DOCUMENT RESUME

ED 155 024 SE 024 279

TITLE Mathematics for Junior High School, Pilot Edition, Chapters 1-7.
INSTITUTION Boston Univ., Mass.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 75
GRANT NSF-SED-74-18105
NOTE 212p.; For related documents, see SE 024 280-283

EDRS PRICE MF-$0.83 HC-$11.37 Plus Postage.
DESCRIPTORS Activity Units; *Curriculum; Fractions; Geometric Concepts; *Instructional Materials; *Junior High Schools; Map Skills; *Mathematics Materials; Measurement; Number Concepts; Problem Sets; Ratios (Mathematics); *Secondary School Mathematics; *Textbooks; Whole Numbers
IDENTIFIERS Angles; Area; *Boston University Mathematics Project

ABSTRACT

This book contains the first seven chapters of a pilot mathematics sequence for the seventh and eighth grades. The content of the sequence is to serve as a vehicle for the development of relevant computational skills, mathematical reasoning, and geometric perception in three dimensions and is to reflect the application of mathematics to the social and natural sciences. The material is divided into five types of sections: (1) activities by the whole class, small groups, or individuals; (2) short reading sections; (3) questions; (4) sections for the student with a weaker background; and (5) sections for the strongly motivated student. The material in the first seven chapters includes: simplified maps, length--whole numbers and fractions, angles and their measurement, enlarging and reducing, similar figures, reading maps, quotients and ratios, and area. (MN)

MATHEMATICS

FOR JUNIOR HIGH SCHOOL

pilot edition ——— chapters 1 - 7

Boston University Mathematics Project

PREFACE

This book contains the first seven chapters of a mathematics sequence for the seventh and eighth grades. The content for the sequence is being selected to serve as a vehicle for the development of relevant computational skills, mathematical reasoning, and geometric perception in three dimensions. The application of mathematics to the social and natural sciences is also an important factor in the selection of material.

The style of the sequence encourages individual as well as group work, thus developing the communication skills in the context of mathematics. Strong emphasis is placed on student activities, many of which are manipulative.

To serve a broad spectrum of students in heterogeneous classes, the material is divided into five types of sections. Three types constitute the main core:

Activities by the whole class, small groups or individuals;

Short reading sections, to be assigned and discussed or to be read in class; and

Questions to be worked out at home or in class.

Sections indicated by H are intended to help the student with a weaker background, and sections indicated by e are to provide extra challenge and pleasure for the strongly motivated student.

The development of this project is supported by a grant from the National Science Foundation.

CONTENTS

1. SIMPLIFIED MAPS

SECTION 1. SKETCHING A BIKE TRIP

A boy who was spending his vacation on a small island wrote a letter to his sister. He wanted to tell her about a bike trip he had taken. Here is part of the letter.

"The town where we are staying is in the center of the island. There is one road from town that leads to a deserted village on the shore of the island. Yesterday we decided to go there on our bikes. On the way out of town we came to a fork in the road by a tall tree. We went to the right. And, as you can guess, the road to the village was the one on the left. Anyway, we were lost.

"After a while we came to an intersection. We were not sure if we should keep going straight ahead, turn left, or turn right. We decided to turn right, and soon came to a forest. (Later we found out that the road to the left leads to a nice beach. Going straight ahead would have taken us to some cliffs overlooking the ocean.)

"Soon we came to another fork in the road and decided to go straight. (We should have turned right. That would have taken us back to town!) We ended up at the boat pier on the other side of the island. By then we were tired, so we wanted to go back to town. We knew there was a direct road from the boat pier to town, but we missed it. Instead we ended up going along the shore of the island all the way to the deserted village. So we visited the village after all, but we sure went the long way.

"What I don't understand is, how we got to the village from the pier and never crossed any other roads. After all, the village and the pier are almost on opposite sides of the island."

His sister, reading the letter, couldn't make head or tail out of the description of her brother's trip. So, she decided to draw a sketch of the island, its roads, and its landmarks.

Can you draw a sketch of the island and the trip using the description given in the letter?

Compare your sketch with your classmates' sketches. In what ways are they alike? Different?

SECTION 2 COMPARING TWO SKETCHES OF THE SAME THING

You probably found that in some ways your map of the island was the same as your classmates', but in other ways it was different. All the sketches, however, were based on the same description. It is possible that all the sketches were correct illustrations of the bike trip.

We will now try to find out, by comparing two sketches, just how different they can be and yet describe the same thing. Figure 1 and Figure 2 on page 5 are maps of an island drawn by two different students. The island here is not the one you drew, but these sketches were made from the same kind of information. Let's use both figures to answer the following questions.

1. (a) Look at Figure 1. Going along Lighthouse Lane, which is closer to the first-aid station, the trees or the flag?

   (b) Now look at Figure 2. Do you get the same answer?

   (c) Which sketch do you think is correct? Why?

   (d) Do you think that either of the two sketches gives the distances between places accurately?

2. Compare the corner of South Street and Airport Road on the two maps.

   (a) How are they different?

   (b) Do you think that either sketch tells you how sharply you must turn, going from one street onto the other?

3. (a) How many roads come together at the flagpole?

(b) Is the number the same for both sketches?

(c) Is the number of roads at each intersection the same on both maps?

(d) Does it make any difference which sketch you use to find the number of roads at an intersection?

4. (a) In Figure 1 the lighthouse is between which two intersections?

(b) Is it between the same intersections in Figure 2?

(c) How many streets come into Airport Road between the shark lookout and the boat dock?

(d) Does it matter which map you use to answer these questions?

5. (a) Use Figure 1 to describe Airport Road.

(b) Now describe the same road using Figure 2.

(c) What is the biggest difference between the two drawings of Airport Road?

(d) Do you think that the sharpness of curves has been correctly shown on either sketch?

6. Which of the following are the same or different for the two sketches?

(a) Distance between any two points.

(b) Number of roads coming together at any one intersection.

(c) Angle that two roads make when they meet.

(d) Order of landmarks along a road between any two places.

(e) Sharpness of curves on any road.

7. (a) Using Figure 1, write down instructions on how to get from the airport to the lighthouse.

(b) If someone were to use your instructions with Figure 2, would they get to the right place?

8. By how many different ways can you get from the lighthouse to the dock?

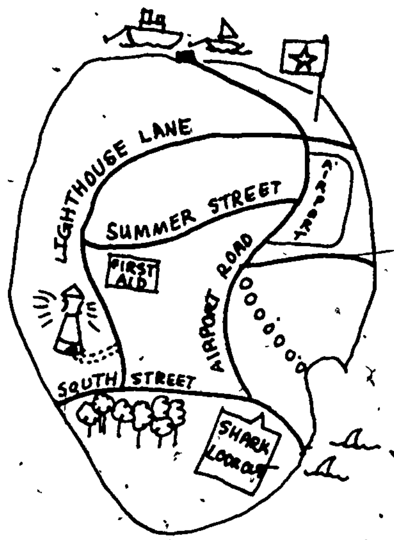


Figure 1

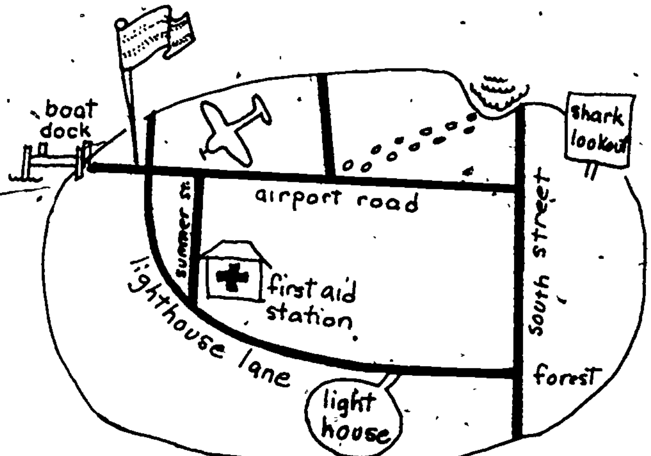


Figure 2

SECTION 3 HOW MUCH CAN YOU CHANGE A MAP AND STILL HAVE IT BE USEFUL?

In the previous section we compared two sketches of the roads on an island. We found that when we look at either of the sketches, we cannot tell accurately the distances, angles between roads at intersections, or sharpness of curves but we can find out:

(a) The number of roads at each intersection.

(b) The order of points along lines.

(c) The number of routes connecting two places.

Often maps or diagrams ignore correct distances, angles, and straightness of roads or routes, but carefully show intersections, connections, and the order of points along a route. We will call such maps or diagrams <u>network</u> maps or <u>network</u> diagrams.

Figure 3 shows such a network diagram for some of the routes of an international airline.

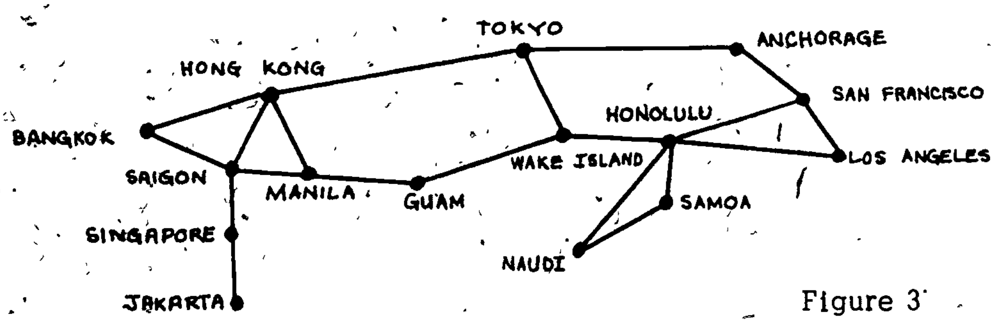


Figure 3

Neither distance, angle, nor the straightness of lines needs to be correct in such a diagram. Therefore, all these things can be changed in any way we wish without changing the important information given by the diagram. Someone could even redraw the network to make it look like an aircraft as shown in Figure 4. This might be done as part of an airline's advertising campaign.

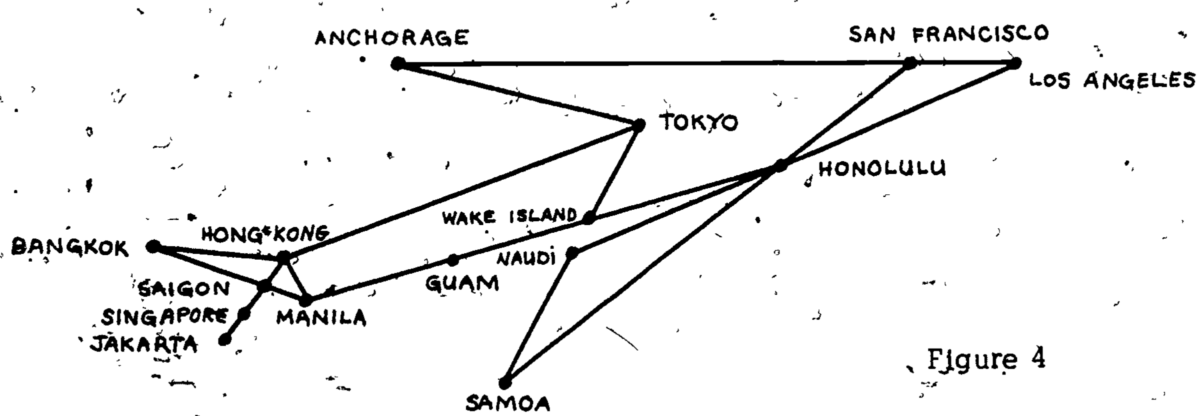


Figure 4

Both diagrams provide the same information to a traveler. Either one can be used for the purpose of choosing a route from Los Angeles to Saigon and back.

9. Are the two diagrams in Figure 5 diagrams of the same airline routes, or of different ones? Give reasons for your answer.

Figure 5(a)

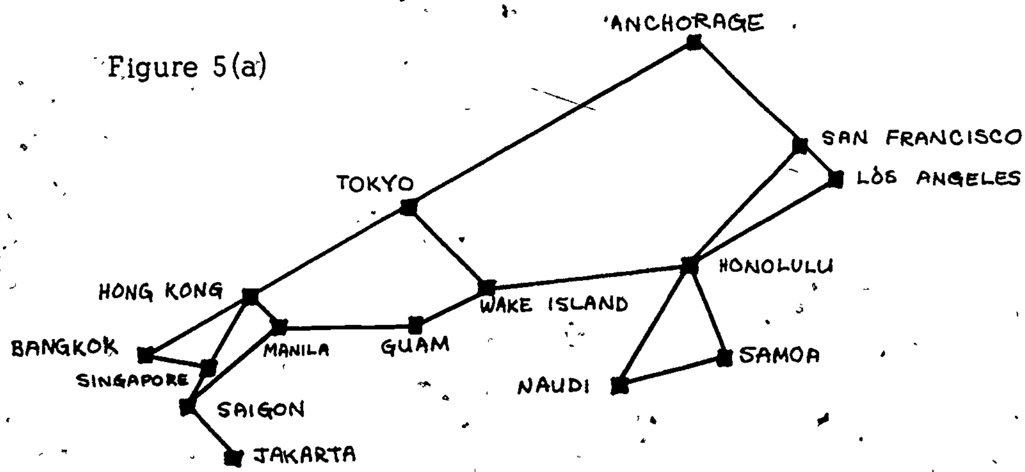
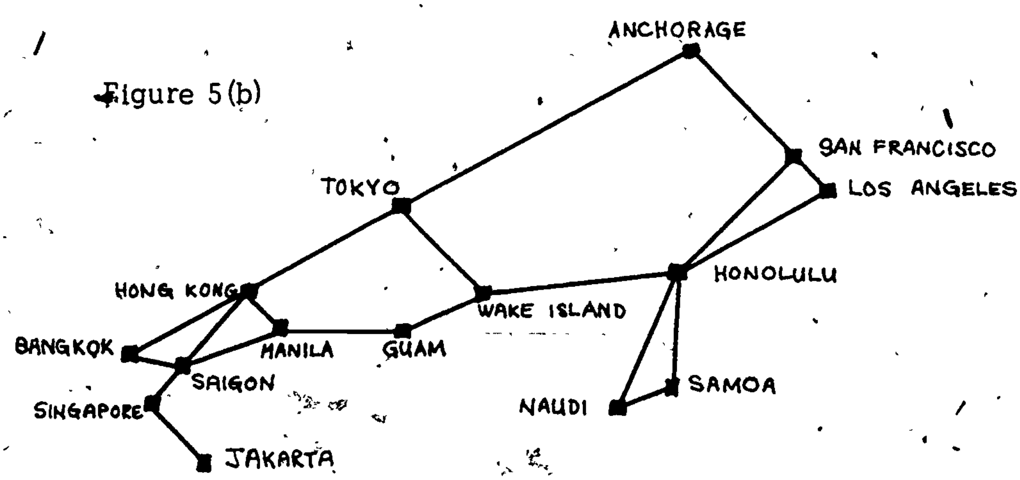


Figure 5(b)



10. Are the airline routes shown in Figure 5(a) the same as those shown in Figures 3 and 4?

11. In Figure 6, find intersections like those in (a), (b), and (c) on the road diagram in (d).

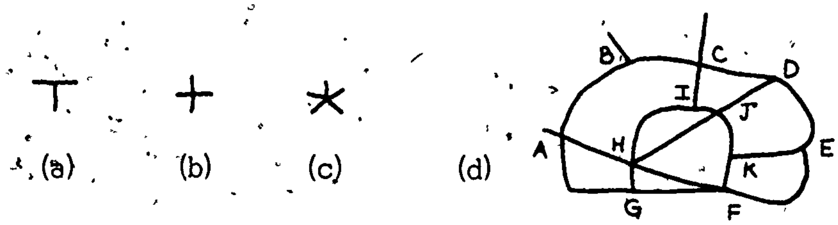
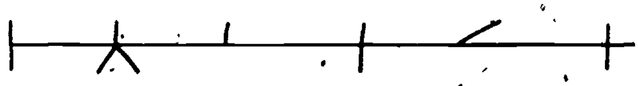


Figure 6

12. Find the path shown in Figure 7 on the road diagram of Figure 6(d) and label the intersections.

Figure 7



13. Figure 8 is a highly simplified version of Figure 9. Copy Figure 8 and label the cities on your copy.

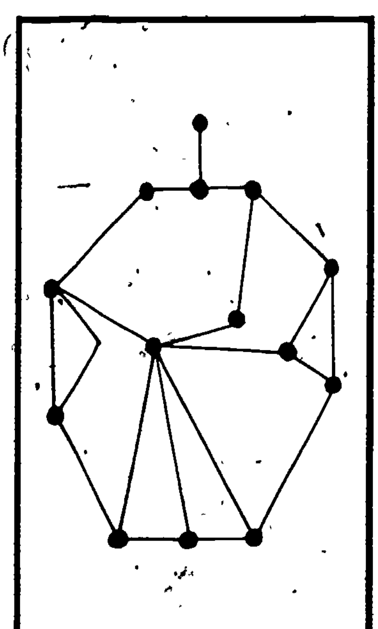


Figure 8

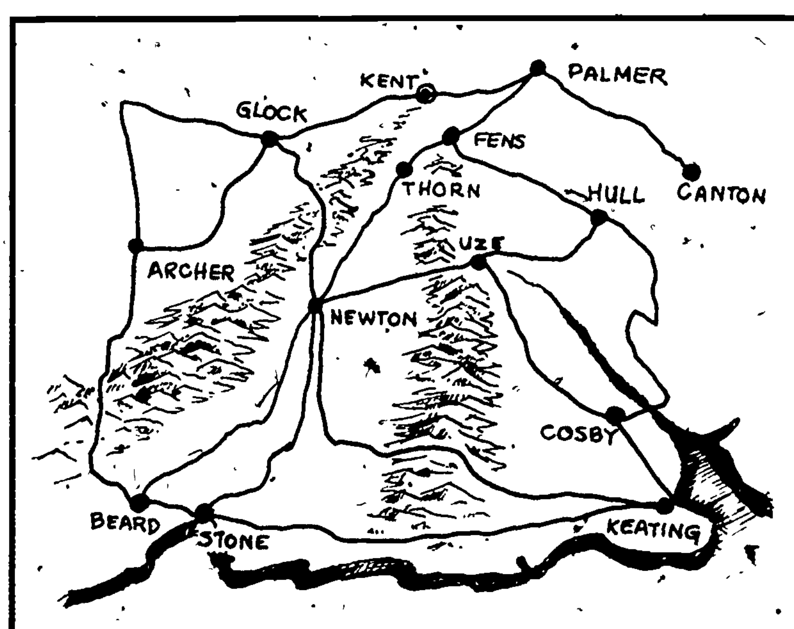


Figure 9

14. Simplify the map in Figure 10 by drawing it so that all roads are straight.

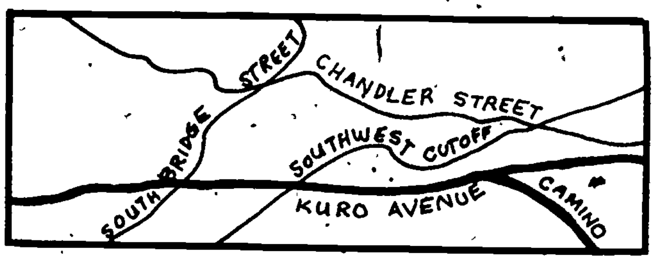


Figure 10

15. Figure 11 is a diagram of roads in a neighborhood. Sketch a simplified diagram of the same neighborhood by making each street appear as a single straight line.

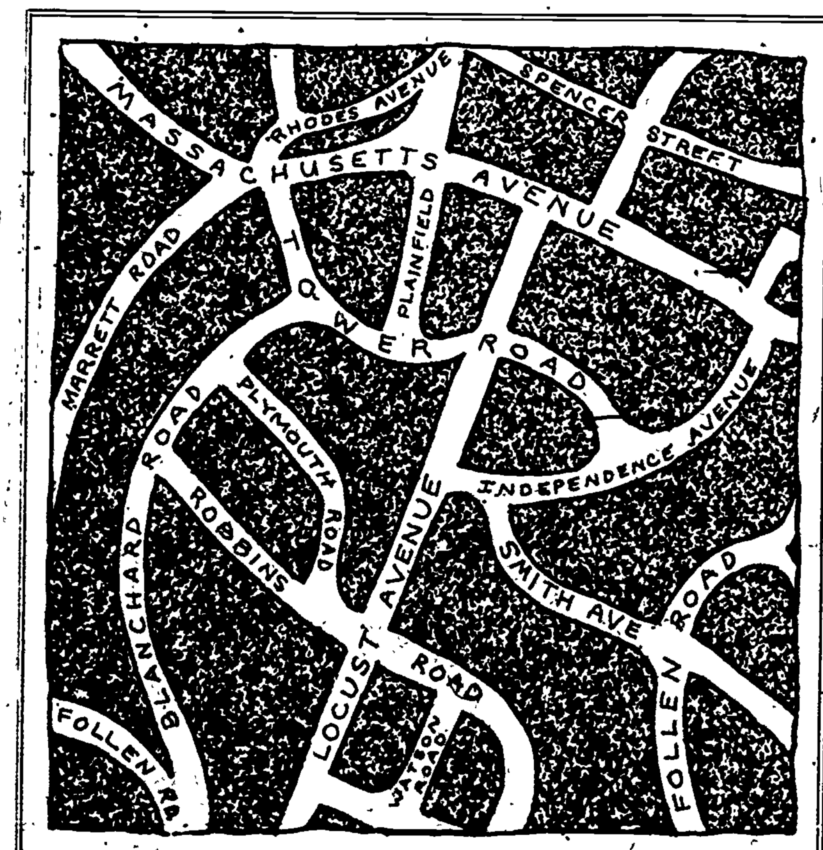


Figure 11

SECTION 4 WHAT DOES A NETWORK MAP TELL US?

We have seen that network maps are often used to show airline routes. Another common use of such maps is to show bus or subway routes. A network map of a subway system can supply much useful information. However, there are some questions that such a map cannot answer.

Use the Boston Rapid Transit map shown in Figure 12 to find the answers to as many of the following questions as you can. Decide what you would need to know in order to be able to answer the others. The stations mentioned in the following questions are underlined on the map.

16. (a) If you got on at Quincy Center and traveled to Airport, how many stations would you pass through?

    (b) How many different routes could you take from Quincy Center to Haymarket?

    (c) Which stations would you pass through just before and just after you pass through Maverick?

    (d) Do you need to change trains in order to go from Quincy Center to Symphony? If so, where?

    (e) Which two stations are nearest to each other?

    (f) Which is faster, to walk from Arborway to Forest Hills or to take the rapid transit?

    (g) Which is farther from Forest Hills, Essex or Dover?

17. (a) Write down instructions on how to go from Quincy Center to Airport.

    (b) Write down instructions on how to go from Quincy Center to Charles, using four subway lines.

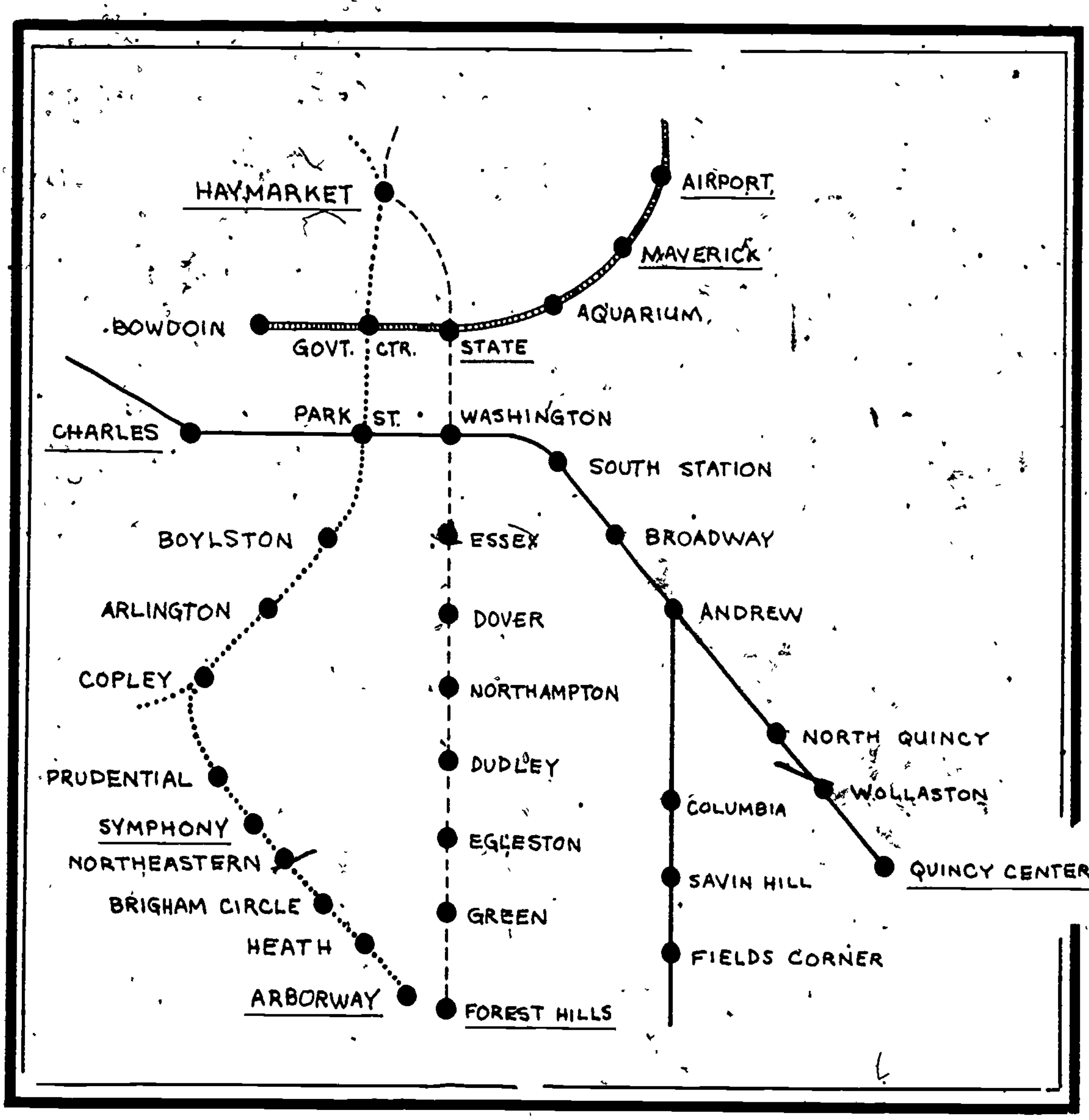


Figure 12

Figure 12 A network map of part of the rapid transit system of Boston, Mass. Many stops that are next to each other along a line seem to be the same distance apart. In fact, most of them are not. They are shown the same distance apart to make the map simple, small, and easy to read. Also, the different transit lines are far from straight.

18. Which of the maps of Figure 13 show the same connections between different points?

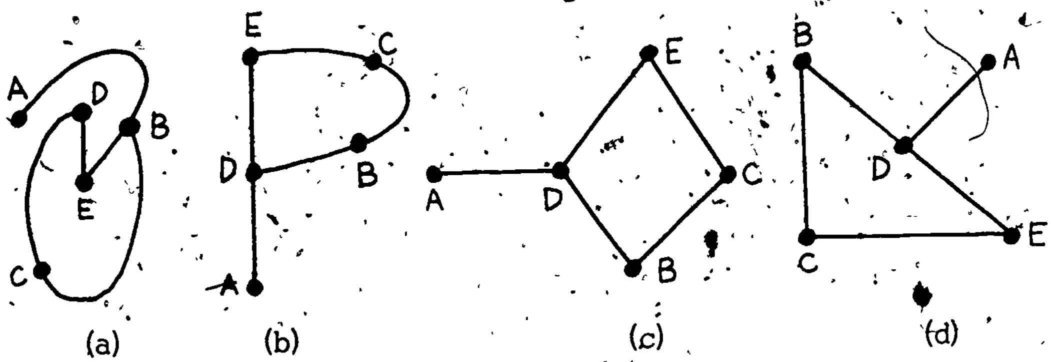


Figure 13

19. Figure 14 represents a diagram of a small network of telephone lines.

(a) The connection from A to D is broken. A cannot talk to B. If there is one additional break in the system, where must it be?

(b) If the connections from A to D and from B to C are broken, can A talk to B?

(c) There is a break in the system somewhere. In order to find it D disconnects his lines to B, C, and E. A cannot talk to B. D then proceeds to reconnect his lines to B, C, and E and disconnects his line to A. Now A can talk to B. Where is the break?

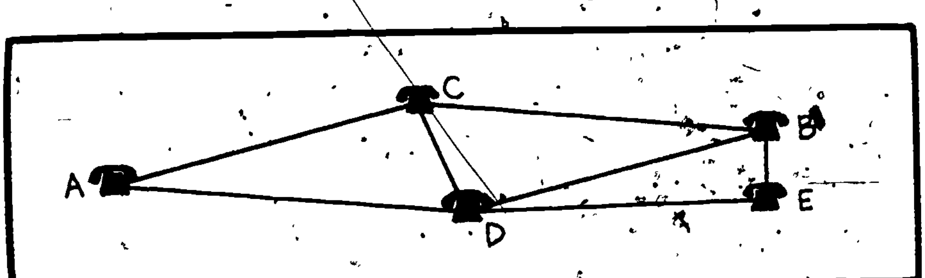


Figure 14

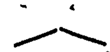
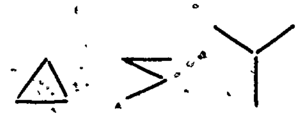
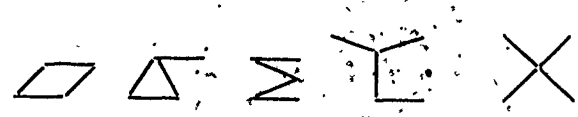
20. Imagine that the letters of the alphabet in Figure 15 are network diagrams. The letters Y and T then represent the same network. How many different networks are represented by the alphabet?

A B C D E F G H I J K L M

N O P Q R S T U V W X Y Z

Figure 15

21. How many different network patterns can you make with five toothpicks? They may not be bent, they must not overlap, and they may touch only at the ends. The table below has been filled out for less than five toothpicks.

Number of Toothpicks	Number of Different Network Patterns	
1	1	
2	1	
3	3	
4	5	

Note: Be careful in doing this problem. The patterns $\wedge$ and — mean the same thing, since only the angle between the toothpicks is changed.

2. LENGTH — WHOLE NUMBERS AND FRACTIONS

In Chapter 1 you learned about simplified maps or diagrams. These maps did not show true distances or directions, but you could use one perfectly well to find your way from one place to another by road or subway. For many purposes, however, you need to know both directions and distances.

In this chapter and the next you will learn about measuring distances and angles. Then you will be able to find distances between places on regular maps.

SECTION 1 UNITS OF MEASUREMENT

?

1. How do you tell people how heavy you are?

2. How do you tell people how tall you are?

3. How do you tell people how old you are?

4. How would you go about measuring how many cups of liquid a small pail will hold?

5. Suppose you wish to measure a table to see if it will go through a doorway before you try moving it. You don't have a ruler or a measuring tape, so you decide to use a pencil as a measuring rod. Tell how you would decide if the table will fit.

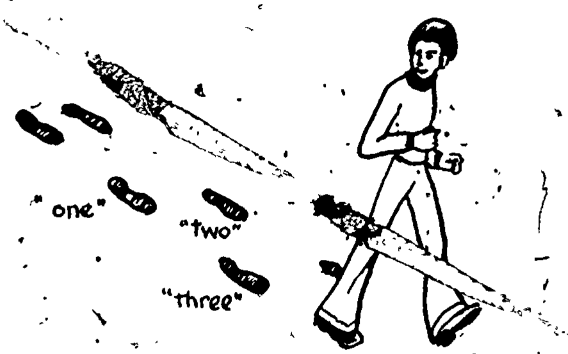
If you wish to know how much money you have, you count the number of pennies, nickels, dimes, and quarters in your pocket or purse, and add up their value in dollars and cents.

So in any other case where you answer the question "how much," you count units. To find the distance between two trees you may count the number of steps it takes to go from one to the other. Sometimes the counting is done for you and you read the result on a scale.

A meter is a unit of length in the metric system of units. This system of measurements is used in most countries of the world. In this book we will measure lengths in meters and other metric units instead of in yards, feet, and inches.

Look at a meter stick. Do you think a meter is longer or shorter than one of your normal steps? Find two or three people in your class who can make their step be close to a meter, and have them pace off the width of the classroom. What result do they get, to the nearest meter?

Now have two people in the class measure, with a meter stick, the width of the classroom to the nearest meter. Does this result agree with the ones found by pacing?

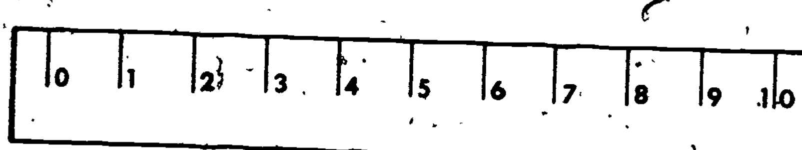


SECTION 2 CENTIMETERS

Meters are useful for measuring long distances, but we need a smaller unit of length for short distances. In the metric system the most common unit of length, next to the meter, is a <u>centimeter</u>. One hundred centimeters equals one meter. Thus if we divide a meter into one hundred equal parts, we will have divided it into centimeters. If you look at a meter stick you will see that it is divided into centimeters.

Look at a pair of centimeter markers that are next to each other. The distance between them is a little greater than the width of a pencil. A centimeter gets its name because <u>centi</u> means "one hundredth of," just as a <u>cent</u> is one hundredth of a dollar. Figure 1 is a life-size picture of a centimeter scale going from 0 to 10 centimeters.

Figure 1



When you use a meter stick to measure short lengths, you do not have to count centimeters, because the centimeter marks are numbered right on the meter stick. These numbers do the counting for you up to 100 centimeters. The abbreviation for centimeter is cm, so we can write "100 cm" instead of "100 centimeters."

6. Why doesn't a meter stick start with the 1 cm mark on the left-hand edge?

7. Describe how you would use a meter stick to measure, in centimeters, a distance that is greater than 1 m. (The abbreviation for meter is m.)

SECTION 3 ESTIMATING LENGTHS IN CENTIMETERS

Here is an estimation contest that will help you to think of the length of objects in terms of centimeters. Separate into about six teams of equal size. Your teacher will pick out several objects to measure. Everyone writes down an estimate of the object's length. Two students from different teams then measure the object to the nearest centimeter, using a centimeter scale.

Each student finds how many centimeters too large or too small his or her estimate is.

Each team adds up its members' errors for its score for that object. On the board, keep score for the game. The team with the lowest total wins.

SECTION 4 HOW TO SUBTRACT

H Finding your group's score in Section 3 calls for subtraction. There are several ways to subtract. One way to subtract involves "borrowing." Here is another way that you may find easier.

Sometimes the larger digits in a subtraction are on top as in 87 − 53:

$$\begin{array}{r} 87 \\ -\underline{53} \\ 34 \end{array}$$

Trouble appears when larger digits are beneath smaller digits as in

$$\begin{array}{r} 572 \\ -\underline{\ 38} \end{array}$$

Here you cannot subtract the bottom digit from the top. To get around this we add 10 to both the top and the bottom numbers. We can do this because the difference between the two new numbers will still remain the same. On top we add 10 by making the 2 in the ones place into a 12. On the bottom, we add 10 by changing the 3 in the tens place to 4. It looks like this:

$$\begin{array}{r} 12 \\ 57\cancel{2} \\ 4 \\ -\underline{\ \cancel{3}8} \end{array}$$

Now we can subtract:

$$\begin{array}{r} 12 \\ 57\cancel{2} \\ 4 \\ -\underline{\ \cancel{3}8} \\ 534 \end{array}$$

To check our answer, we just add to see that 38 + 584 = 572. Here is another example, 1738 − 487:

$$\begin{array}{r} 1738 \\ -\ 487 \\ \hline \ldots 1 \end{array}$$

We can subtract 7 from 8, but not 8 from 3. So we place a 1 in front of the 3 making it 13, and increase the 4 in the hundreds place to 5 like this:

$$\begin{array}{rcccc} & & & 13 & \\ & 1 & 7 & 3 & 8 \\ & & 5 & & \\ - & & \cancel{4} & 8 & 7 \\ \hline & 1 & 2 & 5 & 1 \end{array}$$

We can check the answer by adding. In this case, adding 487 + 1251 we get:

$$\begin{array}{r} 487 \\ 1251 \\ \hline 1738 \end{array}$$

8. You give a salesperson 75¢ for a 59¢ item, how much change should you get?

9. In each of the following cases, how far off is your estimate?

(a) Suppose your estimate of a length is 87 cm, and the measured length is 123 cm.

(b) Your estimate is 85 cm, and the measured length is 63 cm.

(c) Your estimate is 39 cm, and the measured length is 57 cm.

(d) You estimate from a map of the United States that you would have to drive 210 miles to get from Charleston, West Virginia, to Richmond, Virginia. Actually, it is a 196-mile drive.

SECTION 5. DOES IT FIT?

Look at the drawing on the sheet of paper your teacher gives you. It doesn't look like a trail, but we will call it that. As you go from start to finish, the trail becomes narrower and narrower. Your teacher has the same trail cut in a board and sliders 1 cm, 2cm, etc., up to 10 cm long. Each slider can easily move along the straight sections of the trail until finally it gets stuck in a corner too narrow for it to get around (see Figure 2).

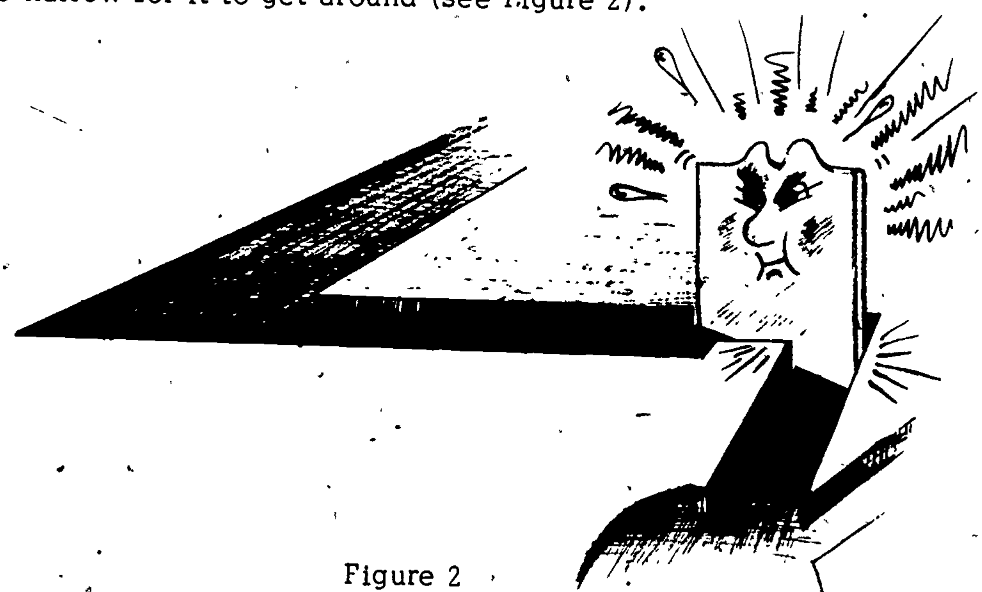


Figure 2

Using a centimeter scale, make measurements on your copy of the trail to decide where you think each slider will get stuck. Write on your sheet the number of the slider at its sticking point. The sliders are so thin that you don't have to worry about their thickness.

Check your predictions by trying the real sliders on the real trail.

SECTION 6 DIVIDING A LINE SEGMENT INTO EQUAL PARTS

You will recall that we defined the centimeter by dividing a meter into 100 equal parts. You may wonder how a given length, such as a meter, can be divided into equal parts. Here is how.

Take an index card and cut it into strips a centimeter wide. Using a sheet of ruled paper (notebook paper will do just fine) divide one of the strips into two parts as shown in Figure 3. Are the two parts equal?

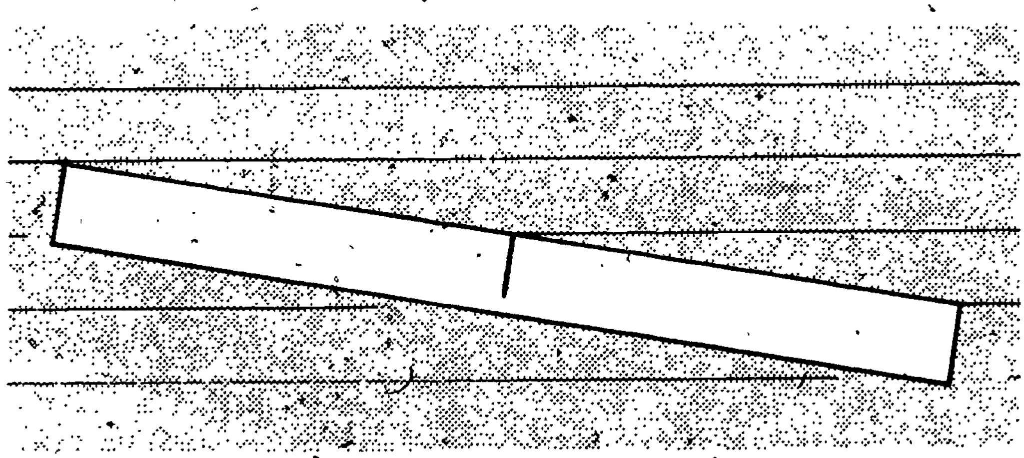
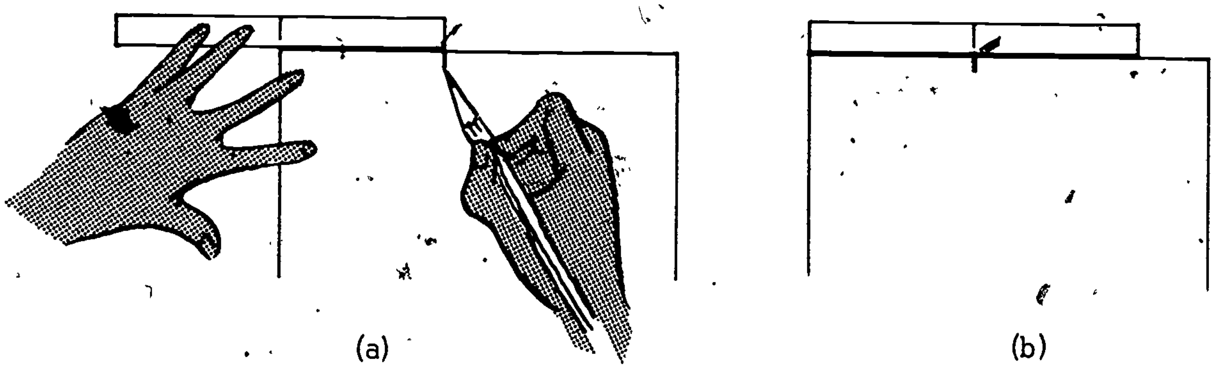


Figure 3

You could check to see if the two parts are equal by measuring them with a centimeter ruler. However, we do not care how long each part is in units of centimeters, or inches, or anything else. All we wish to know is whether the two parts are equal, so you can use a "ruler" without any markings. Figure 4 shows how a sheet of paper can be used to check if the two parts are equal.


(a)

(b)

Figure 4

First mark off the length of one of the parts of the strip on a sheet of paper as shown in (a). Then compare the marked-off length with the other part as shown in (b). Are the two parts equal in length?

?

10. How must the strip be placed on the ruled paper to divide it into three equal pieces?

11. (a) Divide a strip into three pieces, using the same ruled paper you used to divide a strip into two pieces.
    (b) Are the three pieces equal?

12. How many lines would you need to divide a strip into
    (a) four pieces?
    (b) five pieces?
    (c) ten pieces?
    (d) one hundred pieces?

Divide a strip into three pieces, using only every other line on the ruled paper.

Are the three pieces equal? Are they equal to the pieces you made in Question 11? Does it matter how far apart the lines are on the ruled paper?

Now divide a strip into two pieces, using lines next to each other on the ruled paper. Divide the same strip using every fourth line of the ruled paper. Do you get the same result in both cases?

By folding the strips on the divisions you can find out which of the two divisions is more accurate. Why do you think one way is more accurate than the other?

SECTION 7 NAMING SUBDIVISIONS OF A UNIT

Using the parallel lines on a sheet of paper, divide 10 strips as follows:

(a) Divide each of two strips into two equal parts.

(b) Divide the remaining strips into three, four, five, six, seven, eight, nine, and ten equal parts.

Consider one of the strips that you divided into two equal parts. We consider the whole strip to be one unit long. Because each part is one of two equal parts, we can write its size as $\frac{1}{2}$, meaning $\frac{1}{2}$ of the unit length.

In the same way, for the strip divided into three equal parts, each part is $\frac{1}{3}$ and so on with the remaining strips.

The size of the smallest piece when a unit is divided into any number of pieces is called a <u>unit fraction</u>. Therefore $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{10}$ are examples of unit fractions since they show the sizes of the smallest pieces when a unit is divided into two, three, four, etc, pieces.

Since it takes two parts of size $\frac{1}{2}$ to obtain a whole strip, we can write

$$\frac{1}{2} + \frac{1}{2} = 1 \quad \text{and} \quad 2 \times \frac{1}{2} = 1$$

where the $\times$ means "times" or "multiply." So 2×3 means "2 times 3," or "3 multiplied by 2."

[?]

13. What number would you put in each box of the following equations to make the equations correct?

(a) $\square \times \frac{1}{3} = 1$

(b) $5 \times \square = 1$

(c) $10 \times \frac{1}{10} = \square$

(d) $\square \times \frac{1}{100} = 1$

14. Take the two strips you divided into two parts each and place them end to end. Complete the following equations by looking at your strips.

(a) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \square$

(b) $\square \times \frac{1}{2} = 2$

15. Without making any additional strips, complete the following equations.

(a) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \square \times \frac{1}{2}$

(b) $\square \times \frac{1}{2} = 4$

(c) $16 \times \frac{1}{2} = \square$

(d) $\square \times \frac{1}{5} = 10$

(e) $100 \times \frac{1}{10} = \square$

16. How many strips would you have to divide into equal parts to check your answer to part (c) of Question 15?

17. By placing the two-piece and three-piece strips together as shown in Figure 5, decide which is the greater, $\frac{1}{2}$ or $\frac{1}{3}$.

