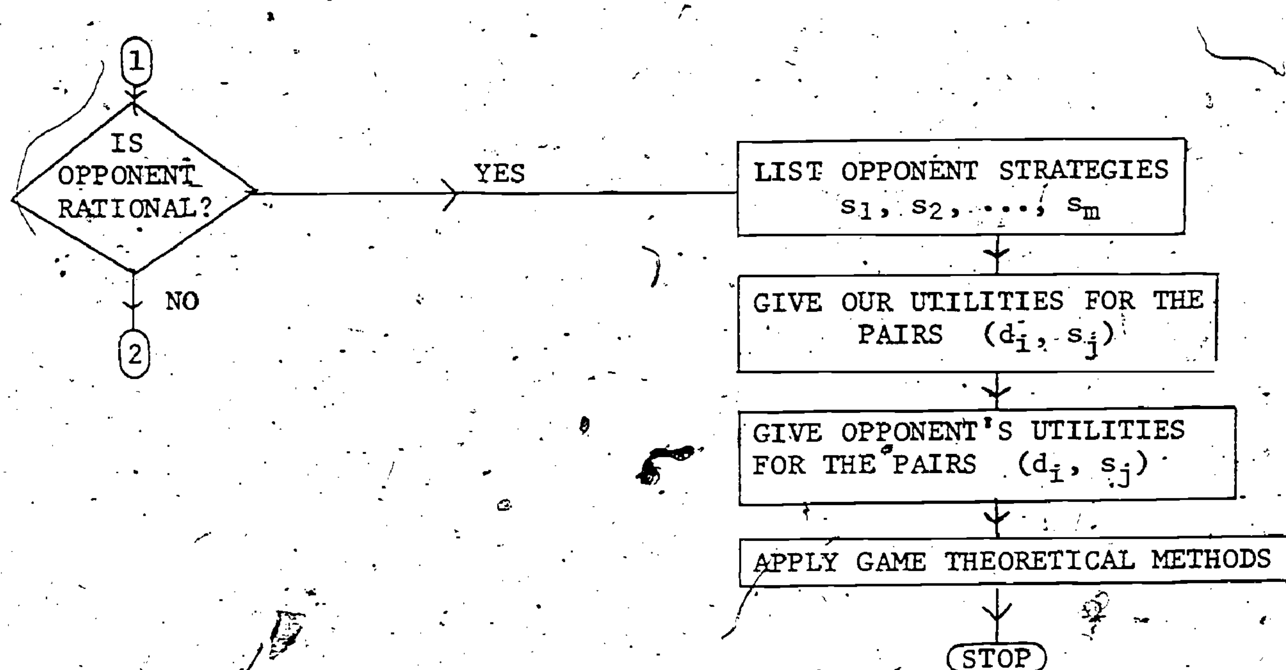


Figure 6.3

In our final group of decision models, we assume that the outcome of a decision is influenced by a "state of nature". By this we mean any non-rational agency. For example, we might be trying to decide whether or not to carry an umbrella. The decisions are d_1 = carry umbrella, d_2 = leave umbrella home. The states of nature might be s_1 = will rain today, s_2 = won't rain today. In any case the possible states of nature $S = \{s_1, s_2, \ldots, s_m\}$ are assumed to be specified. There are three types of knowledge that might help us make a decision against nature:

1) The utilities U_i, for the outcome O_{ij}, specified by a decision d_j and a state of nature s_j may be specified.

2) The probability p_j that a given state of nature will occur may be given.

3) We may be able to carry out some experiment whose outcome is influenced (depends upon) the state of nature in effect at the time the experiment was conducted.

In a given case we might have any or all of these types of knowledge. Hence there are $2^3 = 8$ alternatives possible for decision theory where the decisions are against nature. The eight alternatives are:

A) Utilities are Known, No Probabilities Known, No Experiment Available.

B) Utilities and Probabilities are Known, No Experiment Available.

C) Utilities and Probabilities are Known, Experiment is Available.

D) Utilities are Known, No Probabilities Known, but an Experiment is Available.

E) No Utilities or Probabilities Known, but an Experiment is Available.

F) No Utilities Known but Probabilities are Known and an Experiment is Available.

G) No Utilities Known and No Experiment is Available, but Probabilities are Known.

H) Nothing is Known.

We consider each of these in turn in the order shown above.

A. Utilities are Known, No Probability Known, No Experiments Available.

In decisions against nature the case when only utilities are known is called "decision under uncertainty". There appears to be no universally acceptable solution to this problem, and a rather thorough survey is given in Luce-Raiffa, Chapter 13. We shall confine ourselves here to a brief presentation of some of the more common methods of treating decisions under uncertainty.

Example 5. (Luce-Raiffa) Consider the utility matrix.

	s_1	s_2
d_1	0	100
d_2	1	1

Which decision should we make?

The conservative approach is to retain the game theory criterion of Maxi - min Utility. Since the first row has minimum 0 and the second row has minimum 1 , the maxi - min choice is the second row, i.e., decision d_2 .

Since we are not competing with a rational opponent, the maxi - min criterion may be unduly pessimistic. This is especially true in a case such as that shown in the above table where we pass up all chance of a gain of 100 to assure a gain of only 1 . An alternative decision criterion is "Mini- Max Regret" . Again referring to the above example, if we make decision d_1 and s_1 is the true state of nature, we have a "regret" of 1 , since we could have received a payoff of 1 greater by deciding d_2 . Similarly if we decide d_2 and s_1 is the state, we have no "regrets", but if s_2 turns out to be the true state our regret is 99 . Thus we can replace the utility matrix of example 5 by the *regret matrix*

$$B(J) = P(A_j \mid B), \quad j = 1, 2, \ldots, n$$

In addition the program computes and prints;

$$P(B) = \sum_{j=1}^{n} P(B \mid A_j) * P(A_j)$$

P("tuberculosis" | "healthy x-ray") = 0.00

Notice that tuberculosis is a rare disease, and the failure t

a stiff penalty. Therefore, if we are to err it should be on

healthy patient with a unhealthy x-ray.

000505

to detect it carries

n ~~the~~ side of a

	s_1	s_2
d_1	1	0
d_2	0	99

Now if we decide d_1 our maximum regret is 1 and if we decide d_2 our maximum regret is 99. Therefore, we choose the smaller of these maximum regrets (1) and make decision d_1.

In general the regret r_{ij} corresponding to the utility U_{ij} is given by

$$r_{ij} = \max_j (U_{ij}) - U_{ij} .$$

Although the mini - max regret criterion appears attractive in this example, it is no ponacea as the following example shows.

Example 6. (Luce-Raiffa) Consider the utility matrices

	s_1	s_2	s_3
d_1	0	10	4
d_2	5	2	10

	s_1	s_2	s_3
d_1	0	10	4
d_2	5	2	10
d_3	10	5	1

Note that the matrices have the same first two rows - i.e., decisions d_1 and d_2 have the same utilities in each matrix.

Analysis: For the first utility matrix, we have the following regret matrix

	s_1	s_2	s_3
d_1	5	0	6
d_2	0	8	0

Hence d_1 has maximum regret 6 and d_2 has maximum regret 8. Thus the mini-max regret criteria selects d_1 as optimal and d_2 is non-optimal.

Now consider the second utility matrix which in effect merely adds a new decision d_3. The corresponding regret matrix is

	s_1	s_2	s_3
d_1	10	0	6
d_2	5	8	0
d_3	0	5	9

Hence d_1 has maximum regret 10, d_2 has maximum regret 8 and d_3 has maximum regret 9. Thus the mini - max regret criterion now selects d_2 as optimal! That is adding an "irrelevent alternative" d_3 has changed d_2 from non-optimal to optimal.

Luce-Raiffa (p. 288) give the following humorous illustration of this type of incongruous result:

DOCTOR: Well, Nurse, That's the evidence. Since I must decide whether or not he is tubercular, I'll diagnose tubercular.

NURSE : But, Doctor, you do not have to decide one way or the other, you can say you are undecided.

DOCTOR: That's true, isn't it? In that case, mark him not tubercular.

NURSE : Please repeat that!

The final criterion we shall consider for making decisions under uncertainty is the "principle of insufficient reason". Here one asserts that if one is "completely ignorant" as to which state $s_1, \ldots, s_m$ obtains, then one should behave as if all states are equally likely. This means that we assign probability $1/m$ to each state of nature. Since decisions when both utilities and probabilities of the states of nature are given is considered in the next section, we do not pursue this criterion here.

To repeat the axioms for the model of this section are:

1) A set $D = \{d_1, d_2, \ldots, d_n\}$ of decisions is specified.
2) A set $S = \{s_1, s_2, \ldots, s_n\}$ of states of nature is specified.
3) The outcomes are determined solely by the pair (d_i, s_j).
4) For each outcome a utility U_{ij} is specified.

The procedure is to select and use a "decision under uncertainty" criteria such as those discussed in this section.

