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ABSTRACT

In this guide for teachers using the SMSG text materials for grade 5, five chapters on numeration systems, factors and primes, multiplication and division, and congruency of geometric figures are considered. The purpose is stated for each unit and mathematical background for the teacher is presented. Teaching procedures are then detailed through specific activities, statements, questions, and anticipated responses. Exercise sets and answers are also included. (MS)

Mathematics for the Elementary School, Grade 5

Teacher's Commentary, Part I

REVISED EDITION

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

CONTENTS

Chapter

PREFACE

As one of its contributions to the improvement of mathematics in the schools of this country, the School Mathematics Study Group has prepared a series of sample text materials for grades 4 through 6. These are designed to illustrate a kind of mathematics curriculum that we believe appropriate for elementary schools.

This volume is a portion of these materials which were prepared by a group of 30 individuals, divided almost equally between distinguished college and university mathematicians and master elementary teachers and consultants. A strong effort has been made on the part of all to make the content of this text material mathematically sound, appropriate and teachable. Preliminary versions were used in numerous classrooms both to strengthen and to modify these judgments.

The content is designed to give the pupil a much broader concept, than has been traditionally given at this level, of what mathematics really is. There is less emphasis on rote learning and more emphasis on the construction of models and symbolic representation of ideas and relationships from which pupils can draw important mathematical generalizations.

The basic content is aimed at the development of some of the fundamental concepts of mathematics. These include ideas about: number; numeration; the operations of arithmetic; and intuitive geometry. The simplest treatment of these ideas is introduced early. They are frequently re-examined at each succeeding level and opportunities are provided throughout the texts to explore them more fully and apply them effectively in solving problems. These basic mathematical understandings and skills are continually developed and extended throughout the entire mathematics curriculum, from grades K through 12 and beyond.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that these texts may greatly assist all pupils and teachers who use them to achieve this goal, and that they may experience something of the joy of discovery and accomplishment that can be realized through the study of mathematics.

Chapter 1

EXTENDING SYSTEMS OF NUMERATION

PURPOSE OF UNIT

This unit is an extension of the work of Chapters 2 and 10 of Fourth Grade.

(a) The decimal system of numeration, with its principle of place-value, is extended to involve numerals for whole numbers larger than those considered in Chapter 2.

(b) The decimal system of numeration, with its principle of place-value, is extended to the right of the ones' column to embrace the writing of numerals in decimal form for tenths, hundredths, and thousandths.

(c) Non-decimal systems of numeration, with a principle of place-value, are extended to cover the writing of three-place numerals. This is introduced primarily as a means to a greater understanding of the decimal system particularly and the nature of numeration generally. Only when the decimal system is studied in the context of place-value systems do certain of its properties emerge clearly.

In addition to the mathematical background which follows, you will find it helpful to study Chapter 2 (pages 17-49) of Number Systems (SMSG Studies in Mathematics, Volume VI).

MATHEMATICAL BACKGROUND

Principles of numeration cannot be developed effectively if confusion exists regarding the terms number and numeral. These are not synonymous. A number is a concept, an abstraction. A numeral is a symbol; a name for a number. A numeration system is a numeral system (not a number system), a system for naming numbers.

Admittedly, there are times when making the distinction between "number" and "numeral" becomes somewhat cumbersome. However, an attempt has been made in this unit to use terms such as number, numeral, and numeration with precise mathematical meaning.

This may be an appropriate time to comment on our use of the equals sign (=). For example, when we write

$$5 + 2 = 8 - 1$$

we assert that the symbols "5 + 2" and "8 - 1" are each names for the same thing - the number 7. In general, when we write

$$A = B$$

we do not mean that the letters or symbols "A" and "B" are the same. They very evidently are not! What we do mean is that the letters "A" and "B" are synonyms. That is, the equality

$$A = B$$

asserts precisely that the thing named by the symbol "A" is identical with the thing named by the symbol "B". The equals sign always should be used only in this sense.

The naming of numbers is a problem that has received attention over a period of many, many years. Sources such as the one mentioned earlier (Studies in Mathematics, Volume VI) give interesting and helpful information in this connection. For our immediate purposes it will suffice to consider only the underlying nature of the scheme for naming numbers that we use commonly today.

We are so familiar with our decimal system of numeration that we may fail to sense clearly that it is only one instance of the class of numeration systems. These are called place-value systems because they use the same idea of place-value.

We learn, for example, that the symbol 213 (read "two one three") means

$$2(\text{ten} \times \text{ten}) + 1 \text{ ten} + 3 \text{ ones}.$$

It is because the base of our numeration system is by convention ten and not nine that we give 213 this interpretation and not

$$2(\text{nine} \times \text{nine}) + 1(\text{nine}) + 3(\text{ones}).$$

Both interpretations belong to what can be called place-value numeration systems. In any such system 213 would designate

$$2(n \times n) + 1(n) + 3(\text{ones}).$$

The different systems correspond to the possible choices of the number n, called the base of the numeration system.

Because the numeral 213 has different meanings in different place-value systems, it is necessary to indicate the base of the system which is intended. We do this by writing the word name for the base as a subscript if the base is not ten. Thus

$$213 = 2(\text{ten} \times \text{ten})(\text{hundreds}) + 1(\text{ten}) + 3(\text{ones}),$$
$$213_{\text{nine}} = 2(\text{nine} \times \text{nine}) + 1(\text{nine}) + 3(\text{ones}),$$
$$213_{\text{eight}} = 2(\text{eight} \times \text{eight}) + 1(\text{eight}) + 3(\text{ones}).$$

(The symbol 213_{nine} is read "two one three, base nine".)

In any place-value system arbitrary symbols are needed as numerals for whole numbers less than the base of the system. These numbers are called the digits of the numeral system. Since there are available conventional symbols for the digits of the decimal system, we can adopt these as the numerals for the digits of other systems. No new symbols will be needed provided we restrict consideration to systems with bases no greater than ten. Thus in the base eight system we name the digits 0, 1, 2, 3, 4, 5, 6, and 7. In the base five system we name the digits 0, 1, 2, 3, 4.

Since any symbol such as 3, whenever used as the numeral for a digit will name the same number in every system in which it appears, this convention is unambiguous. The numerals for digits therefore require no subscript. As is often done in this chapter the subscript may, however, be added as a reminder of the system under consideration.

In giving the interpretation of a place system numeral like 213_{nine} it can be confusing to use numerals from another place system. Thus

$$213_{nine} = (2 \times 81) + (1 \times 9) + 3$$

involves the decimal numerals 9 and 81, and the latter requires for its interpretation the very idea it is assisting to explain. This difficulty arises because all place systems derive from the same principle of construction and because one of these systems, the decimal system, is our "native" system.

In such a situation it seems preferable to restrict the explanatory use of numerals to the single digit numerals common to all place systems under consideration. The other numbers involved are named by words which are not part of any of the systems being discussed. Thus we prefer to write

$$213_{nine} = 2(\text{nine} \times \text{nine}) + 1(\text{nine}) + 3(\text{ones})$$

or $$213_{nine} = 2(\text{eighty-ones}) + 1(\text{nine}) + 3(\text{ones}).$$

This is of course just the sort of explanation we are compelled to give for decimal numerals, and it therefore has the added advantage of revealing without bias the common aspects of all place systems.

A word about the distinction between symbols and names may be in order. In any context where a symbol is used in more than one way it is important to distinguish the symbol as an object in itself from the symbol as a name of something. That is why the <u>symbol</u> 213 is read "two one three" and not "two hundred thirteen". The latter is appropriate only when the symbol is employed as a decimal numeral. Similarly, to read 213_{nine} as "two hundred thirteen, base nine" would be to suggest a decimal interpretation which is not intended. That is why we read 213_{nine} "two one three, base nine". It is important that such distinctions be made from the beginning in any discussion of numeral systems.

A chart such as the following one is helpful in sensing better the numeral sequence for place-value numeration systems with different bases.

Base							
Ten	Nine	Eight	Seven	Six	Five	Four	Three
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	10
4	4	4	4	4	4	10	11
5	5	5	5	5	10	11	12
6	6	6	6	10	11	12	20
7	7	7	10	11	12	13	21
8	8	10	11	12	13	20	22
9	10	11	12	13	14	21	100
10	11	12	13	14	20	22	101
11	12	13	14	15	21	23	102
12	13	14	15	20	22	30	110
13	14	15	16	21	23	31	111
14	15	16	20	22	24	32	112
15	16	17	21	23	30	33	120
16	17	20	22	24	31	100	121
17	18	21	23	25	32	101	122
18	20	22	24	30	33	102	200
19	21	23	25	31	34	103	201
20	22	24	26	32	40	110	202
21	23	25	30	33	41	111	210
22	24	26	31	34	42	112	211
23	25	27	32	35	43	113	212
24	26	30	33	40	44	120	220
25	27	31	34	41	100	121	221

As seen from the chart, the base numeral always appears as <u>10</u> when written in that particular base system. Similarly, in a particular base system the numeral <u>100</u> always designates the base squared i.e., the base times itself. In the chart, all numerals in the same row name the same number.

Extension of a place value system of numeration to the right of the ones' column is not restricted to a system whose base is ten. As before, a numeral such as 13.24 may be interpreted in various ways depending upon the base used.

$$13.24 = (1 \times 10) + (3 \times 1) + (2 \times \frac{1}{10}) + (4 \times \frac{1}{100})$$

$$13.24_{nine} = (1 \times 9) + (3 \times 1) + (2 \times \frac{1}{9}) + (4 \times \frac{1}{81})$$

$$13.24_{eight} = (1 \times 8) + (3 \times 1) + (2 \times \frac{1}{8}) + (4 \times \frac{1}{64})$$

$$13.24_{seven} = (1 \times 7) + (3 \times 1) + (2 \times \frac{1}{7}) + (4 \times \frac{1}{49})$$

$$13.24_{six} = (1 \times 6) + (3 \times 1) + (2 \times \frac{1}{6}) + (4 \times \frac{1}{36})$$

$$13.24_{five} = (1 \times 5) + (3 \times 1) + (2 \times \frac{1}{5}) + (4 \times \frac{1}{25})$$

Notice that, for symbolic simplicity, we have used decimal numerals in explaining other place-value numerals. In some respects it may be clearer to write

13.24_{nine} = 1(nine) + 3(ones) + 2(one-ninths) + 4(one-eighty firsts).

and

13.24 = 1(ten) + 3(ones) + 2(one-tenths) + 4(one-hundredths).

Since the decimal system is in nearly universal use the value in introducing any other base may be questioned. The principle object in doing so is to improve understanding of the properties of the decimal system by relating it to a general scheme. This provides a perspective which should promote useful insights such as:

(1) the distinction between properties of numbers and properties of <u>numerals</u>. For example the statement $3 + 7 = 7 + 3$ reflects a <u>number</u> property which is independent of the language (numeral system) in which it is expressed.

(2) the distinction between general properties of <u>all</u> place-value systems and particular decimal properties. For example, the statement $3 + 7 = 10$ is peculiar to the decimal system, while the <u>procedures</u> for adding, subtracting, multiplying, and dividing are the same in any place system.

Such insights should help to reinforce the learning of both number properties and computational skills.

TEACHING PROCEDURES

UNDERSTANDING OUR SYSTEM OF NUMERATION

Objective: To review the structure of the decimal numeral system

Materials: Place-value chart

Exploration:

> The numerals used in the following discussion should be written on the chalkboard and a place-value chart should be used.
>
> In a numeral such as 436, there are two things to be stressed in relation to the idea of place-value. One of these deals only with the place-value associated with each digit. For example, in 436 the 4 is in the hundreds' place, the 3 is in the tens' place, and the 6 is in the ones' place. The other thing to be stressed is the number represented by each digit in relation to place-value. For example, in 436 the 4 represents 4 hundreds, or 400, the 3 represents 3 tens or 30, and the 6 represents 6 ones or 6. Both of these ideas are stressed in the following discussion.

Let us review our decimal system of numeration. Look at the numeral 936,427. In what position is the 9 located? (hundred-thousand's place) In what position is the 3 located? (ten-thousand's place) In what position is the 6 located? (thousand's place) In what position is the 4 located? (hundred's place) In what position is the 2 located? (ten's place) In what position is the 7 located? (one's place) What is the value of the place in which the 3 is written? (ten thousand) What is the value of the place in which the 2 is written? (ten) What is the value of the place in which the 4 is written?. (One hundred) What is the value of the place in which the 9 is written? (One hundred thousand) What is the value of the place in which the 6 is written? (One thousand) What is the value of the place in which the 7 is written? (One)

> Write the numeral, 444,444 on the chalkboard. Point to each four and ask: "What number is represented by this 4?" (4; 40; 400; 4,000; 40,000; 400,000) Point to two separated fours, for instance, the 4 in the thousands' place and

the 4 in the tens' place. This 4 means 4,000 and this 4 means 40. Point to the 4 in the thousands' place and to the 4 in the tens' place. The number represented by this 4 is how many times as large as the number represented by this 4? (100) Follow this with other examples.

Suppose we write a 1 to the left of the numeral 936,427. Can you read this numeral? (One million, nine hundred thirty-six thousand, four hundred twenty-seven) In what place is the digit 1? (million) What number is represented by this 1? (1,000,000) If you had 936,427 and wrote a 1 to the right of the numeral (9,364,271), or wrote a 1 to the left of the numeral (1,936,427), which numeral would represent the larger number? (9,364,271) Why? (When the 1 is written to the left of the numeral, it is in the millions' place, and the place-values of the rest of the digits remain the same value. When the 1 is written to the right of the numeral, all the digits represent numbers that are ten times as large as they were before.)

Provide further practice in analyzing other seven-place numerals.

Chapter 1

EXTENDING SYSTEMS OF NUMERATION

UNDERSTANDING OUR SYSTEM OF NUMERATION

Place Value Name	Hundred Millions	Ten Millions	One Millions	Hundred Thousands	Ten Thousands	One Thousands	Hundreds	Tens	Ones
Digits			1,	2	3	4,	5	6	7

In our decimal system each place or position in a numeral has a name. This name tells its value - ones, tens, hundreds, etc. For instance, in 24, the 4 means 4 ones. In 421, the 4 means 4 hundreds.

Look at the chart above. Tell what number is represented by each digit in the numeral 1,234,567.
(1,000,000; 200,000; 30,000; 4,000; 500; 60; 7)

If the 5 in the numeral above is changed to 9, how much was added to the original number? (400)

What happens to the number, if the 3 is replaced with a 0? (30,000 is subtracted)

READING LARGE NUMBERS

Objective: To learn the reason for the use of "periods" in marking off groups of ones and thousands

Vocabulary: Period

Exploration:

When we read small numerals as in "one hundred sixty-seven", we use the place-value name with each digit.

However, this is not convenient with very large numerals. To make the reading of large numerals easier, we group the numerals in sets of three. These groups of three digits are called <u>periods</u> and are separated by commas as shown:

12,406,037.

To make reading easier and clearer this numeral is interpreted as

12 millions and 406 thousands and 37 ones.

In the reading of the numeral this is modified slightly to:

"twelve million, four hundred six thousand, thirty-seven".

> Write <u>1834695</u> on the chalkboard without commas. Ask a child to read it. Rewrite the numeral, using commas. Ask a child to read it. Discuss which is easier to read.

READING LARGE NUMBERS

Period	Million			Thousand			Units		
Place Name	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones
Digits			1,	2	7	4,	3	6	5

To make it easier to read numerals for large numbers, the names of the digits, the place-value name, and the period name are used. To read the numeral in the table above begin with the period on the left. Read the digit or digits in the first period as one numeral, followed by the name of the period, as "one million".

Then read the second group of digits as one numeral, followed by the name of the period, as "two hundred seventy-four thousand".

Now read the third group of digits as one numeral without the period name, as "three hundred sixty-five".

The complete numeral is read, "one million, two hundred seventy-four thousand, three hundred sixty-five".

P3

In what place is each digit written in the numeral 1,274,365?
(1 in the millions' place, 2 in the hundred thousands' place, 7 in the ten thousands place, 4 in the thousands place, 3 in the hundreds place, 6 in the tens place and 5 in the ones place)
How many commas were used in writing this numeral? (2)

Why is each period separated by a comma? (for ease in reading.)

Explain how to place the commas to help you read a numeral.
(Group the numerals in sets of three starting at the ones place and going to the left.)

Read each of the following numerals.

7,862,419	18,771	5,440,103
275,002	9,030,210	4,564,300

7,862,419: (seven million, eight hundred sixty-two thousand, four hundred nineteen)

275,002: (two hundred seventy-five thousand, two.)

18,771: (eighteen thousand, seven hundred seventy-one)

9,030,210: (nine million, thirty thousand, two hundred ten)

5,440,103: (five million, four hundred forty thousand, one hundred three)

4,564,300 (four million, five hundred sixty-four thousand, three hundred)

Exercise Set 1

1. What number is represented by the symbol 3 in each numeral below?

a) 234,600 (30,000)   d) 413,062 (3000)
b) 98,532 (30)   e) 6,371,524 (300,000)
c) 3,827,129 (3,000,000)   f) 9,317 (300)

2. Write the decimal numeral for each of these.

a) Six thousand, nine hundred thirty-seven (6,937)
b) Nine hundred eight thousand, thirteen (908,013)
c) Four hundred thirty thousand, nine hundred ninety-nine (430,999)
d) Eight million, three hundred five thousand, two hundred fifty-four (8,305,254)
e) Two million, eight hundred twenty thousand, one (2,820,001)

3. Write the name of each numeral in Exercise 1.

BRAINTWISTERS.

4. Write the decimal numeral for each of these.

a) Twenty-two million, four hundred seven thousand, three hundred sixty-one (22,407,361)
b) Seven hundred thirty-six million, five hundred twenty-five thousand, two hundred thirteen (736,525,213)
c) Three hundred million, forty thousand, six (300,040,006)

5. Write the largest possible nine-place decimal numeral using the digits 3, 4, and 6 just once, and as many zeros as necessary. (643,000,000)

EXPANDED NOTATION

Objective: To introduce the writing of numerals in expanded notation

Vocabulary: Expanded notation

Exploration:

You learned in the fourth grade that 634 means 600 + 30 + 4. What is the meaning of 600, 30, and 4? (600 means 6 hundreds, 30 means 3 tens, and 4 means 4 ones.) Six hundred is the product of 6 times what? (100) 30 is the product of 3 times what? (10) 4 is the product of 4 times what? (1)

> Write on the chalkboard each part as it is discussed. The complete chart will be as follows.
>
> 600 = 6 hundreds = (6 × 100)
> 30 = 3 tens = (3 × 10)
> 4 = 4 ones = (4 × 1)

When we write 634 = (6 × 100) + (3 × 10) + (4 × 1), we are writing 634 in expanded notation.

Let us see how we would write a four-place numeral such as 8,172 in expanded notation.

8,172 = (8 × 1000) + (1 × 100) + (7 × 10) + (2 × 1)

Suppose we are writing the expanded notation for 3,206. We will first write (3 × 1,000) + (2 × 100). What will be written next? (0 × 10) Is it always necessary to write (0 × 10)? (No) Why? (0 × n = 0) We could write

3,206 = (3 × 1000) + (2 × 100) + (0 × 10) + (6 × 1) or
3,206 = (3 × 1000) + (2 × 100) + (6 × 1).

> Pupils should have practice in writing other four-place numerals in this way. Additional practice in writing numerals in expanded notation should include five-, six-, and seven-place numerals.

EXPANDED NOTATION

To better understand a number, we learned to add the numbers represented by each digit in the numeral for that number. For example, we learned that 352 can be thought of as 300 + 50 + 2.

Since 300 means 3 hundreds, we can write it as (3 × 100). 50 means 5 tens, which can be written as (5 × 10). 2 ones can be written as (2 × 1). Writing 352 as (3 × 100) + (5 × 10) + (2 × 1) is called expanded notation.

Look at the numerals in the chart below. Place values are written at the top of the chart. Use the chart to help you see how these numerals are written in expanded notation.

	1,000,000	100,000	10,000	1,000	100	10	1	
a				4	2	8	3	= (4 × 1000) + (2 × 100) + (8 × 10) + (3 × 1)
b			2	3	5	8	4	= (2 × 10,000) + (3 × 1,000) + (5 × 100) + (8 × 10) + (4 × 1)
c		6	2	8	7	3	9	= (6 × 100,000) + (2 × 10,000) + (8 × 1,000) + (7 × 100) + (3 × 10) + (9 × 1)
d	7	9	4	3	2	1	5	= (7 × 1,000,000) + (9 × 100,000) + (4 × 10,000) + (3 × 1,000) + (2 × 100) + (1 × 10) + (5 × 1)

Exercise Set 2

1. Write the decimal numeral for each of these following in expanded notation.

   a) 8,134 (8x1000)+(x100)+(3x10)+(4x1)

   b) 2,236 (2x1000)+(2x100)+(3x10)+(6x1)

   c) 14,892 (1x10,000)+(4x1000)+(8x100)+(9x10)+(2x1)

   d) 2,591,622 (2x1,000,000)+(5x100,000)+(9x10,000)+(1x1000)+(6x100)+(2x10)+(2x1)

   e) 49,525 (4x10,000)+(9x1000)+(5x100)+(2x10)+(5x1)

   f) 835,731 (8x100,000)+(3x10,000)+(5x1000)+(7x100)+(3x10)+(1x1)

2. Write the decimal numeral for each of these.

   a) (4 × 1,000) + (2 × 100) + (2 × 10) + (3 × 1) (4,223)

   b) (5 × 1,000) + (8 × 100) + (1 × 10) + (7 × 1) (5,817)

   c) (2 × 10,000) + (2 × 1,000) + (9 × 100) + (6 × 10) + (5 × 1) (22,966)

   d) (9 × 10,000) + (3 × 1,000) + (7 × 10) + (4 × 1) (93,074)

   e) (8 × 100,000) + (1 × 10,000) + (6 × 1,000) + (5 × 100) + (9 × 10) + (2 × 1) (816,592)

3. Write the decimal numeral for each of these. Look carefully at this exercise.

   a) (6 × 10) + (3 × 100) + (5 × 1) (365)

   b) (4 × 100) + (1 × 1,000) + (7 × 1) + (3 × 10) (1,437)

   c) (6 × 1) + (9 × 1,000) + (2 × 10) (9,026)

   d) (4 × 10,000) + (8 × 10) + (2 × 1) + (2 × 100) + (7 × 1,000) (47,282)

   e) (8 × 1,000) + (3 × 10) + (4 × 100,000) + (5 × 1) + (6 × 100) (408,635)

4. BRAINTWISTERS. Fill in the blanks so these mathematical sentences are true.

a) (4 × 100) + (5 × 10,000) + (6 × 1,000) + (8 × 1) + ((7×10)) = 56,478.

b) (9 × 1,000) + (8 × 1) + ((5×100)) + (1 × 10,000) + (8 × 10) = 19,588.

c) (9 × 10) + ((3×1000)) + (8 × 100) + (6 × 10,000) + ((7×1)) + (2 × 100,000) = 263,897.

d) (5 × 10) + ((4×100,000)) + (2 × 10,000) + ((3×100)) + (8 × 1) = 420,358.

(When two parts are missing in Exercise 4, it is not necessary to have the answers in the order given.)

RENAMING LARGER NUMBERS

Objective: To provide practice in renaming five-, six-, and seven-place numerals in a variety of ways

Exploration:

Discuss with the children that in grade four they learned that a number has many names. They should be able to express three- and four-place numerals in a variety of ways.

Begin by writing 1,000 on the chalkboard. Ask the pupils to give some of the ways it can be renamed. For example,

1,000 = 1,000 ones
1,000 = 100 tens
1,000 = 10 hundreds

Continue renaming these powers of ten: 10,000, 100,000, 1,000,000.

Practice renaming multiples of the powers of ten such as 60,000, 490,000, 5,000,000, 2,700,000, etc.

Now consider the four-place numeral, 8,456. Ask the class to give some of the ways it can be renamed. For example,

8,456 = 8 thousands + 4 hundreds + 5 tens + 6 ones
8,456 = 84 hundreds + 5 tens + 6 ones
8,456 = 845 tens + 6 ones
8,456 = 8,456 ones
8,456 = 8,000 + 400 + 50 + 6
8,456 = 8,400 + 50 + 6

Then discuss various ways to express five- and six-place numerals. Although it is important to explore the numerous ways for renaming a number, it is not necessary to exhaust all possibilities. You might, however, point out that an interpretation like 7,000 + 1400 + 40 + 16 is often used in subtraction problems.

RENAMING LARGER NUMBERS

Below are examples showing some of the ways a number can be named.

A.
25,000 = 2 ten thousands + 5 thousands
25,000 = 25 thousands
25,000 = 25,000 ones
25,000 = 250 hundreds
25,000 = 2,500 tens

B.
426,315 = 4 hundred thousands + 2 ten thousands + 6 thousands + 3 hundreds + 1 ten + 5 ones
426,315 = 42 ten thousands + 6 thousands + 3 hundreds + 1 ten + 5 ones
426,315 = 426 thousands + 3 hundreds + 1 ten + 5 ones
426,315 = 425 thousands + 13 hundreds + 15 ones
426,315 = 400,000 + 20,000 + 6,000 + 300 + 10 + 5

Exercise Set 3

1. Write four different names for each of these numbers.

a) 14,651 c) 230,000

b) 27,748 d) 632,110

(there are many possibilities)

2. Write the decimal numeral for each of the following.

a) Twelve thousands + three hundreds + seventeen ones

(12,317)

b) Thirty-eight ten thousands + eight thousands + ninety-four tens + two ones

(388,942)

c) Four ten thousands + twenty-eight hundreds + fifty-three ones

(42, 853)

3. Write each of the following as a decimal numeral.

a) 365 tens + 7 ones

(3,657)

b) 46 hundreds + 2 tens + 5 ones

(4,625)

c) 16 thousands + 12 hundreds + 14 tens

(17,340)

d) 29 ten thousands + 3 thousands + 73 tens + 16 ones

(293,746)

4. Write each of the answers in Exercise 3 in expanded notation.

3,657 = (3x1000) + (6x100) + (5x10) + (7x1)

4,625 = (4x1000) + (6x100) + (2x10) + (5x1)

17,340 = (1x10,000) + (7x1000) + (3x100) + (4x10) + (0x1)

293,746 = (2x100,000) + (9x10,000) + (3x1000) + (7x100) + (4x10) + (6x1)

DECIMAL NAMES FOR RATIONAL NUMBERS

Objective: To develop understanding and skill in reading and interpreting decimal numerals corresponding to fractions with denominators 10 or 100

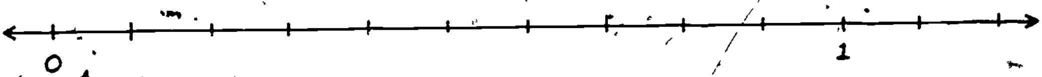
Materials: Place-value chart

Vocabulary: Rational number, fraction, decimal, decimal point

> It is important at the outset for you to understand clearly the way in which certain terms are used in this and subsequent chapters.
>
> There are two methods of naming rational numbers in common use. The first uses fractions (symbols of the form $\frac{a}{b}$) and has already been introduced in Chapter 10 Grade Four. The second is an extension of the place-value concept in the decimal system and uses numerals like .47 and 31.8 which we will call decimals. Since we prefer the term "fraction" to "common fraction" the term "decimal" is preferable to "decimal fraction", because the latter in our terminology does not name a fraction. Thus the numeral $\frac{3}{2}$ and the numeral 1.5 both name the same rational number. The former is a fraction name and the latter a decimal name for that number. Both are names for numbers and therefore numerals.

Exploration:

Let us consider the part of the number line from 0 to a little beyond 1. We can divide the segment of length 1 by 10 points from 0 to 1 into 10 segments of the same length. What are the names for these points? ($\frac{0}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, and so on.)



The length 1 is how many times the length $\frac{1}{10}$? (10 times.) We shall therefore say that the number 1 is 10 times the number $\frac{1}{10}$.

> This amounts to anticipating, in this case, the way multiplication will later be defined for rational numbers:
>
> $$1 = 10 \times \frac{1}{10}.$$
>
> At this stage, however, there is no need to introduce this product notation.

With these facts in mind, let us try to think how our place-value system of notation might be made to include a digit with place-value $\frac{1}{10}$. If we still wish each digit in a numeral to have place-value just 10 times the place-value of the digit to its right, where should a digit with place-value $\frac{1}{10}$ appear? (Just to the right of the digit with place-value 1.) Why? (1 is the same as 10 times $\frac{1}{10}$.) A digit with place-value $\frac{1}{10}$ in a numeral is in the <u>tenths' place</u>.

Another way to name the number $\frac{1}{10}$ is to use the <u>decimal</u> .1. Both are read "one tenth". The dot in the decimal is called the <u>decimal point</u>. It is needed so that we do not confuse [illegible] or .01 with .1.

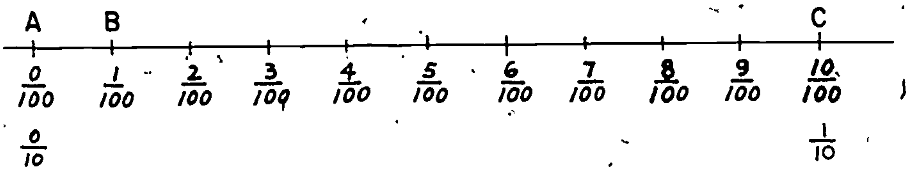
> At this point the teacher might write on the chalkboard several decimals involving tenths (but not yet hundredths or thousandths) and ask the children to read them aloud.
>
> In talking about decimals keep in mind that we are using this word as an abbreviation of "decimal numeral". Any numeral in the place-value, base ten, numeration system will be called a decimal. Thus the numerals 25, 6, 4.3, and .17 are all decimals.

How would you write a <u>decimal</u> name for the number three tenths? (.3) What <u>fraction</u> would name this same number? ($\frac{3}{10}$) The numerals .3 and $\frac{3}{10}$ are just two ways of naming the same rational number. That is,

$$.3 = \frac{3}{10}$$

How would you write .4 as a fraction? ($\frac{4}{10}$) How would you write $\frac{7}{10}$ as a decimal? (.7)

We are now going to talk about decimals which have a digit with place-value $\frac{1}{100}$. Let us first draw the segment of the number line from 0 to $\frac{1}{10}$ and divide it by points into 10 segments of the same length. We label these points $\frac{0}{100}$, $\frac{1}{100}$, $\frac{2}{100}$, and so on.



The length of $\overline{AC}$ is how many times the length of $\overline{AB}$? (10 times) We shall therefore say that the number $\frac{1}{10}$ is 10 times the number $\frac{1}{100}$.

> Again this amounts to anticipating the way multiplication will later be defined:
>
> $$\frac{1}{10} = 10 \times \frac{1}{100}.$$

Where should a digit with place-value $\frac{1}{100}$ appear? (Just to the right of a digit with place-value $\frac{1}{10}$.) Why? ($\frac{1}{10}$ is 10 times $\frac{1}{100}$) A digit with place-value $\frac{1}{100}$ in a numeral is in the <u>hundredths' place</u>. We write $\frac{1}{100}$ in decimal form as .01. We read the decimal as "one hundredth". Why must we write one hundredth as .01 and not as .1?

> At this time the teacher might write on the chalkboard several decimals involving hundredths and ask the children to read them aloud.

How do you read the fraction $\frac{23}{100}$? (Twenty-three hundredths) How would you write the decimal name for this number? (.23) The numerals .23 and $\frac{23}{100}$ are two ways of representing the same number, twenty-three hundredths, so

$$.23 = \frac{23}{100}.$$

In the decimal .23 the 2 is written in the tenths' place and the 3 is written in the hundredths' place.

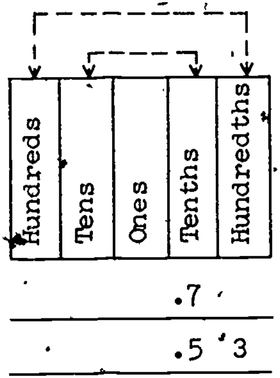
How would you read .47? (Forty-seven hundredths) How would you write .47 as a fraction? ($\frac{47}{100}$)

> Continue giving examples of fractions and ask the children to write the decimal name.
>
> Also give particular attention to the fact that numerals such as .4 and .40 name the same rational number. Have pupils explain why such numerals name the same number.

DECIMAL NAMES FOR RATIONAL NUMBERS

We have learned how to name rational numbers using symbols such as $\frac{2}{7}$ and $\frac{11}{12}$, called fractions. When a fraction has a denominator 10 or 100, as in $\frac{7}{10}$ or $\frac{53}{100}$, there is another way in which we can write its name.

The chart below shows how we can extend the idea of place-value to the right of the ones' place. Using this idea we can name rational numbers like $\frac{7}{10}$ and $\frac{53}{100}$ in a new way.



The name .7 and the name $\frac{7}{10}$ are names for the same rational number. Both names are read in the same way: "seven tenths".

The name .53 and the name $\frac{53}{100}$ are names for the same rational number. Both names are read in the same way: "fifty-three hundredths".

Names like $\frac{7}{10}$ and $\frac{53}{100}$ are called <u>fractions</u>. Names like .7 and .53 are new examples of decimal numerals. We will usually shorten "decimal numeral" to "decimal".

The dot (.) in a decimal is called the <u>decimal point</u>.

In .7, the 7 is written in the tenths' place. In .53, the 5 is written in the tenths' place and the 3 is written in the hundredths' place.

1. Are .7 and $\frac{7}{10}$ names for the same number? (yes)
    a) Which name is a decimal? (.7)
    b) Which name is a fraction? ($\frac{7}{10}$)
2. Are $\frac{53}{100}$ and .53 names for the same number? (yes)
    a) Which name is a decimal? (.53)
    b) Which name is a fraction? ($\frac{53}{100}$)
3. Are .3 and .03 names for the same number? (no) Check your answer by writing each name as a fraction. ($\frac{3}{10} \neq \frac{3}{100}$)
4. Are .7 and .70 names for the same number? yes Check your answer by writing each name as a fraction. ($\frac{7}{10} = \frac{70}{100}$)

Exercise Set 4

1. Rename each of these as a decimal.

$\frac{1}{10}$	$\frac{29}{100}$	$\frac{75}{100}$	$\frac{8}{10}$	$\frac{4}{100}$	$\frac{2}{10}$	$\frac{30}{100}$
(.1)	(.29)	(.75)	(.8)	(.04)	(.2)	(.30)

2. Rename each of these as a fraction.

.15	.9	.1	.82	.05	.4	.60
$(\frac{15}{100})$	$(\frac{9}{10})$	$(\frac{1}{10})$	$(\frac{82}{100})$	$(\frac{5}{100})$	$(\frac{4}{10})$	$(\frac{60}{100})$

3. Copy and finish the following counting chart using decimals.

.01	.02	.03	(.04)	(.05)	(.06)	.07	.08	.09	.10	(.1)
.11	.12	(.13)	(.14)	(.15)	(.16)	(.17)	.18	.19	.20	(.2)
.21	(.22)	(.23)	(.24)	(.25)	(.26)	(.27)	(.28	(.29)	(.30)	(.3)
(.31)	(.32)	.33	(.34)	(.35)	(.36)	(.37)	(.38)	(.39)	(.40)	(.4)
(.41)	(.42)	(.43)	.44	(.45)	(.46)	(.47)	(.48)	(.49)	(.50)	(.5)
(.51)	(.52)	(.53)	(.54)	.55	(.56)	(.57)	(.58)	(.59)	(.60)	(.6)
(.61)	(.62)	(.63)	(.64)	(.65)	.66	(.67)	(.68)	(.69)	(.70)	(.7)
(.71)	(.72)	(.73)	(.74)	(.75)	(.76)	.77	(.78)	(.79)	(.80)	(.8)
(.81)	(.82)	(.83)	(.84)	(.85)	(.86)	(.87)	.88	(.89)	(.90)	(.9)
(.91)	(.92)	(.93)	(.94)	(.95)	(.96)	(.97)	.98	.99	(1.00)	(1.0)

4. Look at the decimals in the last column of the chart you just completed (.10, .20, .30, etc.) Each of these decimals may be replaced by another decimal. (For example, .1 is another name for .10.) To the right of the chart, write another decimal for each decimal in the last column.

5. Complete each of these.

a) .16, .18, .20, (.22), (.24), (.26).

b) .24, .27, .30, (.33), (.36), (.39).

c) .37, .39, .41, (.43), (.45), (.47).

d) .43, .48, .53, (.58), (.63), (.68).

e) .90, .80, .70, (.60), (.50), (.40).

f) .85, .75, .65, (.55), (.45), (.35).

g) .68, .64, .60, (.56), (.52), (.48).

h) .58, .55, .52, (.49), (.46), (.43).

6. Write T if the mathematical sentence is true. Write F if it is false.

a) $.50 = .5$ (T)

b) $.7 < .07$ (F)

c) $\frac{23}{100} > .23$ (F)

d) $\frac{4}{100} \neq .4$ (T)

e) $\frac{45}{100} < .54$ (T)

f) $.72 > .8$ (F)

g) $\frac{9}{10} < .65$ (F)

h) $\frac{50}{100} \neq .05$ (T)

BRAINTWISTERS

Can we rename $\frac{2}{5}$ as a decimal? Can we rename $\frac{9}{25}$ as a decimal? We can if first we are able to rename it as a fraction with a denominator of 10 or 100.

We can rename $\frac{2}{5}$ as $\frac{4}{10}$. We can rename $\frac{2}{5}$ as the decimal, (.4). Also, we can rename $\frac{9}{25}$ as $\frac{(36)}{100}$. So we can rename $\frac{9}{25}$ as the decimal, (.36).

. Now rename each of these as a decimal.

$\frac{1}{2}$ (.5) $\frac{9}{20}$ (.45) $\frac{47}{50}$ (.94) $\frac{3}{5}$ (.6) $\frac{18}{25}$ (.72) $\frac{10}{40}$ (.25)

RENAMING DECIMALS

We have learned to think about a decimal like .73 as 73 hundredths. We also know that in .73, the 7 is in the tenths' place and the 3 is in the hundredths' place. This gives us another way to name .73:

.73 = 7 tenths and 3 hundredths.

In the same way,

.49 = (4) tenths and (9) hundredths.

We also can say

8 tenths and 2 hundredths = .82.

In the same way,

3 tenths and 6 hundredths = (.36).

Exercise Set 5

1. Finish each of these.

a) .29 = (2) tenths and (9) hundredths.

b) .58 = (5) tenths and (8) hundredths.

c) .41 = (4) tenths and (1) hundredths.

d) .80 = (8) tenths and (0) hundredths.

e) .04 = (0) tenths and (4) hundredths.

f) .36 = (6) hundredths and (3) tenths.

2. Write the decimal for each of these.

a) 5 tenths and 7 hundredths = (.57).

b) 9 tenths and 3 hundredths = (.93).

c) 1 tenth and 6 hundredths = (.16).

d) 2 tenths and 0 hundredths = (.20).

e) 0 tenths and 4 hundredths = (.04).

f) 5 hundredths and 3 tenths = (.35).

DECIMALS WITH THOUSANDTHS

Objective: To extend the understanding of decimal fractions to include thousandths.

Materials: Place-value chart

Vocabulary: Thousandths

Exploration:

What are the names of the places to the right of the ones' place? (Tenths' place and hundredths' place) What value does $\frac{1}{100}$ have in relation to $\frac{1}{10}$? ($\frac{1}{10}$ is 10 times $\frac{1}{100}$) What do you think the name of the third place to the right of the ones' place is? (Thousandths) Is $\frac{1}{100}$ ten times $\frac{1}{1000}$? (Yes) How do you know? (In the decimal system of numeration each digit has a place-value ten times the place-value of the digit to its right.)

We can write the fraction $\frac{1}{1000}$ as the decimal .001. Both are read "one thousandth".

$\frac{3}{1000}$ and .003 are two ways of naming the same rational number. What number do they name? (Three thousandths)

Read these decimals.

.005 (five thousandths)
.014 (fourteen thousandths)
.297 (two hundred ninety-seven thousandths)

> Use counting at difficult places so that the pupils will become more familiar with three-place decimals. Such sequences as .008, .009, .010, .011; .098, .099, .100, .101, etc. are hard and need careful teaching.

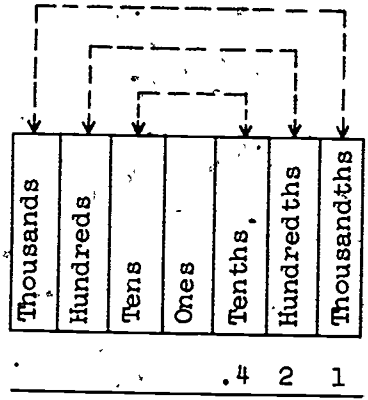
Count by thousandths from 1 thousandth to 10 thousandths. Write the decimals on the chalkboard.

Count from 35 thousandths to 45 thousandths. Write the decimals.

Count from 138 thousandths to 142 thousandths. Write the decimals.

DECIMALS WITH THOUSANDTHS

We have learned how to extend place-value for decimals from tenths to hundredths. Using what we have learned, let us extend the place-value chart another place to the right. This is called the thousandths' place.



The name .421 and the name $\frac{421}{1000}$ are names for the same rational number. Both names are read as "four hundred twenty-one thousandths".

In .421 the 4 is written in the tenths' place, the 2 is written in the hundredths' place, and the 1 is written in the thousandths' place.

1. Are $\frac{421}{1000}$ and .421 names for the same number? (yes)
   a) Which name is a decimal? (.421)
   b) Which name is a fraction? ($\frac{421}{1000}$)

2. Which is largest, .2, .02, or .002? Check your answer by naming each number as a fraction.
($\cdot 2$ is largest $\frac{2}{10} > \frac{2}{100} > \frac{2}{1000}$)

3. Are .2, .20, and .200 all names for the same rational number? (yes) Check your answer by writing each as a fraction. ($\frac{2}{10} = \frac{20}{100} = \frac{200}{1000}$)

Another way to think about and name .421 is 4 tenths and 2 hundredths and 1 thousandth.

In the same way,

.582 = (5) tenths and 8 hundredths and (2) thousandths.

Finish each of these.

a) .138 = (1) tenth and (3) hundredths and (8) thousandths.
b) .140 = (1) tenth and (4) hundredths and (0) thousandths.
c) .306 = (3) tenths and (0) hundredths and (6) thousandths.
d) .374 = (37) hundredths and (4) thousandths.
e) .009 = (0) tenths and (0) hundredths and (9) thousandths.

Exercise Set 6

1. Rename each of these as a decimal.

$\frac{32}{1000}$ $\frac{5}{1000}$ $\frac{9}{10}$ $\frac{492}{1000}$ $\frac{18}{1000}$ $\frac{174}{1000}$ $\frac{8}{1000}$ $\frac{18}{100}$

(.032) (.005) (.9) (.492) (.018) (.174) (.008) (.18).

2. Rename each of these as a fraction.

.475 .011 .8 .023 .62 .729 .007

($\frac{475}{1000}$) ($\frac{11}{1000}$) ($\frac{8}{10}$) ($\frac{23}{1000}$) ($\frac{62}{100}$) ($\frac{729}{1000}$) ($\frac{7}{1000}$)

3. Write T if the mathematical sentence is true. Write F if it is false.

a) $.6 = .600$ (T)

b) $.9 > .009$ (T)

c) $\frac{23}{1000} > .23$ (F)

d) $\frac{8}{10} < .85$ (T)

e) $\frac{52}{100} \neq .052$ (T)

f) $.79 = \frac{79}{1000}$ (F)

g) $.008 > \frac{8}{1000}$ (F)

h) $.072 < .72$ (T)

4. Arrange the three numbers in each group in order of size. Name the smallest number first in each case.

a) .003 .3 .03 (.003, .03, .3)

b) .37 .037 .3 (.037, .3, .37)

c) .402 .42 .042 (.042 .402, .42)

d) .560 .506 .056 (.056, .506, .560)

5. Complete each of these.

a) .058 .060 .062 (.064) (.066) (.068)

b) .007 .012 .017 (.022) (.027) (.032)

c) .550 .450 .350 (.250) (.150) (.050)

d) .755 .760 .765 (.770) (.775) (.780)

e) .042 .142 .242 (.342) (.442) (.542)

6. Complete

a) .729 = (9) thousandths and (2) hundredths and (7) tenths.

b) .402 = (4) tenths and (0) hundredths and (2) thousandths.

c) .519 = (5) tenths and (1) hundredth and (9) thousandths.

d) .052 = (2) thousandths and (5) hundredths and (0) tenths.

e) .530 = (5) tenths and (3) hundredths and (0) thousandths.

7. Write the decimal for each of these.

a) 5 thousandths and 3 hundredths and 4 tenths = (.435).

b) 0 thousandths and 2 hundredths and 3 tenths = (.320).

c) 6 thousandths and 4 hundredths and 8 tenths = (.846).

d) 5 tenths and 0 hundredths and 5 thousandths = (.500).

e) 4 thousandths and 2 hundredths and 0 tenths = (.024).

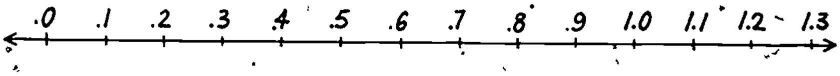
OTHER DECIMALS

Objective: To learn to read, write, and analyze decimals with digits on both sides of the decimal point

Materials: Number lines drawn on the chalkboard, place-value chart

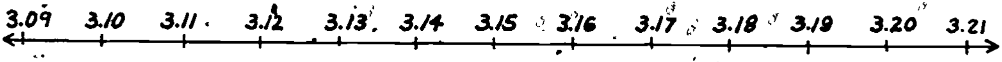
Exploration:

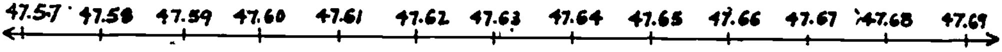
Draw the number line on the board and record the numeral as a child counts by tenths from zero to 1.3.



As soon as the child counts ten tenths, ask for another name for ten tenths. (1) Since we are to continue counting by tenths, indicate that 1 may be written "1.0" to show there are no tenths in the tenths' place. "When we count by tenths, what is the next number?" (one and one tenth) In decimal form this is written 1.1. The 1 to the left of the decimal point is in the ones' place and the 1 to the right of the decimal point is in the tenths' place. The decimal point is read "and." (N.B. 1.1 should be read "one and one tenth" and not as "one point one"). Continue counting and recording to give practice in reading similar decimals with tenths.

The same development may be followed to introduce the reading of other decimals. Draw other number lines on the chalkboard. Put the first numeral on it and ask the child to count by hundredths and record the numeral as he counts. Use these number lines and others if needed.





After counting and recording on number lines, write 14.6 on the chalkboard. Ask children to read it and analyze according to place-value. (14.6 is read fourteen and six tenths. 14.6 = 1 ten, 4 ones, and 6 tenths.) If it is difficult for the children to analyze these decimals, use a place-value chart. Also introduce the mixed form for 14.6($14\frac{6}{10}$) and ask the children for similar translations. Continue reading, analyzing, and renaming decimals like the following:

72.35
19.72
85
64.003
182.294
.781

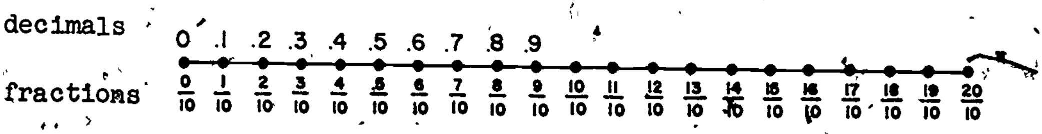
Pages 20-21 in the pupil text summarize this work. Read them carefully with the pupils. Be sure to emphasize by the end of this section that numerals like 35, 6.7, and .72 are all <u>decimals</u>.

OTHER DECIMALS

We have been learning how to read and interpret decimals such as .7 and .39 and .561. We already knew the meaning of decimal numerals such as 82, 7, or 356. Many times we need to use rational numbers which are greater than one but are not whole numbers. We already have fraction names for some of these numbers, names like $\frac{11}{10}$, $\frac{12}{10}$, $\frac{21}{10}$, or $\frac{125}{100}$. Since these all have denominators which are 10 or 100 we should be able to find decimal names for them and for numbers like them.

We might begin by thinking of counting by <u>tenths</u>.

The number line below shows counting by tenths with decimals and with fractions. We need decimal numerals to complete the top line.



$\frac{10}{10} = 1$

$\frac{11}{10}$ = eleven tenths = one and one tenth.

We express this as a decimal numeral by writing 1.1. The numeral 1 on the left stands for 1 <u>one</u>. The numeral 1 on the right stands for 1 <u>tenth</u>.

1. Use this idea to copy and complete the number line shown above. When we are thinking in tenths we usually write 1.0 (one and 0 tenths) instead of 1 and 2.0 instead of 2.

2. Write a decimal for each of the following:

   a) 1 ten and 1 one (11.0)

   b) 1 tenth and 1 hundredth (.11)

   c) 1 one and 1 hundredth (1.01)

We read 2.3 as "two and three tenths", and 1.25 is read as "one and twenty-five hundredths". The chart below should help us to read and interpret other decimals.

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
		2	6 .	3	4	5

We read 26.345 as "twenty-six and three hundred forty-five thousandths". In reading a decimal with digits on either side of the decimal point, the decimal point is read as "and".

3. Read each of the following.

   a) 263.45

   b) 2634.5

   c) 2.6345

Sometimes a kind of numeral is used which combines decimals and fractions. The numeral $1\frac{3}{10}$ is an example. It names one and three tenths or 1.3 (decimal) or $\frac{13}{10}$ (fraction). Such a numeral is called a <u>mixed form</u>.

4. a) Read $7\frac{5}{100}$.

   b) What is a decimal name for this number?

   c) Write a mixed form for 7.5.

Exercise Set 7

1. Choose the largest number in each column.

A	B	C	D	E
3.4	8.50	.002	45.405	.209
3.8	8.56	1.92	35.405	.287
2.4	[✓ 8.65]	2.2	[✓ 45.5]	[✓ .291]
[✓ 4.4]	8.05	[✓ 2.22]	45.05	.289

2. Copy and complete each of these.

a) 7.5 8.0 (8.5) 9.0 (9.5) (10.0) (10.5) (11.0)

b) 3.40 3.30 (3.20) (3.10) (3.00) (2.90)

c) .20 .40 .60 .80 1.00 1.20

d) 4.75 4.80 4.85 4.90 4.95 5.00

3. Write these as decimals.

$2\frac{3}{10}$ $15\frac{7}{100}$ $32\frac{64}{100}$ $148\frac{37}{1000}$ $52\frac{184}{1000}$

(2.3) (15.07) (32.64) (148.037) (52.184)

4. Write a mixed form name for each of these.

22.3 72.15 18.047 459.003 78.39

$(22\frac{3}{10})$ $(72\frac{15}{100})$ $(18\frac{47}{1000})$ $(459\frac{3}{1000})$ $(78\frac{39}{100})$

5. Tell the number represented by each numeral 3.

Tell the number represented by each numeral 5.

a) 321.59 $\binom{300}{.5}$ b) 71.03 (.03) c) 421.36 (.3)

d) 720.513 $\binom{.003}{.5}$ e) 49.035 $\binom{.03}{.005}$ f) 795.309 $\binom{.3}{5}$

6. Write a decimal for each of these.

a) 27 and 9 tenths (27.9)
b) 364 and 57 hundredths (364.57)
c) 70 and 41 thousandths (70.041)
d) 38 and 7 hundredths (38.07)
e) 3 and 0 hundredths (3.00 or 3)
f) 5 and 429 thousandths (5.429)
g) 83 and 4 tenths (83.4)
h) 480 and 5 hundredths (480.05)
i) 20 and 64 hundredths (20.64)
j) 6 and 7 thousandths (6.007)
k) 75 and 2 tenths (75.2)

BASE FIVE NOTATION

Objective: To gain increased understanding of the decimal system by considering systems of notation using bases other than ten

Materials:

1. Flannel board and cut-outs
2. Packets of twenty to thirty objects which can be counted (A demonstration set should be large enough so it may be seen from all parts of the classroom. The students may have smaller objects suitable for work at their desks.)
3. A place-value chart may be made that will show groupings of twenty-fives, fives, and ones.

Exploration:

Our decimal system uses groups of ten. However, the decimal system has not always been in use. Long ago the Mayans of Yucatan counted by groups of twenty. Some tribes of Eskimos used groups of five. Each numeration system made use of grouping in counting.

Let us pretend we are Eskimos and count in groups of five. What name could we give to our new system of numeration? (Base five system) How many symbols would we use in the base five system? (5) What symbols could we use? (0, 1, 2, 3, 4) Why must we always have a symbol to represent zero? (In using the idea of place-value, there must be a numeral to represent the empty set or the set of no members.)

During the discussion the teacher should make a chart showing the numeral, the meaning, and a picture of each number. Follow this form.

Picture	X	XX	XXX	XXXX	XXXXX
Meaning	one one	two ones	three ones	four ones	one five and no ones
Numeral	1_{five}	2_{five}	3_{five}	4_{five}	10_{five}

Let us use our objects to illustrate the base five system. Start with a single object and write the symbol "1_{five}". (The subscript "five" on digits is not really necessary. Use it for emphasis as you wish.) Add one object. What symbol would we use to name the number of objects we now have? (2_{five}) Add another object to the set. What symbol would we use? (3_{five}) Add another object to the set. What symbol would we use? (4_{five}) Add another object to the set. How many do we have? (one set of five) We have written 1_{five}, 2_{five}, 3_{five}, 4_{five}. How do we write one five and no ones? (10_{five})

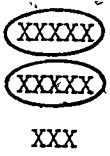
Take time to discuss the idea of "five ones" and "one five" as we did earlier with "ten ones" and "one ten". The final notation should be "10_{five}". (read "one five and no ones" OR "one zero base five"). Continue adding a single object each time and writing the name of the number represented using base five notation. We can discover in this way that the easiest way to count large numbers of objects is to group them first, and then count the groups.
NOTE: It might be advisable to limit the number of objects to twenty-four for this discussion. (to count twenty-five objects using the base five notation, we would need to know about three-digit numerals.)

The next step is to discuss with the pupils the two examples in their text. Read the examples with the pupils and explain unfamiliar mathematics and vocabulary to them.

BASE FIVE NUMERALS

At the beginning of this chapter, you reviewed grouping and regrouping by tens. This is the idea behind our decimal numeral system. However, there are many ways of grouping objects. One of these ways is grouping in sets of five. This gives us the idea of a numeral system based on grouping by fives.

Example A

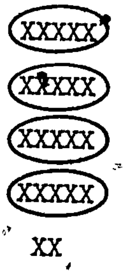


Here is a picture of a set of thirteen X's. This set can be grouped into 2 sets of five and 3 ones. We shorten this to 23 (read "two three") to name the number of X's in the set. The set can also be grouped into 1 set of ten and 3 ones. We shorten this to 13 to get our ordinary decimal numeral. To show that 23 comes from grouping by fives and not by tens we will write the word "five" to the right and slightly below the numeral.

23_{five} means 2 sets of <u>five</u> and 3 ones.
13 means 1 set of <u>ten</u> and 3 ones.

We call 23_{five} a <u>base five</u> numeral and we read it "two three, base five".

Example B



Look at this picture.

How many sets of five X's are there? (4)

How many X's remain? (2)

How would you write the base five numeral? (42 five)

How would you read this base five numeral? (4 fives and 2 ones, <u>or</u> four two, base five)

Exercise Set 8

1. Draw the following sets of X's. Group in fives and answer these questions for each set.

   How many sets of five are there?

   How many ones remain?

   How would you write the base five numeral?

Use this form.

Nine X's XXXXX XXXX — 1 five and 4 ones 14_{five}

a) twelve X's (2 fives and 2 ones 22_{five})

b) nineteen X's (3 fives and 4 ones 34_{five})

c) four X's (0 fives and 4 ones 4_{five})

d) twenty-three X's (4 fives and 3 ones 43_{five})

2. Draw a picture that will represent X's for

a) 30_{five} XXXXX XXXXX XXXXX

b) 42_{five} XXXXX XXXXX xx XXXXX XXXXX

c) 14_{five} XXXXX xxxx

d) 10_{five} XXXXX

3. Name the largest number with a base five numeral having two digits. (44_{five})

4. Name, in base five, the number which will come just before each of these numbers.

a) 4_{five} (3_{five})

b) 20_{five} (14_{five})

c) 32_{five} (31_{five})

d) 40_{five} (34_{five})

PLACE VALUE IN BASE FIVE

Exploration:

Previously we have been limiting our discussion of base five to two-place numerals. Now we are ready to introduce the third place in base five. Use a place-value chart and bundles of cardboard strips to show groupings of twenty-fives, fives, and ones.

Twenty-fives	Fives	Ones

Write the base five notation on the chalkboard as the children count the cardboard strips. For example:

Base Five Counting Chart

1	2	3	4	10
11	12	13	14	20
21	22	23	24	30, etc.

Note: When a title is used, the word "five" does not have to be written beside each numeral.

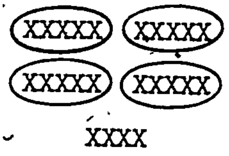
Count out four cardboard strips, writing the base five notation as you count, and put them in the ones' place of your place-value chart. Add one more. "How many sets of five are there?" (One) Bundle the set of five ones and put them in the fives' place. "What notation do we use to show one group of five?" (10_{five}) Continue counting, grouping, and recording until you have put 4 fives and 4 ones in the chart. "What base five numeral do we have?" (44_{five}) Add one more strip in the ones' place. You now have 4 sets of five and five ones. "How many sets of five do we have?" (five) There are five sets of five or 1 set of twenty-five. Bundle the five fives and put this set of twenty-five in the twenty-fives' place. "What notation do we use to show 1 group of twenty-five?" (100_{five}). This is read "one twenty-five, no fives, no ones," or "one zero zero, base five".

Continue grouping, reading, and recording other three-place numerals in the base five system. Note that 111_{five} would be read "one twenty-five, one five, and one one" or "one one one, base five". In base five 124 is read "one twenty-five, two fives, and four ones", or "one two four, base five". Avoid using base ten numerals like 5 and 25 in discussing grouping by fives.

PLACE VALUE IN BASE FIVE

In the base ten system, the number named 99 is the largest with a two-place numeral. This is because 9 is one less than the base.

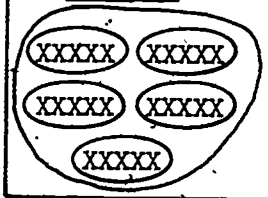
In the base five system, the number named 44_{five} is the largest with a two-place numeral. This is because 4 is one less than the base, as shown in the diagram below.

Picture	Meaning	Notation
	4 fives and 4 ones	44_{five}

There is no two-place symbol in our base ten system to mean ten tens. We give ten tens the name 1 hundred. We write this as the three-place numeral 100.

When we are thinking in base five we think of five groups of five as 1 group of five fives. We can use the name twenty-five for five fives.

How would the base five numeral for five fives or twenty-five be written?

Picture	Meaning	Notation
	1 twenty-five, 0 fives, and 0 ones	100_{five}

Exercise Set 9

1. Copy the X's below and group them in fives and five fives. Write the number of X's in base five notation.

a) XXX
XXX
XXX
XXX
(22_{five})

b) XXXXXXX
XXXXXXX
XXXXXXX
(41_{five})

c) XXXXXXXXX
XXXXXXXXX
XXXXXXXXX
XXXXXXXXX
(121_{five})

2. Copy and complete the following:

a) 33_{five} means (3) fives and (3) ones.

b) 142_{five} means (1) twenty-fives and (4) fives and (2) ones.

c) 104_{five} means (1) twenty-fives and (0) fives and (4) ones.

3. Write the base five numeral for the number that is one larger than each of these.

a) 4_{five} (10_{five}) c) 43_{five} (44_{five}) e) 144_{five} (200_{five})

b) 13_{five} (14_{five}) d) 132_{five} (133_{five}) f) 204_{five} (210_{five})

4. Write these numbers in base five notation

a) The number of this page in this book (102_{five})

b) The number of cookies in 4 dozen (143_{five})

c) The total number of pages in this book (1120_{five})

5. Make a base five chart of the numerals from 1_{five} to 200_{five}.

Base Five Counting Chart

1	2	3	4	10
11	12	13	14	20
21	22	23	24	30
31	32	33	34	40
41	42	43	44	100
101	102	103	104	110
[illegible]	112	113	114	120
[illegible]	122	123	124	130
131	132	133	134	140
141	142	143	144	200

BASE FIVE AND BASE TEN NUMERALS

Exploration:

At first the child compares base five and base ten numbers by grouping objects in base five and then grouping these same objects in base ten. He repeats the same process with drawings.

Another way to see that we are using different numerals to represent the same number is to write in a column the first few counting numbers in base ten notation, and then, in a parallel column, the same numerals in base five notation.

Base Ten	Base Five
1	1
2	2
3	3
4	4
5	10
6	11
7	12
8	13
9	14
10	20
11	21
12	22
13	23
14	24
15	30
16	31
17	32
18	33
19	34
20	40
21	41
22	42
23	43
24	44
25	100
26	101
27	102
28	103
29	104
30	110

Notice that the numeral written "13" in the base ten system is quite different in meaning from the numeral written "13" in the base five system (which is actually a name for the number "eight"). Let us therefore agree that whenever we write just "13", base ten will automatically be understood; and when we want base five to be understood instead, we shall always write "13_{five}", read "one five and three ones or one three, base five". We then know that this stands for one five and three ones, i.e., the number "eight". **The decimal numeral 13 is of course read "thirteen" although "one three base ten" has certain advantages in showing how base ten fits the general pattern.**

The pupil can change from base five to base ten if he knows how to read the base five numerals. He should be able to think through the transferring from base five to base ten in much this way. For example,

$$23_{\text{five}} = (2 \text{ fives} + 3 \text{ ones})$$
$$= (2 \times 5) + (3 \times 1)$$
$$= 10 + 3$$
$$= 13$$

$$114_{\text{five}} = (1 \text{ twenty-five}) + (1 \text{ five}) + (4 \text{ ones})$$
$$= (1 \times 25) + (1 \times 5) + (4 \times 1)$$
$$= 25 + 5 + 4$$
$$= 34$$

Some children may want to show in writing how this change is made. If so, they may use the above form.

BASE FIVE AND BASE TEN NUMERALS

Numeral in Base Five System	Picture in Base Five System	Picture in Base Ten System	Numeral in Base Ten System
a) 22_{five}	XXXXX XXXXX XX	XXXXXXXXXX X X	12
b) 33_{five}	XXXXX XXXXX XXXXX XXX	XXXXXXXXXX XXXXXXXX	18
c) 114_{five}	XXXXX XXXXX XXXXX XXXX XXXXX XXXXX XXXXX	XXXXXXXXXX XXXXXXXXXX XXXXXXXXXX XXXX	34

Study the chart above. What does the numeral 22_{five} tell us? (There are 2 fives and 2 ones which is read, "two two, base five.")

What does the numeral 12 tell us? (12 means 1 ten and two ones which is read "twelve.")

Are 12 and 22_{five} names for the same number? (yes)

Why are 33_{five} and 18 names for the same number? (33 five means 3 fives and 3 ones or eighteen objects. 18 means 1 ten and 8 ones or eighteen)

Why are 114_{five} and 34 names for the same number?

(114 five means 1 twenty five, 1 five, and 4 ones or thirty four objects. 34 means 3 tens and 4 ones or thirty four objects.)

The procedure below shows how we may think to change a base five numeral to a base ten numeral.

a) $22_{\text{five}} = (2 \text{ fives} + 2 \text{ ones})$

$= (2 \times 5) + (2 \times 1)$

$= 10 + 2$

$= 12$

b) $33_{\text{five}} = (3 \text{ fives} + 3 \text{ ones})$

$= (3 \times 5) + (3 \times 1)$

$= 15 + 3$

$= 18$

c) $114_{\text{five}} = (1 \text{ twenty-five} + 1 \text{ five} + 4 \text{ ones})$

$= (1 \times 25) + (1 \times 5) + (1 \times 4)$

$= 25 + 5 + 4$

$= 34$

MORE ABOUT BASE FIVE AND BASE TEN NUMERALS

Exploration:

In changing base ten numerals to base five numerals, select the largest place-value of base five (that is, power of five) contained in the number. Divide the number by this power of five and find the quotient and remainder. This quotient is the first digit in the base five numeral. Divide the remainder by the next smaller power of five. The quotient is the second digit. Continue to divide remainders by each succeeding, smaller power of five to determine all the remaining digits in the base five numeral.

For example, to change 113 to a base five numeral, we must first see that 113 is less than five twenty-fives. Then we find how many groups of twenty-five are in 113. We can do this by division or repeated subtraction. In either case we find there are 4 twenty-fives. This 4 becomes the first digit in the base five numeral. There are 13 ones left to be grouped.

Now find how many groups of five there are in 13. We find there are 2 fives and 3 ones. The two becomes our second digit and the 3 our third digit. Therefore, 113 = 423_{five}.

$$\begin{array}{r} 4 \\ 25\overline{)113} \\ 100 \end{array}$$

$$\begin{array}{r} 5\overline{)13}\ \underline{|2} \\ \underline{10} \\ 3 \end{array}$$

$$\begin{aligned} 113 &= (4 \times 25) + 13 \\ &= (4 \times 25) + (2 \times 5) + 3 \\ &= 4(\text{twenty-fives}) + 2(\text{fives}) + 3 \\ &= 423_{five} \end{aligned}$$

If a group is not contained in a remainder, remember to put a zero in that place in the resulting numeral. For example, when changing 104 to base five we find there are 4 twenty-fives, no fives, and 4 ones. Therefore, 104 = 404_{five}.

MORE ABOUT BASE FIVE AND BASE TEN NUMERALS

?	Twenty-fives	Fives	Ones

So far, when we have written numerals in base five, we have used the place-values that are shown above. Can you tell what the next place-value will be? (Five twenty-fives or one hundred twenty-fives)

For numerals we will be using right now, the only place-values we will work with are twenty-fives, fives, and ones.

Suppose we want to change 111 to a base five numeral.

How many groups of twenty-five are there in 111? (4)

|| The pupil may use any method he may know to solve this problem ||

What is the remainder? (11)

Write the mathematical sentence for this division process.
$(111 = 4 \times 25 + 11)$

Find how many fives there are in 11. (2)

How many ones remain? (1)

Write the mathematical sentence for this division process.
$(11 = (2 \times 5) + 1)$

Put both mathematical sentences together in a mathematical sentence which shows how 111 can be grouped by fives and twenty-fives. $(111 = (4 \times 25) + (2 \times 5) + 1)$

What is the base five numeral for 111? (421_{five})

Try changing the following base ten numerals to base five numerals. In each part write the mathematical sentence which shows why your answer is correct.

a) 12 (22_{five})
$(12 = (2 \times 5) + 2)$

b) 36 (121_{five})
$(36 = (1 \times 25) + (2 \times 5) + 1)$

c) 44 (134_{five})
$(44 = (1 \times 25) + (3 \times 5) + 4)$

d) 52 (202_{five})
$(52 = (2 \times 25) + (0 \times 5) + 2)$

Exercise Set 10

1. Draw a set of 21_{five} X's. Separate these X's into groups of ten. How many X's are there? Write your answer as a base ten numeral.

XXXXX
XXXXX X (11)

2. Draw a set of 134_{five} X's. Separate these X's into groups of ten. How many X's are there? Write your answer as a base ten numeral.

XXXXX XXXXX XXXX
XXXXX XXXXX
XXXXX XXXXX
XXXXX XXXXX

(44)

3. Change the following base ten numerals to base five numerals.

a)	14 (24_{five})	c)	23 (43_{five})	e)	42 (132_{five})
b)	51 (201_{five})	d)	60 (220_{five})	f)	33 (113_{five})

4. Change the following base five numerals to base ten numerals.

a)	23_{five} (13)	c)	34_{five} (19)	e)	42_{five} (22)
b)	141_{five} (46)	d)	340_{five} (95)	f)	204_{five} (54)

5. Which is greater?

a)	210_{five} or [✓ 201]	c)	33_{five} or [✓ 23]
b)	[✓ 134_{five}] or 42	d)	40_{five} or 20 (They are equal)

USING GROUPING BY FIVES

Exploration:

> Since our system of money uses groupings of five, some experience in this area will help develop this idea more completely. At first the teacher may need to use play money with some children. Proceed from these experiences to expressing the groupings in a table as follows. Encourage the children to use the smallest number of coins in separating the money into quarters, nickels, and cents. Draw the chart on the chalkboard. Complete the chart with the children.

How much money?	How many quarters?	How many nickels?	How many pennies?	Base-five notation
7 cents	(0)	(1)	(2)	(12 five)
12 cents	(0)	(2)	(2)	(22 five)
34 cents	(1)	(1)	(4)	(114 five)
58 cents	(2)	(1)	(3)	(213 five)
87 cents	3	2	2	(322 five)
122 cents	4	4	2	(442 five)

USING GROUPING BY FIVES

We use some groupings of five in our everyday life. Let us look at our system of money. Suppose we have 34 cents. If we use only quarters, nickels, and pennies and the fewest coins, we have one quarter, one nickel, and four pennies. How could we write this using base five notation?

Exercise Set 11

Separate the following amounts of money into quarters, nickels, and cents. Use the smallest number of coins.

How much money?	How many quarters?	How many nickels?	How many pennies?	Base five notation
Examples: 14 cents	0	2	4	24_{five}
43 cents	1	3	3	133_{five}
1) 23 cents	(0)	(4)	(3)	(43 five)
2) 26 cents	(1)	(0)	(1)	(101 five)
3) 29 cents	(1)	(0)	(4)	(104 five)
4) 33 cents	(1)	(1)	(3)	(113 five)
5) 42 cents	(1)	(3)	(2)	(132 five)
6) 57 cents	(2)	(1)	(2)	(212 five)
7) 73 cents	(2)	(4)	(3)	(243 five)
8) 97 cents	(3)	(4)	(2)	(342 five)
9) 124 cents	(4)	(4)	(4)	(444 five)

THINKING ABOUT NUMBERS IN OTHER BASES

Exploration:

> We are now going to discuss number bases other than ten and five. The next experiences involve facts that may be deduced from a definite pattern that you help children to discover. The following is one of many procedures that may be used. Again, the students will need their manipulative materials. The teacher may wish to duplicate Exercises Set 12.