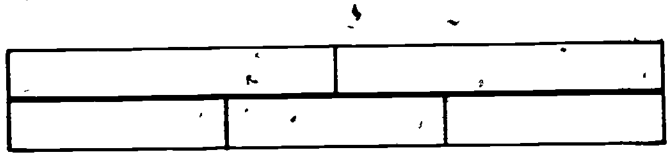


Figure 5

18. Using other strips arranged as in Figure 5, decide which fraction in the following pairs of unit fractions is the greater of the two.

(a) $\frac{1}{3}, \frac{1}{5}$

(b) $\frac{1}{10}, \frac{1}{5}$

19. (a) Arrange the following unit fractions from the smallest to the largest: $\frac{1}{2}, \frac{1}{5}, \frac{1}{3}, \frac{1}{4}, \frac{1}{10}$.

(b) When you increase the number of equal parts into which a unit is divided, what happens to the size of each part?

SECTION 8 MULTIPLES OF UNIT FRACTIONS

How much is $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$? Since $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 5 \times \frac{1}{2}$ we can answer the question by finding out how much $5 \times \frac{1}{2}$ is. We know that the number $4 \times \frac{1}{2}$ is less than $5 \times \frac{1}{2}$ and $5 \times \frac{1}{2}$ is less than $6 \times \frac{1}{2}$.

The symbol "<" is used to stand for "less than." Thus $7 < 8$ is read "seven is less than eight" and $7 < 8 < 10$ is read "seven is less than eight and eight is less than 10."

Using the symbol < for "less than" we write

$$4 \times \frac{1}{2} < 5 \times \frac{1}{2} < 6 \times \frac{1}{2}$$

or, since $4 \times \frac{1}{2} = 2$ and $6 \times \frac{1}{2} = 3$, we can write

$$2 < 5 \times \frac{1}{2} < 3$$

Therefore $5 \times \frac{1}{2}$ is between 2 and 3 and we can say that $5 \times \frac{1}{2}$ is "bracketed" by 2 and 3.

Writing $5 \times \frac{1}{2}$ is one way of saying that we have five halves of a unit. An easier way is to write $\frac{5}{2}$. Thus, $2 \times \frac{1}{3} = \frac{2}{3}$, $4 \times \frac{1}{5} = \frac{4}{5}$, $8 \times \frac{1}{12} = \frac{8}{12}$, etc.

Expressions like $\frac{4}{5}$ or $\frac{8}{12}$ are called fractions. The top of a fraction is called the numerator and the bottom is called the denominator.

Note that every fraction is a multiple of some unit fraction. For example,

$$\frac{4}{5} = 4 \times \frac{1}{5}; \quad \frac{8}{12} = 8 \times \frac{1}{12}$$

A fraction like $\frac{5}{3}$ whose numerator is greater than its denominator stands for more than 1 whole unit.

We can write

$$\frac{5}{3} = \overbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}^{\frac{3}{3}} + \overbrace{\frac{1}{3} + \frac{1}{3}}^{\frac{2}{3}}$$

$$= \frac{3}{3} + \frac{2}{3}$$

$$= 1 + \frac{2}{3}$$

So we see that $\frac{5}{3}$ is $\frac{2}{3}$ more than 1. That is,

$$\frac{5}{3} = 1 + \frac{2}{3}$$

We usually write $1 + \frac{2}{3}$ as $1\frac{2}{3}$, omitting the plus sign. (Note that $1 \times \frac{2}{3} \neq 1\frac{2}{3}$.) Since $1\frac{2}{3}$ involves both a whole number and a fraction, we say that it is written in mixed notation.

20. Bracket the following products with whole numbers.

(a) $7 \times \frac{1}{3}$

(b) $11 \times \frac{1}{10}$

(c) $12 \times \frac{1}{5}$

(d) $3 \times \frac{1}{4}$

21. Bracket the following fractions in the same way as in Question 20.

(a) $\frac{5}{4}$

(b) $\frac{4}{5}$

(c) $\frac{11}{2}$

22. Write the following fractions in mixed notation.

(a) $\frac{5}{4}$

(b) $\frac{11}{2}$

(c) $\frac{23}{10}$

23. What fraction is represented by the following numbers?

(a) $1\frac{5}{10}$

(b) $3\frac{3}{4}$

(c) $2\frac{2}{3}$

24. (a) Which of the following fractions can you write in mixed notation? $\frac{3}{2}, \frac{1}{2}, \frac{2}{3}, \frac{3}{3}, \frac{8}{3}, \frac{6}{3}$

(b) Can you write a fraction in mixed notation if the numerator and the denominator of the fraction are equal? Can you write a fraction in mixed notation if the numerator is a multiple of the denominator?

(c) Note that the fractions $\frac{7}{3}, \frac{8}{3}, \frac{5}{2}$ are all greater than 1, while $\frac{1}{3}, \frac{2}{5}, \frac{7}{12}$ are all less than 1. How must the sizes of the numerator and the denominator of a fraction compare when the fraction can be written in mixed notation?

SECTION 9 EQUAL FRACTIONS

We can label the lines on our strips to show how large a part of the whole strip we have up to that line. For example, the first strip you made can be labeled as in Figure 6(a).

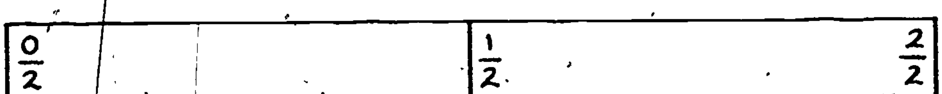


Figure 6(a)

Complete labeling the strip in Figure 6(b), and then label all your strips in this way.

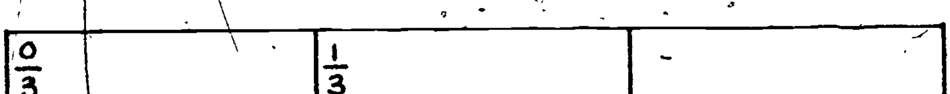


Figure 6(b)

If you put the two strips in Figure 7 together as shown, you can see that $\frac{0}{2} = \frac{0}{4}$; $\frac{1}{2} = \frac{2}{4}$; $\frac{2}{2} = \frac{4}{4}$.

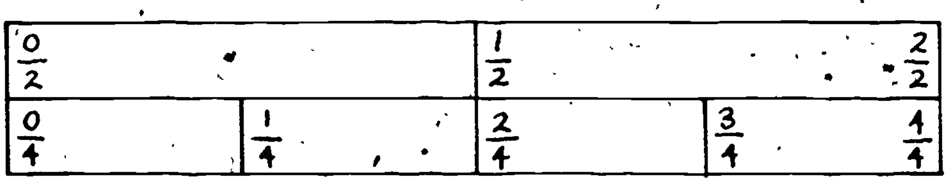


Figure 7

25. (a) Place the strip divided into fifths together with the strip divided into tenths. What equalities like the ones above do you notice?

(b) Using your strips as above, find all the fractions that are equal to $\frac{1}{3}$.

(c) On your strips, which fractions are equal to $\frac{1}{10}$? $\frac{2}{10}$?

26. Complete the following equations.

(a) $\frac{1}{3} = \frac{\square}{6} = \frac{\square}{9} = \frac{\square}{30}$

(b) $\frac{1}{5} = \frac{2}{\square} = \frac{3}{\square} = \frac{5}{\square}$

(c) If you double the numerator of a fraction, what must you do to the denominator of the same fraction in order to keep the value of the fraction unchanged?

27. Complete the following equations.

(a) $\frac{6}{12} = \frac{\square}{6} = \frac{\square}{4}$

(b) $\frac{6}{16} = \frac{3}{\square}$

(c) If you divide the numerator of a fraction by 3, what must you do to the denominator of the same fraction in order to keep the value of the fraction unchanged?

28. If the lengths of the two strips were not equal and one strip was divided into halves while the other strip was divided into fourths, would $\frac{1}{2}$ of the first strip equal $\frac{2}{4}$ of the second strip? Use two strips of unequal length to check your answer.

We can use fractions to divide things other than lengths. In music we need to have a way of showing how long a note is held. We use different symbols to represent different lengths of time.

𝅝 = whole note

𝅗𝅥 = half note

♩ = quarter note

♪ = eighth note

𝅘𝅥𝅯 = sixteenth note

As the names of the symbols suggest, each note represents half as much time as the previous one. We have, for example,

𝅗𝅥 + 𝅗𝅥 = 𝅝

Notes are placed on a staff (a set of five lines) and are marked off in measures (groups of notes separated by vertical lines) as in Figure 8.

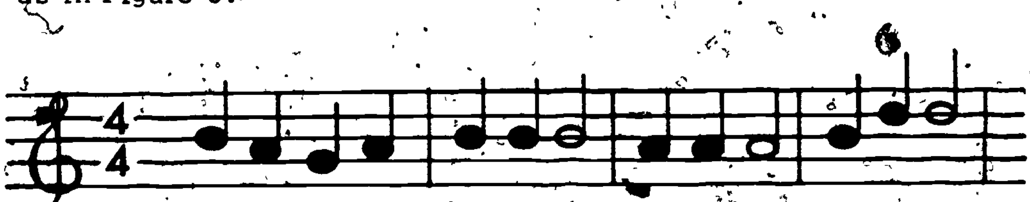


Figure 8

The symbol $\frac{4}{4}$ at the beginning of the staff is called the time signature and means that every measure must contain the equivalent of four quarter notes. Note that this is the same as saying that each unit (measure) is divided into four fourths.

The first measure shows

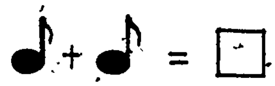
$$4 \times \frac{1}{4} = 1 \text{ measure}$$

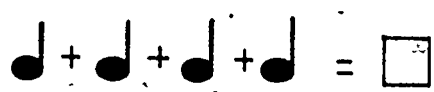
The second measure shows

$$2 \times \frac{1}{4} + 1 \times \frac{1}{2} = 1 \text{ measure}$$

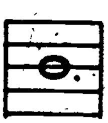
29. Complete the following equations, using a single note as the answer.

(a) 

(b) 

(c) 

30. Express each of the following measures as an equation giving the sum of numbers equal to 1.



(a)

(b)

(c)

(d)

31. Show a measure that represents

(a) $2 \times \frac{1}{2} = 1$

(b) $4 \times \frac{1}{4} = 1$

(c) $\frac{1}{2} + \frac{1}{2} = 1$

(d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

(e) $2 \times \frac{1}{4} + 4 \times \frac{1}{8} = 1$

SECTION 10 DECIMAL NOTATION

You have seen that fractions such as $\frac{14}{10}$ or $\frac{135}{100}$ can also be written in mixed notation as $1\frac{4}{10}$ and $1\frac{35}{100}$. Whenever we have fractions that involve tenths, hundredths, or thousandths, we can write a fraction yet another way. We simply do not write the denominator of the fraction. Instead we use a point (.), called a <u>decimal point</u>, to separate the fraction from the whole number.

Look at the following examples of decimal notation.

One and three-tenths $= 1\frac{3}{10} = 1.3$

Fifteen and one-hundredth $= 15\frac{1}{100} = 15.01$

Seven-tenths $= \frac{7}{10} = 0.7$

Five-thousandths $= \frac{5}{1000} = 0.005$

Note that the decimal point is a device to indicate the place of the ones. The number immediately to the left of the decimal gives you the number of ones and the tens place is one step to the left of the ones. Similarly, one step to the right of the ones gives you the number of tenths. The place of the hundredths is two steps to the right of the ones, etc.

638.156

hundreds
tens
ones
tenths
hundredths
thousandths

32. Write these fractions in words.

(a) 0.2

(b) 0.064

(c) 25.35

33. Write the following in decimal notation.

(a) $4\frac{7}{10}$

(b) $81\frac{1}{100}$

(c) $\frac{21}{100}$

34. Write the following as fractions or in mixed notation.

(a) 4.6

(b) 0.001

(c) 16.36

(d) 16.306

(e) 16.360

SECTION 11 MATCHING FRACTIONS

Spread the cards that your teacher gives you face down in a rectangular arrangement. Two to four players per deck can play this game. A player turns over one card and then another, reading aloud the fraction on each card. If they are equal, the player keeps them and plays again. If the fractions are not equal, the cards must be replaced, face down, and it is then the next player's turn. When there are no cards left on the table, the player who has the most cards wins the game.

SECTION 12 MEASURING TO TENTHS OF A CENTIMETER

The smallest divisions on a meter stick or centimeter ruler divide a centimeter into 10 equal parts. (See Figure 9.) Each part is, therefore, $\frac{1}{10}$ cm = 0.1 cm. Each 0.1 cm is about $\frac{1}{2}$ the thickness of a pencil lead.

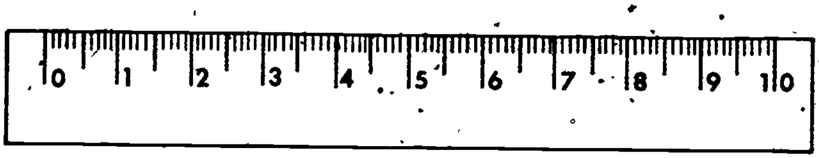


Figure 9

To become familiar with making measurements to $\frac{1}{10}$ cm you can try the "trail" again. This time you will need to measure the trail to 0.1 cm.

The first sliders you will use have their length marked on them but this time they are given to the nearest 0.1 cm. They are 2.4, 3.9, 4.5, 6.6, 7.6, and 9.4 cm long.

As before, mark the length of each slider on a copy of the trail at the turn where you think it will get stuck.

Check your prediction with the real sliders on the real trail.

Now repeat what you have just done with the sliders but use the blocks of different length and width shown in Figure 10.

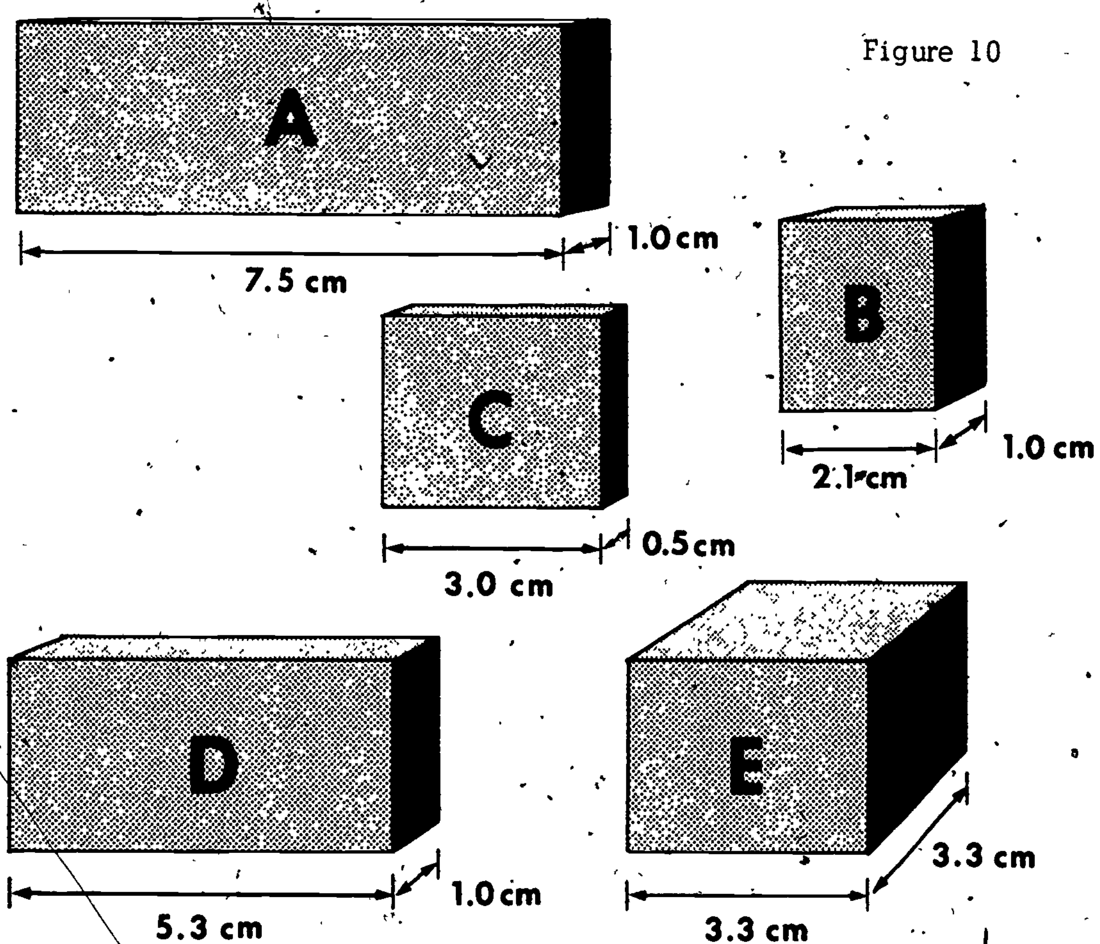


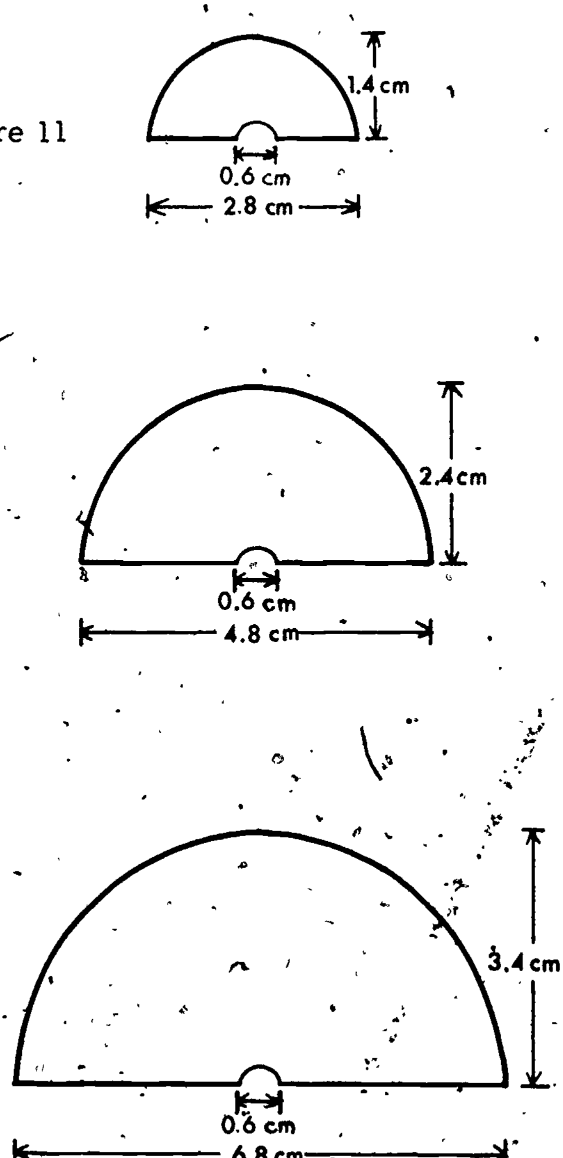
Figure 10

e

Make measurements of your copy of the trail to see where the large, middle-size, and small semicircles (Figure 11) will get stuck.

Check your predictions.

Figure 11



How wide can the block in Figure 12 be and still get through?

To find out cut out a rectangle of paper or cardboard 5.0 cm long and whose thickness is small enough so this slider paper can just get through the first turn. Try it in the real trail. Is it snug?

What is the greatest length you could have for the block in Figure 13 and still have it get through the first turn after the big circle? Cut out a piece of paper with your answer and try it.

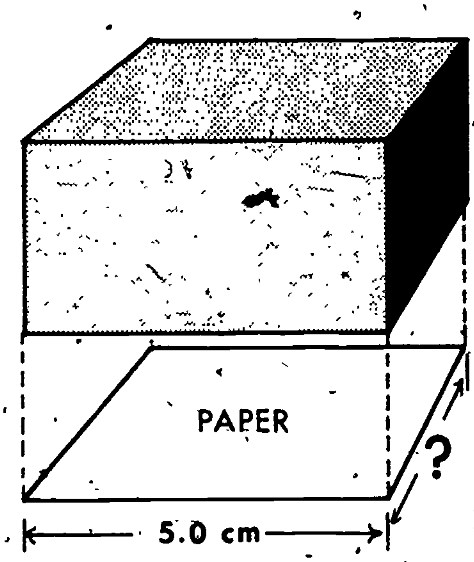


Figure 12

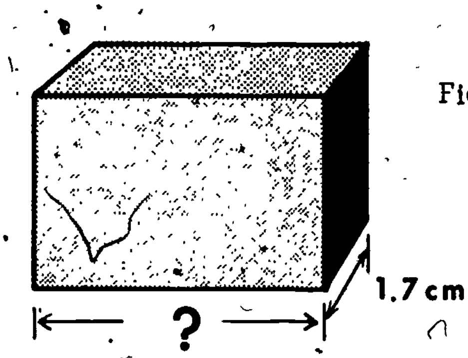


Figure 13

SECTION 13 MAKING PARALLEL LINES

To solve the puzzles in the next section you will have to draw parallel lines. Here is an easy way to make a line that is parallel to a given straight line. The five steps that follow are shown in Figure 14.

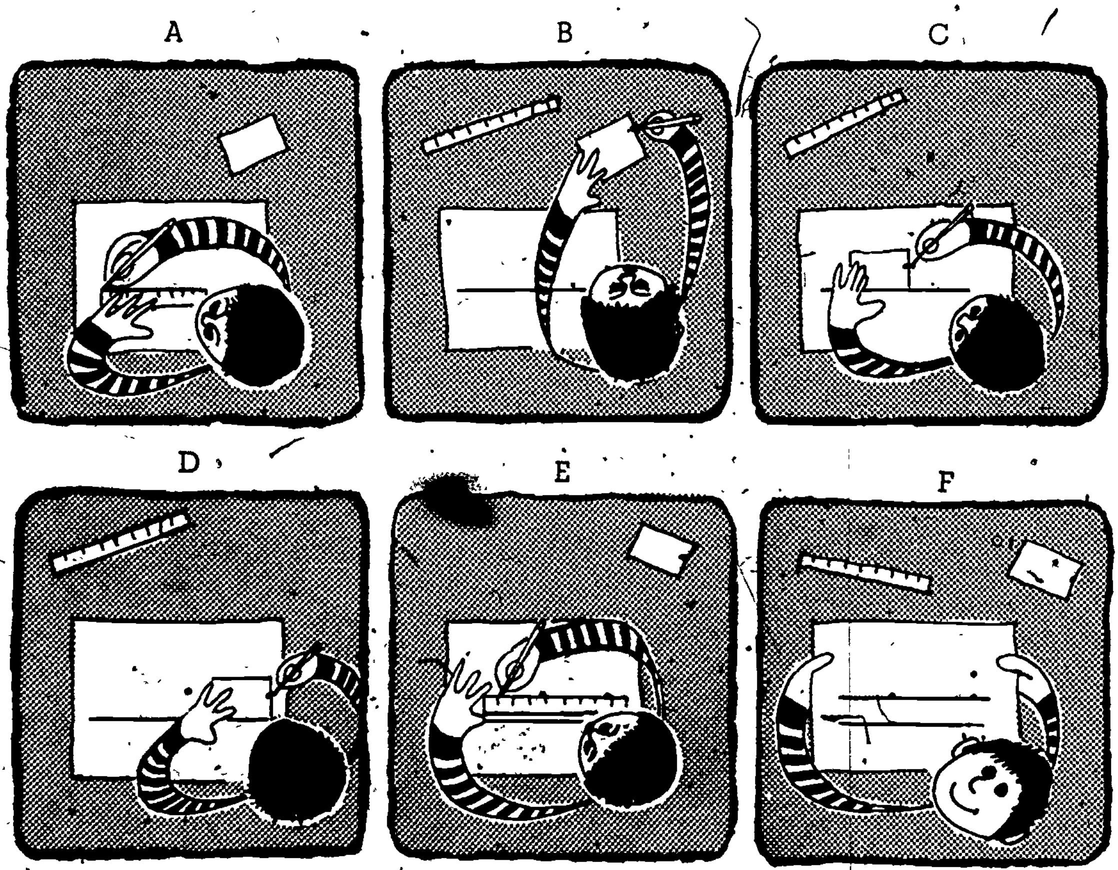


Figure 14

A. First, draw a straight line on a sheet of paper.

B. Next, place a mark somewhere along a short edge of an index card or another sheet of paper.

C. Now, place an edge of the index card on the line you drew on the sheet of paper and make a mark on the sheet of paper next to the mark on the index card.

D. Move the card along the straight line to the right and again make a mark on the paper next to the mark on the card.

E. Now, remove the card and draw a straight line through the two marks you made on the paper.

If two lines are parallel, they are the same distance apart everywhere. Are the two lines you have drawn on the sheet of paper the same distance apart everywhere? If so, ~~what is~~ this distance?

35. Draw two parallel lines that are 2.0 cm apart.

36. Draw two parallel lines that are 3.3 cm apart.

37. Have a classmate measure the distance between the parallel lines you drew in Questions 35 and 36.

38. (a) Draw a rectangle that is twice as long as it is wide.

(b) Fold your rectangle in half, end to end. What is the name of the figure you now have?

SECTION 14 SCRAMBLED SQUARES

Carefully measure and cut out pieces of paper with the dimensions shown in Figure 15. (All lengths are given in centimeters.) Can you fit the pieces together to make a square? Don't turn the pieces over, just use them as they are pictured.

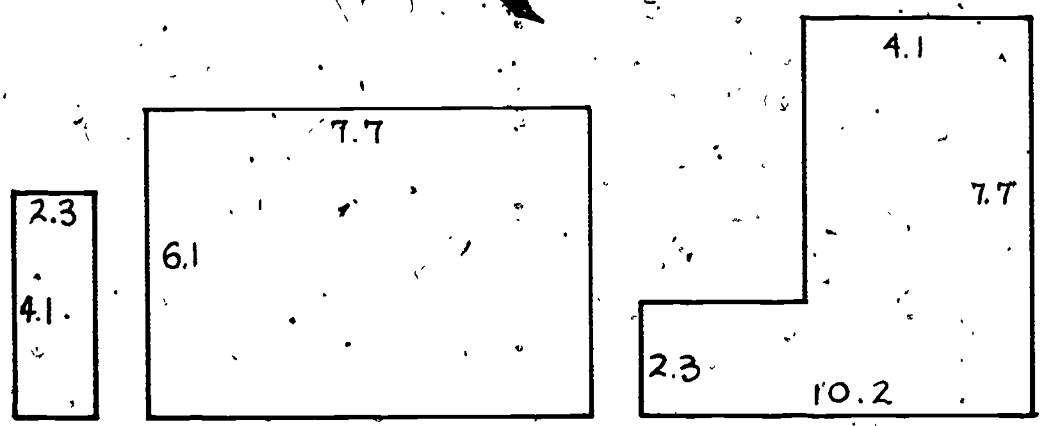


Figure 15

Can you make a square from the pieces in Figure 16(a)? From those in Figure 16(b)?

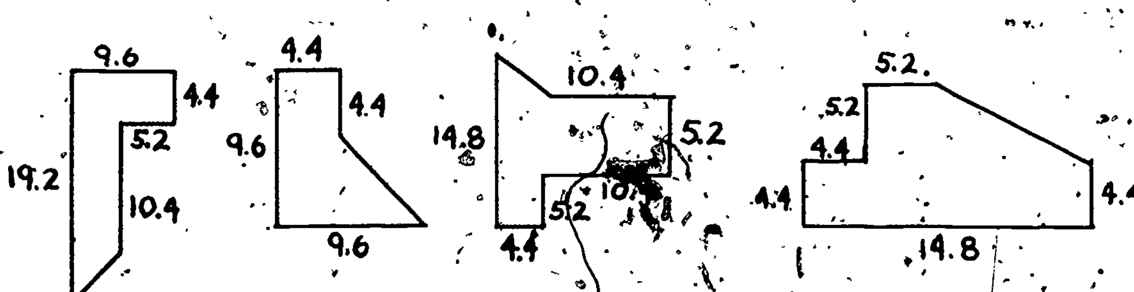


Figure 16(a)

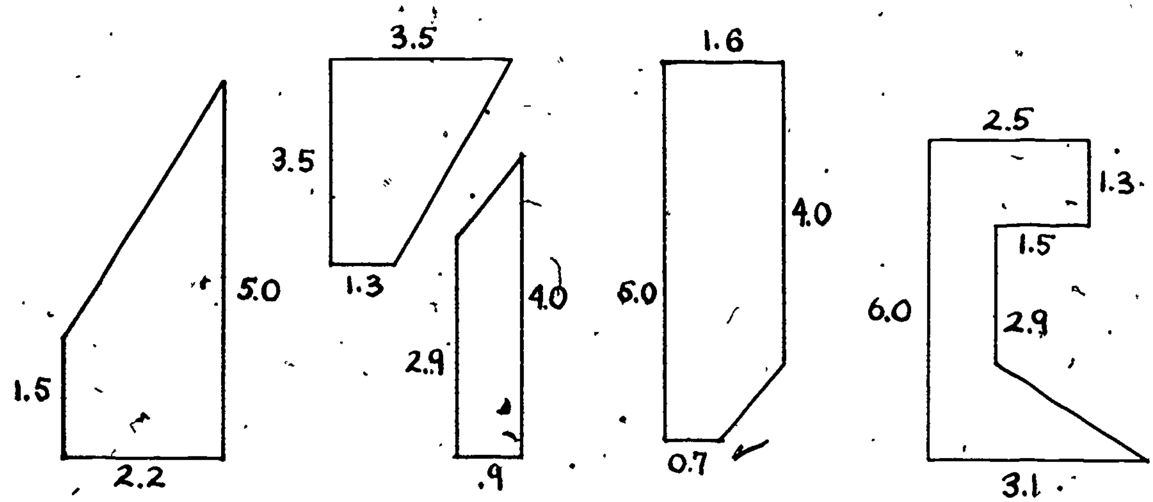


Figure 16(b)

SECTION 15 MEASURING MINIATURE MANHOLES

Imagine that the holes in the sheet of cardboard you are given are miniature manholes. You are going to make covers for the manholes and use them to find out something about the size of the holes.

You can make a cover by fastening two discs together (Figure 17) so that one disc fits inside a hole while the other covers the hole. Make three manhole covers, one for each of the three holes.

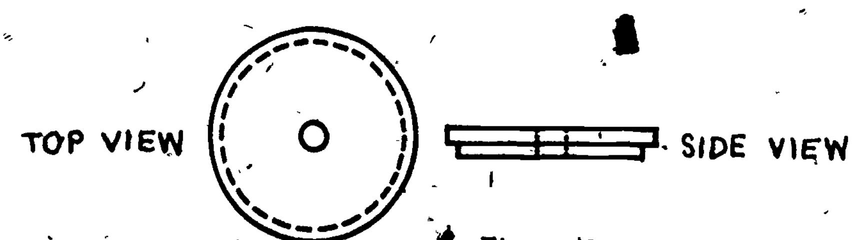


Figure 17

Each disc is marked with its size. From the sizes marked on the covers you have made, what can you say about the size of each hole? Can you bracket the size of each hole?

Try using different discs, if necessary, to bracket the size of the holes as closely as possible. Write down your results using the "less than" (<) sign as in the following example.

Suppose we have a cover whose two parts are 5.7 cm and 6.1 cm. We don't know the exact size of the hole, but we do know that it is between 5.7 cm and 6.1 cm. In other words, we can "bracket" the hole size between 5.7 cm and 6.1 cm. We can write this as

$$5.7 \text{ cm} < \square < 6.1 \text{ cm}$$

where the $\square$ can be filled in with the size of the hole. We leave it open, however, because we don't know exactly how big the hole is.

How closely have you bracketed the three hole sizes? To find out, subtract the lower bracketing value from the upper bracketing value.

Which hole size are you most sure of? Which hole size are you least sure of?

Now measure and write down the size of each hole. Does the size of each hole lie between the brackets you found for it?

39. Suppose you made a cover for a manhole from discs marked 4.8 cm and 5.3 cm. Use ☐ for the hole size and the "less than" symbol to bracket the hole size.

40. (a) A boy has an older sister who is 19 years old. He also has a younger brother who is 13 years old. What might be the age of the boy?

(b) Use ☐ for the boy's age and use the symbol "<" for "less than" to bracket the boy's age.

41. A particular block is too wide to slide through the part of the trail that is 1.5 cm wide. It can, however, slide through the part that is 2.4 cm wide. Letting ☐ stand for the width of the block, bracket the width of the block.

SECTION 16 SUBTRACTING DECIMALS

In the last section you had to subtract the value of the smaller bracket from the value of the larger bracket. Since these values were given to tenths of a centimeter, you had to subtract decimals. The important thing in subtracting decimals is to always line up the decimal points. For example, 100.0 — 93.6 should be written

$$\begin{array}{r} 100.0 \\ -\ \underline{93.6} \end{array}$$

Now you can subtract by using borrowing or the way described in Section 4 of this chapter.

42. What is the difference between 17.8 and 20.0?

43. Subtract 74.5 from 423.7.

44. What is 79.2 minus 43.0?

45. The two discs of a manhole cover are 6.1 cm and 5.7 cm in diameter. What is the difference in their diameters?

46. Do the following subtractions.

(a) 37.95 — 4.4

(b) 198.1 — 95.3

(c) 105.0 — 96.4

(d) 37.6 — 21.0

(e) 91.15 — 7.047

47. Suppose you have $7.00 in your pocket and you owe someone $3.06. How much money do you have left after paying what you owe?

SECTION 17 OPTICAL ILLUSIONS

Optical illusions are drawings and other things that trick your eye. Figure 18 is an example. Do the heavy lines appear straight or curved? Use your ruler to check your eye.

Optical illusions often fool your eye about size. In Figure 19 estimate which of the two long, straight line segments is the longer. Check your estimate by measuring the lines.

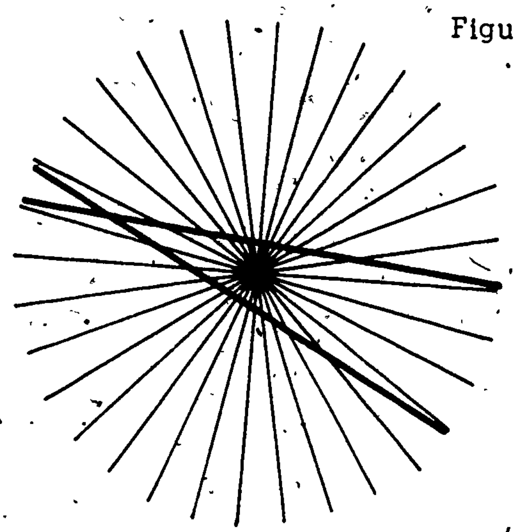


Figure 18

Figure 19

Is the hat in Figure 20(a) taller than it is wide? In Figure 20(b) which pencil is longer, the left or the right?

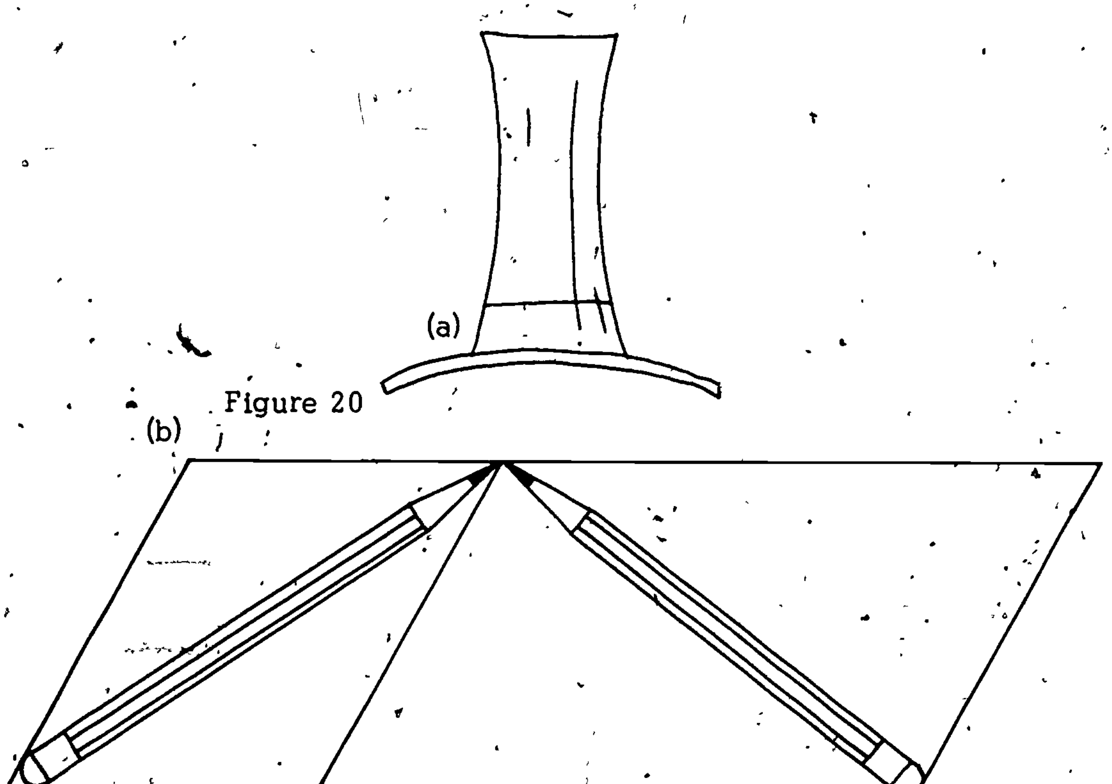
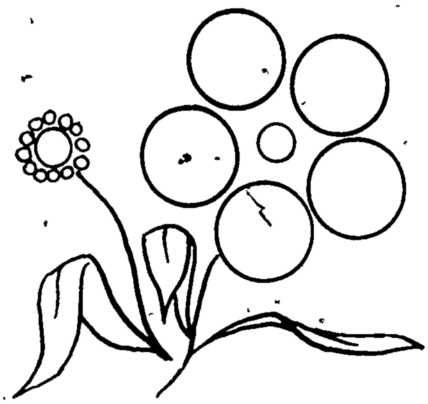


Figure 20

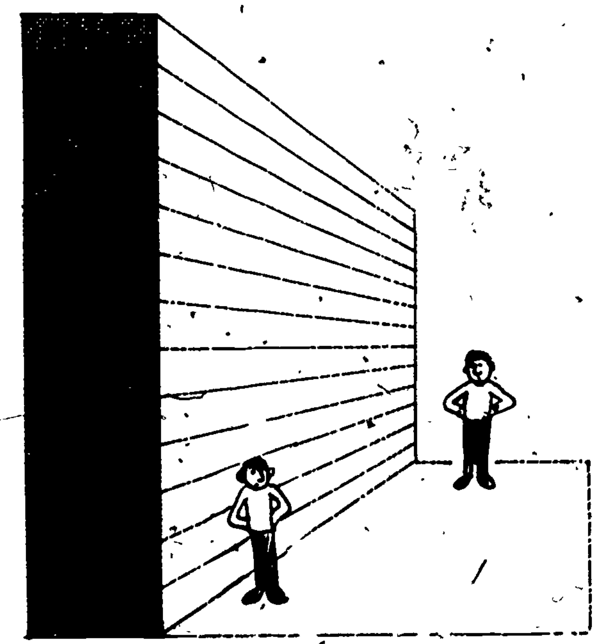
Which is greater in Figure 21(a), the man's arm spread or his height? In Figure 21(b), which flower has the bigger center? Which boy is taller in Figure 21(c)? (What do you mean by taller?)

Figure 21

(a)



(b)



(c)

3. ANGLES AND THEIR MEASUREMENT

SECTION 1 TURNS AND ANGLES

Two line segments that meet at a point, as in Figure 1, form an angle. One way of thinking of angles is to imagine a line segment turning with one end stationary. The two lines of the angle indicate the starting and stopping positions.

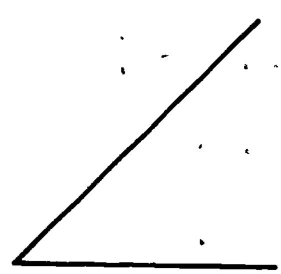


Figure 1

If we think of the horizontal line in Figure 1 as the starting position, there are two ways the line could have turned to get to the other position. We can put a curved arrow in the diagram of the angle so that we will know which way the line turned. (See Figure 2.)

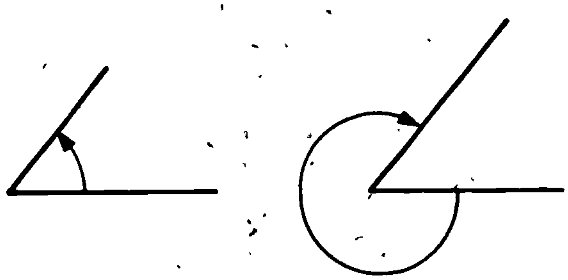


Figure 2

Look at Figure 3. Because a line has to turn farther in angle a than in angle b, we will say that angle a is greater than angle b. To measure angles we measure the amount of turning.

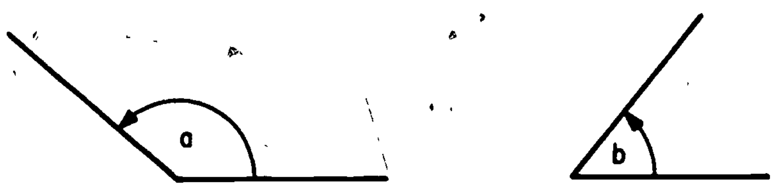


Figure 3

In Figure 4 note that the arrows in angles c and d indicate opposite directions of turning. However, the amount of turning is the same. In angle c the line is turning in the same direction as the hands of a clock turn, so we call this a clockwise turn or rotation. The turn in angle d is a counterclockwise turn or rotation.

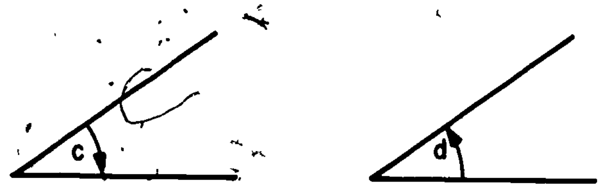


Figure 4

If a line turns so that it comes back to its starting position, we say it makes a full turn. Figure 5 shows three diagrams of full turns starting from different positions. Note that in each case, although the line has made a full turn, it doesn't look as if an angle was formed because the starting and stopping positions of the line are the same.

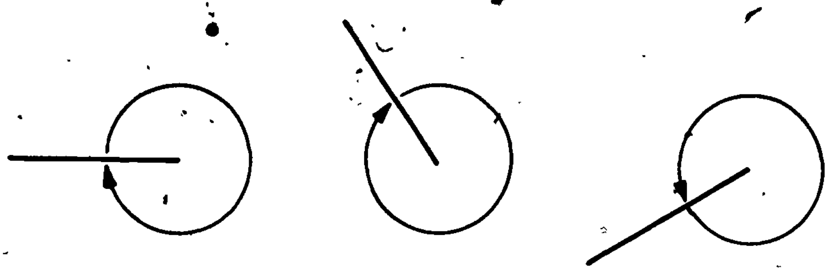


Figure 5

To measure angles we can use one full turn as a unit for measuring angles.

1. If an angle representing one full turn is taken as a unit for measuring angles, what is the size of the angles in Figure 6(a), Figure 6(b), and Figure 6(c)?

2. The angles in Figure 6(b) are so common that they have a special name. They are called <u>right angles</u>. Name some examples of right angles you find in your classroom.