Our flow chart can be continued as shown in Figure 6.4

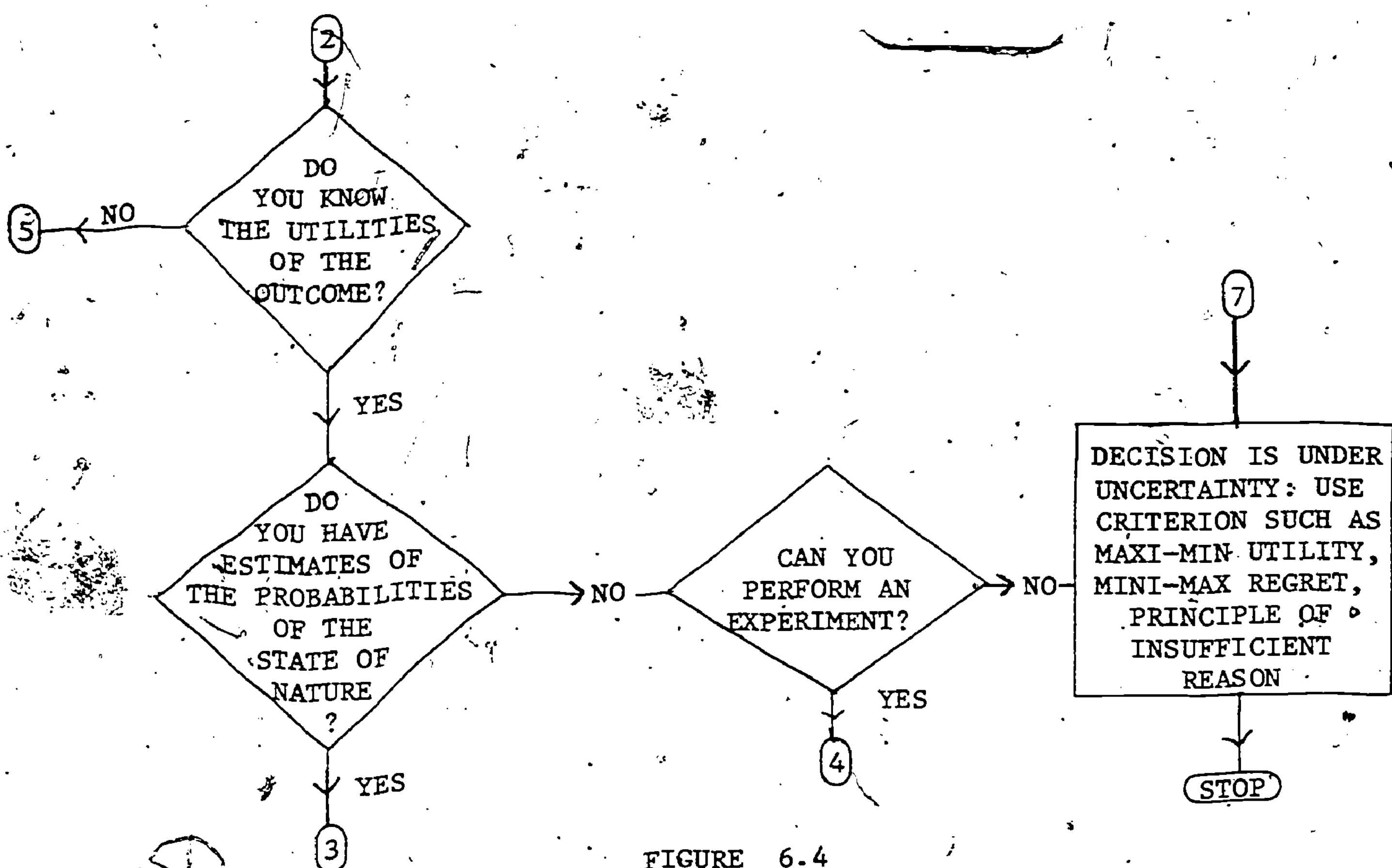


FIGURE 6.4

B. Utilities and Probabilities are Known, No Experiment Available.

In this case (known as "decisions under risk") we can compute the *expected utility* of each decision d_i :

$$E[d_i] = \sum_{j=1}^{m} U_{ij} P_j$$

where P_j = probability that state of nature s_j occurs and U_{ij} is the utility for the outcome corresponding to the decision d_i and state s_i . A straight forward and sensible procedure is to then choose the decision d_i which has the greatest expectation. The axioms for this model are

1) A set $D = \{d_1, d_2, \ldots, d_n\}$ of decisions is specified,
2) A set $S = \{s_1, s_2, \ldots, s_m\}$ of states of nature is specified,
3) The outcomes are determined solely by the pair (d_i, s_j) ,
4) For each outcome a utility U_{ij} is specified,
5) For each state of nature s_j a probability P_j of it occurring is known.

The procedure is then to choose the decision with maximum expected utility.

In practice the probability P_j might involve subjective judgments or might only be approximately known. Nevertheless in order to use this model, we must accept these probabilities as final.

Example 7. (Feibes) We return to our three investments A, B, C of Example 1 . Each investment costs $500, and they have respective returns of 3%, 5% and 6% . Now, however, we assume that the returns are not certain. It is possible for each investment to result in either the given percentage return or a loss of $5 . Specifically, the probabilities of a positive return for A, B, C are .9 , .5, and .2 respectively. Which investment should we make?

Analysis: The utility matrix is

	A Sound(.9)	A Unsound(.1)	B Sound(.5)	B Unsound(.5)	C Sound(.2)	C Unsound(.8)
A	\$15	-\$15	\$ 0	\$ 0	\$ 0	\$ 0
B	\$ 0	\$ 0	\$25	-\$ 5	\$ 0	\$ 0
C	\$ 0	\$ 0	\$ 0	\$ 0	\$30	-\$ 5

Expected value for $A = (15)(.9) + (-5)(.1) = \$13.00$

Expected value for $B = (25)(.5) + (-5)(.5) = \$10.00$

Expected value for $C = (30)(.2) + (-5)(.8) = \$\ 7.00$

Using expected value as a decision criterion, we choose investment A.

Our flow chart is extended as shown in Figure 6.5.

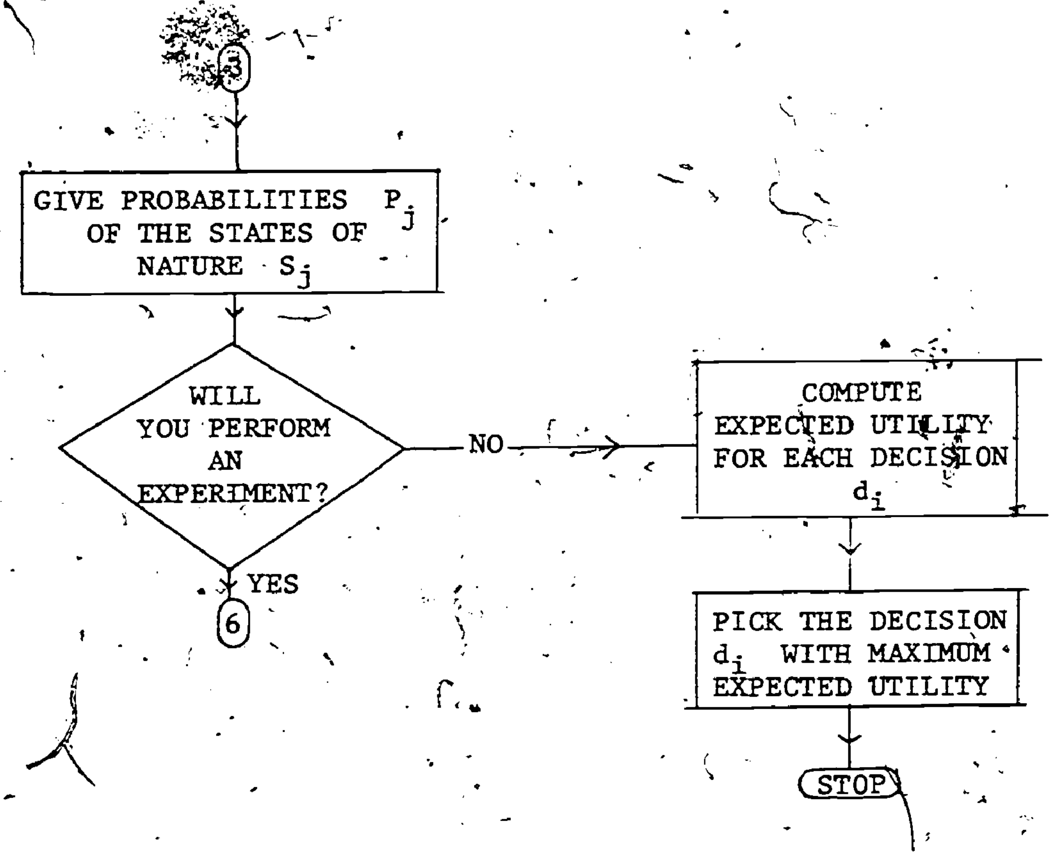


Figure 6.5

C. Utility and Probabilities are Known, Experiment is Available.

In this case the experiment is used to revise the given probabilities of the states of nature. One may wonder why we should bother with the experiment if the probabilities are "given". There are two reasons, first of all as mentioned above, the given probabilities may only have been obtained through subjective judgments or approximations. Secondly, we may have the probabilities for a population as a whole, while we are faced with making a decision involving a particular sample from this population.

Example 8.

An organization wishes to decide whether or not people will succeed in a certain job classification. It is known from past experiences that 40% of all people who apply for this job have been successful. A screening test has now been developed, and it has been found that 70% of the successful candidates pass the test while only 30% of unsuccessful candidates pass the test. A new candidate for the job passes the test. What is the probability that he will be successful in the job?

Analysis: The candidate's probability of being successful prior to taking the test is 0.4. In the light of passing the test, his (posterior) probability of success can be evaluated by Bayes' formula (see Section 6.2, eq. (6.9)):

$$P[\text{Success} \mid \text{Pass}] = \frac{P[\text{Pass} \mid \text{Success}]\ P[\text{Success}]}{P[\text{Pass} \mid \text{Success}]\ P[\text{Success}] + P[\text{Pass} \mid \text{Not Success}]\ P[\text{Not success}]}$$

$$= \frac{(.7)(.4)}{(.7)(.4) + (.3)(.6)}$$

$$= .61$$

Thus passing the test raises the probability of eventual success from 0.4 to 0.61.

The general procedure for re-estimating the probabilities that various states of nature occur is similar to the process developed in Example 8. We assume that a random experiment has a specified set of outcomes $A = \{a_1, a_2, \ldots, a_\ell\}$ and that we know the conditional probabilities $P[a_k \mid s_j]$, of a given experimental outcome a_k if the state of nature is s_j. We then compute the revised probabilities, P_{jk}, that the state of nature is s_j if the outcome of an experiment is a_k. From Bayes' theorem these revised probabilities are:

$$P_{jk} = P[s_j \mid a_k] = \frac{P[a_k \mid s_j]\, P_j}{\sum_{j=1}^{m} P[a_k \mid s_j]\, P_j}$$

At this point, we have "improved" estimates for the probabilities of the state of nature s_j and can proceed as in Section B above to compute the expected utility of decision d_i given the experimental result a_k:

$$E[d_i \mid a_k] = \sum_{j=1}^{m} U_{ij}\, P_{jk}$$

Then for a fixed experimental outcome a_k, just as before, we choose the decision d_i with greatest expection.