1. Ask the students to count out fifteen objects.

2. Separate the set of fifteen objects into groups of nine.
   a) How many groups of nine are there? (one)
   b) How many objects remain? (six)
   c) How would you express this number using base nine notation? (16_{nine})

3. Separate the set of fifteen objects into groups of eight.
   a) How many groups of eight are there? (one)
   b) How many objects remain? (seven)
   c) How would you express this number using base eight notation? (17_{eight})

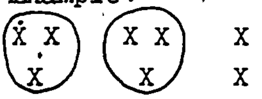
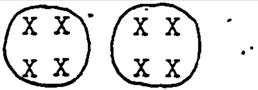
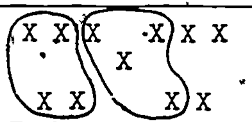
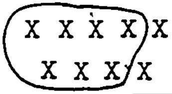
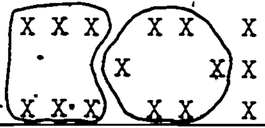
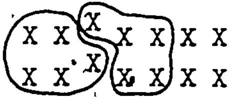
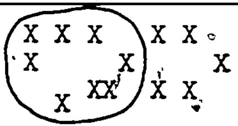
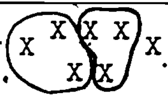
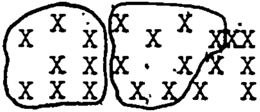
4. Separate the set of fifteen objects into groups of seven.
   a) How many groups of seven are there? (two)
   b) How many objects remain? (one)
   c) How would you express this number using base seven notation? (21_{seven})

5. Separate the set of fifteen objects into groups of six.
   a) How many groups of six are there? (two)
   b) How many objects remain? (three)
   c) How would you express this number using base six notation? (23_{six})

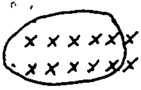
THINKING ABOUT NUMBERS IN OTHER BASES

Exercise Set 12

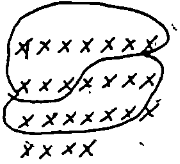
Copy and complete this chart.

	Arrange in groups of	How many groups?	How many remain?	Notation
Example: 	three	2	2	22_{three}
1. 	four	(2)	(0)	(20_{four})
2. 	four	(2)	(3)	(23_{four})
3. 	seven	(1)	(2)	(12_{seven})
4. 	six	(2)	(3)	(23_{six})
5. 	five	(2)	(4)	(24_{five})
6. 	eight	(1)	([illegible])	(15_{eight})
7. 	three	(2)	(1)	(21_{three})
8. 	eight	(2)	(6)	(26_{eight})

9. Draw a set of 20_{six} objects. Separate these objects into groups of ten. How many objects are there? Write your answer in base ten notation. (12)



10. Draw a set of 34_{seven} objects. Separate these objects into groups of ten. How many objects are there? Write your answer in base ten notation. (25)



11. Each mathematical sentence below shows how to change a decimal numeral into a numeral in another base. Write that numeral in the blank as shown in a).

a) $21 = (1 \times 16) + (1 \times 4) + 1$

$21 = \underline{(111\text{ four})}$

b) $50 = (1 \times 36) + (2 \times 6) + 2$

$50 = \underline{(122\text{ six})}$

c) $26 = (1 \times 16) + (1 \times 8) + (1 \times 2)$

$26 = \underline{(11010\text{ two})}$

d) $82 = (1 \times 81) + 1$

$= \underline{(101)}_{nine}$

$= \underline{(10001)}_{three}$

PLACE VALUE IN OTHER BASES

Exploration:

Base	Group Names		
Ten	Hundreds	Tens	Ones
Five	Twenty-fives	Fives	Ones
Three			
Four			

We will now develop the idea of a three-digit numeral in bases other than five or ten. Place on the chalkboard the chart shown above. Fill in the group names of base three and base four as they are discussed. During the discussion use a place-value box labeled with the group names of the base being studied.

Review with the children that in base ten each successive place to the left represents a group ten times that of the preceding place. The first place tells us how many groups of one there are. The second place tells us how many groups of ten, or (10 × 1). The third place tells us how many groups of ten times ten there are, (10 × 10), or one hundred.

Continue with base five, noting that the first place tells us how many groups of one there are. The second place tells us how many groups of five there are. The third place tells us how many groups of five times five there are. Each successive place to the left represents a group five times that of the preceding place.

Using the place-value box and the bundles of cardboard strips, develop place-value in base three.

The teacher should lead the pupils to discover that when numerals represent whole numbers, the last digit on the right indicates the number of ones (or units) in base three, the second digit from the right indicates the number of groups of three. The third digit from the right indicates the number of groups of nine. In writing numerals in base three, the value of each place in the numeral is three times the value of the place to its right.

nines	three	ones
1	2	2_{three}

Continue with base four.

sixteens	fours	ones
2	3	3_{four}

In base four notation, the value of each place is four times the value of the place to its right.

The number of basic symbols necessary to write numerals in a numeration system depends upon the base used. For example, base three uses three symbols, base ten uses ten symbols, base five uses five symbols, etc.

The base we use will determine the value of each place in the numeral. As the number increases in size, the number of places in its numeral increases faster when one uses a small base than when one uses a larger base.

Example: $17_{ten} = 32_{five}$

$17_{ten} = 122_{three}$

The child should be able to compare numerals in other bases with base ten numerals. For example, 15_{eight} is read one eight and five ones, which is 13 in the decimal system. Likewise, 43_{seven} is read four sevens and three ones which is 31 in the decimal system.

When the children count in different bases, they will discover many interesting facts.

The following chart, Exercise Set 13, may be duplicated by the teacher.

Exercise Set 13

Copy this chart. Write the numeral for the first twenty-four counting numbers using base eight, base six, base three, and base four.

Base Ten	Base Eight	Base Six	Base Three	Base Four
1	(1)	(1)	(1)	(1)
2	(2)	(2)	(2)	(2)
3	(3)	(3)	(10)	(3)
4	(4)	(4)	(11)	(10)
5	(5)	(5)	(12)	(11)
6	(6)	(10)	(20)	(12)
7	(7)	(11)	(21)	(13)
8	(10)	(12)	(22)	(20)
9	(11)	(13)	(100)	(21)
10	(12)	(14)	(101)	(22)
11	(13)	(15)	(102)	(23)
12	(14)	(20)	(110)	(30)
13	(15)	(21)	(111)	(31)
14	(16)	(22)	(112)	(32)
15	(17)	(23)	(120)	(33)
16	(20)	(24)	(121)	(100)
17	(21)	(25)	(122)	(101)
18	(22)	(30)	(200)	(102)
19	(23)	(31)	(201)	(103)
20	(24)	(32)	(202)	(110)
21	(25)	(33)	(210)	(111)
22	(26)	(34)	(211)	(112)
23	(27)	(35)	(212)	(113)
24	(30)	(40)	(220)	(120)

Exercise Set 14

Complete the table.

Base Ten Numeral	Sixteens	Fours	Ones	Base Four Numeral
31	(1)	(3)	(3)	(133 four)
17	(1)	(0)	(1)	(101 four)
59	(3)	(2)	(3)	(323 four)
Base Ten Numeral	Thirty-sixes	Sixes	Ones	Base Six Numeral
34	(0)	(5)	(4)	(54 six)
90	(2)	(3)	(0)	(230 six)
215	(5)	(5)	(5)	(555 six)
Base Ten Numeral	Nines	Threes	Ones	Base Three Numeral
26	(2)	(2)	(2)	(222 three)
9	(1)	(0)	(0)	(100 three)
22	(2)	(1)	(1)	(211 three)
Base Ten Numeral	Forty-nines	Sevens	Ones	Base Seven Numeral
60	(1)	(1)	(4)	(114 seven)
290	(5)	(6)	(3)	(563 seven)
99	(2)	(0)	(1)	(201 seven)
Base Ten Numeral	Twenty-fives	Fives	Ones	Base Five Numeral
46	(1)	(4)	(1)	(141 five)
103	(4)	(0)	(3)	(403 five)
89	(3)	(2)	(4)	(324 five)
Base Ten Numeral	Sixty-fours	Eights	Ones	Base Eight Numeral
31	(0)	(3)	(7)	(37 eight)
80	(1)	(2)	(0)	(120 eight)
54	(0)	(6)	(6)	(66 eight)

Exercise Set 15

1. Fill in blanks as shown in the example.

43_{five} The numeral 4 stands for 4 fives.

a) 301_{four} The numeral 3 stands for 3 (sixteens).

b) 423_{five} The numeral 4 stands for 4 (twenty-fives).

c) 63_{seven} The numeral 6 stands for 6 (sevens).

d) 85_{nine} The numeral 8 stands for 8 (nines).

e) 300_{six} The numeral 3 stands for 3 (thirty-sixes).

2. Change these numerals into base ten numerals as shown in a).

a) $23_{five} = (2 \times 5) + 3$

$= 10 + 3$

$= 13.$

b) 202_{three}

$(202_{three} = (2 \times 9) + (0 \times 3) + 2 = 20)$

c) 106_{seven}

$(106_{seven} = (1 \times 49) + (0 \times 7) + 6 = 55)$

d) 210_{four}

(210_{four} = [illegible]16) + (1×4) + 0 = 36)

e) 18_{nine}

$(18_{nine} = (1 \times 9) + 8 = 17)$

f) 34_{eight}

$(34_{eight} = (3 \times 8) + 4 = 28)$

g) 440_{five}

$(440_{five} = (4 \times 25) + (4 \times 5) + 0 = 120)$

h) 122_{three}

$(122_{three} = (1 \times 9) + (2 \times 3) + 2 = 17)$

i) 312_{four}

$(312_{four} = (3 \times 16) + (1 \times 4) + 2 = 54)$

3. Copy and complete this counting chart.

a) Base five 133_{five} $\left(\frac{134}{five}\right)\left(\frac{140}{five}\right)\left(\frac{141}{five}\right)\left(\frac{142}{five}\right)\left(\frac{143}{five}\right)\left(\frac{144}{five}\right)$
$\left(\frac{200}{five}\right)\left(\frac{201}{five}\right)\left(\frac{202}{five}\right)$

b) Base seven 56_{seven} $\left(\frac{60}{seven}\right)\left(\frac{61}{seven}\right)\left(\frac{62}{seven}\right)\left(\frac{63}{seven}\right)\left(\frac{64}{seven}\right)$
$\left(\frac{65}{seven}\right)\left(\frac{66}{seven}\right)\left(\frac{100}{seven}\right)\left(\frac{101}{seven}\right)$

c) Base four 31_{four} $\left(\frac{32}{four}\right)\left(\frac{33}{four}\right)\left(\frac{100}{four}\right)\left(\frac{101}{four}\right)\left(\frac{102}{four}\right)$
$\left(\frac{103}{four}\right)\left(\frac{110}{four}\right)$

d) Base six 125_{six} $\left(\frac{130}{six}\right)\left(\frac{131}{six}\right)\left(\frac{132}{six}\right)\left(\frac{133}{six}\right)\left(\frac{134}{six}\right)$
$\left(\frac{135}{six}\right)\left(\frac{140}{six}\right)\left(\frac{141}{six}\right)$

4. In what base are we counting?

a) 1, 2, 3, 4, 10, 11, 12, 13, ... (base five)

b) 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 30, ...(base seven)

c) 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, ...(base four)

d) 11, 12, 20, 21, 22, 100, 101, 102, 110, ...(base three)

5. Copy the work below. Use the "greater than", "less than", or "equals" sign to complete a true mathematical sentence.

a) 44_{five} (>) 102_{three}

b) 100_{seven} (=) 54_{nine}

c) 32_{six} (<) 25_{eight}

d) 211_{three} (>) 21_{four}

e) 77_{eight} (=) 223_{five}

6. A place value system of numeration has twenty digits. What is the base? (twenty or 20)

7. Count by tens in base five from 20_{five} to 400_{five}.

(20_{five}, 40_{five}, 110_{five}, 130_{five}, 200_{five}, 220_{five}, 240_{five}, 310_{five}, 330_{five}, 400_{five})

8. Are these odd or even numbers?

a) 12_{three} (odd)

b) 21_{three} (odd)

c) 101_{three} (even)

d) 111_{three} (odd)

e) 121_{three} (even)

f) 102_{three} (odd)

BRAINTWISTERS

9. Copy and fill in the blanks.

   a) $33_{five} = \underline{(24)}_{seven}$

   b) $14_{eight} = \underline{(110)}_{three}$

   c) $25_{six} = \underline{(101)}_{four}$

   d) $128_{nine} = \underline{(412)}_{five}$

10. What is n in each of these mathematical sentences?

    a) $n_{five} + 2_{five} = 11_{five}$ ($n_{five} = 4_{five}$)

    b) $23_{four} + 10_{four} = n_{four}$ ($n_{four} = 33_{four}$)

    c) $n_{eight} - 42_{eight} = 25_{eight}$ ($n_{eight} = 67_{eight}$)

    d) $123_{six} + n_{six} = 130_{six}$ ($n_{six} = 3_{six}$)

11. Suppose a base three system used the symbol A for the number zero, B for one, and C for two. In this numeral system count from zero through ten.

    (A, B, C, BA, BB, BC, CA, CB, CC, BAA, BAB)

12. Change each of the following to decimal numerals.

    a) BBB (13)

    b) CAB (19)

    c) CBA (21)

    d) ABC (5)

Chapter 2

FACTORS AND PRIMES

PURPOSE OF UNIT

The most fundamental objective of this unit is to investigate what might be called the multiplicative structure of the counting numbers. We try to find out something about how new numbers are "constructed" as products of given numbers and how a given number can be "broken up" into products of smaller numbers. Because a given number does not have every smaller number as a factor, the situation is not as simple as it is in addition where every smaller number is an addend. There are, in fact, simple statements about multiplicative structure which remain unsettled.

While the study of multiplicative structure can be approached as a game of intrinsic interest, it should also be of substantial value in reinforcing the learning of multiplication facts by emphasizing their interrelations.

The immediate aim of this unit is (1) to develop the techniques of expressing a number as a product of prime numbers and to put this to use in (2) finding all factors of a number, and (3) finding the greatest common factor of two numbers. These techniques will be used later as manipulative tools in operating with fractions. At that time they can be reviewed and the necessary proficiency developed.

Special Note to the Teacher: If this is the first time that you have taught this unit, you will find it most helpful, before you present the unit, to study first all of the pupil pages and the background accompanying them. Then your study of the Mathematical Summary at the end of the chapter will be much more rewarding. After you have seen the arrangement of the chapter as a whole, your teaching of the material will be more effective.

ARRANGEMENT OF CHAPTER

The materials of this unit are organized and presented somewhat differently than in other units. The basic pattern for each section of the unit is as follows:

1. Background material for the teacher including comments on ideas and possible lines of discussion
2. An outline of suggestions for classwork
3. Pupil pages containing examples and a summary of the language, ideas, or techniques which have been developed in classwork
4. Pupil pages containing exercises involving the ideas of the section

At the end of the unit the mathematical ideas which appear in it are summarized briefly in a section headed Mathematical Summary. In this summary, more attention is paid to deductive explanations than in the background material in the body of the unit.

TEACHING THE UNIT

FACTORS AND PRODUCTS

Objective: To review some of the basic ideas involving factors and products.

Materials: Five arrays (1 by 10, 2 by 5, 1 by 20, 2 by 10, 4 by 5)

Vocabulary: Factor, product, multiplication sentence, product expression, commutative property, associative property.

Background:

Special Note: It is imperative that children have a strong knowledge of the basic multiplication facts through 9 × 9. If they do not, then you must spend some time in review, using both mental arithmetic and written work.

It is important also that they know the division algorism. In this unit we use one developed in Chapter 7, Grade Four; but if the class did not study SMSG in the fourth grade, then the algorism they know will be sufficient.

For your own information, you will want to review the basic properties of multiplication and division as they are presented in Chapter 4, Grade Four. This does not mean to go back and teach all these ideas to the children, but each teacher needs an understanding of that unit.

From the outset of this unit it is important for you to keep in mind the distinction between <u>prime</u> and <u>composite</u> numbers, even though this distinction is not needed specifically and explicitly until the later section on "Prime Numbers."

A <u>prime</u> number is a counting number greater than 1 that has no factor (among the counting numbers) other than itself and 1. (e.g.: 2, 3, 5, 7, 11, 13, 17, and 19 are the prime numbers less than 20.)

A <u>composite</u> number is a counting number greater than 1 that <u>has factors</u> (among the counting numbers) other than itself and 1.

By definition, <u>1</u> is neither a prime number nor a composite number.

It is well to keep in mind that we are interested in factors that are counting numbers and not just any factors. For example, although 5 can be factored (since $\frac{5}{2} \times 2 = 5$) it can not be factored using only counting numbers.

Let us use the number 24 to illustrate several different kinds of things children may be asked to find in terms of factors associated with a composite number.

1. Children may be asked to find a product expression for a composite number such as 24.

   a. The product may be expressed as two factors; e.g.,

$$24 = 3 \times 8$$
$$24 = 4 \times 6$$
$$24 = 1 \times 24$$
etc.

   b. The product may be expressed as three (or more) factors; e.g.,

$$24 = 2 \times 3 \times 4$$
$$24 = 2 \times 2 \times 6$$
$$24 = 1 \times 3 \times 8$$
etc.

2. Children may be asked to express a composite number such as 24 as a product of prime factors (i.e., as a product of factors which are prime numbers). Without regard for the order in which the factors are stated, there is only one way in which a particular composite number can be expressed as a product of prime factors. In the case of 24:

$$24 = 2 \times 2 \times 2 \times 3$$

3. Children may be asked to find the set of all factors of a composite number. In the case of 24 this is {1, 2, 3, 4, 6, 8, 12, 24}. Each member of this set is a factor of 24.

Special mention should be made of the use of 1 as a factor in connection with each of the three preceding situations.

In connection with 1a and 1b, beginning work permits the use of 1 as a factor. Ultimately it is shown that writing 1 as a factor in many product expressions gives no additional information regarding the factors of a number; hence, it need not be written.

In connection with situation 2 (expressing a composite number as a product of prime factors), 1 is never included as a factor since 1 is not a prime number (by definition).

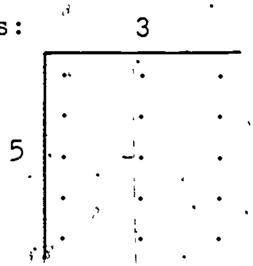
> In connection with situation 3 (listing the set of all factors of a number), 1 is always included, along with the number itself. Both 1 and n are factors of n (a composite number), but neither is a prime factor.
>
> Some children will need your help at times in sensing clearly which one of the three preceding situations is under consideration.

Every whole number has many names. In this chapter, we will use this idea again. Take the number 20. Many names can be given for 20 (10 + 10, 22 - 2, 2 × 10, 1 × 20, 4 × 5, etc.). If we list only names which show multiplication for 20, we include only product expressions. (1 × 20, 2 × 10, 4 × 5, 5 × 4, 10 × 2, 20 × 1.) It will be noted in the next section that if we remember the commutative property, three of these product expressions for 20 are sufficient.

By using the commutative property, we get the last three from the first three.

$$1 \times 20 = 20 \times 1$$
$$2 \times 10 = 10 \times 2$$
$$4 \times 5 = 5 \times 4$$

Each product expression for a number corresponds to an array. An array may be described by a number pair like 5, 3. The first number named gives the number of rows, and the second number named gives the number of columns in the array. An array describing 5, 3 looks like this:



Suppose there are 10 objects with which to construct arrays. If all objects are used, how many different arrays can be formed? A 1 by 10, 2 by 5, 5 by 2, or a 10 by 1 array can be formed. Each of the arrays is different from the others if they are not to be moved about.

Again you will notice that we have considered every pair of factors whose product is 10. (1 × 10, 2 × 5, 5 × 2, and 10 × 1) Actually, we have used only two pairs of numbers, but have four different expressions *if* we consider order.

Take 36. 1 × 36 is a product expression for 36. Since 1 is a factor of all numbers, every number has a product expression of this type. Other product expressions for 36 are 2 × 18, 3 × 12, 4 × 9, and [illegible] × 6. Here there are five different expressions. By applying the commutative property of multiplication to them, we can arrive at four more: 9 × 4, 12 × 3, 18 × 2, and 36 × 1. There are not five more expressions because when the commutative property is applied to 6 × 6 we arrive at the same product expression.

For a small number, knowledge of the multiplication facts enables us to find every product expression with two factors for the number. For a large number, another method must often be used to find factors and product expressions for the number.

Suppose the problem is to find whether a number has a factor 3, and to write a product expression for it if it has. This might be done by two methods.

METHOD A:

(1) Is 3 a factor of 37?

3) 37	
30	10
7	
6	2
1	12

37 = (12 × 3) + 1 (a remainder of 1)

3 *is not* a factor of 37.

(2) Is 3 a factor of 57?

3) 57	
30	10
27	
27	9
	19

57 = 19 × 3 (no remainder)

3 *is* a factor of 57.

METHOD B: (Here we must use the multiplication facts and mathematical sentences.)

Is 7 a factor of 67? I know $9 \times 7 = 63$,

$$67 - 63 = 4,$$

therefore $(9 \times 7) + 4 = 63 + 4 = 67$.

When we divide 67 by 7 there is a remainder of 4. The only way that 7 could be a factor of 67 would be if there were no remainder.

Suggestions for Exploration:

Review many names for the same number.

Review multiplication language and ways of writing.

Review arrays and their relation to product expressions.

Find several product expressions for several numbers.

Introduce Methods A and B to find whether a number has a factor, thereby making it possible to write a product expression.

Chapter 2

FACTORS AND PRIMES

FACTORS AND PRODUCTS

Let's think of two numbers, for example 4 and 5. Use multiplication to get a third number, 20. We write this

$$4 \times 5 = 20.$$

4 is called a *factor* of 20.

5 is called a *factor* of 20.

20 is called the *product* of 4 and 5.

If we use the name, 4×5, for 20, we are *writing 20 as a product of two factors*. Sometimes we call 4×5 a *product expression* for 20.

The multiplication sentence

$$30 = 2 \times 3 \times 5$$

says that

30 is the product of 2 and 3 and 5.

It also says that

2 is a factor of 30, and 3 is a factor of 30

and 5 is a factor of 30.

A product expression for 30 is $2 \times 3 \times 5$.

Exercise Set 1

1. List three different names for each of the following who[illegible] numbers: (Use product expressions.)

   a. ten (10x1, 1x10 2x5, 5x2)

   b. twelve (4x3, 3x4 2x6, 6x2 12x1, 1x12)

   c. sixteen (8x2, 2x8 4x4, 16x1, 1x16)

   d. twenty-one (3x7, 7x3 21x1, 1x21)

   e. nine (3x3, 9x1, 1x9)

2. Copy the following statements and fill in the blanks.

   a. 5 is a factor of 15 because 15 = (5 x 3 or 3x5)

   b. 15 = 5 × 3 shows that (3) is another factor of 15.

   c. 24 is the product of 6 and (4).

   d. (1) is a factor of every number.

   e. Every number greater than 1 has at least (2) different factors.

3. How many different arrays can be formed with

   a. 10 objects? 4. (2×5, 5×2, 10×1, 1×10)

   b. 20 objects? 6 (4×5, 5×4, 2×10, 10×2, 1×20, 20×1)

   List the number of rows and columns in each array. (Remember that the number of rows is always named first.)

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Exercise Set 2

1. Express the following numbers as a product of two factors. Find three different ways for each.

   a. 24 (2 × 12, 4 × 6, 3 × 8)

   b. 30 (5 × 6, 2 × 15, 3 × 10)

   c. 28 (4 × 7, 1 × 28, 2 × 14)

2. Write the decimal numeral for each product.

   a. 6 × 9 = (54)

   b. 7 × 6 = (42)

   c. 9 × 7 = (63)

   d. 8 × 8 = (64)

   e. 7 × 7 = (49)

   f. 5 × 9 = (45)

   g. 8 × 6 = (48)

   h. 9 × 8 = (72)

   i. 7 × 8 = (56)

   j. 6 × 6 = (36)

3. Complete each mathematical sentence below to make a true statement.

   a. 3 × (7) = 21

   b. (7) × 8 = 56

   c. 4 × (1) = 4

   d. 9 × (9) = 81

   e. (8) × 9 = 72

   f. (7) × 4 = 28

   g. 8 × (4) = 32

   h. 4 × (9) = 36

   i. 4 × 6 = 24

   j. 7 × (9) = 63

4. Express each of the following numbers as a product of two factors in every possible way.

   a. 12 (There are 6 ways.) (12 × 1, 6 × 2, 4 × 3, 3 × 4, 2 × 6, 1 × 12)

   b. 35 (There are 4 ways.) (5 × 7, 35 × 1, 7 × 5, 1 × 35)

   c. 42 (There are 8 ways.) (42 × 1, 21 × 2, 14 × 3, 7 × 6, 1 × 42, 2 × 21, 3 × 14, 6 × 7)

   d. 18 (There are 6 ways.) (1 × 18, 2 × 9, 3 × 6, 6 × 3, 9 × 2, 18 × 1)

   e. 45 (There are 6 ways.) (1 × 45, 3 × 15, 5 × 9, 9 × 5, 15 × 3, 45 × 1)

   f. 24 (There are 8 ways.) (1 × 24, 2 × 12, 3 × 8, 4 × 6, 6 × 4, 8 × 3, 12 × 2, 24 × 1)

TESTING NUMBERS AS FACTORS

Is 3 a factor of 57? Is 3 a factor of 37? We may see by using division (Method A).

$$\begin{array}{r|r} 3\,)\,57 & \\ \underline{30} & 10 \\ 27 & \\ \underline{27} & 9 \\ \hline & 19 \end{array} \qquad\qquad \begin{array}{r|r} 3\,)\,37 & \\ \underline{30} & 10 \\ 7 & \\ \underline{6} & 2 \\ \hline 1 & 12 \end{array}$$

$57 = (19 \times 3)$ $\qquad\qquad$ $37 = (12 \times 3) + 1$

3 <u>is</u> a factor of 57. $\qquad$ 3 <u>is</u> <u>not</u> a factor of 37.

Here is another method we may use to see if one number is a factor of another (Method B). Is 7 a factor of 67?

I know $9 \times 7 = 63$

and $67 = 63 + 4$.

Therefore $(9 \times 7) + 4 = 63 + 4 = 67$.

Since 4 is less than 7, 4 is the remainder when 67 is divided by 7. This shows that 7 is not a factor of 67.

P45

Exercise Set 3

1. Use Method A to answer each of these.

   a. Is 8 a factor of 81? (No, 8 is not a factor of 81.)

   b. Is 4 a factor of 52? (Yes, 4 is a factor of 52.)

   c. Is 7 a factor of 59? (No, 7 is not a factor of 59.)

2. Use Method B to answer each of these. Write your answer in a complete sentence.

   a. Is 7 a factor of 58? (No, 7 is not a factor of 58.)

   b. Is 9 a factor of 75? (No, 9 is not a factor of 75.)

   c. Is 8 a factor of 56? (Yes, 8 is a factor of 56.)

3. Use either Method A or Method B to answer these. Write your answer in a complete sentence.

   a. Is 3 a factor of 51? (Yes, 3 is a factor of 51.)

   b. Is 9 a factor of 138? (No, 9 is not a factor of 138)

   c. Is 6 a factor of 73? (No, 6 is not a factor of 73)

   d. Is 7 a factor of 217? (Yes, 7 is a factor of 217.)

   e. Is 8 a factor of 94? (No, 8 is not a factor of 94)

DIFFERENT PRODUCT EXPRESSIONS FOR THE SAME NUMBER

Objective: To help children understand that one number can be named by more than one product expression

Vocabulary: Product expression, associative property, commutative property

Background:

> Before beginning this unit, the teacher should study Chapter 4, Grade Four, particularly the material on the associative and commutative properties of multiplication.

In the review on P46 and P47, we attempt to show that for our purposes, it is unnecessary to distinguish between $(2 \times 3) \times 5$ and $2 \times (3 \times 5)$ or between 2×3 and 3×2 in writing product expressions.

Once we know

$$2 \times 3 \times 5 = 30,$$

we also know that any rearrangement of 2, 3, and 5 gives another product expression for 30. We may, of course, find it helpful to think of the rearrangements; but we will not regard them as different product expressions for 30; and we will write any one as a representative of them all. The essential point is that, by remembering the commutative and associative properties, we can get as much information about factors and product expressions of 30 from

$$2 \times 3 \times 5 = 30$$

as we can from all possible groupings and rearrangements of the factors shown.

Suggestions for exploration:

1. Review the associative property of multiplication with the class before introducing pupil page 46. Use examples similar to the one given on that page.
2. Review the commutative property in the same way.

THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

A. Starting from $6 \times 5 = 30$, we can get

$$(2 \times 3) \times 5 = 30.$$

B. Starting from $2 \times 15 = 30$, we can get

$$2 \times (3 \times 5) = 30.$$

The associative property also shows us how to get B from A.

$$6 \times 5 = 30$$

$$(2 \times 3) \times 5 = 30$$

$$2 \times (3 \times 5) = 30 \quad \text{(Associative Property)}$$

$$2 \times 15 = 30.$$

If we show no grouping and just write

$$2 \times 3 \times 5 = 30,$$

we see clearly that 2, 3, and 5 are factors of 30.

By thinking of both groupings, we see that 6 and 15 are also factors of 30, because we get

$$2 \times 15 = 30 \quad \text{and}$$

$$6 \times 5 = 30.$$

Writing the product expression of 3 or more factors without parentheses can give us as much information as writing all possible groupings. We will use parentheses only when we want to show particular groupings.

THE COMMUTATIVE PROPERTY OF MULTIPLICATION

When we know $6 = 2 \times 3$, we also know

$$6 = 3 \times 2.$$

If we know that $24 \times 32 = 768$, then we know that

$$32 \times 24 = 768.$$

If we know $30 = 2 \times 3 \times 5$, then we also know

$$30 = 2 \times 5 \times 3,$$
$$30 = 5 \times 2 \times 3,$$
$$30 = 3 \times 2 \times 5,$$
$$30 = 3 \times 5 \times 2, \text{ and}$$
$$30 = 5 \times 3 \times 2.$$

Any one of these ways of expressing 30 as a product of three factors tells us that 2, 3, and 5 are factors of 30. When we know one way, we can list all six; but we will find nothing new from the other five ways.

From now on in this unit we will not say two ways of writing a product expression are different ways unless they show a different set of factors.

Background:

On pupil page 47 six ways were found to use the same factors to express the product, 30. They are all considered as one way of expressing 30 as a product. We will not say that two ways of writing a product expression are different ways unless they show different factors. For example:

$$6 = 1 \times 6$$
$$6 = 2 \times 3.$$

There are two different ways of expressing 6 as a product because each product expression involves a different set of factors. There are five different ways to express 30 as a product of three factors.

$$\begin{aligned} 30 &= 2 \times 3 \times 5 \\ &= 1 \times 2 \times 15 \\ &= 1 \times 5 \times 6 \\ &= 1 \times 3 \times 10 \\ &= 1 \times 1 \times 30 \end{aligned}$$

To write the product expression for 12 using three factors and beginning with the expression 12 = 4 × 3, we have:

$$12 = 4 \times 3,$$
$$12 = 2 \times 2 \times 3, \quad \text{and}$$
$$12 = 1 \times 4 \times 3.$$

If we begin with 12 = 2 × 6, then 12 = 2 × 2 × 3, and
12 = 1 × 2 × 6.

Each product expression shows certain factors of the number it names. Other factors can be obtained by multiplying two factors or by multiplying three factors, etc.

For example, from $12 = 2 \times 2 \times 3$,

we know that
1 is a factor of 12. (1 is a factor of every number)
2 is a factor of 12. (Shown)
3 is a factor of 12. (Shown)
4 is a factor of 12, (2×2)
6 is a factor of 12, (2×3)
12 is a factor of 12, $(2 \times 2 \times 3)$.

We also know that 12 is a factor of 12 because every number has itself as a factor. So, we know that 1, 2, 3, 4, 6 and 12 are factors of 12. This happens to be the set of all factors of 12.

If we had written

$$12 = 1 \times 2 \times 6$$

then, from this expression we would have found only

1, 2, 6, 12 as factors of 12.

We could have found out just as much from

$12 = 2 \times 6$ as from $12 = 1 \times 2 \times 6$.

This is a good reason for not always specifically including 1 as a factor in a product expression.

Expressing a number as a product of 2, 3, 4, 5 or even more factors does not always give all factors of the number. In this section children are learning that different product expressions for the same number may lead to different sets of factors. For example, these different product expressions for 60 lead easily to recognition of different sets of factors for 60.

$$60 = 1 \times 2 \times 30$$

Easily seen set of factors of 60: {1, 2, 30, 60}.

$$60 = 2 \times 3 \times 10$$

Easily seen set of factors of 60: {1, 2, 3, 6, 10, 20, 30, 60}.

$$60 = 2 \times 5 \times 6$$

Easily seen set of factors of 60: {1, 2, 5, 6, 10, 12, 30, 60}

Nothing assures us that the union of all these sets of factors of 60 is the set of all factors of 60. Indeed, from the three given factorizations it is clear that not all factors of 60 are obtained since 4 and 15 are not in the sets of factors and clearly they are factors of 60. This raises the following question: From which product expressions can we find all factors? The answer depends on the idea of prime numbers and is given in the section FINDING ALL FACTORS.

Suggested Exploration:

Discuss the different ways a product expression having two factors may be written as a product expression with three factors. Emphasize the role of multiplication facts.

Discuss the way in which product expressions can be used to find factors.

Show by example that different product expressions for the same number lead readily to some factors of the number. Some of these factors might not be seen at all if we began with a different product expression. This is quite evident in our example which used different product expressions for 60. For example:

60 = 2 × 3 × 10	60 = 2 × 5 × 6
We get the factors:	We get the factors:
2 (given)	2 (given)
3 (given)	6 (given)
10 (given)	5 (given)
6 (2 × 3)	12 (2 × 6)
20 (2 × 10)	10 (2 × 5)
30 (3 × 10)	30 (6 × 5)

1 is a factor because 1 is a factor of every number. 60 is a factor because every number has itself as a factor.

If 60 = 2 × 3 × 10,	If 60 = 2 × 6 × 5,
the factors of 60 are:	the factors of 60 are:
1, 2, 3, 6, 10, 20, 30, 60	1, 2, 5, 6, 10, 12, 30, 60

Show by example that using 1 as a factor to extend a product expression does not give more information about factors. Discuss the answers to exercises 1, 2, and 3 in Exercise Set 4 in the light of these ideas.

WAYS TO WRITE DIFFERENT PRODUCT EXPRESSIONS FOR THE SAME NUMBER

There are two different ways to express 6 as a product of two factors. We can use the factors 1 and 6, or 2 and 3.

$$6 = 1 \times 6$$

$$6 = 2 \times 3$$

There are five different ways to write 30 as a product of three factors. The factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Using these factors, name the 5 different ways.

$$\begin{aligned} 30 &= 1 \times 1 \times 30 \\ &= 1 \times 2 \times 15 \\ &= 1 \times 3 \times 10 \\ &= 1 \times 5 \times 6 \\ &= 2 \times 3 \times 5 \end{aligned}$$

P49

The factors we get depend upon the way we write the product expression. If we write 60 = 2 × 3 × 10, we will find one set of factors. If we write 60 = 2 × 6 × 5, we will get a different set of factors:

60 = 2 × 3 × 10		60 = 2 × 6 × 5	
The factors are:		The factors are:	
2	(given)	2	(given)
3	(given)	6	(given)
10	(given)	5	(given)
6	(2 × 3)	12	(2 × 6)
20	(2 × 10)	10	(2 × 5)
30	(3 × 10)	30	(6 × 5)

1 is a factor because 1 is a factor of every number. 60 is a factor because every number has itself for a factor.

If 60 = 2 × 3 × 10,	If 60 = 2 × 6 × 5,
the factors are:	the factors are:
1, 2, 3, 6, 10, 20, 30, 60	1, 2, 5, 6, 10, 12, 30, 60

Exercise Set 4

Answers to the Exercise Set are on T.C. pages 91 and 92

1. Each number below is written as a product of two factors. Use this to write the number as a product of <u>three</u> factors.

   a. $12 = 4 \times 3$

   <u>Answer</u>: $12 = 1 \times 4 \times 3$ or $12 = 2 \times 2 \times 3$

   b. $8 = 4 \times 2$

   c. $18 = 9 \times 2$

   d. $16 = 4 \times 4$

   e. $18 = 6 \times 3$

   f. $36 = 6 \times 6$

   g. $36 = 4 \times 9$

2. Write two different product expressions for each of these numbers. Use three factors in each product expression. Then use each product expression to find as many different factors of the number as you can. Part a. is done for you.

   a. 12

   Answers: $12 = 2 \times 2 \times 3$ Factors we can find: 2, 3, 4, 6, 12

   $12 = 1 \times 2 \times 6$ Factors we can find: 1, 2, 6, 12

   b. 18

   c. 36

   d. 16

3. In exercise 2, when we used $12 = 2 \times 2 \times 3$, we find that if we put 1 in our list we have <u>all</u> of the factors of 12. Find whether this is true for each of the product expressions in exercise 2.

4. How can we express a number as a product of three factors in all different ways? We might first express the number as a product of two factors in different ways.

a. 10

10 = 2 × 5, so 10 = 1 × 2 × 5

10 = 1 × 10, so 10 = 1 × 1 × 10

I can find two different ways.

b. 12

12 = 3 × 4, so 12 = 1 × 3 × 4, and
12 = 3 × 2 × 2 .

12 = 2 × 6, so 12 = 1 × 2 × 6, and
12 = 2 × 2 × 3 (already found).

12 = 1 × 12, so 12 = 1 × 1 × 12, and
12 = 1 × 2 × 6, (already found)
and 12 = 1 × 3 × 4 (already found).

I can find four different ways.

1 × 3 × 4
2 × 2 × 3
1 × 2 × 6
1 × 1 × 12

Use the method shown in a and b to find as many ways as you can to express these numbers as products of three factors.

c. 16
d. 18
e. 20
f. 11
g. 44
h. 42

Answers Exercise Set 4

1. b. 1 × 4 × 2 or 2 × 2 × 2 (possibly different order)
   c. 1 × 9 × 2 or 3 × 3 × 2 " " "
   d. 1 × 4 × 4 or 2 × 2 × 4 " " "
   e. 1 × 6 × 3 or 2 × 3 × 3 " " "
   f. 1 × 6 × 6 or 2 × 3 × 6 " " "
   g. 1 × 4 × 9 or 2 × 2 × 9 or 4 × 3 × 3

2. b. 18 = 1 × 9 × 2 or 3 × 3 × 2 or 1 × 3 × 6
   1, 2, 9, 18   1, 2, 3, 6, 9, 18; 1, 3, 6, 18

   c. 36 = 1 × 6 × 6 or 2 × 3 × 6 or 1 × 4 × 9
   1, 6, 36   1, 2, 3, 6, 12, 18, 36; 1, 4, 9, 36

   or others

   d. 16 = 1 × 4 × 4 or 2 × 2 × 4 or 1 × 2 × 8
   1, 4, 16   1, 2, 4, 8, 16; 1, 2, 8, 16

(answers continued on next page)

3\. b. If your answer was 18 = 3 × 3 × 2, then by adding 1 to your list of factors, you would have the set of all the factors of 18, i.e. {1, 2, 3, 6, 9, 18}.

c. Not true f and g.

d. If your answer was 16 = 2 × 2 × 4, then by adding 1 to your list of factors, you would have the set of all factors of 16, i.e. {1, 2, 4, 8, 16}.

There is no way that 36 = 4 × 9 could be expressed as a product of 3 factors that would aid you in finding all the factors of 36. You must use 36 = 2 × 2 × 3 × 3.

4\. c. 2 × 2 × 4, 4 × 4 × 1, 2 × 8 × 1

d. 3 × 3 × 2, 6 × 3 × 1, 9 × 2 × 1, 18 × 1 × 1

e. 2 × 2 × 5, 4 × 5 × 1, 10 × 2 × 1, 20 × 1 × 1

f. 11 × 1 × 1

g. 2 × 2 × 11, 4 × 11 × 1, 22 × 2 × 1, 44 × 1 × 1

h. 2 × 3 × 7, 6 × 7 × 1, 14 × 3 × 1, 21 × 1 × 2, 42 × 1 × 1

ONE AS A FACTOR

Using 1 as a factor in a product expression tells us nothing we don't know about the factors of the number. For example:

a. We know that 1 and 15 are factors of 15, since every number has as factors, itself and 1. Writing 15 = 1 × 15 tells us nothing more about the factors of 15.

b. If we write 12 = 4 × 3 × 1, we know no more about the factors of 12 than if we write 12 = 4 × 3.

c. If we write 36 = 9 × 4 × 1 or 36 = 1 × 4 × 1 × 9, we know no more about the factors of 36 than if we write 36 = 4 × 9.

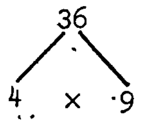
Because of this, when we want to know more about the factors of a number, we look for factors greater than 1 but less than the number itself.

FACTOR TREES

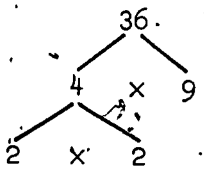
Background:

The process used to express a number as a product of more than two factors can be pictured in a diagram. This diagram may help children see how a number is "built up" from smaller numbers by multiplication.

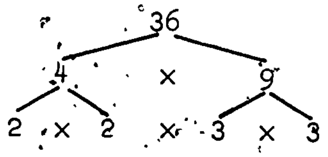
A factor tree is a way to picture factors. 36 = 4 × 9 is represented by drawing:



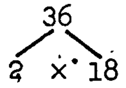
We picture 36 = (2 × 2) × 9 by extending this drawing to make:



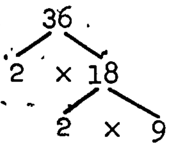
Finally 36 = (2 × 2) × (3 × 3) is shown as:



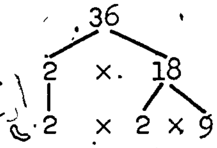
If, for example, a different pair of factors had been chosen; such as, 36 = 2 × 18. The drawing would have been started:



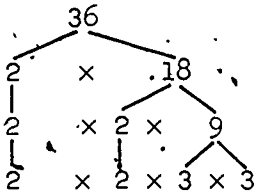
36 = 2 × (2 × 9) would be added to the diagram:



Since 2 has only 1 and 2 as factors, it would not be written as a product. To show this we would draw:



The picture would be:



This unit is good readiness for study of prime numbers. You will notice when a factor tree is completed, the last row is a product expression showing all the prime factors of the product.

Suggested Exploration:

Use the examples shown in the background for an explanation of factor trees.

Follow each step carefully. Do not omit any of the procedure.

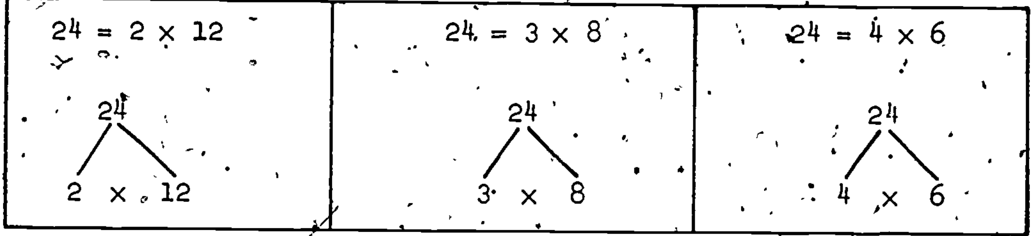
Use several examples (16, 15, 40) as necessary for the class.

FACTOR TREES

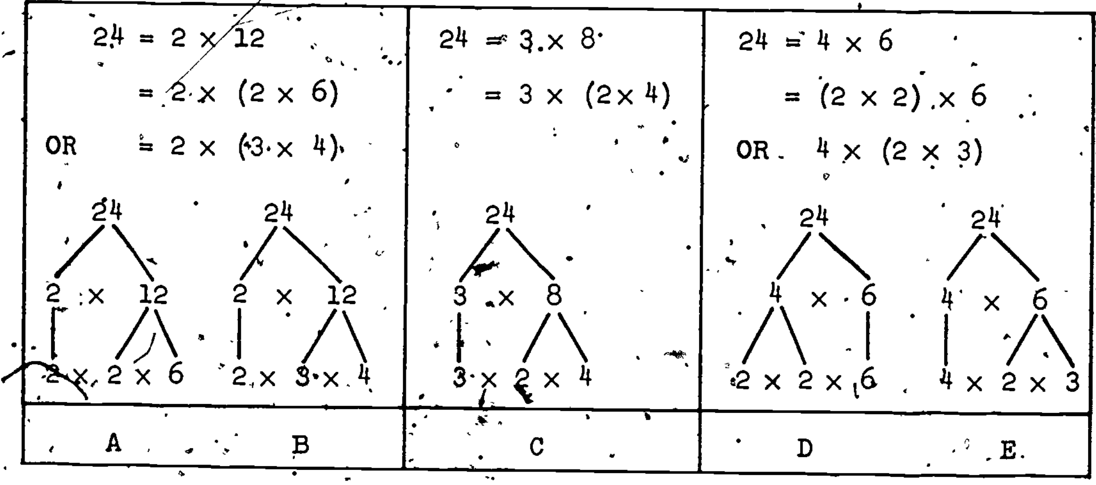
A "factor tree" is a diagram which shows factors of a given number. Let's look at the number 24. We can give product expressions with two factors (each one greater than 1) as follows:

$$24 = 2 \times 12$$
$$24 = 3 \times 8$$
$$24 = 4 \times 6$$

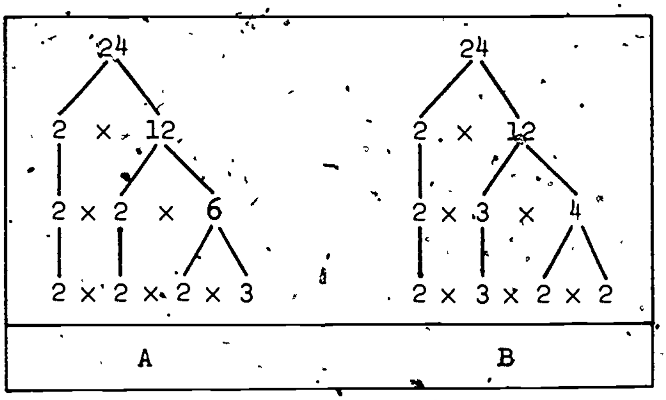
These product expressions can be pictured by "factor trees" which look like this.

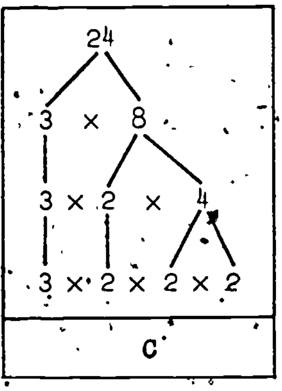


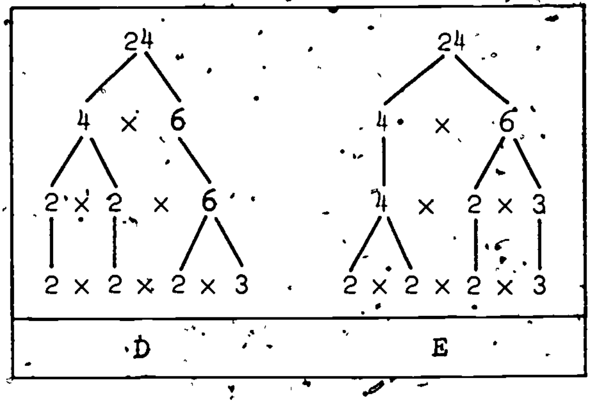
We can picture each product expression using 3 factors (each > 1) by using the "factor trees."



We can extend the factor trees at the bottom of page 96 to picture how 24 can be expressed as a product of 4 factors.







Is it possible to extend the factor tree to another row that would show 24 as a product of 5 factors (not using 1 as a factor)? No

What do you notice about the last row in the factor trees in A, B, C, D, and E above? (The last row of each tree expresses 24 as the product of 4 factors using the same set of 4 factors).

Exercise Set 5

1. Draw two factor trees (if there are two) for each of the following numbers. Extend each tree as far as possible. Do not use the factor 1.

   a. 24
   b. 30
   c. 28
   d. 35
   e. 60
   f. 23
   g. 48
   h. 72

   (answers will vary)

2. List the smallest number which has all of these numbers as factors.

   a. 2, 3, 5 (30)
   b. 2, 5, 7 (70)
   c. 2, 4, 8 (8)
   d. 2, 6, 12 (12)
   e. 2, 3, 4 (12)
   f. 4, 6, 8 (24)
   g. 5, 7 (35)
   h. 2, 5, 7, 10 (70)

BRAINTWISTERS

3. 6 is a factor of 678. This means that 678 must have other factors. What are they? (Some of them are 2, 3, 6, 113, 226) There are others such as 678, 339

4. 12 is a factor of 2,844. What other factors must 2,844 have? (2, 3, 4 and 6 are factors because they are factors of 12. There are others such as 2844, 1422, 948, 711, 474)

PRIME NUMBERS

Objective: To help children understand prime numbers and the role they play in multiplication

Vocabulary: Prime number (prime), composite number, Sieve of Eratosthenes

Background:

> Note: The process illustrated with factor trees always terminates, perhaps after many steps, perhaps after one or two. It may happen that it cannot be begun, as for 5, 7, 17, 23, since numbers like this cannot be expressed as a product of two smaller factors. In this chapter, factors shall always be whole numbers. Of course, it is just prime numbers such as these that appear in the last level of a factor tree. They are the "bricks" from which all other numbers are "constructed" by multiplication. If one is to answer questions involving factors or product expressions, we must become familiar with the properties of these numbers, called prime numbers. Our study will also have some very practical consequences for the computation of greatest common factors and least common multiples. Least common multiples will be reserved for a later chapter.

It is not possible for a number to appear in the last level of a factor tree if it can be expressed as a product of two whole numbers less than itself. For example, 6 cannot appear, because it can be expressed as the product 2 × 3. The numbers in the last level are those which cannot be written as a product of two smaller factors. These numbers in the last level are called <u>prime numbers</u>.

A <u>prime number</u> is a number which is greater than 1 but cannot be written as the product of two smaller factors that are whole numbers greater than 1. Take the number 3. 3 is greater than 1 but cannot be written as the product of two smaller factors. Therefore, 3 is a prime number.

On the other hand, the number 4 is larger than 1 but can be written as the product of two smaller factors, 2 × 2. So 4 is not a prime number. It is a composite number.

There are other ways to define prime numbers:

(1) A prime is a number which is greater than 1 but which cannot be written as a product without using 1 as a factor. (You may use a prime to mean a prime number.)

(2) A prime (or, a prime number) is a number with exactly two factors, itself and one. For instance, 3 is prime (or, is a prime number) because its only factors are 1 and 3. 4 is not prime (or, is not a prime number) because it has factors 1, 2, and 4.

(3) A prime number is a number with no factor which is smaller than itself but greater than 1. 37 is prime because there is no factor of 37 that is smaller than 37 but greater than 1. 6 is not prime because it has the factor 2 that is smaller than 6 but greater than 1.

All these definitions of prime numbers are saying the same thing: "A prime number is a number which is greater than 1 but cannot be written as the product of two smaller factors, each of which is smaller than the number."

A whole number which is not prime and is greater than 1 is called a composite number.

$$4 = 2 \times 2$$

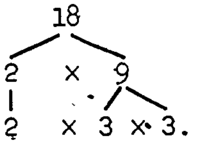
$$6 = 2 \times 3$$

$$9 = 3 \times 3$$

show that 4, 6, and 9 are composite numbers.

Suggested Exploration:

Write several factor trees on the board, for example:



Ask when we know we have finished a factor tree.

Introduce the idea and terminology of prime and composite numbers. Use many examples.

Define a prime number as a number which is greater than 1 but which cannot be written as the product of two numbers each smaller than the number.

Define a composite number as a number which is not prime and is greater than 1. It can be written as the product of two numbers each smaller than the number.

After discussion, have children study pupil page 56.

PRIME NUMBERS

A prime number is a whole number which is greater than 1 but cannot be expressed as the product of two smaller factors.

2, 3, 5, 7, 11 are examples of primes.

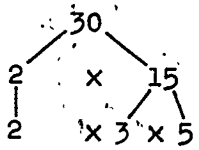
The name "prime number" is usually shortened to "prime".

A whole number which is not prime, and is greater than 1, is called a composite number.

A composite number is one which can be expressed as a product of two smaller factors.

4, 6, 8, 9, 10 are examples of composite numbers.

A "factor tree" can picture prime numbers. This factor tree tells us that 2, 3, and 5 are prime numbers.

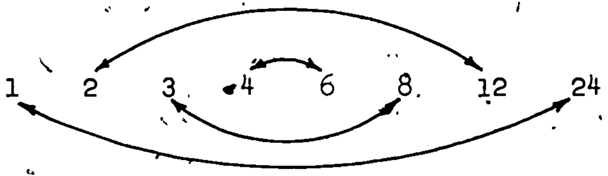


PAIRS OF FACTORS

> This section is included as preparation for the next section, "Testing for Primes."

Background:

The set of all factors of 24 is {1, 2, 3, 4, 6, 8, 12, 24}.

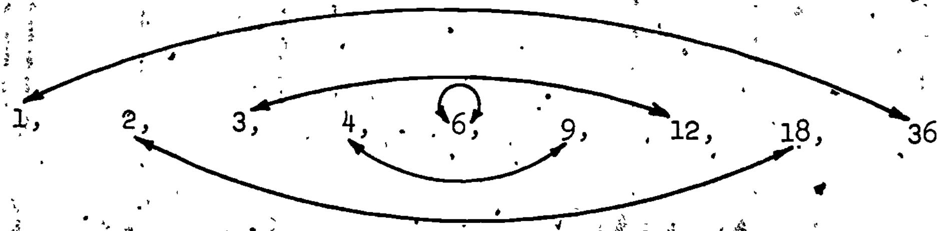


The diagram shows that the factors of a number belong together in pairs. 24 can be expressed as a product of pairs of factors in these ways:

$$\left.\begin{array}{ccc}4 & \times & 6\\ 3 & \times & 8\\ 2 & \times & 12\\ 1 & \times & 24\end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{ccc}6 & \times & 4\\ 8 & \times & 3\\ 12 & \times & 2\\ 24 & \times & 1\end{array}\right.$$

> If one of a pair of factors of 24 is less than 5 then the other factor of the pair must be greater than 5. For example, if one factor is 4 (the factor less than 5) the other factor of the pair must be greater than 5 since $4 \times 5 = 20$ and $20 < 24$. If both factors in a pair were 5, then their product would be 25 and if both factors were greater than 5, then their product would be greater than 25. This may be summarized briefly in the following way. Select all the whole numbers each of whose "squares" (the "square" of a number n is $n \times n$) is less than or equal to 24. These numbers are possible factors of 24. Then test each number to determine if it is a factor. Such of these that are factors will be one factor of a pair of factors. The other factor of the pair can be found easily. In this manner all factor pairs can be obtained.

If one of a pair of factors of 36 is greater than 6, the other is less than 6. This diagram shows the pairs of factors of 36.



36 can be expressed as a product of pairs of factors in these ways. The pairs of factors are:

1 × 36		36 × 1
2 × 18		18 × 2
3 × 12	OR	12 × 3
4 × 9		9 × 4
6 × 6		6 × 6

In each case if one of the factors is less than 6, the other factor must be greater than 6. In any case, if each factor is greater than 6, the product would be at least 7 × 7 = 49.

Suggested Exploration

Use diagrams as those shown on Pages 103 and 104 to illustrate pairs of factors.

Using P57 for class discussion, help the children make observations similar to these:

1. If one of a pair of factors of 24 is less than 5, the other is greater than 5. If it weren't, the product would be no greater than 4 × 5 = 20.
2. If both factors in a pair are 5 or more, then their product will be at least 5 × 5 = 25; and 25 > 24.

P57

Questions for Class Discussion

1. In each classroom in a school, the seats form an array. There are never more than 7 rows of 5 seats each. What is the largest number of seats there can be in a classroom? 35

2. I am thinking of two numbers. One is no greater than 8, and the other is no greater than 7. What do you know about their product? Their product will be 56 or less.

3. A number is no greater than 4. If it is multiplied by itself, how great can the product be? 16 or less.

4. The product of two numbers is 64. One of them is greater than 8. What do you know about the other? The other number is less than 8.

5. The product of two numbers is 100. One is less than 10. What do you know about the other? The other is greater than 10.

6. A certain factor of 144 is greater than 12. What do you know about the unknown factor? The unknown factor is less than 12

BRAINTWISTER

7. The number 6 is equal to the sum of its factors, not including 6 itself. 6 = 1 + 2 + 3. There is another whole number less than 30 which is equal to the sum of its factors, not including itself. Find it. (28)

TESTING FOR PRIMES

Background:

The idea brought out on page 57 can be used to make the work easier in finding factors of any number and in locating primes.

Find the set of the factors of 15. 1 and 15 are both factors of 15. 2 is tested and it is found that 2 is not a factor of 15. 3 is tested and it is found that 3 is a factor of 15. Since 2 is not a factor of 15, then 4 cannot be a factor, because 2 is a factor of 4. If 4 were a factor of 15, then 2 would also be a factor of 15. Also, if 15 had a factor greater than 4, the other factor of the pair belonging together would have to be less than 4, because 4 × 4 = 16 and $16 > 15$. Without testing further than 3, a factor from each pair of factors of 15 is found. The remaining factors can be found from known multiplication facts or by division. For example, 3 × 5 is 15, so the set of all factors of 15 is {1, 3, 5, and 15.} This method greatly reduces the work in finding factors of larger numbers and in finding primes.

Take the number 23. In every pair of factors, one would have to be less than 5. Otherwise their product would be at least 5 × 5. The only proposed factors necessary to test will be 2, 3, and 4. Multiplication facts demonstrate that neither 2 nor 3 is a factor of 23. Therefore, 4 is not a factor of 23. Since none of these is a factor, then the only factors of 23 are 1 and 23. This tells us that 23 is a prime number.

As another example consider 67. We wish to determine if 67 is a prime number. Consider the number whose "squares" are 67 or less than 67. These numbers are 2, 3, 4, 5, 6, 7, and 8. If none of these is a factor then 67 is a prime number. The testing of these possible factors can be shortened in this way. Test 2 and find that 2 is not a factor; then it follows that neither 4 nor 6 nor 8 is a factor because each of these has 2 as a factor. Then test 3 and find that 3 is not a factor. (We already know that 6 is not a factor.)

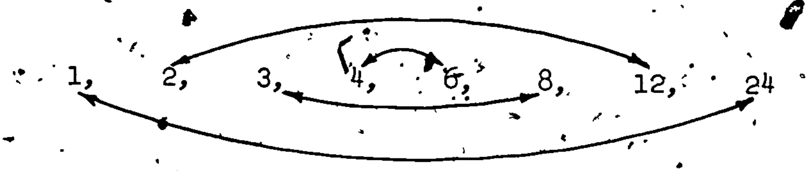
Consequently only 5 and 7 remain to be tested and testing shows neither is a factor. Consequently 67 is a prime number. Now observe that of all the possible numbers whose "squares" are less than 67, namely 2, 3, 4, 5, 6, 7 and 8, it was necessary to test only the ones of these that are prime numbers, i. e., 2, 3, 5 and 7.

Suggested Exploration:

From the specific problems, help children discover the generalization that to test a number for "primeness", we need consider only the prime factors whose squares are less than the number. One should not expect a statement of this idea until further work is done. Children can be aware of the notion and use it without being able to express it in words.

TESTING FOR PRIMES

The factors of a number are arranged in pairs. This diagram shows these pairs of factors of 24.



If one of a pair of factors of 24 is less than 5, the other is greater than 5. Why?

If one of a pair of factors of 36 is greater than 6, the other is less than 6. Why?

At least one factor in every pair of factors of 48 is less than 7. Why?

We can use this idea to make the work easier in finding factors. It also helps in locating primes.

Suppose we want to find factors of 23. We can test 2, 3, 4 by dividing or by knowing multiplication facts. None of these is a factor of 23. We know, then, that 23 is prime because: if 23 had a factor greater than 4, the other factor would have to be 4 or smaller. Otherwise, their product would be at least 5 × 5 = 25.

To know that 23 is prime, we do not need to test any other numbers as factors. We do not even need to test 4. Do you see why?

Exercise Set 6

1. To find whether 41 is prime or composite, what numbers must we test as possible factors? 2, 3, 5

2. Use division to find whether 41 is prime.

Test the following numbers as you did 41. If the number is composite, express it as a product of prime factors. If it is prime, write "prime."

Example: 19 prime

21 composite, 21 = 3 × 7

3. 22 = 2 × 11

4. 27 = 3 × 3 × 3

5. 31 prime

6. 33 = 3 × 11

7. 39 = 3 × 13

8. 53 prime

9. 55 = 5 × 11

10. 67 prime

11. 69 = 3 × 23

12. 83 prime

13. 87 = 3 × 29

14. 143 = 11 × 13

THE PRIME FACTOR CHART

Background:

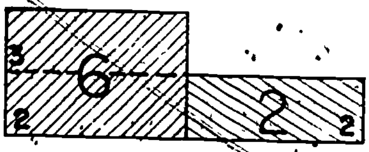
The role of primes in multiplication may be better understood with the aid of an analogy. Like all analogies, it requires judicious use and firm resistance to addiction. Experience indicates that this analogy may be best appreciated if it is read several times.

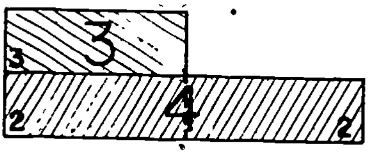
The essence of the analogy is the viewing of a number as a compound structure, say a wall. The wall is built from several different colors of bricks. By a brick, we mean a structure consisting of just one indecomposable unit. The analogy requires that we think of prime numbers as bricks. The process of putting bricks together to construct a wall corresponds to multiplying primes to form composites. Given a pile of "number bricks" of different colors; so many 2's, so many 3's, and so on, many different walls can be constructed using some or all of each color. Since

$$60 = 2 \times 2 \times 3 \times 5,$$

"the wall" (60) is made of 2 bricks of one color (2) and one each of two other colors, (3 and 5).

Suppose, on the other hand, that we are given a finished number wall, e.g. 12, and wish to determine how it is constructed. We can break the wall apart into smaller parts, which we must also think of as walls, in several ways. (In the wall analogy, "factor" corresponds to "part of".) The wall 12 breaks up into the wall 6 and the wall 2. It also breaks into the wall 4 and the wall 3.





However for number walls as for actual walls, no matter how we break up the wall into smaller walls, if we continue breaking pieces until each piece is a single brick, then we must always finish with the same collection of bricks. That is to say, two different sets of bricks, say 2, 2, 3, 7 and 2, 3, 3, 5 can never form exactly the same wall. This is the meaning of the uniqueness in the representation of a number as a product of primes.

Notice that the analogy does not provide a counterpart to the commutativity of multiplication. For the number wall, it does not matter in what order the bricks are laid, the result is the same. In an actual wall, a white brick over a red brick produces a different wall than the reverse.

Nevertheless the analogy can be extended to some of the properties of primes. For example, if an actual wall contains a red brick, and if the wall is broken into two parts, then one of the two parts contains a red brick. This is the analogy of a useful property of primes: If a prime divides a product, then it divides at least one of the factors.

> Note: The wall analogy is suggested as a possibly useful way to illustrate the process of factorization and the rule of primes. It is strictly optional for classroom use, and no reference is made to it in the pupils' text.

Finding the prime factors of a number by testing smaller primes as factors has several disadvantages. First of all we must already know the primes smaller than a certain number. To test 9,997 we might have to try all primes less than 100. (100 × 100 = 10,000) and so we must already know them. Secondly, the process is extremely tedious. It is particularly poorly adapted to the very problem whose prior solution it requires; namely that of finding all primes smaller than 50 or 100 or 200.

A much better process for systematically discovering primes derives from the observation that it is relatively easy to write down the composite numbers less than 100. Each composite number less than 100 has 2, 3, 5, or 7 as a factor. Therefore, if we list the numbers 2 to 100 and then strike out the numbers which are larger than 2 and have 2 as a factor, the primes must be among those left.

2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~, 17, ~~18~~, 19, ~~20~~, 21, ~~22~~, 23, ~~24~~, 25, ~~26~~, 27, ~~28~~, 29, ~~30~~, 31, ~~32~~, 33 ...

If we now also strike out the numbers greater than 3 with 3 as a factor, the primes still remain. Then we can eliminate in order those with 5 or 7 as a prime factor. The remaining numbers must be the primes less than 100.

This idea suggests the physical analogy of a series of sieves through which a heterogeneous bunch of particles is passed in succession. The method we have described is actually called the Sieve of Eratosthenes, after a man of ancient Greece who used it. Using the sieve analogy we can describe what we have done in the following terms:

First we put the numbers 2 through 100 onto a "2 sieve". This "2 sieve" holds only numbers larger than 2 with 2 as a factor and allows the rest to pass through. These passing through then fall onto the "3 sieve" which retains only numbers larger than 3 with 3 as a factor. The numbers passed by the 3 sieve fall through onto the "5 sieve" and then the "7 sieve". Those which pass through the final "7 sieve" are the primes.

> Note: The actual process of finding primes in this way can be made to reveal more than the immediate objective, and is something the children can do themselves. It is suggested that the chart which is shown on page 60 be duplicated and distributed to the children. Some children may be interested enough to extend the chart through 100. The chart can be extended to 120 using only the primes 2, 3, 5, 7. The columns showing prime factors up to 7 can be filled in now. The column showing each number as a product of primes should be filled in at an appropriate point in the work of the next section.

Suggestions for Exploration:

The wall analogy is included primarily for teacher background. If it seems appropriate to use with pupils, do so. Explore the ways in which a number is like a wall, factors are like parts of a wall, primes are like bricks, and finding prime number expressions is like finding the number of each color brick that makes up the wall.

The Sieve of Eratosthenes offers a systematic process for discovering primes. Discuss with children the meaning of the word, sieve.

Use the last paragraph of the teacher background (P.112) as a guide.

Distribute duplicated copies of the chart shown on P 60. Either ask the children to fill in the prime factor part of the chart individually, or do it as a class project. Keep the charts. The final column should be completed later. Here children use the chart in their discussions in Exercise Set 7.

P60

THE PRIME FACTOR CHART

	Prime Factors						Prime Factors				
No.	2	3	5	7		No.	2	3	5	7	
2	2				prime	26	2				2 × 13
3		3			prime	27		3			3×3×3
4	2				2 × 2	28	2			7	2×2×7
5			5		prime	29					prime
6	2	3			2 × 3	30	2	3	5		2×3×5
7				7	prime	31					prime
8	2				2×2×2	32	2				2×2×2×2×2
9		3			3 × 3	33		3			3 × 11
10	2		5		2 × 5	34	2				2 × 17
11					prime	35			5	7	5 × 7
12	2	3			2×2×3	36	2	3			2×2×3×3
13					prime	37					prime
14	2			7	2 × 7	38	2				2 × 19
15		3	5		3 × 5	39		3			3 × 13
16	2				2×2×2×2	40	2		5		2×2×2×5
17					prime	41					prime
18	2	3			2×3×3	42	2	3		7	2×3×7
19					prime	43					prime
20	2		5		2×2×5	44	2				2×2×11
21		3		7	3 × 7	45		3	5		3×3×5
22	2				2 × 11	46	2				2 × 23
23					prime	47					prime
24	2	3			2×2×2×3	48	2	3			2×2×2×2×3
25			5		5 × 5	49				7	7 × 7
						50	2		5		2×5×5

Exercise Set 7 (Oral)

Using your prime factor chart, answer the questions.

1. Look at all the primes in the chart that are greater than 2. There is always at least one number between any two of them. Why? (all prime numbers except 2 are odd and there is at least one even number between any two odd numbers.)

2. Look at the numbers between 7 and 49 with 7 as a prime factor. Each number also has 2, 3, or 5 as a factor. Why must this happen? Each composite number is expressed as a product of primes. 2, 3, and 5 are the only primes less than 7.

3. Can the numbers from 2 to 50 have prime factors which are not shown on the chart? Give an example if there is one. (Yes. 22 has the factor 11)

4. What numbers in the chart are prime numbers in addition to the numbers 1, 3, 5, and 7? (2, 11, 13, 17, 19, 23 29, 31, 37, 41, 43, 47)

TESTING 2, 3, AND 5 AS FACTORS OF A NUMBER

Background:

Below is a list from the factor chart of the composite numbers that have 2 as a factor.

4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
46
48
50

The unit digit in each numeral shows that a definite pattern exists in those numbers having 2 as a factor. It is: 4, 6, 8, 0, 2, 4, 6, 8, 0, ... If a numeral ends in 2, 4, 6, 8, or 0, the number will have 2 as a factor. We can draw a conclusion: In the set of all counting numbers, {1, 2, 3, ...}, a number will have 2 as a factor provided the unit digit in its decimal numeral is 2, 4, 6, 8, or 0.

There also is a pattern existing among all the composite numbers having 3 as a factor. Below is a list of all the composite numbers in the chart having 3 as a factor.

6
9
12
15
18
21
24
27
30
33
36
39
42
45
48

There is a pattern in the units' digits but the pattern gives us no clue as the pattern did for selecting the factor 2. All of the ten digits appear as units' digits in the above set of multiples of 3. Certainly we cannot conclude that a number whose units' digit is one of the ten digits has 3 as a factor.

Consequently we must look elsewhere for a clue in determining if 3 is a factor of a certain number. For this purpose consider the following numbers and the corresponding numbers obtained by finding the sum of the digits in the numerals.

Number	Sum of digits	Number	Sum of digits
71	7 + 1 = 8	86	8 + 6 = 14
92	9 + 2 = 11	304	3 + 0 + 4 = 7
96	9 + 6 = 15	522	5 + 2 + 2 = 9
129	1 + 2 + 9 = 12	675	6 + 7 + 5 = 18
135	1 + 3 + 5 = 9	111	1 + 1 + 1 = 3

In the table above consider the digit sums which have 3 as a factor. These sums are the numbers 15, 12, 9, 9, 18, 3. The numbers with these sums are 96, 129, 135, 522, 675, 111. These numbers whose "digit sums" have 3 as a factor also themselves have 3 as a factor. Indeed it is true in general that "If the sum of the digits of a numeral is a number which has 3 as a factor, then the number named by the numeral has 3 as a factor."