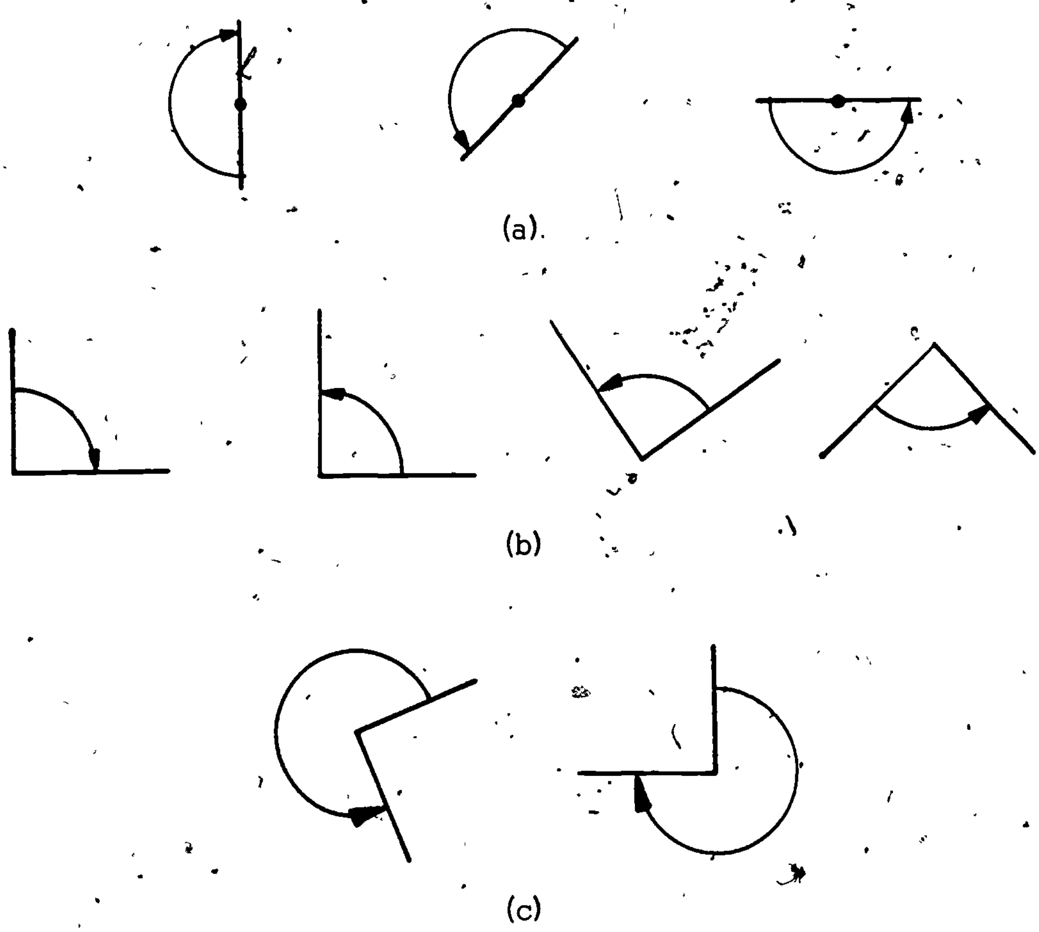


Figure 6

3. In Figure 7 decide, for each turn, whether it is more than, equal to, or less than a right angle.

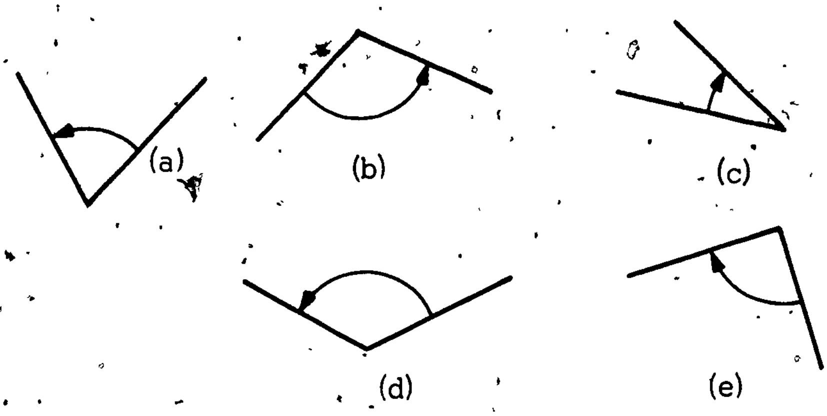
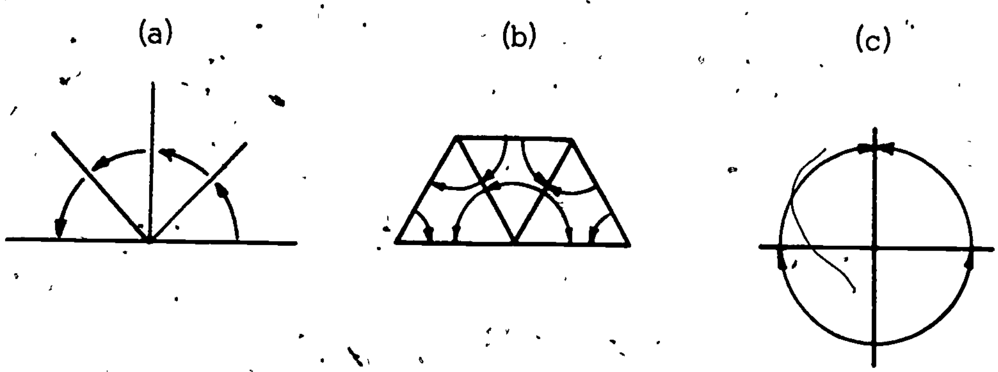


Figure 7

4. What part of a full turn is each of the angles in Figure 8? All the angles in each drawing are the same size.

Figure 8



SECTION 2 A SIMPLE PROTRACTOR FOR MEASURING ANGLES

You can make a simple instrument called a protractor for measuring angles. Trace a circle on a piece of paper. Be sure that the diameter of the circle is about 5 cm. Cut out your circle.

Now fold your circle three times; first in half, then again in half, and finally once more in half. It will look like Figure 9(a). While it is folded, cut off the tip of the point. Now unfold the circle and draw lines along the folds. It should look like Figure 9(b).

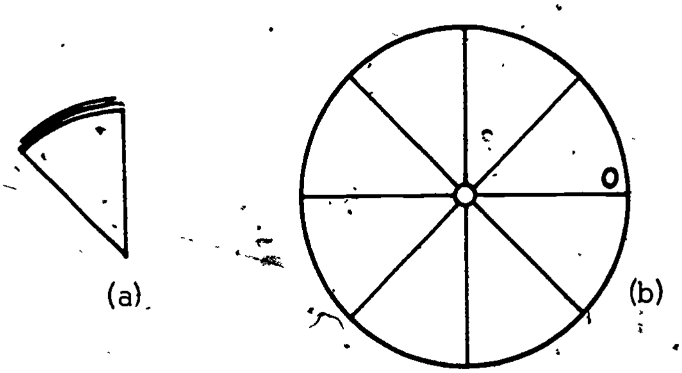


Figure 9

Pick one line and think of it as the starting position for all turns. Label this line with a "0" as shown in Figure 9(b). A complete turn from this line would bring you back to this same line. We can represent a complete turn by the number 1. The other lines form with this line an angle that is some fraction of 1.

Starting at the 0 line and going around counterclockwise, use fractions to label the size of each angle on the paper.

To measure an angle, place the center hole of your protractor over the center of the angle. Place the 0 line on one side of the angle. The place where the other side of the angle falls on the protractor gives the size of the angle.

?

5. Use your protractor to measure the angles in Figure 3 and Figure 7. If the angle falls between two angle measures on your protractor, bracket the measure of the angle. For example, if the angle is between $\frac{1}{4}$ and $\frac{3}{8}$ turn, write

$\frac{1}{4}$ turn < $\square$ < $\frac{3}{8}$ turn.

e You can divide your simple protractor into twice as many angles as you have by folding it again. You would then have angles which are half the size of the ones marked. Sometimes you may wish to divide an angle into two equal parts without folding the paper or page on which it is drawn. Here is how:

You can use a pair of parallel lines to divide an angle into two equal angles.

Lay a narrow straightedge along one side of the angle to be divided as in Figure 10(a). Now draw a line segment like AB that is parallel to the side of the angle along which the rule lies.

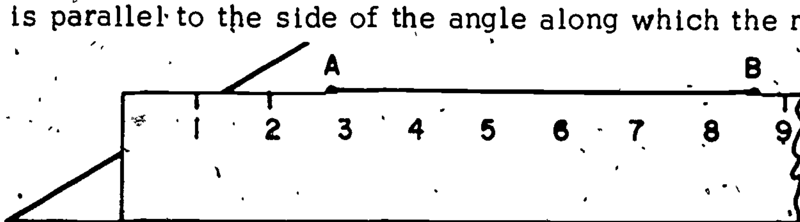


Figure 10(a)

Next, lay the straightedge along the other side of the angle as in Figure 10(b) and repeat what you have just done.

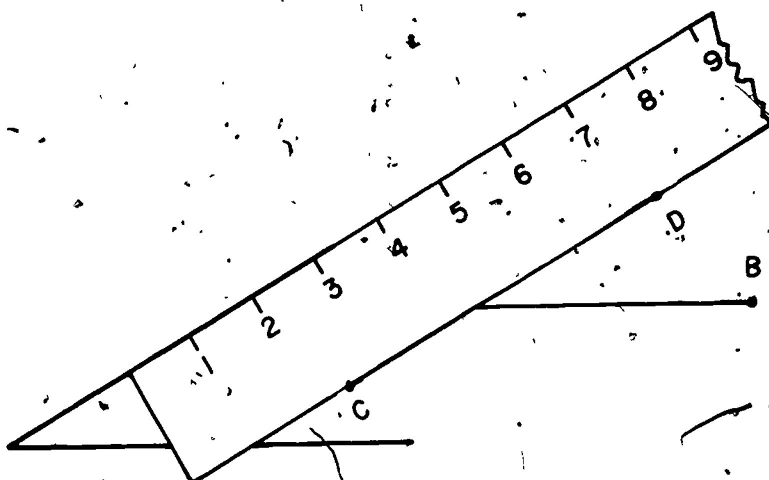


Figure 10(b).

Draw a straight line connecting the point of intersection of the parallels with the point of intersection of the two sides of the angle (called the <u>vertex</u> of the angle) as shown in Figure 10(c). You have now divided the original angle into two equal parts or, as is commonly said, you have <u>bisected</u> the original angle.

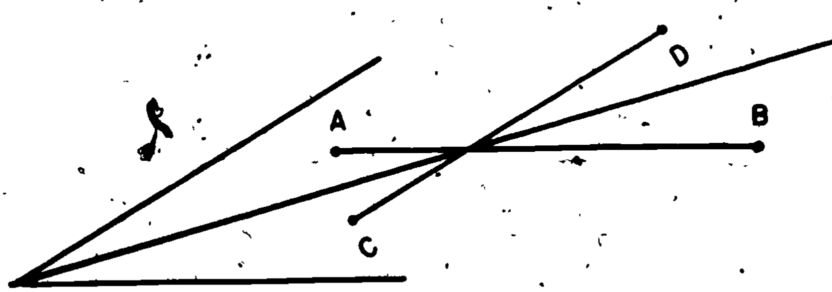


Figure 10(c)

Bisect each of the angles in Figure 11 and give its size.

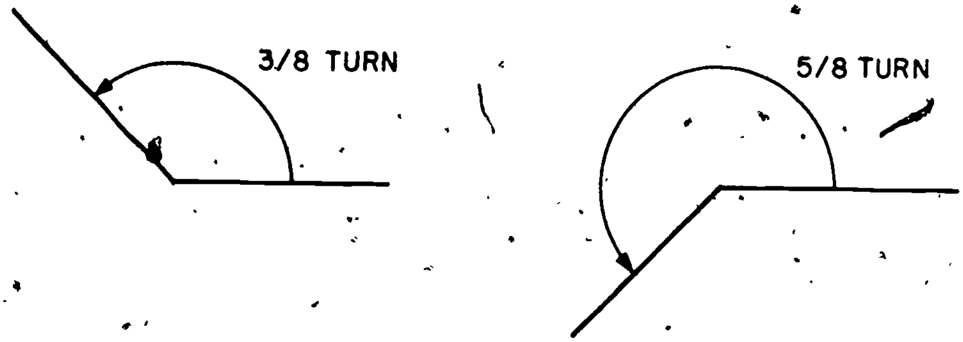


Figure 11

SECTION 3 MEASURING ANGLES IN DEGREES

It is not usually convenient to measure angles as fractions of a full turn. What is commonly done is to divide an angle representing one full turn into 360 equal parts called degrees (written as 360°). The number 360 was probably chosen because it can be divided evenly by many different whole numbers. So now we can name many parts of a full turn with whole numbers instead of fractions.

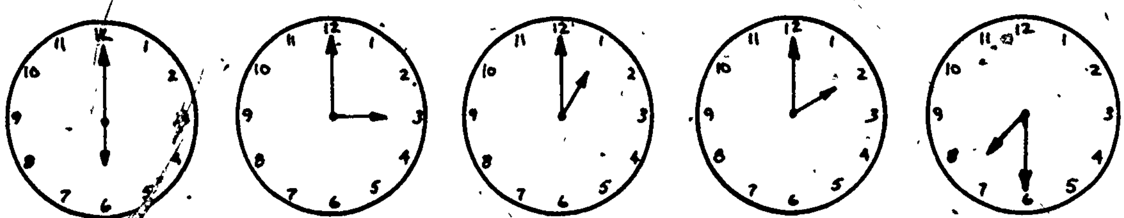
$$2 \times 180 = 360$$
$$3 \times 120 = 360$$
$$4 \times 90 = 360$$
$$5 \times 72 = 360$$
$$6 \times 60 = 360$$
$$9 \times 40 = 360$$
$$10 \times 36 = 360$$
$$12 \times 30 = 360$$
$$15 \times 24 = 360$$
$$18 \times 20 = 360$$

etc.

?

6. Starting at the 0 line, label the angles on your simple protractor in degrees. Write the degree name directly below the fractional name for each angle.

7. What size angle, in degrees, is formed by the hands of each clock in Figure 12?



(a) (b) (c) (d) (e)

Figure 12

8. Use your protractor to measure at least five angles of different size in the bridge in Figure 13. If the angle falls between two angle measures on your protractor, bracket the measure of the angle. For example, if the angle is between 270° and 315°, write $270° < \square < 315°$.

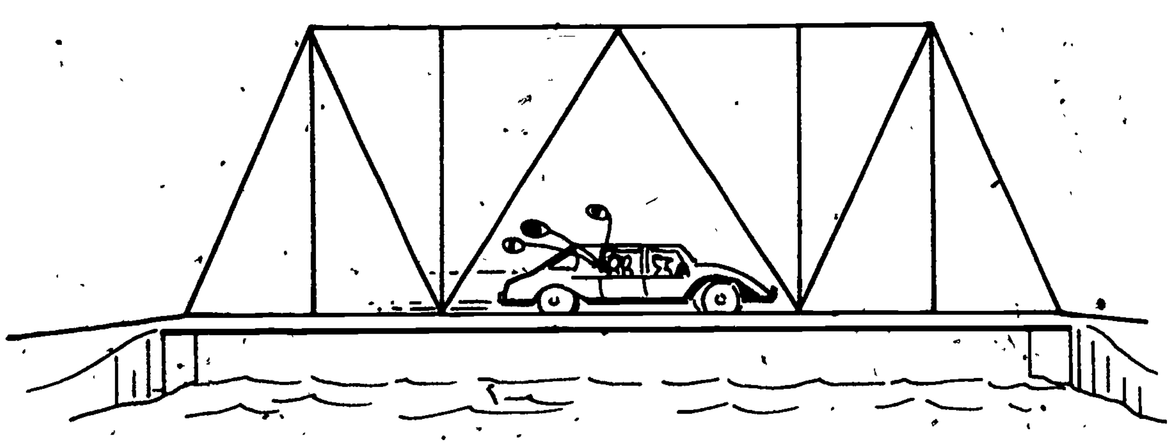


Figure 13

9. Draw angles whose sizes are

(a) 90° (b) 45° (c) 135° (d) 315°

10. Without using a protractor, figure out the measure of the smaller angle formed by the hands of a clock at each of the following times. (Drawing a picture for each case may help.)

(a) 4:00 (b) 9:00 (c) 11:00 (d) 5:00 (e) 4:30

11. When you bracket the angles in Figure 13, how many degrees are there between two neighboring brackets (two neighboring folds)?

e Here's a way to measure some angles without using a protractor. Trace the star in Figure 14 and cut it out. Now cut off the five points of the star. Place them side by side so that the five points all meet. How large is the angle at each of the points of the star?

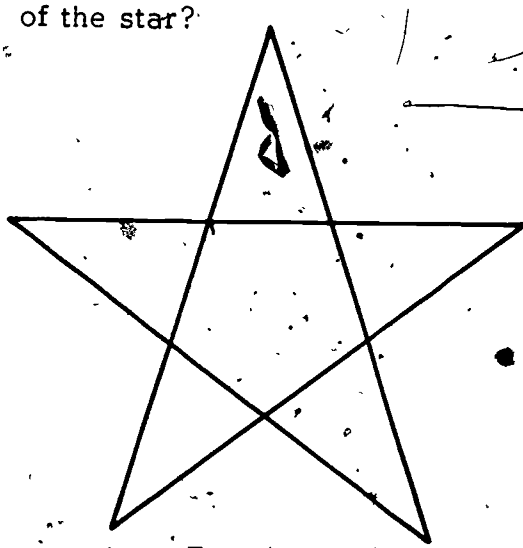


Figure 14

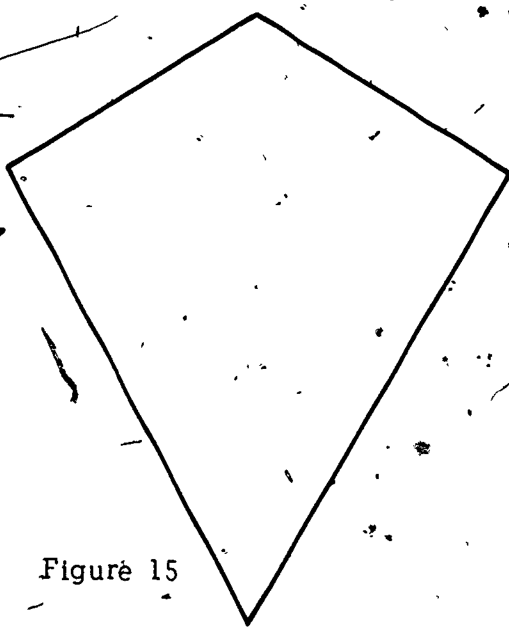


Figure 15

Trace and cut out three identical copies of Figure 15. By placing the three copies carefully together, you can find out what each of the angles must be.

SECTION 4 MEASURING ANGLES TO THE NEAREST DEGREE

The protractor shown in Figure 16 is marked off in degrees from zero to 180. There are two numbers every 10° around the scale. One set of numbers marks degrees counterclockwise from right to left. The other set reads clockwise from left to right. This makes it possible to measure angles from either left to right or right to left, eliminating awkward or upside-down positions of the protractor.

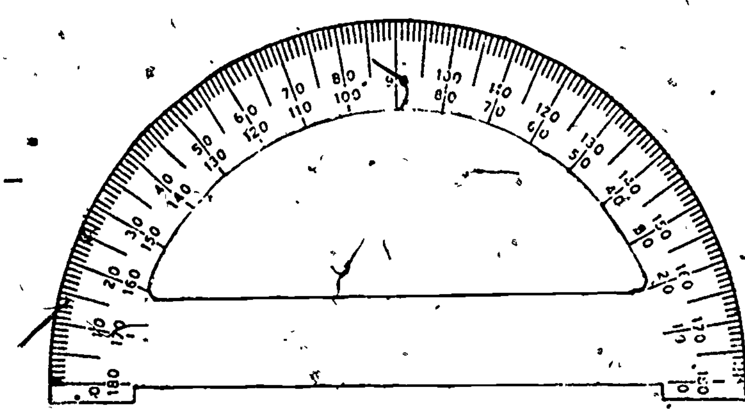


Figure 16

Estimating Angles

Divide into groups of three or four players. Each player draws a card from the deck supplied by your teacher. The player drawing the largest angle plays first. Place the shuffled deck face down in the center of the table. The first player turns over the top card. He estimates the measurement of the angle by bracketing the angles, making the upper bracket 10° greater than the lower bracket.

The player to the left then measures the angle to the nearest degree. If the first player bracketed the angle he gets one point. The next player then draws a card from the top of the deck and gives his estimate. The game continues until all cards have been drawn. The player with the highest total wins.

[?]

12. What angle would complete the half turn in each case in Figure 17? (An angle that represents a half turn is called a straight angle.) This question can be answered without measuring. How?

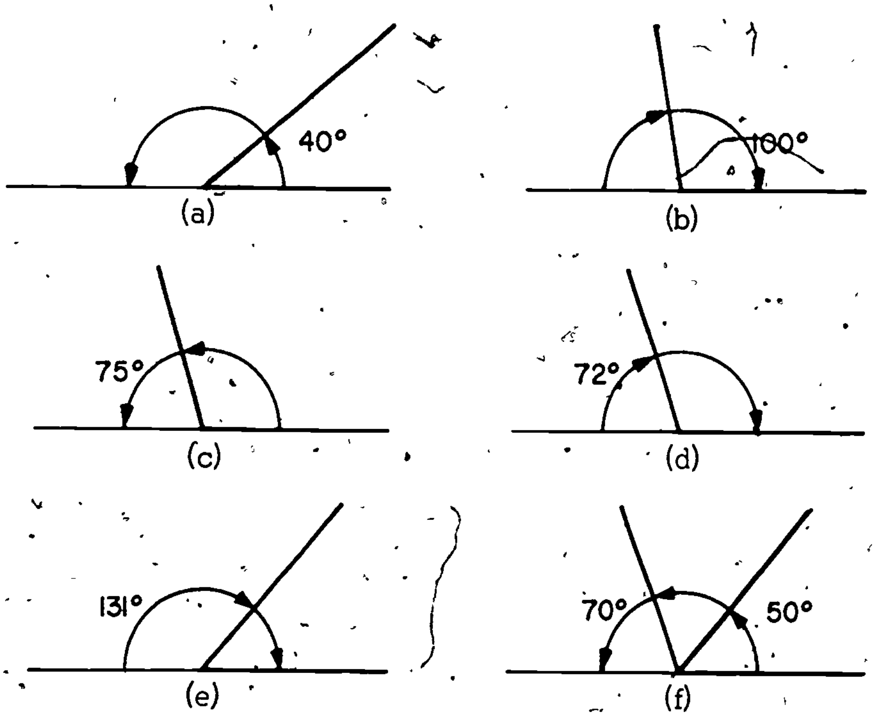


Figure 17

13. In each diagram of Figure 18, what is the size of the angle that completes the full turn?

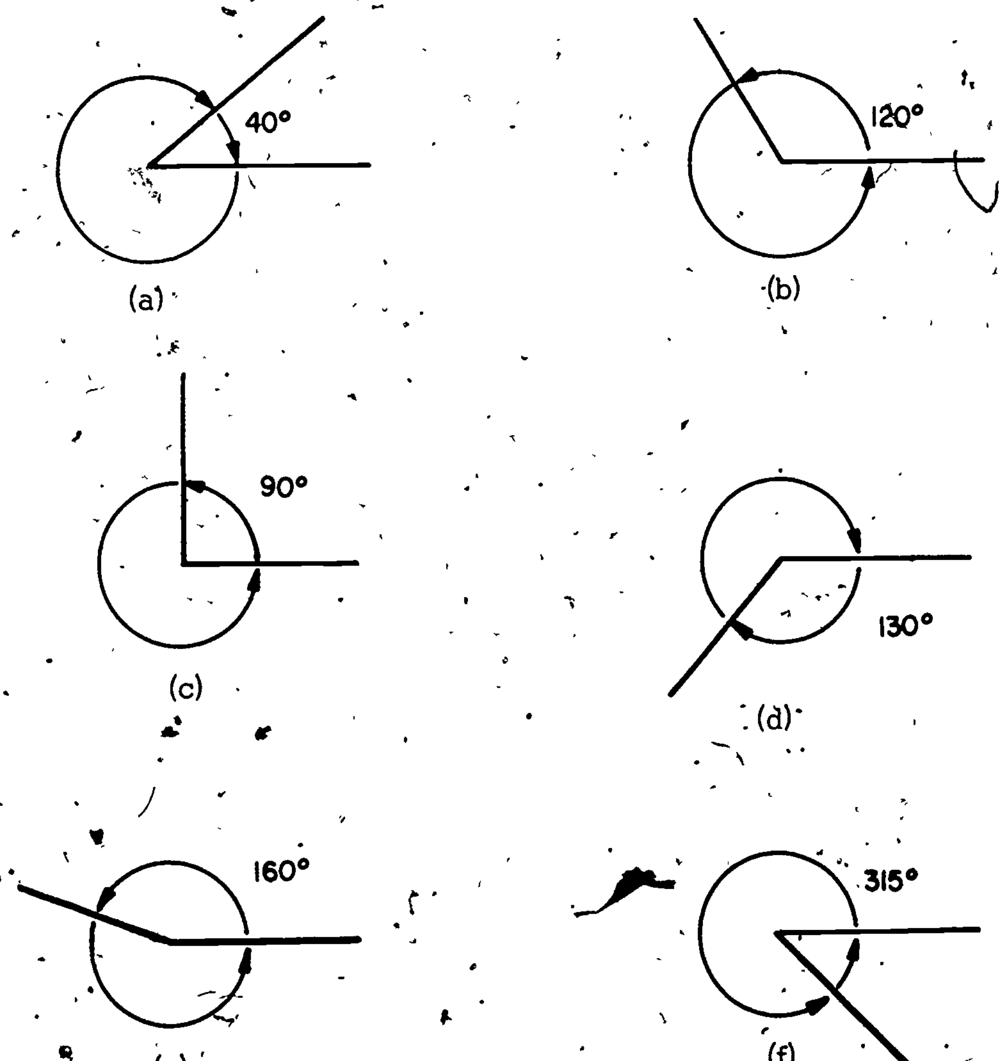
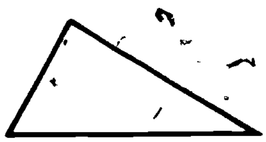


Figure 18

14. Describe how you used your protractor to measure the angles of the triangle in Figure 19 to the nearest degree.

Figure 19 

15. Use a protractor like that in Figure 16 to measure the different angles in Figure 13. Do your results agree with those you found for Question 8 in Section 3?

16. (a) Are the angles a and b in Figure 20 the same size? To find out, measure them with a protractor.

(b) Do the lengths of the sides of an angle have anything to do with the size of the angle?

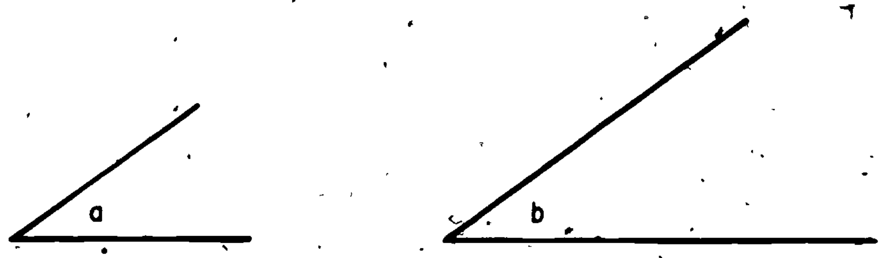


Figure 20

17. Without using a protractor, make sketches of angles whose sizes are:

30°, 45°, 120°, 90°, 155°, 85°, 270°

Check your drawings by actually measuring with a protractor. In each case determine how much your drawing is in error.

18. How many angles are formed by the intersection of two lines? Are they all different?

Vertical Angles

Look at the pair of angles formed by the two intersecting lines in Figure 21. How does the figure change if you flip it over, left-to-right? If you turn it upside down? Does this suggest to you what angles are equal to each other? Why?

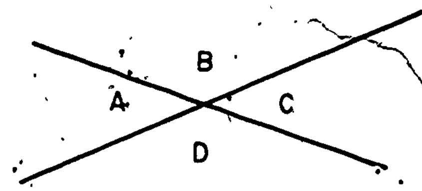


Figure 21

Using the fact that $\angle A + \angle B = 180^\circ$ and $\angle B + \angle C = 180^\circ$ we can show that $\angle A = \angle C$. ($\angle A$, $\angle B$, and $\angle C$ represent the sizes of these angles.)

$$\angle A + \angle B = 180^\circ$$
$$\angle C + \angle B = 180^\circ$$

Note that we can write either equation as

$$\square + \angle B = 180^\circ$$

where we write in the box the number that when added to $\angle B$ gives 180°. But there is only <u>one</u> number that can be added to $\angle B$ to give 180°, and so $\angle A$ and $\angle C$ must be equal.

19. Can you show that $\angle B$ in Figure 21 equals $\angle D$?

20. In Figure 22, without using a protractor find
    (a) the sizes of all the unlabeled angles.
    (b) the sum of the angles of the triangle.

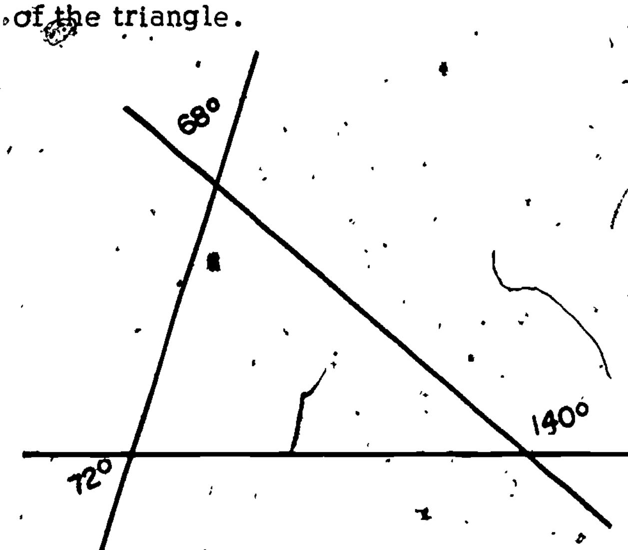


Figure 22

Make the triangles shown in Figure 23 and assemble them into a square.

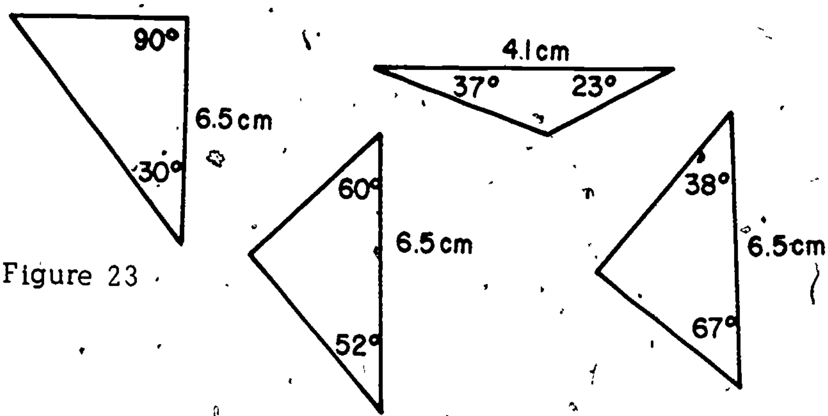


Figure 23

SECTION 5 THE SUM OF THE ANGLES IN A TRIANGLE

Measure, to the nearest degree, the angles of the triangles in Figure 24 and put your results in a table like that shown in Table 1.

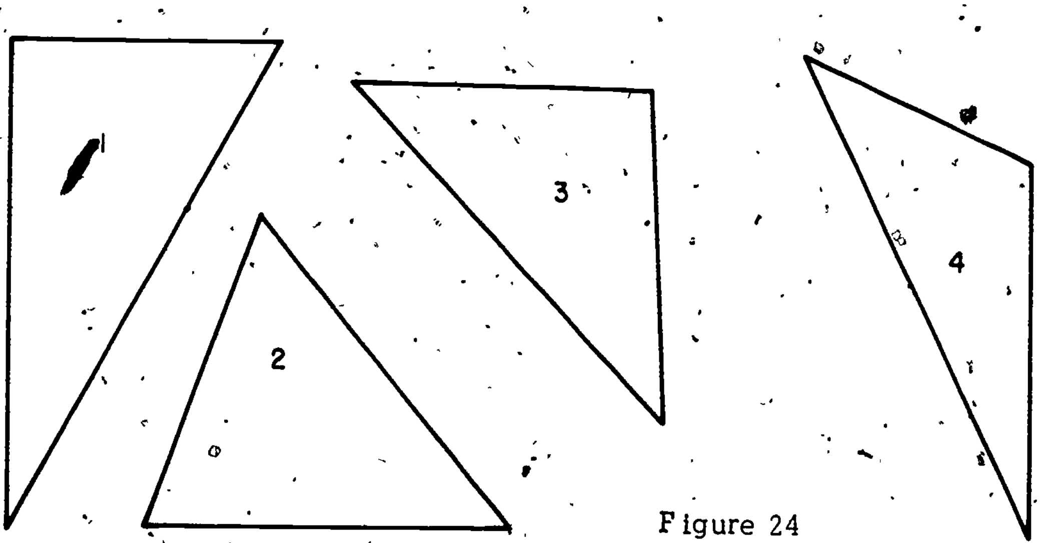


Figure 24

TABLE 1

Triangle	1	2	3	4
Measure of Angles				
Sum of All Three Angles				

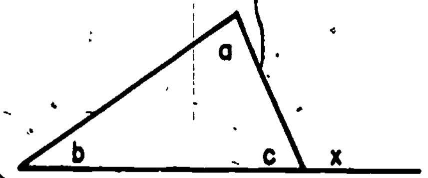
Now look at the sum of the measures of the three angles in each triangle. What does this tell you about the sum of the angles of <u>any</u> triangle?

Now draw <u>any</u> triangle. Measure the three angles. What is the sum of these three angles?

Cut out a triangle and number the angles. By tearing off the corners of the triangle and rearranging the pieces, show that the sum of the angles in your triangle is 180°.

Can you fold the vertices (the corners) of a paper triangle so that they meet at a point on the long side and show that the sum of the angles of a triangle is 180°?

In the triangle below, $\angle a + \angle b + \angle c = 180^\circ$. Why? It is also true that $\angle x + \angle c = 180^\circ$. Why? What can you conclude about $\angle x$ and the sum $\angle a + \angle b$?



SECTION 6 PROTRACTORS AND ACCURACY

Three protractors are shown in Figure 25. How large a gap, in degrees, is bracketed by neighboring marks on each one?

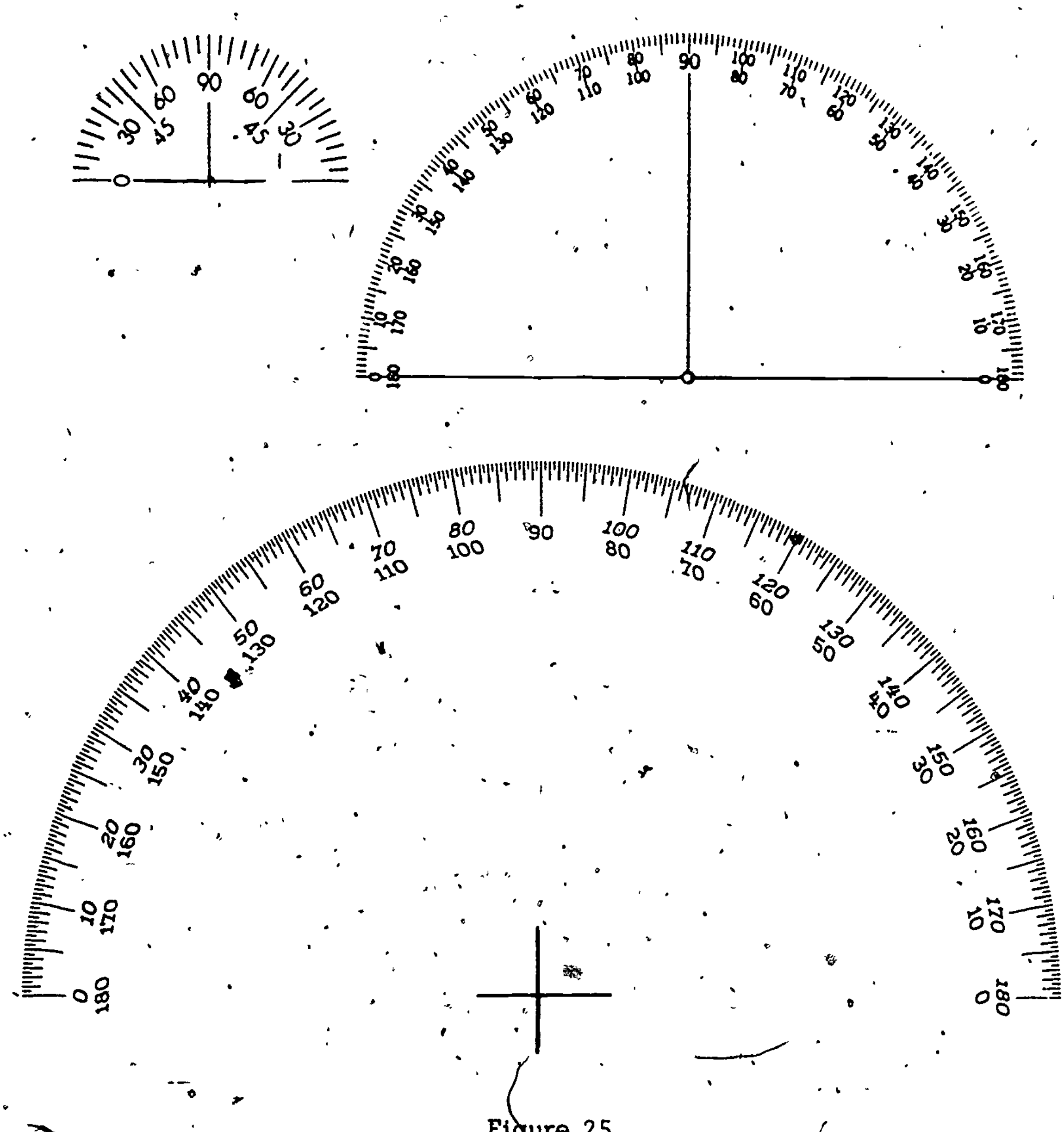


Figure 25

Cut a large triangle from a sheet of paper and place it on top of one of the protractors in Figure 25 and measure one of the angles of the triangle.

Next, measure the same angle with each of the other protractors. Which protractor do you think gives the most accurate measurement of the angle?

If you want to know the size of an angle accurately rounded off to the nearest degree, which protractor (or protractors) would you use?

Use your commercial protractor to estimate the angle in Figure 1 to the nearest 0.1°.

Put your answer on the chalkboard, next to those of your classmates.

How far apart are the largest and smallest measurements made by your class? Does this tell you how closely you can bracket the measurement of an angle, using your commercial protractor?

As you can see from the scales on the three protractors in Figure 25, the larger the protractor, the greater the distance between degree marks. This is not the case with a centimeter scale. On any metric ruler, long or short, 0.1 cm marks are always the same distance apart. By making it longer, we cannot improve the accuracy with which we read a centimeter scale. Special devices are required to measure accurately to hundredths or thousandths of a centimeter.

A protractor is different. We can divide any half circle, no matter how large, into 180 degree marks and fractions of degrees. We can, therefore, make a more accurate protractor by making a larger protractor. There are limits, however. If a protractor is very large, it may be that we cannot accurately extend the sides of an angle we wish to measure. Our error in extending the sides may be greater than the angle bracketed by the smallest divisions on the protractor. Or perhaps we can't line up the protractor with the sides of an angle accurately enough.

Finally, there is a limitation in our ability to make small fractions accurately on a protractor because of its manufacturing process.

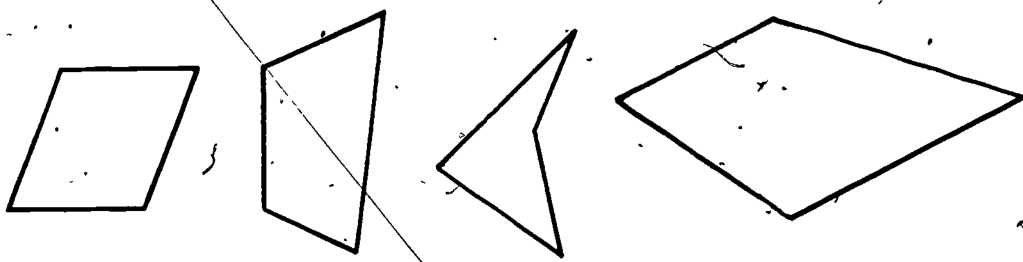
As in the case of measuring lengths, there are special devices that improve the accuracy of measuring angles beyond that obtainable with a simple protractor. In astronomy and surveying, for example, specially designed telescopes make possible the measurement of angles to tiny fractions of a degree.

SECTION 7 ANGLES OF REGULAR POLYGONS

What is the sum of the angles of a four-sided figure?

The sum of the angles of a triangle is 180°, so we can find the sum of the angles of any figure by dividing it into the smallest number of triangles (with no lines crossing).

Divide figures like those below into triangles by drawing one-line segments connecting any pair of opposite corners.

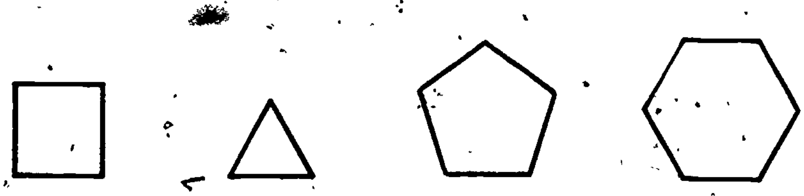


Draw some different four-sided figures and divide them into triangles.

21. Into how few triangles can you divide a four-sided figure?

22. What is the sum of the measures of the angles of a four-sided figure?

23. A figure that has all sides and all angles equal is called a regular polygon. A <u>polygon</u> is any plane figure bounded by straight lines, from "poly" meaning many and "gon" meaning angles. Here are some examples of regular polygons.



How large is each angle in a regular four-sided figure?

24. How could you find the sum of the measures of the angles of a five-sided polygon? What is this sum?

25. Into how few triangles could you divide the polygon in Question 24?

26. What is the sum of the angles in a regular five-sided polygon?

27. What is the measure of each angle in a regular five-sided polygon?

28. Find the measure of each angle in a regular six-sided polygon.

29. Complete Table 2 describing regular polygons.

TABLE 2

Number of Sides	Number of Δ's	Total of Angles	Measure of Each Angle
3	1	$1 \times 180^\circ = 180^\circ$	$\frac{1}{3} \times 180^\circ = 60^\circ$
4	2	$2 \times 180^\circ = 360^\circ$	$\frac{1}{4} \times 360^\circ = 90^\circ$
5			
6			

30. Into how few triangles can an eight-sided polygon be divided?

31. What will be the measure of each angle in an eight-sided regular polygon?

32. Can you write a rule for finding the sum of the angles of any polygon?

33. Can you write a rule for the measure of each angle of a regular polygon?

SECTION 8 TILING WITH REGULAR POLYGONS

You may have seen pictures such as Figure 26, which is a drawing by the Dutch artist Maurits Escher. Notice that he used the outline of a man to fill a surface without leaving any gaps. We could, therefore, design tiles in the outline of a man and use them to tile a floor.

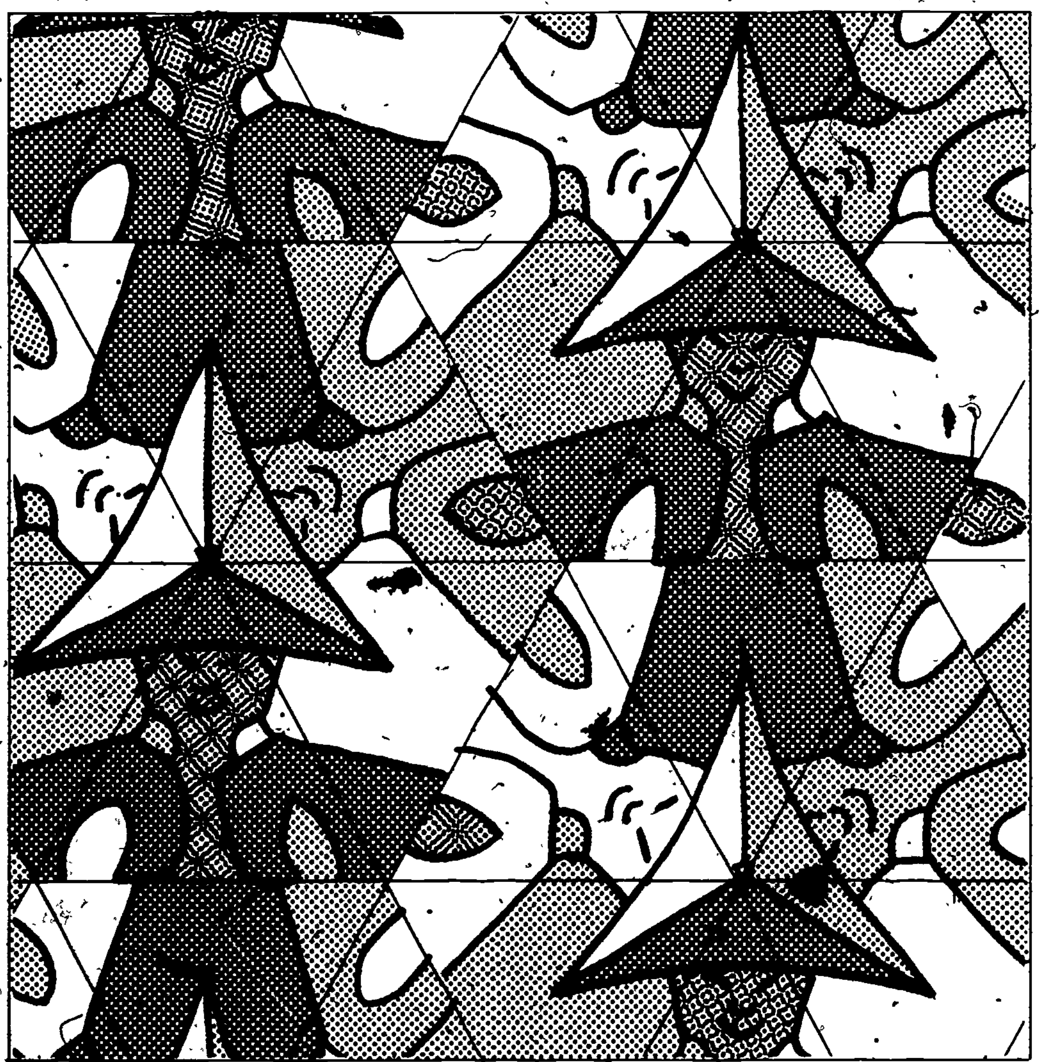


Figure 26

Needless to say, tiles like the ones in Figure 26 are difficult to design. Let us ask a related but somewhat easier question: Which of the regular polygons we studied in the previous section could be used as floor tiles? Remember that to cover a surface such as a floor we must be able to fit the tiles together without leaving any spaces between them.

Using index cards (or stiff pieces of cardboard) draw one of the following regular polygons on each card. Make the sides of each polygon equal to 2 cm.

(a) A regular three-sided figure.

(b) A regular four-sided figure.

(c) A regular five-sided figure.

(d) A regular six-sided figure.

(e) A regular eight-sided figure.

Now cut out each of the polygons as shown in Figure 27 and tape the cut as illustrated.

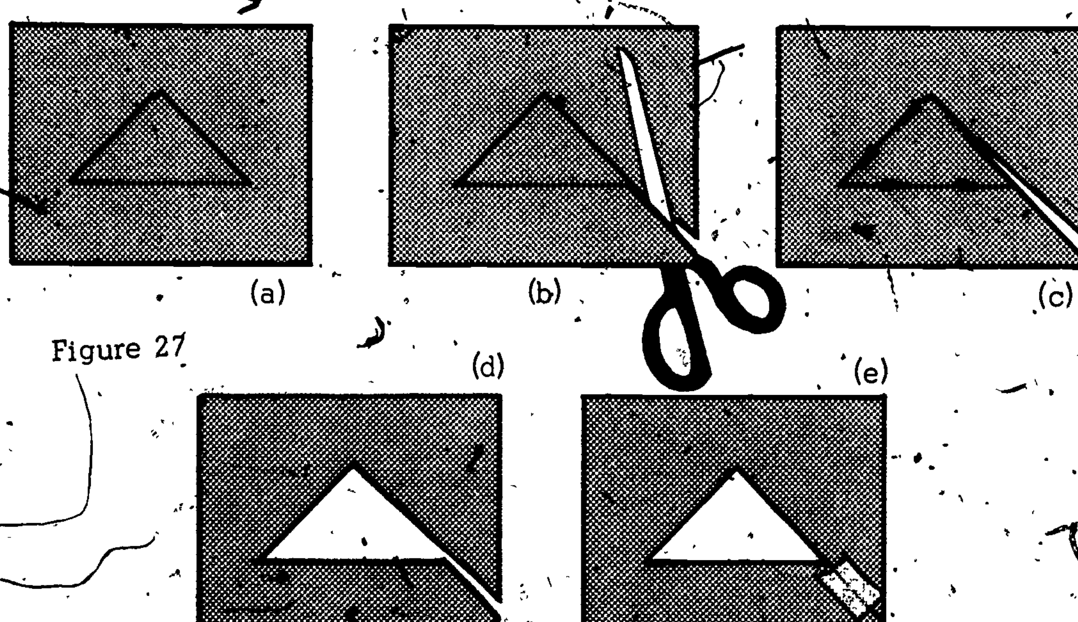


Figure 27

Now, using the template — the hole left by the cutout triangle — as a stencil, determine if a regular three-sided polygon can be used to tile a floor. To do this, trace the triangle, then mark a point on one of the vertices (Figure 28). Begin tiling by filling in the space around that point with triangles.