Example 9. (Modified from Moore & Yackal)

Mr. Smith has a congenital hearing defect, caused by malformation of the bones of the inner ear. A surgeon states that an operation is available to correct this defect, but the operation is not always sucessful. In fact, the operation may correct Mr. Smith's hearing (s_1), have no effect (s_2), or destroy the partial hearing which he now has (s_3). Although the surgeon cannot predict in advance which of these states of nature will hold, he can, from extensive medical data, state that the probabilities P_1, P_2, P_3 of these three states of nature are .9, .05, .05 respectively. Also there is a laboratory test which can be performed which has three outcomes A, B, C. Medical data is available which gives the probabilities of A, B, C when the states of nature (discovered after surgery) are s_1, s_2, s_3:

STATE OF NATURE	TEST OUTCOME A	B	C
s_1	.6	.2	.2
s_2	.3	.4	.3
s_3	.1	.2	.7

The laboratory report on Mr. Smith turns out to be B . Mr. Smith must decide whether to have the operation (d_1) or not have the operation (d_2).

After careful consideration of the inconveniences of his partial hearing loss, the expense of the operation, and the risk of total hearing loss; Mr. Smith draws up the following payoff matrix which reflects his personal feelings:

STATES OF NATURE AFTER OPERATION

Mr. Smith's Decision	s_1	s_2	s_3
d_1	25	-15	-100
d_2	-10	-10	-10

Analysis: We first compute the conditional probabilities of the outcome B given s_1, s_2, and s_3 by Bayes' theorem

$$P[s_1 \mid B] = \frac{(.2)(.9)}{(.2)(.9) + (.4)(.05) + (.2)(.05)} = .858$$

$$P[s_2 \mid B] = \frac{(.4)(.05)}{(.2)(.9) + (.4)(.05) + (.2)(.05)} = .095$$

$$P[s_3 \mid B] = \frac{(.2)(.05)}{(.2)(.9) + (.4)(.05) + (.7)(.05)} = .047$$

We now use these probabilities as the probabilities of the state of nature given the result B of the lab test. Hence we can compute

$$E[d_1 \mid B] = 25(.858) + (-15)(.095) + (-100)(.047) \doteq 15 ,$$

$$E[d_2 \mid B] = -10(.858 + .095 + .047) = -10 .$$

Hence on the basis of maximizing his expected utility and knowing that the lab result was B , Mr. Smith should decide to have the operation.

The criterion used here is known as ~~the~~ *Bayes' Decision Criterion.* A *Bayes' Decision Strategy* is to perform the same computation (i.e. maximize expectation) for each possible experimental outcome a_k and use these Bayes' probabilities to specify the best decision for any outcome of the experiment.

When an experiment is performed, it is always the case that *decision strategies (or decision rules)* are involved. A decision strategy is a function from the set A of experimental outcome to the set D of decisions. Thus if there are l experimental outcomes and n decisions there are n^l possible decision strategies. This number, n^l, can become exceptionally large. For example even in the relatively simple ear operation problem (Example 9) $n = 3$ and $l = 2$ is $n^l = 9$. (This is to be compared with the number of decisions, 2 . The number of decision strategies far exceeds the number of decisions. Hence there is the potential of requiring a choice from among M^l things (decision strategies) rather then from among M things (decisions). It is, of course, to our advantage to try to avoid this increase in the number of possible choices.

In the case under consideration here when utilities *and* prior probabilities are available in addition to the conditional probabilities resulting from an experiment, the Bayes' Decision Strategy is usually taken to be the best of the n^l possible strategies. In this case, the optimal strategy is known and the potential increased difficulty of choosing among n^l decision strategies instead of among the n decisions does not occur. The axioms for a Bayes Decision Strategy Model are:

1) The set $D = \{d_1, d_2, \ldots d_n\}$ of decisions is specified,

2) The set $S = \{s_1, \ldots, s_n\}$ of state of nature is specified,

3) The set $A = \{a_1, a_2, \ldots, a_\ell\}$ of experimental outcomes is specified,

4) The utilities U_{ij} associated with the outcome of d_i and s_j are specified.

5) The (prior) probabilities P_j of each s_j are known,

6) The conditional probabilities $P[a_k \mid s_j]$ of the experimental outcome a_k when the state of nature is s_j are known.

The procedure is then to compute $P[s_j \mid a_k]$ from eq (6.9) and for each a_k choose the d_i so that $E[d_i \mid a_k]$ is maximized.

Our flowchart is now extended as shown in Figure 6.6.

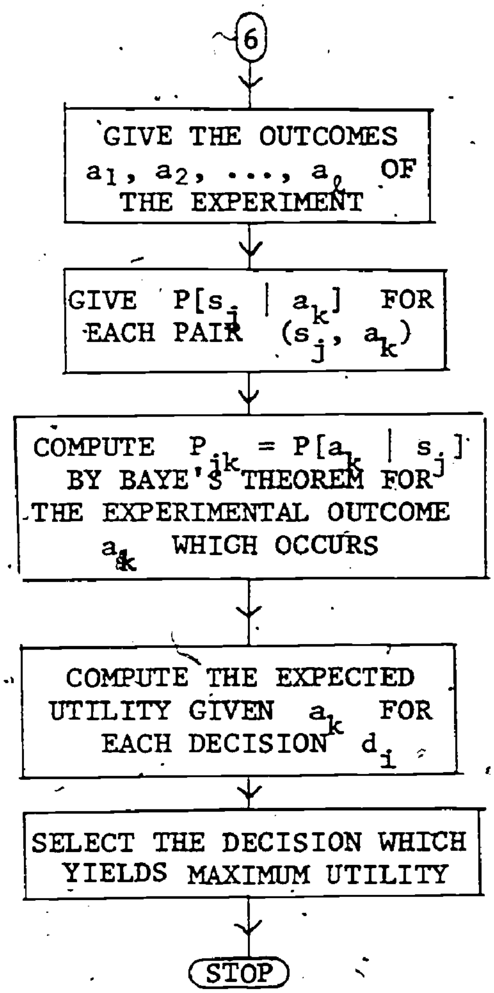


Figure 6.6

D. Utilities are Known, No Probabilities Known, but an Experiment is Available.

This situation arises most commonly in statistical decision theory. Since an experiment is involved, we must consider decision rules. Recall that a decision rule assigns a decision to each experimental outcome. Since we do not have prior probabilities, it will not be possible to reduce the selection of a decision rule to that of selecting a decision as in Section C.

In this case we know the values of the utilities

$$U(d_i, s_j)$$

and we know the conditional probabilities

$$P[s_j \mid a_k]$$

where a_k is an outcome of the experiment.

Let r denote a decision rule and $r(a_k)$ denote the decision which r assigns to the experimental outcome a_k, e.g.

$$d_i = r(a_k)$$

Then for each decision $r(a_k)$ and each state of nature s_j, we know a utility $U(r(a_k), s_j)$ (recall $r(a_k)$ is some particular decision). Since the probabilities, $P[s_j \mid a_k]$ are known, we may average out the experimental outcomes and assign a utility for the decision *rule* r which is independent of the particular experimental outcome as follows

$$U(r, s_j) = \sum_{k=1}^{\ell} U(r(a_k), s_j) \quad P[s_j \mid a_k]$$

We have now eliminated the outcome of the experiment and have only utilities. However, the utilities are for decision *rules*, rather than utilities for the decisions. Thus we have reduced the problem to the case of decision under uncertainty as treated in Section A, and creterion such as Maxi-Min utility may be used. Notice, however, that what we will produce is a rule for making a decision, not the decision itself.

For a fuller treatment of this case in statistical decision theory see Mood, Graybill and Boes, in particular pp. 297-299, pp. 350-351 and pp. 414-416.

Our flow chart is continued as shown in Figure 6.7

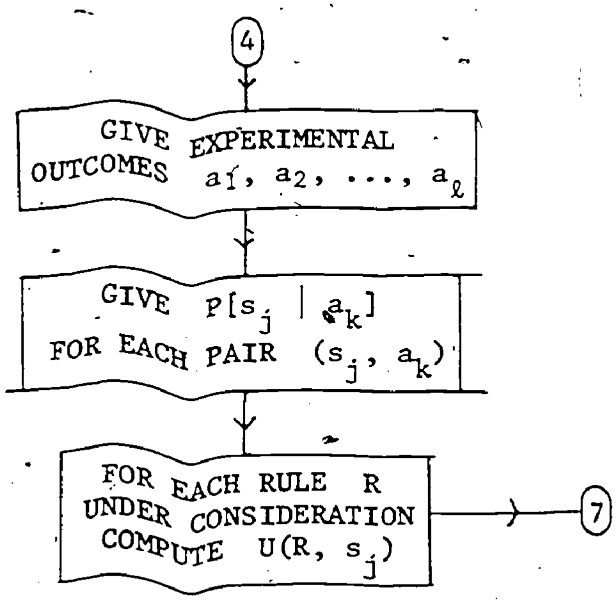


Figure 6.7

E. No Utilities or Probabilities Known, but an Experiment is Available.

The case when only an experiment (typically taking a random sample) is available is the subject of classical statistics. In this situation one adopts criteria such as the principle of maximum liklihood, minimum variance - unbiased estimators, arbitrary confidence levels, arbitrary size of type one error, in place of the missing utilities and/or prior probabilities to evaluate decision rules. The flow chart continuation is shown in Figure 6.8.

