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ABSTRACT

This report presents the results of a teaching experiment which investigated (1) the role of mathematical experiences on the development of counting, addition, subtraction, mental arithmetic, classification, and other arithmetical topics and (2) the role of quantitative comparisons and class inclusion as readiness variables for learning the content in (1). The readiness and achievement variables are discussed in detail and tasks are described carefully. Forty-eight first graders were tested and interviewed, with each of the three interviews videotaped. Data were extracted from the tapes and coded. Twenty-four pupils were given instruction on the concepts for 12 weeks. Multivariate analysis of variance, univariate analysis of variance and discriminant functions as necessary, and correlation matrices were used. Results are presented and discussed in detail. (MS)

PMDC Technical Report
No. 9

Quantitative Comparisons and Class Inclusion as Readiness Variables for Learning First Grade Arithmetical Content

Leslie P. Steffe

pmdc

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FOREWORD

Ed Begle recently remarked that curricular efforts during the 1960's taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better. The mathematician will, quite likely, agree with both parts of this statement. The layman, the parent, and the elementary school teacher, however, question the thesis that the "new math" was really better than the "old math." At best, the fruits of the mathematics curriculum "revolution" were not sweet. Many judge them to be bitter.

While some viewed the curricular changes of the 1960's to be "revolutionary," others disagreed. Thomas C. O'Brien of Southern Illinois University at Edwardsville recently wrote, "We have not made any fundamental change in school mathematics."[1] He cites Allendoerfer who suggested that a curriculum which heeds the ways in which young children learn mathematics is needed. Such a curriculum would be based on the understanding of children's thinking and learning. It is one thing, however, to recognize that a conceptual model for mathematics curriculum is sound and necessary and to ask that the child's thinking and learning processes be heeded, it is quite another to translate these ideas into a curriculum which can be used effectively by the ordinary elementary school teacher working in the ordinary elementary school classroom.

Moreover, to propose that children's thinking processes should serve as a basis for curriculum development is to presuppose that curriculum makers agree on what these processes are. This is not the case, but even if it were, curriculum makers do not agree on the implications which the understanding of these thinking processes would have for curriculum development.

In the real world of today's elementary school classroom, where not much hope for drastic changes for the better can be foreseen, it appears that in order to build a realistic, yet sound basis for the mathematics curriculum, children's mathematical thinking must be studied intensively in their usual school habitat. Given an opportunity to think freely, children clearly display certain patterns of thought as they deal with ordinary mathematical situations encountered daily in their classroom. A videotaped record of the outward manifestations of a child's thinking, uninfluenced by any teaching on the part of the interviewer, provides a rich source for conjectures as to what this thinking is, what mental structures the child has developed, and how the child uses these structures when dealing with the ordinary concepts of arithmetic. In addition, an intensive analysis of this videotape generates some conjectures as to the possible sources of what adults view as children's "misconceptions" and about how the school environment (the teacher and the materials) "fights" the child's natural thought processes.

The Project for the Mathematical Development of Children (PMDC)[2] set out to create a more extensive and reliable basis on which to build mathematics curriculum. Accordingly, the emphasis in the first phase is to try to understand the children's intellectual pursuits, specifically their attempts to acquire some basic mathematical skills and concepts.

The PMDC, in its initial phase, works with children in grades 1 and 2. These grades seem to comprise the crucial years for the development of bases for the future learning of mathematics, since key mathematical concepts begin to form at these grade levels. The children's mathematical development is studied by means of:

1. One-to-one videotaped interviews subsequently analyzed by various individuals.

2. Teaching experiments in which specific variables are observed in a group teaching setting with five to fourteen children.

3. Intensive observations of children in their regular classroom setting.

4. Studies designed to investigate intensively the effect of a particular variable or medium on communicating mathematics to young children.

[1]"Why Teach Mathematics?" The Elementary School Journal 73 (Feb. 1973), 258-68.