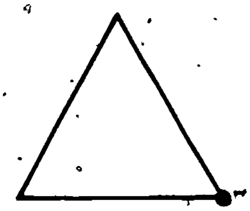


Figure 28

With the same procedure you used with triangles, determine which of the remaining regular polygons can be used to tile a floor.

34. Use your knowledge of the angles of regular polygons and the results of the tiling you have just done to answer the following questions:

(a) Does the triangle tile the plane?

(b) How many triangles meet at the labeled point?

(c) What is the measure of each angle of the triangle?

(d) What is the sum of the angles around the point?

35. Using the tiling results, answer the four questions of Question 34 for the remaining regular polygons.

36. (a) For the regular polygons that tile a floor, what is the sum of the angles around the marked point?

(b) State in your own words how you could determine whether a regular polygon would or would not tile a floor without trying it.

SECTION 9 TILING WITH MORE THAN ONE POLYGON

In the previous section we saw that the only regular polygons with which tiling is possible are triangles, squares, and hexagons. (A six-sided polygon is called a <u>hexagon</u>.) The tiling patterns using these polygons are not very complicated. However, if we use more than one of these polygons we can get rather attractive patterns. Figure 29 shows part of a pattern using two squares and three tri-

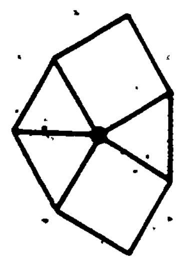


Figure 29

angles meeting at a point. Draw this pattern on your paper using the templates you made earlier. Can you complete the design? Does every point include a corner of two squares and three triangles? What is the sum of the angles at that point?

We can use polygons to tile a floor if the sum of the angles around a point is 360 degrees.

37. There are four other patterns that use two different polygons. Find two of them.

38. There is one pattern that uses three of the polygons you have. What is it?

39. There is a pattern that uses twelve-sided polygons and one other regular polygon. What must the other polygon be?

4. ENLARGING AND REDUCING; SIMILAR FIGURES

SECTION 1 RUBBER BAND ENLARGEMENTS

Look at the sketch of the airplane in Figure 1. How would you enlarge this sketch to twice its size? Would you have to measure the length of each line segment and then double it? Not necessarily. You can make the enlargement with a pencil, a rubber band and some adhesive tape.

Place a sheet of paper on the page containing Figure 1 so that the paper covers the page. Trace the airplane in the figure.

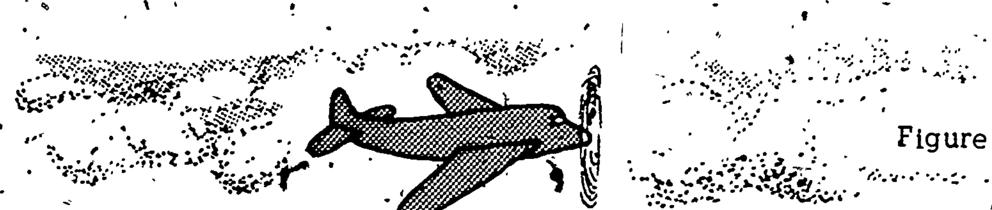
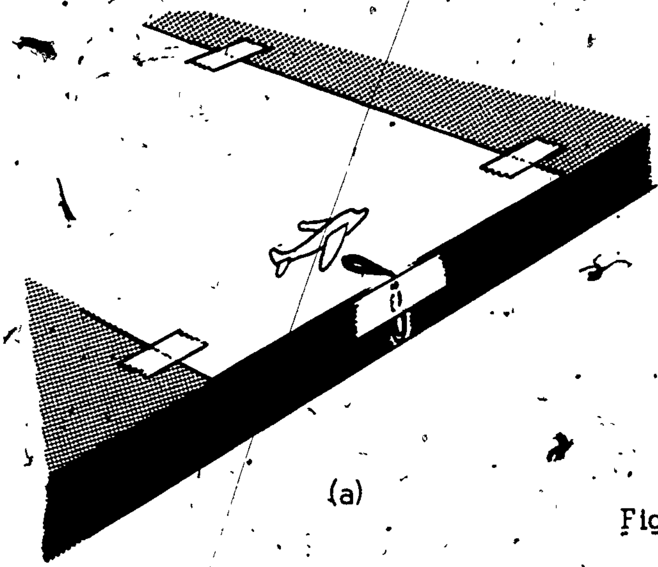


Figure 1

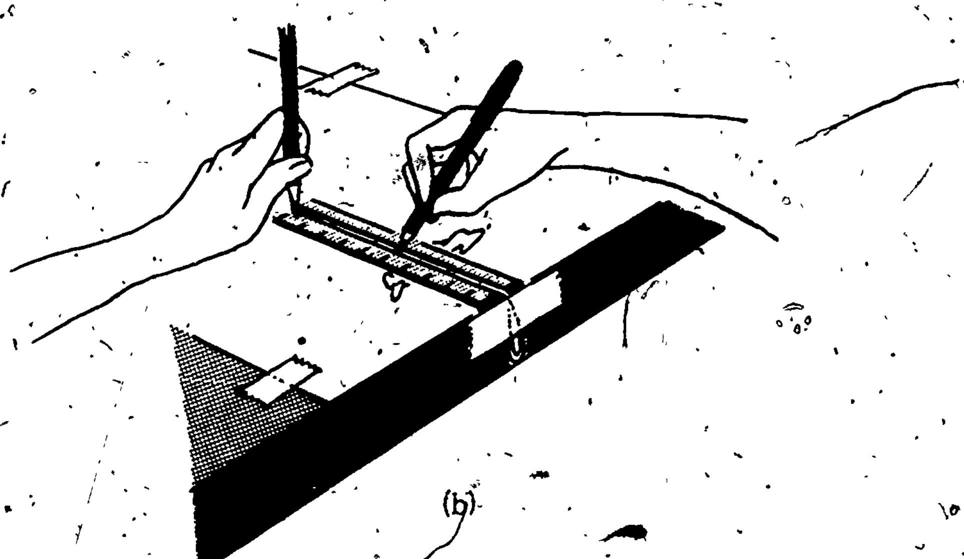
Now tape the tracing to the edge of a table or desk as shown in Figure 2(a). Bury one end of the rubber band under a couple of centimeters of tape along the edge of the desk.



(a)

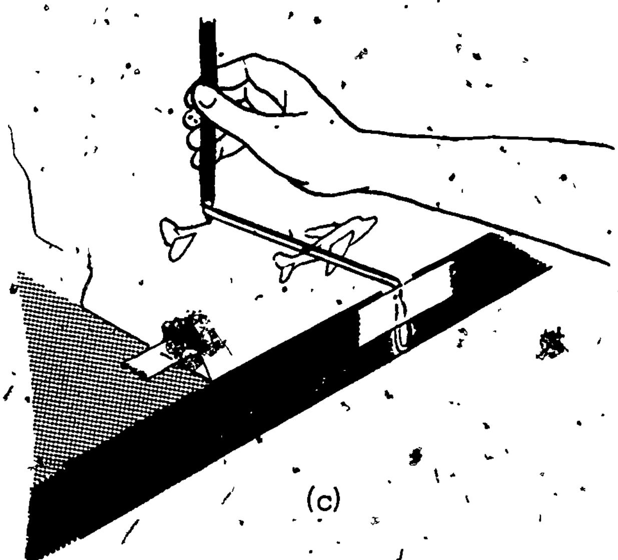
Figure 2

Stretch the rubber band with the tip of a pencil so that it is 10 cm long and have a classmate mark the rubber band at the 5 cm mark, the half-way point, as shown in Figure 2(b).



(b)

Now, with the pencil point firmly on the paper, trace out an enlargement of the airplane. This you can do by moving the pencil tip over the paper while keeping your eye on the half-way mark to make sure it follows the outline of the airplane tracing (Figure 2(c)).



(c)

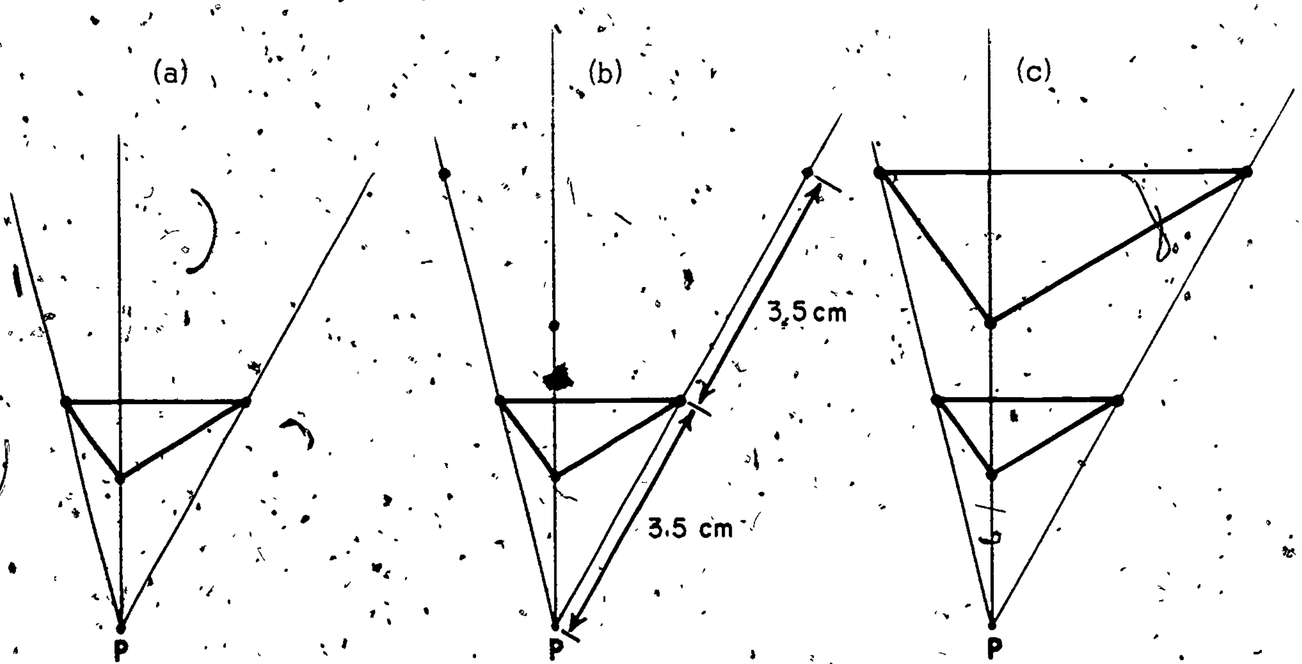
Use the rubber band method of enlargement to enlarge the nose of the airplane in Figure 1 to three times its size.

SECTION 2 ENLARGEMENTS MADE FROM A POINT

When you used a rubber band to double the size of a drawing, you placed a mark in the middle of the rubber band. The rubber band stretched and contracted as you made the enlargement. However, at all times, the distance from the tape at the end of the rubber band to the pencil was twice the distance from the tape at the end to the pencil mark.

Because the stretched length of the rubber band was always twice that of the distance to the mark, you were able to make a two times enlargement. Figure 3 shows another way of enlarging a figure. We pick any point P, called the center of enlargement, and draw straight lines from it through each of the corners in the figure in turn (Figure 3(a)).

Figure 3



Now, to get a two times enlargement, we do what the rubber band would have done. We measure from the center of enlargement to each corner on the drawing. Then we mark equal distances from the extension of the lines beyond the corners (Figure 3(b)).

Finally we connect the new marked points to get a two times enlargement (Figure 3(c)).

A center of enlargement may be located anywhere: below, above, inside, or on the figure itself. In Figure 4, the top corner P of triangle PAB is used as the center. First, we extend lines from P through all other corners of the figure (see Figure 4(a)). Then we locate the new point C twice as far out from the center P as the original point A (see Figure 4(b)). We locate D in the same way.

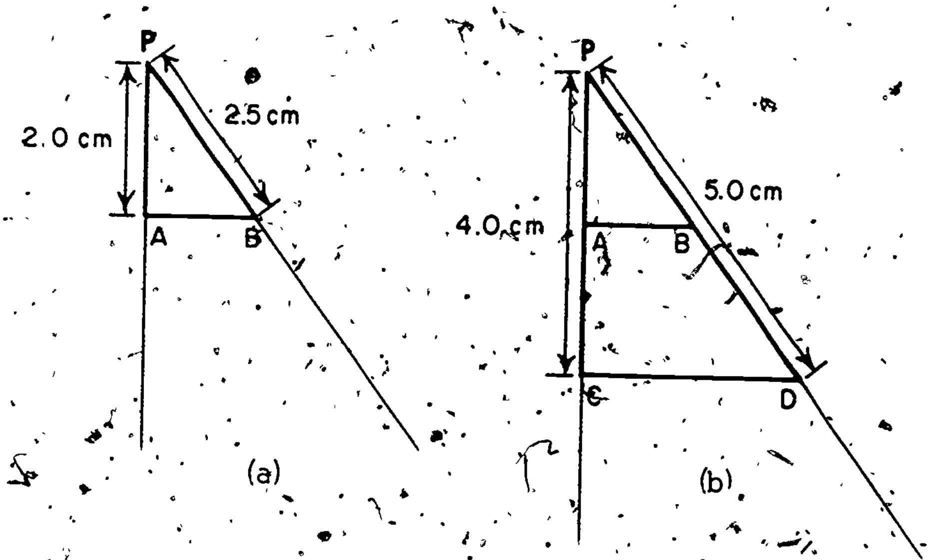


Figure 4

In Figure 4(b) as sides PA and PB get doubled to sides PC and PD, side AB gets enlarged to side CD. Measure to see that CD is two times larger than AB.

Each side and its enlargement are called corresponding sides. When a figure is enlarged each pair of corresponding sides is parallel (or the two sides lie on the same straight line). Check to see that sides AB and CD are parallel by using the procedure described in Section 13 of Chapter 2.

Draw a square, 2 cm on a side, choose a point to use as a center of enlargement and then make a three times enlargement.

Draw another 2 cm square and this time choose a completely different center of enlargement and again make a three times enlargement.

Does it matter what point we choose for a center of enlargement?

Trace the rectangles and points below. Double each rectangle from the indicated center of enlargement. Rectangle 1 should be doubled twice, once from each center labeled "1." What kind of animal do you get?

SECTION 3 REDUCTIONS FROM A CENTER

We can use the method of enlargement developed in the last section to also reduce a figure. We start with the larger flag in Figure 5, then draw lines to any point P chosen as a <u>center of reduction</u>. To get a one-half reduction, we mark the halfway point from the center of enlargement to each corner of the flag (see Figure 5). Connecting these points will give you a smaller flag that is the same shape as the original, but half as large.

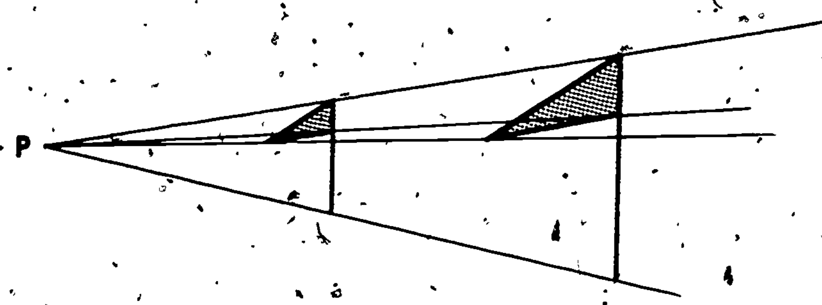


Figure 5

How do we find the halfway point on each line? We can use the same method we used to subdivide the strips in Section 6 of Chapter 2. We place a sheet of ruled paper under the figure, then adjust and mark the halfway point and repeat the process for each segment.

The quadrilateral, B, in Figure 6 is a reduction to one third the size of the quadrilateral A.

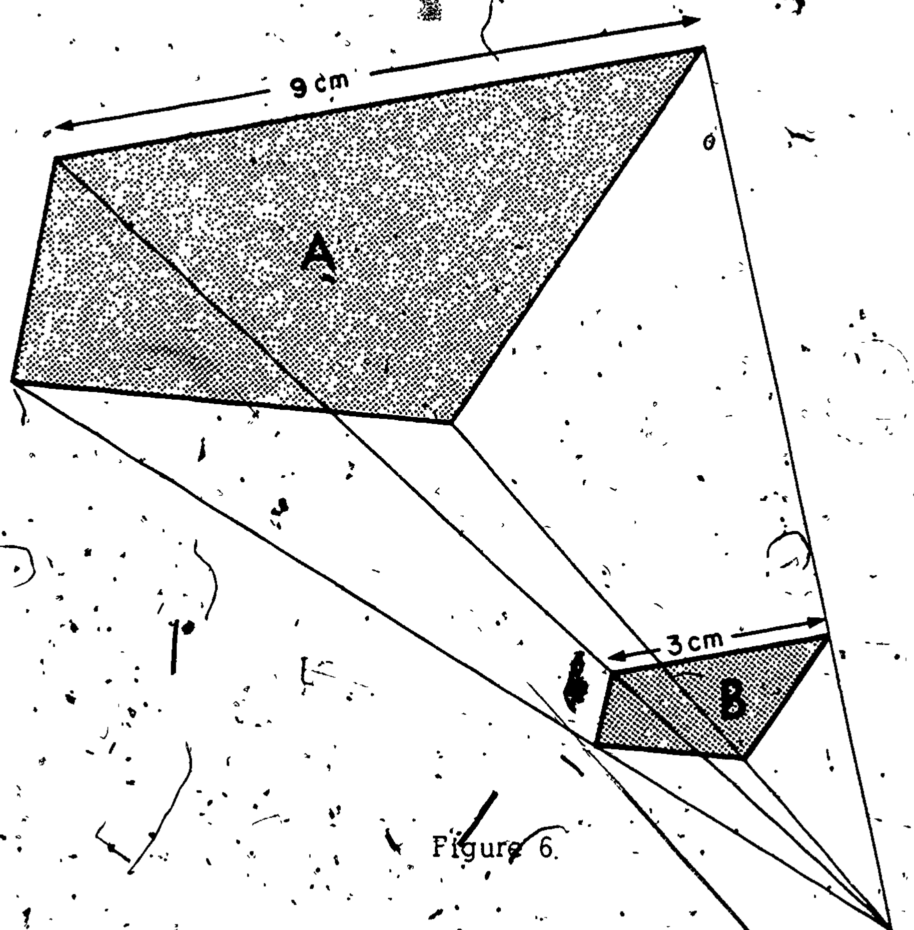


Figure 6.

Show by measuring that each side of quadrilateral B (Figure 6) is one-third of the length of the corresponding side of quadrilateral A.

Trace the figure and the center of reduction in Figure 7 and reduce the figure to one-third its size.

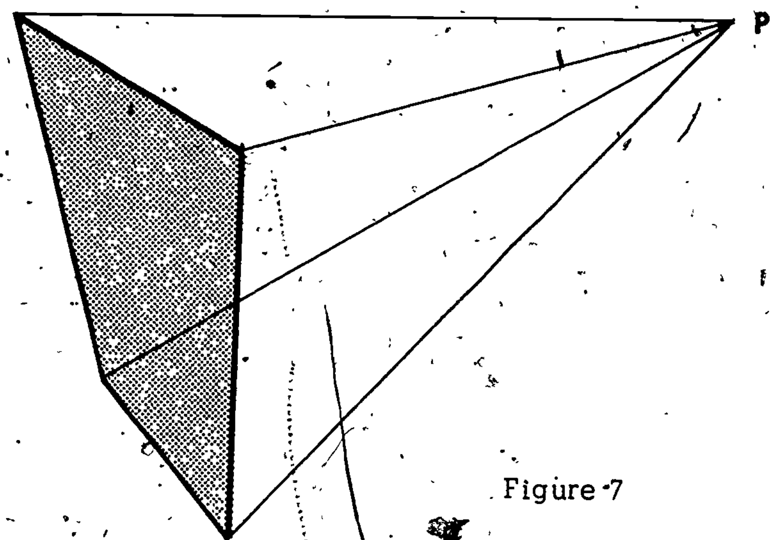


Figure 7

SECTION 4 THE SCALING FACTOR

When a figure is enlarged three times, all lengths of the original figure are multiplied by three. Similarly, when a figure is reduced to one-half its original size, all lengths of the original figure are multiplied by one-half. The number by which we multiply the lengths of the original figure is called the <u>scaling factor</u>:

$$\text{new length} = (\text{scaling factor}) \times (\text{original length})$$

This equation is read from left to right like any other sentence in English. However, when you carry out its instructions, you go from right to left. Thus, the equation says: "Take the original length, and multiply it by the scaling factor. The product is the new length."

Going from right to left in carrying out the instructions given by an equation is standard procedure in mathematics.

1. What scaling factors have you applied in your enlargements and reductions so far?

2. An enlargement of a triangle is made using a scaling factor of 5. If one side of the original triangle is 9 cm long, what is the length of the corresponding side of the enlargement?

3. A scaling factor of 5 is used in making an enlargement of a snapshot. If a tree in the enlargement is 15 cm tall, how tall was the tree in the snapshot from which the enlargement was made?

4. Figure 8 shows three kites.

(a) If the kite in Figure 8(a) is the original, what scaling factors were used to draw Figures 8(b) and 8(c)?

(b) If the kite in Figure 8(b) is the original, what scaling factors were used to draw Figures 8(a) and 8(c)?

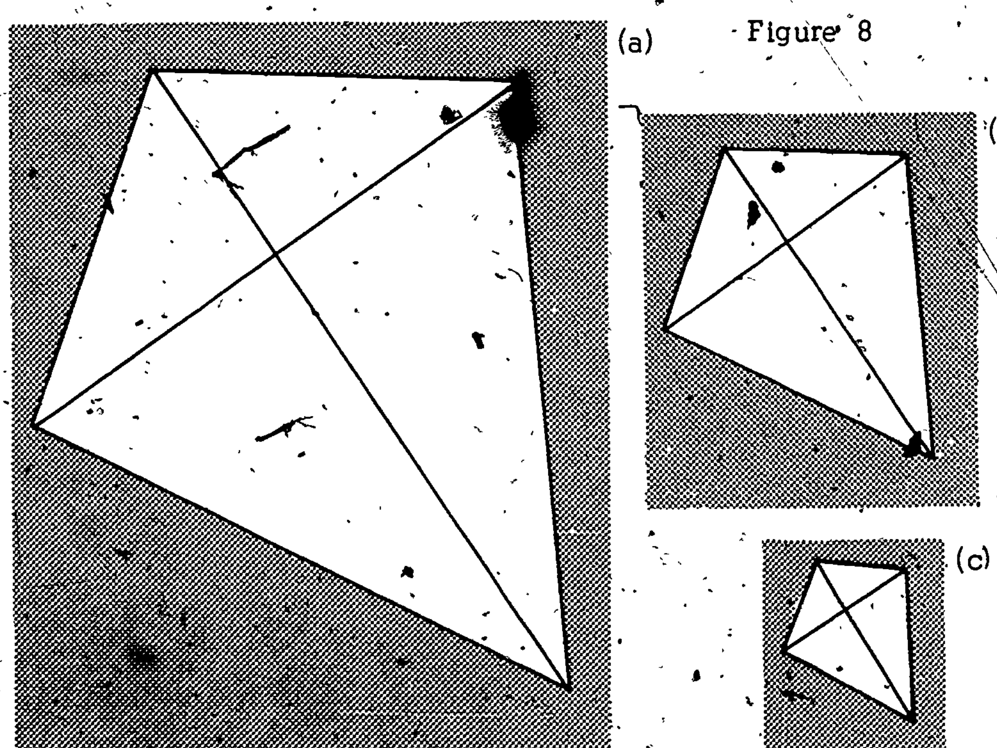


Figure 8

5. A car on a photographic slide is enlarged by a slide projector by a factor of 47. What is the length of the car on the slide if it is 94 cm on the screen?

6. For each pair of triangles in Figure 9 find the scaling factor from the small to the large triangle, and from the large to the small triangle.

7. How does the size of each angle in an enlargement or a reduction compare to each angle in the original? Use a protractor to compare corresponding angles in each pair of triangles in Figure 9.

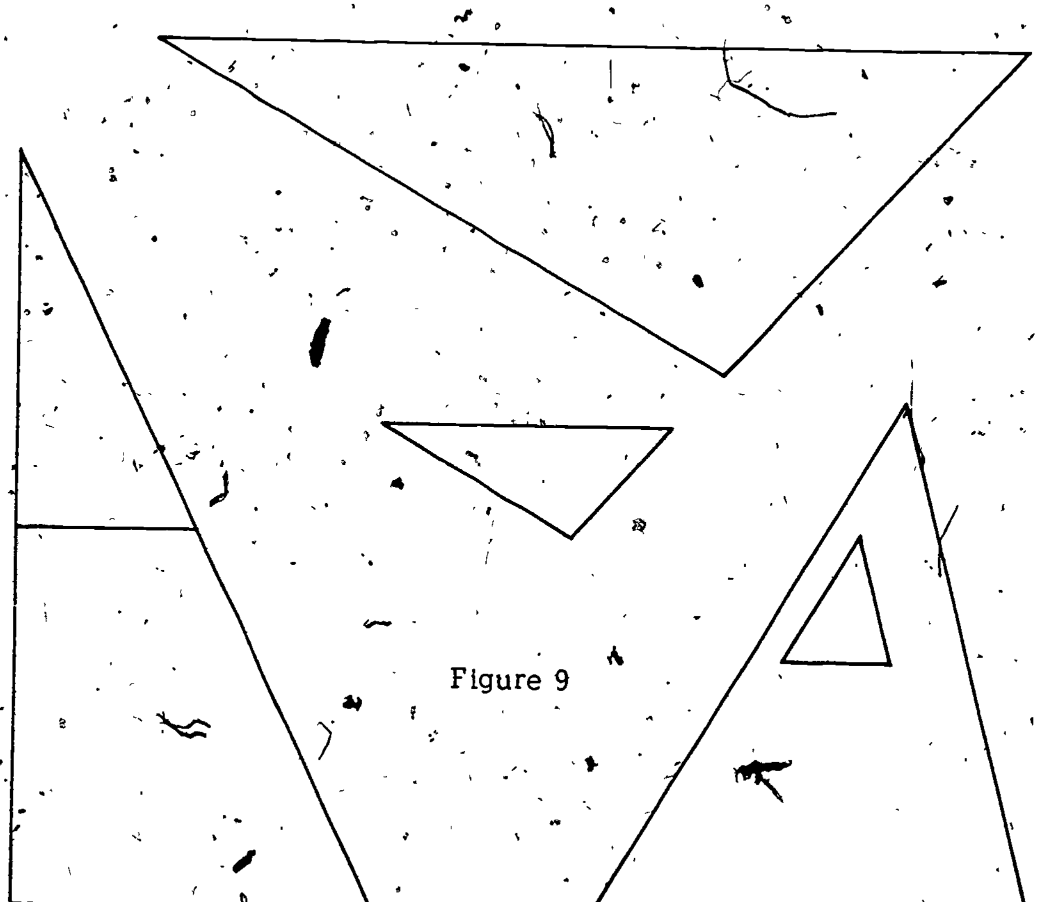


Figure 9

8. When you multiply the length of each side of the triangle in Figure 10(a) by three you get Figure 10(b). When you add 3 cm to the length of each side in Figure 10(a) you get the triangle in Figure 10(c).

(a) Are the angles in Figures 10(b) and 10(c) the same as the corresponding angles in Figure 10(a)?

(b) Are both Figures 10(b) and 10(c) enlargements of Figure 10(a)?

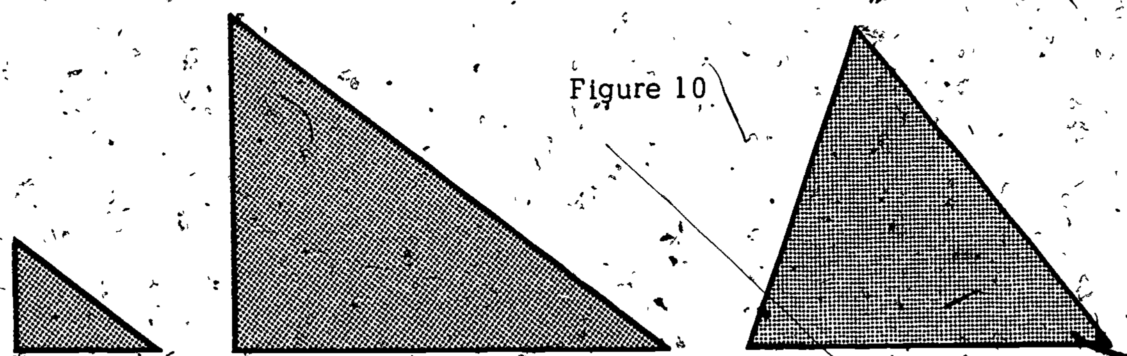


Figure 10

(a) (b) (c)

SECTION 5 A SIMPLE MULTIPLYING DEVICE

In Section 2, Figure 4, a triangle is enlarged using its top, P, as the center of enlargement. The enlarged side CD is parallel to the corresponding side AB.

Instead of beginning with a center of enlargement and enlarging one side to a corresponding side, we can reverse the process. We can take a pair of parallel segments and find a center of enlargement.

Figure 11(a) shows two parallel line segments. The first is 1 cm long; the second is 3 cm long. To find a center from which the cm segment enlarges to the 3 cm segment, we draw lines connecting the endpoints of the segments. The point of intersection of these lines, P, gives us the center we are looking for (Figure 11 (b)). The scaling factor for this enlargement is 3.

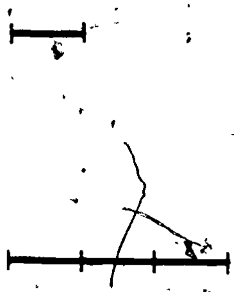
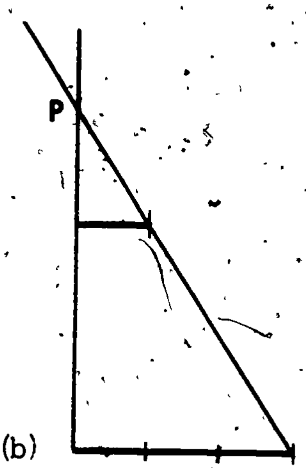


Figure 11

(a)



(b)

This center of enlargement has an interesting property. It can be used to enlarge segments of any length by a factor of 3, as long as the segments are extensions (or parts) of the original segments. Figure 12 shows the two parallel segments extended. We now draw a line from P that crosses both parallel lines (Figure 13). Then CG is three times as long as AF.

Figure 12

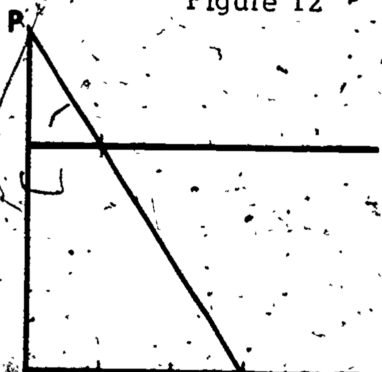
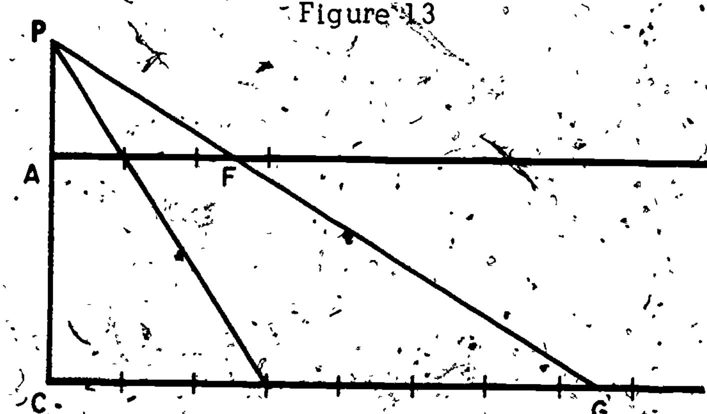
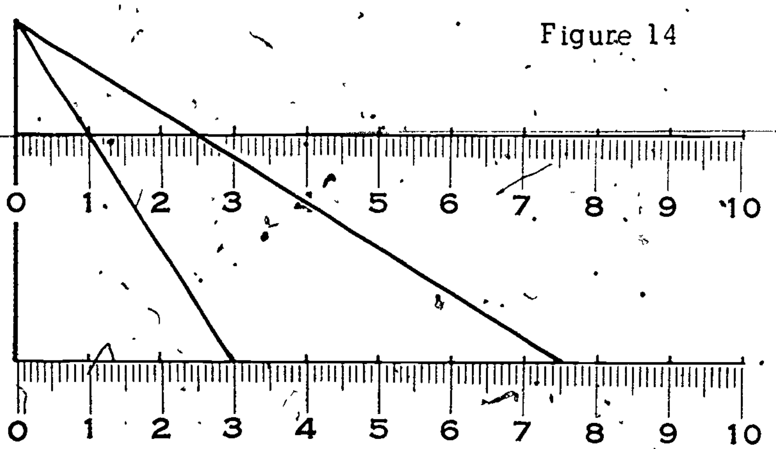


Figure 13



Since both lengths are expressed by numbers, we have constructed a device that multiplies numbers by a factor of 3. To make it easy to use we mark off scales on the two parallel lines. Figure 14 shows how we can use this device to find the product $3 \times (2.5)$. We can draw a straight line from P through the point labeled 2.5 on the upper scale. We continue the line until it hits the lower scale. This occurs at 7.5, which is the product $3 \times (2.5)$.

Figure 14



This process works because we are really enlarging segments so that

(new length) = (scaling factor) × (original length)

We can use this method of constructing a multiplying device for any multiplier. Suppose the multiplier is to be 2.5. You start with the two parallel scales. You connect the endpoints of the 0 to 1 segment on the upper scale with the ends of the 0 to 2.5 segment on the lower scale. Extend the lines until they cross, and you have the new center, P, for multiplying by 2.5.

9. Use a ruler on Figure 14 to find the products $3 \times (2.0)$, $3 \times (0.7)$, and $3 \times (2.9)$.

10. Use a ruler on Figure 14 to solve the equations

$$3 \times \square = 6.0, \quad 3 \times \square = 5.7, \quad \text{and} \quad 3 \times \square = 4.2$$

11. Construct a device to multiply any number between 1 and 5 by $\frac{3}{2}$.

12. One inch equals 2.54 cm. Construct a device to convert inches to centimeters.

13. The perimeter of a circle is given (very nearly) by the formula

$$\text{perimeter} = 3.14 \times (\text{diameter})$$

Construct a device to rapidly find the perimeter of any circle with a diameter between 0 and 5 cm.

14. In 1974 one German mark was equal to about 0.40 American dollars. Construct a device to convert prices in marks to prices in dollars. Use your device to obtain the price in dollars of an item that costs 96 marks?

15. Could you use the device of Question 14 to convert prices in dollars to prices in marks?

16. If the scale of Question 14 extended only to 10 marks, could it be used for conversions up to 100 marks? What would be the price in dollars of an item which costs 96 marks?

17. We connected the segment from 0 to 1 on the upper number line to the segment from 0 to 3 on the lower number line to get the center of enlargement in Figure 12. Are there other segments that could have been used to find this center? Illustrate with a drawing.

18. Is the distance between the two number lines important in the multiplying devices?

SECTION 6 · SIMILAR FIGURES: SCALE MODELS

Imagine that you have a photo and an enlargement of the photo. Certainly the enlargement remains an enlargement regardless of how the page on which it is printed is turned. Just by looking at two photos, you can tell by eye whether one is an enlargement of the other or not.

With simple geometric shapes, however, it is not always so easy to spot an enlargement. If a shape is turned or flipped over, it may be difficult to tell by eye whether one is an enlargement of the other.

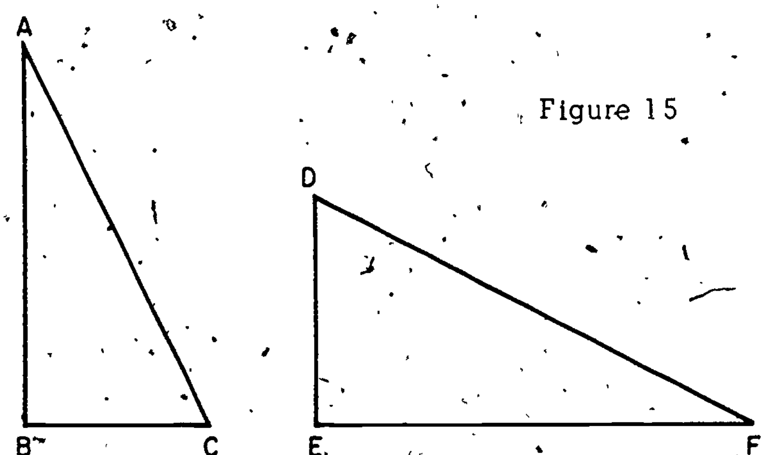


Figure 15

Is one triangle in Figure 15 an enlargement of the other? Apparently it is not. It appears as if AB is reduced to DE, while BC is enlarged to EF!

Actually triangle DEF is an enlargement of triangle ABC. To see that this is so, trace the two triangles, and cut out the copies. Move the copies around (flip over if needed) until you can see that the one is an enlargement of the other.

Two geometric figures, one of which can be shown to be an enlargement of the other, are called similar figures.

19. The pairs of shapes shown in Figure 16 are not similar. Explain why.

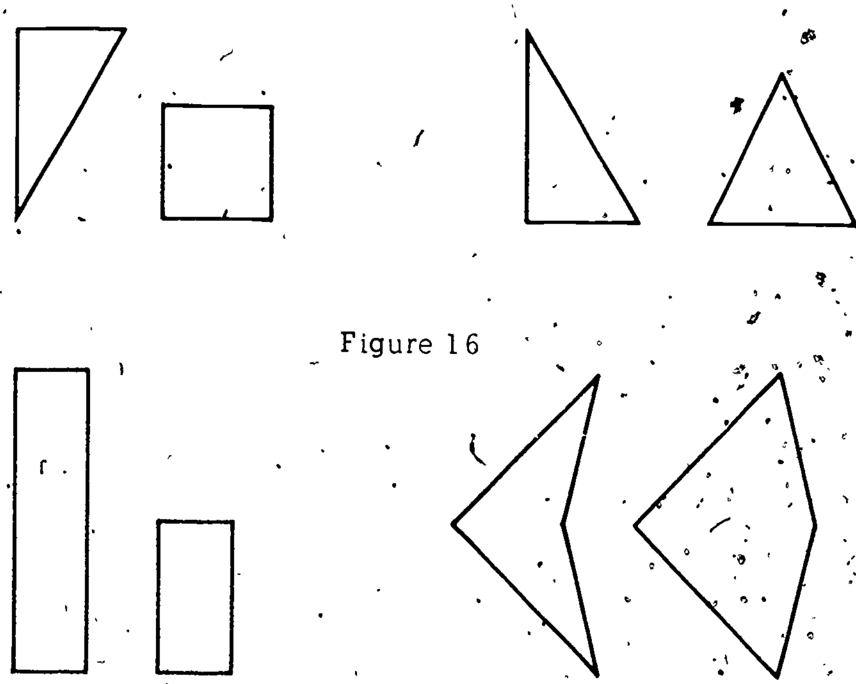


Figure 16

20. The pairs of shapes in Figure 17 are similar. Describe how to move one shape to make it look like an enlargement of the other.

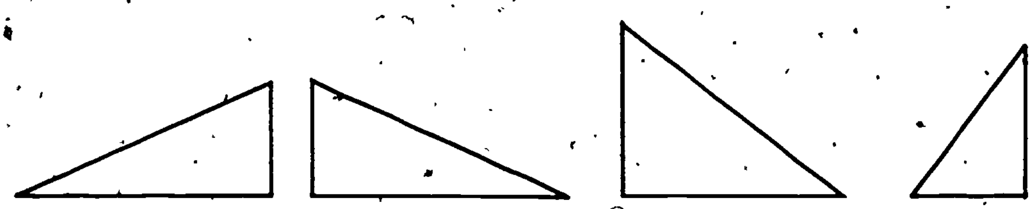
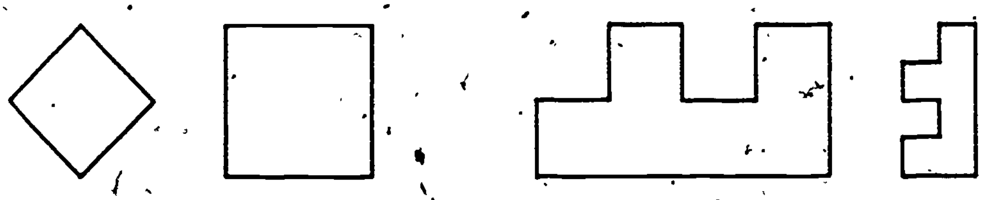


Figure 17



21. Which of the pairs in Figure 18 are pairs of similar figures? Trace and move to find a center of enlargement to test your answer.

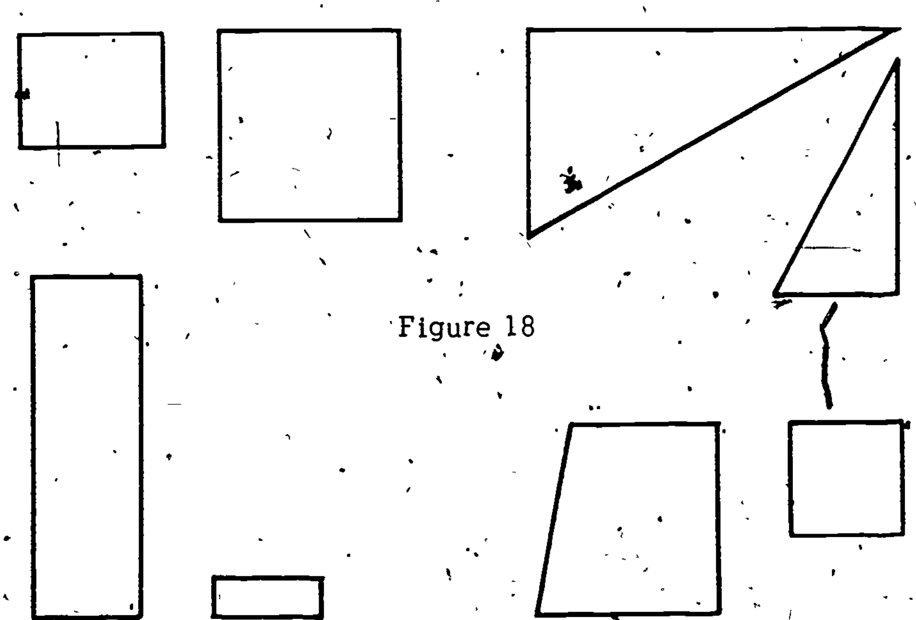


Figure 18

22. There are three triangles in Figure 19; two of these triangles are similar. Which two are they?

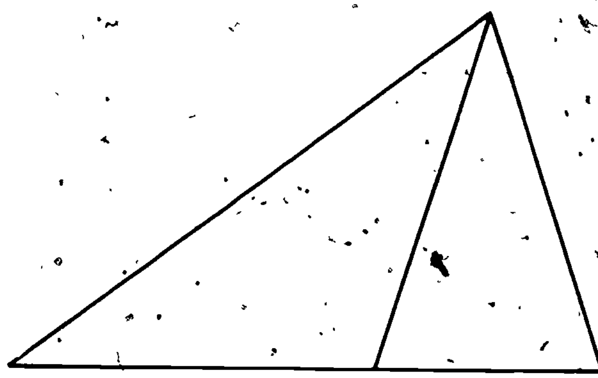


Figure 19

One often makes a small scale model of a ship or a building before it is constructed. This allows people to get an idea of what the real ship or building will look like.

By making length measurements on the scale model and multiplying them by the scaling factor, you can calculate lengths on the real thing.

23. (a) The length of a model car is 19 cm. The real car is 20 times larger. What is the length of the real car?

(b) If the distance from the front bumper to the windshield is 5 cm on the model, what is the same distance on the real car?

24. (a) A scale model boat is $\frac{1}{25}$ the size of the actual boat. If the length of the model is 75 cm, what is the length of the actual boat?

(b) If the height of the model is 26 cm, what is the height of the actual boat?

25. (a) An architect builds a model building which is $\frac{1}{48}$ of the actual size building. The width of the scale model of the building is 56 cm on each side. Can the life size building be constructed on a lot that is 24 meters wide?

(b) If the model has 12 floors, how many floors will the life size building have?

26. An architect builds a scale model home $\frac{1}{24}$ of the actual size. A 12 cm block is added onto the model home for the garage. Can the garage contain two cars?

27. Bouwen Building Company makes billboards that are 80 times bigger than the picture of the billboard on the cover of this book.

(a) About how many meters of wood are needed to make a frame that goes around the billboard?

(b) If diagonal braces are placed along the back of the billboard for extra strength, about how long must one diagonal brace be?

28. A boat is 9 m long, 12 m high, and 2 m wide. You have a scale model $\frac{1}{10}$ the size of the boat. What dimensions would you use to make a rectangular glass case to hold the model?

29. Trace Figure 20 and make four cut-out copies. Arrange the copies to form a larger similar figure.

30. There are eight different size triangles in Figure 21 that are similar to the smallest triangular tile. Can you find them all?