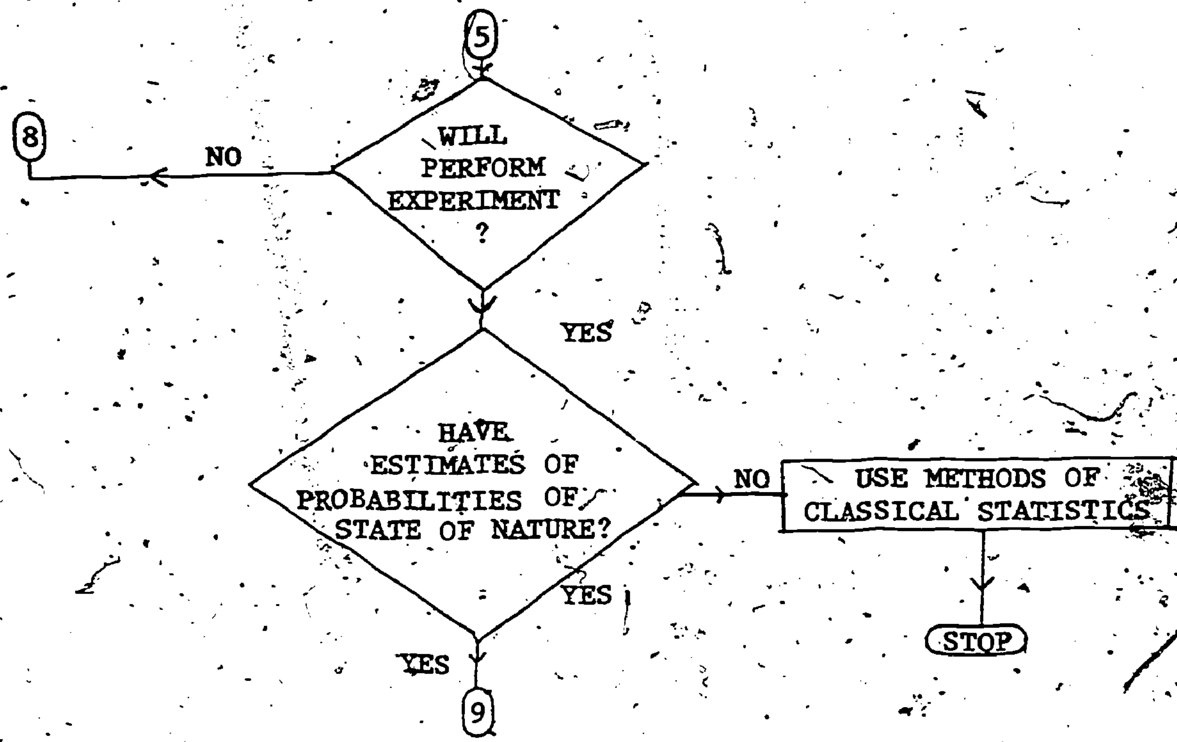


Figure 6.8

F. No Utilities Known but Probabilities are Known and an Experiment is Available.

Here we can use Bayes' theorem to refine our knowledge of the probabilities and hence reduce the problem to one in which there is no experiment. This case is considered in the next sub-section – G. The flow chart addition is given in Figure 6.9.

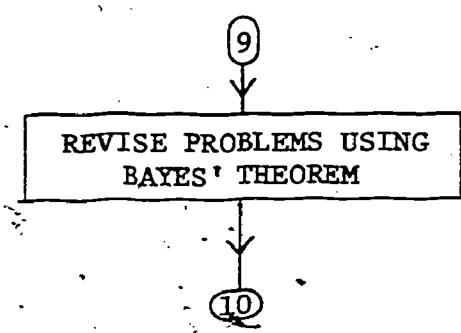


Figure 6.9

G. No Utilities Known and No Experiment is Available, but Probabilities are Known.

Here no global rules can be given, but the general advice to "behave as if the most likely state of nature is in fact certain" may be helpful. We reference Mood, Graybill & Boes pp. 340-343 for an example involving point estimation. Additional examples when this criteria is useful occur whenever the decisions are of the form: d_i = assert s_i is the true state of nature. The flow chart continuation is given in Figure 6.10.

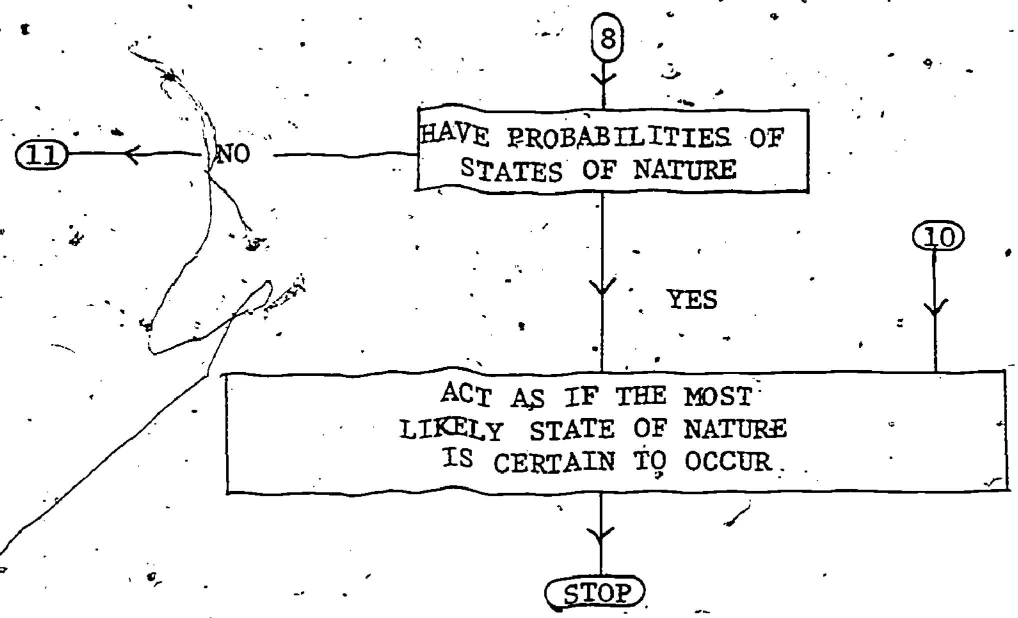


Figure 6.10

H. Nothing is Known.

Our only advice in this case is to guess! See Figure 6.11.

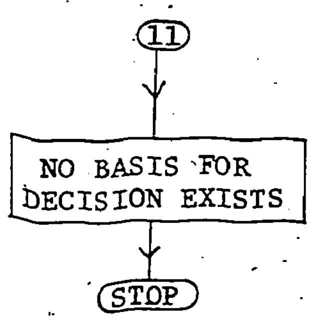


Figure 6.11

* *

Interterm Project.

Piece together the flow chart developed in this section (Figures 6.1 to 6.11) to give a comprehensive set of rules for decision making.

* *

6.4 Markov Chain Models.

We will begin this final topic of Chapter 6 by presenting a simple demographic model in a form suitable for use in a matrix theory course. In order to do this, we will temporarily avoid the nomenclature of probability theory and use a slightly non-standard notation.

We consider the movement of the population of the United States. Our model will be over-simplified, but the generalization to a more realistic model should be apparent.

We divide the nation into four regions: East, Midwest, Mountains and West. Suppose that in any given year one-half of the population in the East remains there. The other half moves as follows: 1/12 of the total moves to the Midwest, 1/6 moves to the Mountains and 1/4 of the total moves to the West. Notice that

$$1/2 + 1/12 + 1/6 + 1/4 = 1$$

Thus the entire population is accounted for. We can express this movement as a column vector

$$\text{(6.11)} \qquad \text{TO} \left\{ \begin{array}{l} \text{East} \\ \text{Midwest} \\ \text{Mountain} \\ \text{West} \end{array} \right. \overset{\text{FROM EAST}}{\begin{bmatrix} 1/2 \\ 1/12 \\ 1/6 \\ 1/4 \end{bmatrix}}$$

Now suppose we have the similar column vectors for the Midwest, Mountains and West:

$$\text{(6.12)} \qquad \overset{\text{FROM MIDWEST}}{\begin{bmatrix} 0 \\ 1/2 \\ 1/4 \\ 1/4 \end{bmatrix}} \qquad \overset{\text{FROM MOUNTAINS}}{\begin{bmatrix} 0 \\ 0 \\ 3/4 \\ 1/4 \end{bmatrix}} \qquad \overset{\text{FROM WEST}}{\begin{bmatrix} 1/4 \\ 0 \\ 1/4 \\ 1/2 \end{bmatrix}}$$

Note that the sum of the components of each of these vectors is 1. We can combine these movement vectors into a 4 x 4 table:

(6.13)

TO:	FROM EAST	FROM MIDWEST	FROM MOUNTAINS	FROM WEST
EAST	1/2	0	0	1/4
MIDWEST	1/12	1/2	0	0
MOUNTAINS	1/6	1/4	3/4	1/4
WEST	1/4	1/4	1/4	1/2

We can raise some interesting questions regarding the movement of population implied by this table. For example:

a) Given some initial population distribution, what is the population distribution after the first year?

b) Assuming that the movement is the same for the second year, then what is the population distribution after the second year?

c) What will the "long run" population distribution be if this pattern of movement continues indefinitely?

d) Is there some population distribution which will be unchanged by the movements discribed by the matrix T?

Assume then that the initial population is 3/4 in the East and the remaining 1/4 in the Midwest (as in the early days of the nation's development). Hence the original population may be represented as the column vector

(6.14)
$$P_0 = \begin{array}{c} \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \text{EAST} \\ \text{MIDWEST} \\ \text{MOUNTAINS} \\ \text{WEST} \end{array}$$

To compute the proportion of the population in the East after the first year, we observe from the first row of our table and our population vector P_0 that $1/2 \times 3/4$ remains in the East, $0 \times 1/4$ move from the Midwest to the East, 0×0 move from the Mountains to the East and $1/4 \times 0$ move from the West to the East, hence a total of

$$1/2 \times 3/4 + 0 \times 1/4 + 0 \times 0 + 1/4 \times 0 = 3/8$$

are in the East after the first year.

Similarly from the third row of the table

$$1/6 \times 3/4 + 1/4 \times 1/4 + 3/4 \times 0 + 1/4 \times 0 = 3/16$$

are in the Mountains after the first year.

These calculations clearly correspond to ordinary matrix multiplication, where we construe our table as a 4×4 matrix. Thus denoting the population after the first period by P_1, we have

(6.15) $$P_1 = \begin{pmatrix} 1/2 & 0 & 0 & 1/4 \\ 1/12 & 1/2 & 0 & 0 \\ 1/6 & 1/4 & 3/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1/4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/8 \\ 3/16 \\ 3/16 \\ 1/4 \end{pmatrix}$$

which agrees with and extends our previous calculations.

This answers question (a). For brevity, let us denote our 4×4 matrix by the letter T. Then a symbolic version of our above result is

(6.16) $$P_1 = T P_0$$

With this notation it is easy to answer our second question. Denoting the population vector after the second year by P_2 and assuming the movement matrix T remains the same, we have

(6.17) $$P_2 = T\,P_1 = \begin{bmatrix} 1/4 \\ 1/8 \\ 5/16 \\ 5/10 \end{bmatrix}$$

We could also compute P_2 as follows, taking advantage of the fact that matrix multiplication is associative:

(6.18) $$P_2 = T\,P_1 = T(T\,P_0) = T^2\,P_0$$

Similarly we find, for the population vector P_k after k years

(6.19) $$P_k = T^k\,P_0$$

These results answer question (b) and its generalization to k years and also shed some light on question (c). For from (6.19), we see that the "long run" behavior is closely bound up with the behavior of T^k for large k.