No proof of this general statement is given here but the following illustration may be of interest to the teacher. Consider 2439, for example. We may write

$$\begin{aligned} 2439 &= 2(1000) + 4(100) + 3(10) + 9 \\ &= 2(999 + 1) + 4(99 + 1) + 3(9 + 1) + 9 \end{aligned}$$

and then by use of the distributive, commutative, and associative properties we can write this as

$$2(999) + 4(99) + 3(9) + (2 + 4 + 3 + 9).$$

It should be clear now from this expanded form of writing 2439 that if 3 is a factor of (2 + 4 + 3 + 9 or 18), then 3 is a factor of 2439.

In summary, among the set of counting numbers, {1, 2, 3, ...}, a number will have a factor of 3 provided the "sum of its digits" has 3 as a factor. 111 has the factor 3 because 1 + 1 + 1 = 3 and 3 is a factor of 3. 1,437 has the factor

3, because 1 + 4 + 3 + 7 = 15 and 3 is a factor of 15. Also, 3 is a factor of 765 because 7 + 6 + 5 = 18. 2 is not a factor of 765 because the last digit is not 2, 4, 6, 8, or 0.

There is one other observation to be made at this time. How can we tell quickly (without dividing) whether a number has a factor of 5? Make a list of all the numbers in the chart that have 5 as a factor. 5 is a factor of 5, so it may be included in the list.

5
10
15
20
25
30
35
40
45
50

The units' digit in the listing is either 5 or 0. This means that if the units' digit is 5 or 0, then it must be divisible by 5 or have a factor of 5. There is no number that ends in 5 or 0 that does not have 5 as a factor.

In the set of all counting numbers, {1, 2, 3, ...}, a number will have 5 as a factor provided the units' digit of its decimal numeral is 5 or 0. 235 has a factor of 5 because its units' digit is 5. 630 has a factor of 5 because its units' digit is 0. 630 has a factor of 2, a factor of 3, and a factor of 5. Since 2, 3, and 5 are each factors of 630, then 2 × 3 × 5, 2 × 3, 3 × 5, and 2 × 5 are each factors of 630. Some of the factors of 630 are 2, 3, 5, 30, 6, 15, and 10.

Tests for divisibility by 2, 3, or 5 can be applied quickly to a number. For example,

734 { The units' digit is 4; so 2 is a factor of 734.
The sum of the digits is 14; so 3 is not a factor of 734.
The units' digit is 4; so 5 is not a factor of 734.

615 { The units' digit is 5; so 2 is not, but 5 is a factor of 615.
The sum of the digits is 12; so 3 is a factor of 615.

Suggestions for Exploration:

Develop rules for divisibility by following the background and referring to the prime factor chart pupils have just completed. Much of the background for rules of divisibility can be drawn from the pupils' observations as they work with the chart.

First, consider numbers divisible by 2. Proceed to 3's and 5's. Then give several examples in which all three are tested as factors of the same number.

TESTING 2, 3, AND 5 AS FACTORS OF A NUMBER

From our study of the Prime Factor Chart we observed:

1. In the set of counting numbers, {1, 2, 3, 4 ...}, a number will have 2 as a factor if the units' digit of its numeral is 0, 2, 4, 6 or 8.

   Examples of counting numbers which have a factor of 2 are: 40, 182, 364, 56, 218.

2. In the set of counting numbers, a number will have 3 as a factor if the sum of the digits in its numeral can be divided by 3.

   Examples of counting numbers which have a factor of 3 are:

   951 (Because 9 + 5 + 1 = 15 and 15 can be divided by 3.)

   543 (Because 5 + 4 + 3 = 12.)

   864 (Because 8 + 6 + 4 = 18. 864 also has 2 for a factor because the units' digit is 4.)

3. In the set of counting numbers, a number will have 5 as a factor if the units' digit of its numeral is 0 or 5.

   Examples of counting numbers which have a factor of 5 are: 4,835, 495, and 860.

   495 would also have 3 as a factor because the sum of the digits of its numeral can be divided by 3.

   860 would have a factor of 2 because the units' digit in its numeral is 0.

P63

Exercise Set 8

Find one prime factor of each of the following numbers.

1. 785 5
2. 7,012 2
3. 8,001 3
4. 7,136 2
5. 4,895 5
6. 4,083 3
7. 67,210 2 or 5
8. 60,105 3 or 5

Find two different prime factors of each of the following numbers.

9. 405 3 and 5
10. 6,780 2 and 5, 3 and 5, or 2 and 3
11. 3,042 2 and 3
12. 5,055 3 and 5
13. 4,314 2 and 3
14. 6,060 2 and 3, 3 and 5, or 2 and 5

Write 2, 3, and 5 in the correct places in this chart. Exercise 15 is done for you.

	Number	These numbers are factors	These numbers are not factors
15.	365	5	2, 3
16.	492	(2, 3)	(5)
17.	835	(5)	(2, 3)
18.	3,681	(3)	(2, 5)
19.	370	(2, 5)	(3)
20.	86,910	(2, 3, 5)	

P64

BRAINTWISTERS:

For each exercise below, what are all the numbers less than 100 which have these numbers and no others as prime factors?

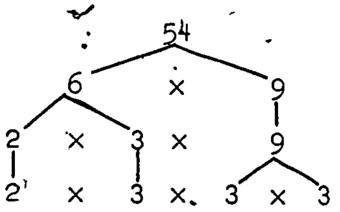
21. 3 and 5 (15, 45, 75)
22. 3 and 7 (21, 63)
23. 5 and 7 (35)
24. 2 and 11 (22, 44, 88)

COMPLETE FACTORIZATION

Background:

When the factor tree can be extended no further, the last row always contains all the prime factors of the number. If a number in the tree is composite (having a factor smaller than itself but greater than 1), two more branches can be drawn.

Example:



A composite number, 54.

Both factors are composite.

In this row, we have 2 primes and one composite number.

All factors in this row are primes.

Every composite number is the product of smaller factors. If one of these factors is composite, then it is the product of smaller factors. If this process is continued, a product will emerge in which no factor is composite, and every factor is a prime. For example, in the tree above --

$54 = 6 \times 9$	(6 and 9 are both composite.)
$= 2 \times 3 \times 9$	(2 and 3 are prime but 9 is composite.)
$= 2 \times 3 \times 3 \times 3$	(All factors are prime.)

Another approach could be taken:

$54 = 6 \times 9$	(6 and 9 are both composite.)
$= 6 \times 3 \times 3$	(6 is composite and 3 is prime.)
$= 2 \times 3 \times 3 \times 3$	(All factors are prime.)

Look at several other numbers:

$24 = 3 \times 8$	(3 is prime and 8 is composite.)
$= 3 \times 2 \times 4$	(3 and 2 are prime and 4 is composite.)
$= 3 \times 2 \times 2 \times 2$	(All factors are prime.)

$36 = 4 \times 9$ (4 and 9 are composite.)

$= 2 \times 2 \times 9$ (9 is composite.)

$= 2 \times 2 \times 3 \times 3$ (All factors are prime.)

This method suggests that every number greater than 1 is either prime or is a product of primes. The expression of a number as a product of primes is the source of much information. Since we will use these product expressions throughout the remainder of this unit, it is important to devise processes for finding them. Sometimes it is possible to begin with a known multiplication fact. For example, to find the product expression, using only primes, for 36 we may begin by remembering

$$36 = 4 \times 9 \quad \text{or}$$
$$36 = 6 \times 6.$$

Now we think of multiplication facts giving 4, 9, or 6 as products:

$$\begin{aligned} 36 &= 4 \times 9 \\ &= (2 \times 2) \times 9 \\ &= (2 \times 2) \times (3 \times 3) \end{aligned} \qquad \begin{aligned} 36 &= 6 \times 6 \\ &= (2 \times 3) \times 6 \\ &= (2 \times 3) \times (2 \times 3) \end{aligned}$$

This way of factoring can be looked upon as a "splitting process". Notice that in the two solutions above, the final product expressions are the same except for order. The splitting process, applied to 42 might lead to any of the following, depending upon what facts are used.

$$\begin{aligned} 42 &= 2 \times 21 \\ &= 2 \times (3 \times 7) \end{aligned} \qquad \begin{aligned} 42 &= 3 \times 14 \\ &= 3 \times (2 \times 7) \end{aligned} \qquad \begin{aligned} 42 &= 6 \times 7 \\ &= (2 \times 3) \times 7 \end{aligned}$$

Again the splitting process was used in 3 different ways. Each time the same prime factors, apart from order were found.

The splitting process requires knowledge of many multiplication facts and is difficult to apply to large numbers. There is a more systematic way of factoring that requires less knowledge. Begin by examining the units' digit to see if it has a factor 2. If it does, then divide the number by 2. If it does not, then check by division the prime number 3, then 5, then 7, then 11, then 13, etc., until all possibilities have been examined. If the number does have the factor 2, then find the unknown factor and proceed to test 2 as a factor of it. Suppose we wish to write the number 156 as a product of primes. Since the last digit is 6, then 156 is divisible by 2. Division gives

$$156 = 2 \times 78.$$

Again check to see if 78 is divisible by 2. It is, and division gives

$$156 = 2 \times 2 \times 39.$$

Look at 39. Because the last digit is not a multiple of 2, 39 is not divisible by 2. Check for divisibility by 3. 3 + 9 = 12 and 12 can be divided by 3, therefore 39 is divisible by 3. Now,

$$156 = 2 \times 2 \times 3 \times 13.$$

13 is not divisible by 2 or 3 (or any other prime number other than 13), therefore 13 is prime. This process might be called the <u>peeling</u> process.

The results of this process as applied to 780 can be summarized as follows:

$$\begin{aligned} 780 &= 2 \times 390 \\ &= (2) \times (2 \times 195) \\ &= (2) \times (2) \times (3 \times 65) \\ &= (2) \times (2) \times (3) \times (5 \times 13) \end{aligned}$$

(We have all primes, so the process is complete.)

> It is convenient to think of factoring as a "splitting" or "peeling" process. However, these two names for the two different ways of factoring may or may not be used with children. It is possible that as children work with these two different methods, they will develop names of their own to suggest the two ways of factoring.

The next goal to be reached with the pupils is the expression of a composite number as the product expression of all the prime factors of the number. Complete factorization of a number means that the number is expressed as the product expression using its prime factors. For example, complete factorization of 24 means 24 = 2 × 2 × 2 × 3.

As well as the complete factorization of a number we shall consider also all the factors of a number. Finding all the factors of a number is studied in later sections in this unit but it may be well to contrast complete factorization and finding all factors at this time. The names of these processes seem to suggest they might have the same meaning but they are quite different and must not be confused. The complete factorization of 24 (for example) is expressing 24 as the product expression using its prime factors. This can be done by the use of the factor tree, or some other way. But, finding ~~all~~ the factors of 24 requires finding all factors (prime and composite, if any) of 24, namely 1, 2, 3, 4, 6, 8, 12, 24.

Examples.

36 = 2 × 2 × 3 × 3 This is complete factorization.
1, 2, 3, 4, 6, 9, 12, 18, 36 is the set set of all factors of 36.

50 = 2 × 5 × 5 This is complete factorization.
1, 2, 5, 10, 25, 50 is the set of all factors of 50.

144 = 2 × 2 × 2 × 2 × 3 × 3 This is complete factorization.
1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144 is the set of all factors of 144.

After complete factorization of 144, for example, the set of all factors of 144 is obtained in the following manner.

From the product expression 2 × 2 × 2 × 2 × 3 × 3 (1) select all the different primes which appear in the product expression. (2) Then from the product expression select all the products of two factors, (3) then of three factors, (4) then of four factors, etc.

These are respectively

(1) $2, 3$

(2) $2 \times 2,\ 2 \times 3,\ 3 \times 3$

(3) $2 \times 2 \times 2,\ 2 \times 2 \times 3,\ 2 \times 3 \times 3$

(4) $2 \times 2 \times 2 \times 2,\ 2 \times 2 \times 2 \times 3,\ 2 \times 2 \times 3 \times 3$

(5) $2 \times 2 \times 2 \times 2 \times 3,\ 2 \times 2 \times 2 \times 3 \times 3$

and finally

(6) $2 \times 2 \times 2 \times 2 \times 3 \times 3$,

the original product expression for 144. (This last one is, of course, not needed as we knew it from the complete factorization.)

From: (1) we get the factors 2, 3

(2) we get the factors 4, 6, 9

(3) we get the factors 8, 12, 18

(4) we get the factors 16, 24, 36

(5) we get the factors 48, 72

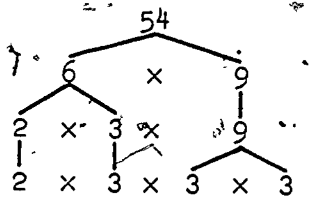
We know 1 and 144 are factors of 144. From the product expression $2 \times 2 \times 2 \times 2 \times 3 \times 3$ we have found that 1, 2, 3, 4, 6, 9, 8, 12, 18, 16, 24, 36, 48, 72, and 144 are factors of 144. It is a consequence of a property of primes that this method yields all factors of 144. Thus the set

$\{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144\}$

is the set of all factors of 144.

Outline for Exploration:

Write a factor tree for 54 on the board.



Analyze the numbers at each level.

Children should see that at the last level, each factor in the product expression is prime.

Continue with other examples (24, 36, 42, etc.)

Be sure children see that, regardless of the first multiplication sentence written, the product expressions at the last level of the factor tree are the same except for order, i.e.

$42 = 2 \times 21$ $\quad$ $42 = 3 \times 14$ $\quad$ $42 = 6 \times 7$

$= 2 \times 3 \times 7$ $\quad$ $= 3 \times 2 \times 7$ $\quad$ $= 2 \times 3 \times 7.$

Two different approaches to factoring were mentioned in the background. Both of them, although not necessarily their names, should be presented to children.

METHOD A (Splitting)

$36 = 4 \times 9$ $\quad$ 36 is written as 4×9.

$= 2 \times 2 \times 3 \times 3$ $\quad$ 4 is written as 2×2 and

9 is written as 3×3.

In this method, multiplication facts are used to write the composite number as a product of smaller and smaller factors until it is expressed as a product of primes.

METHOD B (Peeling)

$140 = 2 \times 70$	2	is a factor of 140 by divisibility test.
$= 2 \times (2 \times 35)$	2	is a factor of 70 by divisibility test. Neither 2 nor 3 is a factor of 35 (by divisibility).
$= (2 \times 2) \times (5 \times 7)$	5	is a factor and the other factor is 7.

In this method, we look for prime factors of the composite number by testing the primes in order, starting with 2; i.e., we try 2, 3, 5, 7, etc.

Several examples of each method may be needed before understanding is realized.

Example:

$252 = 2 \times 126$ (Peeling off 2)

$= 2 \times 2 \times 63$ (Peeling off 2)

$= 2 \times 2 \times 3 \times 21$ (Peeling off 3)

$= 2 \times 2 \times 3 \times 3 \times 7$ (Peeling off 3)

Children and teacher should read and discuss pupil pages 65 and 66

After Exercise Set 9 has been completed, children are introduced to a property of products of primes. This property, stated on pupil page 69 is called the Fundamental Theorem of Arithmetic. This idea should be discussed carefully with pupils.

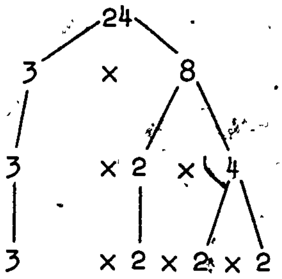
COMPLETE FACTORIZATION

Every composite number is the product of smaller numbers. If one of these numbers is composite, then it also is the product of smaller numbers. If we continue this, we must come to a product expression in which no number is composite and every factor is a prime. Doing this is called <u>complete factorization</u> of a composite number.

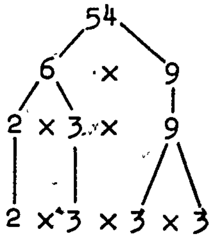
An example of complete factorization:

A picture, using the factor tree is:

$24 = 3 \times 8$	(3 is prime.) (8 is composite.)
$= 3 \times 2 \times 4$	(3 and 2 are prime.) (4 is composite.)
$= 3 \times 2 \times 2 \times 2$	(All are prime.)
$= 2 \times 2 \times 2 \times 3$	(Rearranged for convenience)



$54 = 6 \times 9$	(6 and 9 are composite.)
$= 2 \times 3 \times 9$	(2 and 3 are prime.) (9 is composite.)
$= 2 \times 3 \times 3 \times 3$	(All are prime.)



This suggests that every number greater than 1 is either prime or is a product of primes.

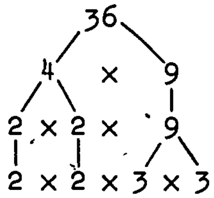
How can we find a way to express any number as a product of primes, for example 36?

We may know some way to express the number as a product.

$$36 = 4 \times 9$$

Then we can write each composite factor as a product expression. Continue until we have only prime factors.

$$36 = 2 \times 2 \times 9$$
$$= 2 \times 2 \times 3 \times 3$$



This product expression $2 \times 2 \times 3 \times 3$ is the complete factorization of 36.

Another way to express a number as a product of primes is by testing small prime numbers such as 2, 3, 5, 7, etc., to see if they are factors of the numbers.

Example:

$$36 = 2 \times 18 \text{ (starting with } 2)$$

Then we look for prime factors of 18 starting with 2.

$$36 = 2 \times (2 \times 9)$$

Then we look for prime factors of 9, starting with 2. Since 2 is not a factor, we next test 3.

$$36 = (2 \times 2) \times (3 \times 3)$$
$$= 2 \times 2 \times 3 \times 3.$$

Either of these ways may be called <u>factoring</u>. Sometimes it is easier to use one process. Sometimes it is easier to use the other process. With practice, you can find shortcuts by combining them.

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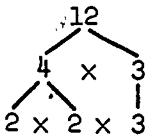
Exercise Set 9

For answers, see T.C. pages 134 and 135.

Express each number below as a product of two smaller factors. If possible, then express one of these factors as a product of smaller factors. Continue until you have expressed the number as a product of primes. This is one factoring process. Show your work by drawing a "factor tree".

Example: $12 = 4 \times 3$

$= (2 \times 2) \times 3$ or



$12 = 2 \times 2 \times 3$

1. 16
2. 18
3. 20
4. 25
5. 27
6. 28
7. 30
8. 35
9. 40

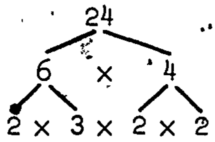
10. Do exercises 1 through 9 again, but this time start with a different pair of factors if there is another pair.

11. Following the example shown, express each number as a product of primes. Draw a factor tree for parts b, d, f.

Example: $24 = 6 \times 4$

$= 2 \times 3 \times 4$

$= 2 \times 3 \times 2 \times 2$



$24 = 2 \times 2 \times 2 \times 3$

a. 30

b. 72

c. 84

d. 96

e. $128 = 8 \times 16$

f. $288 = 12 \times 24$

g. $225 = 15 \times 15$

12. Use any factoring process to write each number as a product expression of primes.

a. 144 Answer: $144 = 2 \times 72$

$= 2 \times 2 \times 36$

$= 2 \times 2 \times 2 \times 18$

$= 2 \times 2 \times 2 \times 2 \times 9$

$= 2 \times 2 \times 2 \times 2 \times 3 \times 3$

b. 225

c. 588

d. 363

e. 385

f. 127

g. 585

h. 189

i. 143

13. Without multiplying, write each number as a product expression of primes.

a. 18×60

b. 42×84

c. 21×78

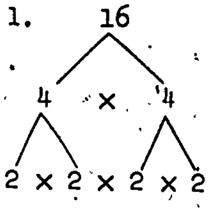
d. 50×50

e. 125×64

f. 25×320

Exercise Set 9, Sample Answers

1.



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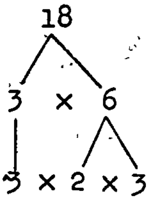

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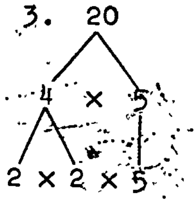

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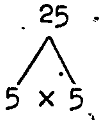

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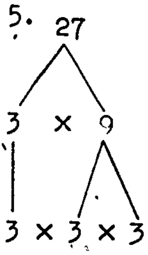

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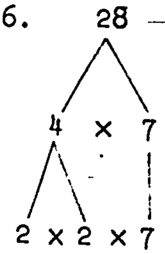

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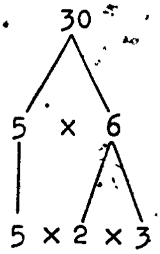

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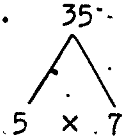

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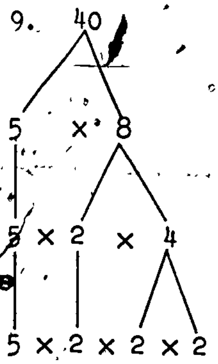

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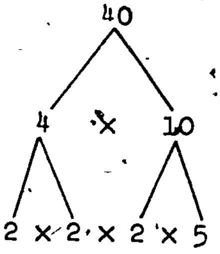


NOTE: We have given only one solution to each exercise. There are other factor trees that can be drawn.

11\. a. $30 = 2 \times 3 \times 5$

b. $72 = 2 \times 2 \times 2 \times 3 \times 3$

c. $84 = 2 \times 2 \times 3 \times 7$

d. $96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$

e. $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

f. $288 = 2 \times 2 \times 3 \times 2 \times 2 \times 2 \times 3$

g. $225 = 3 \times 5 \times 3 \times 5$

Note: Factor trees for b, d, and f will vary.

12\. b. $225 = 3 \times 3 \times 5 \times 5$

c. $588 = 2 \times 2 \times 3 \times 7 \times 7$

d. $363 = 3 \times 11 \times 11$

e. $385 = 5 \times 7 \times 11$

f. 127 is prime.

g. $585 = 3 \times 3 \times 5 \times 13$

h. $189 = 3 \times 3 \times 3 \times 7$

i. $143 = 11 \times 13$

13\. a. $18 \times 60 = (2 \times 3 \times 3) \times (2 \times 2 \times 3 \times 5)$

$= 2 \times 3 \times 3 \times 2 \times 2 \times 3 \times 5$

b. $42 \times 84 = 2 \times 3 \times 7 \times 2 \times 2 \times 3 \times 7$

c. $21 \times 78 = 3 \times 7 \times 2 \times 3 \times 13$

d. $50 \times 50 = 2 \times 5 \times 5 \times 2 \times 5 \times 5$

e. $125 \times 64 = 5 \times 5 \times 5 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

f. $25 \times 320 = 5 \times 5 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5$

A PROPERTY OF PRODUCTS OF PRIMES

The results of the last exercises suggest that we have found a general property. We might state it as:

> Except for the order in which factors are written, a composite number can be expressed as a product of primes in only one way.

You will not find any exceptions to this property because there is a way to show that it is always true. We do not attempt to show in this book why this is true. However, as you use it you should become more sure that it is true. The statement in the "box" is called The Fundamental Theorem of Arithmetic.

FINDING ALL FACTORS

Background:

> Note: The product expression of a number, using prime factors, is always the same regardless of the method by which it is obtained. This fact can be used to justify our methods used in finding:
>
> 1. All factors of a number
> 2. The greatest common factor of a pair of numbers
> 3. The least common multiple of a pair of numbers (in Chapter 5)
>
> These methods are discussed with increasing formality in the pupils' pages (70, 71, 74, 76) the background material (137, 146) and in the mathematical summary at the end of the unit.
>
> It might be helpful now to reread the background on page 126.

If it is known how to express a number as a product of primes, then the <u>set of all factors</u> of the number can be found.

Example: 60 = 2 × 2 × 3 × 5.

A number can be expressed as a product of primes in only one way (disregarding order). Some of the things that can be found are:

1. The prime factors of 60 are 2, 3, and 5.
2. By multiplying in sets of two the factors in the product expression (2 × 2 × 3 × 5), it is apparent that 4, 6, 10, and 15 are factors of 60. (2 × 2, 2 × 3, 2 × 5, and 3 × 5)
3. By multiplying in sets of three the factors in the product expression, it is apparent that 12, 20, and 30 are also factors of 60. (2 × 2 × 3, 2 × 2 × 5, 2 × 3 × 5). There cannot be any other factors except 1 and 60.
4. {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60} is the set of all factors of 60.

In general, if we can write a number as a product of primes then we can find all factors of that number in the manner used in finding all factors of 60.

That we get *all* factors by this method is a consequence of the property of primes stated on pupils' page 69. It is *not true* of other product expressions. For example, from

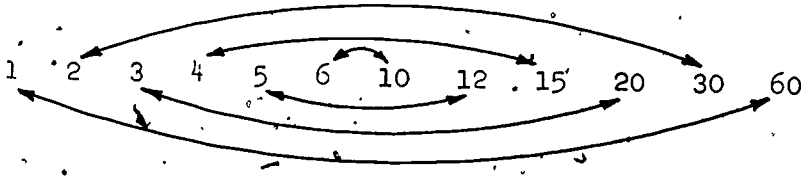
$$36 = 2 \times 3 \times 6$$

we conclude that 2, 3, 6, (2×3), (2×6) and (3×6) are factors of 36. Thus we know that

1, 2, 3, 6, 12, 18 and 36

are factors of 36. Those are *not*, however, all factors, since 4 and 9 are also factors of 36. It is because 6 is not prime that the method failed to give all factors.

Another way of using the complete factorization of a number is in finding all ways to express that number as a product of two numbers. First we find all factors, for example, of 60. These factors can be arranged in pairs so that the product of the factors in each pair is 60. Thus



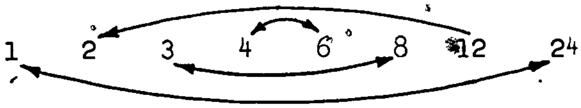
$1 \times 60 = 60$
$2 \times 30 = 60$
$3 \times 20 = 60$
$4 \times 15 = 60$
$5 \times 12 = 60$
$6 \times 10 = 60.$

This shows all the pairs of factors of 60 and gives every way of naming 60 as a product of two factors.

As another example, consider 24. The product expression for 24, using prime factors is 2 × 2 × 2 × 3. This tells us that 2 and 3 are both factors of 24. (2 × 2), (2 × 3), (2 × 2 × 2), and (2 × 2 × 3) also are factors of 24. Every number has itself and 1 as factors; 1 and 24 may be included as factors of 24. Now we may list all the factors of 24 in order, from small to large,

1, 2, 3, 4, 6, 8, 12, 24.

This information can be used to get every way to name 24 as a product of two factors.



$1 \times 24 = 24$

$2 \times 12 = 24$

$3 \times 8 = 24$

$4 \times 6 = 24$

Yet another use of complete factorization is its application in discovering whether one number is a factor of another. First each number is expressed as a product of primes. Then the question can be answered. For example, is 42 a factor of 714?

$42 = 2 \times 3 \times 7$

$714 = 2 \times 3 \times 7 \times 17 = (2 \times 3 \times 7) \times 17$

42 is a factor of 714.

Is 28 a factor of 238?

$28 = 2 \times 2 \times 7$

$238 = 2 \times 7 \times 17$

28 is not a factor of 238 because 2 × 2 does not appear in the complete factorization of 238.

Children will be helped in determining if one number is a factor of another if examples which require rearrangement of the factors are used. Example:

Is 42 a factor of 252?
Is 210 a factor of 3150?

Suggestions for Exploration:

Using the previous background, recall that a number can be expressed as a product of primes in only one way, disregarding order. (This is The Fundamental Theorem of Arithmetic.)

Indicate that if we know how to express a number as a product of primes, we can find the set of all facters of the number by multiplying the factors shown in the product expression in two's, three's, etc. Follow the teacher background using similar examples.

Point out how the set of all factors can be used to find all ways to express the number as a product of two factors. Also point out how complete factorization can be used to find if one number is a factor of another.

Read and discuss pupil page 70, FINDING ALL FACTORS. Then pupils can work Exercise Set 10 independently.

FINDING ALL FACTORS

If we know how to express a number as a product of primes, then we can find the set of all factors of the number.

Suppose we write

$$60 = 2 \times 2 \times 3 \times 5.$$

Here are some of the things we can find:

1. The prime factors of 60 are 2, 3, and 5.

2. By multiplying in pairs the factors shown in the product expression for 60, we see that 4, (2 × 2), 6, (2 × 3), 10, (2 × 5) and 15, (3 × 5) are factors of 60.

3. By multiplying in threes the factors shown in the product expression for 60, we see that 12, (2 × 2 × 3), 20, (2 × 2 × 5) and 30, (2 × 3 × 5) are also factors of 60.

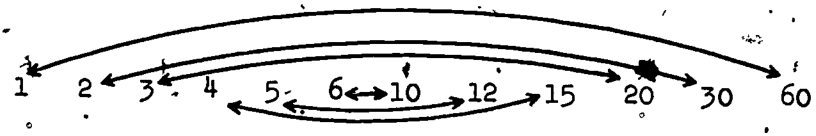
   The factors shown in 2 × 2 × 3 × 5 are primes. For this reason, we must have found by our method, every factor of 60.

4. We know then that

   {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

   is the set of all factors of 60.

5. From the set of all factors of 60, we can get every way of naming 60 as a product of two factors.



$1 \times 60 = 60$

$2 \times 30 = 60$

$3 \times 20 = 60$

$4 \times 15 = 60$

$5 \times 12 = 60$

$6 \times 10 = 60$

Exercise Set 10, Answers

1. b. 30, {1, 2, 3, 5, 6, 10, 15, 30}
   c. 72, {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}
   d. 84, {1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84}
   e. 96, {1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96}
   f. 128, {1, 2, 4, 8, 16, 32, 64, 128}
   g. 225, {1, 3, 5, 9, 15, 25, 45, 75, 225}
   h. 144, {1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144}
   i. 363, {1, 3, 11, 33, 121, 363}
   j. 385, {1, 5, 7, 11, 35, 55, 77, 385}
   k. 89, {1, 89}
   l. 189, {1, 3, 7, 9, 21, 27, 63, 189}
   m. 143, {1, 11, 13, 143}

2. b. 30 = 1 × 30 = 2 × 15 = 3 × 10 = 5 × 6
   c. 72 = 1 × 72 = 2 × 36 = 3 × 24 = 4 × 18 = 6 × 12 = 8 × 9
   d. 84 = 1 × 84 = 2 × 42 = 3 × 28 = 4 × 21 = 6 × 14 = 7 × 12
   e. 96 = 1 × 96 = 2 × 48 = 3 × 32 = 4 × 24 = 6 × 16 = 8 × 12
   f. 128 = 1 × 128 = 2 × 64 = 4 × 32 = 8 × 16
   g. 225 = 1 × 225 = 3 × 75 = 5 × 45 = 9 × 25 = 15 × 15
   h. 144 = 1 × 144 = 2 × 72 = 3 × 48 = 4 × 36 = 6 × 24 = 8 × 18
   = 9 × 16 = 12 × 12
   i. 363 = 1 × 363 = 3 × 121 = 11 × 33
   j. 385 = 1 × 385 = 5 × 77 = 7 × 55 = 11 × 35
   k. 89 = 1 × 89
   l. 189 = 1 × 189 = 3 × 63 = 7 × 27 = 9 × 21
   m. 143 = 1 × 143 = 11 × 13

Exercise Set 10

For answer, see T.C. page 143

1. Find the set of all factors of each number.

   a. 24

   Answer: $24 = 2 \times 2 \times 2 \times 3$

   Set of factors of 24 = {1, 24, 2, 3, 4, 6, 8, 12}
   = {1, 2, 3, 4, 6, 8, 12, 24}

   b. 30
   c. 72
   d. 84
   e. 96
   f. 128
   g. 225
   h. 144
   i. 363
   j. 385
   k. 89
   l. 189
   m. 143

2. Use what you found in exercise 1 to get all of the different ways to write each number in that exercise as a product expression of two factors.

   a. 24

   Answer:

   Set of factors of 24 = {1, 2, 3, 4, 6, 8, 12, 24}
   $24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6.$

3. Find whether each number listed below is a factor of $2 \times 2 \times 3 \times 7 \times 11 \times 11$:

a. 6

<u>Answer:</u>

Yes, because $2 \times 2 \times 3 \times 7 \times 11 \times 11$

$$= (2 \times 3) \times (2 \times 7 \times 11 \times 11)$$

$$= 6 \times (2 \times 7 \times 11 \times 11)$$

The factor belonging with 6 is $2 \times 7 \times 11 \times 11$.

b. 14 Yes, because $(2 \times 7) \times (2 \times 3 \times 11 \times 11) = 14 \times (2 \times 3 \times 11 \times 11)$

c. 28 Yes, because $(2 \times 2 \times 7) \times (3 \times 11 \times 11) = 28 \times (3 \times 11 \times 11)$

d. 210 no, because $210 = 2 \times 3 \times 5 \times 7$ and 5 does not appear in $2 \times 2 \times 3 \times 7 \times 11 \times 11$

e. 242 Yes, because $(2 \times 11 \times 11) \times (2 \times 3 \times 7) = 242 \times (2 \times 3 \times 7)$.

COMMON FACTORS

Objective: To use prime product expressions to find a greatest common factor

Vocabulary: Intersection, common factor, greatest common factor

Background: (Common Factors, Pupil pages 74 and 75.)

First, review an idea developed in the study of sets. Consider Set K and Set L.

$$K = \{11, 12, 13, 14, 15\}$$
$$L = \{11, 13, 17, 19\}$$

The intersection of these two sets is the set of members common to both sets. Specifically, the intersection of Set K and Set L is the set {11, 13}. This can be written as follows using the special symbol, $\cap$, to indicate intersection. $K \cap L = \{11, 13\}$. This is read as "the intersection of Set K and Set L is the set whose members are 11 and 13," or more briefly, "the Set <u>K intersection L</u> is {11, 13}."

Now consider the set of all factors of 12. Call it Set S.

$$\text{Since } 12 = 2 \times 2 \times 3,$$
$$S = \{1, 2, 3, 4, 6, 12\}$$

Next consider the set of all factors of 18. Call it Set R.

$$\text{Since } 18 = 2 \times 3 \times 3,$$
$$R = \{1, 2, 3, 6, 9, 18\}$$

There are some members of Set S that are also members of Set R. The members which are contained in both Set S and Set R are 1, 2, 3, 6. This information can be recorded as $S \cap R = \{1, 2, 3, 6\}$. Since the members of Set S are the factors of 12 and the members of Set R are the factors of 18, we say that the members of $S \cap R$ are the <u>common factors</u> of 12 and 18. The common factors of 12 and 18 are 1, 2, 3, 6.

If Set A is the set of factors of 15, and Set B is the set of factors of 20, what is $A \cap B$?

$A = \{1, 3, 5, 15\}$

$B = \{1, 2, 4, 5, 10, 20\}$

$A \cap B = \{1, 5\}$ These are the common factors of 15 and 20.

Background: (Finding the Greatest Common Factor, Pupil pages 76-78)

There are two observations to be made about the character of the set of all the common factors of any two numbers.

1. If any number is in the set, each of its factors must be also.

   For example, look at the set of common factors we found for 12 and 18.

   {1, 2, 3, 6}

   Each of these numbers has its factors in the set. The factors of 6 are 1, 2, 3, and 6. The factors of 3 are 1 and 3 and each of these is in the set. The factors of 2 are also in the set. Thus {1, 2, 6} cannot be the set of all common factors of any two numbers because every number with 6 as a factor also has 3 as a factor.

2. A set of all common factors of two numbers contains only those numbers which are factors of the largest number in the set.

   Look again at the set of common factors for 12 and 18.

   {1, 2, 3, 6}

   The largest number in the set is 6; and 1, 2, and 3 are factors of 6. We see then, that factors of 6 are the only members in the set. Thus, because 4 is not a factor of 6, {1, 2, 3, 4, 6} cannot be the set of all common factors of any two numbers. That this is always true is not at all obvious.

It is a consequence of The Fundamental Theorem of Arithmetic. For example, the reason that {1, 2, 3, 4, 6} cannot be the set of all common factors of two numbers is that each number must have both 2 × 2 (4 is a factor) and 2 × 3 (6 is a factor) as "pieces" in its complete factorization. But this cannot occur unless 2 × 2 × 3 appears in each prime product expression. Thus every number with both 4 and 6 as factors must have 12 also as a factor. Consequently any set of common factors which includes 3 and 4 must also include 12.

This last observation about pieces of prime product expressions suggests a way to find the greatest common factor without first finding all factors and then finding the greatest among them. We first write each number, say 30 and 42 as a product of primes.

$$30 = 2 \times 3 \times 5$$

$$42 = 2 \times 3 \times 7.$$

Since all factors of 30 and 42 can be found by using "pieces" of these expressions, the greatest common factor must be expressed by the largest piece common to both expressions. Thus 2 × 3 or 6 is the greatest common factor of 30 and 42. (By "pieces" of the expression 2 × 3 × 5, we mean 2, 3, 5, 2 × 3, 3 × 5, 2 × 5, and 2 × 3 × 5. The "pieces" of the expression 2 × 3 × 7 are 2, 3, 7, 2 × 3, 2 × 7, 3 × 7, and 2 × 3 × 7.)

Consider another example, 90 and 84:

$$\begin{aligned} 90 &= 2 \times 3 \times 3 \times 5 \\ 84 &= 2 \times 2 \times 3 \times 7 \end{aligned}$$

By regrouping the factors:

$$\begin{aligned} 90 &= (2 \times 3) \times 3 \times 5 \\ 84 &= (2 \times 3) \times 2 \times 7. \end{aligned}$$

Since 3 × 5 and 2 × 7 have no common prime factors, 2 × 3 is the largest common block shown in both product expressions 90 and 84.

To find the greatest common factor of 90 and 50, first write:

$$90 = 2 \times 3 \times 3 \times 5$$
$$50 = 2 \times 5 \times 5.$$

By regrouping, show the common factors:

$$90 = (2 \times 5) \times 3 \times 3$$
$$50 = (2 \times 5) \times 5,$$

The greatest common factor of 90 and 50 must be 2×5 or 10.

Perhaps a quicker way to find the largest "piece" that is common to both expressions would be this. Write each factor that is common to both expressions the <u>least</u> number of times it appears in <u>both</u> expressions.

$$3{,}150 = 2 \times 3 \times 3 \times 5 \times 5 \times 7$$
$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$$

The largest "piece" is $2 \times 3 \times 3 \times 5$ because 2 appears only <u>once</u> in 3,150, (even though it appears 3 times in 360), 3 appears <u>twice</u> in both expressions, and 5 appears just once in 360 (even though it appears twice in 3,150). Therefore, the greatest common factor is $2 \times 3 \times 3 \times 5$ or 90.

A more complicated example is:

$$10{,}890{,}936 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 7 \times 7 \times 7 \times 7 \times 7$$
$$8{,}820 = 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 7$$

The greatest common factor is $2 \times 2 \times 3 \times 3 \times 7 \times 7 = 1{,}764$.

2 appears at least twice in <u>both</u> expressions.
3 appears at least twice in <u>both</u> expressions.
7 appears at least twice in <u>both</u> expressions.

The 5 appears only in 8,820 so it is <u>not</u> included in the "piece."

Because of the properties noted under 1 and 2 on page 147 once we have found the greatest common factor we can readily find <u>all</u> common factors.

Since 6 is the greatest common factor of 30 and 42 the set of all common factors is the set of all factors of 6 or {1, 2, 3, 6}.

For the same reason {1, 2, 3, 6} is also the set of all common factors of 84 and 90, and {1, 2, 5, 10} lists all the common factors of 50 and 90.

Suggestions for Exploration:

1. Review the idea of intersection of sets. Then apply this to the intersection of the sets of <u>all factors</u> of 12 and 18 as is developed in the background. Other numbers such as 15 and 20, 18 and 28, and 25 and 40 can be used. This will lead to an understanding of <u>common factors</u> of two numbers.

   Pupil page 74 might be quickly noted and pupils can then work Exercise Set 11. These exercises can be discussed after pupils have completed them. Exercise 2 of this Exercise Set leads to work on Finding the Greatest Common Factor. During the discussion of the sets of common factors, the special character of the set may be noticed. Leading questions such as these can be asked:

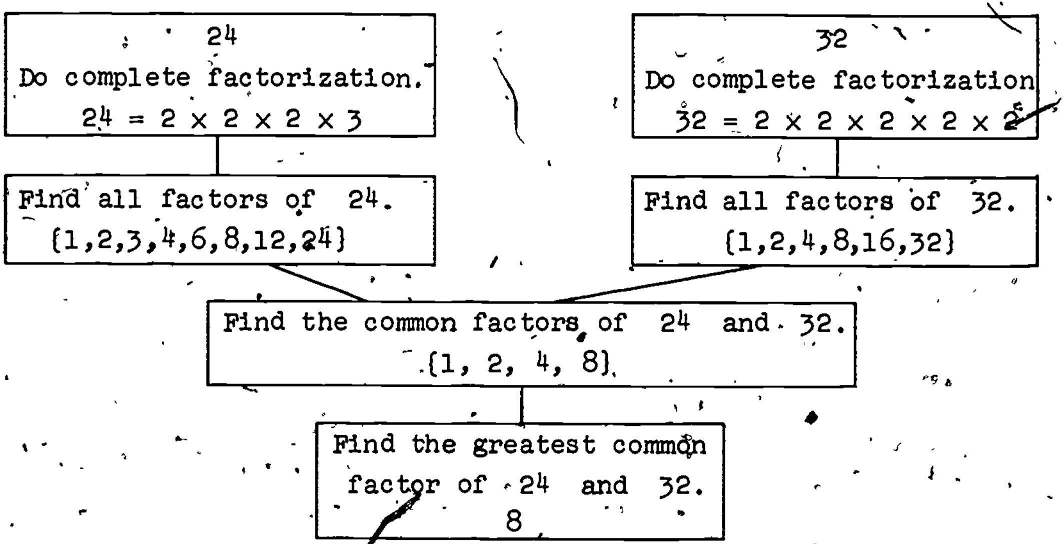
   > What do you notice about the largest number in each set of common factors? Are all the other members of the set also factors of the largest number? Are the factors of each member of the set also in the set? Are there any members in the set which are <u>not</u> factors of the largest member? Do you think {1, 2, 5, 10} can be a set of all common factors? Can {1, 2, 6} be a set of all common factors? Can {1, 2, 3, 4, 6} be a set of all common factors?

   Discuss these questions in the light of the preceding background given for the teacher.

2. Now is the time to introduce the term *greatest common factor*. Children should have no trouble in identifying the largest number in the set of common factors as the *greatest common factor*.

   Draw a diagram on the board to illustrate the way in which the greatest common factor of two numbers has been found, for example 24 and 32.

Diagram



24
Do complete factorization.
24 = 2 x 2 x 2 x 3

32
Do complete factorization
32 = 2 x 2 x 2 x 2 x 2

Find all factors of 24.
{1,2,3,4,6,8,12,24}

Find all factors of 32.
{1,2,4,8,16,32}

Find the common factors of 24 and 32.
{1, 2, 4, 8}

Find the greatest common factor of 24 and 32.
8

Ask if anyone can see a way to get from the first step to the greatest common factor without going through all the other steps. Have children closely examine the product expressions for 24 and 32 given in the first step. Ask questions such as, "What factors of 24 are also factors of 32? How many times does the factor 2 appear in the product expression for 24? for 32? What is the greatest number of times that 2 appears in *both* product expressions? Are there any other prime factors which appear in both expressions?" Try to get pupils to see that there is a way to find the greatest common factor without going through all the steps in the diagram. Use other examples of finding the greatest common factor such as those given in the background for the teacher. Follow the development given in the teacher background to help children find the "piece" that is common to both product expressions. Children will need to find the greatest common factor of two numbers in several examples during this exploration period in order to gain skill and confidence.

After the greatest common factor for two numbers has been found, all common factors of the two numbers can be determined because these are simply the factors of the greatest common factor. In class discussions, have the children determine the set of all common factors after they find the greatest common factor of two numbers. If this is done during the exploration period, children should be prepared to do Exercise Set 12.

Pages 76, 77, and 78 in the Pupils' Book provide material designed to help children understand the meaning and application of greatest common factor. Following the exploration you have done with the pupils, you may want to examine these pages with the children before they begin working independently on the exercise set.

COMMON FACTORS

Suppose Set S is the set of all factors of 12 and Set R is the set of all factors of 18.

$S = \{1, 2, 3, 4, 6, 12\}$

$R = \{1, 2, 3, 6, 9, 18\}$

Then the set of all factors of <u>both</u> 12 and 18 is

$S \cap R = \{1, 2, 3, 6\}$

The members of this set are called the <u>common factors</u> of 12 and 18.

What are the common factors of 16 and 36?

$K = \{1, 2, 4, 8, 16\}$ **is the set of all factors of 16**

and

$L = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ **is the set of all factors of 36,**

$K \cap L = \{1, 2, 4\}$ **is the set of all common factors of 16 and 36.**

The common factors of 16 and 36 are 1, 2, and 4.

Exercise Set 11

1. Two numbers are given in each exercise below. Find all factors of each number; then find the common factors of the two numbers. The first exercise is an example of what you are to do.

   a. 12 and 30.

   Let A = the set of all factors of 12.
   $A = \{1, 2, 3, 4, 6, 12\}$
   Let B = the set of all factors of 30.
   $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$
   $A \cap B = \{1, 2, 3, 6\}$
   1, 2, 3, and 6 are the common factors of 12 and 30.

   b. 40 and 30
   c. 36 and 27
   d. 60 and 40
   e. 52 and 72
   f. 75 and 120
   g. 72 and 108

2. For each intersection in exercise 1:

   a. What is the largest or greatest factor in each set of common factors?

   b. Is each other member of the set of common factors a factor of the largest member?

   c. Are there any members of the intersection set which are not factors of the largest member?

Exercise Set 11, Answers

1\. b. R = the set of factors of $2 \times 2 \times 2 \times 5$
$= \{1, 2, 4, 5, 8, 10, 20, 40\}$
S = the set of factors of $2 \times 3 \times 5$
$= \{1, 2, 3, 5, 6, 10, 15, 30\}$
$R \cap S = \{1, 2, 5, 10\}$

c. C = set of factors of $2 \times 2 \times 3 \times 3$
$= \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$
D = set of factors of $3 \times 3 \times 3 = \{1, 3, 9, 27\}$
$C \cap D = \{1, 3, 9\}$

d. E = the set of factors of $2 \times 2 \times 3 \times 5$
$= \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
F = the set of factors of $2 \times 2 \times 2 \times 5$
$= \{1, 2, 4, 5, 8, 10, 20, 40\}$
$E \cap F = \{1, 2, 4, 5, 10, 20\}$

e. G = the set of factors of $2 \times 2 \times 13$
$= \{1, 2, 4, 13, 26, 52\}$
H = the set of factors of $2 \times 2 \times 2 \times 3 \times 3$
$= \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$
$G \cap H = \{1, 2, 4\}$

f. X = the set of factors of $3 \times 5 \times 5$
$= \{1, 3, 5, 15, 25, 75\}$
Y = the set of factors of $2 \times 2 \times 2 \times 3 \times 5$
$= \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$
$X \cap Y = \{1, 3, 5, 15\}$

g. A = the set of factors of $2 \times 2 \times 2 \times 3 \times 3$
$= \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$
B = the set of factors of $2 \times 2 \times 3 \times 3 \times 3$
$= \{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108\}$
$A \cap B = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

2\. a. b'. The g.c.f. is 10.
c'. The g.c.f. is 9.
d'. The g.c.f. is 20.
e'. The g.c.f. is 4.
f'. The g.c.f. is 15.
g'. The g.c.f. is 36.

b. Yes.

c. No.

FINDING THE GREATEST COMMON FACTOR

If we know the set of common factors of two numbers, we can easily find the greatest common factor of the two numbers. The greatest number in the set of common factors is called the greatest common factor.

The set of common factors of 12 and 18 is

$$\{1, 2, 3, 6\}.$$

The largest among these numbers is 6. It is called the greatest common factor of 12 and 18.

The set of common factors of 16 and 36 is

$$\{1, 2, 4\}.$$

The greatest common factor of 16 and 36 is 4.

There is a way to find the greatest common factor of two numbers without first finding the intersection of the sets of factors of each number.

First we express the numbers, say 30 and 42, as products of primes.

$$30 = 2 \times 3 \times 5$$

$$42 = 2 \times 3 \times 7.$$

The factors of 30 can all be found by forming "pieces" of this expression. Pieces of $2 \times 3 \times 5$ are 2, 3, 5, 2×3, 2×5, 3×5, and $2 \times 3 \times 5$. The factors of 42 can all be found in the same way. The pieces of $2 \times 3 \times 7$ are 2, 3, 7, 2×3, 2×7, 3×7, and $2 \times 3 \times 7$. The common factors of 30 and 42 must be expressed by those pieces which are found in both expressions. The greatest common factor must be the largest piece found in both expressions.

The largest piece in the prime product expressions for both 30 and 42 is 2×3 or 6. Then 6 must be the greatest common factor of 30 and 42.

Here is another example. To find the greatest common factor of 90 and 50 we write:

$$90 = 2 \times 3 \times 3 \times 5$$
$$50 = 2 \times 5 \times 5.$$

By rewriting $90 = (2 \times 5) \times (3 \times 3)$ we see that 2×5 is the largest piece that can be found in both expressions. The expression $2 \times 5 \times 3$ can be found in one and $2 \times 5 \times 5$ in the other. But neither can be found in *both*. We know then that 10 is the greatest common factor of 90 and 50.

If we have found the greatest common factor in this way we can quickly find all common factors. Do you see how? The common factors must be those which can be expressed as pieces of both prime product expressions. They must then be the *pieces of the largest piece.* This means that the common factors are simply the *factors of the greatest common factor.*

Since 6 is the greatest common factor of 30 and 42, the set of common factors is {1, 2, 3, 6}.

Since 10 is the greatest common factor of 90 and 50, the set of common factors is {1, 2, 5, 10}.

Now try 24 and 60.

$$24 = 2 \times 2 \times 2 \times 3$$
$$60 = 2 \times 2 \times 3 \times 5.$$

The pieces which these expressions have in common are 2, 3, 2 × 2, 2 × 3, and 2 × 2 × 3. This last is the largest, so 12 is the greatest common factor of 24 and 60. The set of all common factors is {1, 2, 3, 4, 6, 12}.

Exercise Set 12

1. Find the greatest common factor by first finding the intersection of the sets of factors. Exercise a. is answered for you as an example.

   a. 12 and 40

   12 = 2 × 2 × 3

   All factors of 12 A = {1, 2, 3, 4, 6, 12}

   40 = 2 × 2 × 2 × 5

   All factors of 40 B = {1, 2, 4, 5, 8, 10, 20, 40}

   A ∩ B = {1, 2, 4}

   The greatest common factor of 12 and 40 is 4.

   b. 16 and 6 (2)

   c. 90 and 12 (6)

2. Find the greatest common factor by first writing each number as a product of primes.

   a. 2 and 6 (2)

   b. 7 and 35 (7)

   c. 16 and 8 (8)

   d. 20 and 36 (4)

   e. 48 and 30 (6)

   f. 60 and 45 (15)

   g. 72 and 60 (12)

   h. 2 × 2 × 2 × 3 × 3 × 5 and 2 × 3 × 5 × 7
   (2×3×5) (30)

   i. 3 × 3 × 3 × 3 × 7 × 7 × 11 and 2 × 3 × 3 × 13
   (3×3) (9)

   j. m = 2 × 2 × 3 × 3 × 3 × 3 × 7 × 7 × 7 × 7 × 7
   and n = 2 × 2 × 3 × 3 × 7 (2×2 ×3×3 ×7)
   (292)

BRAINTWISTER

3. a. Can a pair of numbers, with 2, 3, and 5 among their common factors have 20 as a greatest common factor? (no) Why? (If 20 was the greatest common factor, 3 could not be a common factor. Also, if 2, 3 and 5 are common factors then 2×3×5 is a common factor and 30 > 20)

b. If 2 and 3 are among the common factors of a pair of numbers, name one other common factor which the pair must have. (6)

Answer the same question if the common factors are:

c. 3 and 5 (15)
d. 9 and 5 (45)
e. 9 and 4 (36)
f. 4 and 6 (12)
g. 6 and 14 (42)
h. 12 and 9 (36)

4. a. The greatest common factor of 728 and 968 is 8. Write the set of common factors of 728 and 968.
{1, 2, 4, 8}

b. The greatest common factor of 330 and 294 is 6. Write the set of common factors of 330 and 968.
{1, 2, 3, 6}

FACTORING AND FRACTIONS

Vocabulary: Measure, numerator, denominator

Suggestions for Exploration:

This section is an application of what has been learned in this chapter.

The presentation on the pupil pages can be followed.

The Braintwisters of Exercise Set 13 should be discussed in class after pupils have had an opportunity to work on them independently.

FACTORING AND FRACTIONS

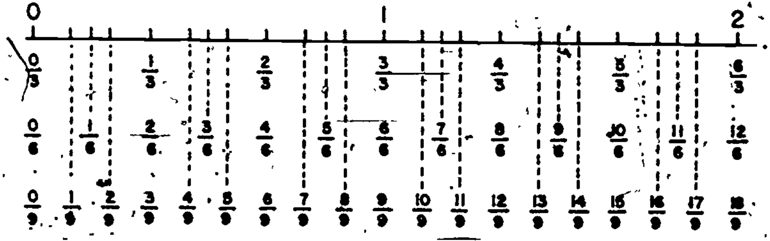
When we studied fractions we learned that there are many fractions which name the same rational number. For example

$$\frac{2}{3},\ \frac{4}{6},\ \text{and}\ \frac{6}{9}$$

are all names for the same number.

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}.$$

This number line may help to remind you why this is so.



The diagram shows scales in units, thirds, sixths, and ninths. It shows that if a segment has a measure $\frac{2}{3}$ then it also has measure $\frac{4}{6}$ and $\frac{6}{9}$. By studying the diagram you should be able to answer the following questions:

1. John has a pencil $\frac{1}{3}$ of a foot long. Mary has a piece of chalk $\frac{1}{6}$ of a foot long. John measures the side of a large book with his pencil. Mary measures the same side with her chalk. John finds that the edge measures 4 in pencil lengths. What does it measure in feet? ($\frac{4}{3}$ ft or ($1\frac{1}{3}$ ft) What number should Mary find as the measure of the edge in chalk lengths? (8) How would she probably express this length in feet? ($\frac{8}{6}$ or $1\frac{1}{3}$ ft)

2. List the two other names for $\frac{5}{3}$ shown on the diagram. $\left(\frac{20}{12}\right)\left(\frac{25}{15}\right)$ List two more names not shown on the diagram. Is there a name for $\frac{1}{2}$ shown on the diagram? (yes) If there is, what is it? $\left(\frac{3}{6}\right)$ What scales would you add to the diagram to show two other names for $\frac{1}{6}$? (twelfths) (eighteenths)

In using fractions it is often very important to be able to answer questions like these:

a. Is $\frac{30}{48} = \frac{25}{40}$?

b. Is $\frac{15}{25} < \frac{24}{30}$?

We can answer questions like these if we can tell when two fractions are names for the same number. We know that

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{1 \times n}{2 \times n}$$

and that

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{2 \times n}{3 \times n}.$$

We can also use this idea to find smaller numerators and denominators.

$$\frac{18}{24} = \frac{2 \times 9}{2 \times 12} = \frac{9}{12} = \frac{3 \times 3}{3 \times 4} = \frac{3}{4}$$

$$\frac{18}{24} = \frac{3 \times 6}{3 \times 8} = \frac{6}{8} = \frac{2 \times 3}{2 \times 4} = \frac{3}{4}$$

Thus $\frac{18}{24} = \frac{9}{12} = \frac{6}{8} = \frac{3}{4}$.

This suggests that we can answer our question about $\frac{30}{48}$ and $\frac{25}{40}$ by factoring. We can start by writing both 30 and 48 as products of primes.

$$30 = 2 \times 3 \times 5$$

$$48 = 2 \times 2 \times 2 \times 2 \times 3$$

Now $\frac{30}{48} = \frac{2 \times 3 \times 5}{2 \times 2 \times 2 \times 2 \times 3} = \frac{(2 \times 3) \times 5}{(2 \times 3) \times (2 \times 2 \times 2)}$

$= \frac{6 \times 5}{6 \times 8} = \frac{5}{8}$.

Also $\frac{25}{40} = \frac{5 \times 5}{2 \times 2 \times 2 \times 5} = \frac{5 \times 5}{5 \times (2 \times 2 \times 2)}$

$= \frac{5 \times 5}{5 \times 8} = \frac{5}{8}$

We find then that $\frac{30}{48} = \frac{25}{40} = \frac{5}{8}$.

Now for our second question, b).

$$\frac{15}{25} = \frac{3 \times 5}{5 \times 5} = \frac{3}{5}$$

$$\frac{24}{30} = \frac{2 \times 2 \times 2 \times 3}{2 \times 3 \times 5} = \frac{(2 \times 3) \times (2 \times 2)}{(2 \times 3) \times 5}$$

$$= \frac{2 \times 2}{5} = \frac{4}{5}.$$

Since we know that $\frac{3}{5} < \frac{4}{5}$, we also know that $\frac{15}{25} < \frac{24}{30}$.

Exercise Set 13

1. Find the fraction with the smallest possible denominator for each of the following.

Example: $\frac{60}{350} = \frac{2 \times 2 \times 5 \times 3}{2 \times 5 \times 5 \times 7} = \frac{(2 \times 5) \times (2 \times 3)}{(2 \times 5) \times (5 \times 7)} = \frac{2 \times 3}{5 \times 7}$.

Since 2×3 and 5×7 have no common factors except 1, $\frac{6}{35}$ must be the fraction we wanted to find.

a. $\frac{6}{16}$ $\left(\frac{3}{8}\right)$

b. $\frac{7}{19}$ $\left(\frac{7}{19}\right)$

c. $\frac{12}{20}$ $\left(\frac{3}{5}\right)$

d. $\frac{21}{35}$ $\left(\frac{3}{5}\right)$

e. $\frac{26}{14}$ $\left(\frac{13}{7}\right)$

f. $\frac{16}{27}$ $\frac{16}{27}$

g. $\frac{2 \times 3 \times 5 \times 5 \times 7}{2 \times 5 \times 7 \times 11}$ $\left(\frac{15}{11}\right)$

h. $\frac{3 \times 5 \times 7}{2 \times 11}$ $\left(\frac{105}{22}\right)$

i. $\frac{9 \times 4 \times 5}{16 \times 3 \times 7}$ $\left(\frac{15}{28}\right)$

2. Find each of the measures given below. Express each using the smallest possible denominator.

Example: The measure of 5 days in <u>weeks</u> is $\frac{5}{7}$. This is the expression with the smallest denominator.

a. The measure of 36 seconds in <u>minutes</u>. $\left(\frac{3}{5}\right)$

b. The measure of 14 hours in <u>days</u>. $\left(\frac{7}{12}\right)$

c. The measure of 30 days in <u>years</u>. $\left(\frac{6}{73}\right)$

d. The measure of 6 ounces in <u>pounds</u>. $\left(\frac{3}{8}\right)$

e. The measure of 42 inches in <u>yards</u>. $\left(\frac{7}{6}\right)$

BRAINTWISTERS

3. Suppose that m and n are counting numbers. Mark T for true or F for false for each of the following sentences about $\frac{m}{n}$.

   a. If m and n are both even then $\frac{m}{n}$ can always be expressed using a denominator smaller than n. (T)

   b. If m and n are both odd then $\frac{m}{n}$ cannot be expressed using a smaller denominator. (F)

   c. If no prime is a factor of both m and n, then the greatest common factor of m and n is 1. (T)

   d. If no prime is a factor of both m and n, then $\frac{m}{n}$ cannot be expressed using a smaller denominator. (T)

   e. If $\frac{m}{n} = \frac{4}{6}$ then 4 is a factor of m and 6 is a factor of n. (T)

   f. If $\frac{m}{n} = \frac{2}{3}$ then 2 is a factor of m and 3 is a factor of n. (T)

SUPPLEMENTARY EXERCISES

These supplementary exercises are set up to challenge the more-able thinkers. They are not arranged so every student should, or would even be able to work all of them.

Below are several suggestions concerning the use of these sets of exercises.

1. Save the complete set of exercises until later in the school year. You may want to return to them as a review discussion.
2. Allow your more able students to work together through these exercises.
3. You, as a teacher, can study the exercises. There are some ideas here that may help you in your own understanding of factors and primes.
4. After you have studied these exercises, you may wish to use some of them to add to your class discussions as you study individual parts of this unit.

CAUTION: If you attempt to use these exercises in total class discussion, you may find your time to teach the unit will extend beyond the 3 to 4 week period, normally needed for this chapter.

Additional Information for Supplementary Exercise Set A:

2. Observe that neither 2 nor 3 is a factor of 7,075.

3. Pick two multiples of 9 for which the factors other than 9 have no common prime factor, e.g. 2×9 and 3×9, or 3×9 and 5×9.

4. A composite number less than 13×13 must have 2, 3, 5, 7, or 11 as a factor. Therefore n has 11 as a factor. The other factor must be 11 also because $11 \times 13 > 125$.

5. Find the g.c.f. of 6 and 9. This is 3. Then find the g.c.f. of 3 and 30.

Supplementary Exercise Set A

1. Write as a product of primes:
   a. 63 × 120 (3 × 3 × 7) × (2 × 2 × 2 × 3 × 5) = (2 × 2 × 2 × 3 × 3 × 3 × 5 × 7)
   b. 65 × 92 (5 × 13) × (2 × 2 × 23) = (2 × 2 × 5 × 13 × 23)
   c. 210 × 180 (2 × 3 × 5 × 7) × (2 × 2 × 3 × 3 × 5) = (2 × 2 × 2 × 3 × 3 × 3 × 5 × 5 × 7)

2. a. How many times does 2 appear if 24 × 7075 is written as a product of primes? (3 times)
   b. How many times does 3 appear? (once)

3. Find three pairs of numbers with the number given as greatest common factor.
   a. 9 (9, 18), (9, 27), (18, 27) etc.
   b. 10 (10, 10), (10, 20), (20, 30) etc.
   c. 12 (12, 24), (12, 36), (24, 36) etc.

4. There is a composite number less than 125. It does not have 2, 3, 5, or 7 as a factor. What is the number? (121)

5. Find the greatest common factor of these triples of numbers.
   a. 6, 9, 30 (3)
   b. 8, 12, 25 (1)
   c. 25, 30, 50 (5)

6. I am thinking of an operation on counting numbers. I will call the result of operating on m and n, $m \cdot n$ ("m dot n"). Here are some facts about the operation "dot."

$6 \cdot 4 = 2$ $4 \cdot 3 = 1$ $5 \cdot 15 = 5$ $8 \cdot 12 = 4$

$n \cdot 1 = 1$ $10 \cdot 15 = 5$ $18 \cdot 26 = 2$ $42 \cdot 25 = 1$

a. What is a rule for finding $m \cdot n$? (greatest common factor)

b. Is the operation "dot" commutative? (yes)

c. Is it associative? (yes)

Additional Information for Supplementary Exercise Set B:

1. Every prime except 2 is odd. Since n is odd, n + 1 is even. Except for 2 and 3 no two primes can ever be adjacent.

2. In *any* base 30 = 3 × 10. In base five, 10 is prime.

   In b, note that in *any* base 100 = 10 × 10.

   In b conversion to base ten is necessary.

3. Make a list comparing base five and base ten numerals, or use a list made previously. The even numbers are 2, 4, 11, 13, 20, 22, 24, 31, ... etc. Notice that in any base, the *base* is a factor of a number only if its unit digit is 0.

4. These questions can all be answered by converting to base ten. Notice that 13 is prime in base ten but not in base 7; that 15 is composite in base 10 but not in base 8. Of course, 10 will be prime only if the base is prime, and 100 is always composite.

5. In any even base, one-half of the base will have the same sort of divisibility test as 5 does in the decimal system.

Supplementary Exercise Set B

1. Suppose you know a large prime number, n. Then you can be sure that n + 1 is not a prime. Why? (n + 1 must be even)

2. In this exercise write only base five numerals. Write as a product of primes, if possible.

   a. $(30)_{five} = (15 = 5 \times 3 = 10_{five} \times 3)$

   b. $(131)_{five} = (25 + 15 + 1 = 41 = 131_{five}$ is prime$)$

   c. $(100)_{five} = (10_{five} \times 10_{five})$

3. a. Using base five numerals, is there a simple test to find whether 2 is a factor of a number? (The sum of the digits must be even)

   b. Is there a simple test for 3 as a factor? (No)

   c. Is there a simple test for $(10)_{five}$? (The unit digit must be 0.)

4. Which are prime and which are composite?

   a. $(10)_{four}$ (2×2 composite)

   b. $(10)_{seven}$ (7, prime)

   c. $(13)_{seven}$ (5×2 composite)

   d. $(10)_{eight}$ (2×4 composite)

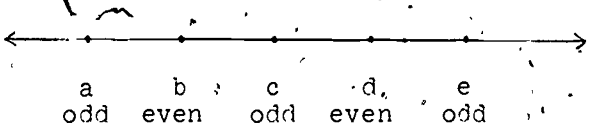
   e. $(15)_{eight}$ (13, prime)

   f. $(100)_{seventeen}$ (17×17 composite)

5. Find a rule for testing 3 as a factor using base six numerals. (The units digit must be 0 or 3.)

Additional Information for Supplementary Exercise Set C:

1. There can be no other triplets because at least one of any set of 3 successive odd numbers is a multiple of 3.



If neither a nor c is a multiple of 3, then b must be. But in that case e is also.

2. Test each number in order.

3. Notice that there is sometimes more than one way to write an even number as a sum of odd primes.

Supplementary Exercise Set C

1. Primes with only one number between them are called twin primes. 11 and 13 are twins, so are 17 and 19.

   a. What are the next two pairs of twin primes? (29 and 31
   41 and 43)

   The primes 3, 5, and 7 might be called triplet primes. If 15 were prime then 11, 13, 15 would be triplets.

   b. Do you know any other triplets besides 3, 5, and 7?

   c. In your chart of prime factors, find one other triplet other than 3, 5, and 7, if you can. (There are no others)

2. The number 6 has an interesting property noticed by Greek mathematicians over 2,000 years ago. It is this: the number 6 is the sum of all of its factors except 6.

$$1 + 2 + 3 = 6.$$

The Greeks admired this rare property and called such numbers perfect numbers. No one has ever been able to find a way to get all perfect numbers. No one knows whether there are any odd perfect numbers.

Find the next perfect number greater than 6.

(28 = 1+2+4+7+14)

3. All primes except 2 are odd. The sum of any two odd primes is even. Suppose we ask what even numbers are sums of two (perhaps equal) odd primes? The smallest number which could be is 6. It is, because 3 + 3 = 6. Also 8 = 3 + 5, 10 = 3 + 7, 12 = 5 + 7.

Show that every number from 6 through 30 is a sum of two odd primes.

No one has ever found an even number greater than 4 which is not the sum of two odd primes. Most mathematicians believe that every such even number is the sum of two odd primes. No one has been able to show that there cannot be any exceptions.

14 = 7 + 7
16 = 5 + 11
18 = 5 + 13 = 7 + 11
20 = 7 + 13
22 = 11 + 11 = 5 + 17

24 = 11 + 13 = 5 + 19 = 7 + 17
26 = 13 + 13 = 7 + 19
28 = 11 + 17 = 5 + 23
30 = 13 + 17 = 7 + 23 = 11 + 19

MATHEMATICAL SUMMARY

Studying Operations

When a mathematical operation, like addition or multiplication, is first studied, attention is usually directed toward

(1) finding and learning basic facts, like $2 + 3 = 5$ or $2 \times 3 = 6$;

(2) knowing, or at least using, the properties, like commutativity, associativity, and distributivity, which underlie the general process for operating on any two numbers.

This is the approach to addition in Chapter 3 of Grade 4 and Chapter 6 of Grade 4 and to multiplication in Chapter 4 of Grade 4 and Chapter 7 of Grade 4.

To organize and extend knowledge of an operation it is often valuable to change the point of view. One can, for instance, invert the usual approach by beginning with a number and asking how it can be obtained by operating on other numbers. This is the attitude toward multiplication taken in this unit. When we express 12 as a product in several ways or ask for all of its factors, we are taking this inverted view of multiplication. The same general questions applied to addition are not so interesting. For example, every number smaller than 12 is an addend for 12.

An Inverted View of Multiplication

We begin our study of the way numbers can be "broken up" into factors by recalling the ways a number (say 12) can be written as a product (1×12, 12×1, 2×6, 6×2, 3×4, 4×3). Next we point out that these factors may themselves be written as a product, thus "breaking up" 12 into more "pieces." For example, from

$$12 = 6 \times 2 \quad \text{we might get}$$

$$12 = (2 \times 3) \times 2.$$

Continued application of this process in several cases should suggest these observations:

(1) The "breaking up" can continue indefinitely if 1's are used as factors, but using 1's as factors does not contribute additional information.

$$12 = 3 \times 4 \times 1 \times 1 \text{ might as well be}$$
$$12 = 3 \times 4 .$$

(2) If 1 is <u>not</u> used as a factor, then the process must end.

$$12 = (2 \times 3) \times 2$$

terminates the "breakup" of 12.

The process ends when each factor <u>cannot be written as a product of smaller factors</u>. At this point we have reached the "bricks" or "atoms" from which the number is "constructed" by multiplication. These are called <u>prime numbers</u> or simply <u>primes</u>. Products of primes are called <u>composite</u> numbers.

(3) It appears that, for a given number, no matter how the "breaking up" process is undertaken, when the "bricks" (primes) are reached, there are always the same numbers of each type of "brick" (each prime).

$$12 = 6 \times 2 = (2 \times 3) \times 2 \text{ and}$$
$$12 = 3 \times 4 = 3 \times (2 \times 2).$$
$$60 = 5 \times 12 = 5 \times (3 \times 4) = 5 \times (3 \times (2 \times 2))$$
$$60 = 6 \times 10 = (2 \times 3) \times (2 \times 5)$$

Another way to say this is: If we ignore distinctions in the order and grouping of factors, there is only one way to write a number as a product of primes.