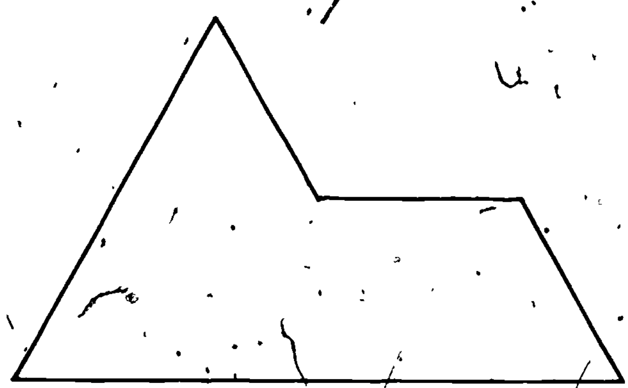


Figure 20

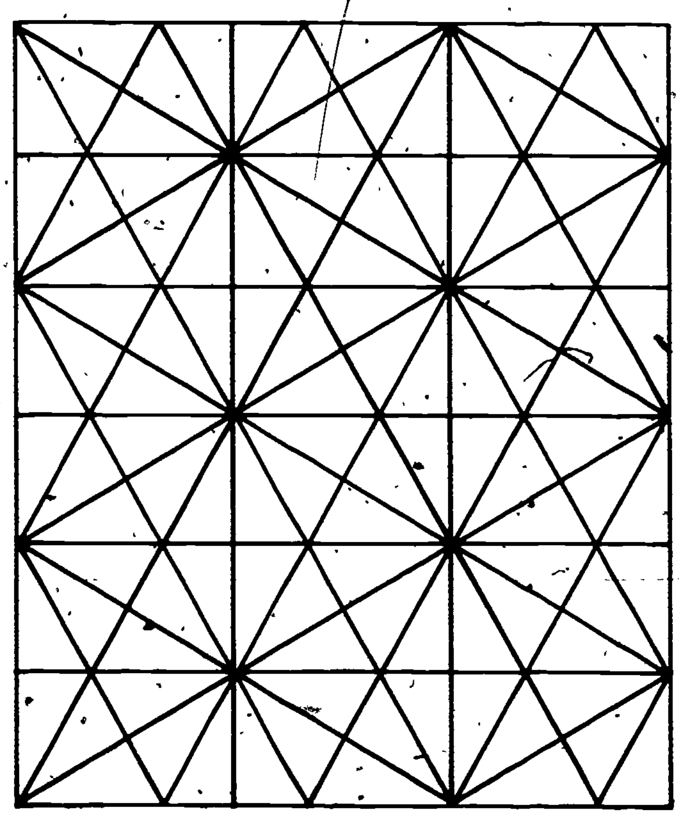
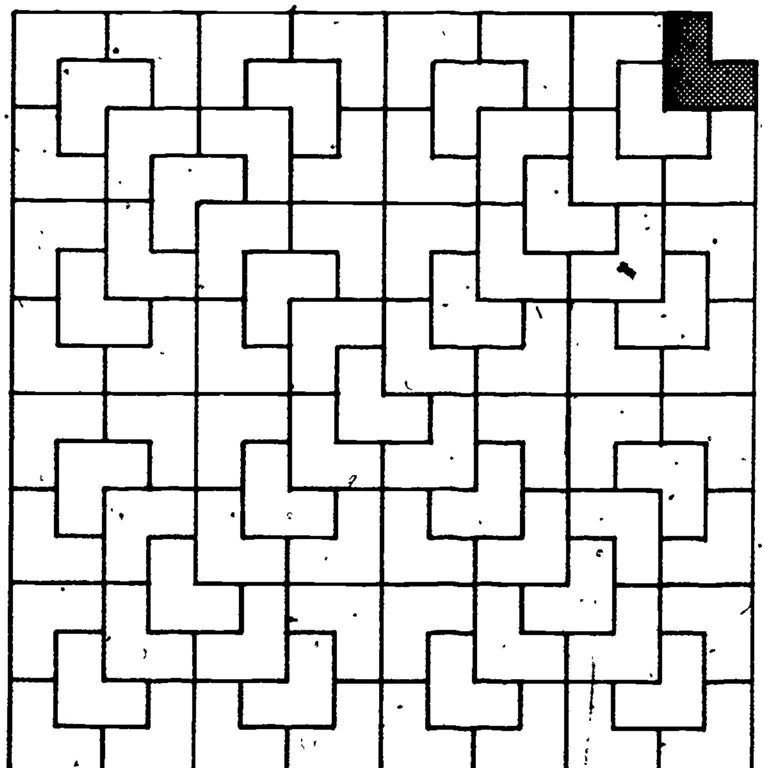


Figure 21

31. How many different size figures can you find that are similar to the small shaded tile in Figure 22?

Figure 22



SECTION 7 STAIRCASE MULTIPLICATION

If you make mistakes in multiplication because you cannot remember where to place the numbers on the paper, here is a way to help you. The idea is to draw "staircases" on paper before you begin to multiply, so that the numbers will automatically fall in the right places. You also do not have to worry about carrying except when you add up columns at the end of the multiplication.

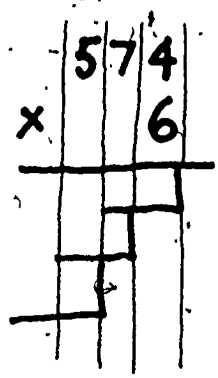
Suppose you wish to multiply 574 by 6. First write the numbers down as shown in Figure 23(a).

Figure 23 (a)

$$\begin{array}{r} 574 \\ \times \quad 6 \\ \hline \end{array}$$

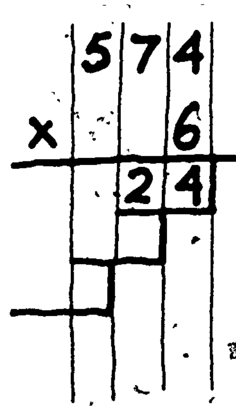
Next, make some vertical columns and draw a staircase as shown in Figure 23(b). The staircase has three steps because the top number, 574, has three digits.

(b)



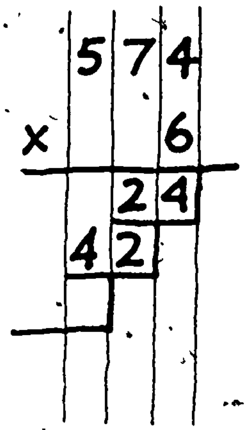
Now start multiplying from the right. First, multiply 4 by 6 to get 24 and place the number 24 on the top step as shown in Figure 23(c).

(c)



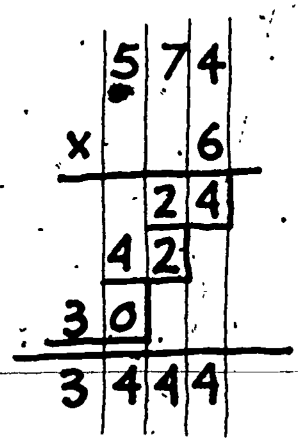
Next multiply 7 by 6 to get 42 and place it on the second stair as shown in Figure 23(d).

(d)



Finally, multiply 5 by 6, put the result, 30, on the third step and then add up the columns to get the answer (Figure 23(e)).

(e)

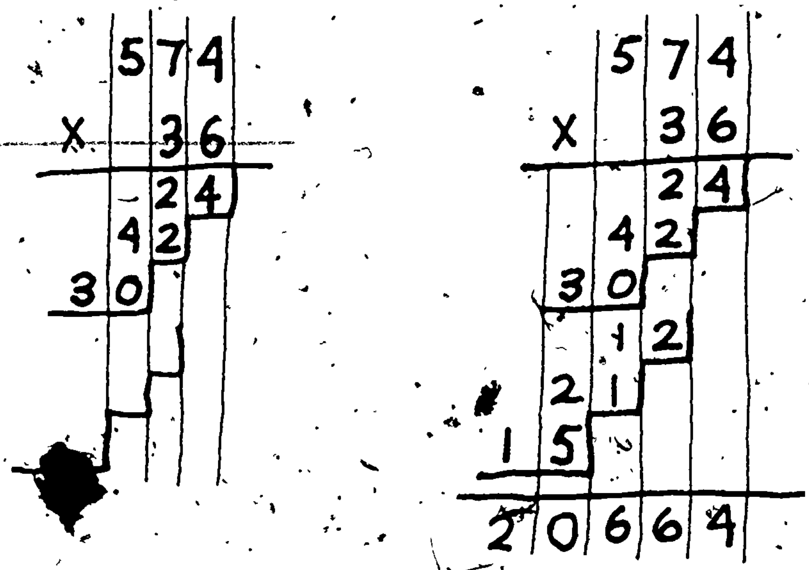


Suppose you wish to multiply 574 by 36, a two-digit number.

Because there are two digits in the bottom number, you need two staircases — one for each of the two digits. The first staircase starts in the column under the bottom digit on the right. The second staircase starts in the column under the bottom digit on the left.

Figure 24(a) shows the two staircases and the multiplication by the first digit on the right. This multiplication has already been shown in Figure 23(e). It is shown again in Figure 24(a), but the columns have not been totaled because the problem is not finished.

Figure 24



(a) (b)

In Figure 24(b), in addition to multiplication by the first digit, the multiplication by the digit on the left is shown. It is done the same way as before. First, you multiply 4 by 3 and put the product, 12, on the first step of the second staircase as shown in Figure 24(b). Then you place the product of $3 \times 7 = 21$ on the second step and the product $3 \times 5 = 15$ on the last step.

Adding the columns gives the answer.

Use staircase multiplication to find the answers to the following questions.

32. Which would you rather have, nine quarters or 275 pennies?

33. In doing the multiplication

$$\begin{array}{r} 375 \\ \times\ \underline{24} \end{array}$$

(a) Why will you use two staircases?

(b) Why will each staircase have three steps?

(c) Do the multiplication and have a classmate check your answer.

34. Multiply 34 times 56.

35. Which would you rather have, 55 quarters or 265 nickels?

36. Find the product of 574 × 306.

37. A newspaper press prints 735 newspapers in one minute. How many does it print in an hour?

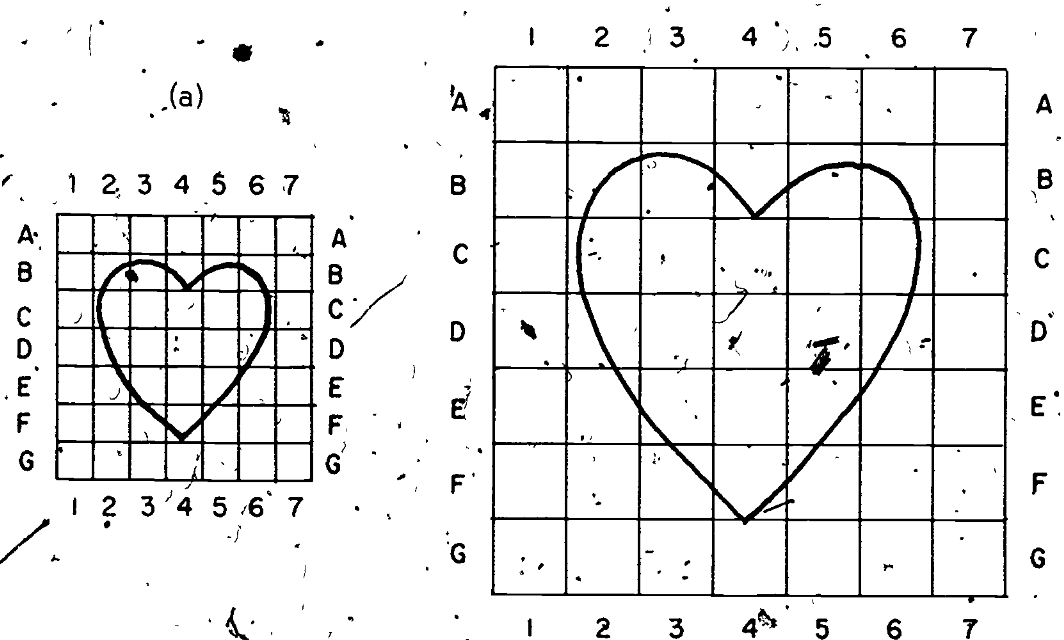
SECTION 8 GRID ENLARGEMENTS

Here is a way to make enlargements and reductions without using a center of enlargement. We draw a grid on the figure we wish to enlarge as shown in Figure 25(a).

The enlargement in Figure 25 was made by copying the part of the heart in each square of the small grid in Figure 25(a) onto the corresponding square of the larger grid (Figure 25(b)). The numbers and letters on the grid are to make it easier to find each square on the large grid that corresponds to each square on the small grid.

Figure 25

(a) (b)



Homemakers occasionally copy designs for stuffed toys and pillows by this technique. First they enlarge the design onto newspaper, and then they cut through the newspaper and cloth at the same time to get the pieces that can be sewed together to make the toy or pillow.

Enlarge the pattern for making a stuffed elephant that is shown in Figure 26. Begin by making an enlarged grid on a sheet of graph paper. Label the grid squares as in Figure 25. Copy each square of the picture on the corresponding enlarged square of your grid.

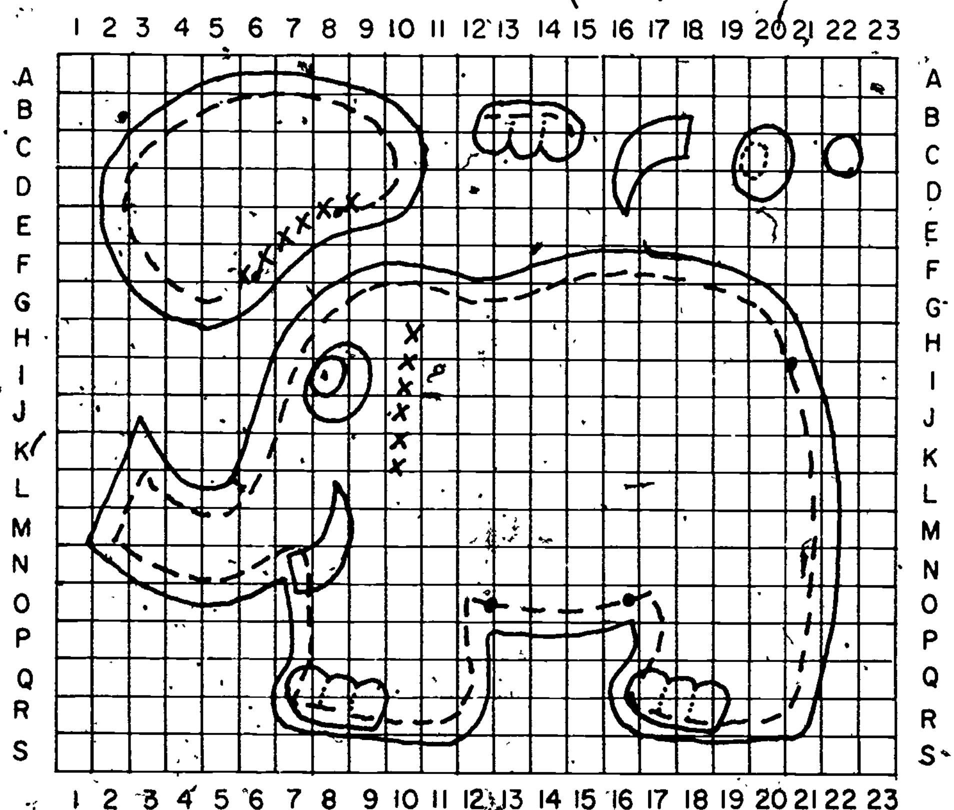


Figure 26

5. READING MAPS

SECTION 1 A FLOOR PLAN: A KIND OF MODEL

Before people construct a building they draw a reduced floor plan to show size of rooms, corridors, etc. Figure 1 is such a floor plan. It shows part of the building where this book is being written.

A floor plan and the real thing are not similar figures in every detail. All the lengths and widths of room and corridors are scaled reductions. The thickness of the walls is also reduced by the same scaling factor. However, the doors and windows themselves are simply indicated by lines and pairs of lines, giving their widths without regard to their thickness.

The scaling factor used in drawing Figure 1 is $\frac{1}{100}$. That is, 1 m or 100 cm in the real building appears as $\frac{1}{100} \times 100$ cm = 1 cm in the drawing. A length of 2 m or 200 cm in the real building will appear as $\frac{1}{100} \times 200$ cm = 2 cm, and so on. Thus, a scaling factor of $\frac{1}{100}$ is very convenient; the multiplication by $\frac{1}{100}$ is carried out simply by replacing meters in the real building by centimeters on the drawing.

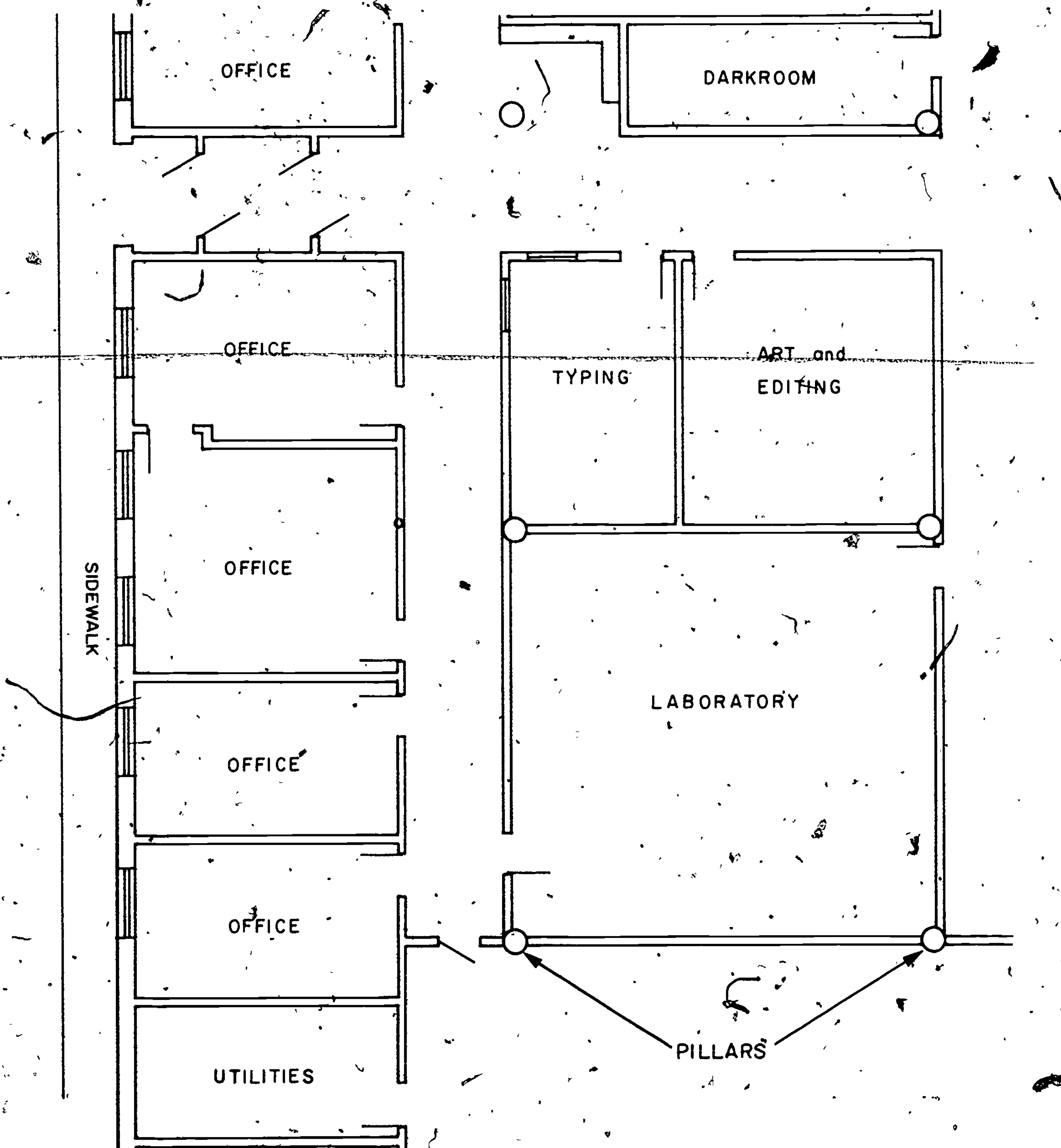

OFFICE
DARKROOM
OFFICE
TYPING
ART and EDITING
OFFICE
SIDEWALK
LABORATORY
OFFICE
OFFICE
UTILITIES
PILLARS

Figure 1

Going from the drawing to the real building we have an enlargement by a scaling factor of 100. That is, every centimeter on the drawing equals 100 cm or 1 m in the building. Here the multiplication by 100 is done by replacing centimeters on the drawing by meters in the real building. This is true for any length in centimeters, whole numbers or fractions.

Trace the floor plan of the Laboratory in Figure 1. Suppose the Laboratory in the floor plan is to be used as a classroom. Measure the length and width of your teacher's desk and your desk in meters. How large would they be on the floor plan? Make pencil sketches or cut out the scaled pieces of furniture to show how you would arrange the classroom. (Don't forget the aisles.)

The typing office has two desks 1.50 m long and 0.75 m wide, and two typewriter tables 0.75 m long and 0.50 m wide. Each desk has a typewriter table beside it. How would you arrange the desks and tables in the most practical way?

?

1. A table is 1.5 m long and 0.8 m wide. How long and wide would you draw it on the floor plan?

2. A window is 1.3 cm wide on the drawing. How wide is the real window?

3. How thick are the inner walls in the building? The outer walls? The pillars?

4. The darkroom has a 0.75 m wide counter along one wall. If a similar counter is placed on the opposite wall, how wide will the aisle between them be?

5. One office has two outside windows. What is the distance between them?

6. Could three display tables 0.80 m wide and 1.80 m long be placed in the hall next to the darkroom and still allow space for walking? How much?

7. The L-shaped figure next to the darkroom is a bench. About how many students will it seat?

8. A student wishes to draw a plan of his neighborhood. The area in which he is interested is a rectangle of length 250 m and width 150 m.

   (a) How large a sheet of paper does he need if he chooses a scale of $\frac{1}{100}$?

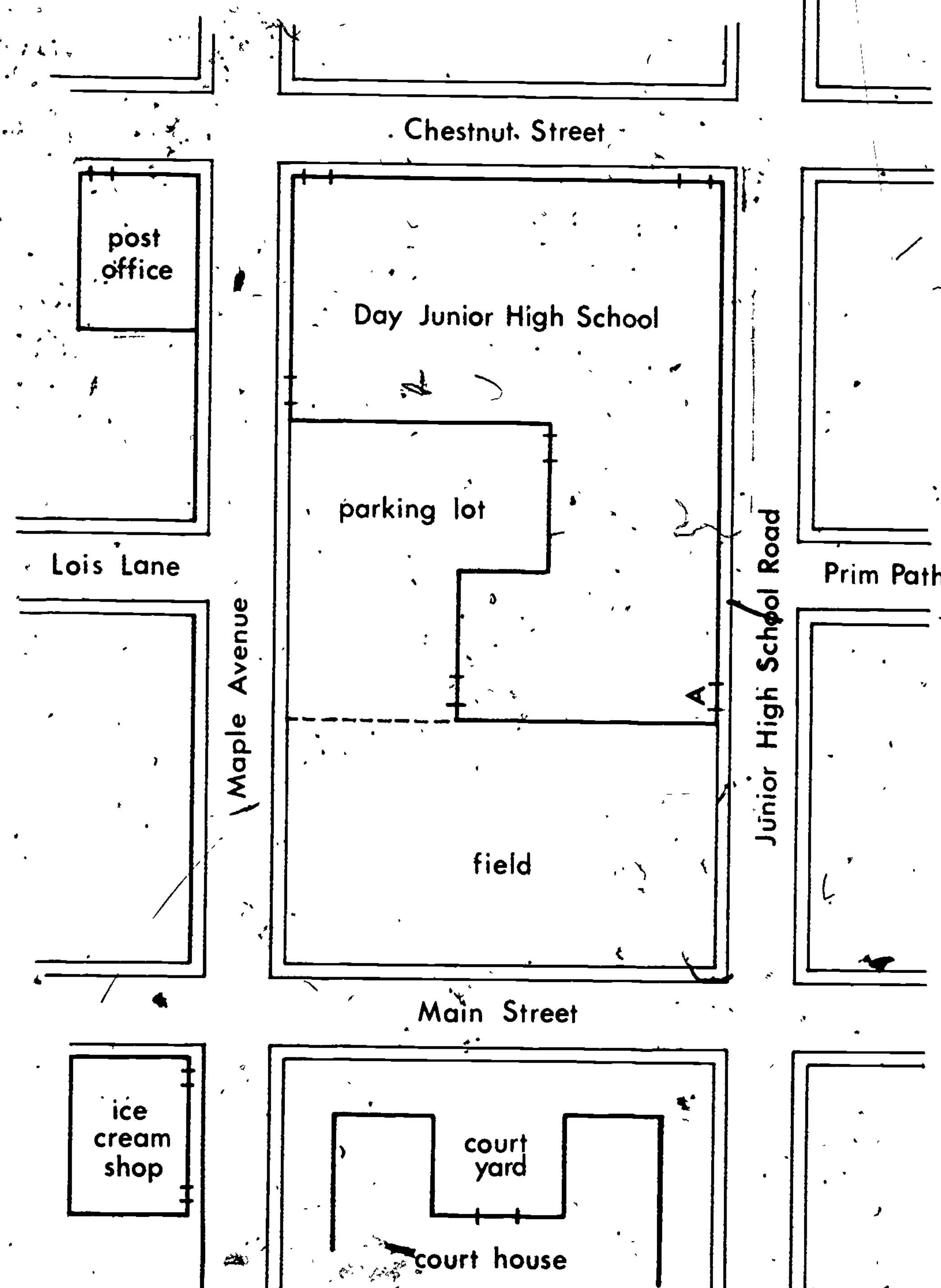
   (b) Would a scaling factor of $\frac{1}{1000}$ be small enough to get the whole neighborhood on a sheet in your notebook?

SECTION 2 A SCALED MAP OF A NEIGHBORHOOD

To get the floor plan of part of a building (Figure 1) on a page of this book, a reduction by a scaling factor of $\frac{1}{100}$ was needed. To get a scaled plan of a larger area on a page of the same size will require a larger reduction. That is, the lengths on the drawing must be less than $\frac{1}{100}$ of the real lengths.

Figure 2 is a scaled map of Day Junior High School and other buildings nearby. Just as in the case of the floor plan (Figure 1) this map and the real thing are not similar figures in every detail. The lengths and widths of buildings, sidewalks and streets are scaled reductions. However, trees, lamp posts, stairs, and other details are not shown at all.

Figure 2


Chestnut Street
post office
Day Junior High School
parking lot
Lois Lane
Maple Avenue
A
Junior High School Road
Prim Path
field
Main Street
ice cream shop
court yard
court house

The scaling factor used in drawing Figure 2 is $\frac{1}{1200}$. Thus, to find a real length we have to measure the corresponding length on the map and multiply it by 1200. Multiplying a length in centimeters by 1200 is best done in two steps; first multiply by 100 and then by 12. The multiplication by 100 can be done by replacing centimeters by meters. Now you are ready for the second step, which is to multiply by 12 to get the real length.

Here is an example: the length of the post office (along Maple Avenue) on the drawing is 2.3 cm. How wide is the real building? Multiplying 2.3 cm by 100 is done by replacing 2.3 cm by 2.3 m. Now multiply by 12 which gives $12 \times 2.3 = 27.6$ m.

?

9. (a) How wide is Main Street?
(b) Will a marching band with eight people abreast be able to march down Main Street?

10. How far is it from door A on the Junior High School to the door of the post office going along the sidewalk on Junior High School Road and Chestnut Street?

11. (a) What is the shortest distance from door A to the ice cream shop?
(b) How many meters do you save by going diagonally across the field rather than along the sidewalks?

12. If you run around the block where Day Junior High School is located, how far will one trip be?

13. Can you make a 150 m loop around the edge of the parking lot with your bike? How about a 250 m loop?

14. A football field measures 50 yds by 120 yds. This is about 48 m by 108 m. Will one fit in the space marked "field"?

15. How many meters of fencing would be needed to enclose the field?

16. Could Chestnut Street be a two-lane street with parking on both sides?

17. Which is wider, the sidewalk outside your school, or the sidewalk on Chestnut Street? By how much?

18. In Figure 2 the block between Junior High School Road and Maple Avenue is 7.0 cm long. What is the real distance expressed in centimeters? Would you ever express such a large distance in centimeters? Why or why not?

SECTION 3 METERS AND DOLLARS

Lengths on a scaled map are conveniently expressed in centimeters. Lengths of building or distances along streets are expressed in meters. When enlarging from a neighborhood map to the real world, we change from centimeters to meters. This saves us the trouble of handling large numbers. We can avoid large numbers in a similar way when multiplying amounts of money expressed in cents. The reason is that 1 cent is $\frac{1}{100}$ of a dollar, just as 1 centimeter is $\frac{1}{100}$ of a meter. Here is an example.

A pencil sells for 14 cents. A box contains 200 pencils. How much does the box of pencils cost? The 200 acts like a scaling factor. To find the cost of 200 pencils, we multiply the cost of one pencil by 200.

As we did in Section 2 we shall do the multiplication in two steps; first multiply by 100 and then by 2. The multiplication by 100 is done by replacing cents by dollars. Now multiply by 2 to get the answer.

$$200 \times 14 \text{ cents} = 2 \times 14 \text{ dollars} = 28 \text{ dollars}$$

19. Express in dollars

    (a) 600×4¢

    (b) 300×7¢

    (c) 1000×19¢

    (d) 6700×80¢

    (e) $14,000 \times 33$¢

    (f) $31,000 \times 74$¢

20. Find these products in dollars

    (a) 600×2.5¢

    (b) 300×1.2¢

    (c) 400×0.2¢

    (d) 1000×0.3¢

    (e) $200 \times \frac{1}{4}$¢

    (f) $1200 \times \frac{1}{3}$¢

21. If BIC pens sell for 19 cents each, how much will 1200 pens cost?

22. At a school book store, ruled paper costs 0.25¢ for each sheet when you buy 100-sheet pads.

    (a) How much does each pad cost?

    (b) How much will it cost for your class to have one pad (of 100 sheets) for each student?

23. Nails are sold in large numbers. Each nail may cost less than one cent. A certain nail costs 0.3 cent.

(a) How much will a bag of 15,000 cost?

(b) How many nails can be bought with three cents? With three dollars?

SECTION 4 APPROXIMATING PRODUCTS OF WHOLE NUMBERS

Mistakes in multiplication happen. Therefore it is useful to be able to spot at least those which produce unreasonable results.

For example, is it possible that $23 \times 380 = 1900$? To find out round off 23 to 20. Similarly round off 380 to 400. The product 23×380 must be close to 20×400. Here is a good way of multiplying 20×400 without paper and pencil. Start with $2 \times 4 = 8$. Now, what about the zeros?

The zeros in each number tell us whether the digit in front of them is in the tens, hundreds, or thousands place. Thus the 2 in 20 stands for 2 tens. The 4 in 400 stands for 4 hundreds. To complete the multiplication of 20×400 we note that (tens) $\times$ (hundreds) = thousands. Therefore $20 \times 400 = 8000$, and we expect 23×380 to be close to 8000. Clearly 23×380 cannot possibly equal 1900.

Here are the main steps to use:

(1) Round off each of the factors to get one non-zero digit and the correct number of zeros.

(2) Multiply the two non-zero digits.

(3) Use the number of zeros in each factor to tell you the place value of each non-zero digit.

(4) Use Table 1 to find the place value of the product.

TABLE 1

X	ones	tens	hundreds	thousands
ones	ones	tens	hundreds	thousands
tens	tens	hundreds	thousands	ten thousands
hundreds	hundreds	thousands	ten thousands	hundred thousands
thousands	thousands	ten thousands	hundred thousands	millions

Here is another example: What is an approximate answer to 112 × 5200?

112 rounds off to 1 hundred

5200 rounds off to 5 thousand

1 × 5 = 5 (hundreds) × (thousands) = hundred thousands

Hence 112 × 5200 is roughly 500,000.

Make a table like Table 1 using numbers instead of words.

24. Round off the following numbers to one non-zero digit and the correct number of zeros:

(a) 782

(b) 780

(c) 843

(d) 103

(e) 91

(f) 85

25. Round off the following numbers to one non-zero digit and the correct word for place value.

(a) 6432

(b) 86100

(c) 47

(d) 978

(e) 945

(f) 3

26. Approximate the following products. (Try to do it in your head.)

(a) 43×22

(b) 19×51

(c) 238×37

(d) 6342×149

(e) 3340×2700

(f) 965×3009

27. The number 1500 can be read as "one thousand five hundred" or as "fifteen hundred." Similarly, read

(a) 2800 as "---- thousand ---- hundred" or as "---- hundred".

(b) 15,000 as "---- thousand" or as "---- hundred".

(c) 60,000 as "---- thousand" or as "---- ten thousand".

(d) 700,000 as "---- thousand" or as "---- hundred".

(e) 3,200,000 as "---- million ---- hundred thousand" or as "---- hundred thousand".

Examples:

(1) $37.5 \times 4.81 = ?$

Ignoring the decimal point we have

$$375 \times 481 = 180375$$

Approximating the product gives $40 \times 5 = 200$. Hence we place the decimal point so that the answer is close to 200; that is $37.5 \times 4.81 = 180.375$.

(2) $40.6 \times 0.021 = ?$

Without decimals $406 \times 21 = 8526$. Roughly, the factors are 40 and 0.02. From Table 2, (tens) × (hundredths) = tenths. Hence the approximate answer is 8 tenths or 0.8; and the exact answer must be 0.8526.

TABLE 2

X	ones	tenths	hundredths
ones	ones	tenths	hundredths
tens	tens	ones	tenths
hundreds	hundreds	tens	ones
thousands	thousands	hundreds	tens

Make two tables like Table 2 replacing the words with numbers. In one table use fractions $(\frac{1}{10}, \frac{1}{100})$ and in the other use decimals (0.1, 0.001).

Examples:

(1) $37.5 \times 4.81 = ?$

Ignoring the decimal point we have

$$375 \times 481 = 180375$$

Approximating the product gives $40 \times 5 = 200$. Hence we place the decimal point so that the answer is close to 200; that is $37.5 \times 4.81 = 180.375$.

(2) $40.6 \times 0.021 = ?$

Without decimals $406 \times 21 = 8526$. Roughly, the factors are 40 and 0.02. From Table 2, (tens) × (hundredths) = tenths. Hence the approximate answer is 8 tenths or 0.8; and the exact answer must be 0.8526.

TABLE 2

X	ones	tenths	hundredths
ones	ones	tenths	hundredths
tens	tens	ones	tenths
hundreds	hundreds	tens	ones
thousands	thousands	hundreds	tens

Make two tables like Table 2 replacing the words with numbers. In one table use fractions $(\frac{1}{10}, \frac{1}{100})$ and in the other use decimals (0.1, 0.001).

?

31. Explain in your own words why
    (a) (hundreds) × (tenths) = tens.
    (b) (thousands) × (hundredths) = tens
    (c) (thousands) × (tenths) = hundreds

32. Explain in your own words why
    (a) (tens) × (hundredths) = tenths
    (b) (tens) × (thousandths) = hundredths

33. Calculate the following products. Use Table 2 if you wish.
    (a) 7 × 0.4
    (b) 20 × 0.03
    (c) 80 × 0.05
    (d) 600 × 0.1
    (e) 2000 × 0.04

34. Where do the decimal points go?
    (a) 4.9 × 5.7 = 2793
    (b) 82 × 3.7 = 3034
    (c) 9.6 × 0.54 = 5184
    (d) 11.7 × 0.46 = 5382
    (e) 0.88 × 4.5 = 3960
    (f) 19 × 0.017 = 323

35. In the following products all the digits are correct. However, in some cases the decimal point is misplaced. Find them.

(a) $3.56 \times 0.03 = 0.1068$

(b) $560 \times 0.8 = 44.8$

(c) $0.04 \times 9.6 = 3.84$

(d) $800.0 \times 0.025 = 2$

(e) $1.6 \times 3.8 = 6.08$

(f) $0.06 \times 12 = 0.72$

36. Find the values of the following products:

(a) 360×0.15

(b) 2.2×0.06

(c) 7.6×0.58

(d) 5.7×8.6

(e) 105×0.027

(f) 2100×0.3

37. Without doing the actual multiplications, which of the following products have the same value?

$\frac{35}{10} \times 2.6$, $\quad 35 \times \frac{26}{1000}$, $\quad 0.35 \times 26$, $\quad 3.5 \times \frac{26}{100}$

Separate into about six teams of equal size. Your teacher will give you a list of products like those in Question 37. Some products are equal to others. Group equal products together. You will receive one point for each correct group. The team with the highest score wins.

SECTION 6 PLAN-A-PARTY

Use the prices from the price list to answer the following questions.

38. You plan a party for five people. Each person has a 12-oz Coke. How much will the Cokes cost?

39. How much do you save by buying the 8-pack rather than eight single cans of Coke?

40. There are 25 persons. You wish to have two hot dogs for each of them. How much will the hot dogs cost?

41. How much will you spend on six quarts of lemonade?

42. Which is cheaper to serve for a party of five, hot dogs or hamburgers? How much cheaper?

43. If you were planning a party for twice as many people, would you have to buy twice as much of everything?

Plan a party for your whole math class. Get prices from local stores. Compare prices to decide on the best buys for each item. Find the total cost. How much will have to be spent on items that are not edible? What will the cost be for each student if the total cost is shared equally?

PRICE LIST

Coke	22¢ for one 12-oz can
	$1.61 for eight 12-oz cans
	68¢ for one 48-oz bottle
Potato chips	59¢ for eight $\frac{1}{2}$-oz packages
	79¢ for one 10-oz bag
Lemonade mix	35¢ for two 1-quart mixes
Hot dogs	$1.00 for 10
Hamburger patties	$2.19 for 12
Hot dog rolls	42¢ for six
Hamburger rolls	52¢ for eight
Ketchup	55¢
Relish	40¢
Mustard	29¢ for 9 oz
	47¢ for 24 oz
Ice cream bars	89¢ for six
Paper plates	49¢ for 40
Napkins	25¢ for 70
Paper cups	85¢ for 100

SECTION 7 THE KILOMETER

When it comes to expressing lengths longer than a few city blocks, a larger unit than a meter is convenient. A common unit for this purpose is the kilometer, which is 1,000 meters. It is abbreviated by km: (1 km = 1,000 m).

To develop a feeling for the length of a kilometer, you can do a variety of things. Here are a few suggestions. Try them and report the result to the class.

(1) If you have a long, straight street near your school, your teacher will assign groups of students to post themselves at 200 m intervals up to 1 km. This will give you the opportunity to note how persons, cars, windows, etc. appear to you at different distances. In particular, at which distances can you still see arms and legs of a person, wheels of a car, or doors and windows of a building?

(2) There is a way of making things appear farther away than they really are. Look "the wrong way" through a telescope or one lens of a pair of binoculars. The telescope acts as a scaling device. For example, a telescope with a magnifying power of 7 will make objects appear seven times farther away when viewed "the wrong way." A person standing 150 m away from you will appear as being 7 × 150 m, or just about 1 km away.

(3) If your school has a field with a track, find out what the perimeter of the track is. Calculate how many times you have to walk around it to cover 1 km. Walk 1 km at your normal pace and time it. You can then think of a kilometer as the distance you walk in so many minutes.

SECTION 8 A ROAD MAP

A road map is commonly used by persons making trips by car or bus. Such a map often covers whole states or several states. To get a scaled map of such a large area on a page will require much larger reductions than a neighborhood map.

Figure 3 is a portion of a scaled map of the state of Michigan. (We purposely left out many small towns and country roads.) The scaling factor used in drawing Figure 3 is $\frac{1}{1,000,000}$. Thus, to find a real length we have to multiply a measure on the map by 1,000,000! For the neighborhood map (Section 2), the scaling factor was 1200. There we did the multiplication in two steps – by 100 and by 12. Here we shall break up the multiplication by 1,000,000 into three steps – first by 100, then by 1000, and finally by 10.

Suppose we measure a length of 1 cm on a map for which the scaling factor is $\frac{1}{1,000,000}$. To what distance does this correspond? One centimeter multiplied by 100 gives 1 m. In the next step, 1m multiplied by 1000 gives 1 km. We have left only the multiplication by 10. Therefore, 1 cm on the map corresponds to 10 km of real distance. In short

$$1,000,000 \times 1 \text{ cm} = 10,000 \times 1 \text{ m} = 10 \times 1 \text{ km} = 10 \text{ km}$$

Once we have this result we can use it directly without going through the in-between steps. For example, the distance between the two markers on Route 21 in Figure 3 is 1.3 cm. The real distance is

$$10 \times 1.3 \text{ km} = 13 \text{ km}$$

44. What is the real distance corresponding to 0.1 cm in Figure 3?

45. What is the (real) distance between the intersection of Routes 75 and 23 and the intersection of Routes 75 and 78?

46. When speaking of maps, the distance between two points may have one of two meanings: (1) the distance "as the crow flies," that is, the distance along a straight line, (2) the distance along a given route.

Find the distance between the intersection of Routes 21 and 94 and the intersection of Routes 21 and 78

(a) as the crow flies.

(b) along Route 21.

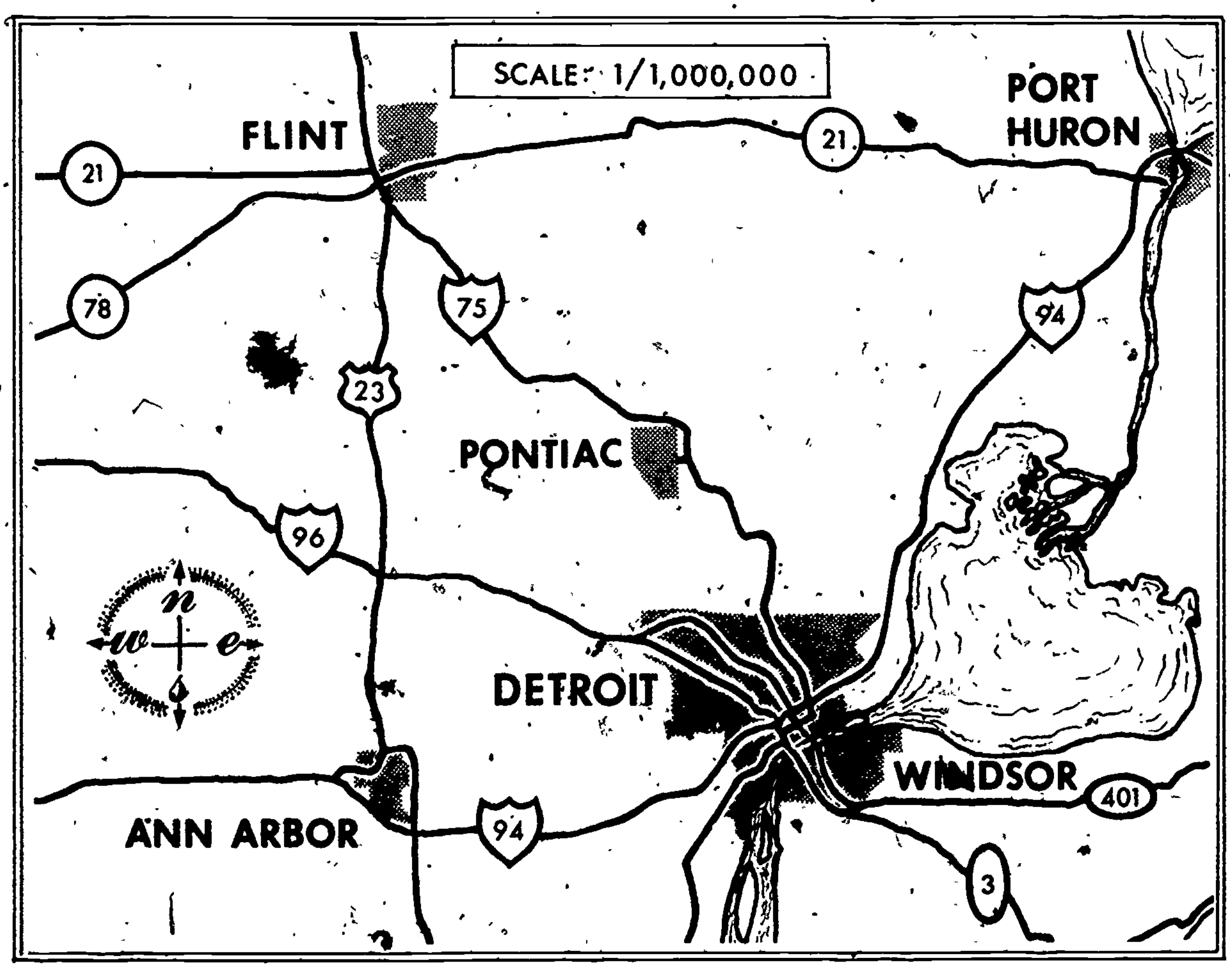


Figure 3

47. Find the distance between the intersection of Routes 23 and 75, and the intersection of Routes 23 and 96

(a) along a straight line

(b) along Route 23.

48. Look at your answers to Questions 46 and 47: Under what conditions is the difference between the distance along the road and along a straight line unimportant? Important?

SECTION 9 BRACKETING DISTANCES ON A MAP

So far we have used Figure 3 to find only distances between intersections of roads. What about distances between cities? Cities, unlike road intersections, often cover large areas. In Figure 3 the rough shape of the cities is indicated by shaded areas. Note also the directions of north, east, south, and west.

?

49. Suppose you fly over Detroit. What is the largest distance you fly over the city in a straight line?

50. (a) What is the distance from the north-west corner of Detroit to the south-east corner of Detroit?

(b) What is the distance from the north-west corner of Pontiac to the south-east corner of Pontiac?

(c) What is the distance from the south-east corner of Pontiac to the north-west corner of Detroit?

(d) Does it make sense to speak of "the distance from Detroit to Pontiac"?

51. (a) How long is the part of Route 21 that goes through Flint?

(b) How far is it from the intersection of Routes 94 and 21 along Route 21 to the eastern end of Flint?

(c) Use your answers to parts (a) and (b) to bracket the distance from the intersection of Routes 94 and 21 to Flint.

52. (a) How "wide" is Port Huron?

(b) Use your answers to part (a) of this question and part (a) of Question 51 to bracket the distance from Port Huron to Flint.