A BASIC program to successively compute $P_1, P_2, \ldots, P_{30}$ from

$$P_{k+1} = T\,P_k \qquad k = 0, 1, 2, \ldots$$

and to print every tenth result is

```
10   DIM T(4,4),X(4,1),Y(4,1)
20   MAT READ T(4,4)
30   MAT READ X(4,1)
40   FOR J=1 TO 3
50     FOR K=1 TO 10
60       MAT Y=T*X
70       MAT X=Y
80     NEXT K
90     PRINT
100     MAT PRINT X
110  NEXT J
120  DATA .5,0,0,.25
130  DATA .0833333,.5,0,0
140  DATA .166667,.25,.75,.25
150  DATA .25,.25,.25,.5
160  DATA .75,.25,0,0
170  END
```

The result of running this program is

```
.16691122716
.02851009612
.47124620573
.33333325419

0.16666709l2
.02777890753
.47222154574
.33333373890

.16666693635
.02777781027
.47222313118
.33333390557
```

Notice that the population vector appears to be settling down to some unchanging vector. The implication is that this vector will be the answer to question (c) above.

Before pursuing question (c) further, however we answer question (d) in the affirmative by producing a vector which remains unchanged. Consider an initial population vector of

$$P_E = \begin{bmatrix} 1/6 \\ 1/36 \\ 17/36 \\ 1/3 \end{bmatrix} \tag{6.20}$$

Then

$$P_1 = T\,P_E = P_E \tag{6.21}$$

and hence

$$P_k = P_E\ , \quad k = 1, 2, 3, \ldots \tag{6.22}$$

Now if we compute successive powers of the matrix T, we will find that the resulting matrices tend towards the following matrix

$$T_E = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 \\ 1/36 & 1/36 & 1/36 & 1/36 \\ 17/36 & 17/36 & 17/36 & 17/36 \\ 1/3 & 1/3 & 1/3 & 1/3 \end{bmatrix} \tag{6.23}$$

Notice that each column of T_E is identical with every other column and in turn is equal to the vector P_E which remained fixed. Notice also that

$$T_E \, P_0 = P_E \tag{6.24}$$

for *any* column vector P_0 which consists of positive components, summing to one. Thus since

$$T^k \rightarrow T_E \quad \text{as} \quad k \rightarrow \infty \tag{6.25}$$

we have that

$$P_k = T^k P_0 \rightarrow T_E \, P_0 = P_E \, . \tag{6.26}$$

Thus question (c) concerning the long run behavior is settled for the particular matrix T given by (6.13). Let us note that there was no need to pull the vector P_E "out of the hat". If we seek a vector P_F which is unchanged (fixed) by T, we have the equation

$$T \, P_F = P_F \tag{6.27}$$

or

$$(T - I) \, P_F = 0 \tag{6.28}$$

where I is 4×4 identity matrix.

Since the equation is homogeneous, we have either just the trivial solution or infinitely many solutions. The latter situation obtains and since we demand that the components of P_F sum to one, we can pick a unique population distribution vector from the infinitely many solutions of (6.28). Thus the material at hand serves as an application of the usual linear equation solving that is performed in linear algebra courses.

A BASIC program to form eqs. (6.28), replace the last equation (which in this case is dependent upon the others) by

$$p_1 + p_2 + p_3 + p_4 = 1$$

and find the unique solution of the resulting 4×4 system is

```
10  DIM T(4,4),I(4,4),R(4,4),X(4,1),B(4,1)
100  MAT READ T
110  MAT I=IDN
120  REM  **  SUBTRACT 1 FRØM DIAGØNAL  **
130  MAT T=T-I
140  REM  **  PLACE ALL 1'S IN LAST RØW  **
150  FØR K=1 TØ 4
160    LET T(4,K) = 1
170  NEXT K
180  REM  **  PUT ZERØES ØN RIGHT SIDE EXCEPT FØR LAST RØW  **
190  MAT B = ZER
200  LET B(4,1) = 1
210  REM  **  CØMPUTE AND PRINT SØLUTIØN  **
220  MAT R = INV(T)
230  MAT X = R*B
240  MAT PRINT X
300  DATA .5,0,0,.25
310  DATA .0833333,.5,0,0
320  DATA .166667,.25,.75,.25
330  DATA .25,.25,.25,.5
400  END
#
```

If we run this program the results are:

```
.16666663333
.02777776111
.47222233889
.33333326667
```

Notice that this is the vector P_E given in equation (6.20).

In any case we find that

$$P_F = P_E \tag{6.29}$$

is our particular numerical example given by (6.13). That is, the vector which is "fixed" is the same as the "equilibrium" on "limiting" vector.

Before enunciating any general theorems, let us investigate some different T matrices. First consider a model in which the country is divided into two parts: East and West. Suppose that all of the people in the East move to the West after one year and vice versa, then

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6.30}$$

and

$$T^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T$$

Thus the columns of T^k do not become identical in this case and as a consequence P_k does not approach a fixed vector. For example, if

$$P_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then

$$P_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = P_0$$

and so on. Thus there is no vector P_E which arbitrary P_0's will approach. However, there is a fixed vector

(6.31) $$P_F = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

As a third example, let there be three regions: East, Midwest and West. Of those in the East, 1/2 stay there and 1/2 move to the West. All of the people in the Midwest stay in the Midwest. Of those in the West, 1/2 move to the Midwest and 1/2 stay in the West. Then T becomes

(6.32) $$T = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

We find in this case that

$$T^2 = \begin{bmatrix} 1/4 & 0 & 0 \\ 1/4 & 1 & 3/4 \\ 1/2 & 0 & 1/4 \end{bmatrix}$$

$$T^4 = \begin{bmatrix} 1/16 & 0 & 0 \\ 11/16 & 1 & 15/16 \\ 4/16 & 0 & 1/16 \end{bmatrix}$$

$$T^8 = \begin{bmatrix} 1/256 & 0 & 0 \\ 247/257 & 1 & 255/256 \\ 8/256 & 0 & 1/256 \end{bmatrix}$$

Thus it appears that

(6.33) $$T_E = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

(6.34) $$P_E = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P_F$$

Notice that all of the population is eventually "absorbed" into the Midwest region.

Rather then say 1/2 of the population in the East moves to the West in any given year, we could have said that for any person in the East, the probability that he will move to the West is 1/2 . Had we done so, we would have been led to the same matrix equations and the same general results regarding equilibrium and fixed vectors. Thus we now turn our attention to stochastic matrices (we can think of (6.13) as one such matrix) and Markov chains.

We will carefully state the axioms for the Markov Chain Model using the standard probabilistic nomenclature and notation. We then will give some general theorems about Markov Chains and finally cite some additional examples which will make the wide scope of applicability of this model more apparent.

The basic assumptions of the Markov Chain Model are:

1) There exists a sample space consisting of a sequence of trials (eg. the population distributions at periods 1, 2, 3, ...),
2) The outcome of each trial is one of a finite set of states $s_1, s_2, \ldots, s_n$ (e.g., East, Midwest, Mountain, West) with respective probabilities $p_1, p_2, \ldots, p_n$,
3) The probability of each outcome s_j depends upon the outcome s_i of the immediately preceeding trial, but not on the earlier trials (e.g., the probability of a person moving to the East depends on whether the person is *now* in the East, Midwest, Mountains or West but not on where he was earlier). The conditional probability of s_j given s_i is denoted by t_{ij}.

We make two helpful definitions. A *probability vector* is a vector $(p_1, \ldots, p_n)$ whose components satisfy

$$\text{(6.35)} \qquad \begin{aligned} &\text{i)} \quad 0 \le p_i \le 1 \\ &\text{ii)} \quad p_1 + p_2 + \ldots + p_n = 1 \end{aligned}$$

In the population movement example, the component p_i represents the probability of being in state i (or the proportion of the population which is in state i). A *transition matrix* T is a matrix whose *rows* are probability vectors. In the population movement example the component t_{ij} of a transition matrix represents the conditional probability of transition to state s_j at the next trial, if the current state is s_i (or the proportion which will move from state i to state j at the next trial).

CAUTION: In adopting this interpretation for t_{ij} (as is standard in the social sciences) we are dealing with the *transpose* of the matrix used earlier - in particular, it is now the *rows* of T which will sum to one.

We trust that the above remarks serve to make clear the notation and the two possible interpretations of the Markov Chain model, and henceforth we confine ourselves to the probabilistic interpretation.

The key result of Markov Chain Theory is the one obtained earlier.*

Theorem 1: If the initial probability vector is P_0 then the probability vector after k trials is given by

$$P_k = P_0 \cdot T^k .$$

Corollary: The conditional probability that a system is state s_i initially is in state s_j after k steps is given by the (i, j) element of T^k.

We now turn to the investigation of the long-run behavior of a Markov Chain. A Markov Chain with transition matrix T is said to be *regular* if

$$T^k \to T_E \text{ as } k \to \infty$$

where T_E is a matrix each of whose *rows* consists of a *common* probability vector $\underline{e} = (e_1, e_2, \ldots, e_n)$ with *positive* (i.e., no zeros) components.

Recall that of our three population movement matrices (which now must be transposed) only the first corresponded to a regular chain. The second failed to be regular since T_E did not exist, and the third failed because $\underline{e} = (0, 1, 0)$ had zero components. The vector $\underline{e}$ is called the equilibrium or *stable* vector.

Theorem 2. T is regular if and only if there exists a power r such that T^r

*Technically, we should check that P_k is actually a probability vector, that is, satisfies (6.35).

has all positive (i.e. no zeros) components.

Proof: That regularity implies the existence of r ~~is~~ immediate (recall T_E has no zeroes in this case). The converse is fairly tedious, see e.g. Kemeny, Snell & Thompson or Maki & Thompson.

Theorem 3:* Let T be the transition matrix of an n-state regular Markov Chain and let r be the smallest power such that T^r has all positive elements, then

(6.36) $$r \leq (N - 1)^2 + 1$$

Theorem 4: If T is a transition matrix of a regular Markov Chain then the stable vector e satisfies

i) $p P^k \to e$ for any probability vector p

ii) $e P = e$ (i.e. the stable vector is a fixed vector).