This property, whose consequences are manifold, is called the <u>fundamental theorem of arithmetic</u>.

Primes and Products of Primes

While this property of primes can be proved, we ask the children to assume it as a probable generalization of their experience. It means that with every number there is associated a certain set of prime factors (types of bricks) and a certain number of repetitions of each prime (number of each type of brick). For example, 36 has two prime factors, 2 and 3. Each is repeated once:

$$36 = 2 \times 2 \times 3 \times 3.$$

This leads us to a computational problem. (1) Can we find a method for writing any number as a product of primes? Less comprehensive objectives are (2) to find a way to determine whether or not a given number is a prime or (3) to find all primes smaller than some given number. Any answer to (1) must include answers to (2) and (3). We begin with these more modest aims because they lead us to a solution of the original problem (1).

There is an obvious but tedious way to find the factors of a number, say 97. Beginning with 2, we divide 97 in order by each number to test its even divisibility. If we already know the primes less than 8 we can shorten our work in finding all prime factors of 97 or prove that 97 is prime. If we are interested only in deciding whether or not 97 is prime, this method can be greatly improved. We need to observe that factors come in pairs; for example (1, 12), (2, 6), (3, 4) are the paired factors of 12. It follows that:

a) for any number less than 5 × 5, if one factor is greater than 5 the other of the pair is less than 5.

b) for any number less than 7 × 7, if one factor is greater than 7 the other is less than 7.

c) for any number less than 10 × 10, if one factor is greater than 10, the other is less than 10.

This principle implies that if 97 has no prime factor less than 10, then 97 is itself prime. For if 97 has a factor greater than 10 then it also has a factor less than 10. If 97 has a factor less than 11, then it has 2, 3, 5, or 7 as a factor. We therefore need only test 2, 3, 5, and 7 for even divisibility to prove that 97 is prime.

Testing 2, 3, or 5 can be simplified by noting characteristic properties of the decimal numerals of numbers with one of these primes as a factor.

The method outlined above is a reasonably effective process for reaching objective (2). But if we wish to find all primes up to 100, testing each number would be tedious. It is more efficient and revealing to find the composite numbers up to 100. This can be systematized by finding the numbers with a given prime factor. First we can write down in order the numbers greater than 2 with 2 as a factor.

4, 6, 8, 10, ...

These are composite. Then we can include the numbers greater than 3 with 3 as a factor, getting

9, 15, 21, ...

in addition to those already written. Each of these is composite. If we add to our list the numbers with 5 or 7 as a factor, we will have listed all composites less than 100. The numbers not listed must be the primes.

This process suggests the passing of a material through a series of selective filters or sieves. In this analogy, each successive "sieve" retains only the numbers with a certain factor and passes the rest. The numbers passed by the final "filtration" will be the primes less than 100. This then, is a way to reach objective (3).

We now have the ingredients of a workable method for writing reasonably large numbers as products of primes, that is for reaching objective (1). The method is to test the number for even divisibility by the primes in order. For example, to apply the method to 1092 we note that 2 is a factor and get, by division

$$1092 = 2 \times 546.$$

Now we apply the method to 546, again beginning with 2, and getting

$$1092 = 2 \times 2 \times 273.$$

Since 273 is not divisible by 2, we test 3, getting

$$1092 = 2 \times 2 \times 3 \times 91.$$

91 is not divisible by 3 or by 5, so we test 7, getting

$$1092 = 2 \times 2 \times 3 \times 7 \times 13.$$

Because 13 is prime, we have achieved our goal.

A Property of Primes

The fact that every number can be written as a product of primes in just one way has many implications. One of these is a particularly significant property of primes which can be used to justify many assertions in the subsequent part of the unit. It is derived from a very useful observation; namely, to write $m \times n$ as a product of primes, we simply bring together the separate expressions for m and n as products of primes.

From $110 = 2 \times 5 \times 11$ and

$78 = 2 \times 3 \times 13$,

we get $8580 = (2 \times 5 \times 11) \times (2 \times 3 \times 13)$

$= 2 \times 2 \times 3 \times 5 \times 11 \times 13$.

This means that any prime factor of a product $m \times n$ is a factor of either m or n.

It is easy to show by example that this property is not shared by composite numbers. While 4 is a factor of $8,580 = 110 \times 78$, it is a factor of neither 110 nor 78.

The observation made above has a direct application in justifying the process for finding all factors of a number from its expression as a product of primes. Suppose r is a factor of 8,580. Then

$$8,580 = r \times s.$$

If r and s are expressed as products of primes, we will have the expression for 8,580 as a product of primes if we bring together these expressions. It follows that r must be a product of some of the factors shown in

$$2 \times 2 \times 3 \times 5 \times 11 \times 13.$$

We conclude that by making all possible product expressions using some of: 2, 2, 3, 5, 11, 13, we get all factors. Given time, we can actually write all of them down.

Common Factors

The most practical benefits of the work on factoring to this point are its applications to the determination of the greatest common factor and the least common multiple of two numbers. The calculation of these quantities is necessary in "reducing" fractions and in adding rational numbers.

To begin, we examine the set of all common factors of two numbers. To get the set of common factors of 12 and 20 we find:

set of factors of 12 is $\{1, 2, 3, 4, 6, 12\}$

set of factors of 20 is $\{1, 2, 4, 5, 10, 20\}$.

The set of common factors is defined to be the intersection of these two sets, namely

$$\{1, 2, 4\}.$$

Now it is not a coincidence that this is the set of all factors of 4. The set of common factors is always the set of all factors of some number. It can never happen that

$$\{1, 2, 3, 4, 6\}$$

is the set of common factors of two numbers. If 6 is the greatest common factor, then

$$\{1, 2, 3, 6\}$$

will be the set of common factors. Why?

To see the answer, suppose that m and n are two numbers with both 4 and 6 as common factors. Then,

$$m = 4 \times \text{[illegible]} = 6 \times q = 3 \times 2 \times q$$

and $\quad n = 4 \times s = 6 \times t = 3 \times 2 \times t.$

This means that 3 is a factor of $4 \times p$. But for a prime to be a factor of $4 \times p$, it must be a factor of either 4 or p. However, 3 is not a factor of 4, hence it is a factor of p. For the same reason, 3 is a factor of s. Thus

$$m = 4 \times 3 \times r. \quad (p = 3 \times r)$$

$$n = 4 \times 3 \times u. \quad (s = 3 \times u)$$

But now 12 is a common factor of m and n. Thus, whenever 4 and 6 are common factors, so is 12.

A general argument of this sort shows that every common factor of two numbers is a factor of the greatest common factor. The problem then reduces to determining the greatest common factor without first having to determine all common factors. Writing each number as a product of primes enables us to find the greatest common factor efficiently.

From $150 = 2 \times 3 \times 5 \times 5$ and

$$420 = 2 \times 2 \times 3 \times 5 \times 7$$

we can pick out the largest common "piece" in the "construction" of 150 and 420 from primes,

$$150 = (2 \times 3 \times 5) \times (5)$$

$$\text{and } 420 = (2 \times 3 \times 5) \times (2 \times 7).$$

Clearly $2 \times 3 \times 5 = 30$ is a common factor. Any greater common factor must be of the form $30 \times$ (common factor of 5 and 2×7). Because the greatest common factor of 5 and 14 is 1, 30 is the greatest common factor of 150 and 420.

The definition, computation, and use of "least common multiple", will be treated in Chapter 6 in connection with the work on the addition of rational numbers.

Chapter 3

EXTENDING MULTIPLICATION AND DIVISION I

PURPOSE OF UNIT

The purpose of this unit is to help children develop greater skill in

(1) multiplying whole numbers, and

(2) dividing whole numbers.

Based on an understanding of relevant properties associated with each operation, emphasis is given to the use of progressively more mature and more efficient algorisms.

Skills and techniques develop at different rates for different children, and not all children can be expected to perform at the same level at any given time. However, each child should be encouraged to progress to as high a level of performance as possible--but not at the expense of understanding.

MATHEMATICAL BACKGROUND

MULTIPLICATION ALGORISMS

When multiplying two numbers such as 12 and 26, it generally is not convenient to remember all of one's thinking used to arrive at the correct product, 312. Rather, it usually is helpful to record some of this thinking in a written way.

Various forms for multiplying may be used, depending upon the pattern of thinking used and the extent to which a record of parts of this thinking is made in writing. Consequently, some forms of recording (or algorisms) are considered to be shorter or more efficient than others. In any event, an algorism must be based upon recognized operational properties and numeration principles.

Examples of algorisms for multiplying two numbers such as 12 and 26 follow.

Examples of Algorisms

20	6	20	6	26	26	26
× 10	×10	× 2	×2	×12	×12	×12
200	60	40	12	260	52	52
				52	260	26
		200		312	312	312
		60				
		40				
		+ 12				
		312				
		A		B	C	D

The fundamental basis for each algorism is found in the distributive property of multiplication over addition, coupled with the commutative and associative properties of multiplication. For example:

Explanation for Algorism A:

$$\begin{aligned}
12 \times 26 &= (10 + 2) \times 26 \\
&= (10 \times 26) + (2 \times 26) \\
&= [10\times(20+6)] + [2\times(20+6)] \\
&= [(10\times20)+(10\times6)] + [(2\times20)+(2\times6)] \\
&= (200 + 60) + (40 + 12) \\
&= 260 + 52 \\
&= 312
\end{aligned}$$

Explanation for Algorism B:

$$\begin{aligned}
12 \times 26 &= (10 + 2) \times 26 \\
&= (10 \times 26) + (2 \times 26) \\
&= 260 + 52 \\
&= 312
\end{aligned}$$

Notice that, in effect, Algorism B is an abbreviated form of Algorism A.

Algorism C is similar to Algorism B, except that 12 is expressed as 2 + 10 rather than as 10 + 2.

Finally, Algorism D is an abbreviated form of Algorism C. In Algorism D the "place value" principle is used explicitly so that by its position the 26 indicates "26 tens" or 260.

DIVISION ALGORISMS

We have recognized that, generally, it is not convenient for a person to remember all of his thinking when multiplying larger numbers. It is even less convenient to remember his thinking when dividing larger numbers. Consequently, the need for a written record of at least some of this thinking is even greater in division.

What is meant by an expression such as "69 divided by 4" or "57 divided by 3"? We may interpret any expression of this kind in two quite different ways.

(1) Expressions like "69 divided by 4" and "57 divided by 3" may be interpreted in relation to the <u>operation</u> of division within the set of whole numbers. We may write: $69 \div 4 = n$; so, $4 \times n = 69$ and $n \times 4 = 69$. Also: $57 \div 3 = n$; so, $3 \times n = 57$ and $n \times 3 = 57$. In each instance we are asked to determine the "unknown" factor, if one exists, within the set of whole numbers.

There clearly is no whole number n such that $4 \times n = 69$ (or $n \times 4 = 69$). In a sense, then, the expression "$69 \div 4$" has no meaning as an operational expression within the set of whole numbers. The set of whole numbers is <u>not</u> closed under division.

In the other instance, however, there <u>is</u> a whole number n such that $3 \times n = 57$ (or $n \times 3 = 57$). That number is 19, since $3 \times 19 = 57$ (or $19 \times 3 = 57$). We also may write: $57 \div 3 = 19$.

(2) Expressions such as "69 divided by 4" or "57 divided by 3" may be interpreted in relation to the partitioning of sets into equivalent subsets as described by mathematical sentences of the form:

$$69 = (n \times n) + r \qquad \text{or} \qquad 69 = (n \times 4) + r$$

$$57 = (3 \times n) + r \qquad \text{or} \qquad 57 = (n \times 3) + r$$

in which n and r are whole numbers, and n is as large as possible.

In the first instance we may write:

$$69 = (4 \times 17) + 1 \quad \text{or} \quad 69 = (17 \times 4) + 1 .$$

In the second instance we may write:

$$57 = (3 \times 19) + 0 \quad \text{or} \quad 57 = (19 \times 3) + 0 .$$

Note that this second instance is analogous to the case in which 19 was found to be the "unknown" factor in the sentence, $3 \times n = 19$.

Solutions such as those illustrated in (1) and (2) above usually cannot be determined easily, by inspection, when larger numbers are involved. Consequently, an algorism -- a way of processing, or of recording one's thinking --is helpful.

Let us illustrate the preceding discussion with an algorism (shown in several alternative forms) that could be used in relation to the expression, "862 divided by 6."

```
       143
         3
        40
       100                                          143
    6 )862          6 )862 |                     6 )862
       600             600 | 100                    600
       262             262 |                        262
       240             240 |  40                    240
        22              22 |                         22
        18              18 |   3                     18
         4               4 | 143                      4
```

In each form we often use special names to refer to specific parts of the algorism:

862 may be called the dividend.
6 may be called the divisor.
143 may be called the quotient.
4 may be called the remainder.

(1) First let us consider the information given by the algorism in relation to the mathematical sentence, $862 \div 6 = n$. We have found that there is no whole number n such that $6 \times n = 862$ (or $n \times 6 = 862$). We therefore know that 6 is not a factor of 862.

(2) Now let us consider the information given by the algorism in relation to the mathematical sentences:

$862 = (6 \times n) + r$ or $862 = (n \times 6) + r$.

We now may write:

$862 = (6 \times 143) + 4$ or $862 = (143 \times 6) + 4$.

We may think of this in relation to:

(a). partitioning a set of 862 objects into 6 equivalent subsets. There will be 143 members in each of the 6 subsets, with a set of 4 members remaining.

(b) partitioning a set of 862 objects into equivalent subsets of 6 members each. There will be 143 such subsets, with a set of 4 members remaining.

Now ~~let~~ us examine the mathematical bases for our commonly used division algorism.

In the preceding volume of Mathematics for the Elementary School, the distributive property of division over addition

$$(a + b) \div c = (a \div c) + (b \div c)$$

was used to explain the basis for the division process. However, the basis for a division algorism can be seen more clearly at times in terms of the distributive property of multiplication over addition:

$$a \times (b + c) = (a \times b) + (a \times c).$$

Think of dividing 760 by 20 using one or the other of these forms:

$$\begin{array}{rr} & \underline{38} \\ & 8 \\ & 30 \\ 20 & \overline{)760} \\ & \underline{600} \\ & 160 \\ & \underline{160} \\ & 0 \end{array} \qquad\qquad \begin{array}{rr|r} 20 & \overline{)760} & \\ & \underline{600} & 30 \\ & 160 & \\ & \underline{160} & \underline{8} \\ & 0 & 38 \end{array}$$

This division could have been indicated by the sentence $760 \div 20 = n$, which may be re-expressed as $20 \times n = 760$.

We know that n must be greater than 10 but less than 100, since $20 \times 10 = 200$ and $20 \times 100 = 2000$, and 760 is between 200 and 1000. We then may think of n as being in the form $b + c$, where b is the largest possible multiple of 10. So $20 \times n = 20 \times (b + c)$.

Using the distributive property of multiplication over addition, we may write:

$$20 \times (b + c) = 760$$

$$(20 \times b) + (20 \times c) = 760$$

$$(20 \times 30) + (20 \times 8) = 760$$

200

Each form of the algorism shows that we have determined b to be 30 and c to be 8. So, the "unknown" factor n is $30 + 8$, or 38.

But how can we determine, for example, that b is 30? We could think:

$$20 \times 10 = 200$$
$$20 \times 20 = 400$$
$$20 \times 30 = 600$$
$$20 \times 40 = 800$$

We see that $800 > 760$ and $600 < 760$. Since $20 \times 30 = 600$, $b = 30$.

In a shorter way, we can use our knowledge of the multiplication "facts" $2 \times 3 = 6$ and $2 \times 4 = 8$ to help us infer that 30 will be the largest multiple of 10 to use as a factor with 20 so that the product will not exceed 760.

By a similar inference we can determine that c is 8. Knowing that $2 \times 8 = 16$ helps us determine that $20 \times 8 = 160$.

Finally, mention should be made of the fact that it is through a more explicit application of the principle of "place value" that we may condense either of the preceding forms to ones such as these:

$$\begin{array}{r} 38 \\ 20\overline{)760} \\ \underline{600} \\ 160 \\ \underline{160} \end{array} \qquad \text{or} \qquad \begin{array}{r} 38 \\ 20\overline{)760} \\ \underline{60} \\ 160 \\ \underline{160} \end{array}$$

This chapter is organized in the following way.

1. There are teaching suggestions and exploration which appear only in the teacher's commentary.
2. There are explorations and summaries which appear in the pupil text.
3. There are pupil exercises to be done independently.

It is recommended that the teacher follow the exploration in the teacher's commentary preceding the work with pupils in the pupil text. The pupil text materials are designed to be read and discussed together. These offer pupils a record of review and extension of techniques of multiplication and division. It is not intended that all children do all exercises. Yet, you also may find it necessary to supplement some exercises with additional work.

As background for this unit, pupils should know the multiplication facts through 10×10. Since the properties of multiplication are used extensively in this chapter, teacher familiarity with Chapters 4 and 7 of fourth grade is recommended.

In the previous chapter it was emphasized that product expressions such as 3×4, $(3 \times 2) \times 2$, and 2×6 are different names for the number twelve. In many problems a desired response to a mathematical sentence such as $3 \times 4 = n$ is $n = 2 \times 6$. Since, in this chapter, we are concerned with multiplying and dividing, we try to be explicit by asking for the decimal numeral. (Decimal numerals are numerals using the base ten numeration system. Actually here we will need such symbols only for representing whole numbers. The numerals usually used to name the whole numbers are 0, 1, 2, 3, ..., 10, 11, 12, 13, ..., 85, 86, 87, ...). In the later exercises we shorten the instructions to something like, find n, compute n, etc. Such instructions are to be interpreted as asking for the decimal numeral form for n.

REVIEWING IDEAS OF MULTIPLICATION

Objective: To review the language of multiplication

Materials: Duplicated blank table as suggested in Exercise Set 1 in pupil text

Teaching Suggestions:

Before children begin this chapter, elicit from them what multiplication means and review the vocabulary of multiplication. Note that the product of two numbers, such as 3 and 4 may be named as a product expression, 3 × 4 or as a decimal numeral 12. Determine pupil understanding of the mathematical sentence.

As one way of reviewing multiplication facts through 10 × 10, charts similar to the one given in Exercise Set 1 may be constructed. Forms may be duplicated so that pupils can fill in numbers as needed. Changes in sequence may be made to provide practice material.

After reading with the children the first page of this chapter in the pupil text, have them do Exercise Set 1 independently.

EXTENDING MULTIPLICATION AND DIVISION I

Chapter 3

REVIEWING IDEAS OF MULTIPLICATION

To express the product of two ~~numbers~~ using a mathematical sentence, we can write:

$$5 \times 4 = 20.$$

We read this either as:

5 times 4 is equal to 20.

or

5 times 4 equals 20.

20 is the product of the numbers 5 and 4. 5 and 4 are factors of 20.

$$\underset{\substack{\uparrow \\ \text{factor}}}{5} \times \underset{\substack{\uparrow \\ \text{factor}}}{4} = \underset{\substack{\uparrow \\ \text{product}}}{20}$$

We have found that any number has many names. The expression, 5 × 4, is another name for 20. When we use a name showing multiplication like 5 × 4 for 20, we call it a <u>product expression</u>. Both 20 and 5 × 4 name the product of 5 and 4. In this chapter we will learn ways of finding the decimal name for the products of large numbers.

Exercise Set 1

Copy the following table and fill in the blanks with the products. (Use decimal numerals.)

x	6	8	5	10	4	0	9	2	7	3	1
4	24	32	20	40	16	0	36	8	28	12	4
7	42	56	35	70	28	0	63	14	49	21	7
1	6	8	5	10	4	0	9	2	7	3	1
9	54	72	45	90	36	0	81	18	63	27	9
3	18	24	15	30	12	0	27	6	21	9	3
6	36	48	30	60	24	0	54	12	42	18	6
10	60	80	50	100	40	0	90	20	70	30	10
5	30	40	25	50	20	0	45	10	35	15	5
0	0	0	0	0	0	0	0	0	0	0	0
8	48	64	40	80	32	0	72	16	56	24	8
2	12	16	10	20	8	0	18	4	14	6	2

REVIEWING THE PROPERTIES OF MULTIPLICATION

Objective: To review the properties of multiplication

Materials: Two 4 by 6 arrays, two 3 by 8 arrays and several other sets of arrays containing the same number of elements for class discussion of the commutative property of multiplication

One 7 by 18 array and similar arrays for class discussion of the distributive property

In reviewing the <u>commutative property of multiplication</u>, use two 4 by 6 arrays. Pupils should review that a 4 by 6 array and a 6 by 4 array are different only in the way they are formed. Each has the same number of elements. It might be well to review that by turning a 3 by 8 array, we can place it over an 8 by 3 array; but a 4 by 6 array cannot be placed over a 3 by 8 array, no matter how much turning is done.

It is important that pupils understand the use of the <u>associative property of multiplication</u>. It is desirable that pupils be able to verbalize their understanding of the property, but it is most important that pupils be able to make use of associativity. They should recognize that the way 3 factors are grouped does not affect the product.

After reviewing the associative property, have pupils do Exercise Set 2.

From their work in Chapter 7 of fourth grade, children should know how to multiply using multiples of 10 and 100. In that chapter, pupils found the associative property very useful in multiplying by multiples of 10 and were able to show their work using the mathematical sentence form. Pupils should be able to write products of multiples of 10 without having to use the longer form.

The teacher may need to give pupils additional oral and written practice. Children should be able to explain their way of arriving at the product to insure that their work with multiplication is not merely mechanical.

In Examples 4 and 5 in the pupil text, you will notice that several steps have been combined in order to reach a shorter form. In this note we will include all of the steps as they should be in order for you, and possibly some of the better pupils, to see the complete form.

Example 4:

$60 \times 70 = (6 \times 10) \times (7 \times 10)$ (Rename 60 and 70.)

$= [(6 \times 10) \times 7] \times 10$ (Use associative property.)

$= [6 \times (10 \times 7)] \times 10$ (Use associative property.)

$= [6 \times (7 \times 10)] \times 10$ (Use commutative property.)

$= (6 \times 7) \times (10 \times 10)$ (Use associative property.)

$= 42 \times 100$ (Product of 6 and 7 is 42; product of 10 and 10 is 100.)

$= 4200$ (Product of 42 and 100 is 4200.)

Example 5:

$700 \times 30 = (7 \times 100) \times (3 \times 10)$ (Rename 700 and 30.)

$= [(7 \times 100) \times 3] \times 10$ (Use associative property.)

$= [7 \times (100 \times 3)] \times 10$ (Use associative property.)

$= [7 \times (3 \times 100)] \times 10$ (Use commutative property.)

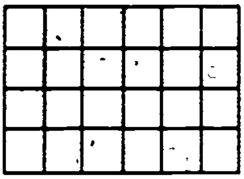
$= (7 \times 3) \times (100 \times 10)$ (Use associative property.)

$= 21 \times 1000$ (Product of 7 × 3 is 21; product of 100 and 10 is 1000.)

$= 21{,}000$ (Product of 21 and 1000 is 21,000.)

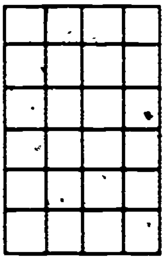
COMMUTATIVE PROPERTY OF MULTIPLICATION

A 4 by 6 array can be turned to form a 6 by 4 array.



4 by 6 array

4 × 6 = 24



6 by 4 array

6 × 4 = 24

This shows that 4 × 6 = 6 × 4 .

A 24 by 35 array can be turned to form a 35 by 24 array. This shows 24 × 35 = 35 × 24. When we write 24 × 35 in place of 35 × 24, we are using the <u>commutative property of multiplication</u>.

By using the commutative property, we have fewer multiplication facts to learn.

If we know 5 × 9 = 45, then we know 9 × 5 = 45.

If we know 7 × 8 = 56, then we know 8 × 7 = 56.

If this property is used, how many multiplication facts are to be learned? (55) How do you know? (We know by observing the multiplication chart. In the first row, there are 9 products; in the next row, 8; in the next, 7; etc. These add to 55.)

What are the properties of 0 and 1 for multiplication? ($0 \times n = n \times 0 = 0$ $1 \times n = n \times 1 = n$)

How can we use these properties so we have even fewer multiplication facts to remember? (Any fact involving 0 or 1 as a factor does not have to be memorized if we can use quickly the properties of 0 and 1.)

ASSOCIATIVE PROPERTY OF MULTIPLICATION

We know that we can multiply three numbers, such as 4 and 2 and 3, in that order, in either of two ways:

$$(4 \times 2) \times 3 = 8 \times 3 = 24$$

$$4 \times (2 \times 3) = 4 \times 6 = 24$$

Each way of grouping the numbers gives the same product. So, we may write:

$$(4 \times 2) \times 3 = 4 \times (2 \times 3).$$

When we replace one way of grouping the numbers by the other way, we are using the associative property of multiplication.

Because of the associative property of multiplication, we can write

$$4 \times 2 \times 3 = 24$$

without using any parentheses. We know that either grouping of the factors will give the same product.

We have learned how to multiply using 10, or 100, or 1000 as a factor in examples like these:

$3 \times 10 = 30$	$7 \times 100 = 700$	$9 \times 1000 = 9000$
$23 \times 10 = 230$	$57 \times 100 = 5700$	$39 \times 1000 = 39,000$

We also know our "multiplication facts," such as:

$4 \times 3 = 12$, $7 \times 5 = 35$, $6 \times 8 = 48$.

Now let us review how we can use these two things along with the associative property of multiplication to find products of numbers such as 4 and 20, or 6 and 700, or 5 and 3000.

Example 1

$4 \times 20 = 4 \times (2 \times 10)$ (Think of 20 as 2 × 10.)

$= (4 \times 2) \times 10$ (Use associative property.)

$= 8 \times 10$ (Product of 4 and 2 is 8.)

$= 80$ (Product of 8 and 10 is 80.)

Example 2

$6 \times 700 = 6 \times (7 \times 100)$ (Think of 700 as 7 × 100.)

$= (6 \times 7) \times 100$ (Use associative property.)

$= 42 \times 100$ (Product of 6 and 7 is 42.)

$= 4200$ (Product of 42 and 100 is 4200.)

Example 3

$5 \times 3000 = 5 \times (3 \times 1000)$ (Think of 3000 as 3 × 1000.)

$= (5 \times 3) \times 1000$ (Use associative property.)

$= 15 \times 1000$ (Product of 5 and 3 is 15.)

$= 15,000$ (Product of 15 and 1000 is 15,000.)

Products of numbers such as 60 and 70, or 700 and 30 can be found using the associative property of multiplication along with the commutative property of multiplication.

Example 4

$60 \times 70 = (6 \times 10) \times (7 \times 10)$ (Rename 60 and 70.)

$= (6 \times 7) \times (10 \times 10)$ (Use the associative and commutative properties.)

$= 42 \times 100$ (The product of 6 and 7 is 42; the product of 10 and 10 is 100.)

$= 4200$ (The product of 42 and 100 is 4200.)

Example 5

$700 \times 30 = (7 \times 100) \times (3 \times 10)$ (Rename 700 and 30.)

$= (7 \times 3) \times (100 \times 10)$ (Use the associative and commutative properties.)

$= 21 \times 1000$ (The product of 7 and 3 is 21; the product of 100×10 is 1000.)

$= 21{,}000$ (The product of 21 and 1000 is 21,000.)

Do you know a way in which you can find the product of numbers like 60 and 70, or 700 and 30 more quickly? If not see if you can find one.

Exercise Set 2

1. Write each of the following products as decimal numerals.

a. 3 × 10 (30)
b. 4 × 100 (400)
c. 1,000 × 7 (7,000)
d. 100 × 12 (1,200)
e. 32 × 1,000 (32,000)
f. 10 × 56 (560)
g. 200 × 4 (800)
h. 33 × 100 (3,300)
i. 4 × 600 (2,400)
j. 800 × 3 (2,400)
k. 8 × 2,000 (16,000)
l. 500 × 6 (3,000)
m. 300 × 2 (600)
n. 7 × 80 (560)

2. Find the product of each of the pairs of numbers by using the commutative and associative properties of multiplication.

Example: 50 and 40

$$\begin{aligned} 50 \times 40 &= (5 \times 10) \times (4 \times 10) \\ &= (5 \times 4) \times (10 \times 10) \\ &= 20 \times 100 \\ &= 2{,}000 \end{aligned}$$

a. 30 and 70
b. 80 and 60
c. 200 and 300
d. 90 and 700
e. 300 and 40
f. 50 and 700
g. 600 and 80
h. 300 and 9,000

Answers to Exercise Set 2

2\. a. $$\begin{aligned} 30 \times 70 &= (3 \times 10) \times (7 \times 10) \\ &= (3 \times 7) \times (10 \times 10) \\ &= 21 \times 100 \\ &= 2{,}100 \end{aligned}$$

b. $$\begin{aligned} 80 \times 60 &= (8 \times 10) \times (6 \times 10) \\ &= (8 \times 6) \times (10 \times 10) \\ &= 48 \times 100 \\ &= 4{,}800 \end{aligned}$$

c. $$\begin{aligned} 200 \times 300 &= (2 \times 100) \times (3 \times 100) \\ &= (2 \times 3) \times (100 \times 100) \\ &= 6 \times 10{,}000 \\ &= 60{,}000 \end{aligned}$$

d. $$\begin{aligned} 90 \times 700 &= (9 \times 10) \times (7 \times 100) \\ &= (9 \times 7) \times (10 \times 100) \\ &= 63 \times 1{,}000 \\ &= 63{,}000 \end{aligned}$$

e. $$\begin{aligned} 300 \times 40 &= (3 \times 100) \times (4 \times 10) \\ &= (3 \times 4) \times (100 \times 10) \\ &= 12 \times 1{,}000 \\ &= 12{,}000 \end{aligned}$$

f. $$\begin{aligned} 50 \times 700 &= (5 \times 10) \times (7 \times 100) \\ &= (5 \times 7) \times (10 \times 100) \\ &= 35 \times 1{,}000 \\ &= 35{,}000 \end{aligned}$$

g. $$\begin{aligned} 600 \times 80 &= (6 \times 100) \times (8 \times 10) \\ &= (6 \times 8) \times (100 \times 10) \\ &= 48 \times 1{,}000 \\ &= 48{,}000 \end{aligned}$$

h. $$\begin{aligned} 300 \times 9{,}000 &= (3 \times 100) \times (9 \times 1{,}000) \\ &= (3 \times 9) \times (100 \times 1{,}000) \\ &= 27 \times 100{,}000 \\ &= 2{,}700{,}000 \end{aligned}$$

Exercise Set 3

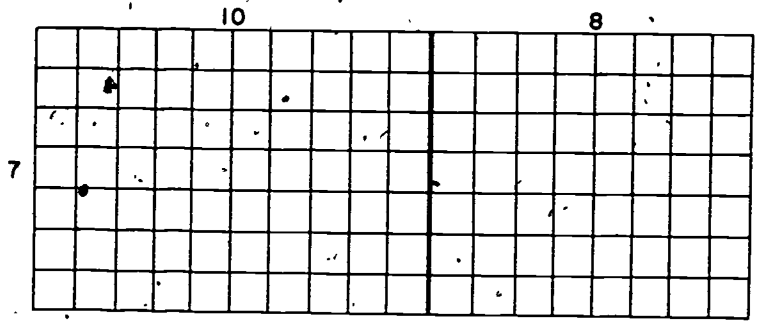
Find n in each sentence. (Use a decimal numeral.)

1. 40 × 30 = n (1,200)
2. 50 × 70 = n (3,500)
3. 60 × 80 = n (4,800)
4. 30 × 50 = n (1,500)
5. 60 × 40 = n (2,400)
6. 20 × 600 = n (12,000)
7. 500 × 30 = n (15,000)
8. 400 × 7 = n (2,800)
9. 70 × 800 = n (56,000)
10. 80 × 900 = n (72,000)
11. 200 × 300 = n (60,000)
12. 500 × 700 = n (350,000)
13. 300 × 800 = n (240,000)
14. 700 × 40 = n (28,000)
15. 30 × 600 = n (18,000)
16. 70 × 90 = n (6,300)
17. 80 × 700 = n (56,000)
18. 90 × 30 = n (2,700)
19. 80 × 50 = n (4,000)
20. 20 × 12,000 = n (240,000)

REVIEW OF PROPERTIES (CONTINUED)

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

A 7 by 18 array may be used to picture the distributive property of multiplication over addition.



When a 7 by 18 array is used, it can be seen that 18 may be renamed in many ways (by making various folds) but most conveniently 18 is named 10 + 8.

From this, pupils should be able to write

$$\begin{aligned} 7 \times 18 &= 7 \times (10 + 8) \\ &= (7 \times 10) + (7 \times 8) \\ &= 70 + 56 \\ &= 126 . \end{aligned}$$

Using the commutative property, we can see that

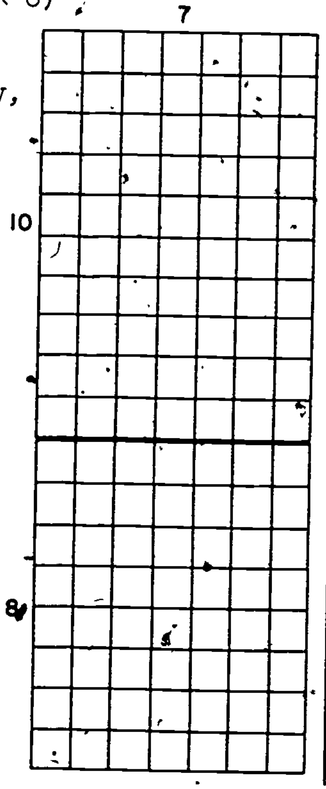
$$7 \times 18 = 18 \times 7 .$$

This could be demonstrated by turning the 7 by 18 array 90° and using the same separations. It becomes apparent that the number of rows are renamed and the number of columns distributed over the rows.

Thus we recognize that:

$$\begin{aligned} 18 \times 7 &= (10 + 8) \times 7 \\ &= (10 \times 7) + (8 \times 7) \\ &= 70 + 56 \\ &= 126 . \end{aligned}$$

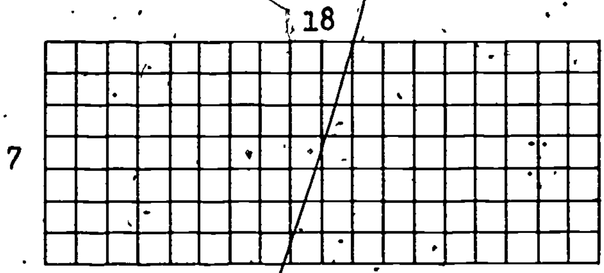
Although many renamings of 18 are possible, it is important to choose the most convenient one; in these examples it is 10 + 8.



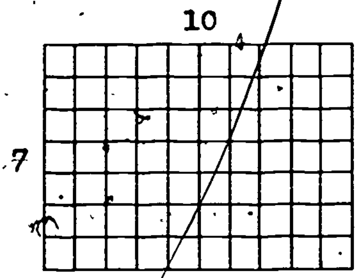
P99

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

To find the product of 7 and 18, think of a 7 by 18 array.

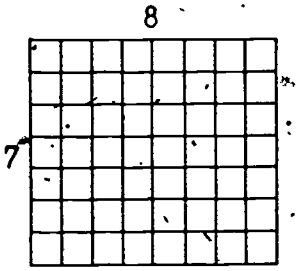


Separate it into two arrays showing products you already know. For example:



7 by 10 array

$7 \times 10 = 70$



7 by 8 array

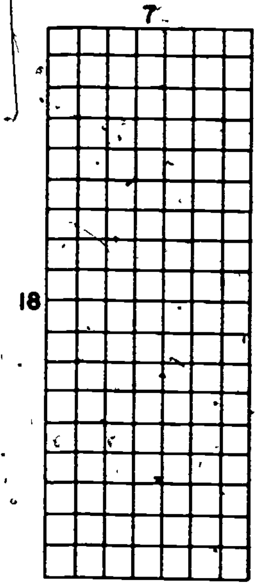
$7 \times 8 = 56$

These arrays help us see that

$$\begin{aligned} 7 \times 18 &= 7 \times (10 + 8) \\ &= (7 \times 10) + (7 \times 8) \\ &= 70 + 56 \\ &= 126 \end{aligned}$$

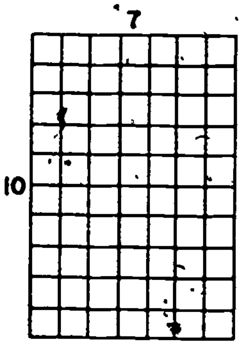
When we write $(7 \times 10) + (7 \times 8)$ in place of $7 \times (10 + 8)$, we are using the <u>distributive property of multiplication over addition</u>.

Now, suppose we find the product of 18 and 7.

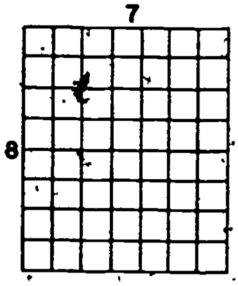


(18 by 7 array)

Can we separate the array like this?



(10 by 7 array)



(8 by 7 array)

Find the products separately and add them to get the total number of elements in the 18 by 7 array.

$$\begin{aligned} 18 \times 7 &= (10 + 8) \times 7 \\ &= (10 \times 7) + (8 \times 7) \\ &= 70 + 56 \\ &= 126 \end{aligned}$$

The commutative property of multiplication tells us that a 7 by 18 array has the same number of elements as an 18 by 7 array, thus: $7 \times 18 = 18 \times 7$

Since $7 \times 18 = 7 \times (10 + 8)$

$= (7 \times 10) + (7 \times 8)$,

and $18 \times 7 = (10 + 8) \times 7$

$= (10 \times 7) + (8 \times 7)$,

then $(7 \times 10) + (7 \times 8) = (10 \times 7) + (8 \times 7) = 126$ elements.

Here are other illustrations of how we may use the distributive property of multiplication over additon.

1. $20 \times 37 = 20 \times (30 + 7)$	(Rename 37 as 30 + 7.)
$= (20 \times 30) + (20 \times 7)$	(Distribute 20 over 30 and 7.)
$= 600 + 140$	(Use multiplication facts and place value.)
$= 740$	(Use addition facts and place value.)
2. $42 \times 30 = (40 + 2) \times 30$	(Rename 42 as 40 + 2.)
$= (40 \times 30) + (2 \times 30)$	(Distribute 30 over 40 and 2.)
$= 1200 + 60$	(Use multiplication facts and place value.)
$= 1260$	(Use addition facts and place value.)
3. $4 \times 285 = 4 \times (200 + 80 + 5)$	(Rename 285 as 200 + 80 + 5.)
$= (4 \times 200) + (4 \times 80) + (4 \times 5)$	(Distribute 4 over 200, 80, and 5.)
$= 800 + 320 + 20$	(Use multiplication facts and place value.)
$= 1140$	(Use addition facts, associative property, and place value.)

Exercise Set 4

1. Using the properties of multiplication, express the following products as decimal numerals. (See answers, page 210)

   Example: $6 \times 21 = 6 \times (20 + 1)$
   $= (6 \times 20) + (6 \times 1)$
   $= 120 + 6$
   $= 126$

   a. 3 × 27
   b. 42 × 6
   c. 2 × 128
   d. 7 × 341
   e. 217 × 8
   f. 4 × 285
   g. 22 × 10
   h. 47 × 30
   i. 20 × 62
   j. 7[illegible] × 30
   k. 40 × 57
   l. 60 × 23
   m. 78 × 10
   n. 20 × 91
   o. 86 × 30
   p. 39 × 50

2. Name the property of multiplication illustrated by each mathematical sentence.

   a. $8 \times 18 = 18 \times 8$ Commutative
   b. $2 \times (9 \times 6) = (2 \times 9) \times 6$ Associative
   c. $10 \times 32 = (10 \times 30) + (10 \times 2)$ Distributive

3. Find n in each mathematical sentence. Use what you know about the properties of multiplication to help you.

   a. $15 \times 30 = (10 \times 30) + (n \times 30)$ $n = 5$
   b. $18 \times 5 = 5 \times n$ $n = 18$
   c. $36 \times (10 \times 2) = 10 \times (2 \times n)$ $n = 36$

4. On your paper, write <u>true</u> if the mathematical sentence is true. Write <u>false</u> if the mathematical sentence is false.

   a. $8 \times (7 + 5) = (8 \times 7) + (8 + 5)$ false

   b. $12 \times 10 = 10 \times 12$ true

   c. $33 \times 42 = (30 + 3) \times (40 + 2)$ true

   d. $(10 \times 3) \times 4 = 10 \times (4 \times 3)$ true

   e. $(10 \times 5) \times 7 = 10 \times (5 + 7)$ false

5. Each of the expressions below is equal to (40×60). Which <u>does</u> <u>not</u> illustrate the distributive property? Write its letter. (c)

   a. $(20 \times 60) + (20 \times 60)$

   b. $(40 \times 30) + (40 \times 30)$

   c. $(4 \times 10) \times (6 \times 10)$

   d. $(25 \times 60) + (15 \times 60)$

Typical answers to Exercise 1, Exercise Set 4:

a. $$\begin{aligned} 3 \times 27 &= 3 \times (20 + 7) \\ &= (3 \times 20) + (3 \times 7) \\ &= 60 + 21 \\ &= 81 \end{aligned}$$

b. $$\begin{aligned} 42 \times 6 &= (40 + 2) \times 6 \\ &= (40 \times 6) + (2 \times 6) \\ &= 240 + 12 \\ &= 252 \end{aligned}$$

c. $$\begin{aligned} 2 \times 128 &= 2 \times (100 + 20 + 8) \\ &= (2 \times 100) + (2 \times 20) + (2 \times 8) \\ &= 200 + 40 + 16 \\ &= 256 \end{aligned}$$

d. $$\begin{aligned} 7 \times 341 &= 7 \times (300 + 40 + 1) \\ &= (7 \times 300) + (7 \times 40) + (7 \times 1) \\ &= 2100 + 280 + 7 \\ &= 2387 \end{aligned}$$

e. $$\begin{aligned} 217 \times 8 &= (200 + 10 + 7) \times 8 \\ &= (200 \times 8) + (10 \times 8) + (7 \times 8) \\ &= 1600 + 80 + 56 \\ &= 1736 \end{aligned}$$

f. $$\begin{aligned} 4 \times 285 &= 4 \times (200 + 80 + 5) \\ &= (4 \times 200) + (4 \times 80) + (4 \times 5) \\ &= 800 + 320 + 20 \\ &= 1140 \end{aligned}$$

g. $$\begin{aligned} 22 \times 10 &= (20 + 2) \times 10 \\ &= (20 \times 10) + (2 \times 10) \\ &= 200 + 20 \\ &= 220 \end{aligned}$$

h. $$\begin{aligned} 47 \times 30 &= (40 + 7) \times 30 \\ &= (40 \times 30) + (7 \times 30) \\ &= 1200 + 210 \\ &= 1410 \end{aligned}$$

i. $20 \times 62 = 20 \times (60 + 2)$

$= (20 \times 60) + (20 \times 2)$

$= 1200 + 40$

$= 1240$

j. $71 \times 30 = (70 + 1) \times 30$

$= (70 \times 30) + (1 \times 30)$

$= 2100 + 30$

$= 2130$

k. $40 \times 57 = 40 \times (50 + 7)$

$= (40 \times 50) + (40 \times 7)$

$= 2000 + 280$

$= 2280$

l. $60 \times 23 = 60 \times (20 + 3)$

$= (60 \times 20) + (60 \times 3)$

$= 1200 + 180$

$= 1380$

m. $78 \times 10 = (70 + 8) \times 10$

$= (70 \times 10) + (8 \times 10)$

$= 700 + 80$

$= 780$

n. $20 \times 91 = 20 \times (90 + 1)$

$= (20 \times 90) + (20 \times 1)$

$= 1800 + 20$

$= 1820$

o. $86 \times 30 = (80 + 6) \times 30$

$= (80 \times 30) + (6 \times 30)$

$= 2400 + 180$

$= 2580$

p. $39 \times 50 = (30 + 9) \times 50$

$= (30 \times 50) + (9 \times 50)$

$= 1500 + 450$

$= 1950$

BECOMING SKILLFUL IN MULTIPLYING

Objective: To develop greater skill in multiplying whole numbers

Vocabulary: Partial product, vertical form of multiplication

Teaching Suggestions:

In this chapter an algorism for multiplication is developed. By using place value, we are able to find a shorter way of recording the process.

Begin class discussion of multiplication by showing the use of the distributive property to find products. Use the mathematical sentence form. For example,

$8 \times 476 = 8 \times (400 + 70 + 6)$	Rename 476 as 400 + 70 + 6.
$= (8 \times 400) + (8 \times 70) + (8 \times 6)$	Distribute 8 over 400, 70, and 6.
$= 3200 + 560 + 48$	Use multiplication facts and place value.
$= 3808$	Use addition facts, associative property, and place value.

Relate the mathematical sentence form with the vertical form below. Children should be able to see that the partial products of the vertical form are the same as those in the mathematical sentence form.

Class discussion could include various orders in which the partial products may be written. (Review from Chapter 7, Grade 4.) For example,

$$8 \times 476 = n$$

$$\begin{array}{r} 476 \\ \times\ 8 \\ \hline 3200 \\ 560 \\ 48 \\ \hline 3808 \end{array} \qquad\qquad \begin{array}{r} 476 \\ \times\ 8 \\ \hline 48 \\ 560 \\ 3200 \\ \hline 3808 \end{array}$$

Have pupils explain the steps in multiplying when they write only the final product. For example, to multiply 6 and 273, the steps are:

$$\begin{array}{r} 273 \\ \times\ 6 \\ \hline 1638 \end{array}$$

6 × 3 = 18. Record the 8 ones, remember 1 ten.
6 × 7 tens = 42 tens. 42 tens + 1 ten = 43 tens. Record the 3 tens, remember the 4 hundreds.
6 × 2 hundreds = 12 hundreds.
12 hundreds + 4 hundreds = 16 hundreds. Record the 16 hundreds.

Certainly, as the process is shortened, place value for each digit of the numeral is emphasized.

The writing of additional numerals to show the regrouping may be used in approaching the level of writing only the final product. For example,

$$\begin{array}{r} \overset{4}{2}\overset{1}{7}3 \\ \times\quad 6 \\ \hline 1638 \end{array}.$$

However, it is expected that when children are ready for this level they will not find the need for this crutch for any length of time. The term "carrying" is not used with children.

It is assumed by fifth grade most children are using the conventional algorism and should be encouraged to continue with it. At the same time it must be recognized that all children are not at the same level of development and may need to use the long form.

You may wish to use such examples as the following for exploration with the class and class discussion before children work independently.

$72 \times 3 = n$ (216)

$7 \times 18 = n$ (126)

$3 \times 78 = n$ (234)

$6 \times 55 = n$ (33[illegible])

It is desirable that all development be done independently of the material in the pupil text. The record in the text then will serve as reference when the child works the exercises and for further study of these ideas. A teacher should develop the exploratory material for his class in light of the needs of his particular group.

BECOMING SKILLFUL IN MULTIPLYING

We have learned that we can use mathematical sentences to show our thinking when we multiply. For example,

$$4 \times 285 = n.$$

We can find the number which n represents in this way.

$$\begin{aligned} 4 \times 285 &= 4 \times (200 + 80 + 5) \\ &= (4 \times 200) + (4 \times 80) + (4 \times 5) \\ &= 800 + 320 + 20 \\ &= 1140 \end{aligned}$$

Then, $4 \times 285 = 1140$.

The numbers 800, 320, and 20, are called <u>partial products</u>.

Here is a shorter way to find the product of 285 and 4. We can write the partial products under each other as we multiply. Then, we can add them. For example, if $4 \times 285 = n$, we find the number which n represents in this way.

$$\begin{array}{r l} 285 & \\ \times\ 4 & \\ \hline 20 & \longleftarrow (4 \times 5) \\ 320 & \longleftarrow (4 \times 80) \\ 800 & \longleftarrow (4 \times 200) \\ \hline 1140 & \end{array}$$

Many of us should be able to write the product in an even shorter way.

$$\begin{array}{r} 285 \\ \times\ 4 \\ \hline 1140 \end{array}$$

Then, $4 \times 285 = 1140$.

What must we remember in order to do this?

Now let us consider this mathematical sentence.

$$3 \times 408 = n$$

We may write:

$$\begin{aligned} 3 \times 408 &= 3 \times (400 + 8) \\ &= (3 \times 400) + (3 \times 8) \\ &= 1200 + 24 \\ &= 1224 \end{aligned}$$

So, $n = 1224$, and $3 \times 408 = 1224$.

If we used shorter ways to find the product, we could write:

$$\begin{array}{rl} 408 & \\ \times \ \ 3 & \\ \hline 24 & \longleftarrow (3 \times 8) \\ 1200 & \longleftarrow (3 \times 400) \\ \hline 1224 & \end{array} \qquad \text{or} \qquad \begin{array}{r} 408 \\ \times 3 \\ \hline 1224 \end{array}$$

In the shorter way at the left, above, why are there just two partial products? (We need write no partial product when there are 0 tens.)

In each of the shorter ways shown above, is there any time when you did or could use the zero property for multiplication? (Yes, $3 \times (0 \times 10) = 3 \times 0 = 0$.)

Exercise Set 5

A. Find n. If you need to, show the partial products.

1. 5 × 63 = n 315
2. 4 × 56 = n 224
3. 6 × 93 = n 558
4. 3 × 256 = n 768
5. 6 × 307 = n 1,842
6. 8 × 209 = n 1,672
7. 9 × 347 = n 3,123
8. 6 × 986 = n 5,916
9. 7 × 837 = n 5,859
10. 8 × 2,609 = n 20,872

B. Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

11. A building has 72 windows. If it takes 3 minutes to wash one window, how many minutes will it take to wash all of them? (3 X 72 = m
216 = m It will take 216 minutes to wash all of the windows.)

12. A traffic light changes its color every 18 seconds. How many seconds will it take for the light to make 7 changes? (7 X 18 = c
126 = c It will take the light 126 seconds.)

13. A phonograph record revolves 33 times a minute. How many revolutions will the record make if it plays for 3 minutes? (3 X 33 = r
99 = r The record will make 99 revolutions.)

14. John and his father went on a fishing trip. It took them 6 hours to get to the lake. John's father was driving 55 miles per hour. How far did they have to drive before they could fish?

(6 X 55 = d
330 = d They had to drive 330 miles before they could fish.)

MULTIPLYING LARGER NUMBERS

Objective: To develop skill in multiplying larger whole numbers

Vocabulary: Vertical form

Materials: One large 17 by 24 array made on material that may be folded while the teacher demonstrates to the class

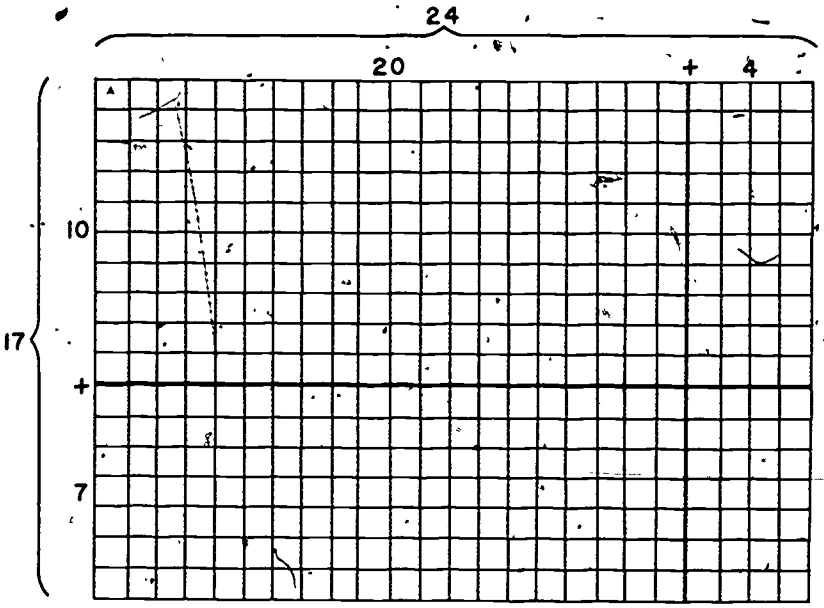
Exploration:

We have learned to find the product of two numbers. When the numeral of one number has one place and the numeral of the other has no more than three places. Now we are ready to consider finding the product of two numbers when both their numerals have two places.

First, let's review the distributive property of multiplication over addition in the example, 5 × 21, which is on the chalkboard.

$$\begin{aligned} 5 \times 21 &= 5 \times (20 + 1) \\ &= (5 \times 20) + (5 \times 1) \\ &= 100 + 5 \\ &= 105 \end{aligned}$$

Now look at this array. How many rows are there? How many columns are there? When we multiply 17 and 24 we will find how many elements there are in this array.



17 by 24 array

How can we rename 24 in a convenient way? (We can rename 24 as 20 + 4.) Can we show this renaming by folding the array? (Yes, we can fold it so there are two arrays. One has 20 columns, and one has 4 columns.) Can you write on the board a mathematical sentence to show what we have done?

$$17 \times 24 = 17 \times (20 + 4)$$
$$= (17 \times 20) + (17 \times 4)$$

Now what can we do to help us find a decimal numeral for 17 × 24? (We can rename 17.) How shall we rename it? (We may think of 17 as 10 + 7.) Let's fold the array to show this. How many smaller arrays have we now? (4) What are they? (10 by 20, 10 by 4, 7 by 20, 7 by 4)

Can we use what we know about multiplication to find the number of elements in each of the smaller arrays? We will record this on the board.

$$\begin{aligned} 10 \times 20 &= 200 \\ 7 \times 20 &= 140 \\ 10 \times 4 &= 40 \\ 7 \times 4 &= 28 \\ 17 \times 24 &= 200 + 140 + 40 + 28 \\ &= 408 \end{aligned}$$

There is a shorter way to find decimal numerals for such expressions as 17 × 24. We could use the vertical form to show what we just did with arrays. Let's look at it.

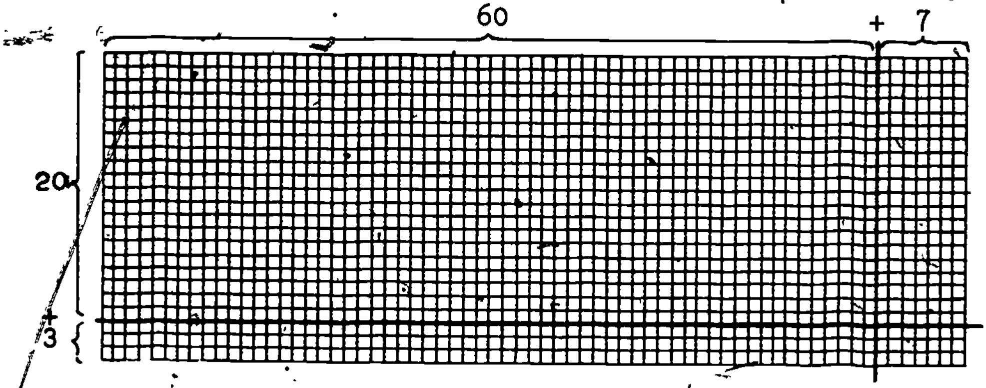
$$\begin{array}{r} 24 \\ \underline{\times\ 17} \\ 28 \\ 140 \\ 40 \\ \underline{200} \\ 408 \end{array}$$

Can you see how the partial products were obtained? (Yes, the 28 = 7 × 4, the 140 = 7 × 20, the 40 = 10 × 4, and the 200 = 10 × 20.)

Proceed in a similar manner with Multiplying Larger Numbers in the pupil text.

MULTIPLYING LARGER NUMBERS

Working Together



23 by 67 array

We can show, by using the distributive property, how to multiply two numbers greater than 10 but less than 100.

$n = 23 \times 67$

$= 23 \times (60 + 7)$ (Think of 67 as 60 + 7.)

$= (23 \times 60) + (23 \times 7)$ (Distribute 23 over 67. The heavy vertical line shows how the array is separated into smaller arrays.)

$= (20 + 3) \times 60 + (20 + 3) \times 7$ (Think of 23 as 20 + 3.)

$= (20 \times 60) + (3 \times 60) + (20 \times 7) + (3 \times 7)$ (The heavy horizontal line then shows how the array is separated into 4 smaller arrays. The heavy lines drawn on the array above illustrate these four arrays.)

$= 1200 + 180 + 140 + 21$ (These show the number of elements in each of the four arrays.)

$= 1541$ (The total number of elements in a 23 by 67 array is 1541.)

The vertical form also can be used with larger numbers. Look at this example.

$23 \times 67 = n$

$$\begin{array}{rl} 67 & \\ \underline{\times 23} & \\ 21 & \longleftarrow (3 \times 7) \\ 180 & \longleftarrow (3 \times 60) \\ 140 & \longleftarrow (20 \times 7) \\ \underline{1200} & \longleftarrow (20 \times 60) \\ 1541 & \longleftarrow (23 \times 67) \end{array}$$

$23 \times 67 = 1541$

See if you can identify each of the partial products shown above with parts of the array.

Using the vertical form, compute the following.

54	25	37
× 32	× 18	×42
8	40	14
100	160	60
120	50	280
1500	200	1200
1,728	450	1,554

Exercise Set 6

A. Compute using the vertical form. Show the partial products.

Example: 32 × 54

$$\begin{array}{r} 54 \\ \times\ 32 \\ \hline 8 \\ 100 \\ 120 \\ 1500 \\ \hline 1728 \end{array}$$

1. 45 × 23
2. 64 × 25
3. 37 × 26
4. 61 × 59
5. 28 × 92
6. 37 × 12
7. 24 × 37
8. 26 × 97
9. 37 × 86
10. 49 × 81
11. 57 × 77
12. 66 × 88
13. 44 × 95
14. 82 × 28
15. 37 × 75
16. 91 × 67

B. Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

17. A set of books weighs 12 pounds. If a school ordered 38 sets, what would be the total weight of the books ordered?
(n = 12×38 n = 456 The total weight would be 456 pounds.)

18. Mr. Jones, a farmer, sent 27 crates of eggs to the market. There were 24 dozen eggs in each crate. How many dozen eggs did he send to market? (n = 27×24 n = 648 He sent 648 dozen eggs to market.)

19. During our vacation last summer, we traveled for 28 hours. We drove at 59 miles per hour. How far did we travel during the 28 hours? (n = 28×59 n = 1,652 We traveled 1,652 miles.)

20. The candy store packed 86 boxes of candy. Each box contained 64 pieces of candy. How many pieces of candy were needed to pack all the boxes? (n = 86×64; n = 5,504 There were 5,504 pieces of candy needed.)

Answers to Exercise Set 6

1.	2.	3.	4.	5.
23	25	26	59	92
×45	×64	×37	×61	×28
15	20	42	9	16
100	80	140	50	720
120	300	180	540	40
800	1200	600	3000	1800
1,035	1600	962	3599	2576

6.	7.	8.	9.	10.
12	37	97	86	81
×37	×24	×26	×37	×49
14	28	42	42	9
70	120	540	560	720
60	140	140	180	40
300	600	1800	2400	3200
444	888	2522	3182	3969

11.	12.	13.	14.	15.
77	88	95	28	75
× 57	× 66	× 44	× 82	× 37
49	48	20	16	35
490	480	360	40	490
350	480	200	640	150
3500	4800	3600	1600	2100
4389	5808	4180	2296	2775

16.
67
× 91
7
60
630
5400
6097

A SHORTER FORM FOR MULTIPLYING

Objective: To lead pupils to use a shorter algorism

As soon as children are ready, develop a shorter algorism. The following is a suggested procedure.

We know that we can think of 23 × 67 as (20 × 67) + (3 × 67). We can use this idea to learn a shorter way of finding the product of 23 × 67. Use the chalkboard to remind children that they know

$$\begin{array}{r} 67 \\ \times\ 3 \\ \hline 201 \end{array} \qquad \text{and} \qquad \begin{array}{r} 67 \\ \times\ 20 \\ \hline 1340 \end{array}$$

Then the same information may be written in this form.

$$\begin{array}{r} 67 \\ \times\ 23 \\ \hline 201 \\ 1340 \\ \hline 1541 \end{array}$$

Ask such questions as:

(1) How did we get 201 ?
(2) How did we get 1340 ?

Continue with many other examples to show the relationship between the longer and the shorter vertical forms.

When it seems appropriate, use the pupil material entitled A Shorter Form for Multiplying.

Children can gain greater insight into multiplication by being reminded of the commutative property. Because of this property, the order in which partial products are written does not change the product.

It may be of value for your more capable children to recognize that the following are other ways of recording partial products.

$$\begin{array}{rl} 67 & \\ \times\ 23 & \\ \hline 1200 & (20\times60) \\ 140 & (20\times7) \\ 180 & (3\times60) \\ 21 & (3\times7) \\ \hline 1541 & \end{array} \quad \begin{array}{rl} 23 & \\ \times\ 67 & \\ \hline 21 & (7\times3) \\ 140 & (7\times20) \\ 180 & (60\times3) \\ 1200 & (60\times20) \\ \hline 1541 & \end{array} \quad \begin{array}{rl} 23 & \\ \times\ 67 & \\ \hline 1200 & (60\times20) \\ 180 & (60\times3) \\ 140 & (7\times20) \\ 21 & (7\times3) \\ \hline 1541 & \end{array} \quad \begin{array}{rl} 67 & \\ \times\ 23 & \\ \hline 1200 & (60\times20) \\ 21 & (3\times7) \\ 140 & (20\times7) \\ 180 & (3\times60) \\ \hline 1541 & \end{array}$$

Children should be able to explain what was done in each example.

A SHORTER FORM FOR MULTIPLYING

Look at this example.

$$25 \times 72 = n$$

Here are two forms for finding the decimal numeral for n:

Longer Form		Shorter Form	
72		72	
× 25		× 25	
10	(5 × 2)	360	(5 × 72)
350	(5 × 70)		
40	(20 × 2)		
1400	(20 × 70)	1440	(20 × 72)
1800		1800	

$$n = 1800$$

$$25 \times 72 = 1800$$

Explain how the partial products in the longer and shorter forms are related to each other.

Exercise Set 7

Compute using a vertical form. Use the shorter form if you can.

Example: 37 × 54

$$\begin{array}{r} 54 \\ \times\ \underline{37} \\ 378 \\ \underline{1620} \\ 1998 \end{array}$$

1. 12 × 34 (408)
2. 21 × 43 (903)
3. 41 × 25 (1,025)
4. 15 × 37 (555)
5. 37 × 18 (666)
6. 24 × 37 (888)
7. 32 × 48 (1,536)
8. 12 × 98 (1,176)
9. 35 × 56 (1,960)
10. 86 × 72 (6,192)
11. 34 × 62 (2,108)
12. 84 × 53 (4,452)
13. 76 × 38 (2,888)
14. 83 × 95 (7,885)
15. 46 × 73 (3,358)
16. 66 × 37 (2,442)
17. 53 × 46 (2,438)
18. 72 × 33 (2,376)
19. 38 × 25 (950)
20. 36 × 49 (1,764)

USING A SHORTER FORM TO MULTIPLY LARGER NUMBERS

Objective: To extend the skills of multiplication to find products of still greater numbers

Teaching Suggestions:

This portion of the chapter should give pupils additional skill with vertical form for multiplying using two-place and three- and four-place numerals.

In examples 1 and 2 on the next pupil page, all of the partial products with the alternative shortened form are shown. It is hoped that children may extend their skills readily so that they may use a shorter form for computing.

Only the vertical form is given for the examples in the pupil book. Some teachers, however, may want to consider the mathematical sentence form which follows in the teacher's commentary. The mathematical sentence form should help pupils understand the multiplication algorism. It should be kept in mind, however, that the teacher's goal is to develop facility with a shorter algorism.

Example 1:

$$\begin{aligned}43 \times 237 &= (40 + 3) \times 237\\ &= (40 \times 237) + (3 \times 237)\\ &= 40 \times (200 + 30 + 7) + 3 \times (200 + 30 + 7)\\ &= (40 \times 200) + (40 \times 30) + (40 \times 7)\\ &\quad + (3 \times 200) + (3 \times 30) + (3 \times 7)\\ &= 8000 + 1200 + 280 + 600 + 90 + 21\\ &= 10{,}191\end{aligned}$$

Example 2:

$$\begin{aligned}34 \times 5432 &= (30 + 4) \times 5432\\ &= (30 \times 5432) + (4 \times 5432)\\ &= 30 \times (5000 + 400 + 30 + 2) + 4 \times (5000\\ &\quad + 400 + 30 + 2)\\ &= (30 \times 5000) + (30 \times 400) + (30 \times 30)\\ &\quad + (30 \times 2) + (4 \times 5000) + (4 \times 400)\\ &\quad + (4 \times 30) + (4 \times 2)\\ &= 150{,}000 + 12{,}000 + 900 + 60 + 20{,}000\\ &\quad + 1{,}600 + 120 + 8\\ &= 184{,}688\end{aligned}$$

USING A SHORTER FORM TO MULTIPLY LARGER NUMBERS

These examples will help you to learn how to find products of larger numbers.

<u>Example 1:</u> n = 43 × 237

237		OR	237	
× 43			× 43	
21	(3 × 7)		711	(3 × 237)
90	(3 × 30)		9480	(40 × 237)
600	(3 × 200)		10191	(43 × 237)
280	(40 × 7)			
1200	(40 × 30)		n = 10,191	
8000	(40 × 200)			
10191	(43 × 237)			

<u>Example 2:</u> n = 34 × 5032

5032		OR	5032	
× 34			× 34	
8	(4 × 2)		20128	(4 × 5032)
120	(4 × 30)		150960	(30 × 5032)
20000	(4 × 5000)		171088	(34 × 5032)
60	(30 × 2)			
900	(30 × 30)			
150000	(30 × 5000)		n = 171,088	
171088	(34 × 5032)			

Exercise Set 8

A. Use a vertical form to compute the following.

1. 26 × 201 (5,226)
2. 41 × 607 (24,887)
3. 42 × 121 (5,082)
4. 64 × 328 (20,992)
5. 270 × 37 (9,990)
   Hint: By using the commutative property of multiplication we know that 270 × 37 = 37 × 270.
6. 863 × 27 (23,301)
7. 96 × 8021 (770,016)
8. 45 × 378 (17,010)
9. 37 × 856 (31,672)
10. 54 × 2805 (151,470)
11. 317 × 47 (14,899)
12. 598 × 36 (21,528)
13. 58 × 4566 (264,828)
14. 638 × 21 (13,398)
15. 956 × 57 (54,492)

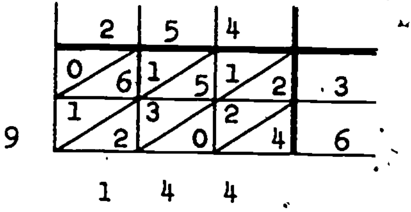
B. Use mathematical sentences to solve the following problems. Express each answer in a complete sentence.

16. If your father earns $840 a month, how much does he earn in a year? ($n = 840 \times 12$, $n = 10,080$ He earns $10,080 in a year.)

17. An automobile averages 16 miles per gallon of gasoline. The gasoline tank holds 17 gallons. How many miles will the automobile go on 17 gallons? ($n = 16 \times 17$, $n = 272$ The automobile will go 272 miles on 17 gallons.)

18. **BRAINTWISTER:** During the time of Columbus, a different multiplication form was used in Europe. This was called the Gelosia or Lattice method.

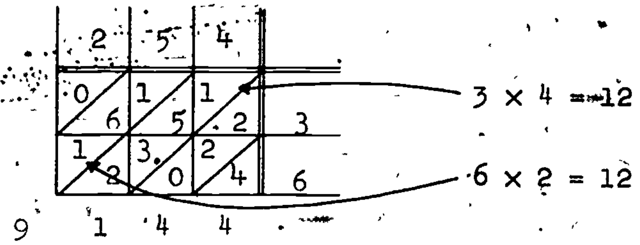
The solution of $n = 254 \times 36$ is shown by the diagram.



Can you find the value of n from the diagram? Test your knowledge of the Gelosia method by showing that

$$56 \times 672 = 37{,}632 .$$

The Gelosia multiplication process is a schematic device using the ideas of positional notation. In each square, the ones' digit of the product is written below the diagonal; the tens' digit of the product is written above the diagonal. (See the diagram below.) The product is found by adding the numbers whose numerals are between that diagonal. We begin in the lower right hand corner. If the sum is 10 or greater, we place the tens' digit in the next diagonal and continue with our addition.



Product is 9144

PROBLEM SOLVING

Objective: To develop the ability to solve "two-step" problems.

Teaching Suggestions:

The purpose of this lesson is to help children learn to use several mathematical sentences to solve one problem, and to combine several sentences into one sentence.

Before using the exploration and development in the pupil text, it is desirable to discuss selected problems. Here are some suggestions.

Example 1:

An auditorium has 48 rows with 26 seats in each row on the main floor. In the balcony there are 16 rows with 23 seats in each row. What is the largest number of people this auditorium can seat?

You might proceed by asking questions as: What do we know about the number of seats in the auditorium? (There are 48 rows of seats on the main floor. There are 26 seats in each row on the main floor.) You may wish to stop here and write a mathematical sentence about the number of seats on the main floor.

$$48 \times 26 = m$$

Now what else do we know about the number of seats in the auditorium? (In the balcony there are 23 seats in each row. There are 16 rows.) Ask in what way we can express this idea. You will hope they will suggest

$$16 \times 23 = b$$

If they don't, try to help them arrive at this sentence.

Then ask for suggestions as to what they should do next to find the number of seats in the auditorium. (They may suggest finding m and b and if they do, proceed in that way.)

Then suggest that they can write still another mathematical sentence for the total number of seats.

$$(m + b = n; \quad 1248 + 368 = n)$$

Also ask if they can write only one sentence for the problem, directing discussion to their suggesting the sentence:

$$(48 \times 26) + (16 \times 23) = n .$$

Upon completing the computation, ask how they can express the answer to the question of the problem, using a complete sentence. (The auditorium can seat 1616 people.)

Here is a second example you may wish to use.

Example 2:

A parking lot has 25 rows with 18 spaces for cars in each row. If 3 rows are removed for a driveway, what is the greatest number of cars which can be parked on the lot?

Suggest they try to think of two ways in which they could solve this problem and tell what mathematical sentences would be written for each way.