53. Bracket the distance from Flint to Ann Arbor along Route 23.

54. Bracket the distance from Port Huron to Detroit.

SECTION 10 UNCERTAINTIES IN MAP READINGS

Figure 3 was drawn with a scaling factor of $\frac{1}{1,000,000}$. Starting from Figure 3 draw a map of Route 23 from Flint to Ann Arbor with a scaling factor of $\frac{1}{2,000,000}$. Include the two cities and shade their shapes the best you can.

Draw the same stretch of the map with a scaling factor of $\frac{1}{3,000,000}$. Can you still recognize the shapes of the two cities?

Figure 4 is a map of part of the southeastern United States. Compare Figure 4 with Figure 3. The roads in the two maps are marked the same way. In Figure 3 the cities are shown by shaded areas which show their shapes. But all the cities in Figure 4 are shown as little circles. Since the scaling factor, $\frac{1}{2,500,000}$, reduces the cities so much on the map, it is not practical to draw the actual shape of the city.

[?]

55. (a) In Figure 3, which city is larger, Flint or Pontiac?

(b) In Figure 4 all cities are represented by the same size circle. Do you think all cities are the same size?

56. Find the real distances that correspond to the following distances in Figure 4:

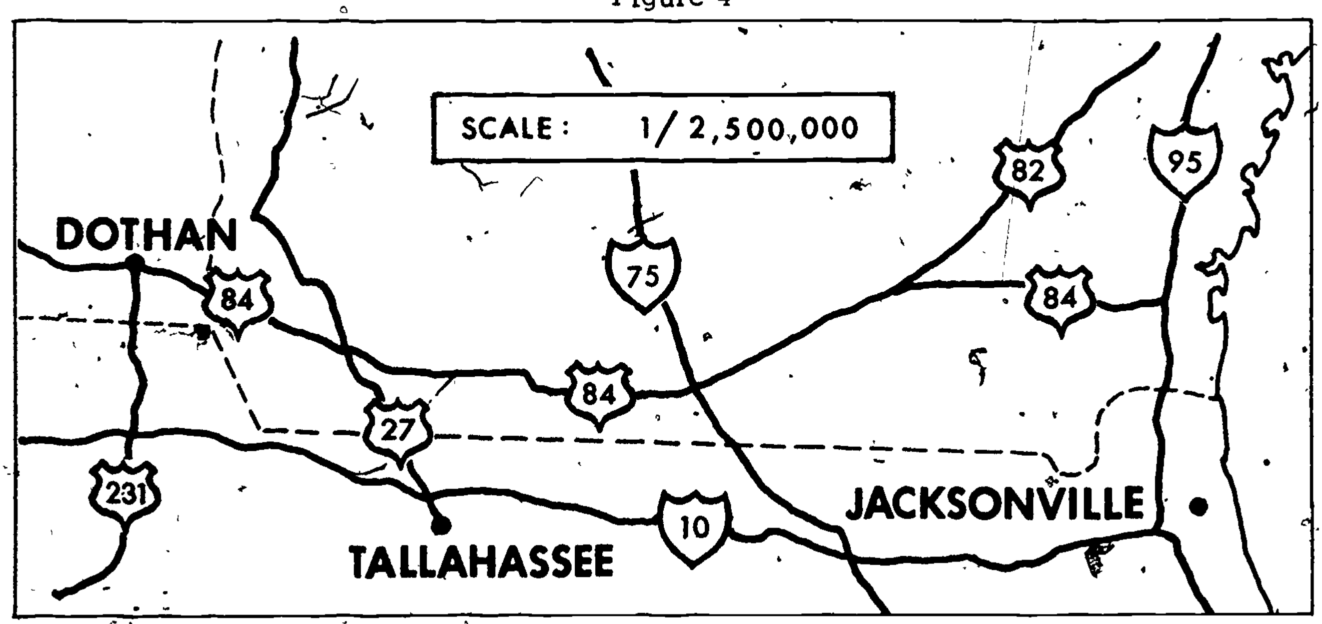
(a) 1.0 cm

(b) 0.1 cm

(c) 0.5 cm

57. Dothan has roughly the shape of a rectangle 11 km wide from east to west and 14 km long from north to south. Show Dothan as a rectangle on the map. Compare your drawing with those of your classmates.

Figure 4



When scaling factors like $\frac{1}{2,000,000}$ or $\frac{1}{3,000,000}$ are used it is impossible to avoid uncertainties in placing cities on a map. Also plain errors in drawing and reading add to the uncertainties. Thus even straight line distances between cities can only be found approximately.

For example, the straight line distance on the map between centers of the circles indicating Columbus and Dothan is 5.8 cm. (See Figure 5.) Multiplying by the scaling factor, we get 145 km. But a measurement error of 0.1 cm would give an error of 2.5 km. Also the cities are each a few kilometers wide. With all of the uncertainties we might expect that answer to be off by as much as 10 km.

One way we could allow for the uncertainity would be to say "The distance is 145 km <u>to within</u> 10 km." That would indicate that the distance is most likely anywhere from 135 km to 155 km. We can also write this as 135 km $< \square <$ 155 km.

Knowing that getting distances from maps is not exact, we often just give answers using the word "about." For example, we could say "The straight line distance from Dothan to Columbus is <u>about</u> 145 km."

Finally, this uncertainty allows us to make our work easier by rounding off the numbers before we multiply. For the map distance from Columbus to Dothan we could use 6 cm instead of 5.8 cm, so we get 6 cm $\times$ 2,500,000 or <u>about</u> 150 km.

58. The straight line distance, on the map, between Savannah and Jacksonville is 6.8 cm (Figure 5).

(a) How far apart are the two cities?

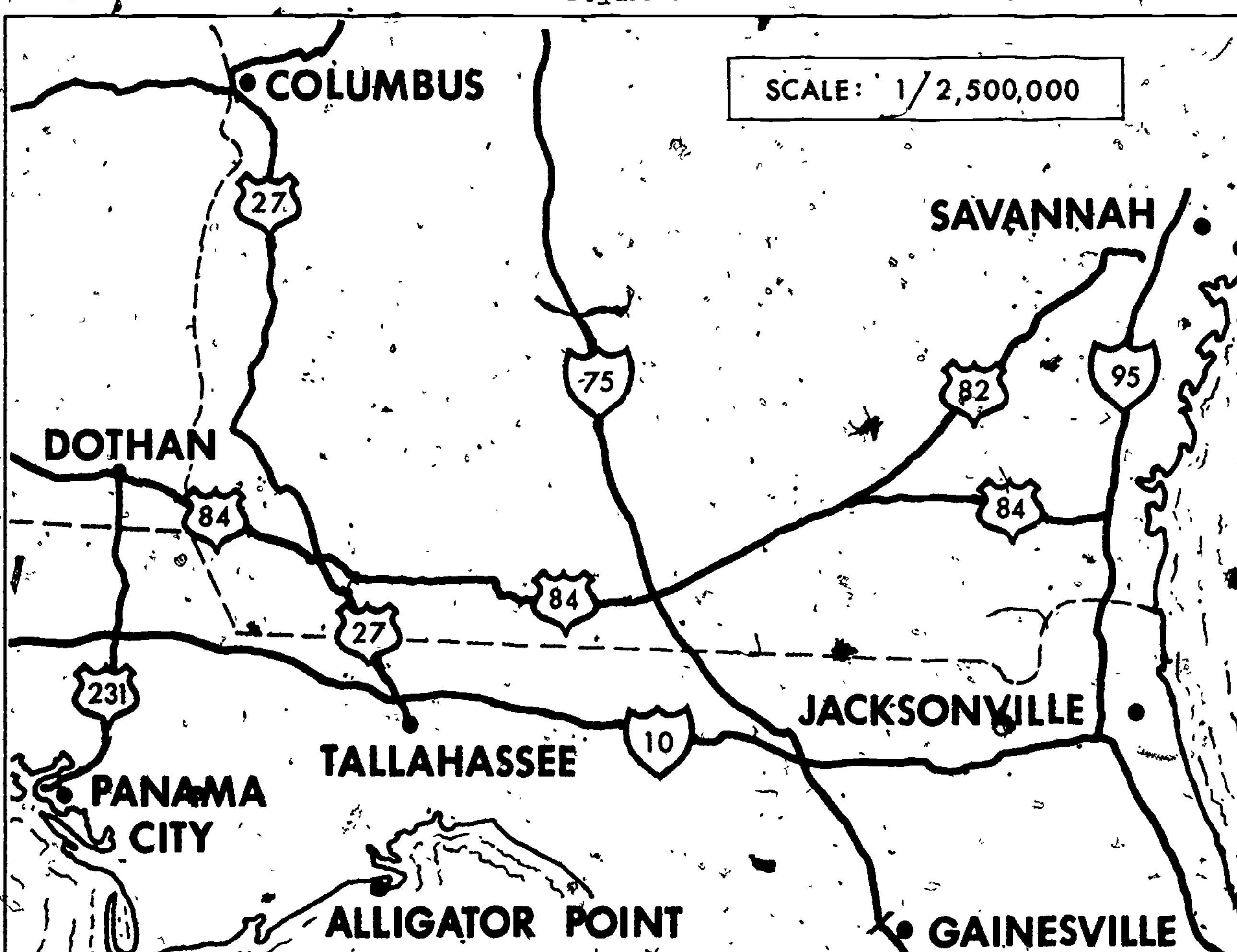
(b) If your answer is accurate to within 10 km, bracket the distance.

59. There are three reasonable routes from Dothan to Gainesville. Measure each route. Which route do you think is shortest? Can you be sure?

60. What is the approximate distance on Route 10 from Tallahassee to Jacksonville?

61. About how many kilometers would you sail from Panama City to Alligator Point?

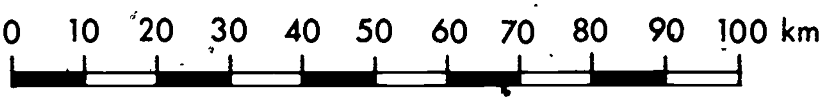
Figure 5



SECTION 11 MAP SCALES

For a map with a scaling factor of $\frac{1}{1,000,000}$, one cm on the map equals 10 km of real distance (Figure 3). We can express the same fact by drawing a scale on the map in which 1 cm is marked as 10 km (Figure 6).

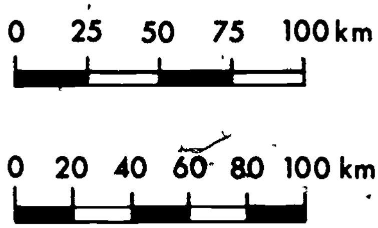
Figure 6



In Figures 4 and 5 one cm of map distance is equal to 25 km in real distance. We can say the same thing by the scales shown in Figure 7.

Figure 7

(a) (b)



In both of these scales 4 cm represents 100 km. So they express the same scaling factor.

Scales such as those in Figures 6 and 7 are very convenient to use for finding approximate distances on maps. All you need to do is mark off the distance on the edge of a sheet of paper. Hold one endpoint at the zero of the scale and read off the distance on the scale at the other endpoint. If the desired distance is too long, you can break it up into several segments, and read each one separately.

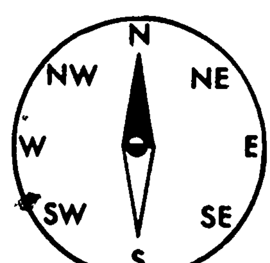
Having a map scale is a great advantage over knowing the scaling factor. You do not need to do any multiplication. This is why almost every map has a scale.

In cooperation with your social studies teacher, do a project involving getting information from a map. The scale on an atlas is likely to be in miles. If you want answers in kilometers remember that 1 mile = 1.63 km. You can make yourself a multiplication device (Chapter 4, Section 5) to convert distances from miles to kilometers.

SECTION 12 BEARINGS AND NAVIGATION

When taking a trip by car, we follow roads. However, when traveling by boat or airplane, there are no roads to follow. Sometimes boat captains and airplane pilots can not see where they are going because of fog or clouds. They use a compass like the one shown in Figure 8 to make sure they are going in the right direction.

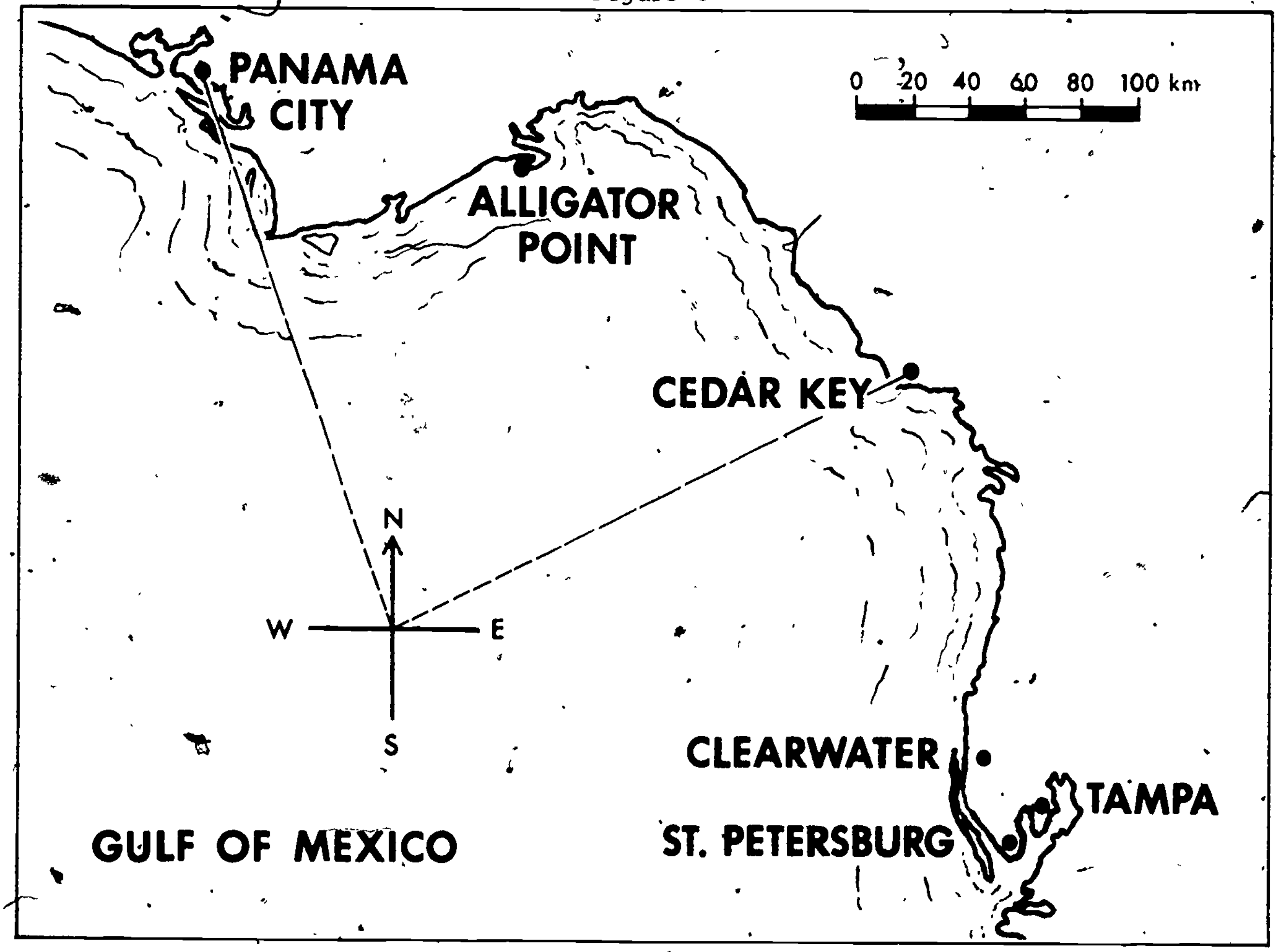
Figure 8



Each compass has a magnetic pointer that points north. The angle between north and the direction you are going, measured clockwise, is called a bearing. If you are going east, your bearing is 90°; if you are going west, your bearing is 270°.

Imagine you are the captain of a boat sailing in the Gulf of Mexico. You are located where the center of the compass is drawn on the map shown in Figure 9. Measure the bearing you must sail along in order to go to Cedar Key. Measure the bearing from the center of the compass to Panama City.

Figure 9

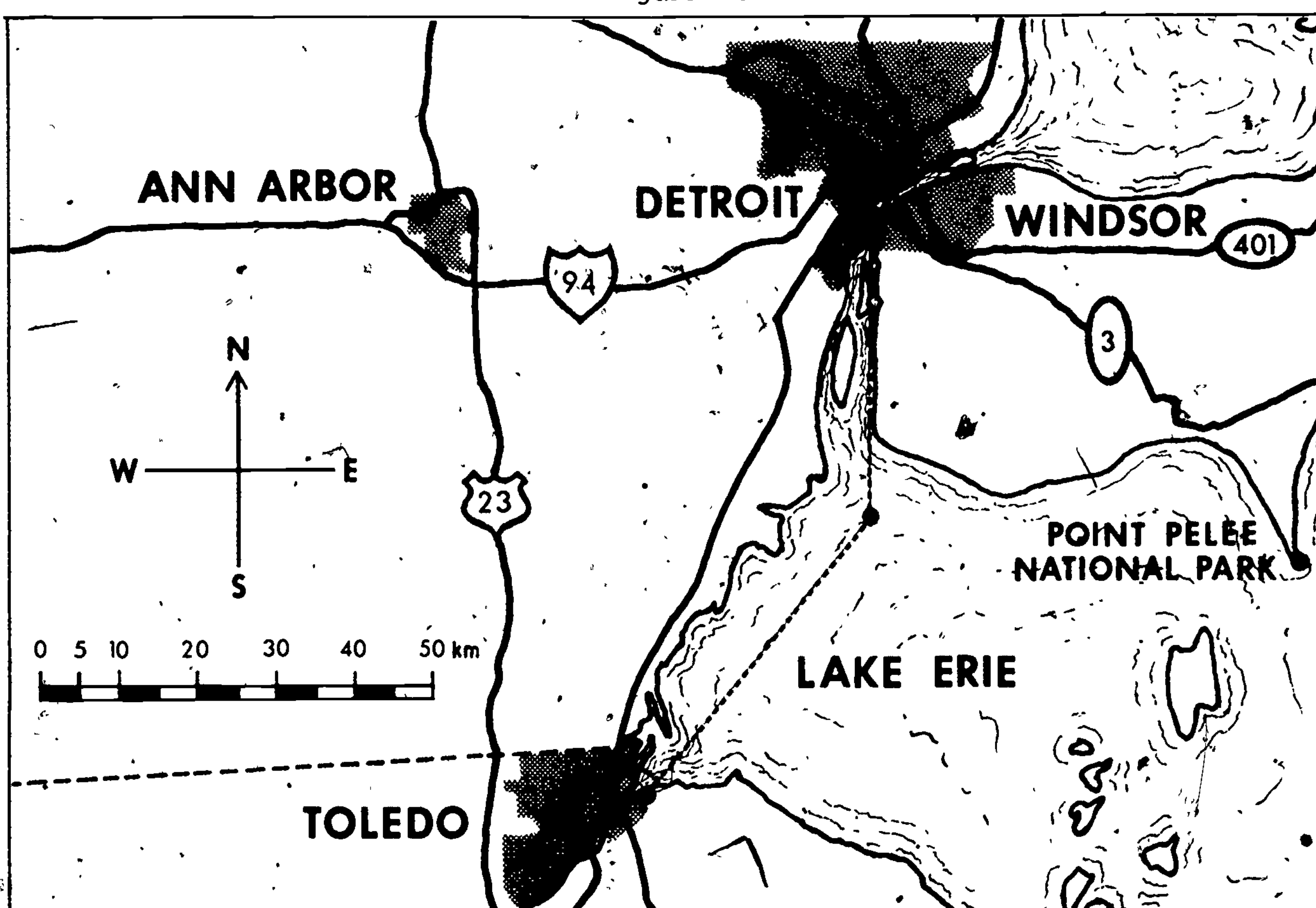


To measure the bearing from any point, you may have to draw a line going north at that point. This line will be parallel to the north direction already drawn on the map. Find the bearing from Alligator Point to Cedar Key. Also find the bearing going back from Cedar Key to Alligator Point.

You are piloting a ship from Toledo to Detroit along the route shown in Figure 10. Along what bearings will you head? How long is the trip in kilometers?

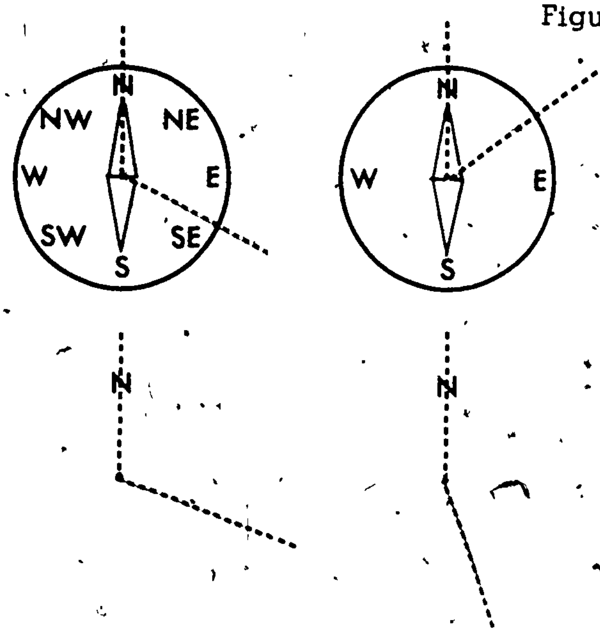
Suppose you want to sail from Detroit to Point Pelee (Canadian) National Park (Figure 10). Choose a course that is short and changes direction only once. Measure to find the bearings and distances. Compare your course with those of others in your class.

Figure 10



62. What bearings are shown in Figure 11?

Figure 11



63. Find the bearings corresponding to E, S, W, N, NE, SE, SW, and NW without using a protractor.

64. For the Gulf of Mexico map (Figure 9), the scale is $\frac{1}{2,500,000}$.

(a) What is the bearing from Alligator Point to Tampa?

(b) What is the bearing from Tampa to Alligator Point?

(c) How far is it between these two places?

65. A navigator on an airplane receives radio signals that tell the bearing from the signaling station to the airplane. The map in Figure 12 shows two such stations.

(a) If you are located on bearing 120° from station A and on bearing 83° from station B, where are you located?

(b) Which station is closer to you?

(c) If the scaling factor is $\frac{1}{500,000}$, how far are you from station A?

66. (a) If you are located at C, in Figure 13, what is the bearing toward D? How can you answer this without a protractor?

(b) The bearing from one town to another is 140°. What is the bearing from the second town back to the first?

67. Imagine you are the captain of a ship sailing from Panama City to Cedar Key (see Figure 9).

(a) Choose and draw the path you would like to follow.

(b) Describe your path as follows: "Start at Panama City. Head in the direction of bearing ____°. Go ____ kilometers. Turn to bearing ____°. Go ____ kilometers. (Continue until your directions take you to Cedar Key.)"

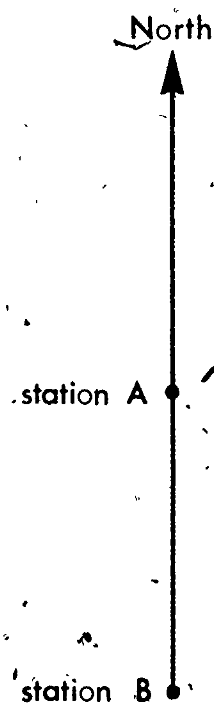


Figure 12

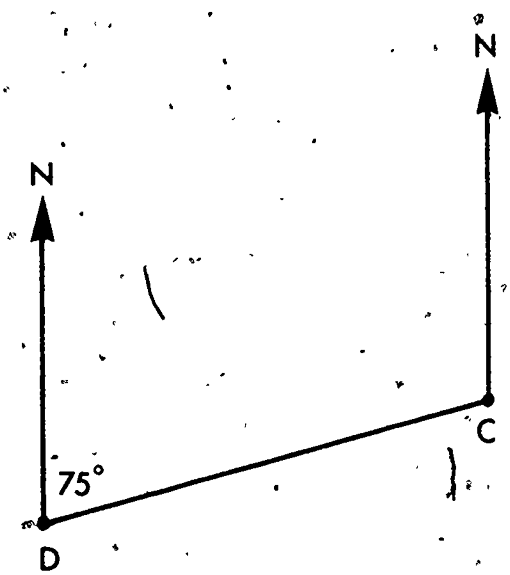
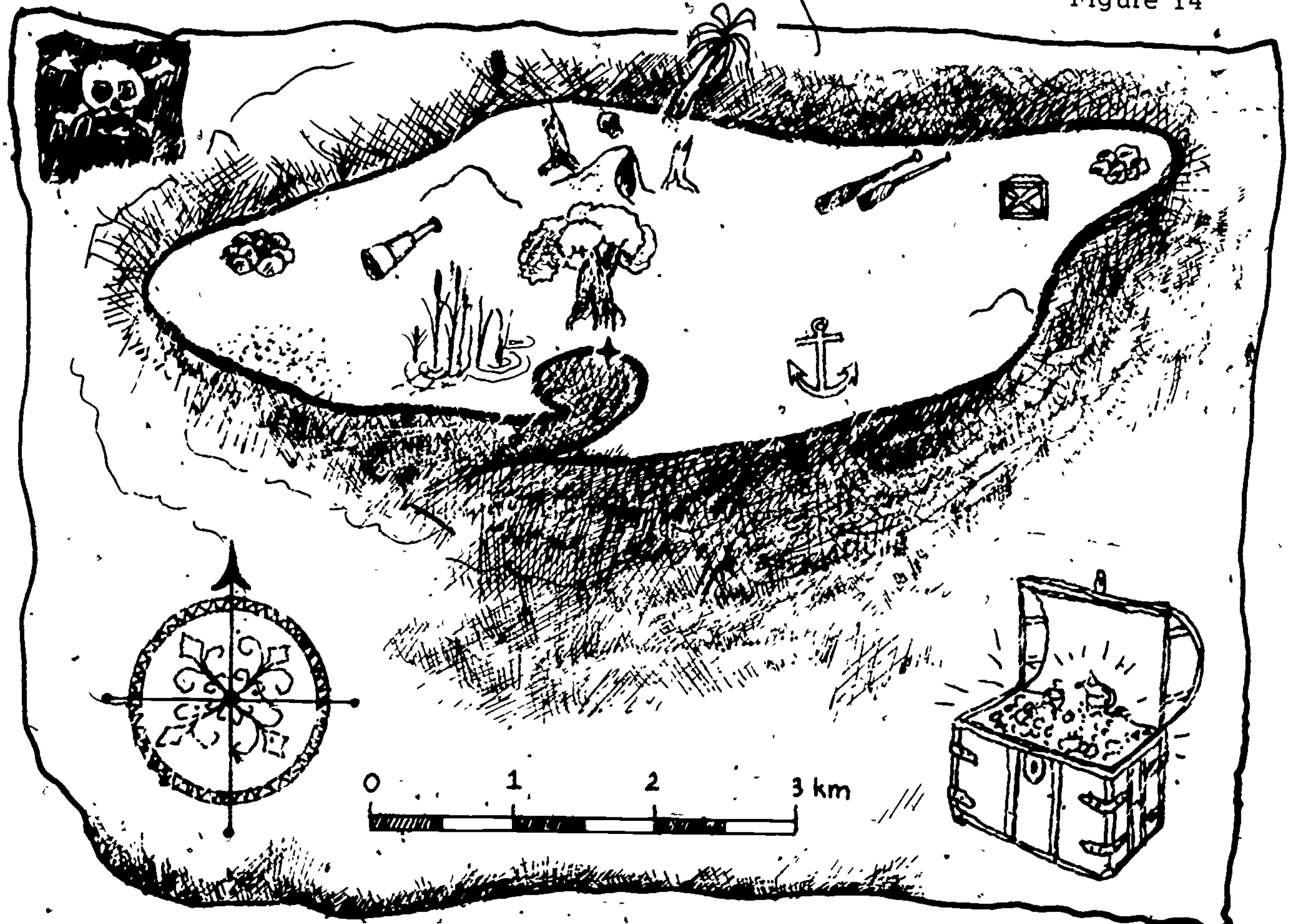


Figure 13

68. Where is the treasure?

Figure 14



Start at . Whale's Mouth Bay (✦).
Go along 62° for 2½ km.
Then along 166° for 1.0 km.
Finally along 292° for 2.5 km.
Treasure is hidden here

69. You and your friend are on Snook's Island at point A (see Figure 15). You want to row to point B on Buffie's Island. But you have to drop your friend off at the mainland. At which numbered point should you drop him off so that you row the shortest distance possible? How long is the total trip?

Measure and compare the two angles formed by your path and the mainland shore. Measure and compare also the angles any other path makes with the shore. What can you say about the two angles of the shortest path that you cannot say about the two angles of any other path?

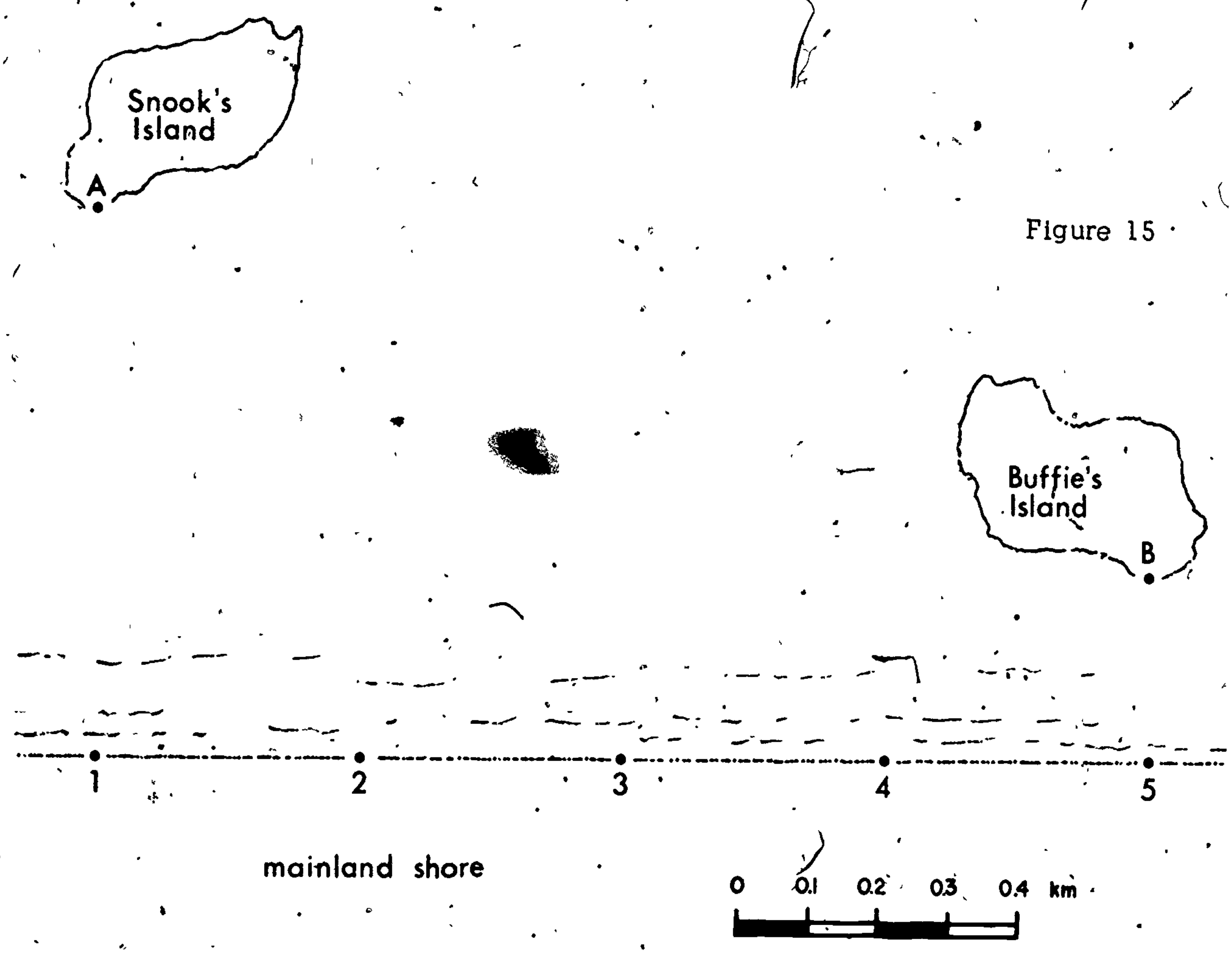


Figure 15

e Suppose you pilot a ship as follows: (1) head along the bearing 180° for 10 km; (2) change your bearing to 270° and travel for 7 km. Suppose you now wish to sail directly back to your starting point. What bearing should you use? How far will you have to sail?

You can find out by drawing a scale model diagram using a scale $\frac{1}{100,000}$ (1 cm represents 1 km). Here is how: First mark a starting point on a sheet of paper. Choose a direction for north. Measure a bearing of 180° from your starting point. Draw a 10 cm line segment to represent 10 km. Measure a bearing of 270°. Draw a 7 cm line segment for the 7 km distance. Now you can measure the bearing and the distance towards the starting point.

70. A sailor sails along 90° for 6 km and then 150° for 10 km. What is his bearing and his distance back to his starting position?

71. Travel in the direction of 120° for 4 km, then 240° for 8 km. What is the bearing and the distance directly back to the starting position?

6. QUOTIENTS AND RATIOS

SECTION 1 WHEN DO WE MULTIPLY?

1. The sides of a triangle are 3, 4, and 5 cm long. If you enlarge the triangle by a factor of 6, how long will the sides be?

2. One shirt costs $3.50. How much will four shirts cost?

3. Choose a number. What mathematical operation do you use to find a number three times as large?

4. You know the weight of one object. How do you calculate the weight of $\square$ such objects?

Questions 1 - 4 have something in common. In all of them you know something about "one" and want to find something about "many." To get the answer we multiply what we know about "one" by the number that tells us "how many." This is true also when the "how many" is less than one. For example, one meter of ribbon costs $2.00. Then $\frac{1}{4}$ m of the ribbon costs

$$\frac{1}{4} \times 2.00\$ = 0.50\$.$$

(Read "one-fourth times two dollars equals half a dollar.")

5. One yard is about 90 cm. A foot is $\frac{1}{3}$ of a yard. How many centimeters are in one foot?

6. One gallon of milk costs $1.40.

(a) How much does $\frac{1}{2}$ of a gallon cost?

(b) How much does $\frac{1}{4}$ of a gallon cost? (A fourth of a gallon is called a quart.)

(c) How much do 2.5 gallons cost?

SECTION 2 WHEN DO WE DIVIDE?

Four apples cost 60¢; how much does one apple cost? Here we know something about <u>many</u> and want to know something about <u>one</u>. In this case:

$$\text{the cost of one apple} = 60¢ \div 4$$

(Read "sixty cents divided by four.")

Sixty cents goes with four apples and $60¢ \div 4 = 15¢$ goes with one apple. In general, if we want to find what goes with "one" when we know what goes with "many," we divide.

Before doing the division, we must decide which number to divide by which. The units asked for in the question will help you decide. Read the question to find the units the answer must have. Divide the quantity with those units by the other number.

?

Write sentences for the answers to Questions 7 - 13. Carry out the division where you know how to do it.

7. Thirty chairs cost $450.00. How much does one chair cost?

8. In an enlarged drawing the side of a rectangle is 30 cm. The scaling factor is 5. What is the original length?

9. A stack of five pennies is 0.8 cm high. How thick is one penny?

10. A good runner runs 100 m in 12 seconds.

(a) How many meters does he run in one second?

(b) How many seconds does it take him to run one meter?

11. A train going at a steady speed travels 480 km in $2\frac{1}{2}$ hours.

(a) How many kilometers does the train travel in one hour?

(b) How many hours does it take the train to travel one kilometer?

12. A jet airplane travels 900 km in 60 minutes.

(a) How many kilometers does it travel in one minute?

(b) How many minutes does it take the plane to travel one kilometer?

13. The cost of $\frac{3}{4}$ of a pound of fish is $1.70. What is the cost of one pound of fish?

In Questions 14 - 17 be sure to decide whether to multiply or divide.

14. My brother is three times as old as I am. I am 12 years old. How old is he?

15. One-third of your day is spent sleeping. How many hours is that?

16. Selling a roll of 100 tickets to the basketball game brought in \$75.00. How much did each one of the tickets cost?

17. A 45 rpm record goes around 45 times each minute (60 seconds). How many times does the record go around in one second?

SECTION 3 QUOTIENTS AND FRACTIONS

In Chapter 2 we used unit fractions to name the size of the smallest piece when a strip was divided into a number of equal pieces. For example, when we divided a strip into three equal parts the size of each part was $\frac{1}{3}$.

Since each strip was one unit long we can write

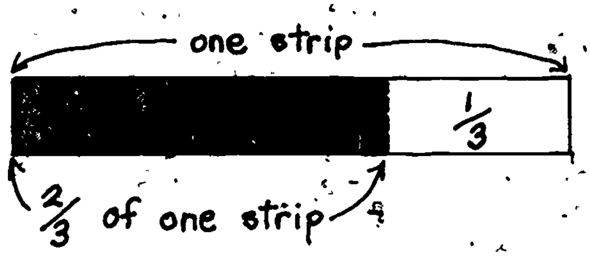
$$\frac{1}{3} = 1 \div 3$$

That is, the unit fraction $\frac{1}{3}$ is the result of dividing 1 by 3.

What can we say about fractions such as $\frac{2}{3}$ which are not unit fractions? In Section 8 of Chapter 2, we saw that $\frac{2}{3}$ was a shorthand way of saying that we have two one-thirds. That is

$$\frac{2}{3} = 2 \times \frac{1}{3} \quad \text{or} \quad \frac{2}{3} = \frac{1}{3} + \frac{1}{3} \qquad \text{(Figure 1)}$$

Figure 1



Is $\frac{2}{3} = 2 \div 3$? To answer this question let us use two strips end-to-end (Figure 2). To represent $2 \div 3$ we divide the strips into three equal pieces. The shaded part of Figure 2 is the result of this divi-

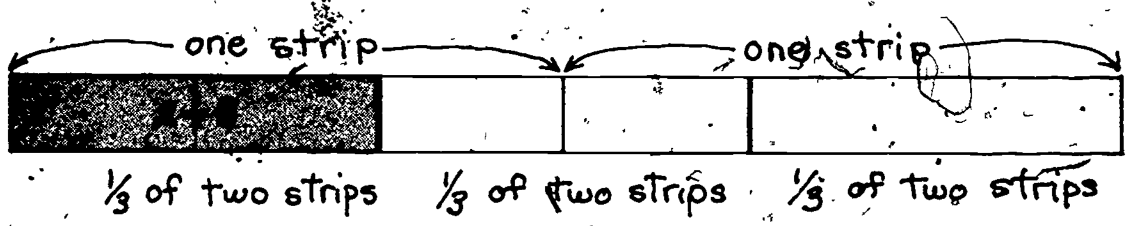


Figure 2

sion. What part of a single strip is this? Comparing the shaded portions of Figure 1 and Figure 2 we conclude that the shaded part of Figure 2 is $\frac{2}{3}$ of a strip. Therefore we can write

$$\frac{2}{3} = 2 \div 3$$

In words, the fraction $\frac{2}{3}$ is equal to the quotient $2 \div 3$.

[?]

18. (a) Draw a picture of three strips lined up end-to-end.

    (b) Think of the strips as one long strip. Divide it into two equal pieces.

    (c) Shade one of the equal pieces.

    (d) How many single strips are shaded?

    (e) What have you shown?

19. How much are

    (a) six halves?

    (b) twenty-one sevenths?

20. By dividing find the value of

(a) $\frac{30}{5}$

(b) $\frac{180}{9}$

(c) $\frac{404}{2}$

(d) $\frac{128}{4}$

(e) $\frac{240}{3}$

21. Three persons share five pounds of sugar equally. How much sugar did each one of them get?

22. What is the result of dividing 4 by 7?

SECTION 4 COMBINING DIVISION AND MULTIPLICATION

It took a car three hours to travel 240 km on an interstate highway. How far would the car travel in five hours? (Assume that the car kept moving at the same speed.)

You can answer this question by combining what you have learned in the preceding two sections. From the distance traveled in three hours, you can find the distance traveled in one hour by dividing

$$\text{distance traveled in one hour} = 240 \text{ km} \div 3 \text{ or } \frac{240}{3} \text{ km}.$$

From the distance traveled in one hour you can find the distance traveled in five hours by multiplying

$$\text{distance traveled in five hours} = \frac{240 \text{ km}}{3} \times 5$$

The reasoning used in this example is quite general. If you can go from many to one by division, and from one to many by multiplication, then you can go from many to a different many by doing both.

23. In a store you can buy four oranges for 80¢.
    (a) What does each orange cost?
    (b) What do six oranges cost?

In Questions 24 and 25, show which numbers you are going to divide and multiply to get the answer. Then calculate the answer.

24. If six candy bars cost 90¢, how much will five candy bars cost?

25. On a car trip you can comfortably go 540 km in six hours. How far can you comfortably go in 14 hours?

In Questions 26 – 28, write sentences describing how to get the answers. You need not carry out the calculations.

26. Eight candy canes cost 90¢.
    (a) What does one candy cane cost?
    (b) What do 15 candy canes cost?

27. A ball is steadily rolling across the floor and goes three meters every two seconds.
    (a) How far will it roll in five seconds?
    (b) How many seconds does it take the ball to go four meters?

28. A city of 100,000 inhabitants uses about 300,000 gallons of gasoline in one week.

(a) How much gasoline would a similar city with 130,000 inhabitants use in one week?

(b) How much gasoline would a similar city with 160,000 inhabitants use in two weeks?

(c) What is meant by the word "similar" in this question?

29. A workman can do a job in five days.

(a) How much of the job can he do in one day?

(b) How much of the job can he do in four days?

(c) How many days will it take him to do $\frac{8}{5}$ jobs?

SECTION 5 RATIOS

30. Figure 3 shows two similar triangles. Consider the smaller one as the original. What was the scaling factor used in drawing the larger figure?

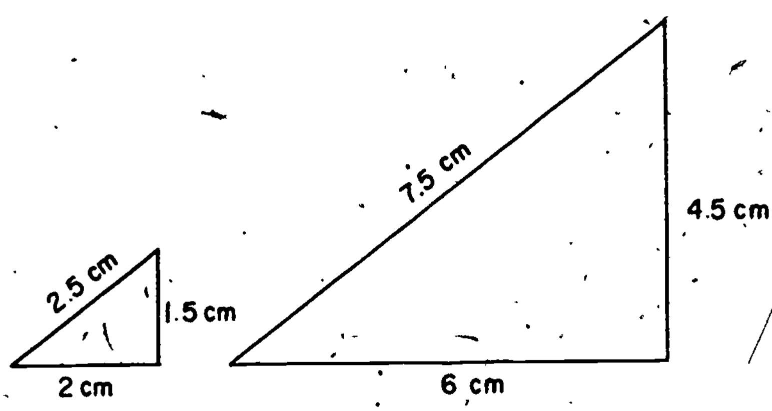


Figure 3

31. A big dog weighs about 80 pounds. A cat weighs about 10 pounds. How many times heavier is the dog?

32. A new car costs $4,000. A new bicycle costs $100. How many bicycles can you buy for the cost of one car?

33. You have 50¢. How many 15¢ candy bars can you buy?

In Questions 30 - 33 you were asked to make a comparison between two quantities. In all four problems the question "how many times" is either said or implied. The answer is found by division. In general, the question "how many times larger is □ than △," has the answer

$$\frac{\square}{\triangle} \quad \text{(Read "}\square\text{ over }\triangle\text{")}$$

Can we compare any two quantities? Consider the question "What is larger, five meters or seven pounds?" Obviously there is no way to compare meters with pounds. So the question makes no sense.

Now consider the question "How many times larger is 8$ than 200¢?" Here both quantities are money. They are expressed in different units, so we cannot compare them as they are. (8 ÷ 200 will certainly not give the right answer.) However, we can convert eight dollars to 800 cents and then divide:

$$\frac{800¢}{200¢} = 4$$

So 800¢ is four times as much money as 200¢.

We could also convert cents into dollars: 200¢ = 2$ and get the same answer:

$$\frac{8\$}{2\$} = 4$$

In summary, the answer to the question "How many times larger is □ than △?" is $\frac{\square}{\triangle}$ if they are both expressed in the same units. The quotient of two quantities expressed in the same units is called a ratio. When we speak of the ratio of □ to △ we always mean $\frac{\square}{\triangle}$.

34. Which of the following quantities can be compared as they are, after proper conversion of units, or not at all?
    (a) Five apples, three quarts of milk
    (b) 6.2 m, 8.5 m
    (c) 1.7 cm, 0.6 m
    (d) 6$, 150¢
    (e) 10$, 20 Swiss francs
    (f) Two pounds, $\frac{1}{2}$ pound

35. A large box of candies costs $3.00. A candy bar costs 20¢. How many times more expensive is the box?

36. How many times more is
    (a) $3.00 than 30¢?
    (b) 2400 cm than 12 m?
    (c) Seven dollars than 35 dimes?

(d) Fifty-six dimes than seven nickels?

(e) Two pads of 100 sheets each than 25 sheets of paper?

(f) Three quarters than three dimes?

SECTION 6 DOING DIVISION BY REPEATED SUBTRACTION

How many 15¢ candy bars can you buy with 50¢? You read in the last section that you can find the answer to this problem by doing the division 50 ÷ 15. You can do the division by using repeated subtraction.

Imagine starting with 50¢ and buying candy bars one by one.

			Number of 15's subtracted
Start with	50¢		
	-15¢	Buy one candy bar.	1
Now you have	35¢		
	-15¢	Buy one more candy bar.	1
Now you have	20¢		
	-15¢	Buy one more candy bar.	1
You have	5¢	left over.	3

You can buy three candy bars because you can subtract 15¢ from 50¢ three times.

How many 15¢ candy bars can you buy with $2.00? You can answer this question by changing $2.00 to 200¢ and then seeing how many times you can subtract 15¢ from 200¢.

	Number of 15's subtracted
200¢	
− 15¢	1
185	
− 15	1
170	
− 15	1
155	
− 15	1
140	
− 15	1
125	
− 15	1
110	
− 15	1
95	
− 15	1
80	
− 15	1
65	
− 15	1
50	
− 15	1
35	
− 15	1
20	
− 15	1
5¢	13

Since you can subtract 15¢ from 200¢ a total of 13 times, you can buy 13 candy bars for $2.00.