Note that Theorems 2 and 3 allow us to decide if T is regular while Theorem 4 part (ii) tells us how to compute e. If any power T^r has all positive elements then it is easy to see that all succeeding powers have all positive elements. A *flow chart* to decide whether P is regular is given in Figure 6.12.

*Reference: Maki & Thompson cite this result on p. 101 without proof.

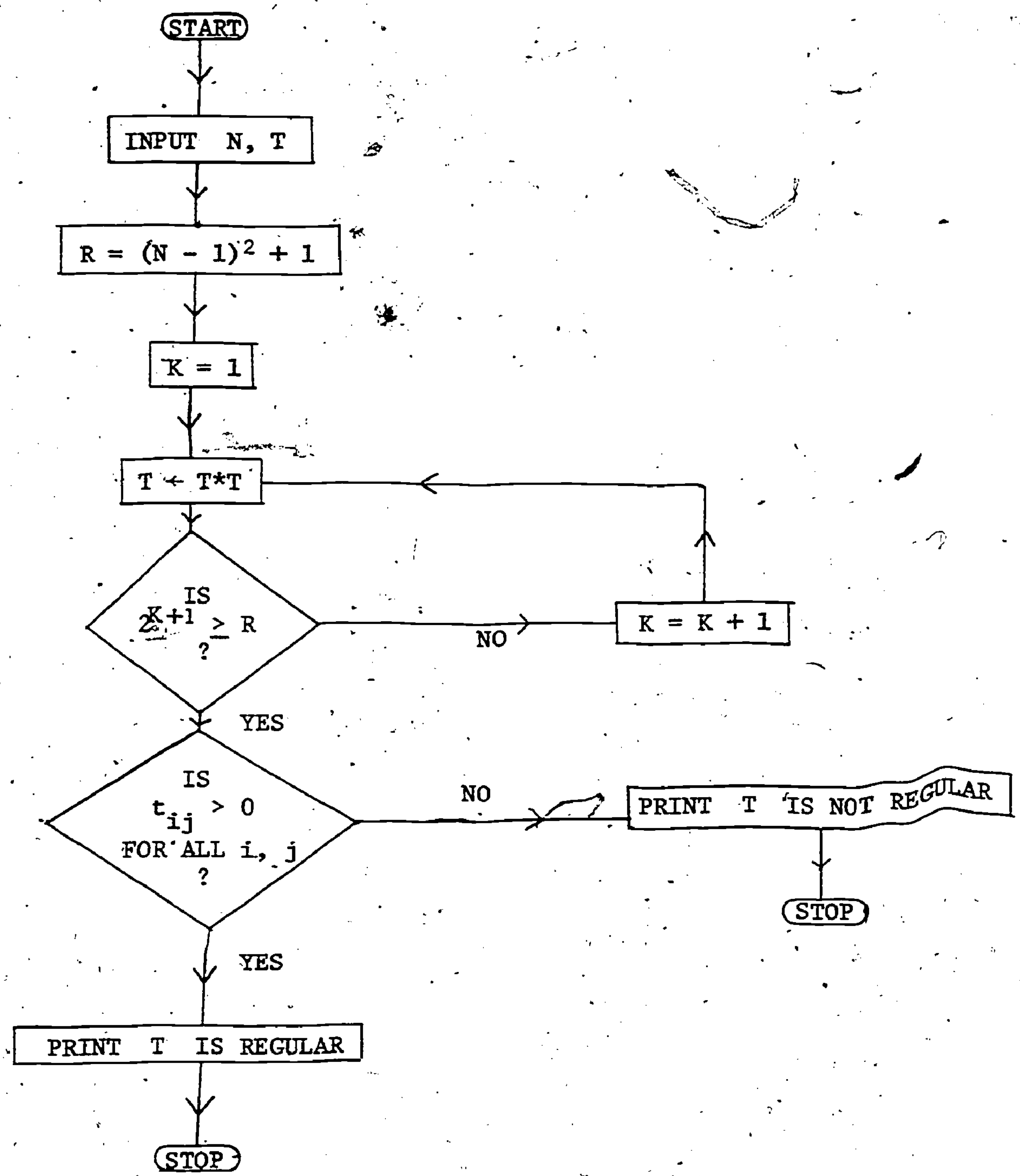


Figure 6.12

Alternatively, we could test $t_{ij} > 0$ after each iteration. If indeed T turns out to be regular, we can compute the stable vector e by solving

$$e\,T = e$$

subject to

$$\sum_{i=1}^{n} e_i = 1$$

by a Gaussian elemination routine.

We now cite some of the better known results for non-regular chains (again Maki & Thompson and Kemeny, Snell & Thompson give proofs and/or discussions). A Markov Chain is *ergodic*, if for every pair of states s_i and s_j, there exists an integer r such that a transition from s_i to s_j has positive probability (i.e. the $(i, j)^{th}$ of T^r is positive for some r - which *now* can depend on i and j). The ergodic property means that we can get from any state to any other state in a finite number of trials.

Of course, a regular chain is always ergodic since for regular chains there exists an r which works for all pairs of states simultaneously. However, there are ergodic chains which are not regular, for example, our second population movement matrix

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Here we can set from state 1 to state 2 and from state 2 to state 1 in only 1 step, while the transitions from state 1 to state 1 and state 2 to state 2 take only 2 steps. Hence T is ergodic, but not regular as noted previously.

<u>Theorem 5</u>: Ergodic Chains have a unique fixed vector, $e = eT$, where e is a probability vector.

Note that now the unique fixed vector is *not* necessarily a *stable* vector. In the above example $e = (1/2, 1/2)$ but T^k does not converge, hence $T^k P_0$ will not converge (unless $P_0 = e$). However, the components of e *do* have a useful interpretation: if $e = (e_1, e_2, \ldots, e_n)$ then e_i is the long run average probability of being in state s_i .

Finally we briefly consider *absorbing chains*. The state s_i of a Markov Chain is an *absorbing state* if $t_{ii} = 1$. A Markov Chain is said to be an *absorbing chain* if there exists at least one absorbing state and if transitions from *each* non-absorbing state to *some* absorbing state are possible in a finite number of steps.

chird population movement matrix (now transposed) provides an example of an absorbing chain. As we saw then,

$$T^k \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which implies that eventually we enter and remain in the absorbing state "middle" with probability one.

This behavior is typical, in fact if we write T in the Canonical form (by reordering the states if necessary)

$$T = \begin{bmatrix} I_m & 0 \\ R & Q \end{bmatrix}, \quad I_m = m \times m \text{ identity matrix}$$

then

$$T^k = \begin{bmatrix} I_m & 0 \\ R_k & Q^k \end{bmatrix}$$

where

$$R_k = R + Q^{k-1} R .$$

<u>Theorem 6:</u>

$$Q^k \to 0$$

This theorem means that we enter an absorbing state with probability one.

Theorem 7: $N = (I - Q)^{-1}$ exists and n_{ij} is the expected number of times that the system is in state s_j given that it started in state s_i and continued until absorbed.

We turn now to some additional examples of Markov Chains.

Example 10: A Learning Model for Simple Tasks (modified from Maki & Thompson).

We assume that such learning is all or none and thus there are only 2 states "learned" and "ignorant". We assume that the subject is "taught" in a sequence of learning sessions and that the probability of learning at any given session is c, where $0 \leq c < 1$. We assume that forgetting never takes place – that is once the subject is in the "learned" state, he remains there.

Analysis: If we denote "learned" as state one and "ignorant" as state two, then our assumptions lead to the transition matrix,

$$T = \begin{bmatrix} 1 & 0 \\ C & 1-C \end{bmatrix}$$

and the initial probability vector,

$$P_0 = (0, 1) .$$

It is easy to see that

$$T^k = \begin{bmatrix} 1 & 0 \\ 1 - (1 - C)^k & (1 - C)^k \end{bmatrix}$$

and hence if we denote the probabilities of being in states 1 and 2 at time k by the vector P_k, we have

$$P_k = P_0 \ T^k = (1 - (1 - C)^k, \ (1 - C)^k) .$$

One consequence of our model is that the probability of learning at or before the k^{th} trial is $1 - (1 - C)^k$.

Example 11: Another Learning Model (from Dorn).

In this model we assume that learning is gradual. Specifically we assume that in each trial, the subject learns a proportion A $(0 \leq A < 1)$ of the unlearned material. Again we assume the subject is totally ignorant of the material initially.

Analysis: Here the states have not been defined and a moment's reflection reveals that there are infinitely many states of partial learning. Hence the Markov Chain Model does *not* apply. An appropriate model is the difference equation

$$L_{k+1} - L_k = A(1 - L_k) \quad , \quad L_0 = 0 \; ,$$

where L_k denotes the proportion of material learned after k trials. The solution of the equation is

$$L_k = 1 - (1 - L_0)(1 - A)^k \; .$$

Moore & Yachel, p. 65 ff present still another learning model which involves two-step learning.

Example 12: (Malkevitch & Meyer)

An electric power company checks its main generator once each quarter year to forestall blackouts due to equipment failure. Assume there are two outcomes, W = generator needs no repair, D = generator is defective and needs repairs. Repairs will be made only in State D. (It is natural to suppose that the transition $D \to D$ is rare and $W \to D$ more common). Assume that if a given inspection yields W then the probabilities that next quarter's inspection yield W or D are .6 and .4 respectively, while if the current inspection yields D then these probabilities are .9 and .1 .

Analysis: The transition matrix is

$$\begin{array}{cc} & \overbrace{\begin{array}{cc} W & \quad D \end{array}}^{\text{TO}} \\ \text{FROM} \left\{ \begin{array}{c} W \\ D \end{array} \right. & \begin{bmatrix} .6 & .4 \\ .9 & .1 \end{bmatrix} \end{array}$$

which is clearly regular. The Markov Chain assumptions are at least plausible here.

<u>Example 13</u>: (Kemeny, Snell & Thompson). In predicting long term trends in Republican - Democrat transitions taking into account only the prior state (Republican or Democrat), one would use a transition matrix of the form

$$\begin{array}{ccc} & R & D \\ R & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{array}$$

$$\begin{array}{c|cc} & R & D \\ \hline R & 1-a & a \\ D & b & 1-b \end{array}$$

where a is the probability of a change from a Republican majority to a Democratic majority (estimated from historical records) and b is the probability of the opposite transition. A refinement which allows a little additional past history to be used is to consider the last two year's results - i.e. the states are now RR, DR, RD, DD where for example RR means voted Republican the last two times. The transition matrix now has the form

$$\begin{array}{c|cccc} & RR & DR & RD & DD \\ \hline RR & 1-a & 0 & a & 0 \\ DR & b & 0 & 1-b & 0 \\ RD & 0 & 1-e & 0 & e \\ DD & 0 & d & 0 & 1-d \end{array}$$

Note that certain transitions are now impossible. This device of enlarging the set of states is commonly employed to partially circumvent the Markov restriction that the transition probabilities can only depend on the current state.