(a) One way might be:

What mathematical sentence can we write to express the number of cars that can be parked in the lot? Then what is the sentence for the number of spaces to be removed for the driveway?

After the decimal numeral is found for each of these numbers, a sentence can be written for the greatest number of cars that can be parked on the lot.

$$25 \times 18 = p \quad \text{(Before driveway)}$$

$$3 \times 18 = d \quad \text{(For driveway)}$$

$$\left.\begin{array}{r} p - d = n \\ \text{or} \\ 450 - 54 = n \end{array}\right\} \text{(After driveway)}$$

(b) Another way might be:

Use (25 × 18) as the number of spaces before making the driveway and (3 × 18) as the number of spaces removed for the driveway. Then the mathematical sentence for the number of cars that can be parked after making a driveway is:

$$(25 \times 18) - (3 \times 18) = n .$$

Ask what computations are necessary. After finding that 25 × 18 = 450 and 3 × 18 = 54, you must subtract 54 from 450.

(c) With either way, you can then answer the question of the problem: There is room for 396 cars on the parking lot.

You may wish to use other examples before going to the materials in the pupil text.

PROBLEM SOLVING.

A coin book has 35 slots for coins on each page. If the book has 12 pages and 287 coins have been placed in the slots, how many more are needed to complete the book?

Here is a way to solve this problem using two mathematical sentences.

$12 \times 35 = p$	$420 - 287 = n$
35	420
$\underline{\times 12}$	$\underline{-287}$
70	133
$\underline{350}$	
420	

There are 133 coins needed to complete this book.

Here is a way to solve this problem using one mathematical sentence.

$$(12 \times 35) - 287 = n$$

35	420
$\underline{\times 12}$	$\underline{-287}$
70	133
$\underline{350}$	
420	

There are 133 coins needed to complete the book.

Exercise Set 9

Use mathematical sentences to help you solve the following problems. Express each answer in a complete sentence.

1. A typewriter prints 12 symbols to an inch across a page. How many symbols can be printed on a sheet of paper 8 inches wide without using spaces between the symbols if there are 65 rows of symbols possible?
2. John bought a notebook for 25¢, a pencil for 7¢, and an arithmetic book for $2.50. He gave the clerk $5.00. How much change did he receive?
3. Jane takes the bus to and from school 5 days per week. The fare each way is 25¢. How much is her fare for the week?
4. The Brown family of six planned to fly to Washington on their vacation. Each person was allowed 40 pounds of free baggage. The Browns had 263 pounds of baggage. What was the number of pounds of extra baggage?
5. There are 24 pages in Mary's stamp album. On each page there is room for 18 stamps. Mary has 279 stamps. How many stamps does she need to fill her album?
6. A parking lot had 25 rows with 16 spaces in each row. The size of the lot was increased with spaces for 225 cars. Since the addition, how many cars can be parked on this lot?

Answers to Exercise Set 9

1. $8 \times 12 = p$ or $65 \times (8 \times 12) = n$
   $65 \times 96 = n$ $n = 6{,}240$
   6,240 symbols can be printed on the sheet of paper.

2. $25 + 7 + 250 = p$ or $500 - (25 + 7 + 250) = n$
   $500 - 282 = n$ $n = 218$
   John received \$2.18 change.

3. $2 \times 25 = p$ or $(2 \times 25) \times 5 = n$
   $5 \times 50 = n$ $n = 250$
   Jane's fare is \$2.50 each week.

4. $6 \times 40 = p$ or $263 - (6 \times 40) = n$
   $263 - 240 = n$ $n = 23$
   They had 23 pounds of extra baggage.

5. $18 \times 24 = p$ or $(18 \times 24) - 279 = n$
   $432 - 279 = n$ $n = 153$
   Mary needs 153 stamps to fill her album.

6. $25 \times 16 = p$ or $(25 \times 16) + 225 = n$
   $400 + 225 = n$ $n = 625$
   625 cars can be parked on the lot.

REVIEWING IDEAS OF DIVISION

Objectives:
1. To review the ideas of division by relating the operation of division to the operation of multiplication
2. To place particular emphasis on the division process
3. To distinguish between ideas associated with the operation of division and the division process

Teaching Suggestions:

The major emphasis in this chapter is upon an understanding of algorisms and developing increasing skill in their use.

Throughout the chapter, two forms of the division algorism will be presented in the pupil text.

Form I:

$$\begin{array}{rr} & \underline{95} \\ & 5 \\ & \underline{90} \\ 5 & \overline{)\,475} \\ & \underline{450} \\ & 25 \\ & \underline{25} \end{array}$$

Form II:

$$\begin{array}{rr|r} 5 & \overline{)\,475} & \\ & \underline{450} & 90 \\ & 25 & \\ & \underline{25} & \underline{5} \\ & & 95 \end{array}$$

> IMPORTANT: This <u>does not mean</u> that pupils should become skillful in using both forms. Pupils should determine which form they prefer and gain skill in <u>just one</u>. While it is not to be expected that all children achieve the same degree of skill or work at the same level, they should be encouraged to move to a more mature form as they are ready. Of course, a more mature form is:
>
> $$\begin{array}{rr} & 95 \\ 5 & \overline{)\,475} \\ & \underline{450} \\ & 25 \\ & \underline{25} \end{array} \qquad\qquad \begin{array}{rl} & \;95 \\ 5 & \overline{)\,475} \\ & \underline{45} \\ & \;\;25 \\ & \;\;\underline{25} \end{array}$$

Before having children read Reviewing Ideas of Division, elicit from them their ideas of the relationship of multiplication and division. Be sure that pupils know the language of division and how to read and write the sentences showing division as an operation, as illustrated in the pupil text.

REVIEWING IDEAS OF DIVISION

Division is the operation we use to find an unknown factor when the product and one factor are known.

The following sentences suggest division.	This is how we can read them.
$n \times 4 = 20$	What number times 4 is equal to 20?
$4 \times n = 20$	4 times what number is equal to 20?
$20 \div 4 = n$	20 divided by 4 is equal to what number?
$20 \div n = 4$	20 divided by what number is equal to 4?

In each case we are to find the unknown factor. We may use the same process.

A form for computing:

$$\underset{\text{Product}}{\underset{\uparrow}{20}} \div \underset{\text{Known Factor}}{\underset{\uparrow}{4}} = \underset{\text{Unknown Factor}}{\underset{\uparrow}{n}} \qquad \begin{array}{r} 5 \\ 4\overline{)20} \\ \underline{20} \end{array}$$

$n = 5$

We have learned to become skillful with multiplication. Now we want to learn ways of making the process of division easier.

WORKING WITH MULTIPLES OF 10 AND 100

Objectives: 1. To develop skill in multiplying with multiples of 10 and 100

2. To develop skill in finding an unknown factor that is a multiple of 10 or 100

Materials: Duplicate tables as in the next section of the pupil text

> Teaching Suggestions:
>
> Children need to be able to recognize and find multiples of numbers--particularly those of 10's, 100's, and 1000's in order to make their work in division easier. The following exploration using the dittoed tables is designed to increase pupils' familiarity with multiples. The chart also serves as a means of demonstrating to children the rapidly increasing size of products of a number and a multiple of 10.

Exploration: (Referring to the table on the page entitled Working with Multiples of 10 and 100 in the pupil text)

What kind of a table is this? (Multiplication)

How do you know? (There is a multiplication sign in the upper left corner.)

Consider the numbers across the top of the table. What do those numbers have in common? (They are multiples of 10.)

We know that this is a multiplication table and that the numbers at the top of the table are multiples of 10.

See that "40" is written in the square which is the intersection of the "2-row" and the "20-column". What can we call the "40". (Product)

40 is the product of what two numbers? (2 and 20)

40 is the product of 2 and what multiple of 10? (20)

Find 420 in the table. 420 is the product of what two numbers (6 and 70)

Let's fill out the "4-row" together.

Now complete the table. You can do this easily if you know how to multiply a number and a multiple of 10.

After the table is completed, discuss it with the pupils as suggested in the pupil text. After pupils have finished Exercise Set 10, have the second duplicated table completed and have a similar discussion. During this, you may want to emphasize the relations between multiplying by 10 and multiplying by 100.

WORKING WITH MULTIPLES OF 10 AND 100

Copy the table and complete it.

x	10	20	30	40	50	60	70	80	90	100
1	10	20	30	40	50	60	70	80	90	100
2	20	40	60	80	100	120	140	160	180	200
3	30	60	90	120	150	180	210	240	270	300
4	40	80	120	160	200	240	280	320	360	400
5	50	100	150	200	250	300	350	400	450	500
6	60	120	180	240	300	360	420	480	540	600
7	70	140	210	280	350	420	490	560	630	700
8	80	160	240	320	400	480	560	640	720	800
9	90	180	270	360	450	540	630	720	810	900
10	100	200	300	400	500	600	700	800	900	1,000

Study the table you have just completed. How did you know to write 1000 in the lower right hand box?

How can this table be used to find the unknown factor in a division example?

Look at this example.

$$150 \div 3 = n$$

We think: $3 \times n = 150$. In the table, find the "3-row" and follow it until you see 150. Then look up the column and find the other factor, 50. Thus, $3 \times \underline{50} = 150$. So, $150 \div 3 = \underline{50}$.

Exercise Set 10

Find n in each of these.

1. 540 ÷ 9 = n (60)

2. 270 ÷ 3 = n (90)

3. 600 ÷ 10 = n (60)

4. 720 ÷ 8 = n (90)

5. 490 ÷ 7 = n (70)

6. 350 ÷ 5 = n (70)

7. 180 ÷ 6 = n (30)

8. 210 ÷ 3 = n (70)

9. 640 ÷ 8 = n (80)

10. 400 ÷ 5 = n (80)

11. 120 ÷ 2 = n (60)

12. 810 ÷ 9 = n (90)

13. 360 ÷ 9 = n (40)

14. 540 ÷ 6 = n (90)

15. 240 ÷ 4 = n (60)

16. 400 ÷ 5 = n (80)

Exercise Set 11

x	100	200	300	400	500	600	700	800	900	1000
1	100	200	300	400	500	600	700	800	900	1000
2	200	400	600	800	1000	1200	1400	1600	1800	2000
3	300	600	900	1200	1500	1800	2100	2400	2700	3000
4	400	800	1200	1600	2000	2400	2800	3200	3600	4000
5	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
6	600	1200	1800	2400	3000	3600	4200	4800	5400	6000
7	700	1400	2100	2800	3500	4200	4900	5600	6300	7000
8	800	1600	2400	3200	4000	4800	5600	6400	7200	8000
9	900	1800	2700	3600	4500	5400	6300	7200	8100	9000
10	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000

After you complete this table, your teacher will discuss it with you.

Find n in the following examples. Use the table you have just completed.

1. 1500 ÷ 5 = n (300)
2. 4900 ÷ 7 = n (700)
3. 6000 ÷ 6 = n (1,000)
4. 3200 ÷ 4 = n (800)
5. 7200 ÷ 8 = n (900)
6. 900 ÷ 3 = n (300)
7. 2700 ÷ 9 = n (300)
8. 10,000 ÷ 10 = n (1,000)
9. 5600 ÷ 7 = n (800)
10. 2400 ÷ 8 = n (300)

Exercise Set 12

Using the tables you just completed, find the unknown factor in each of these mathematical sentences.

1. 80 ÷ 2 = n (40)
2. 280 ÷ 7 = n (40)
3. 5400 ÷ 9 = p (600)
4. 6400 ÷ 8 = s (800)
5. 3500 ÷ 5 = m (700)
6. 490 ÷ 7 = r (70)
7. 810 ÷ 9 = n (90)
8. 320 ÷ 4 = p (80)
9. 270 ÷ 3 = s (90)
10. 1400 ÷ 2 = r (700)
11. 6300 ÷ 7 = n (900)
12. 4200 ÷ 6 = s (700)
13. 640 ÷ 8 = n (80)
14. 270 ÷ 9 = m (30)
15. 6300 ÷ 9 = r (700)
16. 4000 ÷ 8 = m (500)
17. 450 ÷ 5 = n (90)
18. 420 ÷ 7 = s (60)
19. 1200 ÷ 4 = t (300)
20. 5000 ÷ 10 = p (500)

Exploration:

Look at the examples.

1. $3 \times \underline{(3)} = 9$ $\quad 3 \times \underline{(30)} = 90$ $\quad 3 \times \underline{(300)} = 900$
2. $7 \times \underline{(8)} = 56$ $\quad 7 \times \underline{(80)} = 560$ $\quad 7 \times \underline{(800)} = 5600$
3. $8 \times \underline{(9)} = 72$ $\quad 8 \times \underline{(90)} = 720$ $\quad 8 \times \underline{(900)} = 7200$

> As you work the examples in row 1 above, ask the following questions.
>
> 3 times what number equals 9?
>
> 3 times what multiple of 10 equals 90?
>
> 3 times what multiple of 100 equals 900?
>
> As the children give the answer, write it on the chalkboard. Ask the same kind of questions for rows 2 and 3.
>
> When the examples have been worked, discuss them in this manner.

Look at the first example in each row.

Now look at the second example in each row.

Do you see any relationship between the two? (The answer to the second example is 10 times the first.)

How are the products related? (The second product is 10 times the first product.)

In the first example, you used your multiplication facts. How can the first example help you with the second one? (I can think of 3 and 9 to help me with 3 and 90, etc.)

Exercise Set 13

Copy each row of exercises below. Complete the blanks so that each mathematical sentence is true.

	Use the largest whole number.	Use the largest multiple of 10.	Use the largest multiple of 100.
1.	(a) 4 × 3 = 12	(b) 4 × 30 = 120	(c) 4 × 300 = 1200
2.	(a) 6 × 6 = 36	(b) 6 × 60 = 360	(c) 6 × 600 = 3600
3.	(a) 8 × 3 = 24	(b) 8 × 30 = 240	(c) 8 × 300 = 2400
4.	(a) 9 × 5 = 45	(b) 9 × 50 = 450	(c) 9 × 500 = 4500
5.	(a) 5 × 6 = 30	(b) 5 × 60 = 300	(c) 5 × 600 = 3000
6.	(a) 3 × 9 = 27	(b) 3 × 90 = 270	(c) 3 × 900 = 2700
7.	(a) 7 × 8 = 56	(b) 7 × 80 = 560	(c) 7 × 800 = 5600
8.	(a) 4 × 8 = 32	(b) 4 × 80 = 320	(c) 4 × 800 = 3200

Exercise Set 14

1. Copy and complete with the correct multiple of 10.

Example: 70 × 5 = 350

a. 70 × 6 = 420
b. 8 × 60 = 480
c. 30 × 9 = 270
d. 80 × 3 = 240
e. 2 × 90 = 180
f. 90 × 9 = 810
g. 50 × 8 = 400
h. 30 × 6 = 180
i. 7 × 30 = 210
j. 40 × 6 = 240

2. Copy and complete with the correct multiple of 100.

Example: 400 × 4 = 1600

a. 500 × 3 = 1500
b. 400 × 6 = 2400
c. 4 × 800 = 3200
d. 700 × 7 = 4900
e. 200 × 8 = 1600
f. 900 × 5 = 4500
g. 9 × 800 = 7200
h. 800 × 6 = 4800
i. 900 × 7 = 6300
j. 6 × 600 = 3600

3. Copy and complete with the correct multiple of 10 or 100.

Example: 80 × 6 = 480

a. 7 × 900 = 6300
b. 700 × 4 = 2800
c. 900 × 5 = 4500
d. 90 × 3 = 270
e. 10 × 600 = 6000
f. 800 × 2 = 1600
g. 700 × 9 = 6300
h. 800 × 8 = 6400
i. 7 × 800 = 5600
j. 500 × 5 = 2500

Exploration:

$\underline{3} \times 6 < \underline{19}$	$\underline{3}0 \times 6 < \underline{19}7$	$\underline{3}00 \times 6 < \underline{19}74$
$\underline{4} \times 5 < \underline{22}$	$\underline{4}0 \times 5 < \underline{22}5$	$\underline{4}00 \times 5 < \underline{22}56$
$\underline{5} \times 7 < \underline{39}$	$\underline{5}0 \times 7 < \underline{39}2$	$\underline{5}00 \times 7 < \underline{39}28$

Look at the examples on the chalkboard.

What is the largest whole number times 6 that is not greater than 19? (3 because: 3 × 6 = 18, 4 × 6 = 24, and 24 is greater than 19.)

What is the largest <u>multiple of ten</u> times 6 that is not greater than 197? (30 because: 30 × 6 = 180, 40 × 6 = 240, and 240 is greater than 197.)

What is the largest <u>multiple of one hundred</u> times 6 that is not greater than 1974? (300 because: 300 × 6 = 1800, 400 × 6 = 2400, and 2400 is greater than 1974.)

> Ask the same kind of questions for rows 2 and 3.
>
> When the examples have been worked, discuss them in this manner.

In row 1, do you see any relationship among the unknown factors?

How can the result of the first example help you with the second and third?

> Discuss rows 2 and 3 similarly. It would be valuable for the teacher to have pupils tell how they find the largest multiple of ten and one hundred.

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Exercise Set 15

Copy each row of exercises below. Complete the blanks so that each mathematical sentence is true.

	Use the largest whole number.	Use the largest multiple of 10.	Use the largest multiple of 100.
1.	(a) $\underline{4} \times 6 < 25$	(b) $\underline{40} \times 6 < 252$	(c) $\underline{400} \times 6 < 2526$
2.	(a) $\underline{7} \times 4 < 31$	(b) $\underline{70} \times 4 < 315$	(c) $\underline{700} \times 4 < 3158$
3.	(a) $\underline{3} \times 9 < 28$	(b) $\underline{30} \times 9 < 283$	(c) $\underline{300} \times 9 < 2834$
4.	(a) $\underline{5} \times 8 < 44$	(b) $\underline{50} \times 8 < 446$	(c) $\underline{500} \times 8 < 4465$
5.	(a) $\underline{8} \times 3 < 26$	(b) $\underline{80} \times 3 < 263$	(c) $\underline{800} \times 3 < 2639$
6.	(a) $\underline{9} \times 8 < 76$	(b) $\underline{90} \times 8 < 765$	(c) $\underline{900} \times 8 < 7657$
7.	(a) $\underline{7} \times 8 < 60$	(b) $\underline{70} \times 8 < 600$	(c) $\underline{700} \times 8 < 6000$
8.	(a) $\underline{6} \times 7 < 45$	(b) $\underline{60} \times 7 < 456$	(c) $\underline{600} \times 7 < 4568$

Exercise Set 16

Copy each row of exercises below. Complete the blanks so that each mathematical sentence is true.

	Use the largest whole number.	Use the largest multiple of 10.	Use the largest multiple of 100.
1.	(a) $\underline{3} \times 7 < 23$	(b) $\underline{30} \times 7 < 238$	(c) $\underline{300} \times 7 < 2385$
2.	(a) $6 \times \underline{9} = 54$	(b) $6 \times \underline{90} = 540$	(c) $6 \times \underline{900} = 5400$
3.	(a) $\underline{4} \times 5 < 21$	(b) $\underline{40} \times 5 < 219$	(c) $\underline{400} \times 5 < 2197$
4.	(a) $5 \times \underline{7} < 37$	(b) $5 \times \underline{70} < 375$	(c) $5 \times \underline{700} <$ 37[illegible]9
5.	(a) $\underline{7} \times 7 = 49$	(b) $\underline{70} \times 7 = 490$	(c) $\underline{700} \times 7 = 4900$
6.	(a) $8 \times \underline{9} < 78$	(b) $8 \times \underline{90} < 782$	(c) $8 \times \underline{900} < 7828$
7.	(a) $\underline{9} \times 7 < 65$	(b) $\underline{90} \times 7 < 654$	(c) $\underline{900} \times 7 < 6547$
8.	(a) $8 \times \underline{6} < 50$	(b) $8 \times \underline{60} < 500$	(c) $8 \times \underline{600} < 5000$

Exercise Set 17

1. Complete with the largest multiple of 10 that may be used to make the sentence true.

a. 20 × 5 < 103

b. 30 × 6 < 191

c. 30 × 7 < 220

d. 4 × 40 < 175

e. 5 × 60 < 311

f. 8 × 60 < 500

g. 70 × 9 < 650

h. 80 × 7 < 583

i. 9 × 80 < 750

j. 90 × 6 < 549

2. Complete with the largest multiple of 100 that may be used to make the sentence true.

a. 400 × 6 < 2500

b. 100 × 5 < 600

c. 200 × 4 < 1000

d. 6 × 300 < 2000

e. 7 × 500 < 4000

f. 4 × 700 < 3000

g. 500 × 9 < 4852

h. 300 × 3 < 1000

i. 4 × 400 < 1846

j. 2 × 900 < 1946

3. Complete with the largest multiple of 100 that may be used to make the sentence true. If this is not possible then use the largest multiple of 10.

a. 8 × 600 < 5000

b. 500 × 4 < 2196

c. 7 × 80 < 568

d. 6 × 90 < 596

e. 300 × 8 < 2502

f. 4 × 70 < 304

g. 6 × 700 < 4507

h. 50 × 8 < 412

i. 800 × 4 < 3597

j. 9 × 900 < 8200

BECOMING SKILLFUL IN DIVIDING

Objective: To help children use a division algorism more skillfully

Vocabulary: Partial quotient

Teaching Suggestions:

Review ways of finding an unknown factor starting with such an example as $n \times 5 = 365$. Note that we also can write this: $365 \div 5 = n$. Ask questions which will suggest that pupils think about multiples of 10 and 100. For example:

Is $5 \times 10 < 365$?

Is $5 \times 100 < 365$?

Is $5 \times 100 > 365$?

Then ask what does this tell us about the quotient? (We need to think of the largest multiple of 10 so that when it is multiplied by 5, the product is no greater than 365.)

Help children decide what this multiple of 10 is to be. For example:

$7 \times 5 = 35$ so $70 \times 5 = 350$.

$8 \times 5 = 40$ so $80 \times 5 = 400$.

Then help them with whichever form (as completed at the right) is being used by your class to record their thinking. After recording the partial quotient, 70, and subtracting 350 from 365, ask if the work is completed. If it isn't, what must be done? Continue by thinking: What is the largest multiple of 5 which is equal to or less than 15. Record as before. Discuss the result and how it can help us to rewrite our first sentence. Write the sentences:

$73 \times 5 = 365$ (or $365 \div 5 = 73$)

Recall with them how we check our computation by multiplying 73 by 5.

Form I:

```
   73
    3
   70
5 ) 365
    350
     15
     15
      0
```

Form II:

```
5 ) 365 |
    350 | 70
     15 |
     15 |  3
      0 | 73
```

Choose other examples (be sure the remainder is 0) and discuss them with the children. Then proceed to material in the pupil text.

NOTE: We want chil[illegible] write 0 for the remainder in such examples [illegible]ese. This is in preparation for work to follow when we express the result of dividing using the form $a = (b \times n) + r$ in the section "Finding Quotients and Remainders".

BECOMING SKILLFUL IN DIVIDING

We shall use what we know about multiples of numbers to learn more about dividing one number by another.

Suppose we are to find n in either of these sentences.

$$\begin{array}{ccccccc}
n & \times & 4 & = & 332 & \text{or} & 332 \div 4 = n \\
\uparrow & & \uparrow & & \uparrow & & \\
\text{Unknown Factor} & & \text{Known Factor} & & \text{Product} & &
\end{array}$$

To find n in either sentence we divide 332 by 4. We can use one of the forms below. You may select the one you would like to use. Use either Form I or Form II.

Form I: **Form II:**

$$\begin{array}{rcrl}
\underline{83} & & & \\
3 & & & \\
\underline{80} & & & \\
4\overline{)332} & & 4\overline{)332} & \\
\underline{320} & \longleftarrow (80 \times 4) \longrightarrow & \underline{320} & 80 \\
12 & & 12 & \\
\underline{12} & \longleftarrow (3 \times 4) \longrightarrow & 12 & \underline{3} \\
0 & & 0 & 83
\end{array}$$

Mathematical Sentence: $83 \times 4 = 332$ or $332 \div 4 = 83$.

We can check our answer:

$$\begin{array}{r}
83 \\
\underline{\times\ 4} \\
332
\end{array}$$

Exercise Set 18

Find n. Use either Form I or Form II. Check your answers.

1. $n \times 4 = 52$ (n = 13)
2. $n \times 6 = 84$ (n = 14)
3. $n \times 9 = 117$ (n = 13)
4. $5 \times n = 75$ (n = 15)
5. $7 \times n = 98$ (n = 14)
6. $n \times 4 = 84$ (n = 21)
7. $n \times 8 = 560$ (n = 70)
8. $5 \times n = 390$ (n = 78)
9. $n \times 9 = 837$ (n = 93)
10. $9 \times n = 135$ (n = 15)
11. $n \times 4 = 208$ (n = 52)
12. $7 \times n = 217$ (n = 31)
13. $3 \times n = 153$ (n = 51)
14. $n \times 9 = 828$ (n = 92)
15. $n \times 7 = 574$ (n = 82)
16. $7 \times n = 231$ (n = 33)
17. $8 \times n = 448$ (n = 56)
18. $4 \times n = 192$ (n = 48)
19. $n \times 7 = 595$ (n = 85)
20. $n \times 3 = 279$ (n = 93)

FINDING QUOTIENTS AND REMAINDERS

Objective: To help children understand the technique of division with remainder and the mathematical sentence which describes this division process $a = (b \times n) + r$ or $a = (n \times b) + r$ where a is the dividend, b is the divisor, n is the quotient, and r is the remainder.

Teaching Suggestions:

The pupils should be given practice similar to the following examples to stress understanding of mathematical sentences of the form $a = (b \times n) + r$.

$$37 = (7 \times n) + r$$
$$57 = (8 \times n) + r$$
$$89 = (n \times 9) + r$$

For each, pupils are to find n and r so that n will be the greatest whole number possible. In each instance, r then should be less than the "known" factor in the product expression.

Exploration:

We can use the division process to solve problems like this one.

Mr. Smith has 372 oranges which
he wants to pack into 5 crates.
How many can he put in each crate?
How many will he have left over?

As you guide children in solving this problem, lead them first to write the sentence:

$$372 = (5 \times n) + r .$$

Use one of the forms shown to find n and r. Rewrite the sentence as:

$$372 = (5 \times 74) + 2 .$$

Have the children interpret the 74 and the 2 in relation to the problem, and check their work.

$$\begin{array}{r} \underline{74} \\ \underline{4} \\ 70 \\ 5\,\overline{)372} \\ \underline{350} \\ 22 \\ \underline{20} \\ 2 \end{array} \qquad \text{or} \qquad \begin{array}{r|r} 5\,\overline{)372} & \\ \underline{350} & 70 \\ 22 & \\ 20 & 4 \\ \hline 2 & 74 \end{array}$$

This means that Mr. Smith would have 74 oranges in each crate with 2 remaining.

Using the results of either method we can write a mathematical sentence like this:

$$372 = (5 \times 74) + 2.$$

In our work, we call 5 the divisor, 74 the quotient, 372 the dividend, and 2 the remainder. The remainder is less than the divisor.

To check our work, we can multiply 5 and 74. Their product is 370. To this we add the remainder 2. This check may be shown like this:

$$\begin{array}{r} 74 \\ \underline{\times\ \ 5} \\ 370 \\ \underline{+\ \ 2} \\ 372 \end{array}$$

> You may wish to use other problems such as this one before the pupils study the text material. Be sure to select problems with remainder not 0.

FINDING QUOTIENTS AND REMAINDERS

We have used sentences like this

$$47 = (5 \times n) + r$$

in working with story problems.

We have seen how we can find the largest possible n and the smallest r in ways like these.

$$\begin{array}{rl}
 & 9 \leftarrow \text{quotient} \\
\text{divisor} \rightarrow 5 & \overline{)47} \leftarrow \text{dividend} \\
 & \underline{45} \\
 & 2 \leftarrow \text{remainder}
\end{array}
\qquad
\begin{array}{rr|l}
\text{divisor} \rightarrow & 5\overline{)47} & \leftarrow \text{dividend} \\
 & \underline{45} & \underline{9} \\
\text{remainder} \longrightarrow & 2 & 9 \leftarrow \text{quotient}
\end{array}$$

We have found that $47 = (5 \times 9) + 2$.

We can see that this sentence is true by thinking

$$47 = 45 + 2.$$

We can use these same ways to find quotients and remainders when we work with larger dividends.

Now look at this mathematical sentence.

$$437 = (n \times 9) + r$$

$$\begin{array}{rr}
 & \underline{48} \\
 & 8 \\
 & 40 \\
9 & \overline{)437} \\
 & \underline{360} \\
 & 77 \\
 & \underline{72} \\
 & 5
\end{array}
\begin{array}{c}
\\ \\ \\ \\
\longleftarrow (40 \times 9) \longrightarrow \\
\\
\longleftarrow (8 \times 9) \longrightarrow \\
\\
\end{array}
\begin{array}{r|r}
9\overline{)437} & \\
\underline{360} & 40 \\
77 & \\
\underline{72} & \underline{8} \\
5 & 48
\end{array}$$

Which number is the quotient? (48)

Which number is the dividend? (437)

Which number is the divisor? (9)

Which number is the remainder? (5)

Is the remainder less than the divisor? (Yes)

We have found that

$$437 = (48 \times 9) + 5.$$

We can check to see if the sentence is true by multiplying 48 and 9, and adding 5. Our answer should be 437.

$$\begin{array}{r} 48 \\ \times\ 9 \\ \hline 432 \\ +\ 5 \\ \hline 437 \end{array}$$

Exercise Set 19

A. Use either Form I or Form II to find n and r. Then rewrite the sentence using the numbers you found.

1. $600 = (n \times 7) + r$ $600 = (85 \times 7) + 5$
2. $138 = (n \times 9) + r$ $138 = (15 \times 9) + 3$
3. $213 = (7 \times n) + r$ $213 = (7 \times 30) + 3$
4. $450 = (n \times 8) + r$ $450 = (56 \times 8) + 2$
5. $271 = (n \times 3) + r$ $271 = (90 \times 3) + 1$
6. $107 = (3 \times n) + r$ $107 = (3 \times 35) + 2$
7. $230 = (n \times 7) + r$ $230 = (32 \times 7) + 6$
8. $162 = (n \times 6) + r$ $162 = (27 \times 6) + 0$
9. $738 = (9 \times n) + r$ $738 = (9 \times 82) + 0$
10. $200 = (n \times 6) + r$ $200 = (33 \times 6) + 2$
11. $372 = (n \times 9) + r$ $372 = (41 \times 9) + 3$
12. $725 = (8 \times n) + r$ $725 = (8 \times 90) + 5$
13. $373 = (n \times 9) + r$ $373 = (41 \times 9) + 4$
14. $288 = (n \times 8) + r$ $288 = (4 \times 72) + 0$
15. $451 = (n \times 8) + r$ $451 = (56 \times 8) + 3$

B. Use mathematical sentences to solve these problems. Express each answer in a complete sentence.

16. At camp, John made a collection of 176 small stones. He put the same number of stones in each of 4 small boxes. How many did he put in each box? How many were left over?

($176 = (n \times 4) + r$ $n = 44$
John put 44 stones in each box.)
(None left over)

17. There were 256 children visiting the Natural History Museum. Nine guides showed children around the museum. How many groups containing the same number of children could be formed? Are there any children left over?

($256 = (n \times 9) + r$ $n = 28$ $r = 4$
There would be 28 groups, with 4 children left over.)

Teaching Suggestions:

How can we find the answer in this example?

$4\overline{)965}$

First ask children the series of questions

$4 \times 10 = n$ (read "what number")

$4 \times 100 = n$

$4 \times 1000 = n$

$$\begin{array}{r} \underline{241} \\ 1 \\ 40 \\ \underline{200} \\ 4\overline{)965} \\ \underline{800} \\ 165 \\ \underline{160} \\ 5 \\ \underline{4} \\ 1 \end{array}$$

Then ask if we should use the largest multiple of 10, of 100, or of 1000 and how we can decide. Tell what multiple of 100 should be used. Guide their thinking by asking for products of 4 and 100, 4 and 200, etc., seeing that 800 is the largest multiple.

After recording 200 and subtracting 800, continue by determining 4 times what multiple of 10 is not greater than 165. Ask for products of 4 and 10, 4 and 20, 4 and 40, etc. Decide what to use.

or

$$\begin{array}{r|r} 4\overline{)965} & \\ \underline{800} & 200 \\ 165 & \\ \underline{160} & 40 \\ 5 & \\ \underline{4} & 1 \\ \hline 1 & 241 \end{array}$$

Record the 40 as a partial quotient and subtract 160. Now ask if they have completed the computation and ask why they should continue. Complete the problem.

Ask how we can find the quotient, helping them see that the quotient is the sum of the partial quotients. When work is completed, ask children to name the quotient and the remainder. Also, write the mathematical sentence in the form:

$$965 = (241 \times 4) + 1$$

Ask how we can be certain this is true. Suggest that if we find the product of 241 and 4 and then add 1, the result should be 965.

Use other examples.
Select one such as $376 = (n \times 4) + r$.
Continue with computation as before.
Observe that the remainder is 0.
That is $376 = (94 \times 4) + 0$
We can shorten this to $376 = 94 \times 4$.
Bring out the idea that 94 and 4 are factors of 376 and how we know this is true.

Exercise Set 20

1. Name the divisor, dividend, quotient, and remainder for each of the following.

a.

$$\begin{array}{lr}
\text{quotient} \rightarrow & \underline{32} \\
 & 2 \\
 & 30 \\
\text{divisor} \rightarrow 8 & \overline{)258} \\
\text{dividend} \nearrow & \underline{240} \\
 & 18 \\
 & \underline{16} \\
\text{remainder} \nearrow & 2
\end{array}$$

b.

$$\begin{array}{lr|l}
\text{divisor} \nearrow 6 & \overline{)732} & \\
\text{dividend} \nearrow & \underline{600} & 100 \\
 & 132 & \\
 & \underline{120} & 20 \\
 & 12 & \\
 & \underline{12} & \underline{2} \\
\text{remainder} \rightarrow & 0 & 122 \nwarrow \text{quotient}
\end{array}$$

2. Use a number to complete the following so they are true statements.

a. If the remainder is <u>0</u>, then the diviso[illegible]s a factor of the dividend.

b. If the remainder is not <u>0</u>, then the divisor is not a factor of the dividend.

c. If $1026 = (7 \times 146) + 4$, then the remainder is <u>4</u>.

d. If $842 = (6 \times n) + r$ with $r < 6$, then $n =$ <u>140</u>, and $r =$ <u>2</u>.

3. Divide the first number by the second. Then write the mathematical sentence. **For example,** 258 divided by 8 gives a quotient 32 and a remainder 2. The mathematical sentence is $258 = (32 \times 8) + 2$. Check the last 5 sentences.

a. 512 by 8
$512 = 64 \times 8$

b. 382 by 7
$382 = (54 \times 7) + 4$

c. 251 by 4
$251 = (62 \times 4) + 3$

d. 456 by 6
$456 = 76 \times 6$

e. 812 by 9
$812 = (90 \times 9) + 2$

f. 756 by 7
$756 = 108 \times 7$

g. 527 by 3
$527 = (175 \times 3) + 2$

h. 805 by 4
$805 = (201 \times 4) + 1$

i. 927 by 9
$927 = 103 \times 9$

j. 625 by 5
$625 = 125 \times 5$

k. 859 by 3
$859 = (286 \times 3) + 1$

l. 604 by 6
$604 = (100 \times 6) + 4$

m. 2597 by 7
$2597 = 371 \times 7$

n. 2001 by 5
$2001 = (400 \times 5) + 1$

o. 7024 by 8
$7024 = 878 \times 8$

FINDING MULTIPLES OF LARGER NUMBERS

Objective: To help children acquire skill in finding multiples of larger numbers

Materials: Duplicated table as on the next pupil page

Teaching Suggestions:

Have pupils fill in this table as they did earlier ones. Then discuss with them how the table can be used to find quotients. The completion of the table will serve as a review of the facts pupils learned in previous units.

In this section we are concerned with such product expressions as

$$20 \times 30,$$

$$50 \times 70,$$

$$200 \times 30, \text{ etc.}$$

After children have completed Exercise Set 21, use the following mathematical sentence to introduce further work with these multiples.

$$40 \times n < 983$$

Guide children to sense how they can use 4 and 9 as "helpers" to determine the largest multiple of 10 to use with 40 so that the product of 40 and n will be less than 983. In this connection, have them recall how they already have learned how to use 4 and 9 as "helpers" when dividing 98 by 40, for example.

Use further examples as needed. Be sure to include some like

$$30 \times n < 1314$$

in which one of the "helpers", 13, is named by a two-place numeral.

Then have children work independently on Exercise Set 22.

FINDING MULTIPLES OF LARGER NUMBERS

Copy and complete the following table.

x	10	20	30	40	50	60	70	80	90	100
10	100	200	300	400	500	600	700	800	900	1000
20	200	400	600	800	1000	1200	1400	1600	1800	2000
30	300	600	900	1200	1300	1800	2100	2400	2700	3000
40	400	800	1200	1600	2000	2400	2800	3200	3600	4000
50	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
60	600	1200	1800	2400	3000	3600	4200	4800	5400	6000
70	700	1400	2100	2800	3500	4200	4900	5600	6300	7000
80	800	1600	2400	3200	4000	4800	5600	6400	7200	8000
90	900	1800	2700	3600	4300	5400	6300	7200	8100	9000
100	1000	2000	3000	4000	5000	6000	7000	8000	9000	10,[illegible]0

Exercise Set 21

Use your table to find n.

1. 800 ÷ 20 = n (n = 40)
2. 2800 ÷ 40 = n (n = 70)
3. 2800 ÷ 70 = n (n = 40)
4. 20 × n = 1800 (n = 90)
5. n × 70 = 5600 (n = 80)
6. 70 × 90 = n (n = 6300)
7. 4500 ÷ 50 = n (n = 90)
8. n × 100 = 8000 (n = 80)
9. 60 × n = 5400 (n = 90)
10. 2700 ÷ 90 = n (n = 30)
11. n × 50 = 1500 (n = 30)
12. 80 × 80 = n (n = 6400)
13. 4900 ÷ 70 = n (n = 70)
14. 50 × n = 2000 (n = 40)
15. 80 × n = 7200 (n = 90)
16. 6000 ÷ 60 = n (n = 100)
17. 3600 ÷ 40 = n (n = 90)
18. 30 × n = 1800 (n = 60)
19. n × 90 = 6300 (n = 70)
20. n × 100 = 10,000 (n = 100)

Exercise Set 22

1. Complete with the largest multiple of 10 which makes the sentence true.

a. $\underline{30} \times 20 < 720$

b. $\underline{80} \times 10 < 836$

c. $\underline{10} \times 30 < 506$

d. $\underline{10} \times 50 < 918$

e. $20 \times \underline{20} < 432$

f. $\underline{50} \times 60 < 3290$

g. $\underline{40} \times 70 < 3040$

h. $\underline{90} \times 60 < 5500$

i. $\underline{60} \times 80 < 5000$

j. $90 \times \underline{70} < 6500$

k. $80 \times \underline{50} < 4700$

l. $50 \times \underline{60} < 3500$

2. Complete with the largest multiple of 100 which makes the sentence true.

a. $40 \times \underline{200} < 8442$

b. $20 \times \underline{200} < 5591$

c. $10 \times \underline{200} < 2146$

d. $\underline{200} \times 30 < 6723$

e. $\underline{500} \times 6 < 3290$

f. $\underline{900} \times 3 < 2872$

g. $50 \times \underline{700} < 36,012$

h. $\underline{600} \times 70 < 45,000$

i. $20 \times \underline{200} < 5640$

j. $70 \times \underline{300} < 26,500$

k. $80 \times \underline{700} < 60,000$

l. $90 \times \underline{800} < 75,000$

3. Find the largest multiple of 100 which makes the sentence true. If there is no multiple of 100, then find the largest multiple of 10.

a. $20 \times \underline{30} < 731$

b. $\underline{100} \times 46 < 4830$

c. $\underline{20} \times 30 < 742$

d. $30 \times \underline{400} < 12,200$

e. $50 \times \underline{500} < 26,200$

f. $40 \times \underline{60} < 2449$

g. $60 \times \underline{700} < 45,000$

h. $70 \times \underline{400} < 30,000$

i. $\underline{80} \times 90 < 7500$

j. $90 \times \underline{800} < 75,460$

USING DIVISORS THAT ARE MULTIPLES OF 10

Objective: To extend techniques in computation to include dividing by multiples of 10 which are less than 100

> Follow pupil exploration carefully. If you encounter difficulty in terminology, refer to earlier parts of the unit.

USING DIVISORS THAT ARE MULTIPLES OF 10

Exploration

We are going to learn to divide when the divisors are multiples of 10. Look at each of the examples below. Can you tell what was done in each example?

Example 1:

Divide 480 by 20.

$$
\begin{array}{rr c rr|l}
 & \underline{24} & & & & \\
 & 4 & & & & \\
 & 20 & & & & \\
20 & \overline{)480} & & 20 & \overline{)480} & \\
 & \underline{400} & \longleftarrow (20 \times 20) \longrightarrow & & \underline{400} & \underline{20} \\
 & 80 & & & 80 & \\
 & \underline{80} & \longleftarrow (4 \times 20) \longrightarrow & & \underline{80} & \underline{4} \\
 & 0 & & & 0 & 24
\end{array}
$$

$$480 = 20 \times 24$$

We think of n as the largest multiple of 10, so that $(n \times 20)$ is not greater than 480. (n is 20.)

We then think of n as the largest number so that $(n \times 20)$ is not greater than 80. (n is 4.)

We describe the results of the process by the mathematical sentence:

$$480 = (24 \times 20) + 0 \quad \text{or} \quad 480 = 24 \times 20.$$

We can check the work by multiplication:

$$\begin{array}{r} 24 \\ \underline{\times 20} \\ 480 \end{array}$$

Example 2:

Divide 9,285 by 40.

$$\begin{array}{r} \underline{232} \\ 2 \\ 30 \\ 200 \\ 40\,\overline{)9285} \\ \underline{8000} \\ 1285 \\ \underline{1200} \\ 85 \\ \underline{80} \\ 5 \end{array} \qquad\qquad \begin{array}{r|r} 40\,\overline{)9285} & \\ \underline{8000} & 200 \\ 1285 & \\ \underline{1200} & 30 \\ 85 & \\ \underline{80} & \underline{2} \\ 5 & 232 \end{array}$$

We think of n as the largest multiple of 100 so that (n × 40) is not greater than 9,285. (n is 200.)

Next, we think of n as the largest multiple of 10 so that (n × 40) is not greater than 1,285. (n is 30.)

Finally, we think of n as the largest number so that (n × 40) is not greater than 85. (n is 2.)

We describe the results of the process by the mathematical sentence

$$9,285 = (40 \times 232) + 5.$$

We can check our work by multiplication and addition.

$$\begin{array}{r} 232 \\ \underline{\times\ \ 40} \\ 9280 \\ \underline{+\ \ \ \ 5} \\ 9285 \end{array}$$

Exercise Set 23

A. For each of the following exercises, divide the first number by the second. Then write a mathematical sentence which describes how we can express the results.

1. 720 by 30
   $720 = 24 \times 30$
2. 840 by 20
   $840 = 42 \times 20$
3. 680 by 40
   $680 = 17 \times 40$
4. 570 by 10
   $570 = 57 \times 10$
5. 1160 by 40
   $1160 = 29 \times 40$
6. 990 by 90
   $990 = 11 \times 90$
7. 780 by 60
   $780 = 13 \times 60$
8. 3850 by 50
   $3850 = 77 \times 50$
9. 5810 by 70
   $5810 = 83 \times 70$
10. 5360 by 80
    $5360 = 67 \times 80$
11. 783 by 10
    $783 = (78 \times 10) + 3$
12. 1600 by 30
    $1600 = (53 \times 30) + 10$
13. 1956 by 20
    $1956 = (97 \times 20) + 16$
14. 1897 by 40
    $1897 = (47 \times 40) + 17$
15. 3162 by 50
    $3162 = (63 \times 50) + 12$
16. 5599 by 70
    $5599 = (79 \times 70) + 69$
17. 2600 by 60
    $2600 = (43 \times 60) + 20$
18. 8746 by 90
    $8746 = (97 \times 90) + 16$
19. 7543 by 80
    $7543 = (94 \times 80) + 23$
20. 5757 by 70
    $5757 = (82 \times 70) + 17$

B. Solve the following problems.

21. A shipping carton holds 20 books. How many cartons will be needed to ship an order of 900 books?
    ($900 = (n \times 20) + r$ $n = 45$
    There will be 45 cartons needed.)
22. An auditorium can seat 1680 persons. If each row seats 40 persons, how many rows are in this auditorium?
    ($1680 = (n \times 40) + r$ $n = 42$
    There are 42 rows of seats in the auditorium.)
23. How many trips must an elevator (capacity 20 persons) make to carry 254 people? (Hint: One trip may not carry a full load.)
    ($254 = (n \times 20) + r$ $n = 12$ $r = 14$.
    The elevator would make at least 13 trips.)
24. The room mothers are boxing candy to sell at the annual carnival. They bought 2,880 pieces of candy and each box will hold 30 pieces. How many boxes of candy do the room mothers have to sell?
    ($2880 = (30 \times n) + r$ $n = 96$
    The room mothers had 96 boxes of candy.)

A SHORTER FORM FOR DIVIDING

Objective: To develop a shorter division algorism

Teaching Suggestions:

Have the following example worked on the chalkboard, using either Form I or Form II of the algorism:

$$7\,\overline{)\,5934}$$

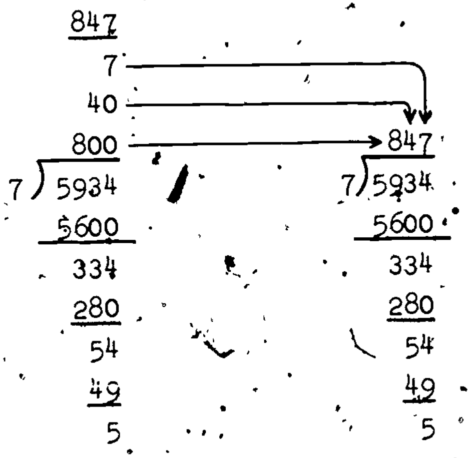
Give pupils whatever guidance is necessary to determine appropriate multiples of 100 and 10 to use in finding the partial quotients.

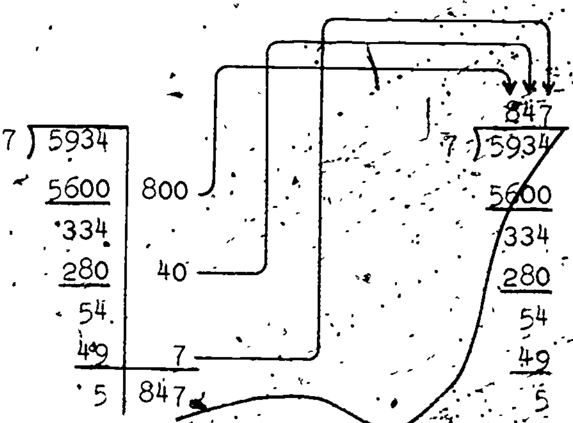
The example should be completed and the results interpreted in terms of an appropriate mathematical sentence:

$$5934 = (847 \times 7) + 5 .$$

Then ask the children to think how they might develop a shorter form for computing. In particular, ask them if they can see how they might use place value as a way to make it easier to record the partial quotients and the quotient.

Diagrams such as the ones illustrated below may be used to help the children see the kind of shorter form that is to be developed.





In your discussion emphasize why, in the shorter form, the 8 indicates 800, the 4 indicates 40, based on the principle of place value.

Continue this exploration using other examples as needed. Help children see that they can determine from the start the number of places there must be in the quotient numeral. Then discuss page 140 in the pupil text with the class.

After the pupils have completed Exercise Set 24, develop with them a similar shorter algorism for divisors such as 20, 30, 60, etc. Take into account the shorter form the children have been using with divisors less than 10.

A Word of Caution:

Children will differ in the time they are ready to move from Form I or Form II to a shorter algorism as developed here. Consequently, this material on A SHORTER FORM FOR DIVIDING will be appropriate for some children at one time and other children at another time.

A SHORTER FORM FOR DIVIDING

There is a shorter way to write your quotient in division. It will allow you to do your work more quickly.

Study the examples below.

a. Longer Form	b. Shorter Form
139	
9	
30	
100	139
6) 836	6) 836
600	600
236	236
180	180
56	56
54	54
2	2

In b, to show the partial quotient 100, we can write 1 in the hundred's place. Instead of writing 30, we can write 3 in the ten's place. Then we can write 9 in the one's place.

We describe the results of either process by the mathematical sentence

$$836 = (139 \times 6) + 2.$$

c. Longer Form

$$\begin{array}{r|l} 6\overline{)836} & \\ \underline{600} & 100 \\ 236 & \\ \underline{180} & 30 \\ 56 & \\ \underline{54} & \underline{9} \\ \hline 2 & 139 \end{array}$$

d. Shorter Form

$$\begin{array}{r} 139 \\ 6\overline{)836} \\ \underline{600} \\ 236 \\ \underline{180} \\ 56 \\ \underline{54} \\ 2 \end{array}$$

In d, to show the partial quotient 100, we can write 1 in the hundred's place. Instead of writing 30, we can write 3 in the ten's place. Then we can write 9 in the one's place.

We describe the results of either process by the mathematical sentence

$$836 = (139 \times 6) + 2.$$

What do you notice about b and d?
(They are shorter than a and c.)

Exercise Set 24

For each of the following, divide the first number by the second. Write a mathematical sentence to describe the result.

1. 963 by 3 963 = 321 X 3
2. 848 by 4 848 = 212 X 4
3. 499 by 3 499 = (166 X 3) + 1
4. 648 by 4 648 = 162 X 4
5. 4882 by 6 4882 = (813 X 6) + 4
6. 6896 by 8 6896 = 862 X 8
7. 4928 by 6 4928 = (821 X 6) + 2
8. 6524 by 9 6524 = (724 X 9) + 8
9. 7932 by 8 7932 = (991 X 8) + 4
10. 3654 by 4 3654 = (913 X 4) + 2

A SHORTER FORM FOR DIVIDING BY LARGER DIVISORS

Study the examples below.

a. Longer Form

$$\begin{array}{r}
\underline{261}\\
1\\
60\\
200\\
30\,\overline{)7833}\\
\underline{6000}\\
1833\\
\underline{1800}\\
33\\
\underline{30}\\
3
\end{array}$$

b. Shorter Form

$$\begin{array}{r}
261\\
30\,\overline{)7833}\\
\underline{6000}\\
1833\\
\underline{1800}\\
33\\
\underline{30}\\
3
\end{array}$$

In b, to show the partial quotient 200, we can write 2 in the hundred's place. Instead of writing 60, we can write 6 in the ten's place. Then we can write 1 in the one's place.

We can describe the results of either process by the mathematical sentence

$$7833 = (261 \times 30) + 3.$$

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c. Longer Form

$$\begin{array}{r|l} 30\overline{)7833} & \\ \underline{6000} & 200 \\ 1833 & \\ \underline{1800} & 60 \\ 33 & \\ \underline{30} & 1 \\ \hline 3 & 261 \end{array}$$

d. Shorter Form

$$\begin{array}{r} 261 \\ 30\overline{)7833} \\ \underline{6000} \\ 1833 \\ \underline{1800} \\ 33 \\ \underline{30} \\ 3 \end{array}$$

In d, to show the partial quotient 200, we can write 2 in the hundred's place. Instead of writing 60, we can write 6 in the ten's place. Then we can write 1 in the one's place.

We can describe the results of either process by the mathematical sentence

$$7833 = (261 \times 30) + 3.$$

What do you notice about examples b and d? (They are shorter than a and c.)

Find the quotient and remainder in each of these, using both ~~a longer~~ form and the shorter form.

$40\overline{)8153}$ $30\overline{)10517}$

For each example, did you get the same quotient and remainder using both forms? You should have!

Exercise Set 25

For each of the following, divide the first number by the second. Write a mathematical sentence to describe the result of the process.

1. 5820 by 10 5820 = 582 X 10
2. 9240 by 40 9240 = 231 X 40
3. 13,440 by 20 13,440 = 672 X 20
4. 17,550 by 30 17,550 = 585 X 30
5. 23,350 by 50 23,350 = 467 X 50
6. 58,980 by 60 58,980 = 983 X 60
7. 57,840 by 80 57,840 = 723 X 80
8. 40,680 by 90 40,680 = 452 X 90
9. 27,760 by 80 27,760 = 347 X 80
10. 21,000 by 50 21,000 = 420 X 50
11. 3,462 by 10 3,462 = (346 X 10) + 2
12. 18,464 by 20 18,464 = (923 X 20) + 4
13. 19,056 by 40 19,056 = (476 X 40) + 16
14. 27,291 by 70 27,291 = (389 X 70) + 61
15. 29,083 by 30 29,083 = (969 X 30) + 13
16. 32,240 by 60 32,240 = (537 X 60) + 20
17. 15,989 by 90 15,989 = (177 X 90) + 59
18. 42,750 by 80 42,750 = (534 X 80) + 30
19. 40,876 by 50 40,876 = (817 X 50) + 26
20. 31,452 by 70 31,452 = (449 X 70) + 22

Practice Exercises

1. Write each of the following as the product of two factors. Write 3 different product expressions for each number.

Example: 30 = 1 × 30 , 2 × 15 , 5 × 6

a) 52 = 1 × 52 , 2 × 26 , 4 × 13

b) 116 = 1 × 116 , 2 × 58 , 4 × 29

c) 128 = 2 × 64 , 4 × 32 , 8 × 16

d) 88 = 2 × 44 , 4 × 22 , 8 × 11

e) 176 = 2 × 88 , 4 × 44 , 8 × 22

f) 90 = 3 × 30 , 5 × 18 , 9 × 10

g) 81 = 1 × 81 , 3 × 27 , 9 × 9

h) 126 = 2 × 63 , 3 × 42 , 6 × 21

i) 110 = 2 × 55 , 5 × 22 , 10 × 11

2. Solve the following.

a) 8 × (9000 + 6) (72,048)

b) (32 + 78) - 41 (69)

c) 9 × 847 (7,623)

d) .6 + .45 + 1.7 + 8 (10.75)

e) (74 × 600) + (74 × 95) (51,430)

f) 835 - 585 (250)

g) 301 ÷ 7 (43)

h) 7 × 7 × 912 (44,688)

i) .61 + .09 + 8.5 + .48 (9.68)

j) 976 ÷ 8 (122)

3. Write the number that n represents.

a) 90 × 370 = n (33,300)

b) 49,003 - n = 39,936 (9,067)

c) n × 9 = 936 (104)

d) 887 + 875 + 699 - n = 0 (2,461)

e) n ÷ 9 = 98 (882)

f) 7 × n = 637 (91)

g) 835 - 257 = n (578)

h) (104 × 9) + n = 950 (14)

i) 97 × 8697 = n (843,609)

j) 2275 = (n × 35) + 0 (65)

4. Solve the following;

a) n ÷ 8 = 5632 (45,056)

b) 52 × (6000 + 40) = n (314,080)

c) 6408 = (8 × n) + 0 (801)

d) 70 × 490 = n (34,300)

e) 7 × n = 672 (96)

f) 32 + n + 41 = 162 (89)

g) n + 184 = 986 (802)

h) 503 = (6 × n) + 5 (83)

i) 764 = (34 × 22) + n (16)

j) 3 × 3 × 465 = n (4,185)

5) Solve:

a)	$997 = (33 \times n) + 7$	(30)
b)	$9076 \times 6 \times 6 = n$	(326,736)
c)	$5472 = (8 \times n) + 0$	(684)
d)	$164 = (41 \times 4) + n$	(0)
e)	$5838 = (6 \times n) + 0$	(973)
f)	$n = (7 \times 906) + 3$	(6,345)
g)	$6 \times 465 \times 3 = n$	(8,370)
h)	$48 \times 7080 = n$	(339,840)
i)	$97 \times 8697 = n$	(843,609)
j)	$2275 = (n \times 35) + 0$	(65)

6. Add

1) $\begin{array}{r}578\\4,549\\496\\\underline{27,083}\\32,706\end{array}$

2) $\begin{array}{r}6,324\\796\\39,137\\\underline{4,034}\\50,291\end{array}$

3) $\begin{array}{r}304\\76,451\\3,517\\\underline{25,064}\\105,336\end{array}$

4) $\begin{array}{r}29\\80\overline{)2320}\end{array}$

5) $\begin{array}{r}13\\50\overline{)650}\end{array}$

Subtract:

6) $\begin{array}{r}58,931\\\underline{6,336}\\52,595\end{array}$

7) $\begin{array}{r}6,719\\\underline{2,480}\\4,239\end{array}$

8) $\begin{array}{r}5,833\\\underline{3,097}\\2,736\end{array}$

9) $\begin{array}{r}121\\60\overline{)7260}\end{array}$

Multiply:

10) $\begin{array}{r}354\\\underline{26}\\9,204\end{array}$

11) $\begin{array}{r}836\\\underline{54}\\45,144\end{array}$

12) $\begin{array}{r}8235\\\underline{35}\\288,225\end{array}$

13) $\begin{array}{r}709\\\underline{61}\\43,249\end{array}$

14) $\begin{array}{r}126\\\underline{16}\\2,016\end{array}$

15) $\begin{array}{r}789\\\underline{56}\\44,184\end{array}$

Subtract:

16) $\begin{array}{r}5837\\\underline{2528}\\3309\end{array}$

17) $\begin{array}{r}25,813\\\underline{5,804}\\20,009\end{array}$

18) $\begin{array}{r}309\\8\overline{)2472}\end{array}$

19) $\begin{array}{r}408\\20\overline{)8160}\end{array}$

20) $\begin{array}{r}70\\60\overline{)4200}\end{array}$

Review

SET I

Part A

1. Write each of these as a decimal. Example a. is done for you.

   a) $\frac{7}{10}$ = .7
   b) $\frac{34}{100}$ = (.34)
   c) $16\frac{9}{10}$ = (16.9)
   d) $25\frac{13}{100}$ = (25.13)
   e) $4\frac{1}{10}$ =(4.1)
   f) $\frac{45}{10}$ = (4.5)
   g) $\frac{102}{100}$ = (1.02)
   h) $5\frac{16}{1000}$ = (5.016)
   i) $2\frac{10}{100}$ = (2.10)

2. Write the decimal numeral for each of these:

   a) (9 × 100) + (8 × 10) + (6 × 1) = (986)
   b) (3 × 1,000) + (4 × 100) + (2 × 10) + (5 × 1) = (3,425)
   c) (4 × 1,000) + (2 × 100) + (2 × 10) + (3 × 1) = (4,223)
   d) (9 × 10,000) + (3 × 1,000) + (1 × 100) + (7 × 10) + (4 × 1) = (93,174)
   e) (6 × 100,000) + (3 × 10,000) + (4 × 1,000) + (7 × 10) + (4 × 1) = (634,074)
   f) (5 × 100,000) + (8 × 10,000) + (9 × 1,000) + (6 × 10) = (589,060)
   g) (1 × 10,000) + (5 × 1,000) + (8 × 10) + (7 × 1) = (15,087)
   h) (8 × 10,000) + (9 × 10) + (4 × 1) = (80,094)

3. Which of these numbers are divisible by 10 ?

   a) 353
   b) 637
   c) 21
   [✓ d) 4,000]
   [✓ e) 30]
   f) 42
   [✓ g) 960]
   h) 16
   i) 462
   [✓ j) 5,800]
   [✓ k) 190]
   l) 382

   Which of these numbers are divisible by 5 ?

   a) 38
   [✓ b) 700]
   [✓ c) 90]
   [✓ d) 3055]
   [✓ e) 105]
   f) 77
   g) 1114
   [✓ h) 680]
   i) 53
   [✓ j) 215]
   k) 23
   [✓ l) 190]

Which of these numbers are divisible by 2 ?

[✓ a) 94]
[✓ b) 1112]
c) 423
[✓ d) 894]
[✓ e) 7,000]
f) 633
g) 201
[✓ h) 50]
[✓ i) 192]
j) 27
[✓ k) 1,128]
l) 729

4. Complete the following to make them true sentences.

a) 68 × 11 = 680 + (68)
b) 28 × 64 = 512 + (1280)
c) 74 × 14 = (74 × 7) + (74 × 7)
d) 571 × 318 = (500 × 318) + (70 × 318) + (1 × 318)
e) 74 × 386 = 21,000 + 5,600 + 420 + 1200 + 320 + 24

5. Use 2 as many times as you can as a repeated factor of each of these numbers. Example a is done for you.

a) 28 = 2 × 2 × 7
b) 16 = (2 × 2 × 2 × 2)
c) 24 = (2 × 2 × 2 × 3)
d) 14 = (2 × 7)
e) 20 = (2 × 2 × 5)
f) 42 = (2 × 3 × 7)
g) 22 = (2 × 11)
h) 6 = (2 × 3)
i) 12 = (2 × 2 × 3)
j) 32 = (2 × 2 × 2 × 2 × 2)

What do you notice about all of the factors above? (They are prime factors.)

6. In each of the following explain what the 4 represents. A sample problem is done for you.

a) In 242_{five} 4 represents 4 sets of five
b) In 40_{eight} (4 sets of eight)
c) In 104_{five} (4 sets of ones or 4 ones)
d) In 47 (4 sets of ten)
e) In 1024_{seven} (4 sets of one or 4 ones)
f) In 542_{six} (4 sets of six)
g) In 432_{eight} (4 sets of sixty four)

7. Write each of the following as decimal numerals.
   a) Twenty-six thousand eight hundred twelve (26,812)
   b) Forty thousand, three hundred sixty (40,360)
   c) Eight hundred fifty-seven thousand, ninety-one (857,091)
   d) Four million, seven hundred sixty-three thousand (4,763,000)
   e) One million, one thousand, one (1,001,001)

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. The Jackson School bought 7 new wall maps. Each map cost $9.95. What was the total cost of the maps? (7 × 9.95 = n
   n = 69.65 The total cost of the maps was $69.65.)
2. Jim had $3.25. Tom had 75 cents more than Jim. How much money did the two boys have together? (3.25 + .75 = t
   t = 4.00 4.00 + 3.75 = n n = 7.25 or (3.25 + .75) + 3.25 = n
   n = 7.25 Together the boys had $7.25.)
3. Joanne went to a party dressed as a witch. She paid 85 cents for black cloth for a dress, 72 cents for a broom, and 29 cents for a mask. How much did she pay for the entire costume? She gave the clerk five dollars. How much change did she get? (5.00 - (.85 + .72 + .29) = n n = 3.14
   or .85 + .72 + .29 = t t = 1.86 5.00 - 1.86 = n n = 3.14
   Joanne paid $1.86 for her costume. She should get $3.14 change from the clerk.)
4. The pupils in Peggy's class are making bookcovers. There were 26 books to cover. They had a dozen and a half sheets of colored paper. How many more sheets of paper will they need in order to have a sheet for each book? (26 - (12 + 6) = n
   or n + (12 + 6) = 26 n = 8 Peggy's class needed 8 more sheets of paper.)

5. The Hoover School was built in 1934. The Lincoln School was built in 1960. The Hoover School is how many years older than the Lincoln School? (1960 - 1934 = n n = 26 The Hoover School is 26 years older than the Lincoln School.)

6. There are 32 children in Mr. Lang's class. For a party each child received 4 cookies. How many cookies did the class have? (4 x 32 = c c = 128 The class had 128 cookies.)

Suggested Activities

Group Activity

Relays - Working with Multiples

The object of the game is to locate points named by multiples of the number on the number line. The first member of each team draws the line and locates the first point, for example using multiples of 7 he would locate and name 7. The next player in each team would go up to locate 14, the third player names 21, and so on. The team that can correctly name the most points in a determined time period wins. This may also be used for counting in other bases.

Individual Project

Prepare and show your class a magic trick with numbers. Tricks with numbers fall into three main groups--lightning calculations, predictions, or mind reading effects. You will find information about number tricks in many books about mathematics. One clue--try looking up some of the "mysteries of nine."

Review

SET II

Part A

1. Using the symbols >, <, or = make the following true sentences.

a) .40 = .4

b) .6 > .06

c) $\frac{34}{100}$ = .34

d) $\frac{5}{100}$ < .5

e) $\frac{54}{100}$ > .45

f) .64 < .7

g) $\frac{8}{10}$ > .65

h) $\frac{5}{100}$ = .05

i) .4 > .36

j) .3 < .40

2. Write these numerals in expanded notation.

a) 114 = (1 × 100) + (1 × 10) + (4 × 1)

b) 2,236 = (2 × 1,000) + (2 × 100) + (3 × 10) + (6 × 1)

c) 7,330 = (7 × 1,000) + (3 × 100) + (3 × 10) + (0 × 1)

d) 5,050 = (5 × 1,000) + (0 × 100) + (5 × 10) + (0 × 1)

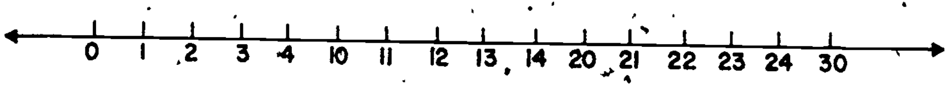
e) 6,803 = (6 × 1,000) + (8 × 100) + (0 × 10) + (3 × 1)

f) 49,527 = (4 × 10,000) + (9 × 1,000) + (5 × 100) + (2 × 10) + (7 × 1)

g) 827,666 = (8 × 100,000) + (2 × 10,000) + (7 × 1,000) + (6 × 100) + (6 × 10) + (6 × 1)

h) 412,305 = (4 × 100,000) + (1 × 10,000) + (2 × 1,000) + (3 × 100) + (0 × 10) + (5 × 1)

3. On the number line below, the points for 0 and 1 are labeled. Label the other points with base five numerals.



Fill in the blanks with the numerals 20_{five} and 24_{five} to make each of the following true sentences.

20_{five} is less than 24_{five} ; 24_{five} is greater than 20_{five} . 20_{five} is to the left of 24_{five} ; 24_{five} is to the right of 20_{five}.

4. A = {1, 3, 5, 7, 9, 11, 13}.

Sets T, S, E and P are subsets of A.

a) The members of Set T are divisible by 3.
T = {3, 9}
b) The members of Set S are divisible by 1.
S = {1, 3, 5, 7, 9, 11, 13}
c) The members of Set E are divisible by 2.
E = { }
d) The members of Set P are prime numbers.
P = {3, 5, 7, 11, 13}
e) Rewrite Set A and rename its members as product expressions. Call it set M.
M = {1 × 1, 1 × 3, 1 × 5, 1 × 7, 1 × 9 or 3 × 3, 1 × 11, 1 × 13}

B = {0, 2, 4, 6, 8, 10, 12, 14, 16}.

Sets F, R, Q and H are subsets of B.

a) The members of Set F are divisible by 2.
F = {2, 4, 6, 8, 10, 12, 14, 16}.
b) The members of Set R are divisible by 3.
R = {6, 12}
c) The members of Set Q are divisible by 1.
Q = {2, 4, 6, 8, 10, 12, 14, 16}
d) The members of Set H are prime numbers.
H = {2}
e) Write the members of the Set A∪B.
A∪B = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16}
f) Write the members of the Set A∩B.
A∩B = { }

5. Rename each of these decimals. The first one is done for you.

a) 6.84 = 6 ones + 8 tenths + 4 hundredths.

b) 12.62 = 12 ones + 6 tenths + 2 hundredths.

c) .07 = 0 ones + 0 tenths + 7 hundredths.

d) 1.01 = 1 ones + 0 tenths + 1 hundredths.

6. This is one way of changing a base five numeral to a base ten numeral.

114_{five} = (1 twenty five) + (1 five) + (4 ones)

$114_{five} = (1 \times 25) + (1 \times 5) + (4 \times 1)$

$114_{five} = 25 + 5 + 4$

$114_{five} = 34$

Using the same procedure change the following base five numerals to base ten numerals.

a) 23_{five} (13) c) 12_{five} (7)

b) 44_{five} (24) d) 123_{five} (38)

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. Roy bought four fish for his aquarium. He paid 60 cents for one, 28 cents for another, 35 cents for another, and 45 cents for the fourth one. How much money did he spend for all the fish? ($60 + 28 + 35 + 45 = n$ $n = 168$
Roy spent $1.68 for the fish.)

2. The Smith family went on a vacation. The first day they drove an average of 41 miles an hour. They traveled 9 hours. How many miles did they drive the first day?
($9 \times 41 = d$ $d = 369$ They had gone 369 miles.)

3. Janis and her sister made 75 pieces of fudge for a party. After the party only 19 pieces of fudge were left. How many pieces of fudge were eaten at the party? ($19 + n = 75$;
$75 - 19 = n$ $n = 56$. There were 56 pieces of fudge eaten at the party.)

4. Mrs. Gray has the milkman deliver 3 quarts of milk each day. The milk costs 26 cents a quart. What is the total milk bill for a week? ($3 \times 7 = 21$ $21 \times 26 = n$ $n = 546$ or $(3 \times 7) \times 26 = n$ $n = 546$ The milk bill is $5.46 for a week.)

5. Shirley has been saving quarters. She now has 10 quarters. If she changes them to nickels, how many will she get? ($5 \times 10 = m$ $m = 50$ Shirley will have 50 nickels.)

6. Mr. Norman pays 16 dollars a month for garage rent. How much rent does he pay in one year? ($16 \times 12 = n$ $n = 192$ Mr. Norman pays $192 rent in one year.)

Braintwisters

1. A frog is climbing out of a well twenty feet deep. He climbs four feet every day and slips down three feet every night. How long does it take the frog to get to the top? ($20 - 3 = 17$ 17 days.)