But there is a better way to do this: subtract more than one 15¢ at a time.

Since one candy bar costs 15¢, ten candy bars cost $10 \times 15¢$ or 150¢. So imagine buying 10 candy bars all at once. Then find out how many more you can buy with what is left over.

	Number of 15's subtracted
200	
-150	10
50	
- 15	1
35	
- 15	1
20	
- 15	1
5	13

You could have done the same problem even quicker like this:

	Number of 15's subtracted
200	
-150	10
50	
- 45	3
5	13

Notice that you subtracted 45¢ from 50¢ because the cost of the three candy bars was $3 \times 15¢$ or 45¢.

37. How many 15¢ candy bars can you buy with $1.00?

38. How many 15¢ candy bars can you buy with $2.50?

39. How many 35's can you subtract from 400?

40. How many 35¢ ice cream cones can you buy with $4.00?

41. How many 25¢ ball point pens can you get for $3.00?

42. How many nickels are there in a roll of $2.00 worth of nickels?

What is 1624 divided by 12?

$$12\overline{)1624}$$

The answer is found by subtracting as many 12's from 1624 as possible. However, subtracting 12's from 1624 one at a time is very slow, so subtract 12's in groups.

Look at 1624. It is 16 hundred and 24. Since we can subtract 12 from 16, we can subtract 12 hundred from 16 hundred. One hundred 12's is 1200, so subtract one hundred 12's:

	Number of 12's subtracted
$12\overline{)1624}$	
-1200	100
424	

Look at 424. It is 4 hundred and 24. You cannot subtract any more groups of one hundred 12's because you cannot subtract 12 hundred from 4 hundred.

Now rename 424 as 42 tens and 4. Since $12 < 42$, we can subtract 12 tens from 42 tens. In fact, we can do that subtracting three times.

	Number of 12's subtracted
12/1624	
−1200	100
424	
−120	10
304	
−120	10
184	
−120	10
64	

(You could have shortened your work by subtracting thirty 12's all at once.)

Finishing, we subtract five 12's.

	Number of 12's subtracted
12/1624	
−1200	100
424	
−120	10
304	
−120	10
184	
−120	10
64	
−60	5
4	

We are done subtracting 12's because no more 12's can be taken away from 4. In fact, 4 is called the remainder.

We find the quotient by adding up the number of 12's that were subtracted.

	Number of 12's subtracted
12/1624	
-1200	100
424	
-120	10
304	
-120	10
184	
-120	10
64	
-60	5
4	135

so 1624 ÷ 24 = 135 with a remainder of 4.

Do the following divisions:

43. 138 ÷ 12

44. 4/530

45. How many 6's are there in 1236?

46. $\frac{63}{11}$

47. 707 is how many times larger than 7?

48. 2939 divided by 8.

49. How many dozen eggs are there in 15,648 eggs?

50. $19\overline{)7700}$

51. $3661 \div 52$

52. $\frac{8700}{42}$

SECTION 7. DIVIDING PAST THE DECIMAL

In order to save money on food bills, people sometimes form a food cooperative (called a co-op). They buy food in large quantities at a cheaper rate, and then divide it up.

53. A co-op bought 40 pounds of rice to be shared equally by eight families. How much rice should each family get?

54. Twelve families are going to share 18 pounds of sugar. How much should each family get?

55. There are 44 gallons of syrup to be shared equally by 16 families. How much syrup should each family get?

All three questions are answered by dividing. For Question 53, the division tells us that each family gets exactly five pounds. However, Questions 54 and 55 do not come out exactly to whole numbers. To share the rest equally, we need to find what part of a pound or gallon each family should get.

For Question 54 we need to divide 18 pounds by 12. Dividing 18 by 12 gives 1 with 6 left over. Therefore, each family gets

1 pound and a fraction part of the 6 pounds left over. The 6 pounds still needs to be divided up among the 12 families. So each family gets 1 pound plus $(6 \div 12)$ pounds. But $6 \div 12 = \frac{6}{12}$ or $\frac{1}{2}$, so each family gets $1\frac{1}{2}$ pounds of sugar.

For Question 55, 44 gallons $\div$ 16 is 2 gallons plus $(12 \div 16)$ gallons or $2\frac{12}{16}$ gallons. Since $\frac{12}{16} = \frac{3}{4}$, each family receives $2\frac{3}{4}$ gallons of syrup.

?

56. If you wish to divide \$10 equally among eight friends, how much will they each get?

57. A 36-cm stick is ~~broken into~~ five sticks of equal length. How long is each stick?

58. You earn \$53 for working 20 hours. How much did you receive for each hour you worked?

59. Twenty oranges cost \$3.00. How much will 30 oranges cost?

60. A hi-fi dealer had a sale on cassette tape players. He sold 24 and took in \$786. How much did he charge for each player?

These questions are best answered by getting the quotient to one or two places past the decimal. We can get quotients in decimal form by extending our division process.

Method I

The quotient of \$786 $\div$ 24 will give you the answer to Question 60. Since you want to know the answer in dollars and cents, you need to calculate to the hundredths place.

Start by getting the whole number part of the quotient.

```
24)786
   720     30
   66.
   48       2
   18      32
```

Since you want the answer to the hundredths place, put a decimal point and two zeroes after the 786. Bring those two zeroes down beside the remainder, 18. Now draw a vertical line on the right. This will remind you that what follows are hundredths.

```
24/786.00
   720       30 |
    66          |
    48        2 |
   1800      32 |
```

Now you divide the 1800 by 24. It will tell us how many hundredths there are in the quotient. Place the answers to the right of the vertical line.

```
24/786.00
   720       30 |
    66          |
    48        2 |
   18 00     32 |
   16 80        |  70
    1 20        |
    1 20        |   5
       0        |  75
```

Therefore the quotient is 32 and $\frac{75}{100}$ or 32.75. The sale price of the cassette player was \$32.75.

(Method I only)

61. The following division is completed to the thousandths place. Explain what happens in each step.

520)16.300			
0			
16 300	0	← STEP 1	
15 600			
700	30	← STEP 2	
520	1	← STEP 3	
180	31	← STEP 4	
Therefore, the quotient is 0.031		← STEP 5	

Method II

The quotient of \$786 ÷ 24 will give you the answer to Question 60. Since you want to know the answer in dollars and cents, you need to calculate to the hundredths place.

Start by getting the whole number part of the quotient as before, except don't add it up yet.

$$
\begin{array}{r|l}
24\overline{)786} & \\
720 & 30 \\
\hline
66 & \\
48 & 2 \\
\hline
18 &
\end{array}
$$

Now, to get the number of tenths, place a zero in the tenths place next to 786 and do the same to the remainder.

```
24/786.0
   720     30
    66
    48      2
    18.0
```

Now do $18.0 \div 24$. Since you want the number of tenths, rename 18.0 as 180 tenths. Then the division is 180 tenths $\div$ 24. You may guess 6 tenths. Put that on the right and multiply $0.6 \times 24 = 14.4$.

```
24/786.0
   720     30
    66
    48      2
    18.0
    14.4   0.6
     3.6
```

Rename 3.6 as 36 tenths. Since $24 < 36$, you can subtract another tenth.

```
24/786.0
   720     30
    66
    48      2
    18.0
    14.4   0.6
     3.6
     2.4   0.1
     1.2
```

Rename 1.2 as 12 tenths and notice that 24 cannot be subtracted from 12. Therefore, you cannot subtract any more tenths.

To get hundredths, put a zero in the hundredths place to the right of 786, and do the same to the bottom remainder, 1.2.

```
24/786.0
   720       30
    66
    48       2
    18.0
    14.4     0.6
     3.6
     2.4     0.1
     1.20
```

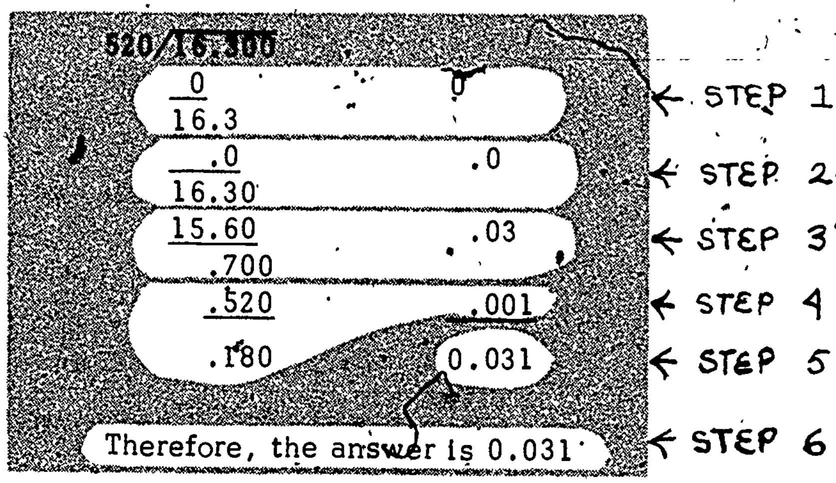
Now 1.20 ÷ 24 is 120 hundredths ÷ 24. You may guess 5 hundredths. Multiplying gives you 0.05 × 24 = 1.20. Adding up gives the quotient.

```
24/786.00
   720       30
    66
    48       2
    18.0
    14.4     0.6
     3.6
     2.4     0.1
     1.20
     1.20    0.05
     0.00   32.75
```

So the cassette players cost $32.75 each.

(Method II only)

62. The following division is completed to the thousandths place. Explain what happens in each step.



63. Write the following fractions in mixed notation:

(a) $\frac{3}{2}$

(b) $\frac{23}{5}$

(c) $\frac{67}{8}$

(d) $\frac{153}{12}$

(e) $\frac{7}{3}$

64. Suppose you want an answer to the nearest tenth. Dividing, you get 14.6 for a quotient. Can you be sure that 14.6 is correct to the <u>nearest</u> tenth? Suppose by dividing to the hundredths place you get 14.66. Then the answer to the nearest tenth is 14.7. Therefore, if you want an answer to the nearest tenth, calculate to the hundredths place and round off.

Find these quotients to the nearest tenth:

(a) $248 \div 33$

(b) $\frac{57.6}{16}$

(c) $7\overline{)28.6}$

65. Find these quotients to the nearest hundredth:

(a) $112 \div 21$

(b) $18\overline{)48}$

(c) $\frac{42.2}{41}$

(d) $\frac{7}{21}$

(e) $\frac{27}{33}$

66. If a car loses \$500 of value in a year, how much does it lose each day? (It's not a leap year.)

67. A bag containing 50 nails costs \$1.30. How much would 55 nails cost?

SECTION 8 . DIVIDING BY A DECIMAL

So far you have not learned to divide by a decimal number. For example,

What is the quotient of $38 \div 1.9$?

Here is a way to answer this kind of division problem. First, rewrite the division as a fraction:

$$38 \div 1.9 = \frac{38}{1.9}$$

Next, find an equal fraction with a whole number in the denominator:

$$\frac{38}{1.9} = \frac{38 \times 10}{1.9 \times 10} = \frac{380}{19}$$

Since $\frac{38}{1.9}$ and $\frac{380}{19}$ are equal fractions, they have the same quotients.

In other words, $380 \div 19$ gives the same answer as $38 \div 1.9$. Therefore do

$$19\overline{)380}$$

Here is another example:

$$1.33\overline{)0.175}$$

Change the division to a fraction:

$$\frac{0.175}{1.33}$$

Find an equal fraction with a whole number in the denominator:

$$\frac{0.175}{1.33} = \frac{0.175 \times 100}{1.33 \times 100} = \frac{17.5}{133}$$

Do the division:

$$133\overline{)17.5}$$

?

68. To change 1.9 into a whole number, we multiplied by 10. To change 1.33 into a whole number, we multiplied by 100. What would you multiply 0.057 by to change it into a whole number?

69. Change the following fractions into equal fractions with a whole number in the denominator:

(a) $\frac{7}{0.2}$

(b) $\frac{1.3}{2.8}$

(c) $\dfrac{0.008}{0.07}$

(d) $\dfrac{4.6}{23}$

(e) $\dfrac{4.15}{2.112}$

(f) $\dfrac{6.83}{0.5}$

70. What division will you do to get the quotients for these divisions:

(a) $4 \div 0.6$

(b) $237 \div 1.42$

(c) $30 \div 0.007$

71. Find the quotient for the following to the nearest tenth:

(a) $\dfrac{36.3}{4.8}$

(b) $0.5239 \div 0.052$

(c) $\dfrac{3.4}{4.91}$

72. Find these quotients to the nearest hundredth:

(a) $4.5 \div 3.02$

(b) $\dfrac{0.073}{0.006}$

The cost of one pound or one gallon of an item at the grocery store is called the unit price. In many states, large grocery stores are required by law to display unit prices. By comparing unit prices, customers can decide which package is the better buy.

[?]

73. A brand of chocolate candies comes wrapped in medium and large size bags. The medium size costs $0.83 for 0.75 pounds and the large size costs $1.05 for one pound.

    (a) What is the unit price for each size?

    (b) Which size is cheaper for one pound of candies?

74. Two different sized packages of the same type of white bread cost

    58¢ for 1.25 lb or
    48¢ for 1 lb

    Which package has the cheaper unit price?

75. You can buy raisins in a 15-ounce box for 69¢ or in a package of six small $1\frac{1}{2}$-ounce boxes (nine ounces total) for 55¢. Which type of packing charges more for each ounce of raisins?

76. If bags of three ounce candy bars cost

    $1.99 for 2 lb.
    $0.88 for 0.8 lb

    which is the better buy?

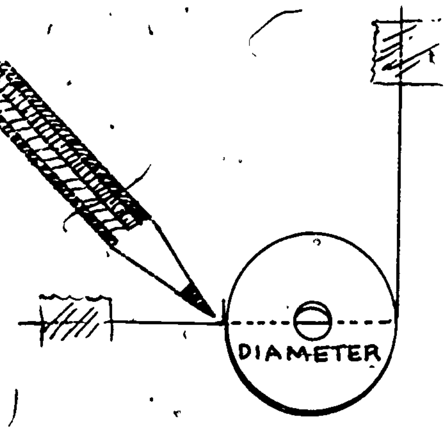
77. Ice cream sandwiches come in two sizes

    Six 3-ounce bars (1.13 lb) for $0.89
    Twelve $2\frac{1}{2}$-ounce bars (2.5 lb) for $1.16

    Which size gives you the cheaper unit price?

SECTION 9 THE FAMOUS RATIO π

Find and record the ratio of the circumference to the diameter of three circles using the steps shown below. You may want to use your manhole covers.

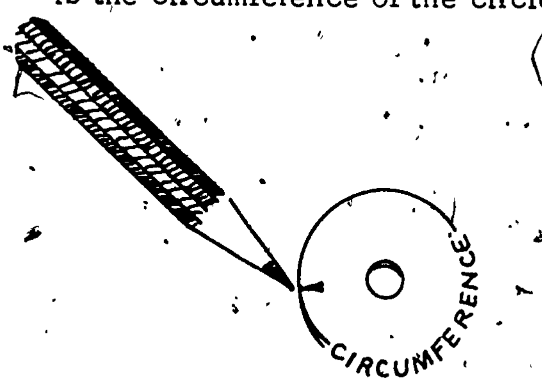


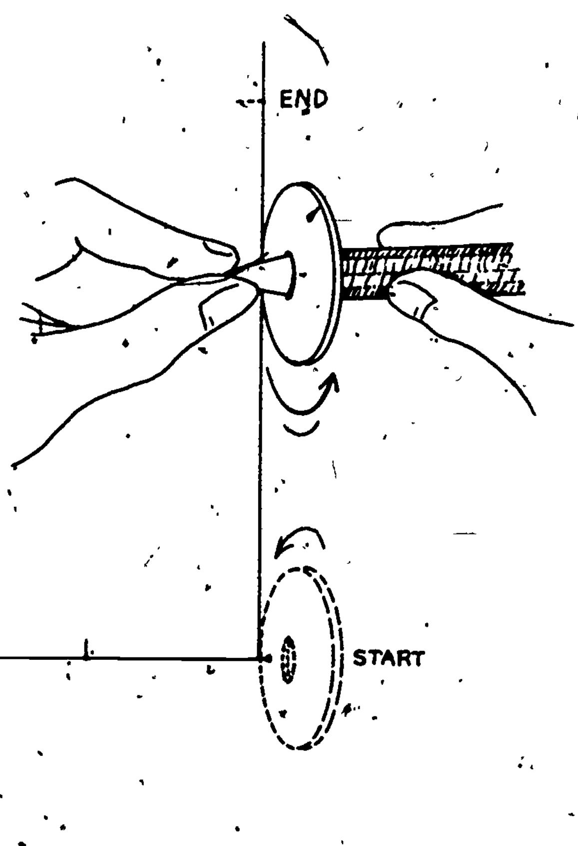
Step 1.

Tape a piece of paper to your desk top. Then place the diameter of the circle at the right corner of the bottom edge of the paper. Mark the diameter length at the edge.

Step 2.

Place a mark on the edge of the circle itself. The distance from the mark all the way around the edge of the circle and back to the mark is the circumference of the circle.



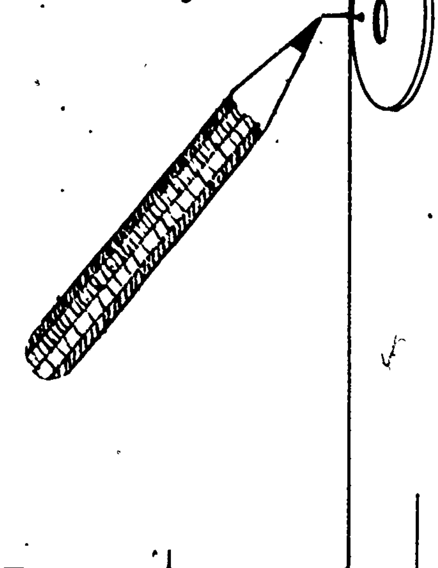


Step 3.

Start with the mark touching the bottom right-hand corner. Roll the circle, without slipping, straight up the side edge until the mark touches the paper again.

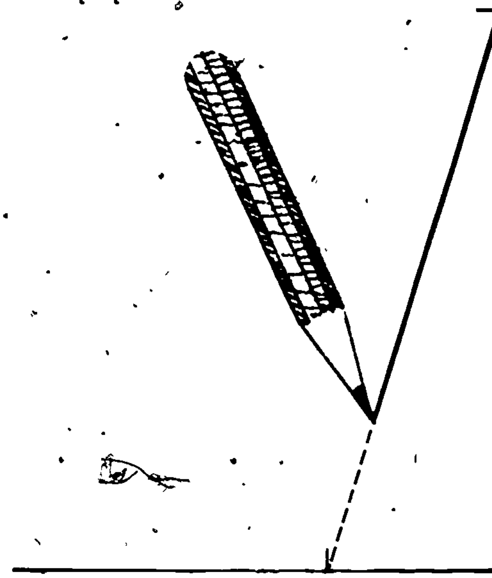
Step 4.

Mark that point at the side edge.



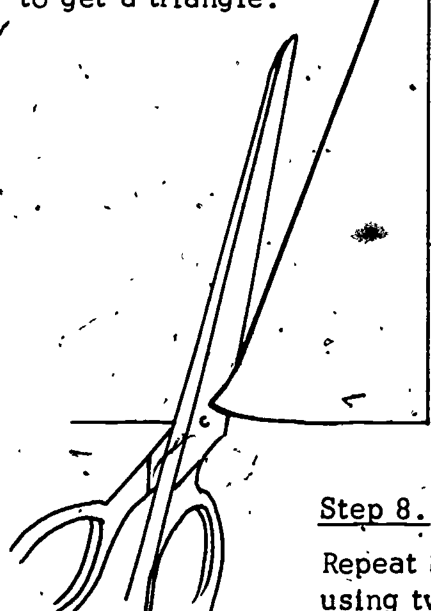
Step 5.

Connect the marks on the paper.



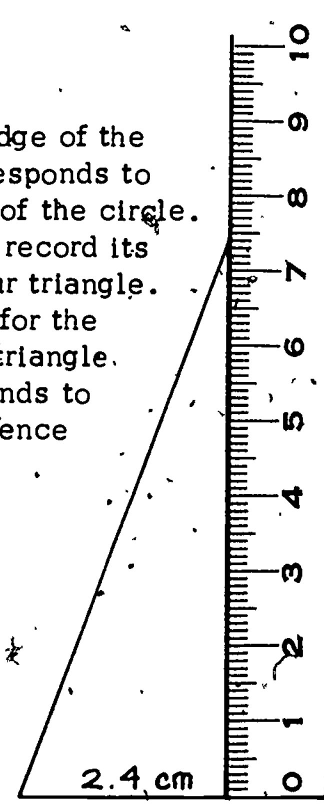
Step 6.

Cut along the diagonal to get a triangle.



Step 7.

The bottom edge of the triangle corresponds to the diameter of the circle. Measure and record its length on your triangle. Do the same for the side of your triangle that corresponds to the circumference of the circle.



Step 8.

Repeat Steps 1 through 7 using two other circles.

78. What do you notice about the three triangles when you place them on top of each other?

79. Calculate the ratio of the circumference to the diameter for the circles you rolled.

80. Are all the ratios equal? Compare your results with your classmates'.

81. What is an approximation of the ratio of the circumference to the diameter for any circle?

The ratio of the circumference of a circle to its diameter is π (read "pie").

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

It means that the circumference of a circle is π times as long as the diameter of that circle.

$$\text{circumference} = \pi \times \text{diameter}$$

Did you find that π is about 3.1? More accurate calculations show that π is approximately equal to 3.14.

82. If a circle has a diameter of 4.0 cm, what is its circumference?

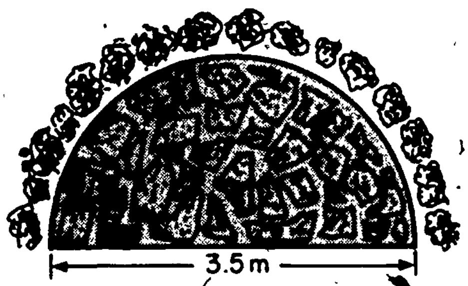
83. Luiz wants to put a fence around his circular flower bed. It measures 6.7 meters across.

(a) How many meters of fencing will he need?

(b) If each meter of fencing costs $1.59, how much will he need to spend?

84. Brenda's family has a patio shaped like a half-circle. It is 3.5 meters across the diameter. How far is it around the curved edge of the half-circle? See Figure 4.

Figure 4



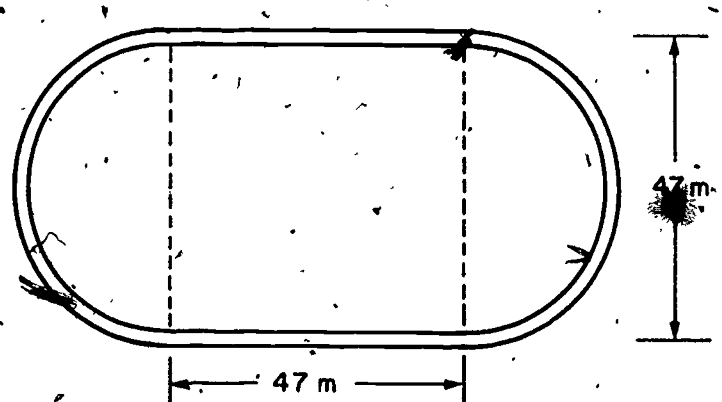
85. Your waist is somewhat like a circle.

(a) Measure the distance around your waist.

(b) How many times longer is your waist than your width?

86. How far is it around the race track shown in Figure 5, which is made up of a square and two half circles?

Figure 5



7. AREA

SECTION 1 UNITS OF AREA

Measurement involves counting units. When measuring lengths, our unit is a length (a centimeter, a meter, a foot, an inch, etc.). When measuring angles our unit is an angle (one degree). Similarly in measuring areas, our unit is an area.

Units of length vary only in size (centimeter, inch, meter). Units of area, however, can vary both in size and in shape. For example, any of the following three units shown in Figure 1(a) can be used to measure the area of the parallelogram in Figure 1(b).

(a) (b)

Figure 1

The result is shown in Figure 2(a), (b), and (c). Notice that copies of each of the three units of area have been arranged on the parallelogram of Figure 1(b) so as to just completely cover it. Then the number of units of area covering the parallelogram in each case has been counted.

Figure 2

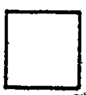
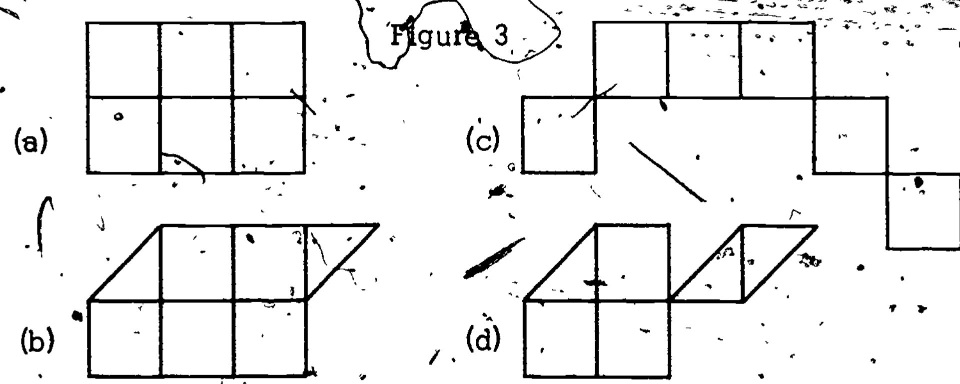
1. Using  as a unit of area, what is the area of each of the shapes in Figure 3?

Figure 3

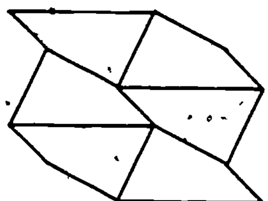


2. Which of the regions in Figure 3 have the same area?

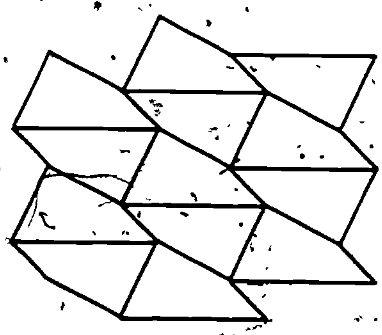
3. Use  as a unit to find the area of the polygons in Figure 4.

Figure 4

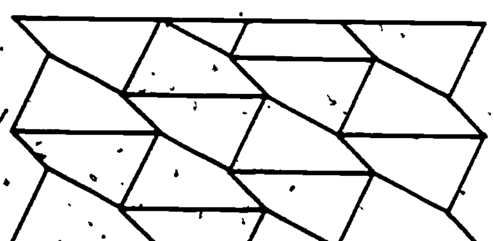
(a)



(b)



(c)



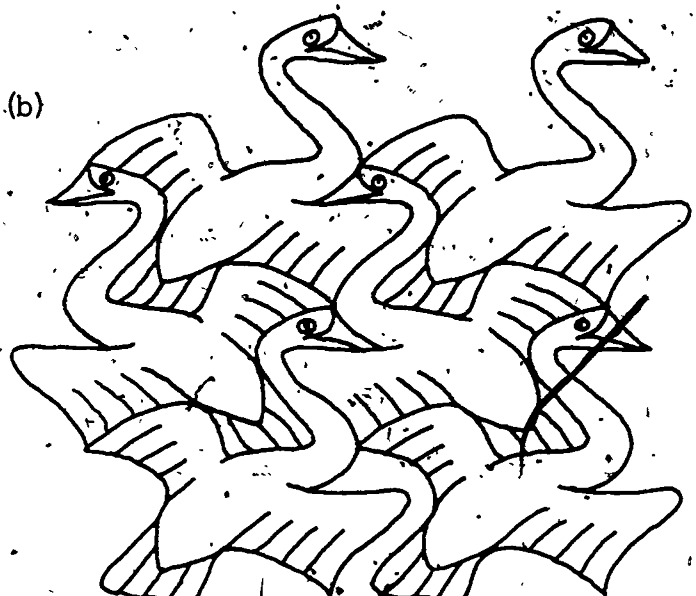
4. Which region has the larger area, the one in Figure 3(a) or in Figure 4(a)?

5. Use the bird in Figure 5(a) as the unit area to find the area of regions of Figures 5(b) and 5(c).

Figure 5

(a)

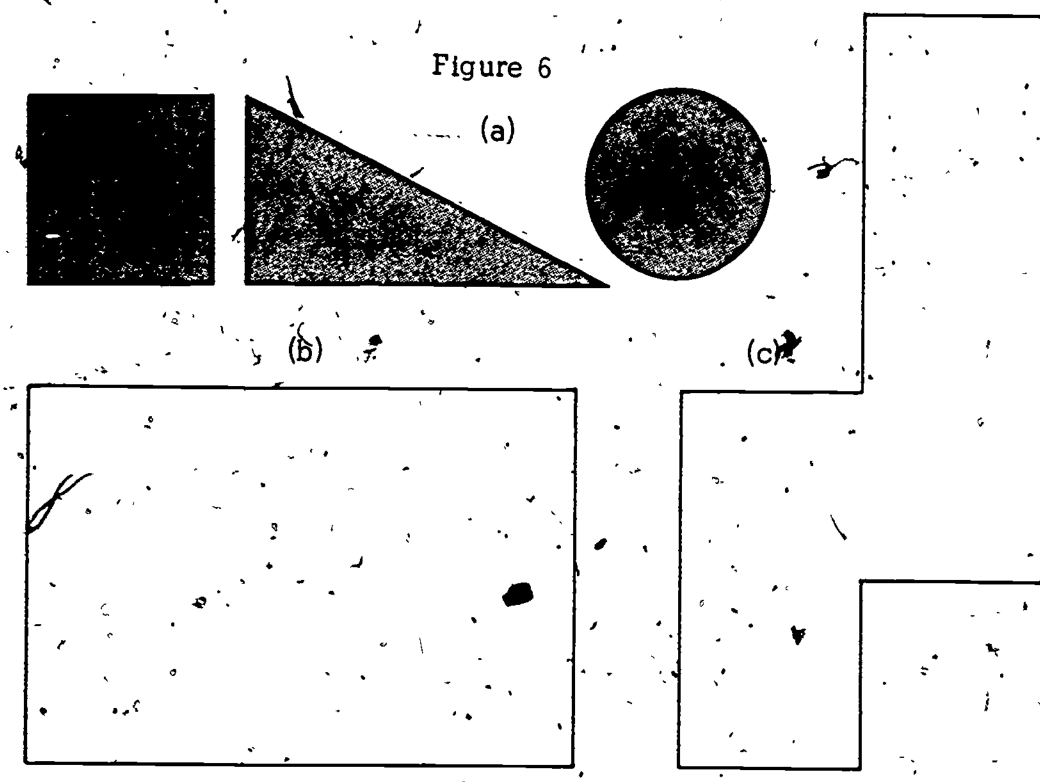
(b)



(c)

Trace each of the shaded unit areas in Figure 6(a) six times and cut each out. Using each shape that will work, find the areas of the polygons of Figures 6(b) and 6(c).

Figure 6



6. Which unit area exactly covers both polygons?

7. What unit area is easiest to count? Why?

8. Which unit area is easiest to divide into smaller equal units? Why?

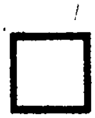
9. Which unit area is easiest to use as a unit of area?

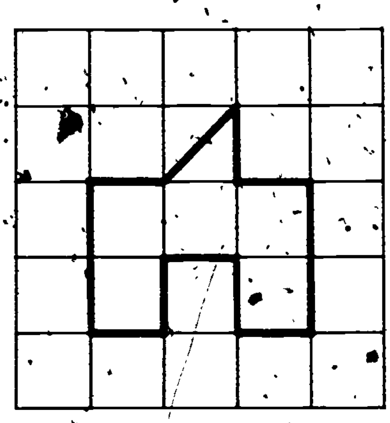
SECTION 2 FINDING AREAS IN SQUARE CENTIMETERS

You have seen that a square unit area is usually best for measuring area. Thus, a common unit of area (for small areas) is the square centimeter. A square centimeter is a square that measures 1 cm on a side (Figure 7(a)). The common abbreviation for square centimeter is <u>cm^2</u>.

To find the area of a region in square centimeters we can either cover the region with cut-out square centimeter pieces or use a centimeter grid as in Figure 7(b). A simple count shows that the area is $5\frac{1}{2}$ cm^2.

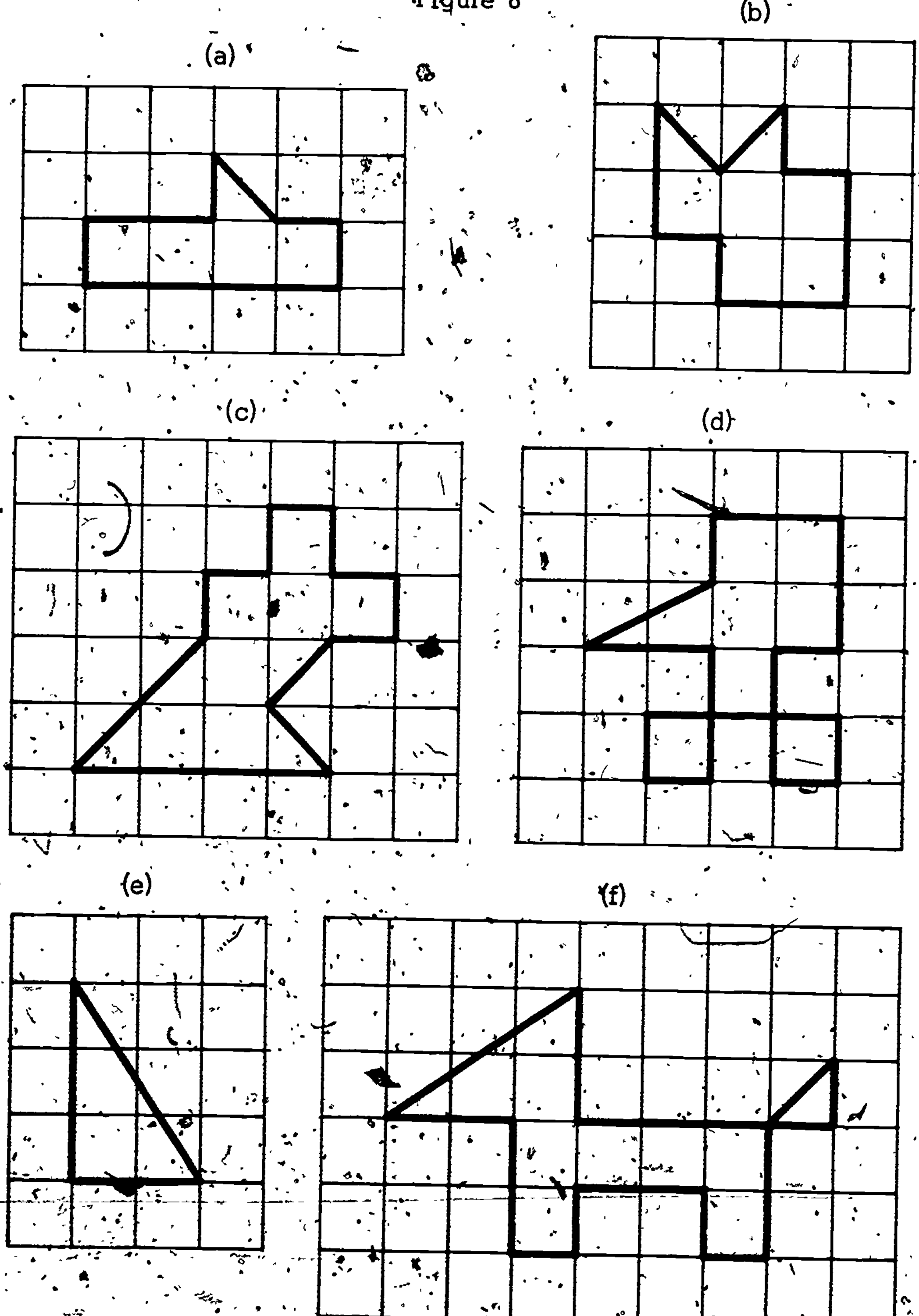
Figure 7

(a) 

(b) 

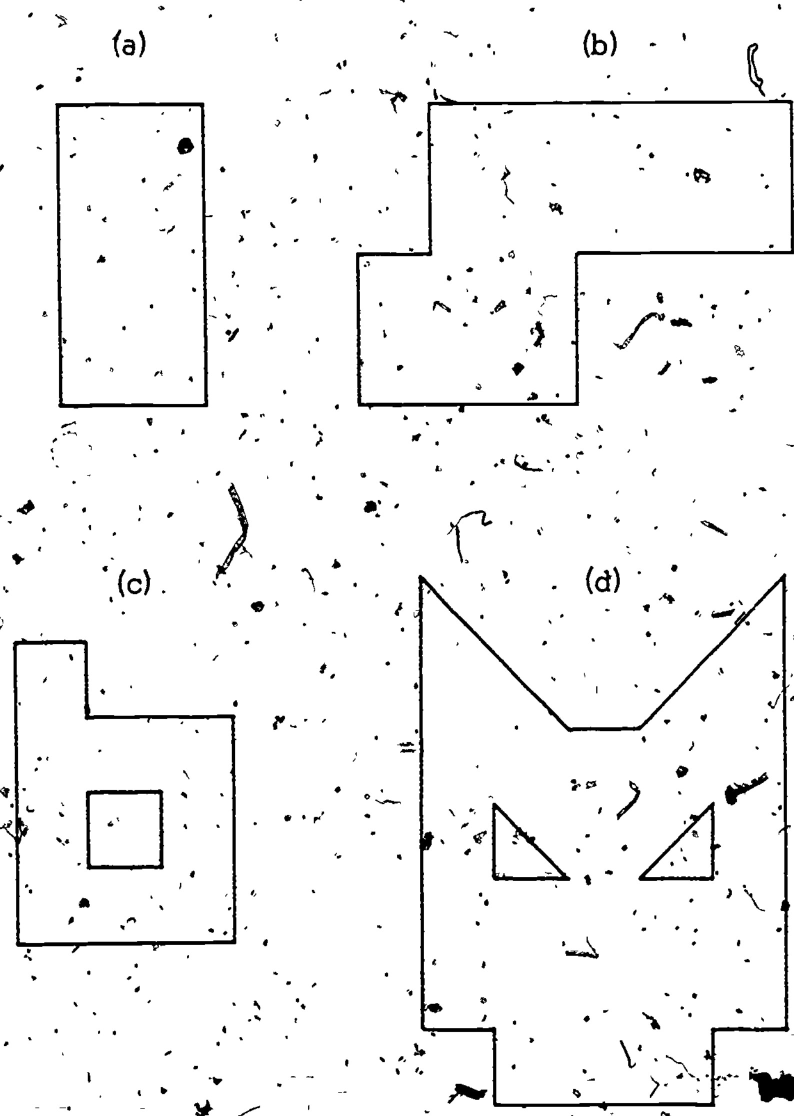
10. Find the areas in square centimeters of the regions in Figure 8. (Centimeter grids have been drawn on each of the regions.)

Figure 8



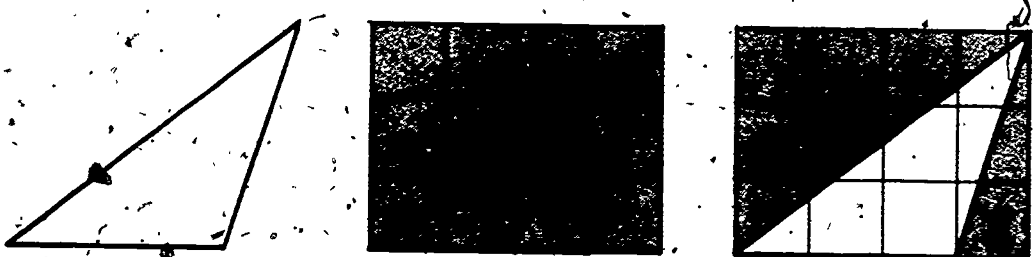
11. Use a centimeter grid to find the area of the shapes in Figure 9.

Figure 9



Sometimes we can use grids to find the exact area of a shape even though we cannot fill it exactly with whole standard units or divide single unit squares with diagonals. For example, to find the area of the triangle in Figure 10(a), we choose a rectangle which completely covers the triangle and whose area we can easily find. Figure 10(b) shows that the shaded 12 cm^2 rectangle covers the triangle. We can get the area of the triangle by subtracting, from the area of the rectangle, the shaded areas, labeled I and II, of Figure 10(c).

Figure 10


(a) (b) (c)

We can do this in two steps. First we find the area of Part I. It is $\frac{1}{2}$ of a rectangle whose area is 12 cm^2. Therefore, the area of Part I is 6 cm^2. Part II is $\frac{1}{2}$ of a rectangle whose area is 3 cm^2. Therefore, the area of Part II is $1\frac{1}{2}$ cm^2. The sum of the area of Part I and the area of Part II is

$$6\ cm^2 + 1\frac{1}{2}\ cm^2 = 7\frac{1}{2}\ cm^2$$

So the area of our region, the unshaded triangle in Figure 10(c), is

$$12\ cm^2 - 7\frac{1}{2}\ cm^2 = 4\frac{1}{2}\ cm^2$$

12. Find the areas of the polygons in Figure 11.

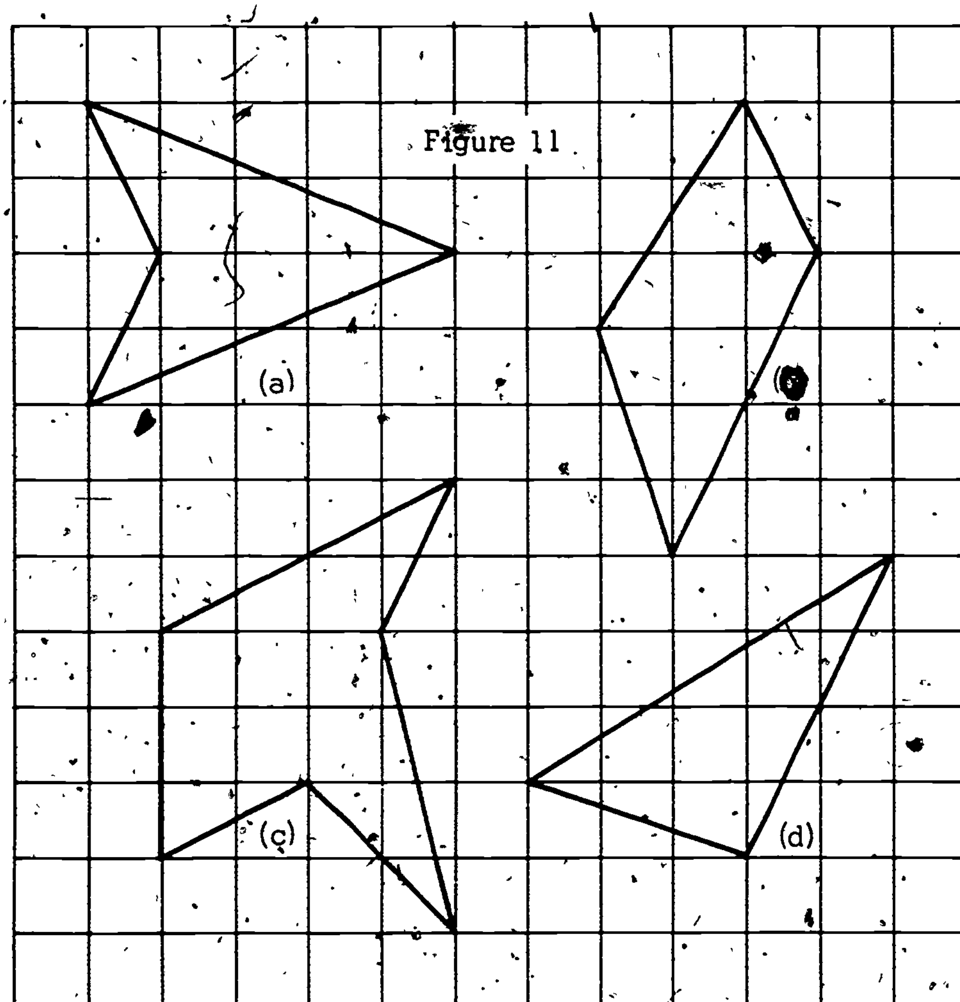


Figure 11

SECTION 3 BRACKETING AREAS

Centimeter squares do not fit exactly into a region enclosed by a curved line (Figure 12). Therefore, we cannot find the area directly by counting unit centimeter squares. We can, however, find an area smaller than the region and an area larger than the region. So we can bracket the area of the region between the smaller and larger areas.