Other examples abound: transitions between parents social states and child's social states; transitions between job catagories; transitions between physical locations; all can often be approximated by the Markov Chain Model. A particularly intriguing model of small group decision making is presented in Maki & Thompson p. 81 ff.

AUTHORS' EVALUATION

(Please circle one of the responses to each question)

1. Did you attend the short course in 1974-75? [illegible] No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ____________________

   ____

   ____

   can you suggest topics to be added? ____________________

   ____

   ____

   If (b), which topics should be abbreviated? ____________________

   ____

   ____

   which topics should be eliminated? ____________________

   ____

   ____

3. Could you read and understand the computer programs?

   (a) always　　　(c) seldom

   (b) sometimes　　　(d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy　　　(b) Too difficult

6. Was the number of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attempt any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Binomial model				
Poisson model				
Bayes' Theorem				
Decision Theory (in general)				
Maximum strategy				
Mini-max regret				
Maximum Expected Value				
Use of posterior probabilities				
Markov Chains				

APPENDIX

DIFFERENCE EQUATIONS

In this appendix we will discuss linear difference equations with constant coefficients. We will find the general solution of such equations when the equations are of the first and second order and have constant coefficients. A fuller discussion as well as a discussion of more general types of difference equations may be found in *Introduction to Difference Equations* by Samuel Goldberg (Wiley, New York).

PART I - LINEAR, FIRST ORDER DIFFERENCE EQUATIONS

1.1 Definitions of Difference Equations and their Solution

The equation

(1.1) $$a_1 y_{k+1} + a_0 y_k = b \qquad k = 0, 1, 2, \ldots .$$

where $a_1 \neq 0$ and $a_0 \neq 0$ is a linear, first order difference equation with constant coefficients. A solution of (1.1) is a function y_k defined over the set of non-negative integers ($k = 0, 1, 2, \ldots$) which reduces (1.1) to an identity. For a given value of y_0 , (1.1) has a unique solution.

As an example consider

$$2y_{k+1} - y_k = 1$$

For $y_0 = -1$ the function

$$y_k = 1 - (1/2)^{k-1}$$

is a solution because

$$2y_{k+1} - y_k = 2 - 2(1/2)^k - 1 + (1/2)^{k-1} = 1$$

for all k .

Self-Study: Problem #A.1

Match the difference equations on the left with the solutions on the right.

(a) $(k + 1)\, y_{k+1} + ky_k = 2h - 3$ (i) $y_k = (k(h-1))/2^k$

(b) $y_{k+1} - y_k = k$ (ii) $y_k = 2^{2k+1} - 1$

(c) $y_{k+1} - 4y_k = 3$ (iii) $y_k = 1 - (2/k)$

Solution to Self-Study: Problem #A.1

(a) - (iii)

(b) - (i)

(c) - (ii)

1.2 Solution of First Order Equations

Rewrite (1.1) as

$$y_{k+1} = -\frac{a_0}{a_1} y_k + \frac{b}{a_1}$$

or

$$y_{k+1} = My_k + C \tag{1.2}$$

where $M = -a_0/a_1$ and $C = b/a_1$. Notice that $M \neq 0$ although C may indeed vanish. Then

$$y_1 = My_0 + C$$

and

$$y_2 = My_1 + C = M(My_0 + C) + C$$

$$= M^2 y_0 + C(1 + M)$$

Then

$$y_3 = My_2 + C = M(M^2 y_0 + C(1 + M)) + C$$

$$= M^3 y_0 + C(1 + M + M^2)$$

Thus we might "guess" that

$$y_k = M^k y_0 + C(1 + M + M^2 + \ldots + M^{k-1})$$

But

$$1 + M + M^2 + \ldots + M^{k-1} = \begin{cases} \dfrac{1 - M^k}{1 - M} & \text{for } M \neq 1 \\ k & \text{for } M = 1 \end{cases}$$

so

$$y_k = \begin{cases} M^k y_0 + C \dfrac{1 - M^k}{1 - M} & \text{for } M \neq 1 \\ y_0 + k\,C & \text{for } M = 1 \end{cases}$$

or

(1.3)
$$y_k = \begin{cases} M^k \left(y_0 - \dfrac{C}{1 - M}\right) + \dfrac{C}{1 - M} & \text{for } M \neq 1 \\ y_0 + k\,C & \text{for } M = 1 \end{cases}$$

To verify that (1.3) is indeed a solution of (1.2) substitute the function defined by (1.3) in (1.2) and show that (1.2) is reduced to an identity.

Self-Study: Problem #A.2

Find solutions for the following difference equations

(a) $y_{k+1} - y_k - 2 = 0$ $\quad y_0 = 1$

(b) $y_{k+1} = y_k$ $\quad y_0 = 2$

(c) $y_{k+1} + y_k = 1$ $\quad y_0 = 2$

(d) $4y_{k+1} - y_k = 2$ $\quad y_0 = 3$

(e) $y_{k+1} - 2y_k = 1$ $\quad y_0 = 2$

(f) $y_{k+1} = 5y_k + 8$ $\quad y_0 = -2$

PART II - LINEAR, SECOND ORDER DIFFERENCE EQUATIONS

2.1 Definitions

The most general second order linear difference equation with constant coefficients may be written

(2.1) $$ay_{k+2} + by_{k+1} + cy_k = d \qquad k = 0, 1, 2, \ldots$$

where $a \neq 0$ and $c \neq 0$.
For given values of y_0 and y_1, (2.1) has a unique solution. The solution is the sum of the general solution of the homogeneous equation

(2.2) $$ay_{k+2} + by_{k+1} + cy_k = 0 \qquad k = 0, 1, 2, \ldots$$

and a particular solution of (2.1). The general solution of (2.2) and a particular solution of (2.1) are discussed below.

2.2 General Solution of the Homogeneous Equation

The general solution of (2.2) is found from the roots of the characteristic equation

$$ax^2 + bx + c = 0$$

Case I: If $b^2 - 4ac > 0$, then both roots are real and unequal. They are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The general solution of (2.2) then is

$$y_k = A(\lambda_1)^k + B(\lambda_2)^k$$

where A and B are constants to be determined from the initial conditions y_0 and y_1.

Solution to Self-Study: Problem #A.2

(a) $M = 1$, $C = 2$ so from (1.3)

$$y_k = 1 + 2k$$

(b) $M = 1$, $C = 0$ so

$$y_k = 2$$

(c) $M = -1$, $C/(1-M) = 1/2$ so

$$y_k = (3/2)(-1)^k + 1/2$$

(d) $M = 1/4$, $C/(1 - M) = 2/3$

$$y_k = (7/3)(1/4)^k + 2/3$$

(e) $M = 2$, $C/(1 - M) = -1$

$$y_k = 3.2^k - 1$$

(f) $M = 5$, $C/(1 - M) = -2$

$$y_k = -2$$

1.3 Behavior of the Solution

Recall first that $M \neq 0$. We will consider two cases: $M < 0$ and $M > 0$.

(i) If $M < 0$, the solution oscillates. The amplitudes of the oscillations increase as k increases if $M < -1$. The amplitude is constant if $M = -1$, and the amplitudes decrease if $0 < M < -1$.

(ii) For $M > 0$ then if $0 < M < 1$, the solution decays exponentially. If $M = 1$, the solution is linear. If $M > 1$ the solution grows without bound. Notice that for $M \neq 1$ if

$$y_0 = C/(1 - M)$$

the solution is a constant.

Finally we note that for $|M| < 1$ then the solution approaches $C/(1 - M)$ for large k.

Case II: If $b^2 - 4ac = 0$ then both roots are real and equal. They are

$$\lambda = -b/2a$$

In this case the general solution of (2.2) is

$$y_k = (A + Bk)\lambda^k$$

Case III: If $b^2 - 4ac < 0$ then both roots are imaginary and are complex conjugates. The roots are

$$\lambda_1 = -\frac{b}{2a} + i\,\frac{\sqrt{4ac - b^2}}{2a} \qquad \lambda_2 = -\frac{b}{2a} - i\,\frac{\sqrt{4ac - b^2}}{2a}$$

Let

$$r = \sqrt{c/a}$$

$$\theta = \cos^{-1}(b/2\sqrt{ac})$$

Then the general solution of (2.2) is

$$y_k = A\,r^k \cos(k\theta + B)$$

where again A and B are constants to be determined from the initial conditions y_0 and y_1.

2.3 Particular Solutions

To find a particular solution of (2.1) where d is a constant there are again three cases.