2. You have 8 sections of silver chain, each of four links. The cost of cutting open a link is 10¢ and of welding it together again is 25¢. What is the least you can pay to have the eight pieces joined together in a single chain? $(6 \times 25) + (6 \times 10) = 210$. $2.10

3. Sally had a piece of ribbon $4_{\underline{?}}$ inches long. She found another piece $4_{\underline{?}}$ inches long. Now she has $13_{\underline{?}}$ inches of ribbon. What number base was Sally using? (Base five $4 + 4 = 8$ $8 = 13_{five}$)

4. Two boys were comparing sticks. One boy had a stick $6_{\underline{?}}$ inches long. The other boy's stick was $3_{\underline{?}}$ inches longer or $12_{\underline{?}}$ inches long. What number base were they using? (Base seven $6 + 3 = 9$ $9 = 12_{seven}$)

Review

SET III

Part A

1. Write each of the following expressions using symbols.

Example: The number n increased by 6 = n + 6.

a) The number n increased by 8 $\quad$ n + 8
b) The number 7 multiplied by n $\quad$ n × 7
c) The sum of n and 9 $\quad$ n + 9
d) The number n decreased by 4 $\quad$ n - 4
e) The product of 6 and n $\quad$ 6 × n
f) The number n divided by 3 $\quad$ n ÷ 3
g) The number which is the result of 10 subtracted from n $\quad$ n - 10

2. What number is represented by each of the expressions in Problem 1 if n = 12?

a) 20 $\quad$ b) 84 $\quad$ c) 21 $\quad$ d) 8 $\quad$ e) 72
f) 4 $\quad$ g) 2

3. Answer each of the following with a complete sentence.

a) How many 4's are there in six 8's?
There are twelve 4's in six 8's.

b) How many 7's are there in three 14's?
There are six 7's in three 14's.

c) How many 6's are there in fifteen 4's?
There are ten 6's in fifteen 4's.

d) How many 3's are there in four 12's?
There are sixteen 3's in four 12's.

e) How many 8's are there in fourteen 4's?
There are seven 8's in fourteen 4's.

4. Find what number y represents in each of these. Tell what operation is needed to find y. Example a is done for you.

a) 108 + y = 144	y = 36	subtraction
b) 87 + 116 = y	y = 203	addition
c) 30 × 74 = y	y = 2,220	multiplication
d) y = 54 × 18	y = 972	multiplication
e) 2563 + y = 8,010	y = 5,447	subtraction
f) 58 × 867 = y	y = 50,286	multiplication
g) y - 2649 = 6763	y = 9,412	addition
h) 30,600 - y = 408	y = 30,192	subtraction

5. Name the first ten members of each of the following sets:

S = {The set of multiples of 100}
S = {100, 200, 300, 400, 500, 600, 700, 800, 900, 1000}
T = {The set of multiples of 1,000}
T = {1,000, 2,000, 3,000, 4,000, 5,000, 6,000, 7,000, 8,000, 9,000, 10,000}

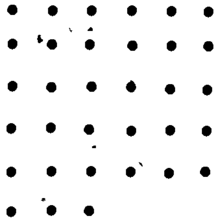
6. Complete these sentences with a multiple of 100 or 1000 needed to make them true sentences. Here are some possibilities. Example: 2,000 × 5 < 12,100

a) 1000 × 6 > 932
b) 9 × 400 < 40,121
c) 1,000 × 4 < 5,210
d) 70 × 200 < 15,316
e) 6 × 5,000 > 27,880
f) 100 × 33 = 3,300
g) 25 × 100 > 2,312
h) 2,000 × 140 < 293,000
i) 30 × 200 = 6,000
j) 200 × 25 = 5,000

7. Complete each of these. Example a is done for you.

a) .58 = 58 hundredths or 5 tenths plus 8 hundredths
b) .33 = 33 hundredths or 3 tenths plus 3 hundredths
c) .07 = 7 hundredths or 0 tenths plus 7 hundredths
d) .70 = 70 hundredths or 7 tenths plus 0 hundredths
e) .09 = 9 hundredths or 0 tenths plus 9 hundredths
f) .99 = 99 hundredths or 9 tenths plus 9 hundredths

8. How many dots are there in this diagram? Write the answer in each of the following number bases.



a) Base ten 33
b) Base five 113
c) Base six 53
d) Base four 201
e) Base nine 36
f) Base seven 45
g) Base eight 41

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. Mark said, "Tonight I am going to sleep 9 hours and 30 minutes. How many minutes will Mark sleep? $(9 \times 60) + 30 = n$
$n = 570$; $(9 \times 60) = t$ $t = 540$ $540 + 30 = n$ $n = 570$
Mark will sleep 570 minutes.

2. An army division has 345 platoons. There are 38 soldiers in each platoon. How many soldiers are there in the division?
$345 \times 38 = d$ $d = 13,110$ There are 13,110 soldiers in the division.

3. Mr. Jones bought 12 gallons of gasoline. He paid 33 cents a gallon. How much money did he spent for gasoline?
$.33 \times 12 = n$ $n = 3.96$ Mr. Jones spent $3.96 for gasoline.

4. Mary and Martha were selling greeting cards at 50 cents a box. The first day Mary sold 16 boxes and Martha sold 10 boxes. How much money did they make altogether that day? (50 x 16) + (50 x 10) = n or (16 + 10) x 50 = n
n = 1300 Mary and Martha made $13.00 altogether.)
5. There were two fifth grade classes in the Marshall School. There were 57 fifth grade pupils in the two classes. 23 of these were girls. How many boys were there? 23 + n = 57
or 57 - 23 = n n = 34 There were 34 boys in the two 5th grade classes.
6. Dick rides his bicycle to and from school in 10 minutes. He walks to and from school in 26 minutes. How much time will he save riding his bicycle to school all week?
(26 - 10) x 5 = y or 26 - 10 = 16, 16 x 5 = y, y = 80
Dick will save 80 minutes each week.

Suggested Activities

Group Project

Column Relays - Have the class choose teams and form team columns facing the board. A dittoed sheet of problems is handed to the first person in line. He moves to the board, reads and works the first problem then returns the problem sheet to the second person in line as he moves to the rear of the line. Each person moves up, works his problem, and returns to line until all members have had a turn. One point is scored for each correct answer.

Example: 16 x _n_ = 212 or 325 30 = _10_ + 25

Other questions may be given on:

a) writing expanded notations

b) changing to other bases

c) writing decimals as fractions and vice versa.

Chapter 4

CONGRUENCE OF COMMON GEOMETRIC FIGURES

PURPOSE OF UNIT

The purpose of this unit is:

1. To review geometric concepts and terms introduced earlier in the fourth grade chapter, Recognition of Common Geometric Figures.

2. To achieve familiarity with the intuitive concept of congruence of geometric figures, particularly as applied to line segments, triangles, and angles.

3. To gain facility in using compass and straightedge in copying and comparing such simple figures as line segments, triangles, and angles.

MATHEMATICAL BACKGROUND

The fact that every object which we see has size and shape suggests that the study of geometry be begun as early as possible in the child's school life. In previous units the child is made aware of representations of geometric figures in his environment. He recognizes the models of some geometric figures and can name them.

We believe that the child can now come to greater understanding and greater enjoyment of his environment through more discriminating observation. He will be provided with guidelines for productive thinking about the figures with which he is now familiar by means of exploratory discussions and developmental exercises. Another major objective is to develop ability to read mathematical material independently. In the preparation of this material, care has been taken to foster achievement of this goal.

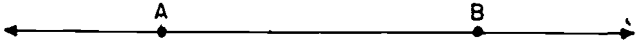
A basic geometric concept developed in this unit is the concept of congruence. Pupils will learn to recognize congruent geometric figures (figures of the same size and shape) by tracing one figure on a sheet of thin paper and determining whether this tracing will fit exactly on another geometric figure. They will learn that two triangles are congruent to each other when three sides of one triangle are congruent to three sides of the other triangle. This is used in copying a triangle with compass and straightedge. Congruent angles will be discussed, using first the method of tracing and then copying an angle using the straightedge and compass. The same two methods will be used to explore inequalities in size of angles.

Even if your last exposure to mathematics was in your early high school years, we think you will enjoy the teaching of informal geometry. It is an intuitive approach and an inductive development of some of the basic understandings and skills of geometry. We do <u>not</u> propose that pupils at this level study a set of formal proofs to reach generalizations about common geometric figures. This material is planned to provide opportunity for observation of common figures and for reaching generalizations about them as a result of this observation.

A geometric figure is a set of points. We know that we cannot make a point on a piece of paper since a mathematical point has no size at all. What we can make is a model or a picture of the point. When we draw a side of a triangle we are drawing a model of this set of points. In this text when we say, "Look at the triangle," we really mean, "Look at this model of the triangle."

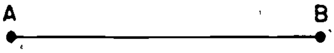
A line (the term "line" means "straight line") is a particular set of points in space with certain properties. One important property is that through any two different points in space there is exactly one line. A second important property is that a line has no end points. We represent a line by a drawing such as this:

If we wish to give it a name we label two points on the line, for example,



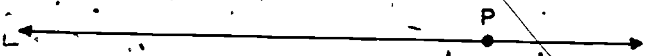
and call it the line AB, written $\overleftrightarrow{AB}$ or the line BA, written $\overleftrightarrow{BA}$. Observe that the order of the letters A and B is immaterial when we are talking about a line.

A segment is the set of points on a line consisting of two points called end-points, and all the points between. We represent a segment by a drawing such as this:

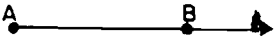


and we name it "segment AB" or "segment BA", written $\overline{AB}$, or $\overline{BA}$. Observe that the order of the letters A and B is immaterial when we are talking about a line segment.

A point on a line, such as point P,



separates the line into three sets of points: the set consisting of the point P and two other sets of points called half lines. The point P is not in either half line. We call a set of points consisting of a half line together with point P a ray. We indicate a ray with endpoint A like this,

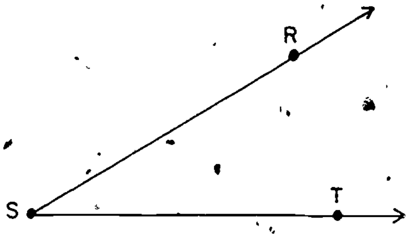


and we name it ray AB, written $\overrightarrow{AB}$, writing first the letter which names the endpoint. It is clear that a ray has only one endpoint and that ray AB is different from ray BA. A model of ray BA looks like this:

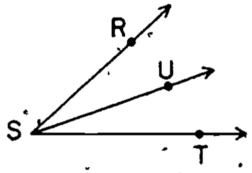
Observe that the order of the letters A and B is very important when we are talking about a ray. We need to use the words line, segment, and ray carefully.

Any flat surface such as the top of a desk or the wall of a room suggests the idea of a plane. Like a line, a plane is thought of as being unlimited in extent. We think of a plane as containing many points and many lines. Just as a line is a set of points that has certain properties, so a plane is a set of points that has certain properties. One important property of a plane is that any three points not on the same line are in one and only one plane. We have seen that a point separates a line into three sets of points, and in the same manner a line separates a plane into three sets of points: the set consisting of the points of the line itself and two other sets of points called half planes. The line of separation is not in either half plane.

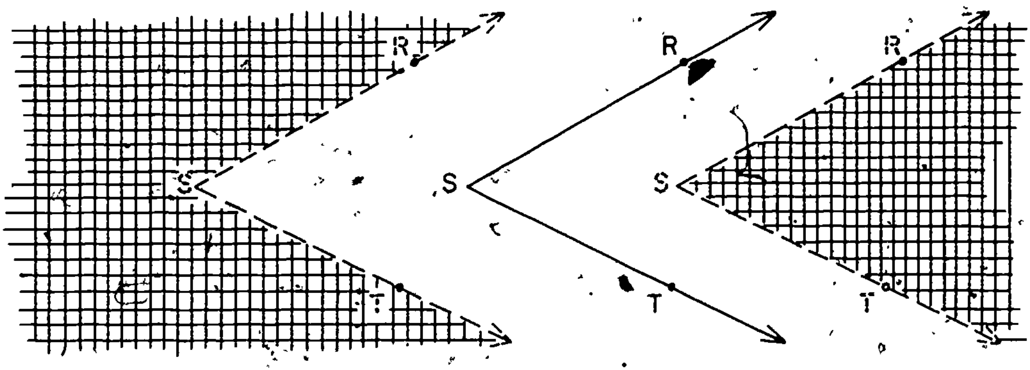
An <u>angle</u> is a set of points consisting of two rays not on the same line but with a common endpoint. We represent an angle by a drawing such as this:



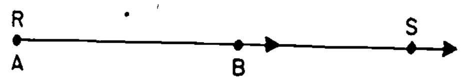
We name this angle: ∠RST or ∠TSR or ∠S. (Many students have suggested the symbol $\overrightarrow{\angle}$ RST, but ∠RST or one of the other variations is quite standard.) It is very important to observe that the endpoint, S, of the rays is named second in both ∠RST and in ∠TSR. On the other hand, the order of R and T is immaterial. If there is no chance for misunderstanding, we just write ∠S. This abbreviation could not be used for a drawing such as



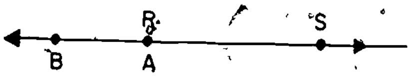
An angle separates a plane into three sets of points: the set consisting of the set of points of the angle itself and two other sets of points called the <u>exterior</u> of the angle and the <u>interior</u> of the angle. These sets are suggested by the following "exploded" model:



The set of points represented by the cross hatched piece in the sketch on the right is called the <u>interior</u> of the angle and the cross hatched piece in the sketch on the left represents the <u>exterior</u>. The angle itself is not included in either the interior or exterior. In order that such concepts as these will have exactly one meaning, we restrict our concept of angle so that the rays will not be on the same line. Thus, situations like these will not be considered (although the rays represented by each drawing do have the same endpoint).

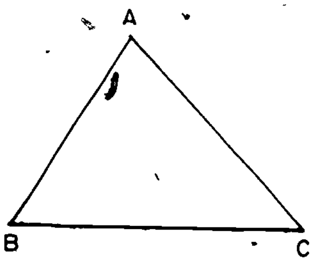


Coinciding rays, $\overrightarrow{AB}$ and $\overrightarrow{RS}$

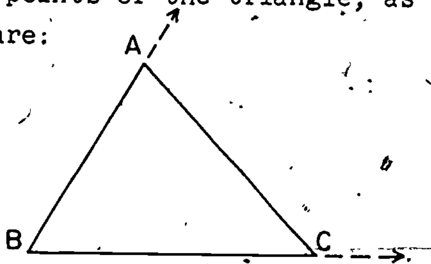


Rays extending in opposite directions $\overrightarrow{AB}$ and $\overrightarrow{RS}$

A <u>triangle</u> is a set of points. It consists of three points not all on the same line and the points on the three segments joining them. Each of the three points (endpoints of the three segments) is called a <u>vertex</u> of the triangle. We label the <u>vertices</u> of a model of a triangle with capital letters, such as A, B, and C, like this:

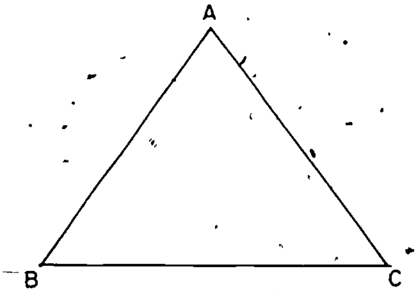


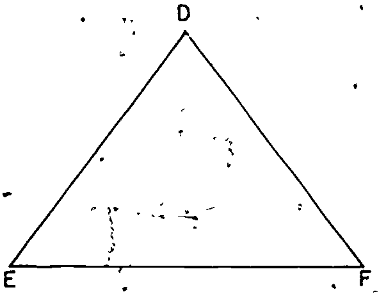
We indicate the segment joining the points A and B as $\overline{AB}$ and call this segment a <u>side</u> of the triangle. The triangle will be named ΔABC or ΔBAC or ΔBCA or with any other arrangement of the letters naming the vertices. A <u>triangle determines three angles</u> called the angles of the triangle. Although ΔABC determines three angles ($\angle ABC$, for example), not all the points of $\angle ABC$ are points of the triangle, as can be illustrated by the following figure:



The side $\overline{BA}$ is just a part of the ray $\overrightarrow{BA}$ and the side $\overline{BC}$ is just a part of the ray $\overrightarrow{BC}$. Remember that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ extend indefinitely from B, Suppose we have another triangle, ΔDEF, which is an exact copy of ΔABC. We cannot say that ΔABC is *equal* to ΔDEF, for this would mean that ΔABC is another name for ΔDEF, and yet the set of points constituting ΔABC is not the same as the set of points forming ΔDEF. But we would like to show that a tracing of ΔABC fits exactly on ΔDEF when we place

vertex A on vertex D,
vertex B on vertex E, and
vertex C on vertex F.





We introduce a new word for this relation and we say that ΔABC (with its vertices named in the order A, B, C) is *congruent* to ΔDEF (with its vertices named in the order D, E, F) and write $\Delta ABC \cong \Delta DEF$. We call the vertices that must be placed together so that ΔABC will fit exactly on ΔDEF, *corresponding vertices*. Note that this correspondence is shown when we write $\Delta ABC \cong \Delta DEF$, since the first vertex, A, named in ΔABC corresponds to the first vertex, D, named in ΔDEF. The second named vertex, B, of ΔABC corresponds to the second vertex, E, named in ΔDEF, and similarly for the third vertex of each triangle.

After the pupils have studied congruent figures by using tracings (which can be "turned over", if needed) for comparison, they will be introduced to reproducing a geometric figure using the straightedge and compass. The pupils should use a *straightedge* and not a ruler in this portion of the chapter. The difficulty with a ruler is that it encourages measuring when such is not desired for the construction involved. The straightedge, can be used only for drawing a line segment. (If only rulers are available then it should be stressed that they are to be used only as a straightedge and not as a measuring device.) The compass is used only for drawing a circle or an arc (that is, a connected piece of a circle). Using these instruments and their knowledge of congruent figures, the pupils will learn how to make congruent segments, congruent triangles, and congruent angles.

Materials Needed:

Teacher: Box of colored chalk, model of a pyramid, model of a cylinder, chalkbox, or other rectangular box, chalkboard compass or string compass, long straightedge (a 36" ruler will do), some type of transparent sheet for tracing triangles at the chalkboard, paper fasteners, cardboard strips, scissors

Pupil: Straightedge, compass, tracing paper (ordinary paper might do), protractor, scissors, paper fasteners, cardboard strips, paper and pencil

TEACHING THE UNIT

The lessons in this unit vary in their composition. Some have three parts which are: first, Suggested Teaching Procedure, second, Exploration, and third, Exercises which the children should do independently. In some lessons the Exploration and Exercises are sufficient to develop the lesson. Some lessons need only the Exploration to clarify the concepts for the children.

The first part Suggested Teaching Procedure provides an overview of the lesson. It is here that the teacher will find suggestions for providing the background the children will need for the understandings and skills to be developed.

Some teachers may prefer to have the children's books closed during this introduction of the concepts. During the second part of the lesson, the Exploration in the pupil's book, the pupils and teacher will read and answer the questions together. She may say, for example "Now turn to page __ and look at the Exploration. Is this what we did? Is this what we found to be true?". A resourceful teacher will be sensitive to the mood of her class and will not extend this part of the lesson beyond the point of interest.

Other teachers may go immediately into the Explorations. The Exploration then serves as a guide for the lesson. Still others may wish to have the pupil's book closed during the presentation and then have the pupils read the Exploration independently for review.

The third part of the lesson is the Independent Exercises. These are designed for the pupil to work independently. They are provided for maintenance and establishment of skill but they are also developmental in nature and help pupils gain additional understandings and skills.

Each teacher should feel free to adapt these ideas in a way that will suit her method of teaching and in a way that meets the particular needs of her class.

The first section of this unit is a review of material covered in the SMSG text for the fourth grade. If the pupils have not studied this material, you will need to spend more time on this section. In either case, you should have a copy of the SMSG text for grade four.

References:

1. School Mathematics Study Group Text for Grade Four.
2. Mathematics for Junior High School, Volume I, Chapter IV, School Mathematics Study Group.
3. Freeman, Mae and Ira, Fun with Figures, New York: Random House, 1946.
4. Ravielli, A., An Adventure in Geometry, New York: Viking Press.
5. Bassetti, F., Solid Shapes Lab, New York Science Material Center.
6. Anderson, R. D., Concepts of Informal Geometry, Volume V, Studies in Mathematics, School Mathematics Study Group.

REVIEW OF GEOMETRIC FIGURES

Objective: To develop the following understandings and skills.

(1) The primary purpose of this section is to recall those understandings previously developed which will be used in this unit.

(2) The idea that plane geometric figures are parts of the solid figures is emphasized.

(3) A review of some of the mathematical vocabulary occurs in a natural setting in which solid figures are manipulated and discussed.

Materials Needed:

Teacher: Any rectangular space figure such as a chalkbox, or a shoe box, or a piece of lumber such as a "two by four;" a pyramid, made of paper (or of wood); a cylinder, such as an unopened soup can; a straightedge for use at the chalkboard; chalkboard and colored chalk; chalkboard compass or string compass

Pupil: Paper and pencil; if practical, examples or models of rectangular solids, pyramids, and cylinders for each pupil

Vocabulary:

Mathematical vocabulary used which has been taught previously includes:

- face
- edge
- vertex
- vertices
- segment
- plane
- point
- square region
- endpoint
- rectangle
- square
- intersection
- union
- pyramid
- base
- circular region
- interior
- triangular
- cylinder
- ray
- half plane
- compass
- measure
- length

Suggested Teaching Procedures:

You may wish to begin by saying to the class something of this nature: "For the next few weeks we are going to be doing things in mathematics that are a little different from what we have been doing." What is meant when we use the term geometric figure? Look around the room. What are some of the geometric figures you see? Can you see any triangles, squares, or rectangles? What shape are the windows? What shape is the door? What figures do you see on your desk? On my desk? There are examples of geometric figures all about us. Can you look anywhere and NOT see examples of them?. We will be studying many of them in our new work."

"Have you ever used a compass? For what can it be used? This is just one of the tools we will be using. Each of you will have one."

Show a compass. Do not take time now to explain its use. Show the rectangular solid, pyramid, and cylinder. Ask whether anyone can tell the names of these figures. Any other type of introduction which gets children thinking about the idea of the unit and provides motivation could, of course, be used. The presentation above gives just one way and the resourceful teacher will no doubt think of many superior introductions.

Use the rectangular solids and have pupils do the activities called for in Exercise 1, page 161. If possible, each pupil should have a rectangular solid.

The meaning of face, edge, vertex, segment, plane, vertices, point, and endpoint are reviewed.

Draw a model of the rectangular solid on the board and label it as in the sketch on page 161. Review the way of writing names of line segments such as $\overline{AB}$, $\overline{DE}$, and $\overline{AH}$. The pupils could, for example, write the symbols for segments DC, BG, and FG. They could also trace on the diagram on the board the segments for which you write the symbols such as $\overline{HG}$ or $\overline{CF}$.

Chapter 4

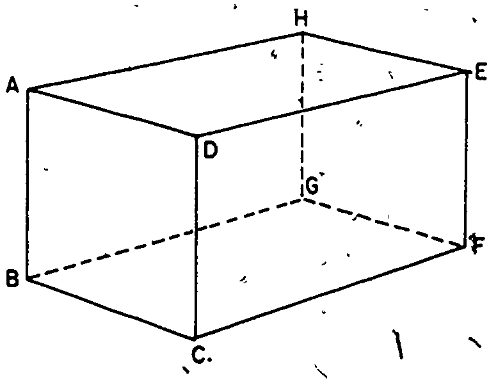
CONGRUENCE OF COMMON GEOMETRIC FIGURES

REVIEW OF GEOMETRIC FIGURES

Rectangular Prism

Exploration

Look at a chalkbox.



1. a) Place your finger on the top face.
   Place your finger on the bottom face.
   How many faces has a chalkbox? (six)

   b) Trace any edge of the box with your finger tip.
   How many edges has the box? (twelve)

   c) Point to a vertex of the box.
   How many vertices has the box? (eight)

2. Suppose we name each corner (vertex) of the box with the letter given in the above sketch.

   a) Name 3 edges of this rectangular prism. (Any three of $\overline{AH}$, $\overline{HE}$, $\overline{ED}$, $\overline{DA}$, $\overline{AB}$, $\overline{DC}$, $\overline{EF}$, $\overline{HG}$, $\overline{BG}$, $\overline{GF}$, $\overline{FC}$, or $\overline{CB}$.)

   b) Name 4 faces of this rectangular prism.
   any four of faces ADEH, DEFC, BGFC, BGHA, HEFG, or ABCD.)

Emphasize in Exercise 3, page P 162, that plane geometric figures are observed in solid figures in the physical world--that solid figures can be used as a source of plane figures. To help the pupils see that the edges form rectangles, the edges may be traced with the fingertips. If the solid is held so that only one face is visible at a time, the outline is a rectangle.

The intersection of two faces is a line segment as illustrated in Example 4, page 162. This can be shown by having the pupils again trace the intersection on the solids with their fingers. The "intersection of the set of points of the bottom face and the set of points of the front face" is the line segment $\overline{BC}$ (We are calling face ABCD the "front" face. If face DEFC is called the "front" face then the intersection is $\overline{CF}$).

In Exercise 5, page P 163 children can best get the idea of intersection by again tracing the edges on the solid figures with their fingers. The idea of <u>point</u> and <u>vertex</u> will be reviewed here.

There are at least two different types of answers to "Name the three sets whose intersection is the point H." One would be the intersection of the line segments AH, EH, and GH. Another would be the intersection of the three faces which are parts of planes. Help the pupils find both of these answers. There are, of course, other answers.

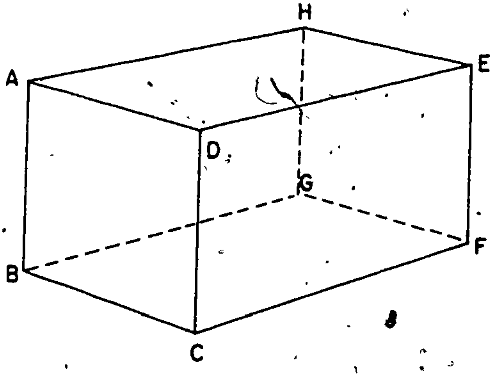
There are other illustrations of the empty set in addition to the intersection of $\overline{AD}$ and $\overline{BC}$. They would include the intersection of any of the segments which are parallel such as $\overline{HE}$ and $\overline{AD}$, $\overline{BG}$ and $\overline{CF}$, and $\overline{AH}$ and $\overline{DE}$. Another illustration is the intersection of $\overline{AD}$ and $\overline{CF}$.

In Exercise 6, other unions of sets which result in rectangles include the union of $\overline{HE}$, $\overline{HA}$, $\overline{AD}$, $\overline{DE}$; $\overline{AD}$, $\overline{DC}$, $\overline{CB}$, $\overline{BA}$; $\overline{BC}$, $\overline{CF}$, $\overline{FG}$, $\overline{BG}$. Children should name all six of the rectangles.

c) You can see that a vertex represents a point; an edge represents a line segment, and a face represents a part of a plane.

Every line segment has two endpoints. We label the endpoints with capital letters.

Then we may name a line segment by using the letters at its endpoints with a bar over them. Thus: $\overline{AD}$ or $\overline{GF}$.



3. What geometric figures can you find that are formed by the edges of the box? (rectangles and possibly some squares)
How many rectangles did you find? How many squares did you find? (six rectangles, two of which are probably squares)

4. Name the intersection of the top face and the front face.
($\overline{AD}$ if face ADCB is called the front face. $\overline{DE}$ if face DEFC is called the front face.)
What is the intersection of the set of points on the bottom face and the set of points on th[illegible]front face?
($\overline{BC}$ if face ADCB is called the front face. $\overline{CF}$ if face DEFC is called the front face.)

5. What is the intersection of $\overline{CF}$ and $\overline{GF}$? $\{F\}$

   What is the intersection of $\overline{AB}$ and the top face? $\{A\}$

   Name three sets whose intersection is the point H.
   $\overline{AH}$, $\overline{EH}$, and $\overline{GH}$ or face BGHA, face AHED, and face GFEH)

   What is the intersection of $\overline{AD}$ and $\overline{BC}$? $\{\ \}$

   Name some other pairs of sets whose intersection is the empty set. ($\overline{AH}$ and $\overline{EF}$; or face CDEF and face BGHA ; or face HEFG and $\overline{AB}$)

6. Name the geometric figure which is the union of the sets $\overline{DC}$, $\overline{DE}$, $\overline{EF}$, and $\overline{CF}$. (rectangle CDEF)

   Name the geometric figure which is the union of the sets $\overline{HG}$, $\overline{GF}$, $\overline{FE}$. $\overline{HE}$. (rectangle or possibly square HGFE)

Pyramid

In guiding the children to recall what they learned about the pyramid, ask them if they have ever seen anything shaped like this as you show a model of a pyramid. Write the word pyramid on the board, and encourage pupils to respond. (They may mention the Pyramids of Egypt and this would be an excellent response. Perhaps a child could show pictures of these Pyramids or make a report about one of them.)

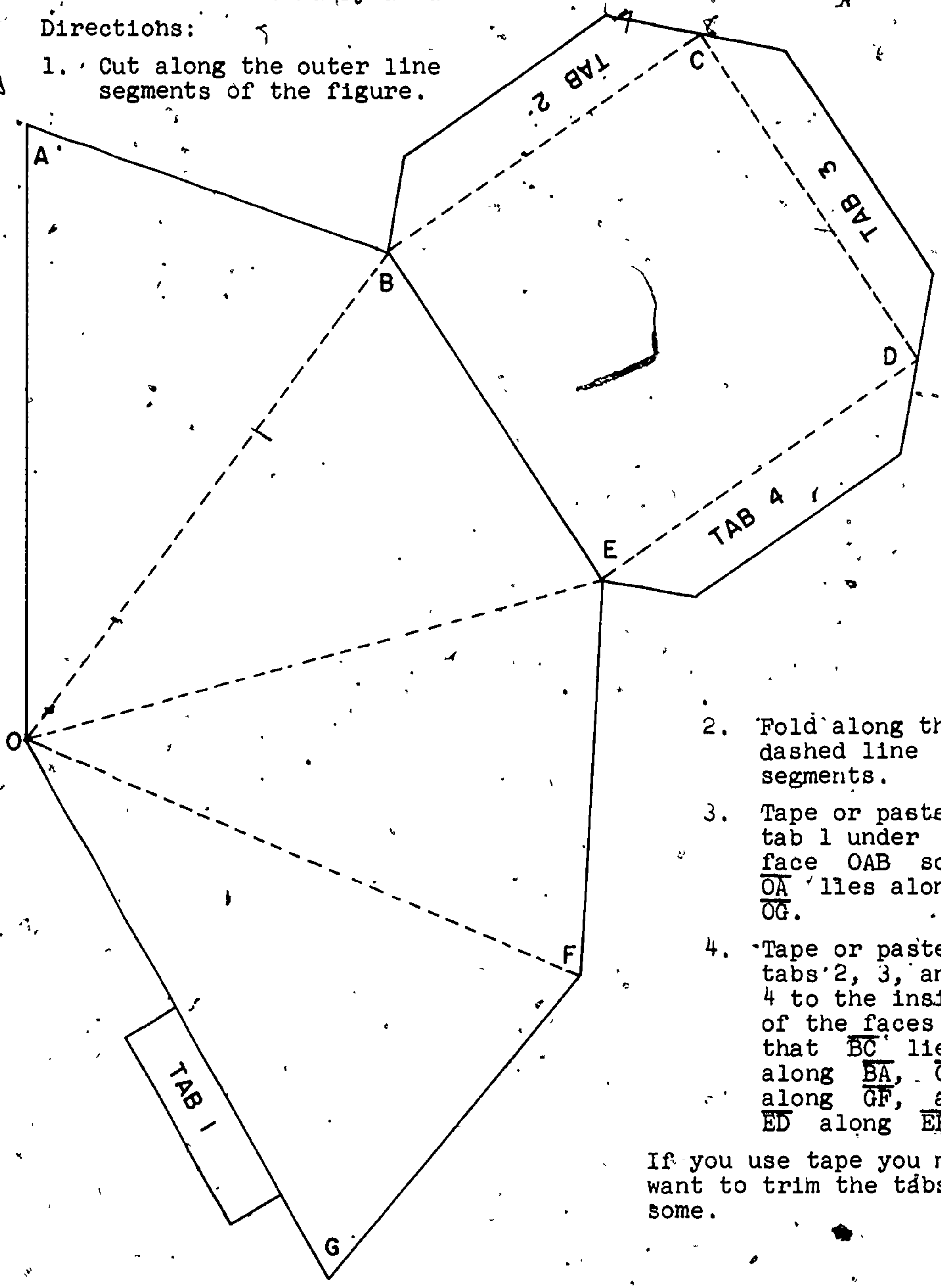
You might use just one pyramid for demonstration and as you show a model of a pyramid have the children handle the model to find the answers to leading questions. Or you can duplicate the pattern for a pyramid, given on the next page, and let each child made a model of it (possibly as a home assignment).

In either case, the pupils can find the answers by handling the pyramid. Ask them to close their eyes and tell what they can "feel" about the pyramid. Have a child describe the pyramid as he handles it with his eyes closed. The base of a pyramid is also called a face.

Pattern for a Pyramid

Directions:

1. Cut along the outer line segments of the figure.



2. Fold along the dashed line segments.

3. Tape or paste tab 1 under face OAB so $\overline{OA}$ lies along $\overline{OG}$.

4. Tape or paste tabs 2, 3, and 4 to the inside of the faces so that $\overline{BC}$ lies along $\overline{BA}$, $\overline{CD}$ along $\overline{GF}$, and $\overline{ED}$ along $\overline{EF}$.

If you use tape you may want to trim the tabs some.

Pyramid

Draw on the chalkboard the pyramid pictured in the pupil text. Use this drawing to answer the questions in Exercise 1, of the Exploration on the pyramid after you have introduced the pyramid. Pyramids must have triangular sides. However they may have bases which are triangular or which have four or more sides.

Look at Exercise 1, and 2 of the Exploration to see what you might conclude at this point. Any two of the faces of a pyramid intersect in a line segment. All faces except the base intersect at O.

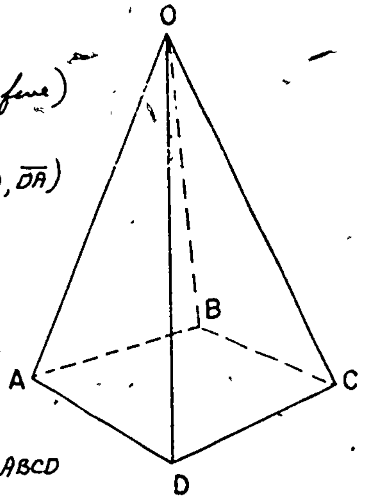
The main idea of Exercise 3, is that the edges of each face--other than the base--of a pyramid form a triangle. In other words, you can see an illustration of a triangle on a pyramid.

The intersection of the edges of the four triangular faces is the set whose only member is the point O. Children can see this by running their fingers along the edges of the pyramid up to the vertex at O.

You may want to refer to the chapter on Recognition of Common Geometric Figures in the text for Grade Four for reviewing the definition and idea about pyramids.

Pyramid

Exploration



1. a) How many faces has this pyramid? (five)

   b) How many edges does the pyramid have? (eight - $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$)

   c) How many vertices has the figure? (five - Vertices O, A, B, C, D)

   d) Which edges outline the bottom face? ($\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$)

   e) Name the figure formed by the edges of the bottom face. (rectangle ABCD or possibly square ABCD)

2. a) Which faces intersect on $\overline{OD}$? (face OAD and face ODC)

   b) Which faces intersect on $\overline{OC}$? On $\overline{OB}$? On $\overline{AB}$? (face OBC and face ODC; face OAB and face OBC, and face OAB and face ABCD respectively)

   c) Do faces OAD, OBC, OAB, ODC, and ABCD represent planes? (yes)

   d) Which of these planes intersect at O? (planes OAB, OBC, OCD, and OAD)

3. a) Name the geometric figure outlined by the edges $\overline{OD}$, $\overline{OC}$, $\overline{DC}$. (triangle)

   b) Trace these edges with your finger tip. Name them. ($\overline{OD}$, $\overline{OC}$, $\overline{DC}$)

   c) Place your finger tip in the interior of ΔOAD.

4. Name the intersection of the edges of the four triangular faces. (Point O.)

5. a) Could a pyramid have just 3 faces? Remember that the base is called a face, too. (No)

   b) Could a pyramid have just 4 faces? (yes).

   c) Could a pyramid have just 999 faces? (yes)

Cylinder

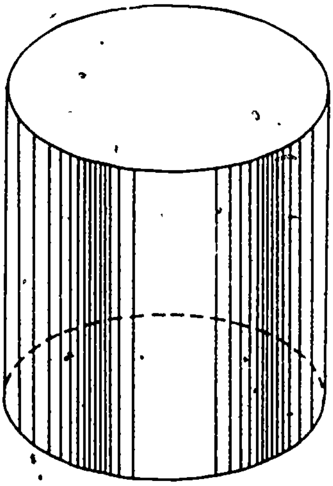
Show the cylinder next, writing ~~the word~~ cylinder on the board. (Remember that a cylinder includes the two bases as well as the "lateral surface", but does not include the interior. That is a cylinder is hollow.) Ask the children what the object is and relate it to its mathematical name on the board. Ask for examples of cylinders. Encourage children to bring examples of cylinders to school. (Be sure the examples have a "top" and a "bottom".) A committee might make a display of these and of other geometric figures.

Show one of the faces ("top" or "bottom") of the cylinder as you ask for the name of the figure which outlines a base. Give children opportunities to handle the cylinder.

You may want to refer to the chapter on Recognition of Common Geometric Figures in the text for Grade Four for reviewing the definition and ideas about cylinders.

Cylinder

Exploration



1. Nearly every time you select a can of food at the store, you are handling an object like a geometric figure called a cylinder.

   a) What are the "top" and "bottom" of a cylinder called? (base)

   b) What is the name of the geometric figure which outlines a base of this kind of cylinder? (a circle)

2. How many such figures are outlined on this cylinder? (two) Trace them with your finger tip.

3. Do the bases of a cylinder have to be circular regions? (No)

4. Could the bases of a cylinder be square regions? (yes)

5. Could each base of a cylinder have 1001 sides? (yes)

Triangle

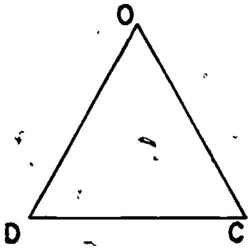
Draw a triangle on the board as shown in the Exploration on the Triangle. Emphasize that an angle is the union of two rays with the same endpoint but not on the same line. Rays of an angle are sometimes called the sides of the angle. Any particular drawing can show only a portion of the rays of an angle. Show more of the rays of the $\angle ODC$, as in Exercise 2, to illustrate this. Show a line segment that (except for its endpoints) is in the interior of each angle of the triangle ODC. (It will be $\overline{OC}$ for $\angle ODC$, $\overline{DC}$ for $\angle DOC$, and $\overline{OD}$ for $\angle DCO$.)

This is a good time to distinguish between a triangle and its interior. Some children may still think the term triangle includes the interior of the triangle. Having them trace with their fingers just the sides of the triangle and then place their finger tip in the interior of the triangle, may help them understand which set of points is the triangle and which set of points is the interior of the triangle.

You may want to refer to the chapter on Sets of Points in the text for Grade Four for reviewing the definition and ideas about triangles.

Exploration

3



1. a) Copy figure ODC on a sheet of paper.

   What set of points form $\triangle$ODC? (the set which is the union of the set of points of $\overline{OD}$, $\overline{OC}$, and $\overline{DC}$)

   b) Trace $\triangle$ODC with your finger tip.

   Place your finger in the interior of the triangle.

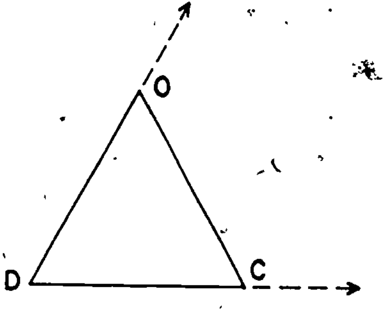
   c) Name the angle whose vertex is at D. ($\angle ODC$ or $\angle CDO$)

   d) Name the angle whose vertex is at O. ($\angle DOC$ or $\angle COD$)

   e) How many names were given for the angle whose vertex is at D? (two)

   f) How many names were given for the angle whose vertex is at O? (two)

2. a) Recall that an angle is the set of points on two rays which have a common endpoint and which are not on the same line.



Trace the rays (that is, part of them) with your finger tip.

b) Name the rays that form ∠ODC. ($\overrightarrow{DO}$ and $\overrightarrow{DC}$)

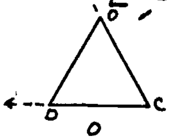
c) Name the common endpoint. (D)

d) Does $\overrightarrow{DC}$ end at C? (No, it continues indefinitely.)

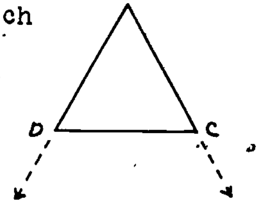
e) How many endpoints does $\overrightarrow{DC}$ have? (one)

f) Why was the letter D placed in the middle (between O and C) in the name, ∠ODC? (Because we have agreed that if we use three letters in the name of an angle, then the letter for the common endpoint of the two rays will be placed between the other two letters.)

3. a) Make another drawing to show the rays which form ∠OCD. Why is the letter C placed between the letters O and D in the name ∠OCD? (Because C is the common endpoint of the rays of the angle.)



b) Make another drawing to show the rays which form ∠DOC. Why is the letter O placed between the letters D and C in the name, ∠DOC? (Because O is the common endpoint of the rays of the angle.)



4. In the drawing for Exercise 2 which line segment (except for its end points) is in the interior of ∠ODC? ($\overline{OC}$)

5. Draw an angle on your paper. Color the interior of the angle red. If only the interior of the angle is to be red, should the rays of the angle be made red? (Rays should not be red.)

Half plane

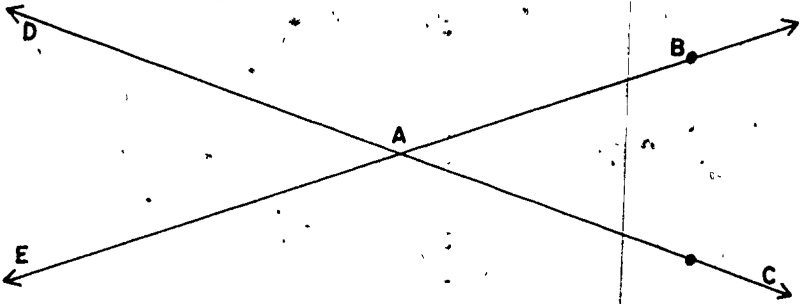
The concept of a half plane may need to be developed here. By first discussing a plane you may make half plane more understandable to the pupil. Show lines of a plane in various positions. Observe that a line separates a plane into three sets of points: the set consisting of the points of the line itself and two other sets of points. Each of these other sets is called a half plane.

The exploration is written for the children to do the indicated steps. You may not want each pupil to do the coloring or make his own models. Instead, you may prefer to imagine that the coloring has been done and then ask the children to point out the sets involved. Alternately, you could do the exploration as a class demonstration and discussion.

Half Plane

Exploration

1. a) Copy the figure below.



b) Color the line EB red.

c) Color the portion of the plane below $\overleftrightarrow{EB}$ (the part which contains C) blue. Do not get any blue on the line EB.

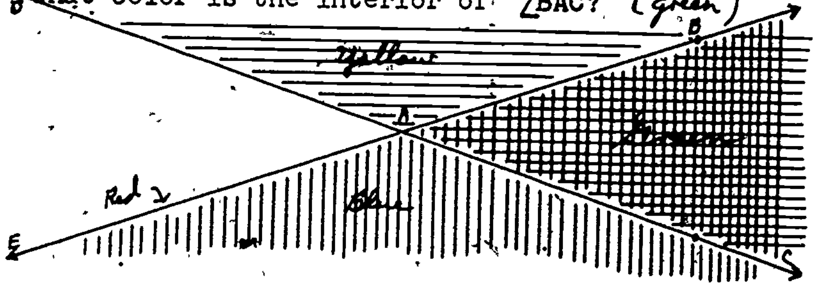
d) What would be a good name for the part of your figure which is colored blue? (a half plane.)

e) What is the name for the part of your figure which is colored red? ($\overleftrightarrow{EB}$)

f) What would be a good name for the part of your figure which is not colored? (a half plane).

2. a) Color the half plane above $\overleftrightarrow{DC}$ (the part which contains E) yellow. Do not get any yellow on line CD.

b) What color is the interior of ∠BAC? (green)



CONGRUENT FIGURES

Objective: To develop the following understandings and skills.

(1) Two geometric figures are called congruent when a tracing (which may be "turned over") of one figure will fit exactly on the other.

(2) Two triangles are congruent only when certain vertices are placed together.

(3) When two triangles are congruent the corresponding angles are congruent and the corresponding sides are congruent.

Materials Needed:

Teacher: Straightedge, sheet of transparent plastic

Pupil: Straightedge, paper suitable for tracing

Vocabulary: Congruent, corresponding

Suggested Teaching Procedure:

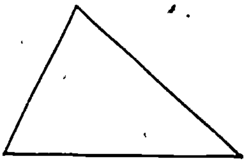
The first paragraph of the pupil text should be read with the class and the distinction between the concept and its representations noted. However, when you draw a triangle on the board say, "Here is a picture of a triangle," and emphasize the fact that you have actually drawn only a model or picture of a triangle. When you draw triangles or other geometric figures on the board, comment frequently that you are really drawing only a <u>picture</u> of a triangle or a <u>model</u> of a geometric figure.

CONGRUENT FIGURES

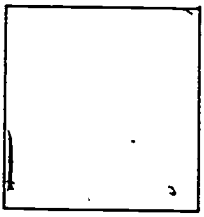
Congruence

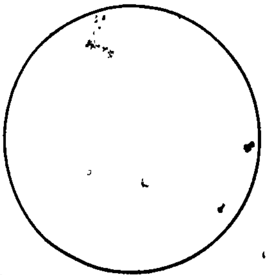
Exploration

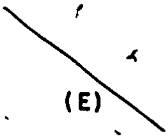
1. Can you find pairs of figures which look as if one of them could fit exactly on the other? (A and I, B and J, D and H)

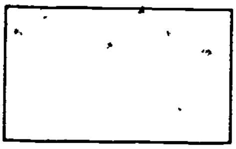

(A)

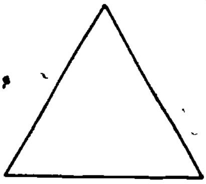

(B)


(C)


(D)


(E)


(F)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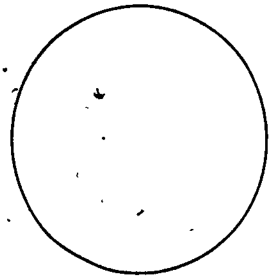
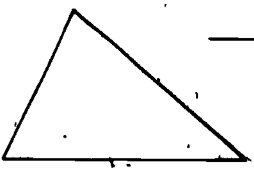
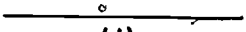
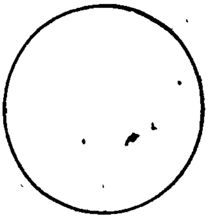
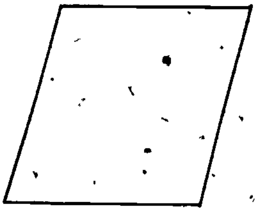
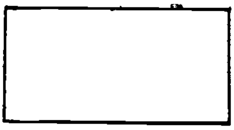

(H)

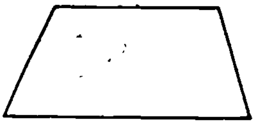

(I)


(J)


(K)


(L)


(M)


(N)

2. Which figure will fit exactly on 

Triangle A (I)	Rectangle F (None)
Segment B (J)	Triangle G (None)
Square C (None)	Figure L (None)
Circle D (H)	Figure N (None)

Figure M (None)

3. How can you use tracing paper to see whether your answers are correct? (Trace one figure on the tracing paper and place the tracing paper over another figure. Your tracing paper may be "turned over" if necessary.)

Summary

A geometric figure is a set of points. We know that we cannot make a point on a piece of paper but only a model or a picture of a point. When we draw a line or a triangle we are drawing a model. In this text when we say, "Look at the triangle," we really mean, "Look at this model of the triangle."

Two geometric figures are <u>congruent</u> to each other if they have exactly the same size and shape. This means that if we make a tracing of one figure and place it on top of the other figure and if it fits exactly, then we say that the two figures are congruent.

Congruent Line Segments

Exploration

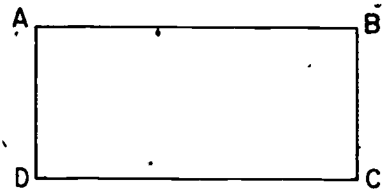
Trace $\overline{AB}$ on a thin sheet of paper. Can you place this tracing of $\overline{AB}$ so that it fits exactly on $\overline{CD}$? Did you place the tracing of the point A on the point C or the point D? Does it matter? (No)

Recall that $A = B$ means A and B are names for the same thing. We cannot write $\overline{AB} = \overline{CD}$ because the points of $\overline{AB}$ are not points of $\overline{CD}$. For example, there is no point on $\overline{CD}$ that is the same point as the point A on $\overline{AB}$. But we would like to write briefly that a tracing of one segment fits exactly on the other. We will write $\overline{AB} \cong \overline{CD}$ to say that the two segments are congruent.

Exercise Set 1

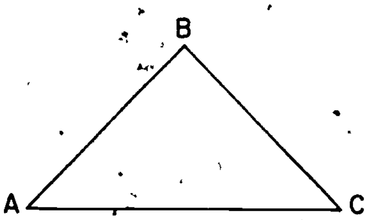
Can you find two congruent segments in each figure? Can you find more than two? Trace segments on a thin sheet of paper to help you decide. Write your answers like this: $\overline{MN} \cong \overline{PQ}$

1.



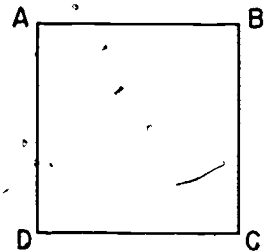
$\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

3.



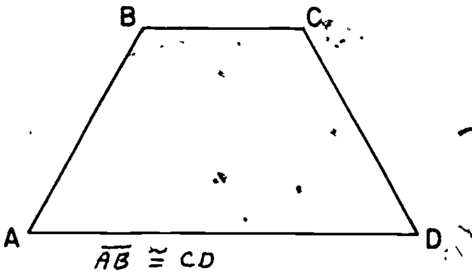
$\overline{AB} \cong \overline{BC}$

2.



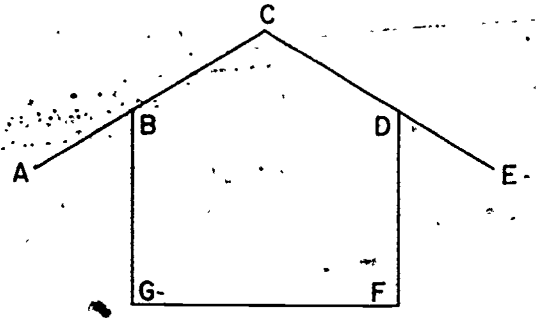
$\overline{AD} \cong \overline{DC} \cong \overline{CB} \cong \overline{BA}$

4.



$\overline{AB} \cong CD$

5.



$\overline{BC} \cong \overline{CD}$, $\overline{BG} \cong \overline{DF}$, $\overline{AB} \cong DE$, $\overline{CA} \cong \overline{CE} \cong \overline{GF}$

Congruent Triangles

By use of the exploration on Congruent Triangles draw congruent triangles ABC, DFE, on the board. You may use straightedge and compass and the method shown in the Exploration on Copying a Triangle (pupil text page 189) to construct the congruent triangles. (Pupils should not see the construction at this time. They will learn it at a later time.) Trace $\triangle ABC$ that you constructed on the board on the sheet of transparent plastic. You might emphasize the corresponding vertices of the congruent triangles by writing the names on the board as follows:

$$\begin{tikzcd}
\triangle & A \arrow[d, leftrightarrow] & B \arrow[d, leftrightarrow] & C \arrow[d, leftrightarrow] \\
\cong \triangle & D & F & E
\end{tikzcd}$$

$$\triangle ABC \cong \triangle DFE$$

Exploration Exercise 8, helps to emphasize that the triangles are congruent only when certain vertices are placed together; that is, $\triangle ABC \cong \triangle DFE$, but $\triangle ABC$ is not $\cong$ $\triangle DEF$. This means, of course, that you will have to be very careful about the order of naming vertices when talking about congruence.

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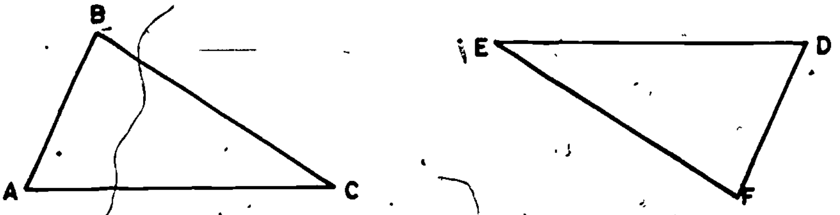
Congruent Triangles

Exploration

You have learned that we call two figures congruent if a tracing of one figure can be placed to fit exactly on the other. (The tracing may be "turned over.") Let us see whether the following two triangles are congruent?



Trace ΔABC on a sheet of thin paper and see whether it will fit exactly on ΔDFE.

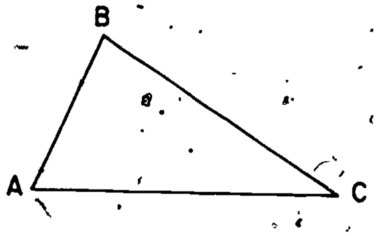
Notice that the triangles will fit exactly if

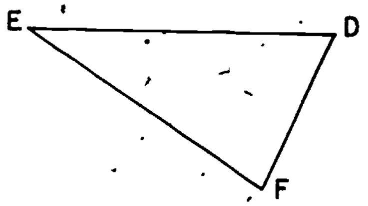
1. Vertex A is placed on vertex D of ΔDFE.
2. Vertex B is placed on vertex F of ΔDFE.
3. Vertex C is placed on vertex E of ΔDFE.

We notice then that when the vertices are matched the sides also match. Complete the following:

4. $\overline{AB}$ is congruent to side DF of ΔDFE.
5. $\overline{AC}$ is congruent to side DE of ΔDFE.
6. $\overline{BC}$ is congruent to side FE of ΔDFE.

We call the vertices A and D, B and F, C and E <u>corresponding vertices</u> since when A is placed on D, B on F, and C on E, one triangle fits exactly on the other. We call sides AB and DF corresponding sides since they join corresponding (matching) vertices.





7. Name the other pairs of corresponding sides.
($\overline{AC}$ and $\overline{DE}$; $\overline{BC}$ and $\overline{FE}$)

We can use the same symbol "≅" that we used for congruent line segments to show that one triangle is congruent to another.

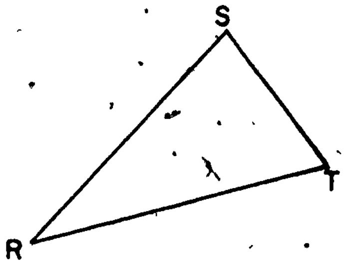
If the triangles fit when

point A is placed on point D,
point B is placed on point F,
point C is placed on point E,

we shall show this by writing

$\Delta ABC \cong \Delta DFE$.

8. Is $\Delta ABC \cong \Delta DEF$? (This means: Can you place the triangles so that A is on D, B is on E, and C is on F?). (No)



9. Use your tracing of ΔABC to see whether the following triangle is congruent to ΔABC. Are the triangles congruent?
($\Delta TSR \cong \Delta ABC$)

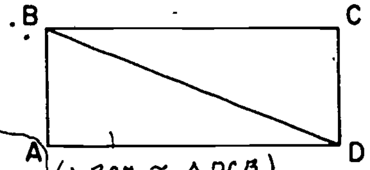
10. List the corresponding vertices.

A and T, B and S, C and R.

Exercise Set 2

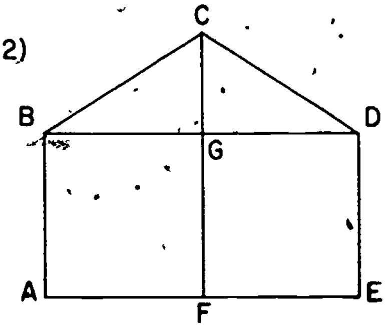
By tracing one triangle on a sheet of thin paper find the triangles which are congruent to each other. Be sure to name corresponding vertices in order. In Exercise I, state your answer like this: $\triangle BAD \cong \triangle DCB$. In Exercises 3, 5, and 6 you may have to trace more than one triangle.

1)



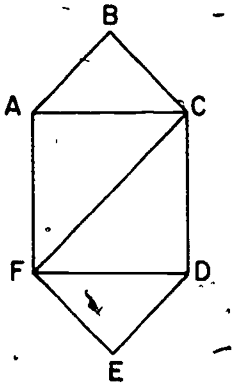
($\triangle BAD \cong \triangle DCB$)

2)



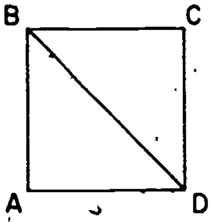
($\triangle BGC \cong \triangle DGC$)

3)



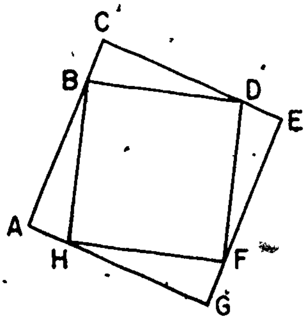
($\triangle ABC \cong \triangle FED$
$\triangle AFC \cong \triangle DCF$)

4)



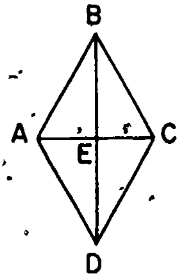
($\triangle BAD \cong \triangle DCB$)

5)



($\triangle DCB \cong \triangle BAH \cong \triangle HGF \cong \triangle FED$)

6)



($\triangle ABE \cong \triangle CBE \cong \triangle ADE \cong \triangle CDE$;
$\triangle ABC \cong \triangle CDA$; $\triangle ABD \cong \triangle CDB$;
$\triangle ABC \cong \triangle ADC$; $\triangle ABD \cong \triangle CBD$)

Congruent Angles

The exploration on Congruent Angles develops the idea that angles may be congruent although the segments shown which are parts of the rays are not congruent. In the previous work the congruent angles have been parts of congruent triangles and consequently have had congruent segments as representatives of the rays. The student should realize that an angle actually consists of two rays and that the segments are parts of the rays. You may wish to discuss the Exploration on Congruence with the children to be sure that they will understand that angles can be congruent although the parts of the rays shown are not congruent. The hands of a large (tower) clock compared with the hands of a small wrist watch (at 3:00 p.m., for example) would provide an illustration of this idea.

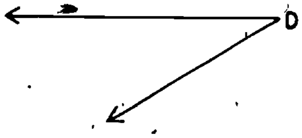
Congruent Angles

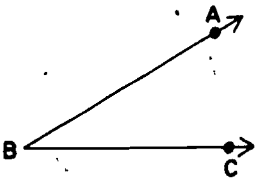
Exploration

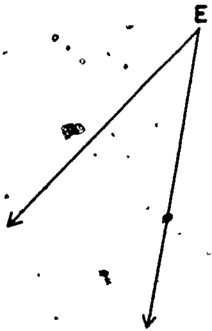
We say two angles are congruent to each other if we can place the vertex of a tracing of one angle on the vertex of the other angle and the rays of the tracing can be placed to lie exactly along the rays of the second angle.

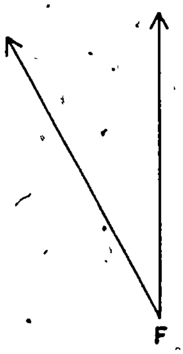
Exercise Set 3

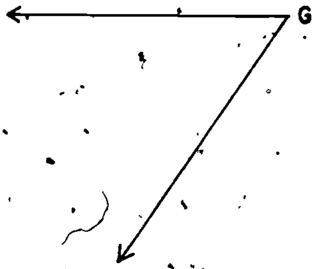
By tracing $\angle ABC$ on a sheet of thin paper, determine which of the following angles are congruent to $\angle ABC$.









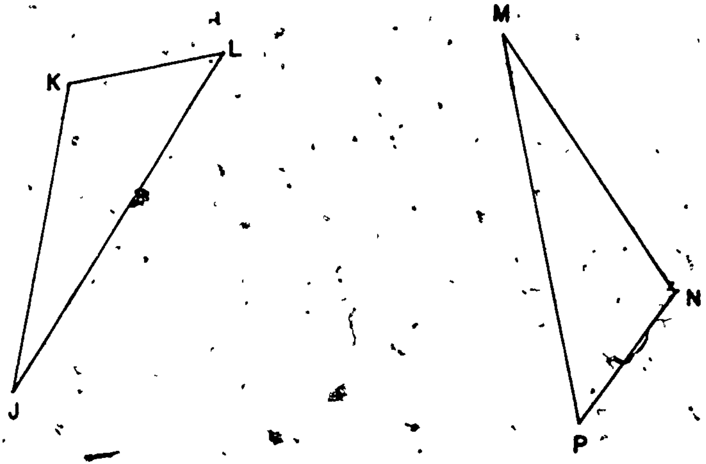


($\angle D \cong \angle ABC$, $\angle F \cong \angle ABC$)

Corresponding Angles

Exploration

Triangles JKL and MNP are congruent.



Trace △MNP and place this tracing so it fits exactly on △JKL.

Where does ∠N fall? (∠N falls on ∠K)

∠N and ∠K are corresponding angles.

Where does ∠L fall? (∠L falls on ∠P)

∠L and ∠P are corresponding angles.

Where does ∠J fall? (∠J falls on ∠M)

∠J and ∠M are corresponding angles.

<u>Corresponding</u> <u>angles</u> of congruent triangles are those which fit together when a tracing of one triangle is placed so it fits exactly on the other.

Summary

In this section we learned some facts about congruent line segments, congruent angles, and congruent triangles. We learned that:

1. Line segments are congruent if a tracing of one can be placed to fit exactly along the other.
2. Triangles are congruent if a tracing of one can be placed to fit exactly along the other. The tracing may be "turned over."
3. In naming congruent triangles, vertices must be named in the proper order.
4. Two angles are congruent if we can place the vertex of a tracing of one angle on the vertex of the other angle, and the rays of the tracing can be made to lie exactly along the rays of the second angle.
5. When two triangles are congruent the corresponding angles are congruent and the corresponding sides are congruent.

Do you agree that this summary tells what we found? Can you think of anything that should be added?

COPYING A LINE SEGMENT

Objective: To develop the following understandings and skills.

(1) Lengths of line segments may be compared with the aid of a compass.

(2) Every point on an arc of a circle is the same distance from its center. The center of an arc is the center of the circle of which the arc is a part.

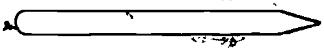
(3) Line segments may be copied with the aid of a straightedge and compass.

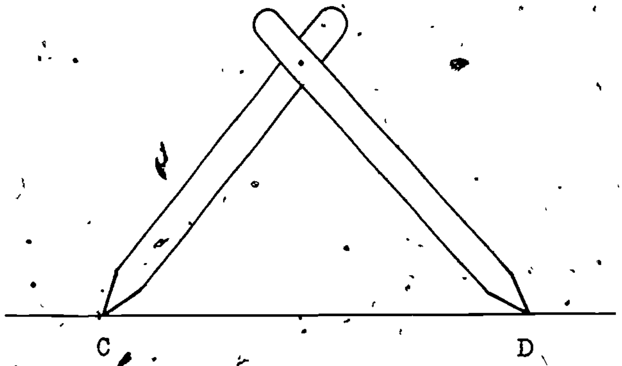
Materials Needed:

Teacher:	Board compass or string compass, yardstick
Pupil:	Straightedge, compass, cardboard strips, paper fasteners (Unlined paper for construction work is preferable.)

Vocabulary: Arc

Suggested Teaching Procedures:

If in the exploration of Comparing Lengths of Line Segments with a Compass, the pupils do not recall from their fourth grade experiences the use of the compass for comparison of line segments, review this here. Actually when a compass is to be used merely for comparing the lengths of line segments, a pair of dividers (which have two points at the ends, and no pencil) is a sufficient substitute. The children can make their own dividers by using two cardboard strips,  and a paper fastener:



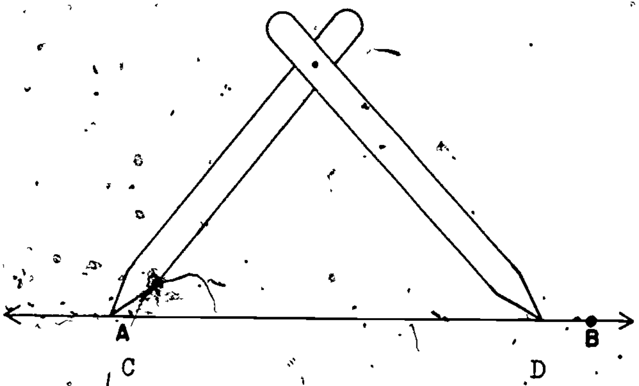
In comparing lengths of line segments, demonstrate on the board:

(1) We place the endpoints of the dividers on the endpoints of one of the line segments.

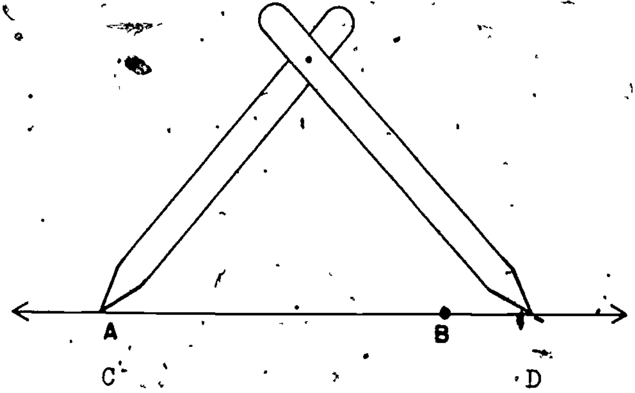
(2) Without changing the setting, move the dividers to the other line segment.

(3) Place one endpoint of the dividers on one endpoint of the second line segment.

(4) If the second endpoint of the dividers falls between the endpoints of the second segment, then the first segment is shorter than the second segment.



(5) If the second endpoint of the dividers falls beyond the second endpoint of the line segment, then the first segment is longer than the second segment.



After you have given the above demonstration on the board, have the children do Exercise Set 4, independently.

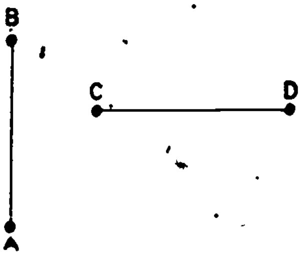
COPYING A LINE SEGMENT

Comparing Lengths of Line Segments

Exploration

1. Do you remember how to use your compass to compare the lengths of two line segments? Look at $\overline{AB}$ and $\overline{CD}$.

   Which appears to be longer, $\overline{AB}$ or $\overline{CD}$? (answers will vary)



2. Use your compass to compare the length of $\overline{AB}$ with that of $\overline{CD}$. What do you observe now? ($\overline{CD}$ is larger)

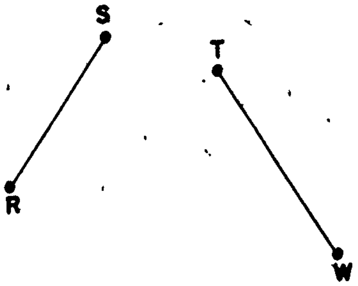
3. Does your observation agree with the guess you made by just looking at the line segment?
   ($\overline{AB}$ appears longer than $\overline{CD}$ but $\overline{CD}$ is longer.)

Exercise Set 4

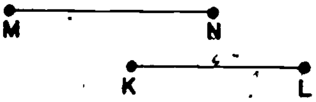
Use your compass to find answers to the following questions.

1. How does the length of $\overline{TW}$ compare with that of $\overline{RS}$? Which is longer? How do you know?

($\overline{TW}$ is longer than $\overline{RS}$. If the compass points are placed on R and S without changing the compass, the sharp point is placed on T, the other point falls between T and W.)

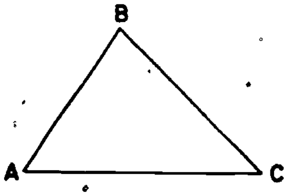


2. Is the length of $\overline{MN}$ greater than, equal to, or less than the length of $\overline{KL}$?

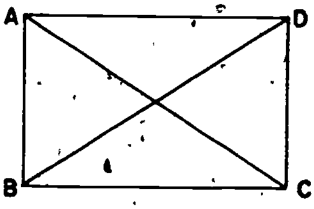


($\overline{MN}$ has greater length than $\overline{KL}$).

3. Which side of $\triangle ABC$ is the longest? ($\overline{AC}$)

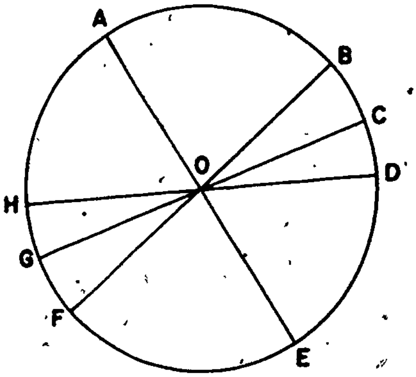


4. Compare the length of $\overline{AC}$ with that of $\overline{BD}$.
($\overline{AC}$ and $\overline{BD}$ have same length.)



5. a) Compare the lengths of $\overline{AE}$, $\overline{FB}$, $\overline{GC}$, $\overline{HD}$.
(The lengths are the same.)
Compare the lengths of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OF}$, $\overline{OH}$.

b) Since O names the center of the circle, do your results agree with what you already knew about circles? (Yes, all points of a circle are the same distance from the center of the circle.)



Copying a Line Segment Using the Compass

Exercise 5, in the Exploration on Copying a Line Segment provided an opportunity to review with the pupils the fact that every point on a circle is equidistant from the center. Follow the exploration in the text to make clear that an arc is part of a circle, and hence every point of an arc is equidistant from the center of the circle. Dividers are no longer satisfactory. We need a pencil point on the compass in order to draw an arc. In the demonstration the teacher may use a string and a piece of chalk instead of the board compass. Discuss with the pupils why this is a satisfactory substitute. In this section a line is named by a small letter, for the first time. The letters k, and l, are most frequently used, but this does not mean that other letters are not acceptable. It is suggested that for this exploration, the teacher work at the board, discussing, and demonstrating.