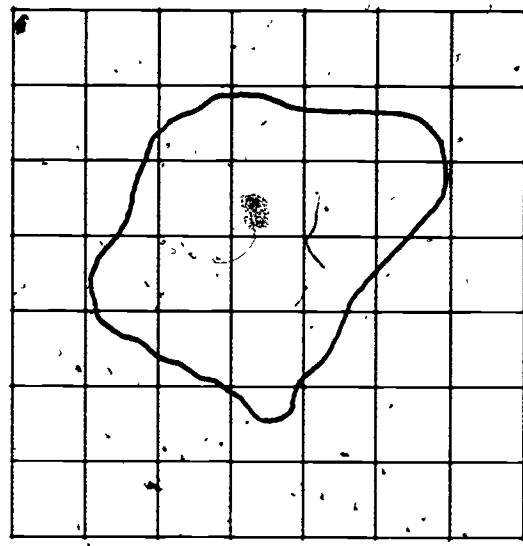


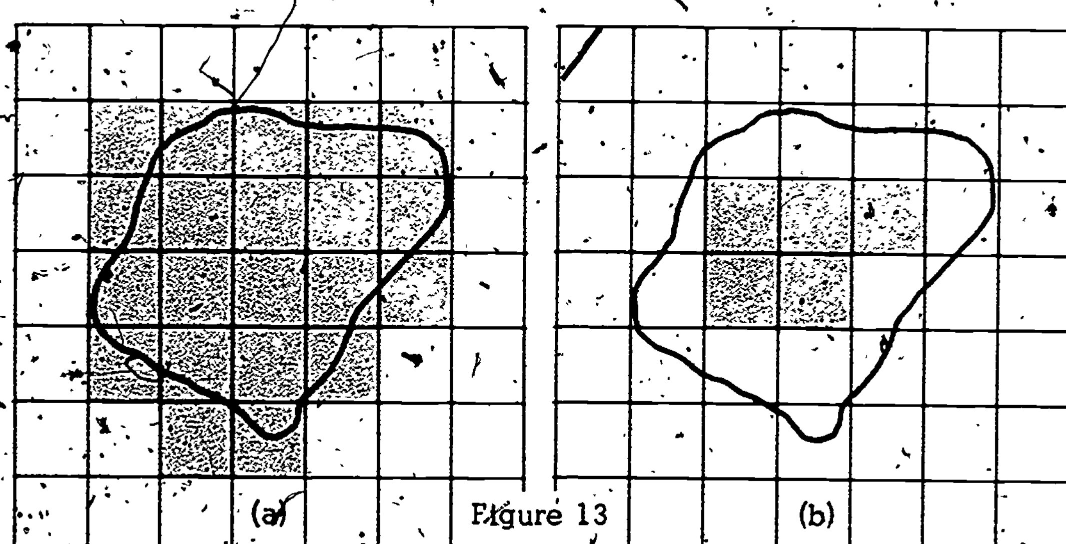
Figure 12

Counting the squares that completely cover the region, we get 21 squares (Figure 13(a)). Each square is 1 cm^2. Therefore

$$\text{area of region} < 21\ cm^2$$

Counting the squares that lie entirely within the region, we get five squares (Figure 13(b)). Therefore

$$5\ cm^2 < \text{area of region} < 21\ cm^2$$



(a) Figure 13 (b)

A value halfway between the smaller and larger bracketing values is usually a good approximation of the area of a region. This halfway value is the average of the lower and upper bracketing values. The average of 5 cm^2 and 21 cm^2 is

$$\frac{5\ \text{cm}^2 + 21\ \text{cm}^2}{2} = \frac{26\ \text{cm}^2}{2} = 13\ \text{cm}^2$$

Since the actual value of the area could be any number between 5 cm^2 and 21 cm^2, our 13 cm^2 approximation could be off by as much as 8 cm^2. This is because the difference between the average and the lower bracket is 8 cm^2 and the difference between the average and the upper bracket is 8 cm^2. To show the uncertainty of the value we say the area is 13 cm^2 <u>to within</u> 8 cm^2.

13. Find the average of the following pairs of numbers.
    (a) 14, 22
    (b) 19, 25
    (c) 112, 137
    (d) 150, 179

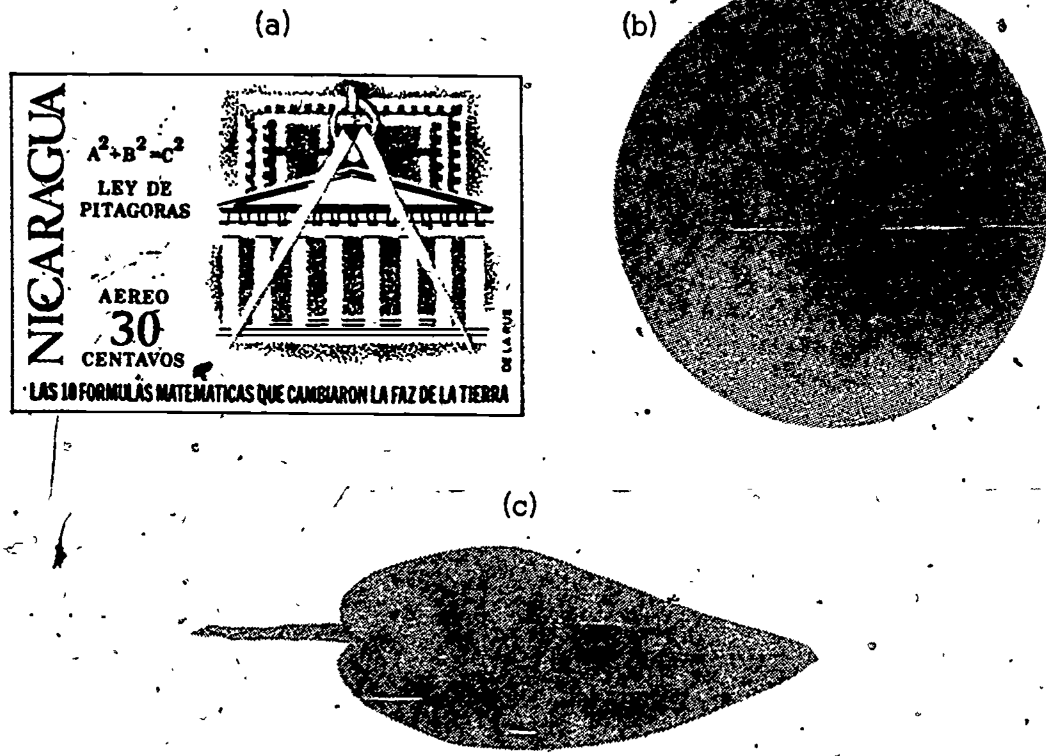
(a) Bracket the area of the regions in Figure 14. (Use your centimeter grid.)

(b) What is the average of the upper and lower brackets in each case?

(c) How good an approximation is each average?

(d) Which area are you most sure of? Why?

Figure 14



SECTION 4 MEASURING AREAS IN FRACTIONS OF A UNIT AREA

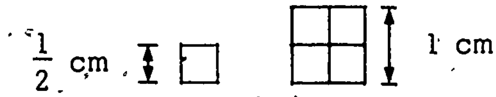
In Section 3 when we used a centimeter grid to bracket the area in Figure 12 we found that

$$5\ \text{cm}^2 < \text{area} < 21\ \text{cm}^2$$

Knowing that the area of the region is between 5 cm^2 and 21 cm^2 gives us only a very rough idea of what the area really is. We can get a better estimate of the area if we use a grid with smaller squares. Let us see what happens when we bracket the area in Figure 12 with squares that are only $\frac{1}{2}$ cm on a side.

A square $\frac{1}{2}$ cm on a side has an area of $\frac{1}{4}$ cm^2 because four such squares are needed to exactly cover a square 1 cm on a side (Figure 15).

Figure 15



Counting the squares in Figure 16(a) that completely cover the region we get 70 of the $\frac{1}{4}$ cm^2 squares. This is

$$70 \times \frac{1}{4}\ cm^2 = 17.5\ cm^2$$

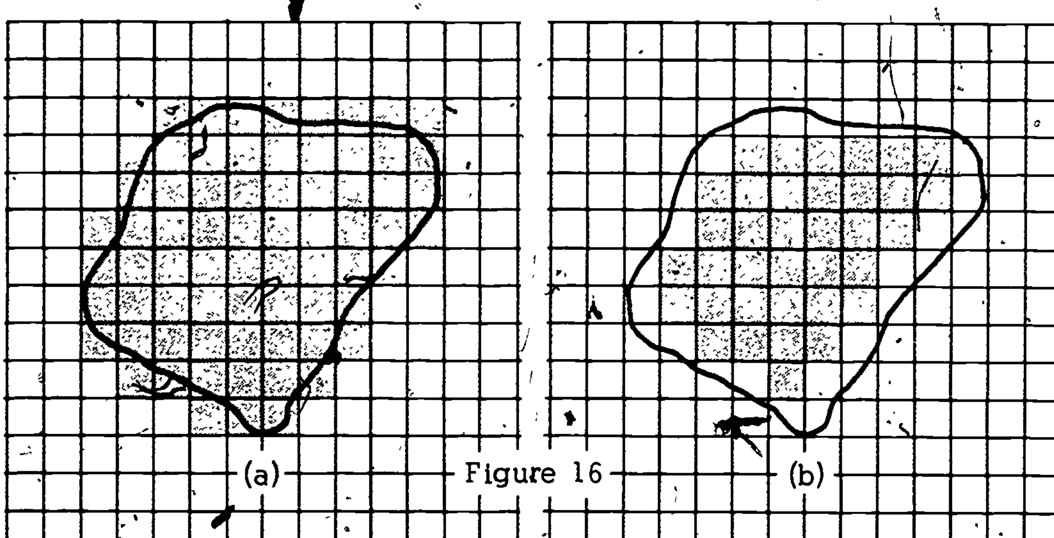
A count of the number of squares that lie entirely within the region (Figure 16(b) gives 36 of the $\frac{1}{4}$ cm^2 squares. This is

$$36 \times \frac{1}{4}\ cm^2 = 9.0\ cm^2$$

Therefore

$$9.0\ cm^2 < \text{area} < 17.5\ cm^2$$

This result is more accurate than the first since now we can say that the area in Figure 12 is $\frac{9+17.5}{2}$ cm^2 = 13.25 cm^2 to within 4.25 cm^2.

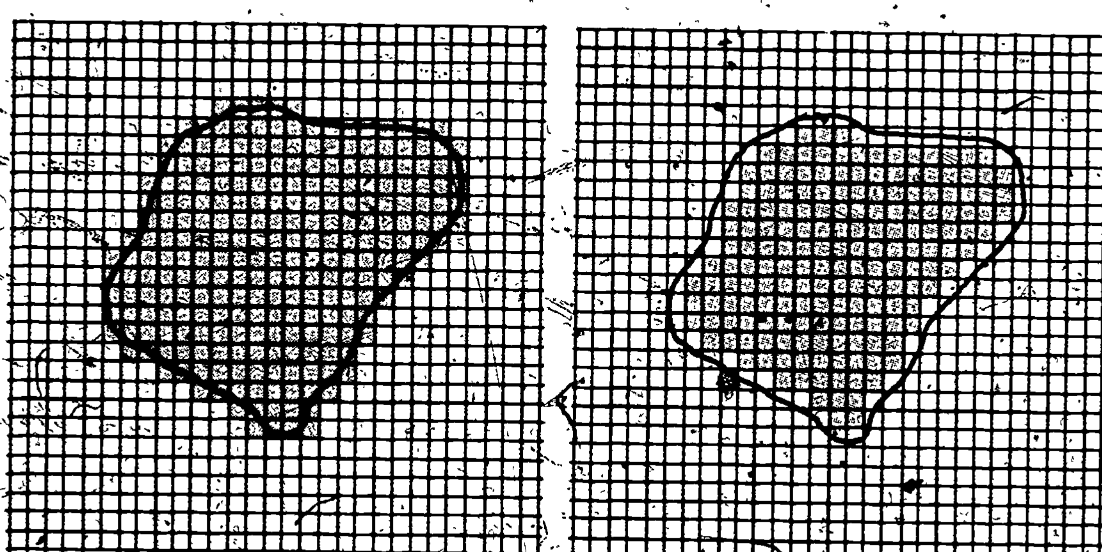


(a) Figure 16 (b)

Figure 17 shows the same region used in Figure 12 but it is covered with squares $\frac{1}{4}$ cm on each side. Because it takes 16 such squares to fill 1 cm^2, each little square has an area of $\frac{1}{16}$ cm^2.

Bracket the area of the region by using Figures 17(a) and 17(b). Since the counting may be tedious, you may want to think of a way to make it easier.

Figure 17



(a) (b)

What bracketing values have you found? What is their average? To within how many square centimeters is your result accurate?

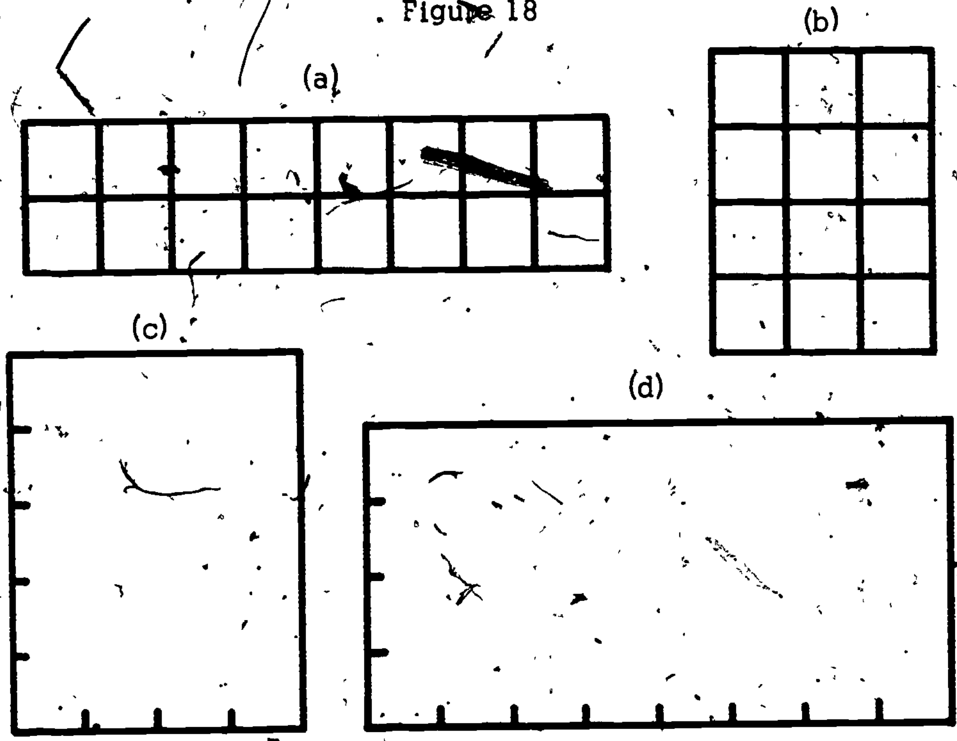
Improve the measurement of the area of the regions in Figure 14 by using one of the smaller grids.

SECTION 5 A SHORTCUT FOR COUNTING AREA UNITS — AREAS OF RECTANGLES

It is tedious to count a very large number of small unit squares as you did in Section 4. Fortunately there is a shortcut that can make the job easier.

Try to find a "shortcut" for counting the number of square units in each of the rectangles in Figure 18. Find their areas using your shortcut. Compare your rule with those of your classmates.

Figure 18



You probably have found a quick way of counting the number of square centimeters in a rectangle.

One way is to count the 1 cm^2-squares in one row and multiply by the number of rows. For example, in Figure 18(b) we have three 1 cm^2-squares in one row. That is, the area of one row is $3 \times 1\ cm^2$. We have four such rows. Therefore, the total area is

$$4 \times 3 \times 1\ cm^2 = 12\ cm^2$$

Since the unit of area (cm^2) is a square, 1 cm on a side, we can write the above multiplication as

$$4\ cm \times 3\ cm = 12\ cm^2$$

We write the unit of length (cm) and the unit of width (cm) after the numbers that are the measure of length and width. We do this as a convenience to make sure that both dimensions are measured in the same units. Suppose we measured the length of a rectangle in meters and the width in centimeters. Then the product would be neither the area in square centimeters nor in square meters. So if the units are the same, the area of a rectangle is the length times the width.

$$\text{area of rectangle} = \text{length} \times \text{width}$$

14. Find the area of the rectangles in Figure 19.

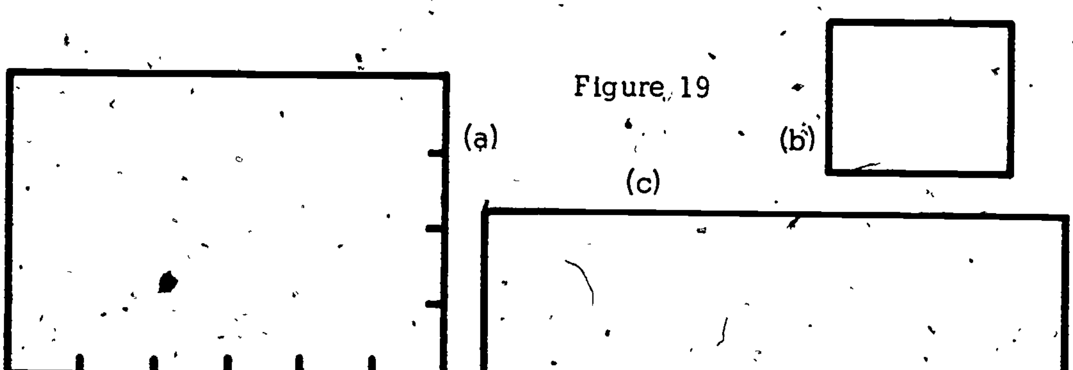


Figure 19

15. (a) How many centimeters are there along a side of a square meter?

(b) How many square centimeters are there in one square meter ($1\ m^2$)?

16. How many square meters are there in a square kilometer (km^2)?

17. Find the area of rectangles with

(a) length, 12 cm
width, 4 cm

(b) length, 23 cm
width, 14 cm

(c) length, 20.5 m
width, 9.7 m

(d) length, 46.7 km
width, 9.2 km

18. Find the area of the following rectangles:

(a) length, 1.2 m
width, 80 cm

(b) 700 m by 2 km

(c) 65 cm by 12.8 m

(d) 8 cm by 5 km

19. You plan to paint a porch floor. It is 40.0 ft long and 12.0 ft wide. You know that one gallon of the paint you are going to use will cover 400 square feet of surface. How many gallons of paint will you need?

20. Draw a rectangle whose area is

(a) $12\ cm^2$

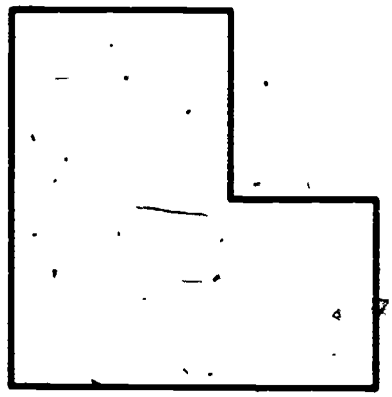
(b) $25\ cm^2$

(c) $15\ cm^2$

(d) $24\ cm^2$

21. In Figure 20 subdivide the enclosed region into rectangles. Find the area of each rectangle, and then find the total area of the enclosed region.

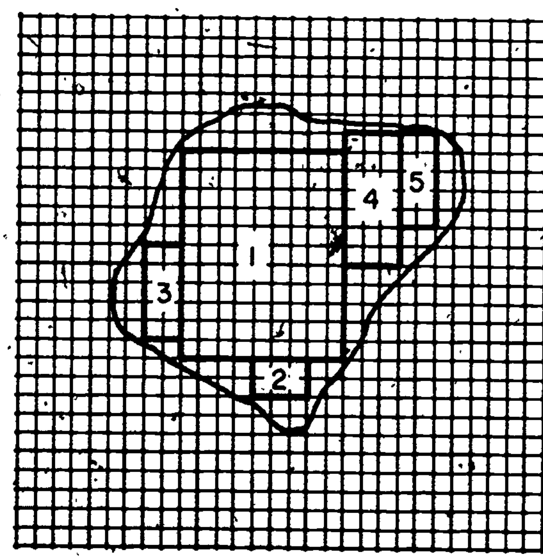
Figure 20



22. (a) Figure 21 shows the region of Figure 17 enclosing five rectangles. What is the total area of the five rectangles?

(b) How would you use your answer to part (a) to find a lower bracket for the area of the region?

Figure 21



SECTION 6 MULTIPLYING FRACTIONS

In finding the area of a rectangle we can use multiplication as a short cut to count area units. For example, to count the 1 cm^2 squares in a 5 cm by 6 cm rectangle we multiply 5 by 6.

So far, however, we have not tried to find areas of rectangles with lengths or widths that are fractions. For example, how do we find the area of a $\frac{2}{3}$ by $\frac{3}{4}$ rectangle?

To get the $\frac{2}{3}$ by $\frac{3}{4}$ rectangle we divide one side of a unit square into $\frac{1}{3}$'s and another into $\frac{1}{4}$'s. This divides the unit square into $3 \times 4 = 12$ equal parts. Each part is $\frac{1}{12}$ of the unit square as shown in Figure 22.

Now we shade a $\frac{2}{3}$ by $\frac{3}{4}$ rectangle (Figure 23). It has two rows and three columns, or $2 \times 3 = 6$ of the 12 equal parts. Therefore, the rectangle is $\frac{6}{12}$ of a unit square. Since the area of a rectangle is length times width we conclude

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$$

Figure 22

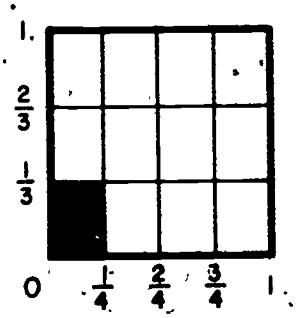
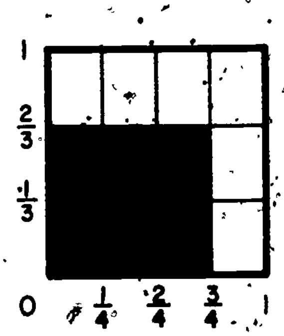
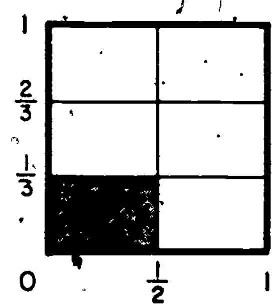


Figure 23



Let us look at another example. What is the area of a $\frac{2}{3}$ by $\frac{5}{2}$ rectangle? In other words, what is $\frac{2}{3} \times \frac{5}{2}$? Since we have $\frac{1}{3}$'s and $\frac{1}{2}$'s we divide a unit square into three rows and two columns or $3 \times 2 = 6$ equal parts (Figure 24).

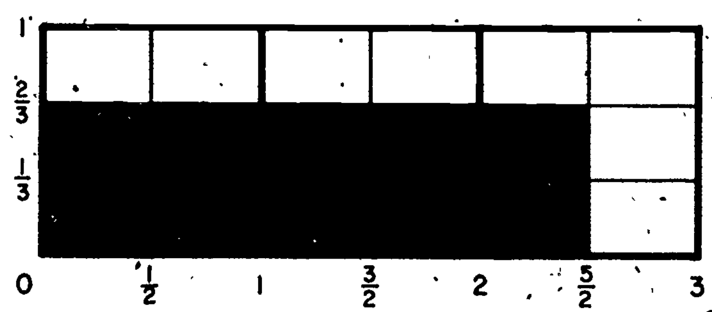
Figure 24



To get a $\frac{2}{3}$ by $\frac{5}{2}$ rectangle, however, we need three unit squares because $2 < \frac{5}{2} < 3$. So we add two unit squares to the one in Figure 24. This is done in Figure 25. The shaded rectangle in this figure has two rows and five columns or $2 \times 5 = 10$ of the six equal parts. Therefore

$$\frac{2}{3} \times \frac{5}{2} = \frac{10}{6}$$

Figure 25

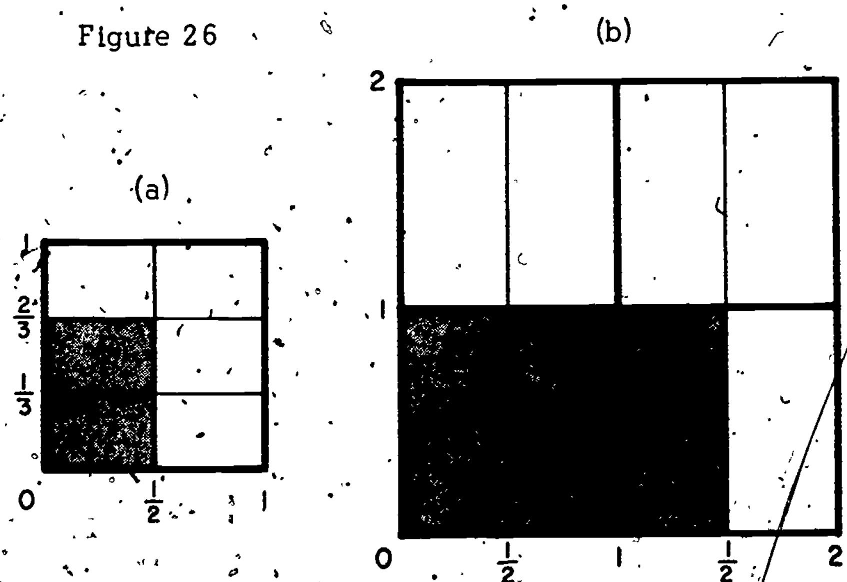


Note that in both examples (1) the denominator of the answer is the product of the denominators of the fractions being multiplied; and (2) the numerator of the answer is the product of the numerators. The examples suggest that we can multiply fractions in two steps. First we multiply the denominators. This product gives us the number of equal parts into which we divide the unit square. Next we multiply the numerators. This gives us the number of the equal parts we need. Therefore to multiply fractions together we multiply numerator by numerator and denominator by denominator. For example

$$\frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{5 \times 4} = \frac{6}{20}$$

23. What products are illustrated by the shaded rectangles in Figure 26?

Figure 26



24. Multiply the following fractions.

(a) $\frac{3}{5} \times \frac{1}{2}$

(b) $\frac{2}{5} \times \frac{3}{4}$

(c) $\frac{7}{12} \times \frac{9}{8}$

(d) $\frac{8}{3} \times \frac{7}{10}$

(e) $\frac{3}{10} \times \frac{4}{10}$

(f) $\frac{7}{10} \times \frac{3}{100}$

25. What is:

(a) $\frac{2}{5}$ of 2?

(b) $\frac{1}{2}$ of $\frac{1}{4}$?

(c) $\frac{3}{5}$ of $\frac{3}{4}$?

(d) $\frac{5}{8}$ of $\frac{1}{2}$?

26. What is

(a) $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$?

(b) $\frac{2}{3} \times \frac{5}{7} \times \frac{4}{3}$?

(c) $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{3}{5}$?

27. Find the following products. (Hint: change each number from mixed notation to a fraction.)

(a) $1\frac{2}{3} \times 1\frac{3}{5}$

(b) $2\frac{1}{2} \times 3\frac{1}{3}$

(c) $7\frac{5}{8} \times 2\frac{1}{4}$

(d) $11\frac{2}{3} \times 12\frac{4}{5}$

28. Which of the following products are equal?

(a) $\frac{7}{8} \times \frac{9}{10}$ and $\frac{7}{10} \times \frac{9}{8}$

(b) $\frac{3}{4} \times \frac{2}{3}$ and $\frac{4}{5} \times \frac{1}{2}$

(c) $\frac{2}{3} \times \frac{4}{5}$ and $\frac{5}{3} \times \frac{1}{5}$

(d) $1\frac{1}{5} \times \frac{3}{4}$ and $\frac{3}{5} \times \frac{6}{4}$

(e) $\frac{10}{7} \times \frac{12}{9}$ and $\frac{5}{9} \times \frac{24}{7}$

(f) $\frac{2}{3} \times \frac{3}{5} \times \frac{6}{2}$ and $\frac{4}{2} \times \frac{9}{15}$

(g) $\frac{2}{3} \times \frac{4}{6}$ and $\frac{2}{3} \times \frac{2}{3}$

SECTION 7 MATCHING PRODUCTS OF FRACTIONS

This is a card game for two to four players. The object of the game is to win the most cards by matching equal fractions.

You win cards by matching one card from your hand with one or more cards from the table. Your card matches one card from the table if the fractions printed on both are equal, and matches two or more cards if their product equals the fraction on your single card.

You can use pencil and paper to multiply fractions if you need to. Whenever you make a match you must convince the other players that your card does match.

The game begins with the dealer dealing four cards to each player, putting four cards face up on the table in the middle and putting the rest of the deck face down in one pile on the table. The play begins to the dealer's left and continues clockwise.

During a turn, a player tries to match one of his cards with one or more cards from the table. If he makes a match he gets to keep his card and the matched cards in a pile of his own. If he cannot make a match for any reason, he must place one of his cards face up in the middle of the table. After matching or placing his card, he draws one card from the unused deck to replace his card.

When the deck is used up, the play continues until all the cards from everyone's hand are used. The last person to make a match then gets all the remaining cards that are face up on the table.

The player with the most cards is the winner.

SECTION 8 MULTIPLYING DECIMALS BY DECIMALS

To calculate the area of a 37.5 cm by 0.52 cm rectangle we need to multiply decimals. We can do this the same way we multiplied a decimal by a whole number. That is, we do it in two steps. (a) We multiply the two numbers as if they were both whole numbers. (b) We approximate the product, with each factor rounded off to one non-zero digit and the correct number of zeros. We can then use Tables 1 and 2 in Chapter 5 (on page 5-10 and 5-13) as well as Table 1 in this section to find the place value of the product.

<u>Examples</u>:

(1) What is 37.5×0.52? Ignoring the decimal point we have

$$375 \times 52 = 19500$$

Approximating the product gives 4 tens × 5 tenths = 20 tens × tenths. From Table 2 on page 5-13 we see that tens × tenths = ones. So our approximation is 20 ones = 20. Therefore, we place the decimal point so that the answer is close to 20; that is

$$37.5 \times 0.52 = 19.5$$

(2) What is 0.51×0.021? Without decimals $51 \times 21 = 1071$. Rounding off the factors we get 0.5 and 0.02. From Table 1 in this section we see that

(tenths) × (hundredths) = thousandths

Therefore the approximate answer is 5 tenths × 2 hundredths = 10 thousandths or 1 hundredth; and the exact answer must be 0.01071.

TABLE 1

X	ones	tenths	hundredths
ones	ones	tenths	hundredths
tenths	tenths	hundredths	thousandths
hundredths	hundredths	thousandths	ten thousandths

Make two tables like Table 1 using fractions $(\frac{1}{10}, \frac{1}{100})$ and decimals (0.1, 0.01) instead of words.

29. Where do the decimal points go in each of the following products?

(a) $0.21 \times 0.32 = 672$

(b) $0.056 \times 0.8 = 448$

(c) $0.96 \times 0.038 = 3648$

(d) $1.6 \times 0.035 = 560$

(e) $0.082 \times 0.057 = 4674$

30. Without doing the calculations, which of the following results are definitely wrong?

(a) $0.36 \times 0.15 = 0.54$

(b) $0.07 \times 0.96 = 0.0662$

(c) $0.88 \times 0.045 = 0.0396$

(d) $0.063 \times 2.92 = 0.184$

(e) $0.011 \times 0.019 = 0.0002$

31. Find the values of the products (a) through (e).

(a) 0.22×0.06

(b) 0.86×0.76

(c) 0.35×0.026

(d) 0.081×0.012

(e) 0.063×0.58

SECTION 9 ESTIMATING AREAS

Here is an area estimating contest. Break up into five or six teams of equal size. Have your teacher pick out a number of different rectangles whose area you will estimate (area of a blackboard, desk top, door, window, postage stamp, sheet of paper, etc.).

Write down your estimates of length and width (in meters for large rectangles and centimeters for small ones). Then calculate your estimate of the area.

After everyone has written their estimate of the area of a rectangle, two students from different teams measure its length and width and calculate the rectangle's area.

Everyone then finds out how large his or her error is.

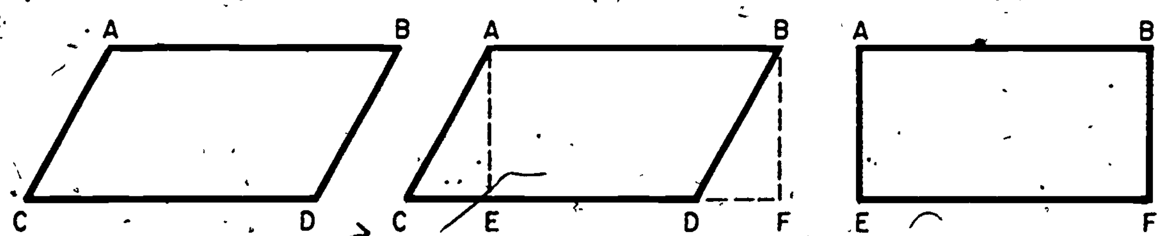
Each team finds the sum of its members' errors. The team with the lowest total wins.

SECTION 10 THE AREAS OF PARALLELOGRAMS AND TRIANGLES

A parallelogram is a quadrilateral that has opposite sides parallel. The angles made by its sides are often not right angles. We can find the area of such a parallelogram by making it into a rectangle.

We start with the parallelogram in Figure 27(a). We cut off the right triangle ACE shown in Figure 27(b) and move it to the other end of the parallelogram. This gives us the rectangle ABEF shown in Figure 27(c). We can easily find the area of this rectangle by multiplying the length of the side AB by the length of the side BF.

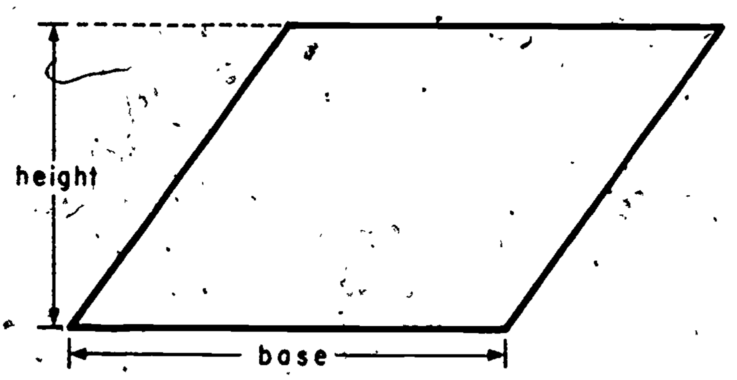
Figure 27



The line segment AE of the parallelogram in Figure 27 is usually called the <u>height</u> of the parallelogram and the line segment EF is called the <u>base</u> (Figure 28). Therefore

area of parallelogram = height × base.

Figure 28



Trace the parallelogram in Figure 29 and cut off a right triangle from either end. Fit the two pieces together to form a rectangle. Measure the sides and find its area.

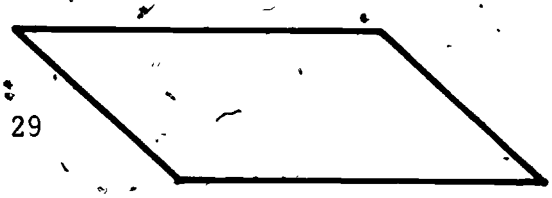


Figure 29

32. Find the area in cm^2 of the parallelograms in Figure 30.

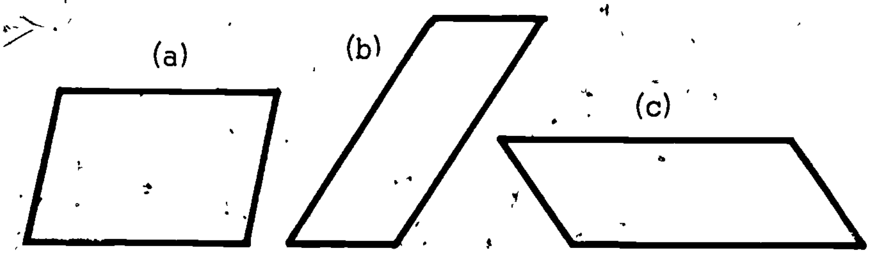


Figure 30

33. Find the area in cm^2 of the parallelogram in Figure 1(b) and the area in cm^2 of the two unit area parallelograms in Figure 1(a).

34. Figure 31 shows a crystal of the mineral calcite. The face ABCD has a base of 3.8 cm and a height of 2.6 cm. What is the area in cm^2 of this face?

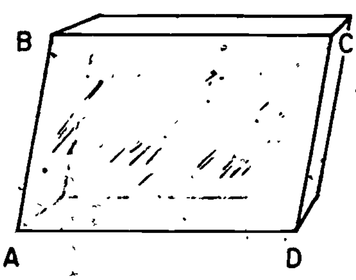
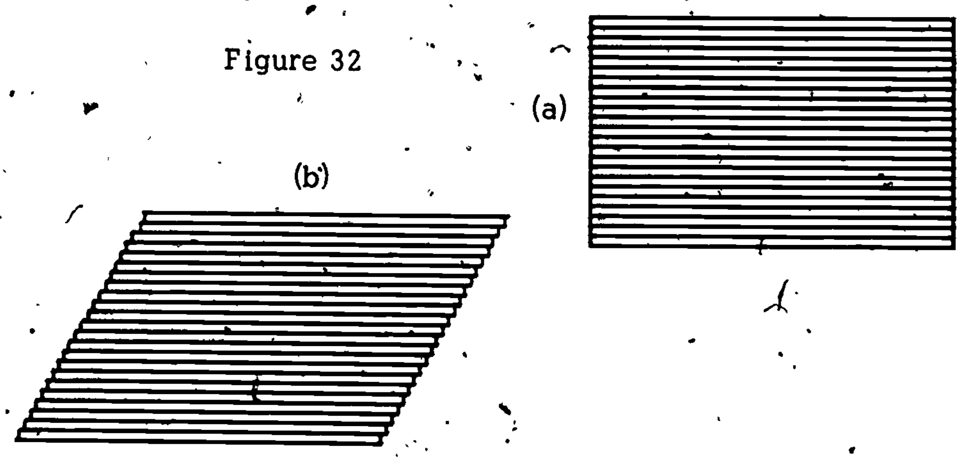


Figure 31

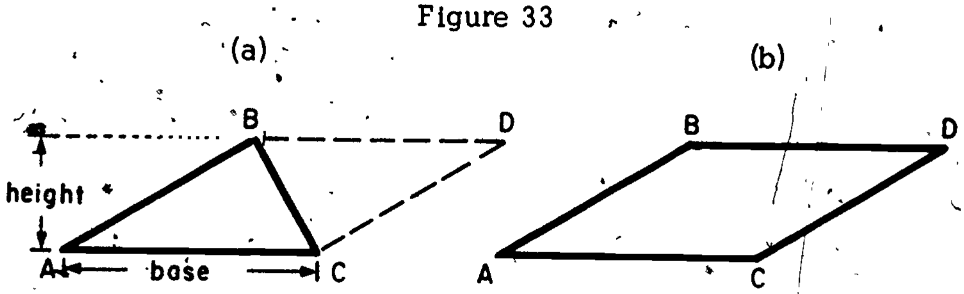
35. A deck of cards is piled up so as to make a rectangle as shown in Figure 32(a). Then the cards are slid one over the other so the deck looks like Figure 32(b) which is a parallelogram. How does the area of the parallelogram in (b) compare with the area of the rectangle in (a)?

Figure 32



To find the area of a triangle we do something much like what we did to find the area of a parallelogram. In this case we make a triangle into a parallelogram. To do this we add another triangle that is the same as the triangle whose area we wish to find. This is shown in Figure 33(a) where the triangle BAC has been flipped over to make the triangle BCD. This gives the parallelogram BACD shown in Figure 33(b).

Figure 33

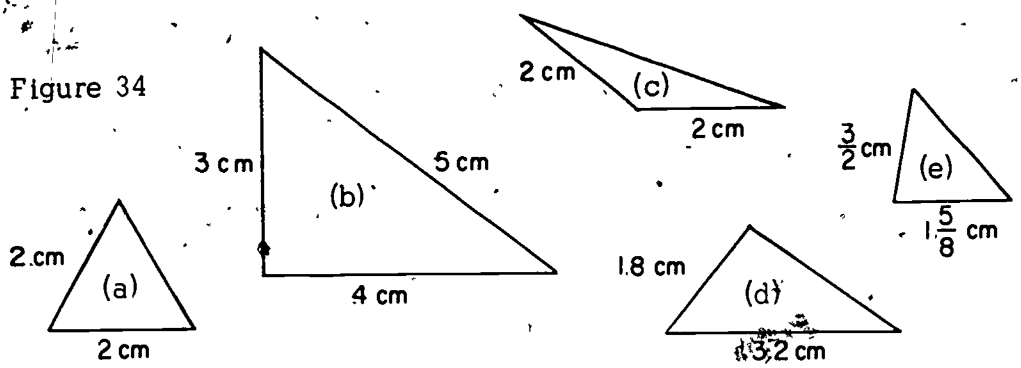


The area of the parallelogram in Figure 33(b) is its height times its base. This area is the same as the area of two triangles like triangle BAC in Figure 33(a). Therefore,

$$\text{area of triangle} = \frac{1}{2} \times \text{height} \times \text{base}$$

36. What is the area of each of the triangles in Figure 34?

Figure 34



37. What is the area of the following triangles?

(a) base = 3.2 m, height = 7.8 m

(b) base = $\frac{1}{3}$ m, height = $\frac{2}{3}$ m.

38. Find the area of the kite in Figure 8(a) on page 10 in Chapter 4.

39. (a) Draw three different triangles with the same base and the same height.

(b) How do their areas compare?

40. (a) Draw three different triangles that have different bases and different heights but all have the same area.

(b) Write down the base and height of each of the triangles in part (a).

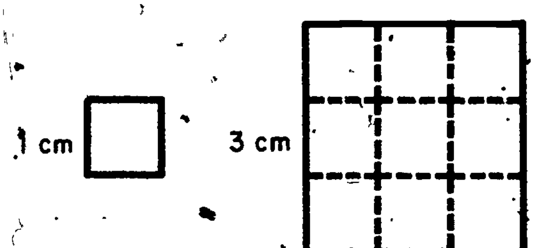
Trace the shaded quadrilateral in Figure 7 on page 9 in Chapter 4 and find its area.

Describe how to find the area of any quadrilateral.

SECTION 11 AREAS OF ENLARGEMENTS

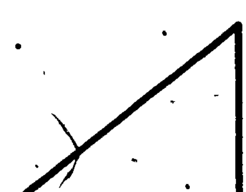
If we are given a square and apply a scaling factor of 3, each side of the enlarged square is three times as long as a side of the small square (Figure 35). But, as you can see, the area of the large square is nine times greater than the area of the small square. Or we can say the ratio of the area of the large square to the area of the small square is 9.

Figure 35



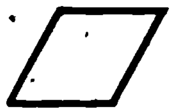
Trace the triangle in Figure 36. Now use the method described on page 4 of Chapter 4 to enlarge the triangle by a scaling factor of 4. What is the ratio of one side of the enlargement to the corresponding side in Figure 36?

Figure 36



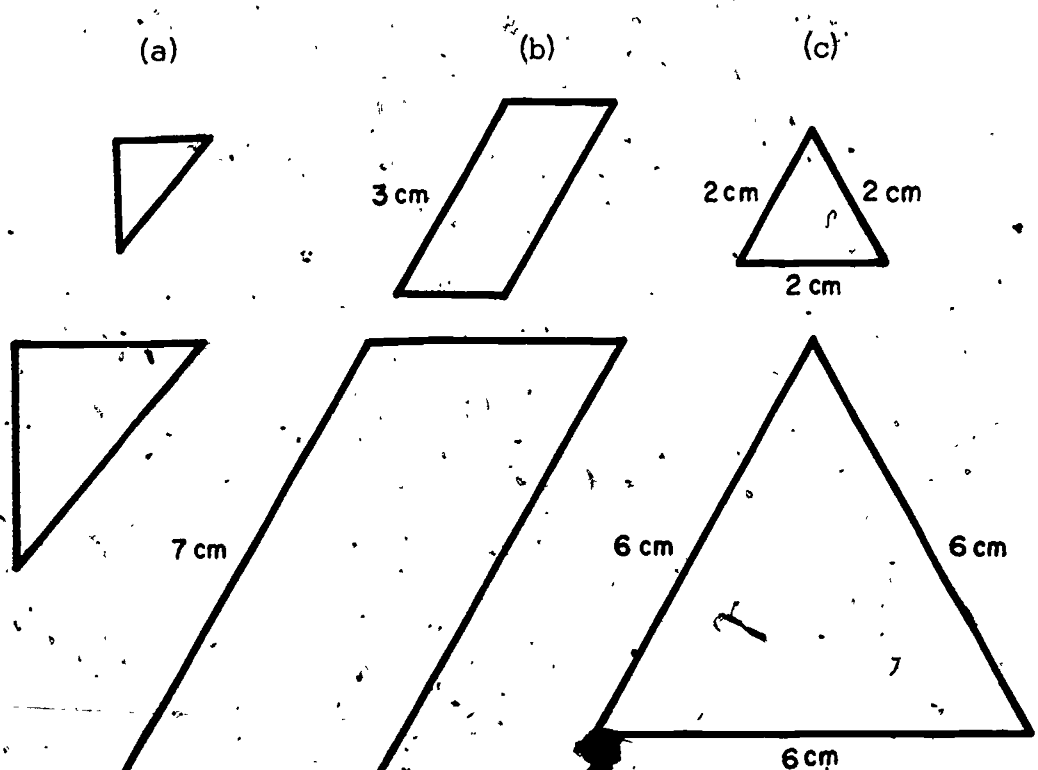
Find the area of each triangle. How many times larger is the area of the new triangle? What is the ratio of the area of the large triangle to the area of the small triangle?

Now make an enlargement of the parallelogram in Figure 37 using a scaling factor of 5. What is the ratio of the area of the large parallelogram to the small one?

Figure 37 

41. Find how many times corresponding lengths are increased and how many times the area is increased in each of the enlargements in Figure 38.

Figure 38



42. The area of a big square is 100 times the area of a smaller square. What is the scaling factor from the small square to the big square?

As you probably have discovered when you make an enlargement of a figure the area of the larger figure does not scale up the same way the lengths of the figure do.

In enlargements, to find out how many times the area is increased we multiply the scaling factor times itself.

Thus, if the scaling factor is 2, all lengths in the enlargement are twice the corresponding lengths in the original. The area of enlargement, however, is four times that of the original.

SECTION 12 AREAS ON MAPS

Scaling factors always apply to lengths, not areas. Consider a map where the scale is $\frac{1}{12,000}$. It is the actual lengths that are reduced by $\frac{1}{12,000}$, not the areas. In fact, the actual areas on such a map are reduced by $\frac{1}{12,000} \times \frac{1}{12,000} = \frac{1}{144,000,000}$.

43. A student is making a map of the lot his house is on. The lot is 20 m by 28 m. He decides to make the scale for his map $\frac{1}{200}$. What is the ratio of each actual length to the corresponding length on his map? What is the ratio of the actual area to the map's area?

44. Turn back in your book to Figure 1 in Chapter 5. To what scale is the figure drawn? Find the area of

(a) the "utilities" room;

(b) the bench next to the darkroom.

45. Using Figure 1 in Chapter 5 find out how many square meters of carpet are needed to cover the floor of the art and editing room.

46. Suppose the field next to the Day Junior High School (Chapter 5, page 5) needs re-seeding. If one box of grass seed will cover about 50 m^2, how many boxes of seed will be needed to re-seed the field?

47. Given a map with a scaling factor of $\frac{1}{1,000,000}$ what would be the actual land area in km^2 of

(a) a square 1 cm on a side on the map?

(b) a square $\frac{1}{2}$ cm on a side on the map?

48. Use the map on page 20 of Chapter 5 to find the area in km^2 of

(a) Flint.

(b) Pontiac.

49. Use a grid to bracket the area of the Detroit-Windsor region on the map on page 29 of Chapter 5.