Case I: If both roots of the characteristic equation are real and distinct and neither root is 1, i.e., $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ then

$$y_k = d/(a + b + c)$$

Case II: If both roots are distinct and real but one, say λ_1, is 1 then $b = -(a + c)$. The general solution is

$$y_k = A + B\lambda_2^k$$

The particular solution is of the form

$$y_k = dk/(2a + b)$$

Case III: If both roots equal 1, i.e., $\lambda_1 = \lambda_2 = 1$, then $b = -2a$ and $c = a$. The general solution is

$$y_k = A + Bk$$

Both a constant and a constant times k are included in the general solution. The particular solution is

$$y_k = dk^2/2a$$

2.4 Examples

Example 1: $$y_{k+2} + y_{k+1} - 6y_k = 4$$

The characteristic equation is

$$x^2 + x - 6 = 0$$

whose roots are $\lambda_1 = 2$, $\lambda_2 = -3$ so the complete solution is

$$y_k = A(2)^k + B(-3)^k - 1$$

If the initial conditions are $y_0 = 2$ and $y_1 = 0$ then

$$y_k = (2)^{k+1} + (-3)^k - 1$$

Example 2: $$y_{k+2} + 3y_{k+1} - 4y_k = 10$$

The characteristic equation is

$$x^2 + 3x - 4 = 0$$

whose roots are $\lambda_1 = 1$, $\lambda_2 = -4$. The complete solution is

$$y_k = A + B(-4)^k + 2k$$

If the initial conditions are $y_0 = 0$ and $y_1 = 7$ then

$$y_k = 1 - (-4)^k + 2k$$

Example 3: $$y_{k+2} - 2y_{k+1} + y_k = -4$$

The characteristic equation is

$$x^2 - 2x + 1 = 0$$

with roots $\lambda_1 = \lambda_2 = 1$. The complete solution is

$$y_k = A + Bk - 2k^2$$

For $y_0 = 2$ and $y_1 = 3$

$$y_k = 2 + 3k - 2k^2$$

Self-Study: Problem #A.3

Find solutions for the following second order difference equations

(a) $y_{k+2} - 6y_{k+1} + 8y_k = 0$; $y_0 = 3$, $y_1 = 2$

(b) $y_{k+2} - 6y_{k+1} + 8y_k = 6$; $y_0 = 5$, $y_1 = 4$

(c) $y_{k+2} - 4y_{k+1} + 4y_k = 0$; $y_0 = 1$, $y_1 = 3$

(d) $y_{k+2} - 2y_{k+1} + 5y_k = 0$; $y_0 = 0$, $y_1 = 1$

(e) $y_{k+2} + y_{k+1} - 2y_k = 12$; $y_0 = 8$, $y_1 = -3$

(f) $y_{k+2} - 2y_{k+1} + y_k = 6$; $y_0 = 5$, $y_1 = 12$

Solution to Self-Study: Problem #A.3

(a) Roots of characteristic equation are 2 and 4.

From Case I of Section 2.2

$$y_k = 5 \cdot 2^k - 2 \cdot 4^k$$

(b) Roots of characteristic equation are 2 and 4.

From Case I of Section 2.3 the particular solution is

$$y_k = 2$$

From Case I of Section 2.2 then

$$y_k = 3 \cdot 2^k - 4^k + 2$$

(c) Roots of characteristic equation are $1 \pm 2i$

From Case III of Section 2.2 then

$$y_k = A(\sqrt{5})^k \cos (k\theta + B)$$

where $\theta = \cos^{-1} (\sqrt{5}/5)$. From the initial conditions

$$0 = A\sqrt{5} \cos B$$

$$1 = A\sqrt{5} \cos (\theta + B)$$

so

$$y_k = (\sqrt{5})^{k-1} \frac{\sin k\theta}{\sin \theta}$$

(d) Roots of the characteristic equation are 2 and 2, so from Case II of Section 2.2

$$y_k = (2 + k)2^{k-1}$$

(e) Roots of the characteristic equation are 1 and -2. The particular solution is from Case II of Section 2.3

$$y_k = 4k.$$

From Case I of Section 2.2 then

$$y_k = 5(-2)^k + 3 + 4k$$

(f) Roots of the characteristic equation are 1 and 1. The particular solution is from Case III of Section 2.3

$$y_k = 3k^2$$

From Case II of Section 2.2 then

$$y_k = 5 + 4k + 3k^2 .$$

APPENDIX

AUTHORS' EVALUATION

(Please circle one of the responses to each question)

1. Did you attend the short course in 1974-75? Yes No

2. Is this chapter

   (a) Too short

   (b) Too long

   (c) About right

   If (a), which topics should be expanded? ____________________

   can you suggest topics to be added? ____________________

   If (b), which topics should be abbreviated? ____________________

   which topics should be eliminated? ____________________

3. Could you read and understand the computer programs?

   (a) always (c) seldom

   (b) sometimes (d) never

4. Did the interim projects seem reasonable? Yes No

5. Were the self-study problems

   (a) Too easy (b) Too difficult

6. Was the <u>number</u> of self-study problems

   (a) Too large

   (b) About right

   (c) Too small

7. Did you attempt any of the self-study problems? Yes No

8. Are the solutions to the self-study problems properly placed (on overleaf from problem)? Yes No

If no, where would you suggest the solutions be placed?

____

____

____

9. For each topic, how solid an understanding do you think you have?

	Excellent	Good	Fair	Poor
Definition of Difference Equations	____	____	____	____
Solutions of First Order Equations	____	____	____	____
Solutions of Second Order Equations	____	____	____	____
Analysis of Qualitative Behavior of Soluitons	____	____	____	____

BIBLIOGRAPHY

A. General Textbooks

In this list are author, title, publisher, year of publication, difficulty rating (1 = easy to 4 = hard), material covered and brief comments for books that might serve as undergraduate texts. We would appreciate receiving comments from our readership concerning our accuracy & judgment in these matters.

(A-1) Adams, William J.; *Finite Mathematics*, Xerox, 1974 (1). Linear Programming Markov, Game Theory, Matrices, Finance. Excellent text, gives extensive criticism of the models developed and also survey uses of the models.

(A-2) Anton, H. & Kalmon, B.; *Applied Finite Mathematics*, Academic Press, 1974 (2). Sets, Linear Programming, Matrices, Probability, Statistics. Has some interesting applications; brief introduction to FØRTRAN.

(A-3) Campbell, H.G. & Spencer, P.E.; *Finite Mathematics*, MacMillan, 1974 (1) Systems (in switching circuits), Logic, Sets, Probability, Matrices, Linear Programming, Game Theory; Has introduction to APL. If you believe in the importance of logic and sets, this is your book. Has interesting references to applications; the treatment of most applications within the text are brief (but good).

(A-4) Dorn, W.S. & Greenberg H.J.; *Mathematics and Computing*, Wiley, 1967 (2) Linear Programming, FØRTRAN Programming, Computer Applications, Probability. Modesty forbids explicit comment.

(A-5) Feibes, Walter; *Introduction to Finite Mathematics*, Hamilton, 1974 (1) Probability, Decision Theory, Linear Programming, Game Theory, Finance. Superb treatment of elementary finite mathematics material. Uses examples to motivate general results.

(A-6) Goodman, A.W. & Ratti, J.S.; *Finite Mathematics with Applications*, 1971 (1) Logic, Sets, Probability, Matrices, Linear Programming, Game Theory; Has applications to social sciences. Should go well in the classroom.

(A-7) Kemeny, J.G., Snell, J.L. & Thompson, G.L.: *Introduction to Finite Mathematics*, Prentice-Hall, 1974 (2). The classic Finite Mathematics textbook, new revised and more modeling oriented (see Chapter 6 for further details).

(A-8) Malkevitch, J. & Meyer, W.; *Graphs, Models and Finite Mathematics*, Prentice-Hall, 1974 (1). Modeling, Graphs, Computers, Statistics, Probability, Games & Decisions, Difference Equations. Best Source for elementary graph theory applications. Good for developing the modeling approach. Has chapter on the theory of elections.

(A-9) Maki, D.P. & Thompson, M., *Mathematical Models and Applications*, Prentice-Hall, 1973 (3). Models, Markov Chains, Linear Programming, Graphs, Growth Models. A wealth of good material. Documents the "respectibility" of applied mathematics.

(A-10) Mizrahi, A. & Sullivan, M.; *Finite Mathematics with Applications*, Wiley, 1973 (2). Logic, Sets, Probability, Models, Linear Programming, Matrices, Graphs, Markov Chains, Statistics, Finance. Has too many rules of thumb without motivation. Nonetheless, a good book - especially the applications from Chapter 7 to the end.

(A-11) Moore, D.S. & Yackel, J.W.; *Applicable Finite Mathematics*, Houghton Mufflin, 1974 (2). Probability, Markov Chains, Linear Programming, Game Theory, Decision Theory. Exceptionally good modeling text. Best elementary reference on Decision Theory. Has computer projects.

(A-12) Negus, R.W.; *Fundamentals of Finite Mathematics*, Wiley, 1974 (1). Sets, Logic, Matrices, Linear Programming, Probability, Markov Chains, Nicely written text, no emphasis on models. Student oriented.

(A-13) Thomas, J.W. & Thomas, A.M.; *Finite Mathematics*, Allen & Bacon, 1973 (1). Logic, Sets, Probability, Matrices (with introduction to Markov processes and Linear Programming). Thoughtfully written; brief applications.

B. Specialized Reading List.

(B-1) Bailey, N.T.J.; *Elements of Stochasite Processes*, Wiley, 1964.

(B-2) Bailey, N.T.J.; *Mathematical Theory of Epidemics*, London: Charles Griffen, 1957.

(B-3) Bartlett, N.S.; *Stochastic Population Models in Ecology and Epidemiology*, Metheuy London, 1960.

(B-4) Coleman, J.P.; *Introduction to Mathematical Sociology*, MacMillan, London, 1964.

(B-5) Feller, W.; *An Introduction to Probability Theory and Its Applications*, Wiley, 1968.

(B-6) Goldberg, S.; *Introduction to Difference Equations*, Wiley (paperback).

(B-7) Hamming, R,; *Computers and Society*, McGraw-Hill (paperback), 1972.

(B-8) Kemeny, J.G. & Snell, J.L.; *Mathematical Modeling in the Social Sciences*, Ginn and Company, 1962.

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(A-15) Parzen, E., *Modern Probability Theory and its Application*, Wiley, 1968.

(A-16) Pielou, E.C.; *Introduction to Mathematical Ecology*, Wiley, 1969.

(A-17) Rainville, E.D. & Bedient, P.E.; *Elementary Differential Equations, 5th Edition*, MacMillan, 1974.

(A-18) Rubenstein, Moshe F.; *Patterns in Problem Solving* (UCLA Notes) 1973 (2) Somewhat engineering oriented. Covers modeling, probability, decision making and statistics. Computing is discussed from a basic hardware point of view with a brief mention of FØRTRAN.

(A-19) Saaty, T.L.; *Topics in Behavioral Mathematics* (Math. Assoc. of America) 1973. Covers theory of model building, optimization problems, disarmament and other conflicts, scheduling problems, inventory control, epidemics, linear programming, game theory, dynamic programming, network theory, probability and decision theory.

(A-20) Selby, Henry A.; *Notes of Lectures on Mathematics in the Behavioral Sciences* (Math. Assoc. of America) 1973. Companion book to Saaty book (see above). Lectures on economics (H.E. Scarf), political science (L.S. Shapley), anthropology (R.G. D'Andrade), sociology (P.F. Lazarsfeld), psychology (R.D. Luce) and measurement (W.S. Torgerson).

(A-21) Singleton, R.R. & Tyndall, W.F., *Games and Programs*, Freeman, 1974.