[2]PMDC is supported by the National Science Foundation, Grant No. PES 74-18106-A03.

5. Formal testing, both group and one-to-one, designed to provide further insights into young children's mathematical knowledge.

The PMDC staff and the Advisory Board wish to report the Project's activities and findings to all who are interested in mathematical education. One means for accomplishing this is the PMDC publication program.

Many individuals contributed to the activities of PMDC. Its Advisory Board members are: Edward Begle, Edgar Edwards, Walter Dick, Renee Henry, John LeBlanc, Gerald Rising, Charles Smock, Stephen Willoughby and Lauren Woodby. The principal investigators are: Merlyn Behr, Tom Denmark, Stanley Erlwanger, Janice Flake, Larry Hatfield, William McKillip, Eugene D. Nichols, Leonard Pikaart, Leslie Steffe, and the Evaluator, Ray Carry. A special recognition for this publication is given to the PMDC Publications Committee, consisting of Merlyn Behr (Chairman), Thomas Cooney and Tom Denmark.

Eugene D. Nichols
Director of PMDC

PREFACE

This publication is a summary of PMDC Technical Report No. 9. That publication is the report of rhe results of a teaching experiment conducted during the academic year 1974-75 with first grade children. The teaching experiment was done to investigate (1) the role of mathematical experiences on the development of counting, addition, subtraction, mental arithmetic, classification, and various other topics in arithmetical curricula and (2) the role of quantitative comparisons and class inclusion as readiness variables for learning the content in (1).

The names of the schools used in this study are fictitious. The study took place in a city in the Southeast with a population of 50,000.

Thanks are expressed to the principals of the two elementary schools, the teachers, and most importantly, the children. Cooperation such as that experienced by the principal investigator is critical in the total enterprise of research and development in mathematics education.

Quantitative Comparisons and Class Inclusion as Readiness Variables for Learning First Grade Arithmetical Content

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This study was conducted in the Project for Mathematical Development of Children, Eugene D. Nichols, Director. The principal investigator and project associates also contributed time to the study as part of their activities in the Georgia Center for the Study of Learning and Teaching Mathematics, Leslie P. Steffe, Director.

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CHAPTER I

Presentation of the Variables

The potential of Piagetian theory as a readiness theory for learning mathematical content seems hardly explored, even though some studies have been directed toward such a purpose. On the face of it, the psychological mechanisms Piaget calls mental operations ought to determine, in a substantive manner, the mathematical content related to cardinal and ordinal number a child is able to acquire within at least a two or three months time span. But whether a child who does not display mental operations, in a Piagetain sense, forms mental operations related to cardinal and ordinal number during the course of instruction is an unanswered question. The issue is simply this--it is not known how children develop mathematically through the course of an instructional program except in the most global of ways. Until the charts of childrens' progress are carefully documented, the best that can be done in the development of instructional programs in mathematics is to guess at the answers to the most basic of questions. An illustration is the introduction of the missing addend problem. During the 1960's, program developers introduced, universally, the missing addend problem in the first grade with the hope that it would connect, for the children, addition and subtraction. Of course, if it did, then a great savings transfer would occur in the learning of subtraction facts. Just define subtraction in terms of addition. But teachers found the missing addend problem a source of great frustration for many children. With such feedback program developers essentially abandoned introduction of the missing addend problem in the first grade. Most decisions made relative to the introduction, then abandonment of the missing addend problem were done in the absence of any data on the way children develop throughout the course of a mathematics program. That such data is desperately needed should be clear from the example given. In fact, this study shows that both decisions are essentially incorrect--that of universal introduction and that of universal abandonment. Moreover, a great deal of information is presented on how one may determine which children are ready for introduction of the missing addend problem and which are not--a very useful piece of information.