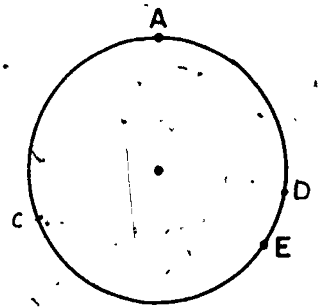
After the development of the procedure for copying a line segment anywhere on another line, have each child do this at his seat. Then illustrate, at the board, copying a line segment when one endpoint of the copy is indicated. Follow this with provision for each child to practice this skill at his seat, under close supervision. Make clear that the intersection of the set of points on the arc made with the compass, and the set of points of the line on which we make the copy, is a set whose only member is a single point. This is an endpoint of the copy. In the exercises which give opportunity to fix the understandings and skills of this subsection, it is assumed that the pupils will make reasonable facsimilies of the figures on their papers and do the construction work there.

Copying a line Segment Using the Compass

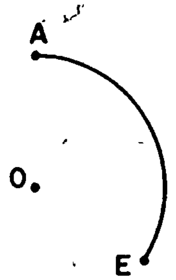
Exploration

Recall that every point on a circle is the same distance from the center of the circle. We call a connected part of a circle an <u>arc</u> of a circle, and we call the center of the circle the <u>center of the arc</u>.

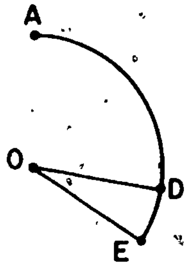
In this picture the part of the circle from A to E which does not include C represents arc AE. The points A and E are the endpoints of the arc. The arc may be named arc AE or arc EA. (If there is a possibility of confusion we name this arc, arc ADE.)



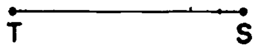
You do not have to draw a complete circle to make an arc of a circle. You could draw arc AE with your compass like this:

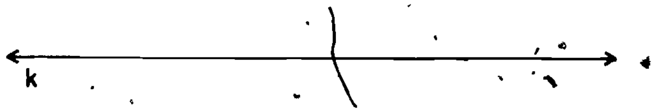


Every point on an arc of a circle is the same distance from its center. The lengths of $\overline{OA}$, $\overline{OD}$, and $\overline{OE}$ are the same, since O names the center.

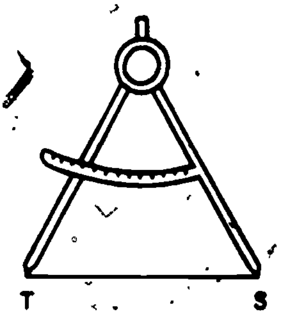


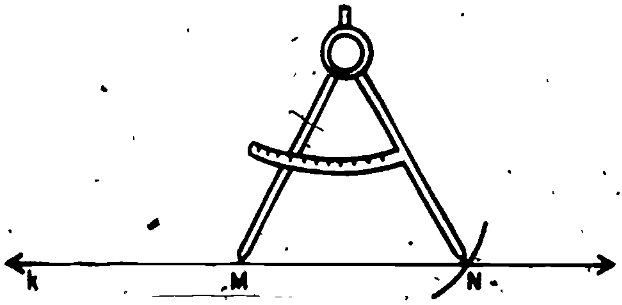
You may use an arc to help make a copy of a line segment. Suppose you are given a line segment TS which you wish to copy on line k. (Sometimes we name a line with a small letter.)





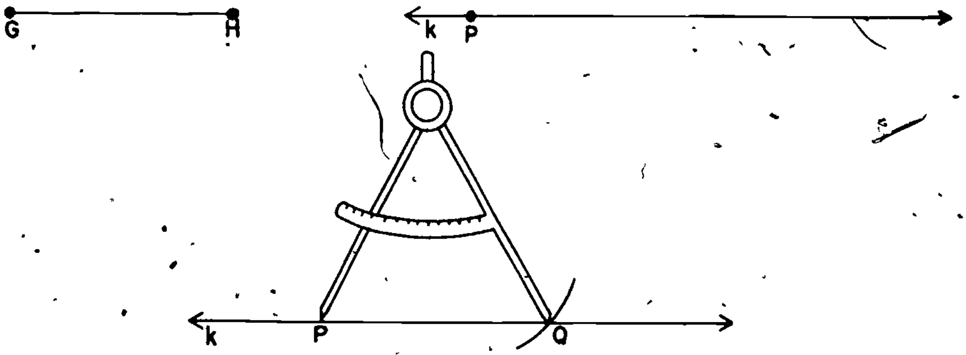
How is the compass placed on $\overline{TS}$?
(One tip at T and the other tip at S.)
Since you haven't been told where on line k to copy $\overline{TS}$ you may place it anywhere on the line.





The sharp metal point of the compass was placed at M. The pencil point of the compass made an arc intersecting the line k at a point we name N. Is $\overline{MN} \cong \overline{TS}$? Why?
($\overline{MN} \cong \overline{TS}$ because the setting of the compass for $\overline{MN}$ was the same as the setting for $\overline{TS}$.)

Sometimes you are asked to copy a line segment at a special place. If you are given $\overline{GH}$, and told to copy it on line k so that one endpoint of the new segment is at point P, then the picture would look like this:



If $\overline{PQ}$ is a copy of $\overline{GH}$, then $\overline{PQ} \cong \overline{GH}$.

Exercise Set 5

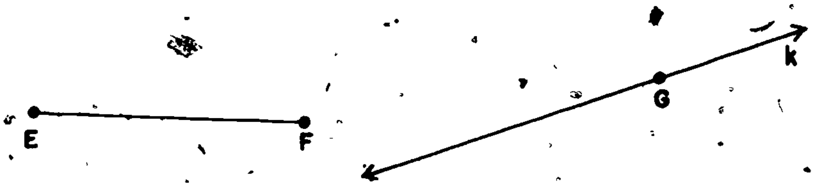
Trace $\overline{AB}$ and $\overleftrightarrow{k}$ on a sheet of paper.

1. Copy $\overline{AB}$ on line k so that one endpoint of the line segment is at Q.

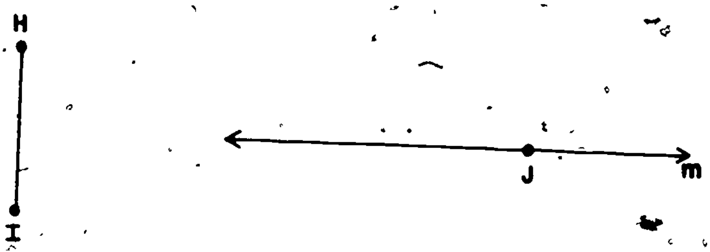
2. Copy each segment so that one endpoint is at the point named on the line.

a.

b.

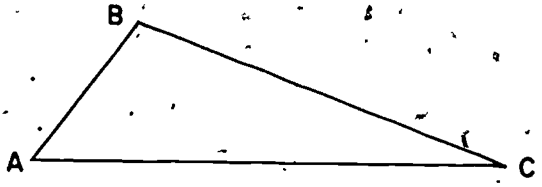


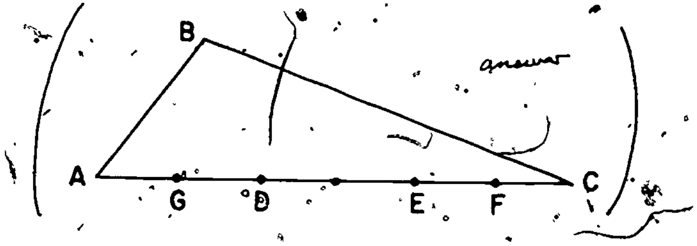
c.



How many segments can you make on line m with one endpoint at J and with the length the same as the length of $\overline{HI}$? (Two segments - one along the right side of J and one along the left of J.)

3. a) Copy this figure on a piece of paper.





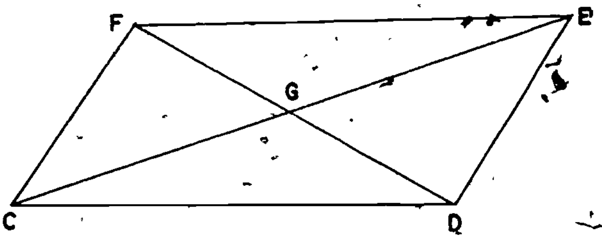
b) Copy $\overline{AB}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at A. Name the other endpoint D.

c) Copy $\overline{AB}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at C. Name the other endpoint E.

d) Copy $\overline{BC}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at A. Name the other endpoint F.

e) Copy $\overline{BC}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at C. Name the other endpoint G.

4. a) Copy this figure on a piece of paper.



b) Copy $\overline{CF}$ on $\overline{CD}$ of your figure using C as an endpoint. Label the other endpoint H.

c) Copy $\overline{FD}$ on $\overline{EC}$ of your figure using C as an endpoint. Label the other endpoint I.

d) Copy $\overline{FG}$ on $\overline{CG}$ of your figure using G as an endpoint. Label the endpoint J.

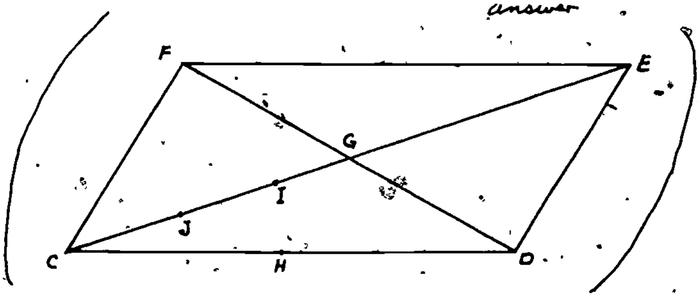
e) Can you copy $\overline{CE}$ on $\overline{FD}$ of your figure using F as an endpoint? (No)

Why? ($\overline{CE}$ is longer than $\overline{FD}$)

Can you do it using D as an endpoint? (No)

Can you do it using any point on $\overline{FD}$ as the endpoint? (No)

answer



TRIANGLES.

Objective: To develop the following understandings and skills:

(1) A triangle is determined if the length of its three sides are given.

(2) We can make a copy of a triangle by copying its three sides.

(3) We can make a triangle if we are given the three line segments whose lengths are the lengths of its sides.

(4) We cannot always make a triangle with sides whose lengths will be those of just any three line segments.

Materials Needed:

Teacher: Board compass or string compass, colored chalk, yardstick

Pupil: Straightedge, compass, paper fasteners, cardboard strips

Vocabulary: determine

Suggested Teaching Procedures:

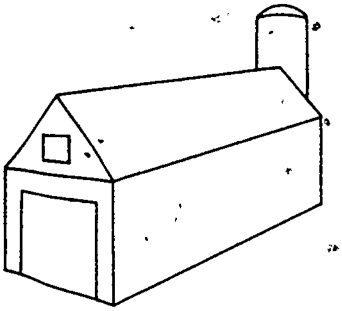
> The brief section on Seeing Triangles in the pupil text will help children "see" triangles in geometric figures. Do this work orally with them as they look at the pictures of the barn, napkin, and star in their texts.

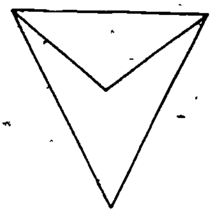
TRIANGLES

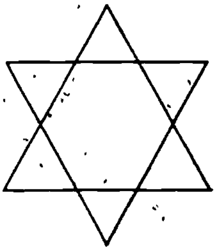
Seeing Triangles

Exploration

Here are sketches of a barn, a folded paper napkin, and a six pointed star.







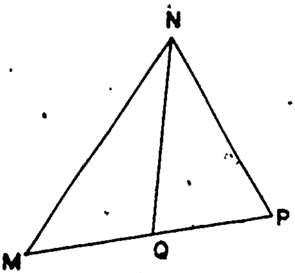
Trace the triangles in each picture with the tip of your finger. How many triangles did you find in the picture of the six pointed star? Did you find as many as eight?

(Yes, there are eight triangles.)

Exercise Set 6

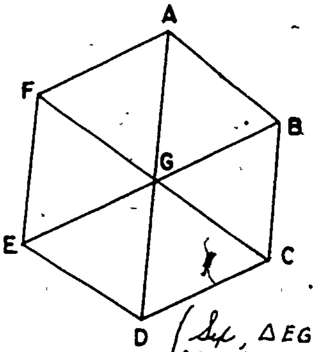
Trace with your finger the triangles in the following figures. Tell how many you found in each case.

1.



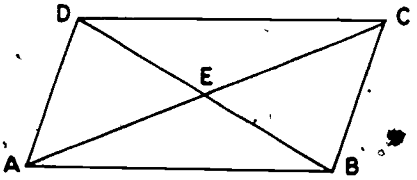
(Three, $\triangle MQN$, $\triangle QPN$, $\triangle MNP$)

2.



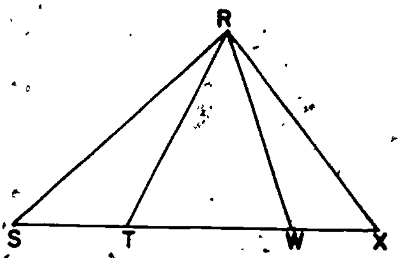
(Six, $\triangle EGD$, $\triangle DGC$, $\triangle CGB$, $\triangle BGA$, $\triangle AGF$, $\triangle FGE$)

3.



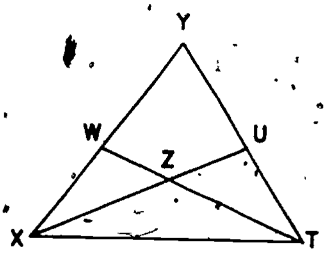
(Eight, $\triangle AEB$, $\triangle BEC$, $\triangle CED$, $\triangle ADC$, $\triangle CBA$, $\triangle DEA$, $\triangle DBA$, $\triangle BDC$)

4.



(Six, $\triangle RSX$, $\triangle RST$, $\triangle RSW$, $\triangle RTW$, $\triangle RTX$, $\triangle RWX$)

5.



(Eight, $\triangle YXT$, $\triangle YXU$, $\triangle YWT$, $\triangle$ [illegible] $\triangle WXT$, $\triangle UXT$, $\triangle WXZ$, [illegible])

Making a Triangle with Strips

At this time there is value in a teacher demonstration lesson showing the Construction of a Triangle with Strips.

Materials Needed:

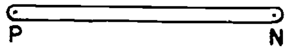
The teacher should have a kit of plastic or cardboard strips and paper fasteners. A kit should have a dozen paper fasteners and at least

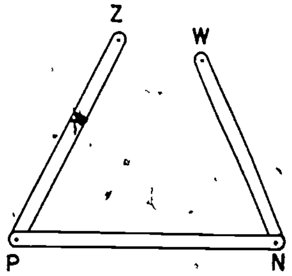
2 strips - 12 inches long
2 strips - 11 inches long
1 strip - 9 inches long
2 strips - 8 inches long
1 strip - 7 inches long
1 strip - 10 inches long

The strips may be an inch wide with holes made a half inch from each end. A compass point makes suitable holes. It should be brought out that when two strips are fastened together and one strip is rotated around the fastener, then the endpoint of that strip traces a circle. Be sure that when the third strip is selected to attach at N, the sum of the lengths of strip PZ and strip WN is greater than the length of strip PN. To assure this, choose one 12 inch strip, one 10 inch strip and one 9 inch strip.

Teacher Demonstration: Making a Triangle with Strips

Choose a 12 inch strip. Name one hole P, and the other hole N. Attach a 10 inch strip at P and a 9 inch strip at N, as shown in the figure.





You may wish to ask the following questions as you proceed with the demonstration lesson.

1. Can I swing strip PZ around P?([illegible])What kind of geometric figure does point Z trace if I swing the strip all the way around? (a circle)

2. How can I make point W trace the same kind of geometric figure? (Swing WN around N)

3. Watch as I swing both strips around at the same time.

   When are the points Z and W farthest apart?
   (when Z, P, N and W are on a line)
   When are they closest?
   (when Z and W are on the same point)

4. Can Z and W fall on the same point? Now I put a single fastener through W and Z. What geometric figure is formed by the three strips?(△)Can I swing either strip around now? (no)

5. Now I choose three other strips whose lengths are the same as PZ, PN, and NW, and attach them to form a triangle.

6. I place these two models of triangles so that one fits exactly on the other. What can you tell me about the two models of triangles? (they are congruent.)

If three sides of one triangle have the same lengths as three sides of another triangle, then the triangles are congruent.

> Make two triangles of different shapes using paper strips. Make a third triangle congruent to one of these triangles. Letter the vertices of the triangles which are congruent to each other.

Is this third triangle congruent to both of the other triangles you watched me construct? (The third triangle will be congruent to only one of the first two triangles.)

List the corresponding vertices of the congruent triangles.

List the corresponding sides of the congruent triangles.

Copying a Triangle

The exploration in Copying a Triangle is in sufficient detail in the pupil text to provide a suitable development for the teacher to follow.

The teacher might carry through the entire construction for copying a triangle at the board with pupil participation whenever indicated. Use one color of chalk to make the arc whose radius is the length of $\overline{AB}$. Use a contrasting color to make the arc whose radius is the length of $\overline{BC}$. This refers to Exploration on Copying a Triangle. Repeat the construction, this time having pupils work at their seats. Have each pupil start with a triangle of the same general shape and size of $\triangle ABC$ in the text.

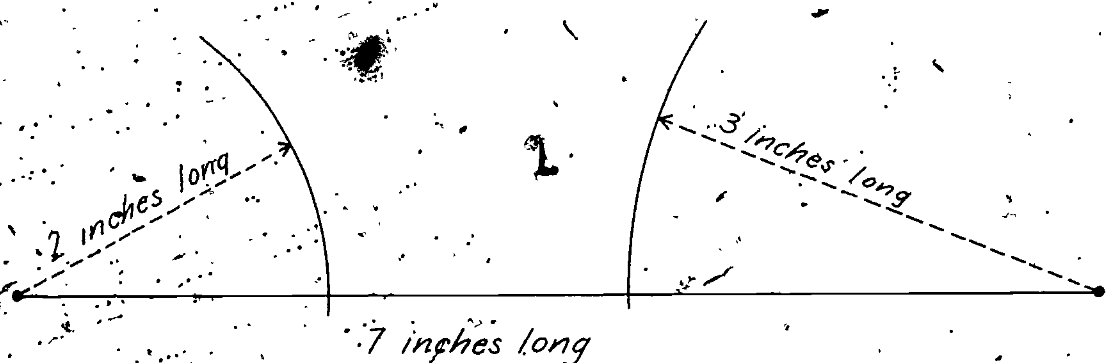
After working through the exploration for constructing a triangle in the Pupil's Text in Constructing a Triangle Given Three Segments and before the pupils attempt the exercises, the teacher should emphasize that it is not always possible to make a triangle with sides whose lengths will be those of just <u>any</u> three line segments.

Teaching Procedure

Do you think we can always construct a triangle when we are given three line segments to use for the sides?

Choose three line segments whose measures, in inches, are 2, 3, and 7. Can we construct a triangle using line segments with these measures? (No)

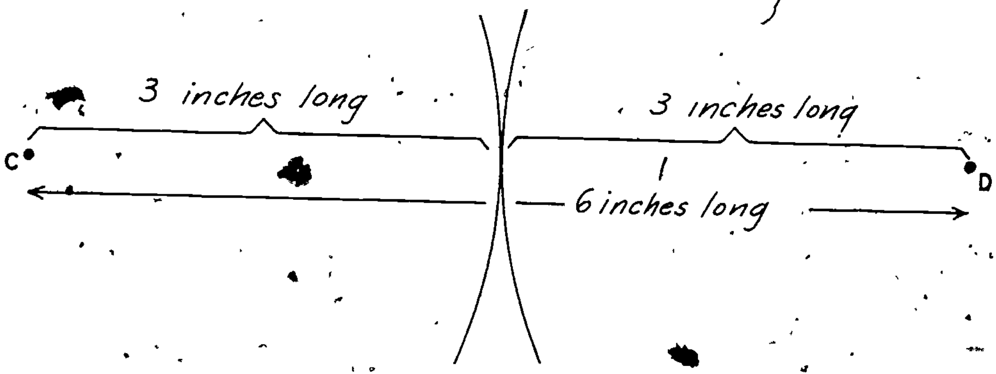
> Let the children experiment to see the difficulty which arises.
>
> Demonstrate at the board, how you would try to draw a triangle using sides whose lengths are 2, 3, and 7 inches. The children will be doing the same work at their seats. For your drawing at the board, use sides four times as great as the 2, 3, and 7. This would give you lengths of 8, 12 and 28 inches with which to work and will be a scale drawing of the shorter segments. Children can see your work better if you use these longer segments.



Why can't we make a triangle with the sides whose measures, in inches are 2, 3 and 7? (because the sum of the measures of two sides of a triangle must be greater than the measure of the third side)

Now let's try this: Make a triangle with sides whose measures, in inches, are 3, 3, and 6.

Again do your demonstration at the board while the children do it at their seats. You might use lengths of 12, 12, and 24 inches at the board.



Have we made a triangle? (no) Why not? (because the sum of the measures of two sides of a triangle must be greater than the measure of the third side)

Bring out that the sum of the measures of two sides of a triangle must be greater than the measures of the third side, otherwise a triangle will not be formed.

The exploration in "How many Sides Determine Exactly One Triangle" is in sufficient detail in the pupil text to provide a suitable development for the teacher to follow.

Bring out that:

(1) two triangles are not congruent if <u>only</u> one pair of corresponding sides are congruent;

(2) two triangles are not congruent if <u>only</u> two pairs of corresponding sides are congruent;

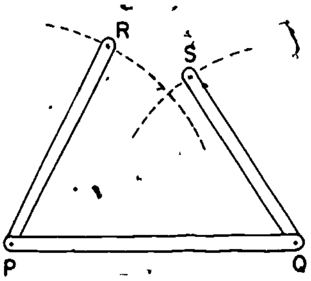
(3) two triangles are congruent if <u>all</u> three pairs of corresponding sides of the triangles are congruent.

Since all triangles with sides congruent to three given line segments are congruent, we say that these three given line segments <u>determine</u> a triangle.

Copying a Triangle

Exploration

When you saw a triangle made with the strips, do you remember that two of the attached strips could be moved around?



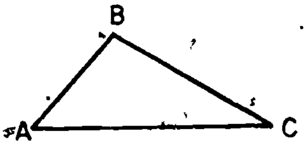
Here are three strips like the ones I used before. I will put the model on the chalkboard, holding strip PQ firmly in place. With the chalkpoint through the hole at R, I will swing PR around. What figure does the chalk point trace? (an arc) Let's do the same thing with the other strip. Do the two arcs I made cross each other? (The lengths will cross unless the length of $\overline{PQ}$ is greater than the sum of the lengths of $\overline{PR}$ and $\overline{QS}$)

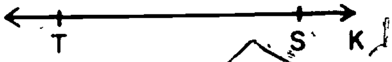
Does this suggest how you might use a compass to copy a triangle? (yes)

Copying a Triangle

Exploration

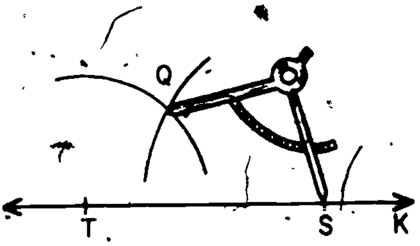
1. Trace $\triangle ABC$ on another sheet of paper. Trace $\overleftrightarrow{K}$ on this same sheet of paper. We may start by copying $\overline{AC}$ on line K. Call the ends of the segment T and S. Your copy should look like this.



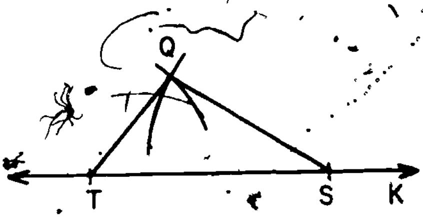


2. Then place the points of your compass at A and B. Move your compass so that the sharp point is on point T. Swing the pencil point to make an arc.

3. Copy $\overline{BC}$. This time put the sharp point of your compass at S and swing the pencil point to make an arc. Label the intersection of the two arcs Q.



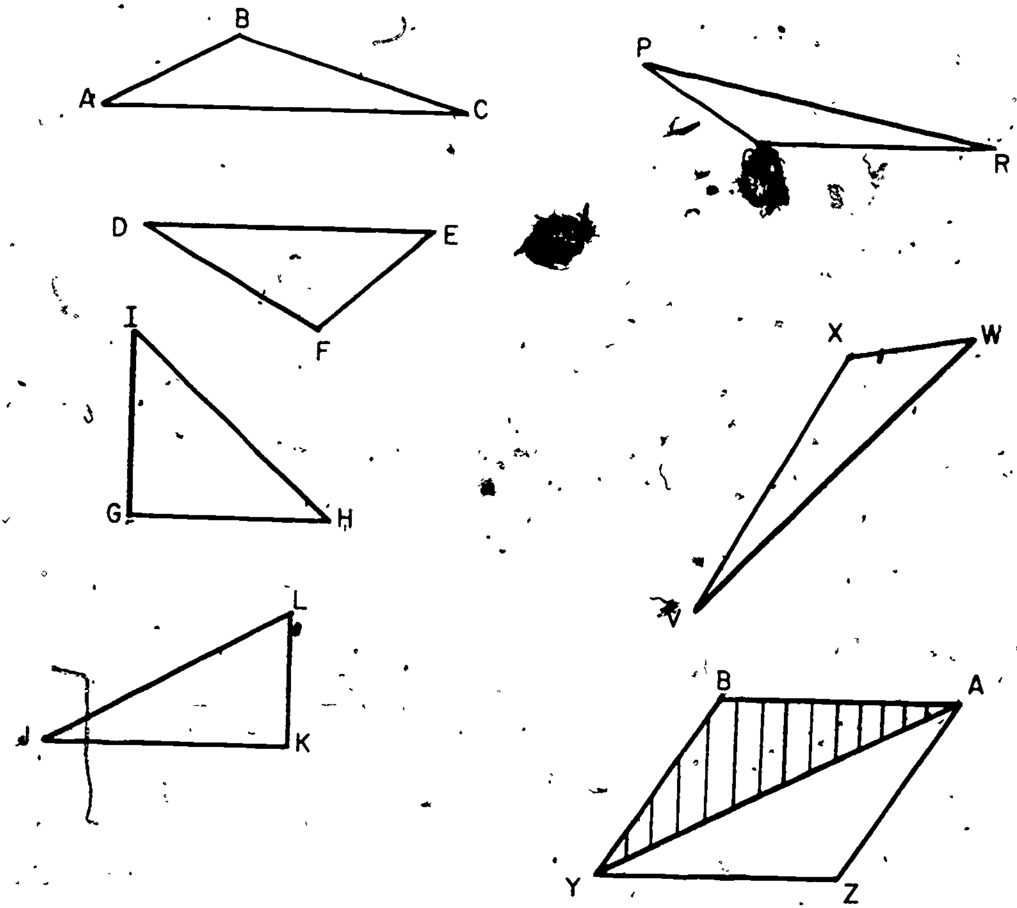
Draw $\overline{TQ}$ and $\overline{QS}$. Your copy of $\triangle ABC$ will be named $\triangle TQS$. Is $\triangle TQS \cong \triangle ABC$? (yes) How can you be sure? ($\overline{TQ} \cong \overline{AB}$, $\overline{TS} \cong \overline{AC}$, $\overline{SQ} \cong \overline{CB}$)



Exercise Set 7

In each of the following exercises draw your own line k and choose some point on it to be an endpoint of the line segment you copy on $\overleftrightarrow{k}$.

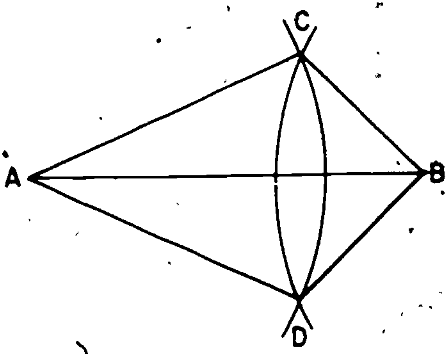
1. Copy each of the following triangles using a compass and straightedge.



Copy the triangle whose interior is shaded.

BRAINTWISTER

2. a) How does the length of $\overline{AC}$ compare with that of $\overline{AD}$ in the figure below?
(AC and AD have the same length.)



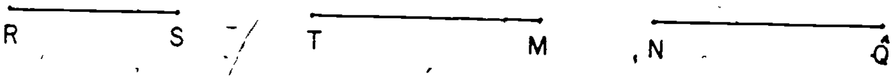
b) How does the length of $\overline{CB}$ compare with that of $\overline{DB}$?
($\overline{CB}$ and $\overline{DB}$ have the same length)

c) What can you predict about ΔABC and ΔABD?
($\Delta ABC \cong \Delta ADB$ because $\overline{AC} \cong \overline{AD}$, $\overline{BC} \cong \overline{BD}$ and $\overline{AB} \cong \overline{AB}$.)

Constructing a Triangle, Given Three Segments

Exploration

You have been copying triangles. However, you might be given these line segments and be asked to construct a triangle



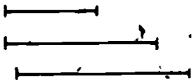
whose sides have the lengths of these segments. Of course, you would need to choose your own line k and point P on it. Does it matter which of the three given segments you copy on line k? (no) If you copy $\overline{RS}$ on line k, which two segments will you use for finding the intersection of the arcs? ($\overline{TM}$ and $\overline{NQ}$) Could you copy $\overline{TM}$ on line k? (yes) Could you copy $\overline{NQ}$ on line k? (yes)

If each child in the class constructs a triangle using $\overline{RS}$, $\overline{TM}$, $\overline{NQ}$ as lengths of sides, what can you predict about all the resulting triangles?
(All will or, at least, should be congruent.)

Exercise Set 8

If possible, in each exercise construct a triangle using the lengths of the given line segments for the lengths of the sides of the triangle. If it is not possible, tell why.

1.



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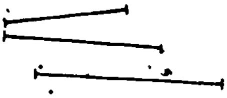

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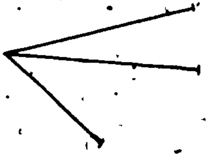

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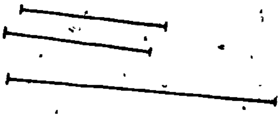

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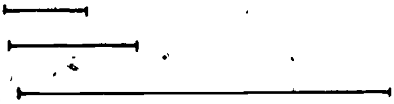
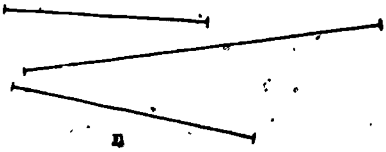


(5) It is impossible because the sum of measures of the shorter segments is less than the measure of the third.

6.



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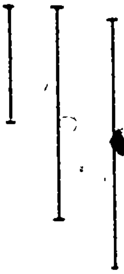

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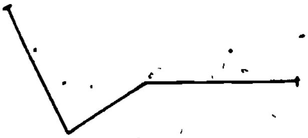

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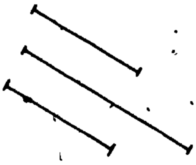

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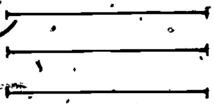


How Many Sides Determine Exactly One Triangle ?

Exploration

Be sure to read all the instructions for each problem before you start. This will help you in arranging your drawings on your paper.

1. a) Draw five congruent line segments, each about four inches long. Call them $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, $\overline{GH}$, and $\overline{KL}$.
   b) Draw a triangle using $\overline{AB}$ for one side.
   c) Draw a differently shaped triangle on each of the other segments.
   d) If you had fifty congruent segments, could you draw a triangle on each of them, each one different in shape and size from the other 49 triangles? (yes)
2. a) Draw five new congruent segments.
   b) Draw a special sixth segment different in length.
   c) On each of the first five segments draw a triangle. This time, make the second side of each triangle congruent to your sixth segment.

      Try to make each triangle different in size and shape from all others. Can you do this? (yes)

3. a) Draw three new congruent segments.

   b) Draw a fourth segment not congruent to any one of the first three.

   c) Draw a fifth segment not congruent to any one of these four segments. Choose the length of this fifth segment carefully. We want to construct a triangle on each of your first three segments with sides congruent to the fourth and fifth segments.

   d) Draw three triangles on the first three segments. In each triangle, make the second side congruent to the fourth segment and the third side congruent to the fifth segment.

   e) Can you make each triangle different in size and shape from any of the others? (No)

   f) What is true about all your triangles? (all are congruent)

   Because all of the triangles are congruent, we say that three sides determine exactly one triangle.

4. Did two sides determine exactly one triangle? (no)

5. Did one side determine exactly one triangle? (No)

COPYING AN ANGLE USING STRAIGHTEDGE AND COMPASS

Objective: To develop the following understandings and skills.

(1) An angle may be copied by making it an angle of a triangle, and then copying that triangle.

(2) It is more convenient to make it an angle of an isosceles triangle, and then copy that triangle.

(3) Skill in using a compass should be increased.

Materials Needed:

Teacher:	Yardstick or meter stick, string and chalk or blackboard compass, colored chalk; plastic sheet for tracing
Pupil:	Compass, straightedge, paper transparent enough to be used as tracing paper

Vocabulary: No new words are included.

Suggested Teaching Procedure:

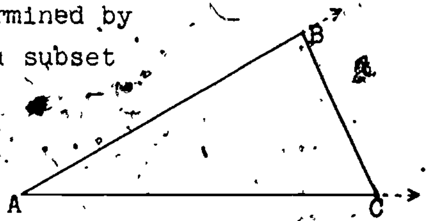
Effective use of this section depends upon certain concepts developed previously. Some of these have been mentioned above. Review what is meant by

(1) An <u>angle</u> (set of points of two rays with same endpoint but not on same line);

(2) a <u>ray</u> (the union of one point (the endpoint of the ray) of a line and the set of all points of the line in one direction from this endpoint);

(3) angle of a triangle.

The sides of a <u>triangle</u> are segments while the sides of an <u>angle</u> are rays. In ΔABC, angle BAC is the angle determined by $\overline{AB}$ and $\overline{AC}$, but $\overrightarrow{AB}$ and $\overrightarrow{AC}$ include points not on $\overline{AB}$ and $\overline{AC}$. Thus an angle of a triangle is determined by the triangle, but the angle is not a subset of the triangle.

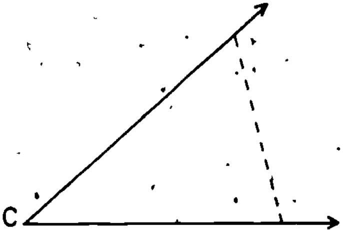


COPYING AN ANGLE USING STRAIGHTEDGE AND COMPASS

Exploration

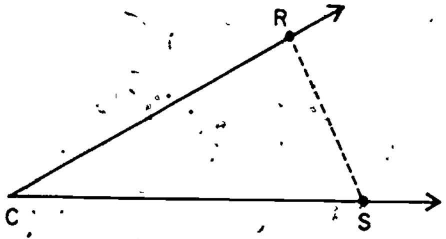
You have learned how to copy line segments and triangles using the straightedge and compass. Now you will learn how to copy an angle using the straightedge and compass.

1. Do you remember how to copy a triangle using the straightedge and compass? Draw a triangle and copy it.
2. When you copied the triangle, did you also copy its angles? (Yes)
3. Suppose you wish to copy ∠C. (When we name an angle by a single letter we mean the angle whose vertex is the point named by that letter.) How could you make part of ∠C two sides of a triangle? Draw a dashed line to complete a triangle. The dashed line will help to keep in mind the angle you are copying.



4. Make a copy of the triangle you made in Exercise 3.
5. Which angle of the triangle that you made in Exercise 4 do you think is congruent to ∠C? Trace this angle and place it on ∠C to see whether it is a copy.

If no pupil thinks of an answer for Exercise 6, ask, "How could you have chosen point R and point S?" Which segments could have been made the same length? Bring out that choosing an isosceles triangle would make the construction simpler.



In discussing the Summary it might be wise to carry out the steps on the board as the pupils do the construction at their seats. Be sure to discuss the questions following Step 5, so that the reasons for the validity of the procedure are understood.

6. In Exercise 3 you made $\angle C$ an angle of a triangle. Would some special triangle have made the construction easier? Can you think of a special triangle which would have required fewer changes in the distance between the points of your compass? (Yes, an isosceles triangle)

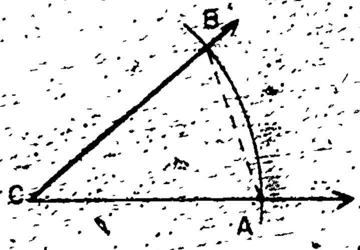
7. List the things you do in copying an angle, and then see how your list compares with the list in the following summary.

Summary

To copy an angle such as $\angle C$ make it an angle of a triangle. Next, copy the triangle by making the three sides the same lengths as the three sides of the first triangle.

The following procedure can be used:

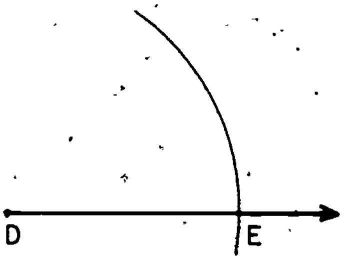
1. The vertex of the angle we wish to copy is point C. With C as a center, construct an arc cutting the sides at points we will call A and B.



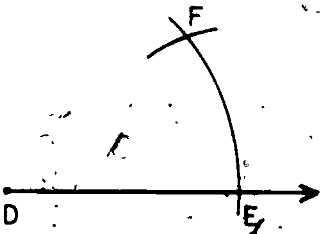
2. Draw the dashed line segment $\overline{AB}$. $\triangle ABC$ is the triangle you are to copy.

3. Draw a ray (leave enough room so you can construct the triangle using part of this ray) and call the endpoint, D.

4. With point D as the center and with the same setting of your compass as in Step 1, construct an arc. Call the point where this arc intersects the ray, point E.



5. Change the setting of your compass so that its point are at points A and B of $\angle BCA$. Keep this setting and place the point of the compass at E and draw an arc which intersects the first arc. Call the point of intersection of the two arcs F.

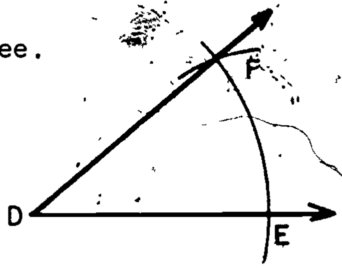


6. Draw $\overrightarrow{DF}$.

Have you made $\angle FDE \cong \angle BCA$? Let us see. Draw $\overline{BA}$ and $\overline{FE}$.

Is $\Delta FDE \cong \Delta BCA$? (yes) Why?

(Because their sides are congruent.)

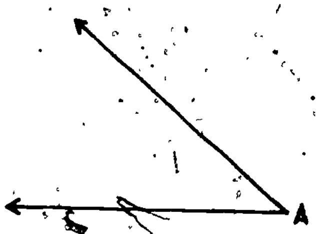


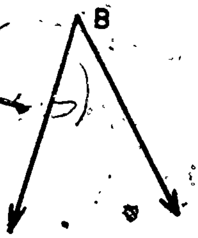
Is $\angle FDE \cong \angle BCA$? Why?

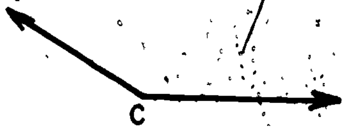
We know $\Delta FDE \cong \Delta BCA$ because we have made three sides of one triangle congruent to three sides of the other triangle. We have chosen two sides the same length for convenience. Now, since we know that corresponding angles of congruent triangles are congruent, we know that $\angle FDE \cong \angle BCA$.

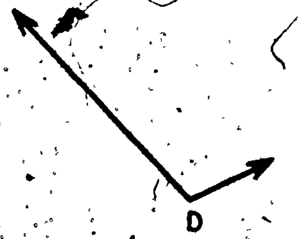
Exercise Set 9

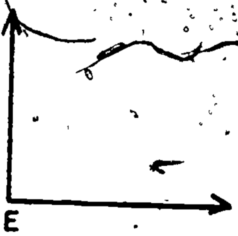
1. Make an angle about like $\angle A$ on your paper. Copy it by using the steps we have outlined. Then do the same for the other angles.











COMPARING SIZES OF ANGLES

Objective: To develop the following understandings and skills:

(1) The sizes of angles can be compared.

(2) The sizes of angles may be compared by use of tracings or compass and straightedge construction.

Materials Needed:

Teacher: chalkboard compass or string compass, meterstick or yardstick, colored chalk, tracing plastic

Pupil: compass, straightedge, tracing paper

Suggested Teaching Procedures:

The definition of an angle as a set of points of two rays suggests that, since a ray has only one endpoint and therefore has no definite length, the idea of the "size" of an angle has no meaning. However, intuition tells us that some angles are "larger in size" than others. In this section we define what is meant by this term, that is, how sizes of angles are compared.

We examine first, the case in which the angles have one ray in common with the second ray of one angle lying in the interior of the other angle. The sketch of the three roads represents such a situation. It will probably be necessary to review the meaning of "interior of an angle" and "exterior of an angle." You may wish to have the pupils observe that all points in a plane are in one of three sets: the set of points in the interior, the set of points in the exterior, and the set of points on the angle itself; and that no point is in more than one of these sets.

We next examine the case in which both rays of one angle lie in the interior of the other angle. The questions in Exercises 11-15, provide practice in identifying angles larger in size and smaller in size than given angles, using the definitions which have been developed.

Exercise 16 shows a case of congruent angles. The pupils should note that the tracing of one angle can be placed to fit exactly on the other; therefore, since a ray of one angle does not fall in the interior of the other, they have the same size.

In Exercise Set 10, Exercise 8, the pupils may fail to recognize that E is in the interior of ∠ABC, since the sides are the rays CA and CB, not the segments CA and CB.

The exploration, "Angles Without a Common Ray," deals with comparing sizes of angles which have no point in common. Most pupils will use the tracing method without difficulty, but some may place perfectly the vertices and one pair of rays of the two angles, but place the second pair of rays in opposite half planes. Note that in Exercise 2, either $\overrightarrow{ED}$ or $\overrightarrow{EF}$ may be placed on either $\overrightarrow{BC}$ or $\overrightarrow{BA}$. The second pair of rays must then be placed on the same side of the first ray. This exploration suggests placing a tracing of one angle on the other. Exercise Set 12 provides practice for this.

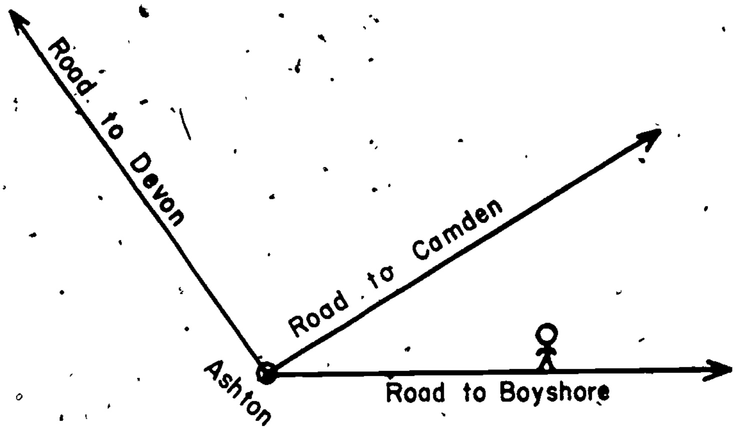
Using the "Congruent Angle Construction," the next exploration, suggests use of the compass construction for congruent angles to make a copy of one of the angles in such a position as to compare their sizes.

In using the compass construction for copying an angle, work through the construction on the board as the pupils work on their papers. Consideration of Exercise 3 and 4 in this exploration, is important for emphasizing the basic idea developed in this section.

The Explorations and Exercises should make it possible in many cases for the pupils to decide which of two angles has the larger size without using either the tracing or the construction procedure. In Exercise Set 13, Exercises 1-5, they should be able, in many exercises, to make the comparison intuitively. This will be more difficult in Exercises 6-11. Furthermore, since the angles to be compared are angles of triangles or of other polygons, some pupils may need help in applying the tracing or construction procedure.

COMPARING SIZES OF ANGLES

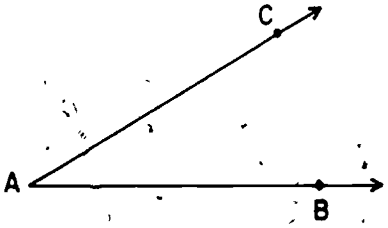
Three roads run from a point in the town of Ashton--one to Bayshore, one to Camden and one to Devon. The man in the sketch is walking toward Ashton. When he comes to the intersection in Ashton, he will choose whether he will follow the road to Camden or the road to Devon. We sometimes say, "The Camden road angles off from the Bayshore road." If he goes to Camden he turns off "at an angle" of one size. If he goes to Devon, he turns off "at an angle" of a different size. Let us see what we mean by the "size" of an angle.

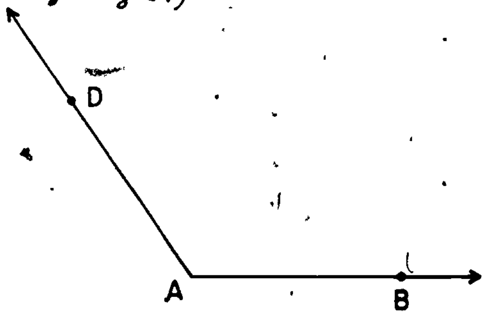


Angles With a Common Ray

Exploration

The first sketch below shows the Bayshore and Camden roads. The second shows the Bayshore and Devon roads. Think of the roads as representing rays with endpoint A. Which angle do you think has the larger size? (The angle named by the road to Devon and the road to Bayshore is the larger angle.)

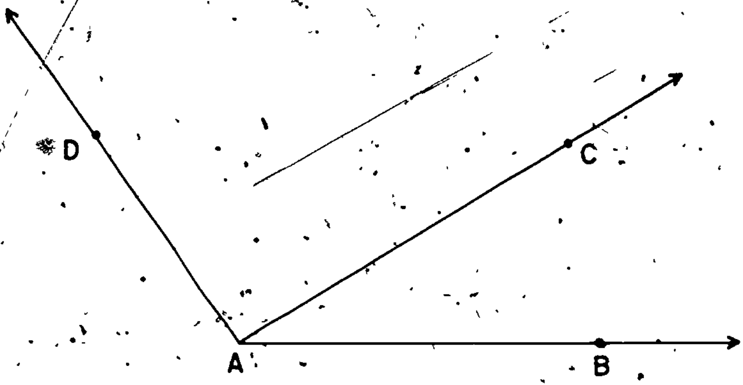




1. Recall what we mean by the word "angle." How have we defined it? (Set of points of two rays with a common endpoint, and not on the same line.)
2. Name the sides of ∠BAC and ∠BAD. Are the sides segments, rays, or lines? ($\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AB}$, $\overrightarrow{AD}$, rays)
3. Do the sides of an angle have a definite length? (No)
4. Do you think the size of an angle depends on the lengths of the sides you actually draw?

   It is clear that the size of an angle cannot depend on the length of its sides, since rays have no definite length.

To see what is meant by "One angle is larger in size than another angle," look at the sketch of the roads to Bayshore, Camden, and Devon.



5. Name the sides of ∠BAC. $(\overrightarrow{AB}, \overrightarrow{AC})$

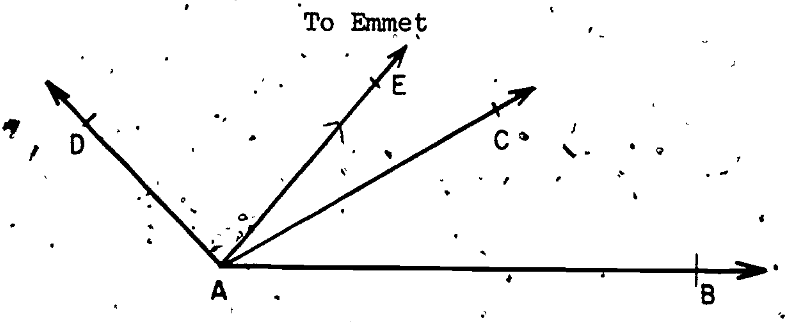
Name the sides of ∠BAD. $(\overrightarrow{AB}, \overrightarrow{AD})$

What ray is a side of both angles? $(\overrightarrow{AB})$

6. Is point C in the interior, or in the exterior of ∠BAD? (Interior)

7. Is $\overrightarrow{AC}$ (except for point A) in the interior, or in the exterior of ∠BAD? (Interior)

Because a) ∠BAD and ∠BAC both have side $\overrightarrow{AB}$, and b) point C is in the interior of ∠BAD, we say that the size of ∠BAD is larger than the size of ∠BAC. (Or we can say that the size of ∠BAC is smaller than the size of ∠BAD.)



8. Name all the angles in the sketch. (There are six.)
(∠BAC, ∠BAE, ∠BAD, ∠CAE, ∠CAD, ∠EAD)

9. Look at ∠CAE. What rays are its sides? ($\overrightarrow{AC}$, $\overrightarrow{AE}$)

10. Are E and C in the interior of ∠BAD? (yes) Because E and C are in the interior of ∠BAD we say, "The size of ∠BAD is larger than the size of ∠CAE." (Or, "The size of ∠CAE is smaller than the size of ∠BAD.")

11. Name an angle whose size is smaller than the size of ∠DAC. (∠DAE) Name another one that appears to be smaller. (∠EAC) How can you be sure your answer is right? (E is in the interior of ∠DAC. Also ∠DAE and ∠DAC have $\overrightarrow{DA}$ in common.) (E is in the interior of ∠DAC. Also ∠DAC and ∠EAC have $\overrightarrow{AC}$ in common.)

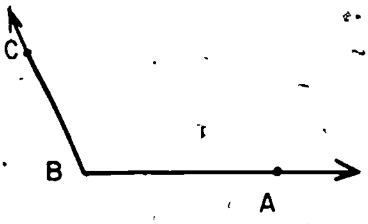
12. Name an angle of larger size than ∠EAD. (∠CAD) Name another one. (∠BAD). How can you be sure? (E is in the interior of ∠CAB, ∠CAD and ∠EAD have $\overrightarrow{AD}$ in common, E is in the interior of ∠CAD; ∠CAD and ∠EAD have $\overrightarrow{AD}$ in common. E is in the interior of ∠BAD; ∠BAD and ∠BAD have $\overrightarrow{AD}$ in common.)

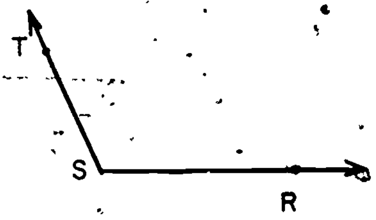
13. Name three angles, each of larger size than ∠EAC.
(∠DAC, ∠EAB, ∠DAB, and ∠DAE)

14. Suppose another town, Farley, is on the Ashton-Camden Road. Copy the sketch and represent Farley by point F.

15. What can you say about the sizes of ∠CAE and ∠FAE? About ∠DAF and ∠DAC? ∠BAC and ∠FAB? (The sizes are the same in each case. In fact, in each case we just have two different names for the same angle.

16. In this sketch, ∠ABC is congruent to ∠RST.





a) Trace ∠ABC on tracing paper. Place B on S and $\overrightarrow{BC}$ on $\overrightarrow{ST}$. Put $\overrightarrow{BA}$ on the R-side of $\overleftrightarrow{TS}$. Must $\overrightarrow{BA}$ lie on $\overrightarrow{SR}$? (yes)

b) Is either of these angles larger than the other? (No)

c) If two angles are congruent, can the size of one be larger than the size of the other? (No)

Summary

The examples above show:

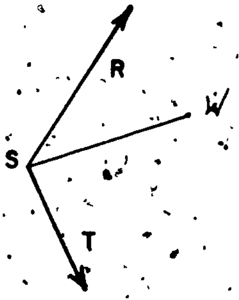
1. The size of one angle is smaller than the size of a second angle:

   a) If the angles have one ray in common, and a point on the other ray of the first angle lies in the interior of the second angle.

   b) If a point on each ray of the first angle lies in the interior of the second angle.

2. Congruent angles have the same size.

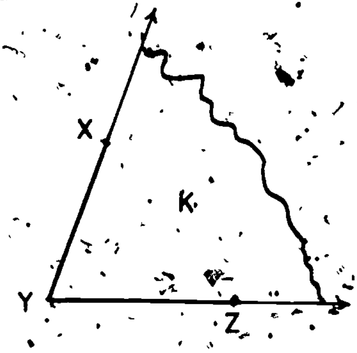
Exercise Set 10



1. a) Trace $\angle RST$. Choose a point in the interior of $\angle RST$. Call this point, W. Draw $\overrightarrow{SW}$.

   b) Compare the size of $\angle RST$ with the size of $\angle RSW$.
   (The size of $\angle RST$ is larger than the size of $\angle RSW$.)

   c) Compare the size of $\angle RST$ with the size of $\angle WST$.
   (the size of $\angle RST$ is larger than the size of $\angle WST$.)



2. a) Trace $\angle XYZ$ and point K. Point K is in the (interior) of $\angle XYZ$. Draw $\overrightarrow{YK}$.

   b) Compare the sizes of $\angle XYZ$ and $\angle XYK$. (The size of $\angle XYZ$ is larger than the size of $\angle XYK$.)

   c) Compare the sizes of $\angle KYZ$, and $\angle XYZ$. (The size of $\angle KYZ$ is smaller than the size of $\angle XYZ$.)

3. a) Cut along $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ and tear along the jagged curve. Fold along $\overrightarrow{YK}$. Does $\overrightarrow{YZ}$ fall along $\overrightarrow{YX}$? (No)

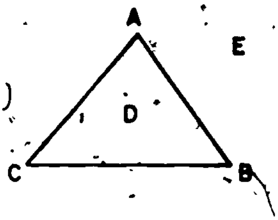
   b) Is $\angle XYK \cong \angle KYZ$? (No.)

4. In the interior of $\angle ZYX$, place a point N near Z and draw $\overrightarrow{YN}$. Fold along $\overrightarrow{YN}$. Which has the larger size, $\angle XYN$ or $\angle NYZ$? ($\angle XYN$)

5. Draw an angle. Name it $\angle MPR$. Choose a point (call it S) so that you can be sure the size of $\angle SPM$ is smaller than the size of $\angle MPR$. Where did you place S?
(S should be placed in the interior of $\angle MPR$.)

6. Using the angle of exercise 5, choose a point (call it T) so you can be sure that the size of ∠TPM is larger than the size of ∠MPR. Where did you place T? (If T is placed in the exterior of ∠MPR and on the opposite side of $\overrightarrow{PR}$ from M, then we can be sure that the size of ∠TPM is larger than the size of ∠MPR. There are other possibilities for T but this is the easiest time to explain.)

7. a) Is point D in the interior of ∠BAC shown in this figure? (yes)



b) Is it in the interior of ∠ABC? (yes) of ∠ACB? (yes)

8. a) Is E in the interior of ∠ACB shown in the figure? (yes)

b) Is it in the interior of ∠BAC? (no) of ∠CBA? (no)

9. a) Draw Δ ABC and label a point D as in the previous sketch. Then draw $\overrightarrow{AD}$.

b) What two angles are smaller in size than ∠CAB? (∠ CAD and ∠ DAB)

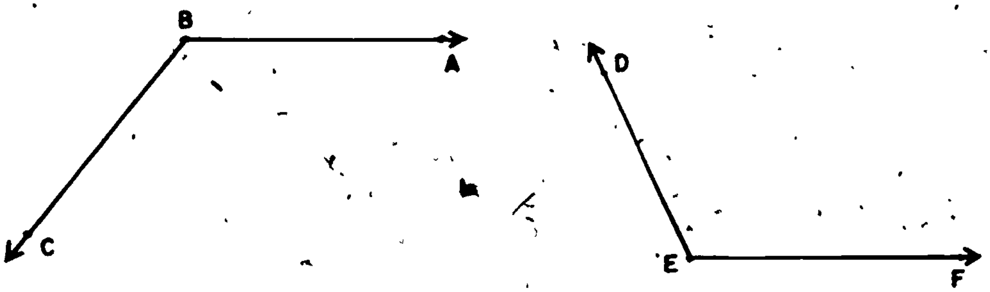
10. a) Draw a Δ ABC and label a point E as in the sketch above. Draw $\overrightarrow{BE}$.

b) What angle of Δ ABC is smaller in size than ∠EBC? (∠ ABC)

Angles Without a Common Ray

Exploration

You know how the sizes of two angles are compared when the two angles have one ray in common, or when the rays (except for the vertex) of one are in the interior of the other. How shall we compare the sizes of two angles which are not placed in either of these ways?



1. Copy $\angle DEF$ by tracing it on thin paper. Copy the letters, too.

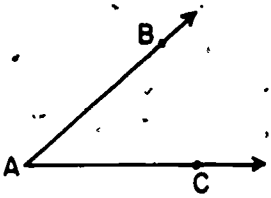
2. a) How should the rays of $\angle DEF$ be placed on $\angle ABC$ to compare the sizes of the angles? You may want to turn your tracing over.
   $\overrightarrow{ED}$ on $\overrightarrow{BA}$ and $\overrightarrow{EF}$ on the C side of $\overleftrightarrow{BA}$
   $\overrightarrow{EF}$ on $\overrightarrow{BA}$ and $\overrightarrow{ED}$ on the C side of $\overleftrightarrow{BA}$
   $\overrightarrow{ED}$ on $\overrightarrow{BC}$ and $\overrightarrow{EF}$ on the A side of $\overleftrightarrow{BC}$
   $\overrightarrow{EF}$ on $\overrightarrow{BC}$ and $\overrightarrow{ED}$ on the A side of $\overleftrightarrow{BC}$

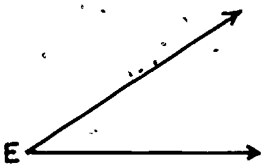
   b) Is there more than one way to place $\angle DEF$ in order to compare its size with that of $\angle ABC$? (yes, see the alternatives listed in 2a.)

3. How do the sizes of $\angle ABC$ and $\angle DEF$ compare?
   (The size of $\angle ABC$ is larger than the size of $\angle DEF$.)

Exercise Set 11

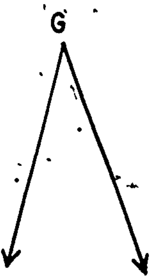
1. Trace ∠CAB on thin paper. Then compare the size of ∠CAB with the size of each angle below.



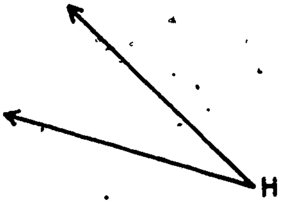


(The size of ∠CAB is larger than the size of ∠E.)

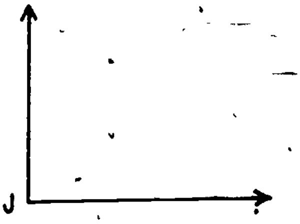
(The size of ∠CAB is smaller than the size of ∠F.)



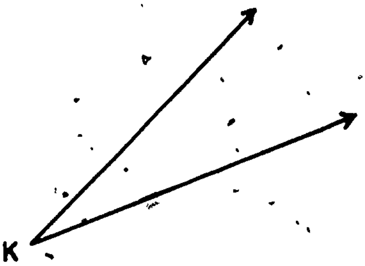
(The size of ∠CAB is larger than the size of ∠G.)



(The size of ∠CAB is larger than the size of ∠H.)



(The size of ∠CAB is smaller than the size of ∠J.)

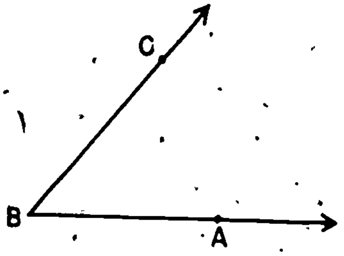


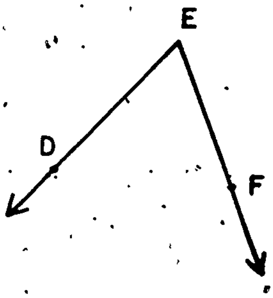
(The size of ∠CAB is larger than the size of ∠K)

Using the Congruent Angle Construction

Exploration

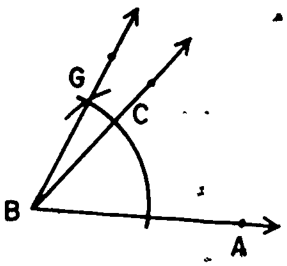
You know how to construct an angle congruent to a given angle, and you know that congruent angles have the same size. Can you use what you know to compare the sizes of two angles, no matter what their positions?

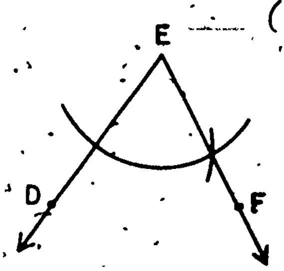




1. a) Look at ∠ABC and ∠DEF. Where should ∠DEF be copied, so as to compare the sizes? What point should you use as vertex?

   b) What ray should you use as one side of the copy? ($\overrightarrow{BA}$ or $\overrightarrow{BC}$)





2. a) In the figures, ∠ABG was constructed congruent to ∠DEF, so they have the same size. What angles can we compare now? (∠ABG and ∠ABC)

   b) What does this tell us about the sizes of ∠ABG and ∠ABC? (The size of ∠ABG is larger than the size of ∠ABC.)

3. a) In what other position could be copy ∠DEF to compare its size with the size of ∠ABC? Could we use some point other than B as vertex? (No)

   b) Could we use a ray different from $\overrightarrow{BA}$ as one side? (yes, $\overrightarrow{BC}$ could be used.)

   c) Could the comparison be the same? (yes)

4. a) Could we copy ∠ABC instead of ∠DEF? (yes)

   b) If so, what point should be the vertex? (E)

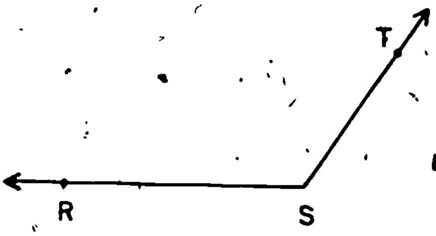
   c) What ray should be a side? ($\overrightarrow{ED}$ or $\overrightarrow{EF}$

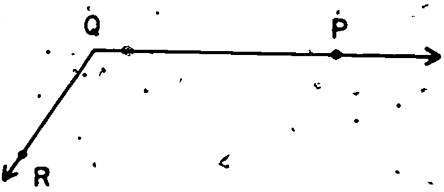
Exercise Set 12

1. Copy ∠ABC and ∠DEF by tracing them on thin paper. Use your compass and straightedge to construct an angle congruent to ∠DEF so you can compare the sizes of the angles.

(The size of ∠DEF is larger than the size of ∠ABC.)

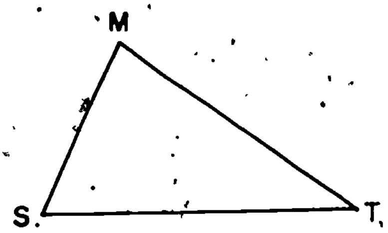
2. Compare the sizes of $\angle RST$ and $\angle PQR$.





(The sizes are the same. That is $\angle RST \cong \angle RQP$.)

3. Compare the sizes of $\angle ABC$ and $\angle MTS$.
(The size of $\angle ABC$ is smaller than the size of $\angle MTS$.)

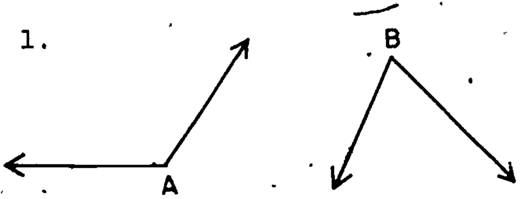


When you understand what is meant by "The size of $\angle A$ is larger than the size of $\angle B$," and what is meant by "$\angle A \cong \angle B$," you can often tell by looking at two angles which has the larger size. You can also tell whether they may be congruent.

Exercise Set 13

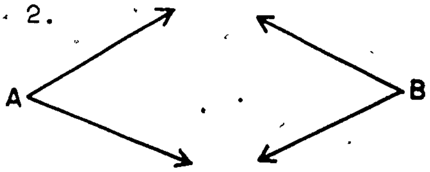
Compare the sizes of ∠A and ∠B in each pair below. If you can't decide which is larger, trace one angle on thin paper and place the tracing on the other angle, or use your compass and straightedge to construct congruent angles.

1.



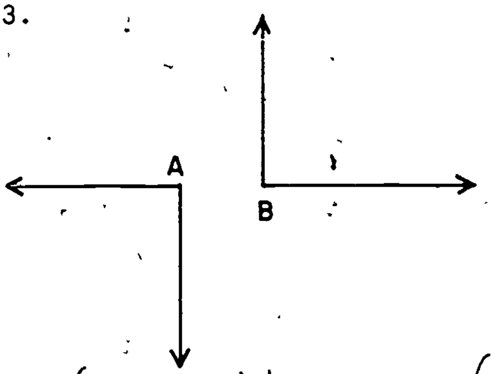
(The size of ∠A is larger than the size of ∠B.)

2.



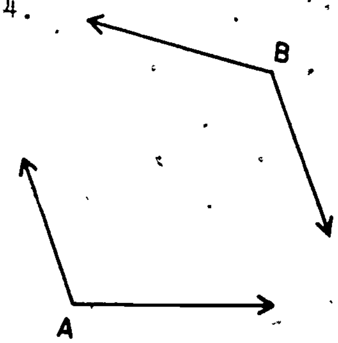
($\angle A \cong \angle B$)

3.



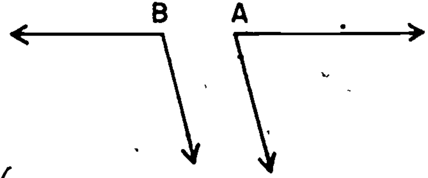
($\angle A \cong \angle B$)

4.



(The size of ∠A is smaller than the size of ∠B.)

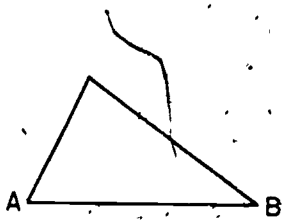
5.



(The size of ∠A is smaller than the size of ∠B.)

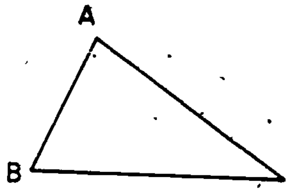
In the figures below, ∠A and ∠B are angles of triangles or angles of other polygons. In each figure, compare the sizes of ∠A and ∠B as you did in Exercises 1 to 5.

6.



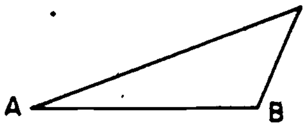
(The size of ∠A is larger than the size of ∠B.)

7.



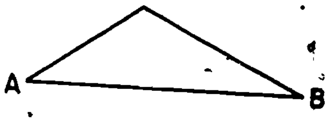
(The size of ∠A is larger than the size of ∠B.)

8.



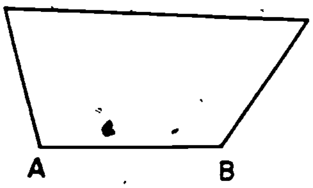
(The size of ∠A is smaller than the size of ∠B.)

9.



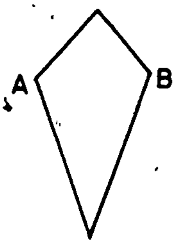
(The size of ∠A is larger than the size of ∠B.)

10.



(The size of ∠A is smaller than the size of ∠B.)

11.



(∠A ≅ ∠B.)

SUGGESTED TEST ITEMS

These sample test questions are meant to serve as suggestions for types of items which the teacher may want to include in a unit test.

1. Choose the item from Column 2 that matches each item in Column 1. Write the word in the space provided.

A. Matching Symbols

Column 1	Column 2
(triangle) $\triangle$	a. ray
(congruent) $\cong$	b. line
ray $\overrightarrow{AB}$	c. segment
(angle) $\angle$	d. angle
(segment) $\overline{DE}$	e. triangle
(a is greater than b) $a > b$	f. a is greater than b
(line) $\overleftrightarrow{GH}$	g. a is less than b
(a is less than b) $a < b$	h. congruent

B. Matching the word with the sentence that describes it.

(isosceles) A triangle with only two sides that are congruent

(arc) A connected part of a circle

(angle) A set of points of two rays which have a common endpoint and which do not lie in the same straight line

(isosceles) A triangle which has at least two sides which are congruent to each other

(segment) A part of a line which includes two endpoints and all points of the line between them

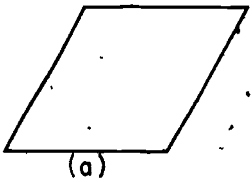
(vertex) The intersection of two sides of a triangle

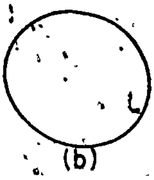
(equilateral) A triangle which has three angles, each congruent to the other two

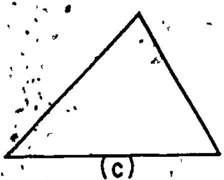
(circle) The set of points in a plane all of which are equidistant from a given point.

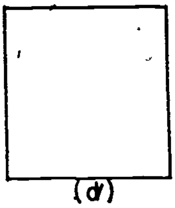
i. angle
j. segment
k. isosceles
l. vertex
m. equilateral
n. arc
o. circle

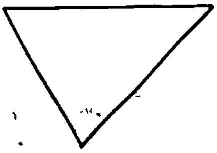
2. Choose the pairs of figures which appear to be congruent.

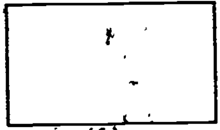

(a)

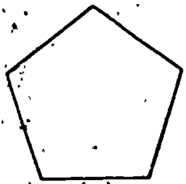

(b)

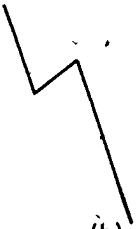

(c)

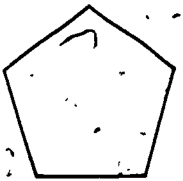

(d)


(e)

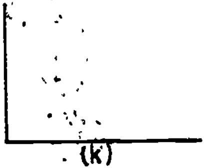

(f)

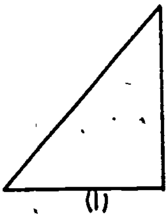

(g)

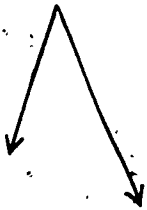

(h)

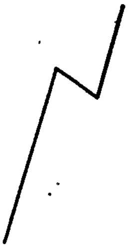

(i)

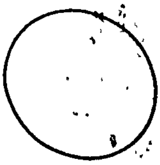

(j)

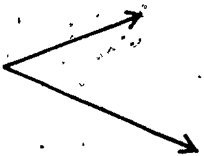

(k)


(l)


(m)


(n)


(o)


(p)

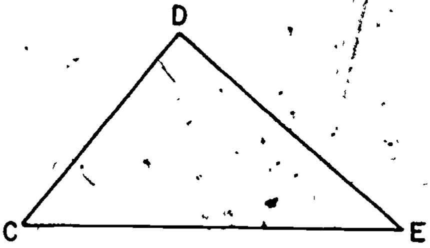
(answer: (b)and(o). (e)and(c). (g)and(i) (h and n). (m)and(p). (j)and(k)

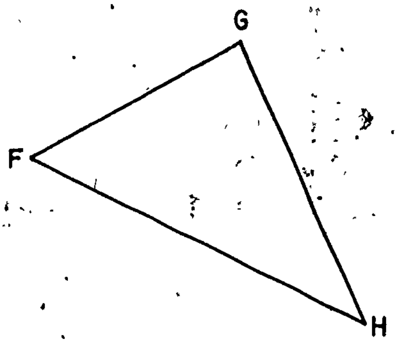
3. Suppose we know that △ PQR ≅ △ STW.

a) List the corresponding vertices. (P and S) (Q and T) (R and W)

b) List the corresponding sides. ($\overline{PQ}$ and $\overline{ST}$ $\overline{QR}$ and $\overline{TW}$ $\overline{RP}$ and $\overline{WS}$)

4. Suppose we know that $\triangle CDE \cong \triangle FGH$. List the congruent angles.





$\left(\begin{array}{l} \angle DCE \cong \angle GFH \\ \angle CED \cong \angle FHG \\ \angle EDC \cong \angle HGF \end{array} \right)$

5. a) Suppose you have two triangles, $\triangle ABC$ and $\triangle DEF$. All you know about them is that $\overline{AB} \cong \overline{EF}$. Can you be certain that the two triangles are congruent? (no)

b) Suppose you have two triangles, $\triangle RST$ and $\triangle XZY$. All you know about them is that

$\overline{RS} \cong \overline{XZ}$,

$\overline{ST} \cong \overline{ZY}$, and

$\overline{RT} \cong \overline{XY}$.

Can you be certain that the two triangle are congruent? (yes)

c) Suppose you have two triangles $\triangle GHI$ and $\triangle JKL$. All you know about them is that

$\overline{HI} \cong \overline{KL}$ and

$\overline{GI} \cong \overline{JL}$.

Can you be certain that the two triangles are congruent? (no)

d) Suppose you have two triangles, $\triangle MNO$ and $\triangle PQR$. All you know about them is that

$\angle M \cong \angle P$,

$\angle N \cong \angle Q$, and

$\angle O \cong \angle R$.

(yes)

Can you be certain that the two triangles are congruent?

e) Suppose you have two triangles, $\triangle STU$ and $\triangle VWX$. All you know about them is that

S corresponds to V,

T corresponds to W, and

U corresponds to X.

(yes)

Can you be certain that the two triangles are congruent?

6. Choose the situations which you think best illustrate the use of the idea of congruence.

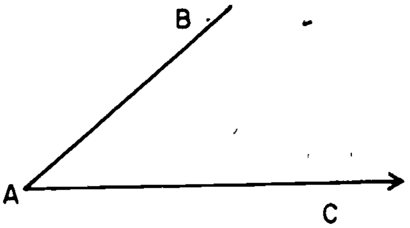
a) Lining shelves of a dish closet with paper.

b) Covering living room floor with wall-to-wall carpeting.

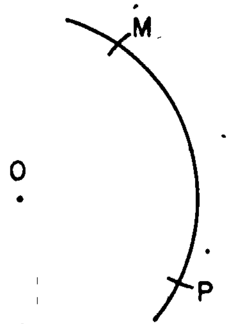
c) Enlarging a photograph.

d) Fitting a coffee table with a glass top.

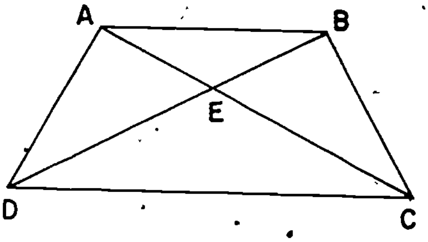
7. Use your compass and straightedge to copy $\overline{AB}$ on $\overrightarrow{AC}$ so that A is one endpoint of the copy.



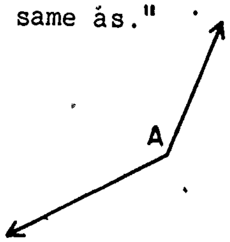
8. Point O is the center of the circle of which MP is an arc. Use only your straightedge to draw three line segments of the same length in this figure.

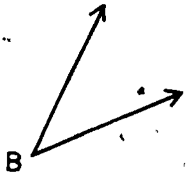


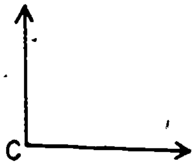
9. Use your compass to find four different pairs of congruent segments in the figure. List your answers. (AD and BC, AE and BE, DE and CE, AC and BD)

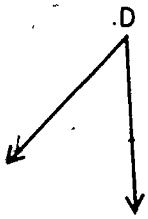


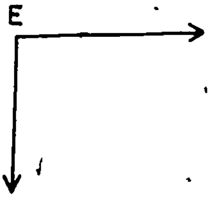
10. Complete the following sentences to compare the sizes of angles: Use "larger than," "smaller than," or "about the same as."

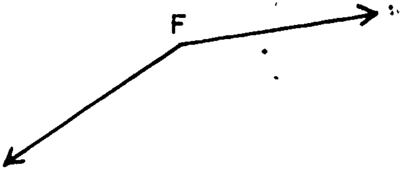






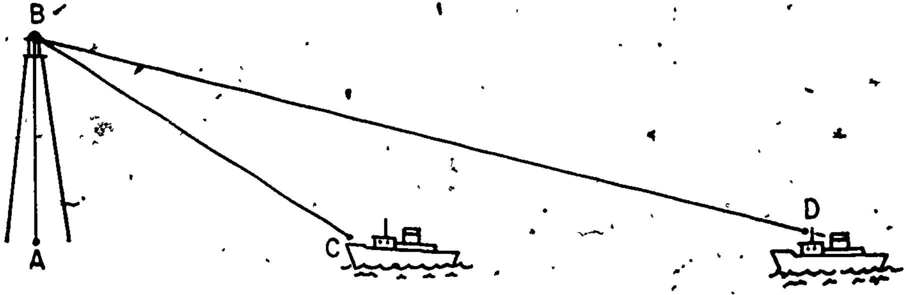




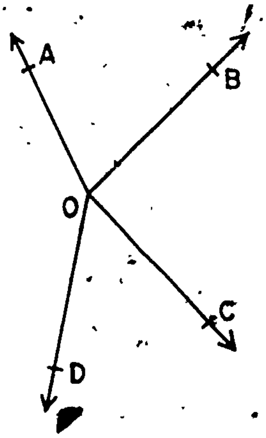


1. The size of ∠A is (greater than) the size of ∠C.
2. The size of ∠B is (smaller than) the size of ∠E.
3. The size of ∠C is (smaller than) the size of ∠F.
4. The size of ∠E is (about the same as) the size of ∠C.
5. The size of ∠A is (smaller than) the size of ∠F.
6. The size of ∠D is (larger than) the size of ∠B.
7. The size of ∠E is (smaller than) the size ~~of~~ ∠A.
8. The size of ∠B is (smaller than) the size of ∠C.

11. Bob is at the top of a lighthouse. He sees two ships C and D as shown below. A, C, and D are on the same line.



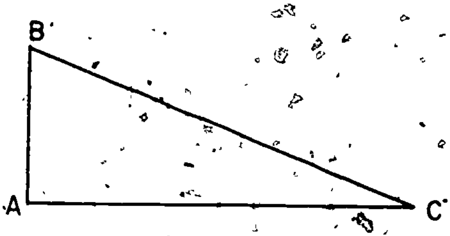
Is the size of $\angle ABD$ greater than, less than, or the same as the size of $\angle ABC$?
(The size of angle ABD is greater than the size of angle ABC)



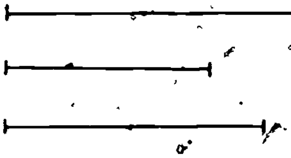
12. a) Use compass and straightedge to show that the size of $\angle AOB$ is smaller than the size of $\angle BOC$.

b) Use compass and straightedge to show that the size of $\angle BOC$ is larger than the size of $\angle DOC$.

13. Use your compass and straightedge to make a copy of $\triangle ABC$ on your paper.

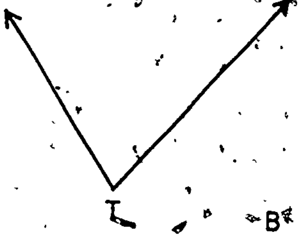


14. Use your compass and straightedge to make a triangle whose sides have the lengths of the three given line segments.

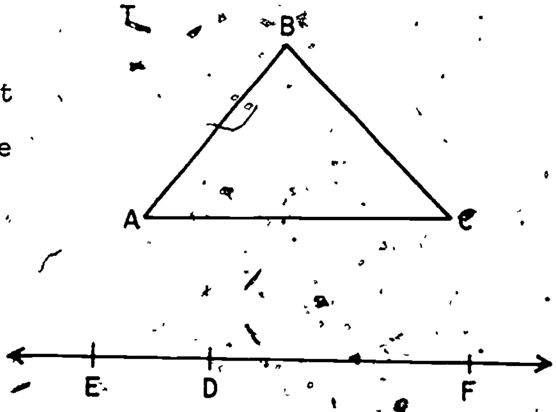


15. Make a triangle whose sides have measures, in inches of 2, 4, and 8, if possible. (Not possible, since the sum of the measures of two sides of a triangle must be greater than the measure of the third side)

16. Use your compass and straightedge to copy $\angle T$ on your paper.



17. Use your compass and straightedge to copy $\angle A$ so that the copy has point D as its vertex; one side shall be $\overrightarrow{DE}$ and the interior of the angle shall be below $\overleftrightarrow{EF}$.



Chapter 5

EXTENDING MULTIPLICATION AND DIVISION II

PURPOSE OF UNIT

The purpose of this unit is to help pupils become more proficient in multiplying and dividing using large numbers.

MATHEMATICAL BACKGROUND

The mathematical background for this unit is presented in Chapter 3.

TEACHING THE UNIT

This chapter is organized in the following way:

1. There are teaching suggestions and explorations which appear only in the teacher's commentary.
2. There are summaries and explorations which appear only in the pupil text.
3. There are pupil exercises to be done independently.

It is recommended that whenever exploration sections appear in the commentary, these should be followed before work is done with pupils on the material in the pupil text.

The explorations in the pupil text are designed to serve as guides to pupil discovery. They are to be read and discussed by teacher and pupils. It is essential that teachers be thoroughly familiar with the teaching suggestions, which usually precede the explorations, as well as the explorations themselves before lessons are undertaken.

In those few instances where additional teaching suggestions are not given, it is recommended that the teacher take time to consider what possible questions or difficulties might arise in his particular class.

The development and utilization of shortened forms in the division process is probably more individual than many other skills which pupils acquire. Therefore, the teacher must be particularly alert to the thinking of his pupils. He must be ready to offer leading questions especially in relation to multiples, place value, and "helpers" to aid children in their own discoveries.

It should be emphasized that pupils shorten their work only to their level of understanding. Pupils should not be encouraged to adopt shorter procedures they are not able to comprehend. On the other hand, when a pupil evidences that he is able to shorten his work with understanding, he should be encouraged to do so.

It must be recognized that some pupils may not be ready to shorten their work as quickly as others during the course of this chapter. Such pupils should not be forced to do so at this time. Rather, they are to be encouraged throughout the rest of the year to shorten their work as they become able.

Maintenance and improvement of techniques of division must not be neglected after the conclusion of this unit; rather, they must be continued throughout the fifth and sixth grades.

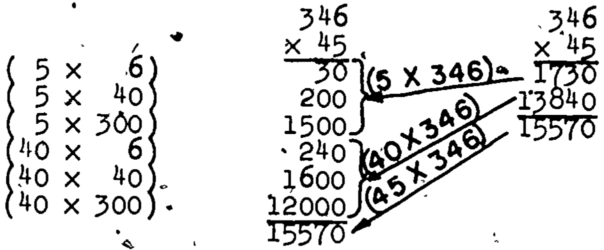
MULTIPLYING LARGE NUMBERS

Objective: To help pupils become more proficient in multiplying using large whole numbers

Teaching Suggestions:

In this chapter pupils learn that a knowledge of place value affords a shorter and more efficient algorism for multiplication.

As an introduction to this chapter, review multiplication as follows. Compare the two forms only if pupils need the review. Pupils who are not using a short form should be encouraged to do so. Yet, the longer form or modification of it may be more desirable for individual pupils.



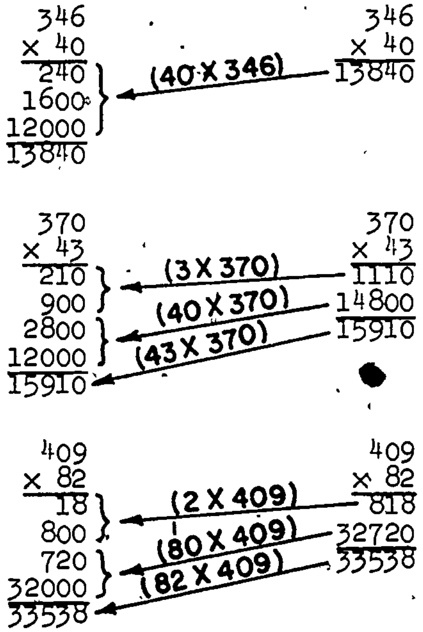
Examples like the ones below sometimes offer unexpected problems to children. For this reason, some like these should be included during an exploration lesson.

40 × 346,

43 × 370,

82 × 409, etc.

These examples may be worked in different ways according to the level of achievement of the pupils.



After review, extend the scope of multiplication examples to include larger numbers. Such exercises as

542 × 836 and

56 × 9578

should be worked together by pupils and teacher.

Attention should be given to the way in which partial products are obtained.

Before assigning Exercise Set 1, read and discuss with pupils the section entitled <u>Multiplying Large Numbers</u> in the pupil text.

After Exercise Set 1 has been completed, read with the pupils the section entitled <u>Multiplying Larger Numbers</u>. Children then should be able to complete Exercise Set 2 independently.

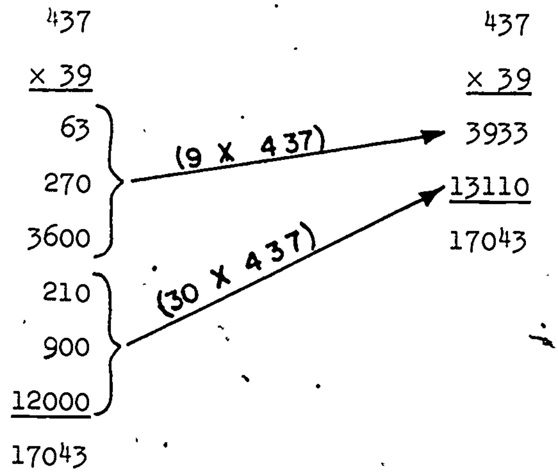
Chapter 5

EXTENDING MULTIPLICATION AND DIVISION II

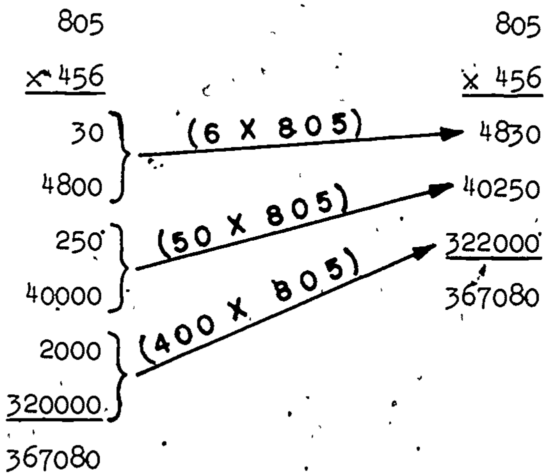
MULTIPLYING LARGE NUMBERS

In Chapter 3 you learned how to find the product of two numbers. Now we want to find shorter ways to find these products. Let's look at these multiplication examples.

Example 1: Multiply 437 and 39.



Example 2 Multiply 456 and 805



Explain how to get each of the partial products in the shorter form of these examples.

Exercise Set 1

Use a vertical form to compute the following.

1. 86 x 923 (79,378)
2. 48 x 654 (31,392)
3. 57 x 874 (49,818)
4. 473 x 52 (24,596
5. 36 x 504 (18,144)
6. 56 x 780 (43,680)
7. 68 x 5346 (363,528)
8. 76 x 3498 (265,848)
9. 4038 x 79 (319,002)
10. 57 x 7239 (412,623)
11. 625 x 834 (521,250)
12. 658 x 762 (501,396)
13. 846 x 648 (548,208)
14. 607 x 546 (331,422)
15. 971 x 356 (345,676)
16. 656 x 750 (492,000)
17. 720 x 856 (616,320)
18. 384 x 507 (194,688)
19. 834 x 720 (600,480)
20. 345 x 637 (219,765)

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

21. There are 64 rows of seats in the auditorium. There are 45 seats in each row. How many people can be seated in the auditorium?
(64 x 45 = n
n = 2,880
2,880 people can be seated in the auditorium.)

22. John kept a record of how much gasoline his family car used on their vacation last summer. They used 167 gallons. If they can travel 18 miles on each gallon of gas, how many miles did they travel during their vacation? (18 x 167 = n , n = 3006 They traveled 3006 miles during their vacation.)

23. A brick wall is 126 bricks long and 42 bricks high. How many bricks are there in the wall?
(42 x 126 = n , n = 5,292 There are 5292 bricks in the wall.)

24. If 76 nails are used in making a shoe, how many nails are needed to make 23 pairs of these shoes?
(2 x 23 = p
46 x 76 = n
n = 3496
or
2 x 23 x 76 = n
n = 3496
3,496 nails are needed to make 23 pairs of shoes.)

25. A helicopter makes a round trip of 102 miles three times daily to collect and deliver mail in the San Francisco Bay area. How many miles does it travel in a year? (Note: Use 365 days.)
(3 x 102 = p
306 x 365 = n
n = 111,690
or
(3 x 102) x 365 = n
n = 111,690
The helicopter travels 111,690 miles in a year.)

MULTIPLYING LARGER NUMBERS

Example 1: Multiply 4365 and 7439.

```
    7439
  × 4365
   37195
  446340
 2231700
29756000
32471235
```

How many partial products are there in this example? (4)

Example 2: Multiply 5063 and 8309.

```
    8309
  × 5063
   24927
  498540
41545000
42068467
```

Notice that there are only 3 partial products in this example. Explain how each of these partial products was obtained.

Multiply the numbers in the following example and compare the product with the product in example 2.

```
  5063
× 8309
```

Are the products the same? (Yes) Why? (Because multiplication is a commutative operation.)

Are the partial products the same? (No) Why? (Because the partial products are found using different factors.)

Exercise Set 2

Use a vertical form to find the product of each of these pairs of numbers.

1. 537 and 4372 (2,347,764)
2. 200 and 317 (63,400)
3. 96 and 897 (86,112)
4. 4569 and 5007 (22,876,983)
5. 957 and 8060 (7,713.420)
6. 357 and 892 (318,444)
7. 5430 and 739 (4,012,770)
8. 709 and 5080 (3,601,720)
9. 101 and 523 (52,823)
10. 3586 and 367 (1,316,062)
11. 3542 and 4673 (16,551,766)
12. 234 and 3112 (728,208)
13. 909 and 673 (611,757)
14. 231 and 706 (163,086)
15. 3570 and 4987 (17,803,590)
16. 8971 and 6173 (55,377,983)
17. 2003 and 2131 (4,268,393)
18. 3672 and 4819 (17,695,368)
19. 8080 and 5599 (45,239,920)
20. 2712 and 3486 (9,454,032)

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

21. A cab driver makes many trips to and from a large city airport. He drives about 315 miles a day. About how many miles does he drive in 28 days?
($28 \times 315 = n$
$n = 8,820$ He drives 8,820 miles in 28 days.)

22. A grapefruit orchard has 32 rows of grapefruit trees with 45 trees in each row. How many trees are there in the orchard? ($32 \times 45 = n$
$n = 1440$ There are 1440 trees in the orchard.)

23. A jet plane travels 485 miles per hour on the average. One month it is flown 114 hours. If that is an average month, how many miles is it flown in a year?
($114 \times 485 \times 12 = n$
$n = 663,480$ or $114 \times 485 = b$
$55,290 \times 12 = n$
$n = 663,480$ The plane is flown 663,480 mile in a year.)

24. The Lincoln family spent $224 for an 8-day trip. If they spent the same amount each day, how much should they plan to save for next year's 21-day trip? ($224 \div 8 = p$
$28 = p$
$28 \times 21 = n$
$n = 588$ or $(224 \div 8) \times 21 = n$
$n = 588$ They should plan to save $588 for next year's trip.)

25. There were 103 passengers on a jet plane going from New York to Toronto. Each passenger was allowed to take 66 pounds of luggage without charge. If each passenger took the full amount, how many pounds of free luggage were carried?
($66 \times 103 = n$
$n = 6798$. 6,798 pounds of free luggage were carried.)

A SHORTER FORM FOR MULTIPLYING

Objective: To develop a shorter form for multiplying

Teaching Suggestions:

At this time it might be well to call attention to a way of shortening the form of recording partial products.

Put several examples on the chalkboard and ask pupils to discover a shorter way which has been used to record the partial products. As in the earlier development, pupils should not be given a rule but should be led through examples to discover one for themselves, although they may not be able to verbalize it precisely. Neither should all pupils be expected to arrive at the same level of achievement at the same time. Of course, pupils should be encouraged to use shorter forms as soon as they appear ready for them.

You may wish to use such examples as the following:

$$\begin{array}{r} 562 \\ \underline{\times\ 47} \\ 3934 \\ \underline{22480} \\ 26414 \end{array} \qquad \begin{array}{r} 562 \\ \underline{\times\ 47} \\ 3934 \\ \underline{2248} \\ 26414 \end{array}$$

$$\begin{array}{r} 362 \\ \underline{\times\ 475} \\ 1810 \\ 25340 \\ \underline{144800} \\ 171950 \end{array} \qquad \begin{array}{r} 362 \\ \underline{\times\ 475} \\ 1810 \\ 2534 \\ \underline{1448} \\ 171950 \end{array}$$

A SHORTER FORM FOR MULTIPLYING

Study the following examples. See what has been done to shorten the way we record the partial products. Why can we do this?

Example 1:

$$\begin{array}{r} 5476 \\ \times\ 3528 \\ \hline 43808 \\ 109520 \\ 2738000 \\ 16428000 \\ \hline 19319328 \end{array} \qquad\qquad \begin{array}{r} 5476 \\ \times\ 3528 \\ \hline 43808 \\ 10952 \\ 27380 \\ 16428 \\ \hline 19319328 \end{array}$$

Example 2:

$$\begin{array}{r} 439 \\ \times\ 605 \\ \hline 2195 \\ 263400 \\ \hline 265595 \end{array} \qquad\qquad \begin{array}{r} 439 \\ \times\ 605 \\ \hline 2195 \\ 2634 \\ \hline 265595 \end{array}$$

Exercise Set 3

Use a vertical form to find the product of each of these pairs of numbers.

1. 47 and 63 (2,961)
2. 92 and 78 (7,176)
3. 478 and 356 (170,168)
4. 4234 and 6209 (26,288,906)
5. 465 and 688 (319,920)
6. 407 and 629 (256,003)
7. 634 and 6070 (3,848,380)
8. 97 and 401 (38,897)
9. 392 and 847 (332,024)
10. 54 and 286 (15,444)
11. 25 and 2359 (58,975)
12. 465 and 750 (348,750)
13. 3049 and 4340 (13,232,660)
14. 89 and 76 (6,764)
15. 7294 and 325 (2,370,550)
16. 58 and 1289 (74,762)
17. 73 and 496 (36,208)
18. 207 and 639 (132,273)
19. 36 and 74 (2,664)
20. 66 and 247 (16,302)

EXPRESSING NUMBERS TO THE NEAREST MULTIPLE OF TEN

Objective: To develop skill in expressing numbers to the nearest multiple of 10

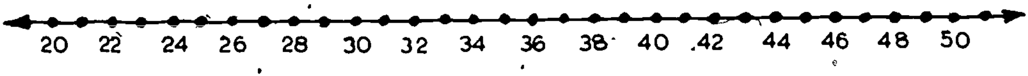
Teaching Suggestions:

Expressing numbers as multiples of 10 and 100 is useful in dividing by larger numbers. Although there are several techniques for expressing numbers as multiples, only one technique is used throughout the unit.

The number line is a helpful visual aid, therefore it is used throughout the exploration. You should have a number line with points labeled from 20 through 50 on the chalkboard before the lesson begins.

It is important that children be led to discover a way to determine the nearer multiple of 10 to any number. The development of formal rules should be avoided because it frequently leads to rote learning rather than understanding.

Exploration: Look at the number line on the chalkboard.



Find 28 on the number line. Is it closer to 20 or 30? (30) If you were asked to express 28 to the nearer multiple of 10, would you choose 20 or 30? (30) Why? (28 is nearer to 30 than 20 on the number line.)

Find 37 on the number line. Is it closer to 30 or 40? (40) If you were to express 37 to the nearer multiple of 10, would you choose 30 or 40? (40)

Is 24 closer to 20 or to 30? (20) How should 24 be expressed to the nearer multiple of 10? (20)

Consider the points shown from 20 through 30. If we were to choose the nearer multiple of 10 to 21, 22, 23, or 24, what number should we choose? (20) If we were to choose the nearer multiple of 10 to 26, 27, 28, or 29, what number should we choose? (30)

What about 25? Is it closer to 20 or 30? (25 is the same distance from 20 as from 30.)

How can we choose the nearer multiple of 10 to 25? Should we choose 20 or 30? (We don't know.)

Except when we have 5 in the ones' place, it is easy to see the nearest multiple of 10 to a number on the number line. How can we know the nearest multiple of ten to a number when we have no number line? Is there a way to discover quickly the nearest multiple of 10 to any number?

Some child will suggest that if the ones' digit is less than 5, think of the next lower multiple of 10. If the ones' digit is greater than 5, think of the next greater multiple of 10. You should not expect the child to state this idea in such precise language, nor is it desirable that he do so. It is important for children to be able to understand and use this knowledge.

When we have a 5 in the ones' place, lead children to see that to find the nearest multiple of 10 we will have to make an agreement that everyone will do the same thing. Lead children to agree to use the next higher multiple of 10.

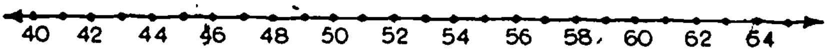
Let's find how well we can use our new way to find the nearest multiple of 10 to a number. What is the nearest multiple of 10 to 42? (40) to 56? (60) to 75? (80) to 49? (50) to 15? (20)

Consider 144. What number would we use as the nearest multiple of 10 to 144? Should it be 140 or 150? (140) Why? (If we had a number line, 144 is nearer to 140 than 150.) Do you think the way we found the nearest multiple of 10 earlier will help us with numbers like 144? (Yes)

What is the nearest multiple of 10 to 279? (280) to 345? (350) to 572? (570)

Read and discuss with the pupils the section in the pupil text entitled Expressing Numbers to the Nearer Multiple of Ten. If you feel that additional practice is necessary, provide other oral or written exercises.

EXPRESSING NUMBERS TO THE NEAREST MULTIPLE OF TEN



We have used a number line to help us see that:

53 is nearer to 50 than 60.

58 is nearer to 60 than 50.

We have discovered a way to find the nearest multiple of 10 to a number without using a number line.

What is the nearest multiple of 10 to each of these numbers?

92 (90)

49 (50)

75 (80)

83 (80)

17 (20)

61 (60)

34 (30)

46 (50)

58 (60)

25 (30)

383 (380)

285 (290)

567 (570)

684 (680)

139 (140)

134 (130)

288 (290)

476 (480)

341 (340)

675 (680)

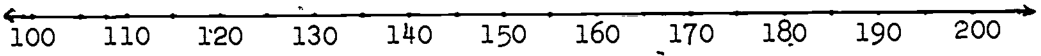
EXPRESSING NUMBERS TO THE NEAREST MULTIPLE OF ONE HUNDRED

Objective: To develop skill in expressing numbers to the nearest multiple of 100

Materials: A number line numbered 100 through 200

Exploration:

Look at the number line on the chalkboard.



Find 160 on the number line. Is it nearer to 100 or 200? (200) What number would we use as the nearer multiple of 100 to 160? (200)

Find 125 on the number line. Is it nearer to 100 or 200? (100) What number should we choose as the nearer multiple of 100 to 125? (100) How could we find these multiples if we did not have the number line? (We could work with multiples of 100 just as we did with multiples of 10 only now we look at the tens' place of our numeral.)

Is 150 nearer to 100 or 200? (It is half-way between them.)

When we were finding the nearest multiple of 10 to numbers like 45, 65, 125, etc., what did we do? (We agreed to choose the higher multiple when the number was half-way between the two multiples.)

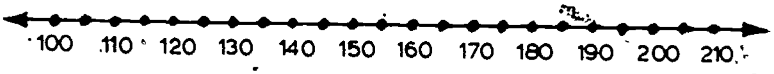
What shall we do here? (Let us again agree to choose the next higher multiple of 100.)

What shall we choose for 150? (200) 350? (400)

What is the nearest multiple of 100 to each of these numbers?

170	(200)	195	(200)
212	(200)	486	(500)
429	(400)	130	(100)
128	(100)	253	(300)
250	(300)	750	(800)

EXPRESSING NUMBERS TO THE NEAREST MULTIPLE OF ONE HUNDRED



We have used a number line to help us see that:

142 is nearer to 100 than 200;

167 is nearer to 200 than 100.

We have discovered a way to find the nearest multiple of 100 to a number without using a number line.

What is the nearest multiple of 100 to each of these numbers?

145 (100)

155 (200)

186 (200

174 (200)

156 (200)

253 (300)

203 (200)

850 (900)

290 (300)

224 (200)

450 (500)

230 (200)

346 (300)

304 (300)

572 (600)

666 (700)

623 (600)

650 (700)

857 (900)

749 (700)

DIVISION

Objective: To help pupils become more proficient in dividing using large numbers

Teaching Suggestions:

The exploration in the pupil text reviews the language and techniques of division. Increased emphasis is placed on the importance of place value in using shorter forms. At the end of the review, it is suggested that pupils try to find a still shorter way of dividing writing only the quotient and remainder. Encourage children to shorten their work. Do not tell them at this time. With understanding, children can develop short cuts on their own.

Read and discuss Review of Division in the pupil text. If some pupils cannot shorten their work readily, reassure them that they will receive help later in the chapter. Do not dwell on a shorter form at this time; rather, be supportive.

REVIEW OF DIVISION

Exploration

In Chapter 3 we learned about a shorter form for dividing. The boxes below show several forms for dividing 836 by 6.

Longer Forms

$$\begin{array}{rr} & \underline{139} \\ & 9 \\ & 30 \\ & 100 \\ 6 & \overline{)836} \\ & \underline{600} \\ & 236 \\ & \underline{180} \\ & 56 \\ & \underline{54} \\ & 2 \end{array}$$

$$\begin{array}{rr|r} 6 & \overline{)836} & \\ & \underline{600} & 100 \\ & 236 & \\ & \underline{180} & 30 \\ & 56 & \\ & 54 & 9 \\ \cline{2-3} & 2 & 139 \end{array}$$

A Shorter Form

$$\begin{array}{rr} & 139 \\ 6 & \overline{)836} \\ & \underline{600} \\ & 236 \\ & \underline{180} \\ & 56 \\ & \underline{54} \\ & 2 \end{array}$$

When 836 is divided by 6, what is the quotient? What is the remainder?

Find a mathematical sentence that tells us that when we divide 836 by 6, the quotient is 139 and the remainder is 2.

We may say that 100 and 30 and 9 are parts of the quotient. Using place value, explain how the shorter form tells us this.

In this chapter we are going to learn about dividing by larger numbers. We also will learn things that can help us become more skillful when we divide.

Can you find a short way to divide 928 by 6 so that you need to write only the quotient and remainder?

If you cannot discover this short way of dividing, this chapter will help you with it later.

Exercise Set 4

For each of the following, divide the first number by the second. Write a mathematical sentence to describe the result.

1. 579 by 8
   $579 = (72 \times 8) + 3$
2. 6847 by 9
   $6847 = (760 \times 9) + 7$
3. 4496 by 8
   $4496 = 562 \times 8$
4. 4701 by 8
   $4701 = (587 \times 8) + 5$
5. 1728 by 9
   $1728 = 192 \times 9$
6. 2505 by 5
   $2505 = 501 \times 5$
7. 4758 by 9
   $4758 = (528 \times 9) + 6$
8. 1690 by 5
   $1690 = (338 \times 5) + 0$
9. 5670 by 6
   $5670 = 945 \times 6$
10. 3549 by 5
    $3549 = (709 \times 5) + 4$
11. 5535 by 7
    $5535 = (790 \times 7) + 5$
12. 6572 by 8
    $6572 = (821 \times 8) + 4$

DIVIDING BY NUMBERS GREATER THAN 10 AND LESS THAN 100

Teaching Suggestions:

Throughout the remainder of this unit, considerable exploratory material has been included in the pupil book. It is important to follow the development carefully. Note that much of it is done by raising questions. These are not necessarily all of the questions that need to be asked about the examples. Indeed, you may need to ask many additional questions. It is hoped that pupils by thinking, discussing, and computing will develop insight into the process.

Essentially, the intent of this unit is to guide through inquiry, rather than to achieve rote learning.

DIVIDING BY NUMBERS GREATER THAN 10 AND LESS THAN 100

Exploration

Let us divide 859 by 23. First, we will use one of the long forms. After we do this, maybe you can see how we can use a shorter form.

$$\begin{array}{r} 30 \\ 23\,\overline{)\,859} \\ \underline{690} \\ 169 \end{array}$$

$$\begin{array}{rr|l} 23 & \overline{)\,859} & \\ & \underline{690} & 30 \\ & 169 & \end{array}$$

A. Will the quotient be at least 10? (Yes, because 23 × 10 = 230 and 230 < 290.)

Will the quotient be as great as 100? (No, because 23 × 100 = 2300 and 2300 > 859.)

What does this information tell us? (The quotient will be greater than 10 but less than 100.)

B. We can use multiples of 10 to help us find part of the quotient.

What are the multiples of 10 that are less than 100? (10, 20, 30, ..., 90)

We try to find the largest multiple of 10 that will be part of the quotient.

What is 10 × 23? (230) What is 30 × 23? (690)

What is 20 × 23? (460) What is 40 × 23? (920)

Have we found the largest multiple of 10 that will be part of the quotient? (yes) What is it? (30)

How do we know that 30 is the largest multiple of 10 that will be part of the quotient? (30 × 23 = 690; 40 × 23 = 920, and 920 > 859.)

Now explain the work shown in the boxes near the top of the page.

C. Now we will find the remaining part of the quotient.

How do we know that the remaining part of the quotient will be less than 10? ($10 \times 23 = 230$, and $230 > 169$.)

We try to find the largest number so that that number times 23 will be no greater than 169. What is it? (7)

How did you find that 7 is the largest number to use? (Answers will vary).

Now explain how the work in the boxes below was completed.

```
        37
         7
        30
  23 ) 859
       690
       169
       161
         8
```

```
  23 ) 859 |
       690 | 30
       169 |
       161 |  7
         8 | 37
```

We divided 859 by 23.

What is the quotient? (37)

What is the remainder? (8)

Write a mathematical sentence that tells us these things.
$[859 = (37 \times 23) + 8]$

Show how to check your work.

```
   23
  ×37
  161
  69
  851
   +8
  859
```

Now let us divide 1724 by 67. Two forms for doing this are shown in the boxes below.

```
         25
          5
         20
67 ) 1724
     1340
      384
      335
       49
```

```
67 ) 1724 |
     1340 |  20
      384 |
      335 |   5
       49 |  25
```

Answer these questions about the division.

How do we know that the quotient must be greater than 10 but less than 100? ($10 \times 67 = 670$, $100 \times 67 = 6700$, Since $670 < 1724$ and $6700 > 1724$, the quotient must be greater than 10 and less than 10)

Multiples of 10 help us find the first part of the quotient. How can we find the largest multiple of 10 to use as the first part of the quotient? What is it? (20)

How do we know that the remaining part of the quotient will be less than 10? (Because $10 \times 67 = 670$ and $670 > 384$.)

How can we find the remaining part of the quotient? What is it? (5)

We divided 1724 by 67.

What is the quotient? (25)

What is the remainder? (49)

Write a mathematical sentence that tells us these things.

[$1724 = (25 \times 67) + 49$]

Exercise Set 5

Divide the first number by the second number. Write a mathematical sentence to describe the result.

1. 604 by 82
   $[604 = (82 \times 7) + 30]$

2. 340 by 41
   $[340 = (41 \times 8) + 12]$

3. 2681 by 39
   $2681 = (39 \times 68) + 29$

4. 2464 by 57
   $[2464 = (57 \times 43) + 13]$

5. 695 by 94
   $[695 = (94 \times 7) + 37]$

6. 4090 by 73
   $[4090 = (73 \times 56) + 2]$

7. 5136 by 66
   $[5136 = (66 \times 77) + 54]$

8. 184 by 27
   $[184 = (27 \times 6) + 22]$

9. 6434 by 75
   $[6434 = (75 \times 85) + 59]$

10. 5103 by 88
    $[5103 = (88 \times 57) + 87]$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

11. It cost \$128 for a bus to take 32 fifth-graders to the state capitol. How much does each pupil have to pay? [$128 = (32 \times n) + r$. Each pupil has to pay \$4.00]
$n = 4$

12. A box holds 24 books. How many boxes will be needed to hold 984 books? ($984 = (24 \times n) + r$ 41 boxes will be needed.)
$41 = n$

13. A store ~~had~~ a sale on one model of a bicycle. 68 bicycles of this model were sold for a total amount of \$2,856. What was the sale price of a bicycle?
[$2856 = (68 \times n) + r$ The sale price of a bicycle was \$42.]
$42 = n$

14. Jane has 630 stamps that she wants to put into envelopes. If she puts 45 stamps in each envelope, how many envelopes will she need?
[$630 = (45 \times n) + r$ Jane will need 14 envelopes.]
$14 = n$

15. An automobile is moving at a speed of 28 feet per second. How many seconds will it take it to move 980 feet? [$980 = (28 \times n) + r$ It will take the automobile 35 seconds to move 980 feet.]
$35 = n$

FINDING SHORTER WAYS OF DIVIDING

Teaching Suggestions:

This pupil exploration contains several shortened forms. Depending upon your class, you may wish to emphasize only part of it at this time. There is some advantage, however, for pupils to have several shortened forms before them. Because developing a shorter form varies with the individual, the display of several forms suggests various possibilities to children.

Throughout the work try to encourage pupils to select the one form that they understand best and then concentrate on it. Children should not be expected to have equal mastery of all shortened forms. They should shorten their work only in-so-far as they understand it.

In this lesson, it will be profitable to write on the chalkboard examples similar to those in the pupil exploration. You may wish to begin by showing either Form I or Form II (whichever your class has used), and comparing it with Form A. When this comparison is made, the work should be put on the board as the discussion unfolds. This causes pupils to focus more directly on the topic under discussion.

As seems advisable, continue comparing the other forms. Remember, it is not expected that all children will attain the level of skill needed for Form C. For some children, the introduction of Form C may need to be delayed until later. In any event, this exploration is one to which you may want to return frequently.

When children are asked to explain the work in examples, they may need to be guided by leading questions provided by the teacher. Such questions should serve to emphasize the importance of place value.

FINDING SHORTER WAYS OF DIVIDING

Exploration

Let us think about dividing 836 by 6.

We have learned how to shorten our work from either one of the two forms at the left to the one at the right.

$$\begin{array}{r} \underline{139} \\ 9 \\ 30 \\ 100 \\ 6\overline{)836} \\ \underline{600} \\ 236 \\ \underline{180} \\ 56 \\ \underline{54} \\ 2 \end{array}$$

$$\begin{array}{r|r} 6\overline{)836} & \\ \underline{600} & 100 \\ 236 & \\ \underline{180} & 30 \\ 56 & \\ \underline{54} & \underline{9} \\ 2 & 139 \end{array}$$

$$\begin{array}{r} 139 \\ 6\overline{)836} \\ \underline{600} \\ 236 \\ \underline{180} \\ 56 \\ \underline{54} \\ 2 \end{array}$$

We divided 836 by 6.

What is the quotient? (139)

What is the remainder? (2)

What mathematical sentence tells us these things?
[836 = (6 x 139) + 2].

Explain how we used place value to shorten the writing of the quotient numeral in the form at the right.

Now let us see how we can shorten our work even more.

$$\begin{array}{rcrl} 139 & & 139 & \\ 6\,\overline{)836} & & 6\,\overline{)836} & \\ \underline{600} & \longrightarrow & \underline{6} & (6\text{ hundreds}) \\ 236 & & 236 & \\ \underline{180} & \longrightarrow & \underline{18} & (18\text{ tens}) \\ 56 & & 56 & \\ \underline{54} & \longrightarrow & \underline{54} & (54\text{ ones}) \\ 2 & & 2 & \end{array}$$

We have used place value to help us shorten the writing of the quotient numeral. In the form at the right we also use place value to help us shorten other parts of our work.

How did we use place value to shorten the writing of 600? (We understand that 6 written in the hundreds position means 600.)

How did we use place value to shorten the writing of 180? (We understand that 18 tens is the same as 180.)

Why is 54 written the same way in both forms? (Because this is the only way we can write 54 ones as a single numeral.)

Can we shorten our work even more than we have already? Look at the forms below.

A.

$$\begin{array}{r} 139 \\ 6\,\overline{)836} \\ \underline{6} \\ 236 \\ \underline{18} \\ 56 \\ \underline{54} \\ 2 \end{array}$$

B.

$$\begin{array}{r} 139 \\ 6\,\overline{)836} \\ \underline{6} \\ 23 \\ \underline{18} \\ 56 \\ \underline{54} \\ 2 \end{array}$$

C.

$$\begin{array}{r} 139 \text{ r } 2 \\ 6\,\overline{)8^{(2)}3^{(5)}6} \end{array}$$

In Form B, explain how you could use each of these "helpers", along with place value, to work the example.

When dividing the hundreds, think:

$8 \div 6$. The quotient is 1; the remainder is 2.

When dividing the tens, think:

$23 \div 6$. The quotient is 3; the remainder is 5.

When dividing the ones, think:

$56 \div 6$. The quotient is 9; the remainder is 2.

Could you use these same "helpers" with Form C? (yes) Explain. What does " r 2 " mean in Form C? (r 2 means we have a remainder of 2.)

If you have a good memory, you don't even have to write the (2) and the (5) in Form C. If you remember them, all you need to write is the quotient and the remainder:

139 r 2.

Let us study together three forms of dividing for the example, 1670 ÷ 7.

A.

238
7) 1670
14
270
21
60
56
4

B.

238
7) 1670
14
27
21
60
56
4

C.

2 3 8 r 4
$7\,\overline{)\,1\ 6^{(2)}\,7^{(6)}\,0}$

Explain how you could use each of these "helpers", along with place value, in forms B and C.

When dividing the hundreds, think:

16 ÷ 7. The quotient is 2; the remainder is 2.

When dividing the tens, think:

27 ÷ 7. The quotient is 3; the remainder is 6.

When dividing the ones, think:

60 ÷ 7. The quotient is 8; the remainder is 4.

Exercise Set 6

Find each quotient and remainder using the shortest form you can.

1. $\begin{array}{r} 26 \text{ r } 1 \\ 3\overline{)79} \end{array}$

2. $\begin{array}{r} 23 \text{ r } 3 \\ 4\overline{)95} \end{array}$

3. $\begin{array}{r} 18 \text{ r } 2 \\ 5\overline{)92} \end{array}$

4. $\begin{array}{r} 47 \text{ r } 1 \\ 2\overline{)95} \end{array}$

5. $\begin{array}{r} 131 \text{ r } 3 \\ 7\overline{)920} \end{array}$

6. $\begin{array}{r} 15 \text{ r } 3 \\ 8\overline{)123} \end{array}$

7. $\begin{array}{r} 222 \text{ r } 2 \\ 6\overline{)1334} \end{array}$

8. $\begin{array}{r} 157 \text{ r } 4 \\ 9\overline{)1417} \end{array}$

9. $\begin{array}{r} 1321 \text{ r } 3 \\ 7\overline{)9250} \end{array}$

10. $\begin{array}{r} 2363 \text{ r } 3 \\ 4\overline{)9455} \end{array}$

11. $\begin{array}{r} 2874 \text{ r } 2 \\ 3\overline{)8624} \end{array}$

12. $\begin{array}{r} 1924 \\ 5\overline{)9620} \end{array}$

13. $\begin{array}{r} 1404 \text{ r } 3 \\ 6\overline{)8427} \end{array}$

14. $\begin{array}{r} 12104 \text{ r } 2 \\ 8\overline{)96834} \end{array}$

15. $\begin{array}{r} 6636 \text{ r } 3 \\ 4\overline{)26547} \end{array}$

Note that remainders are written by the quotients, only because the work is not shown. Whenever pupils show their work, the remainder should be found in the usual position in the algorism.

In most exercises, the pupils' work should appear in this fashion.

$$\begin{array}{r} 54 \\ 30\overline{)1628} \\ \underline{150} \\ 128 \\ \underline{120} \\ 8 \end{array}$$

USING SHORTER FORMS WHEN DIVISORS ARE MULTIPLES OF TEN

Teaching Suggestions:

This exploration closely parallels the preceding one in which the work with divisors less than 10 was shortened. Again you may find it desirable to work and compare some examples on the chalkboard. The following is a suggestion.

Example:

Form A	Form B	Form C
211	211	211
40) 8479	40) 8479	40) 8479
8000	80	80
479	479	47
400	40	40
79	79	79
40	40	40
39	39	39

Questions such as the following should be asked:

How do we use place value to shorten the writing of 8000?

How do we use place value to shorten the writing of 400?

Why is 40 written the same way in both forms?

Explain how we use these "helpers", along with place value, to work the example.

When dividing the hundreds, think:
$8 \div 4$. The quotient is 2.

When dividing the tens, think:
$4 \div 4$. The quotient is 1.

When dividing the ones, think:
$7 \div 4$. The quotient is 1.

In Form C, why do we write in the work just 47 rather than 479?

USING SHORTER FORMS WHEN DIVISORS ARE MULTIPLES OF TEN

Exploration

Here are some of the ways we can shorten our work when we divide 8469 by 30.

A.

$$\begin{array}{r} 282 \\ 30\,\overline{)\,8469} \\ \underline{6000} \\ 2469 \\ \underline{2400} \\ 69 \\ \underline{60} \\ 9 \end{array}$$

B.

$$\begin{array}{r} 282 \\ 30\,\overline{)\,8469} \\ \underline{60} \\ 2469 \\ \underline{240} \\ 69 \\ \underline{60} \\ 9 \end{array}$$

C.

$$\begin{array}{r} 282 \\ 30\,\overline{)\,8469} \\ \underline{60} \\ 246 \\ \underline{240} \\ 69 \\ \underline{60} \\ 9 \end{array}$$

Here are some of the ways we can shorten our work when we divide 9382 by 70.

A.

$$\begin{array}{r} 134 \\ 70\,\overline{)\,9382} \\ \underline{7000} \\ 2382 \\ \underline{2100} \\ 282 \\ \underline{280} \\ 2 \end{array}$$

B.

$$\begin{array}{r} 134 \\ 70\,\overline{)\,9382} \\ \underline{70} \\ 2382 \\ \underline{210} \\ 282 \\ \underline{280} \\ 2 \end{array}$$

C.

$$\begin{array}{r} 134 \\ 70\,\overline{)\,9382} \\ \underline{70} \\ 238 \\ \underline{210} \\ 282 \\ \underline{280} \\ 2 \end{array}$$

Study carefully each set of examples on the preceding page.

What is the quotient and remainder when 8469 is divided by 30? (282) Write a mathematical sentence that tells this.
[8469 = (30 × 282) + 9]

What is the quotient and remainder when 9382 is divided by 70? (134) Write a mathematical sentence that tells this.
[9382 = (70 × 134) + 2]

When dividing 8469 by 30, how could you use each of these as "helpers"?

8 ÷ 3 24 ÷ 3 6 ÷ 3

When dividing 9382 by 70, how could you use each of these as "helpers"?

9 ÷ 7 23 ÷ 7 28 ÷ 7

Which form do you understand best for working each example?

If you can use a shorter form than the ones given on the preceding page, use the chalkboard to show and explain it to other pupils in the class.

Exercise Set 7

Divide. Use the shortest form that you can.

1. $30\overline{)1628}$ 54 r 8

2. $70\overline{)6586}$ 94 r 6

3. $40\overline{)9274}$ 231 r 34

4. $80\overline{)9000}$ 112 r 40

5. $60\overline{)8563}$ 142 r 43

6. $20\overline{)7459}$ 372 r 19

7. $50\overline{)7496}$ 149 r 46

8. $90\overline{)38642}$ 429 r 32

9. $20\overline{)6538}$ 326 r 18

10. $80\overline{)7163}$ 1, 89 r 43

11. $70\overline{)5872}$ 83 r 62

12. $90\overline{)88429}$ 982 r 49

WORKING WITH DIVISORS BETWEEN 10 AND 100

Exploration

We have been working with divisors that are multiples of 10. We have used "helpers" to find parts of the quotient. We can use the same kind of "helper" when working with divisors between 10 and 100.

Here is an example for us to try: 975 ÷ 23.

Our quotient must be between 10 and 100. Why? (10×23 = 230, 100×23 = 2300 We see that 230 < 975 and 2300 > 975.)

Is 23 nearer to 20 or to 30? (20)

Since 23 is nearer to 20, let us use 9 ÷ 2 as a "helper" to try to find the first part of the quotient. For 9 ÷ 2, we think "4".

```
23 ) 975
```

Does the 4 written above the 7 tell us that the first part of the quotient is 40? (yes) Why? (a 4 written above the tens position means 40.) Can the remaining part of the quotient be as great as 10? (No) Explain. (The remaining part of the quotient is 55. 10×23 = 230. 230 > 55.)

```
        4
23 ) 975
     920
      55
```

Now let us use 5 ÷ 2 as a "helper" to find the remaining part of the quotient. For 5 ÷ 2, we think "2". Why is the 2 written above the 5? (Because 2 is the quotient of the remaining part of the dividend 55.) What is the quotient when we divide 975 by 23? (42) What is the remainder? (9) Is the remainder less than the divisor? (yes)

```
       42
23 ) 975
     920
      55
      46
       9
```

Does 975 = (42 × 23) + 9? (yes)

The check at the right will tell us.

```
Check
   23
 x 42
   46
  92
  966
 +  9
  975
```

Now let us try this example; 1939 ÷ 68

Our quotient must be between 10 and 100.

Why? (10 × 68 = 680, 100 × 68 = 6800 Since 680 < 1939 and 6800 > 1939, our quotient must be between 10 and 100)

Is 68 nearer to 60 or to 70? (70)

Since 68 is nearer to 70, let us use 19 ÷ 7 as a "helper" to try to find the first part of the quotient. For 19 ÷ 7, think "2".

```
68 ) 1939
```

Does the 2 written above the 3 tell us that the first part of the quotient is 20? (yes) Why?

Can the remaining part of the quotient be as great as 10? (No) Explain. (The remaining part of the dividend is 579. 10 × 68 = 680 Since 680 > 579, the remaining part of the quotient cannot be as great as 10.)

```
        2
68 ) 1939
     1360
      579
```

Now let us use 57 ÷ 7 as a "helper" to find the remaining part of the quotient.

For 57 ÷ 7, think "8".

Why is the 8 written where it is? (Because 8 is the quotient of the remaining part of the dividend 579.)

What is the quotient when we divide 1939 by 68? (28) What is the remainder? (35)

Is the remainder less than the divisor? (Yes)

```
       28
68 ) 1939
     1360
      579
      544
       35
```

Write the mathematical sentence that goes with this example. [1939 = (68 × 28) + 35]

Show the check for the work.

```
   68
   28
  544
 136
 1904
   35
 1939
```

Exercise Set 8

Divide. Check your answers.

1. $\overset{32 \text{ r } 26}{63\,\overline{)\,2042}}$

2. $\overset{55 \text{ r } 34}{36\,\overline{)\,2014}}$

3. $\overset{67 \text{ r } 19}{29\,\overline{)\,1962}}$

4. $\overset{65 \text{ r } 28}{88\,\overline{)\,5748}}$

5. $\overset{85 \text{ r } 34}{67\,\overline{)\,5729}}$

6. $\overset{43 \text{ r } 59}{73\,\overline{)\,3198}}$

7. $\overset{37 \text{ r } 19}{92\,\overline{)\,3423}}$

8. $\overset{20 \text{ r } 34}{44\,\overline{)\,914}}$

9. $\overset{71 \text{ r } 7}{21\,\overline{)\,1498}}$

10. $\overset{23 \text{ r } 34}{78\,\overline{)\,1828}}$

11. $\overset{14 \text{ r } 53}{55\,\overline{)\,823}}$

12. $\overset{80 \text{ r } 46}{84\,\overline{)\,6766}}$

13. $\overset{69 \text{ r } 38}{49\,\overline{)\,3419}}$

14. $\overset{45 \text{ r } 23}{97\,\overline{)\,4388}}$

QUOTIENTS GREATER THAN 100

We will study these examples together.

$8754 \div 32$

$$\begin{array}{r} 273 \\ 32\overline{)8754} \\ \underline{6400} \\ 2354 \\ \underline{2240} \\ 114 \\ \underline{96} \\ 18 \end{array}$$

How do we know the quotient will be between 100 and 1000? ($100 \times 32 = 3200$, $1000 \times 32 = 32000$ Since $3200 < 8754$ and $32000 > 8754$, the quotient will be between 100 and 1000.)

Is 32 nearer to 30 or to 40? (30)

Explain how we could use each of these "helpers" to find parts of the quotient.

$8 \div 3$. $23 \div 3$. $11 \div 3$.

The first part of the quotient is 200. How do we know that it could not be as much as 300? (Because $300 \times 32 = 9600$ and $9600 > 8754$.)

The second part of the quotient is 70. How do we know that it could not be as much as 80? ($80 \times 32 = 2560$. $2560 > 2354$.)

Explain why each digit of the quotient numeral is placed where it is.

What is the quotient? (273)

What is the remainder? (18)

Is the remainder less than the divisor? (yes)

Write the mathematical sentence for this example.

$[8754 = (32 \times 273) + 18]$

Show the check for your work.

$$\left(\begin{array}{r} 273 \\ 32 \\ \hline 546 \\ 819 \\ \hline 8736 \\ +18 \\ \hline 8754 \end{array}\right)$$

15014 ÷ 57

```
        263
57 ) 15014
     11400
      3614
      3420
       194
       171
        23
```

How do we know the quotient will be between 100 and 1000? (100 × 57 = 5700 , 1000 × 57 = 57000 5700 < 15014 and 57000 > 15014 Therefore the quotient is between 100 and 1000.)

Is 57 nearer to 50 or to 60? (60)

How can we use each of these "helpers" to find parts of the quotient?

15 ÷ 6. 36 ÷ 6. 19 ÷ 6.

How can we know that the first part of the quotient is not as great as 300? (300 × 57 = 17100 and 17100 > 15014.)

How can we know that the second part of the quotient is not as great as 70? (70 × 57 = 3990 and 3990 > 3614).

Explain why each digit of the quotient numeral is placed where it is.

What is the quotient? (263)

What is the remainder? (23)

Is the remainder less than the divisor? (Yes)

Write the mathematical sentence for this example.
[15014 = (57 × 263) + 23]

Show the check for your work.

```
   263
    57
  1841
 1315
 14991
+   23
 15,014
```

Explain the work for these examples.

Be sure to tell why a zero had to be written in each quotient numeral.

17286 ÷ 54

$$\begin{array}{r} 320 \\ 54 \,\overline{)\,17286} \\ \underline{16200} \\ 1086 \\ \underline{1080} \\ 6 \end{array}$$

18376 ÷ 89

$$\begin{array}{r} 206 \\ 89 \,\overline{)\,18376} \\ \underline{17800} \\ 576 \\ \underline{534} \\ 42 \end{array}$$

For each example:

Write the mathematical sentence. 17286 = (54 × 320) + 6
18376 = (89 × 206) + 42

Show a check for the work.

$$\begin{array}{r} 320 \\ \underline{54} \\ 1280 \\ \underline{1600} \\ 17280 \\ \underline{+\ 6} \\ 17286 \end{array} \qquad \begin{array}{r} 206 \\ \underline{89} \\ 1854 \\ \underline{1648} \\ 18334 \\ \underline{+\ 42} \\ 18376 \end{array}$$

Exercise Set 9

Divide. Use the shortest form that you can.

1. $\begin{array}{l} \quad\;\; 186 \text{ r } 26 \\ 38\overline{)7094} \end{array}$

2. $\begin{array}{l} \quad\;\; 143 \text{ r } 6 \\ 82\overline{)11732} \end{array}$

3. $\begin{array}{l} \quad\;\; 129 \text{ r } 61 \\ 65\overline{)8446} \end{array}$

4. $\begin{array}{l} \quad\;\; 982 \text{ r } 79 \\ 93\overline{)91405} \end{array}$

5. $\begin{array}{l} \quad\;\; 296 \text{ r } 42 \\ 47\overline{)13954} \end{array}$

6. $\begin{array}{l} \quad\;\; 398 \text{ r } 54 \\ 56\overline{)22342} \end{array}$

7. $\begin{array}{l} \quad\;\; 811 \text{ r } 12 \\ 74\overline{)60026} \end{array}$

8. $\begin{array}{l} \quad\;\; 888 \text{ r } 17 \\ 18\overline{)16001} \end{array}$

9. $\begin{array}{l} \quad\;\; 456 \text{ r } 49 \\ 75\overline{)34249} \end{array}$

10. $\begin{array}{l} \quad\;\; 461 \text{ r } 6 \\ 21\overline{)9687} \end{array}$

11. $\begin{array}{l} \quad\;\; 930 \text{ r } 40 \\ 89\overline{)82810} \end{array}$

12. $\begin{array}{l} \quad\;\; 435 \\ 53\overline{)23055} \end{array}$

13. $\begin{array}{l} \quad\;\; 446 \text{ r } 18 \\ 27\overline{)12060} \end{array}$

14. $\begin{array}{l} \quad\;\; 245 \\ 32\overline{)7840} \end{array}$

15. $\begin{array}{l} \quad\;\; 657 \text{ r } 27 \\ 67\overline{)44046} \end{array}$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

16. A cattle rancher has 9,792 acres of land. He estimates that it takes 38 acres of land to provide grass for one cow. What is the largest number of cows he can have on his ranch? ($9,792 = (38 \times n) + r$ The ranch can have 257 cows
$n = 257$, $r = 26$ on his ranch.)

17. There are 31 rows of seats on one side of a football field. There are seats for 6,572 people. If each row has the same number of seats, how many seats are in each row. ($6572 = (31 \times n) + r$ There are 212 seats
$n = 212$ in each row.)

18. A machine made 9,503 pencils in 43 minutes. How many pencils did it make in 1 minute?
($9503 = (43 \times n) + r$ The machine made 221 pencils
$n = 221$ in one minute.)

19. A book company can pack 58 books in each box. How many boxes will be needed to pack 39,018 books?
($39,018 = (58 \times n) + r$
$n = 672$ The company will need 672 boxes.)
$39018 = (58 \times 672) + 42$

20. There were 50,902 visitors to a park in 62 days. If the same number of people visited the park each day, how many people visited the park each day?
($50,902 = (62 \times n) + r$ 821 people visited the park each day.)
$n = 821$

MORE ABOUT USING HELPERS WHEN DIVIDING

Teaching Suggestions:

The exploration in the pupil text is for the purpose of indicating to pupils that "helpers" do not always lead to correct partial quotients. As you discuss this with pupils, you may find it necessary to raise other questions which are pertinent to your particular classroom discussion.

Although it seems inadvisable to include more examples in the pupil text, the teacher may wish to use additional ones to the extent that children need them. A procedure similar to that of the pupil exploration is recommended.

Appropriate examples:

$$\begin{array}{r} 261 \text{ r } 12 \\ 23\overline{)6015} \end{array} \qquad \begin{array}{r} 68 \text{ r } 7 \\ 35\overline{)2387} \end{array}$$

$$\begin{array}{r} 43 \\ 35\overline{)731} \end{array} \qquad \begin{array}{r} 296 \text{ r } 12 \\ 65\overline{)19252} \end{array}$$

$$\begin{array}{r} 415 \text{ r } 4 \\ 27\overline{)11209} \end{array}$$

MORE ABOUT USING HELPERS WHEN DIVIDING

Exploration

The "helpers" we use when dividing will not always lead us to a correct part of the quotient.

We will see this in an example, such as:

$$905 \div 24.$$

To try to find the first part of the quotient we can use $9 \div 2$ as a "helper," and think "4."

$$\begin{array}{r} 4 \\ 24\,\overline{)\,905} \\ \underline{960} \end{array}$$

Is 40 the first part of the quotient? (No)

How can you tell that 40 is too great? (960 > 905)

Let us now use 30 as the first part of the quotient.

$$\begin{array}{r} 3 \\ 24\,\overline{)\,905} \\ \underline{720} \\ 185 \end{array}$$

Explain the work in the box.

To try to find the remaining part of the quotient we can use 18 ÷ 2 as a "helper," and think "9."

Is 9 the remaining part of the quotient? (No)

How can you tell that 9 is too great? ($216 > 185$)

```
       39
24 ) 905
     720
     185
     216
```

Let us now use 8 as the remaining part of the quotient.

How do we know that 8 is too great? ($192 > 185$)

```
       38
24 ) 905
     720
     185
     192
```

Is 7 the remaining part of the quotient? (Yes)

How does the work in the box show this? ($168 < 185$ and $17 < 24$)

We divided 905 by 24.

What is the quotient? (37)

What is the remainder? (17)

Is the remainder less than the divisor? (Yes)

```
       37
24 ) 905
     720
     185
     168
      17
```

Now let us work with the example: $1915 \div 36$.

To try to find the first part of the quotient, we can use $19 \div 4$ as a "helper," and think "4." Look carefully at the work in the box. How can we know that 40 is not the greatest multiple of 10 we can use as the first part of the quotient? (Because 10 x 36 = 360 and 360 < 475. This means that 40 is not the greatest multiple of 10 we can use.)

```
        4
36 ) 1915
     1440
      475
```

Let us now use 50 as the first part of the quotient. Is this the greatest multiple of 10 we can use? (yes) Explain. (We can see that 115 is the remaining part of the dividend. Since 10 x 36 = 360 and 360 > 115 we know we have the greatest multiple of 10.)

```
        5
36 ) 1915
     1800
      115
```

To try to find the remaining part of the quotient, we can use $11 \div 4$ as a "helper" and think "2."

How can we tell that 2 is not the greatest number to use for the remaining part of the quotient? (Because the remainder is greater than the divisor. 43 > 36)

```
       52
36 ) 1915
     1800
      115
       72
       43
```

Let us use 3 as the remaining part of the quotient. Is this the greatest number we can use? (yes) Explain (Now the remainder is less than the divisor. 7 < 36.)

We divided 1915 by 36.

What is the quotient? (53)

What is the remainder? (7)

```
       53
36 ) 1915
     1800
      115
      108
        7
```

"Helpers" do not always lead us to correct parts of the quotient.

Exercise Set 10

Divide.

1. $\begin{array}{r}42 \text{ r } 6\\ 75\,\overline{)3156}\end{array}$

2. $\begin{array}{r}92\\ 18\,\overline{)1656}\end{array}$

3. $\begin{array}{r}169 \text{ r } 34\\ 54\,\overline{)9160}\end{array}$

4. $\begin{array}{r}122 \text{ r } 9\\ 38\,\overline{)4645}\end{array}$

5. $\begin{array}{r}68 \text{ r } 23\\ 37\,\overline{)2539}\end{array}$

6. $\begin{array}{r}96\\ 28\,\overline{)2688}\end{array}$

7. $\begin{array}{r}68\\ 21\,\overline{)1428}\end{array}$

8. $\begin{array}{r}43 \text{ r } 8\\ 81\,\overline{)3491}\end{array}$

9. $\begin{array}{r}52 \text{ r } 40\\ 93\,\overline{)4876}\end{array}$

10. $\begin{array}{r}42\\ 37\,\overline{)1554}\end{array}$

11. $\begin{array}{r}38 \text{ r } 5\\ 14\,\overline{)537}\end{array}$

12. $\begin{array}{r}671\\ 58\,\overline{)38918}\end{array}$

13. $\begin{array}{r}435 \text{ r } 6\\ 75\,\overline{)32631}\end{array}$

14. $\begin{array}{r}215\\ 92\,\overline{)19780}\end{array}$

15. $\begin{array}{r}619 \text{ r } 84\\ 94\,\overline{)58270}\end{array}$

16. $\begin{array}{r}455 \text{ r } 24\\ 75\,\overline{)34149}\end{array}$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

17. A machine produces 348 spoons an hour. How many dozen will it produce in 8 hours of continuous operation?

$348 = (12 \times n) + r$

$n = 29$

$s = 8 \times 29$

$s = 232$

OR

$n = 348 \times 8$

$n = 2784$

$2784 = (12 \times s) + r$

$s = 232$

232 dozen, spoons will be produced in 8 hours.

18. An auditorium is to be used for a meeting of 958 persons. If each row seats 21 persons, how many rows will be needed?

$958 = (21 \times n) + r$

$n = 45$ $r = 13$

46 rows will be needed. One row will not be completely filled.

19. Robert reads approximately 96 words a minute. How many minutes will it take him to read a story of 1056 words?

$1056 = (96 \times n) + r$

$n = 11$

It will take Robert 11 minutes to read a story of 1056 words.

20. A grapefruit orchard has 864 trees in 32 rows. How many trees are there in each row?

$864 = (32 \times n) + r$

$n = 27$

There are 27 trees in each row.

SHORTENING OUR WORK

Exploration

We can use place value to shorten our work with division examples when divisors are between 10 and 100.

Think of dividing 17836 by 45.

A.		B.		C.
396		396		396
45) 17836		45) 17836		45) 17836
13500	→	135		135
4336		4336	→	433
4050	→	405		405
286		286		286
270		270		270
1[illegible]		16		16

Does 17836 = (396 × 45) + 16? (yes)

Explain how Form B is shorter than Form A.

Explain how Form C is shorter than Form B.

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Exercise Set 11

Divide. Use the shortest form you can.

1. $\begin{array}{r} 7 \text{ r } 26 \\ 77\overline{)565} \end{array}$

2. $\begin{array}{r} 68 \\ 32\overline{)2176} \end{array}$

3. $\begin{array}{r} 384 \text{ r } 4 \\ 19\overline{)7300} \end{array}$

4. $\begin{array}{r} 128 \text{ r } 17 \\ 58\overline{)7441} \end{array}$

5. $\begin{array}{r} 322 \text{ r } 27 \\ 29\overline{)9365} \end{array}$

6. $\begin{array}{r} 508 \\ 86\overline{)43688} \end{array}$

7. $\begin{array}{r} 378 \\ 18\overline{)6804} \end{array}$

8. $\begin{array}{r} 318 \text{ r } 65 \\ 86\overline{)27413} \end{array}$

9. $\begin{array}{r} 674 \\ 58\overline{)39092} \end{array}$

10. $\begin{array}{r} 546 \\ 28\overline{)15288} \end{array}$

11. $\begin{array}{r} 492 \text{ r } 46 \\ 92\overline{)45310} \end{array}$

12. $\begin{array}{r} 508 \text{ r } 4 \\ 14\overline{)7116} \end{array}$

13. $\begin{array}{r} 573 \text{ r } 20 \\ 25\overline{)14345} \end{array}$

14. $\begin{array}{r} 840 \text{ r } 46 \\ 73\overline{)61366} \end{array}$

15. $\begin{array}{r} 385 \text{ r } 15 \\ 19\overline{)7330} \end{array}$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

16. The committee has 685 tickets for the school play. They put 15 tickets in each package. How many packages of tickets did they have? Were there any left over? If so, how many?

$685 = (15 \times n) + r$

$n = 45 \quad r = 10$

They had 45 packages of ticket

There were 10 tickets left over.

17. Mr. Jones sold 32 television sets for $11,040. If these were all of the same model, what was the price of one set?

$11{,}040 = (32 \times n) + r$

$n = 345 \quad r = 0$

The price of one set was $345

18. Ann wants to make 12 curtains. She needs 42 inches of material for each curtain. How many yards of material does she need?

$C = 12 \times 42 \qquad 504 = (36 \times n) + r$

$C = 504 \qquad n = 14$

She needs 14 yards of material.

19. The Boy Scouts were having a party. Their mothers baked 134 cupcakes for the party. If each of the 67 boys had the same number of cupcakes, how many would each boy eat?

$134 = (67 \times n) + r$

$n = 2 \quad r = 0$

Each boy would eat 2 cupcakes.

20. Jean packed 288 oranges into boxes. If each box holds 36 oranges, how many boxes did she fill?

$288 = (36 \times n) + r$

[illegible] = 8

Jea[illegible]lled 8 boxes.