Before delving into the study, a few preliminary ideas are useful in understanding the nature of the variables. The readiness variables are founded in Piaget's developmental theory, and the achievement variables are founded in the mathematical theory of cardinal and ordinal number. Even though Piaget offers a developmental theory concerning cardinal and ordinal number, the mathematical theory is distinct from the psychological theory. In order to be precise concerning the mathematical theory, a discussion of some important aspects of cardinal and ordinal number is given. Likewise, discussion of Piagetian theory concerning cardinal and ordinal number is offered. For the purpose of the readiness study, Piagetian theory concerning number, quantity, classification, and relations is assumed to be internally consistent. The theoretical interrelationships of number, class inclusion, relations, one-to-one correspondence, and set partitions are discussed in the next few sections.

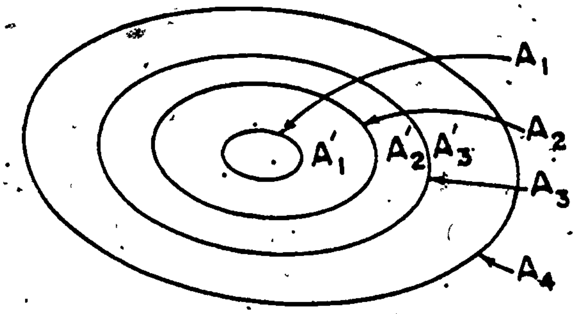
The Readiness Variables

Quantitative Comparisons

Number in Piagetian theory. Piaget, in his classic work *The Child's Conception of Number*, attempted to show that cardinal and ordinal number are developmental, arising in the child as a synthesis of Grouping I, Primary Addition of Classes, and Grouping V, Addition of Connected, Asymmetrical Relations. While the data presented in this book are "old," the basic theory of the Genevans concerning the development of number in the child has not changed substantially over the last three decades (Piaget, 1970; Beth and Piaget, 1966; Sinclair, 1971). Number, for Piaget (1952), "is at the same time a class and an asymmetrical relation" (p. 184).

Even though the relevance of the total grouping structure to cognition of relations has been questioned (Steffe, 1973), literature of the Genevans concerning the development of number can be understood only in the context of the grouping structures. Two essential conditions for the "transformation" of classes into numbers exist (Piaget, 1952, pp. 183-84). Given a class, all of the elements must somehow be regarded as equivalent, but at the same time distinct. To illustrate these two conditions imagine some hierarchical system $\emptyset \subset A_1 \subset A_2 \subset A_3 \subset \ldots \subset \bigcup_i A_i$ of classes where the following classes contain single elements

1. A_1
2. $A_1' = A_2 - A_1$
3. $A_2' = A_3 - A_2$
4. $A_3' = A_4 - A_3$



For example, A_1 could be a bead, A_1' a cube, A_2' a bean, etc.

The first condition given is that all elements must be regarded as equivalent (all qualities of the individual elements are eliminated). But, if condition one holds, then, for example, A_2 would not be a class of two elements, but instead only one, for $A_1 \cup A_1' = A_1$ --which is to say that the quality of the elements are eliminated. If the differences of A_1 and A_1' are taken into account, then they are no longer equivalent to one another except with respect to A_2. This brings the second essential condition into focus. In effect, the equivalent terms must remain somehow distinct, but that distinction no longer has recourse to qualitative differences. Given an object (the bead), then any other object is distinguished from that object by introducing order--by being placed next to, selected after, or etc. "These two conditions are necessary and sufficient to give rise to number. Number is at the same time a class and an asymmetrical relation". . .(Piaget, 1952, p. 184). According to Piaget (1952, p. 184), in qualitative logic, objects cannot be, at one and the same time, classified and seriated, since addition of classes is commutative whereas seriation is not commutative. However, if the qualities of the elements are abstracted, then the two groupings (I and V) no longer function independently, but necessarily merge into a single system.

In Piaget's system, then, number is not to be reduced to one or another of the groupings, but instead is a new construction--a synthesis of Groupings I and V. Elements, from the point of view of their qualities, and either considered from the point of view of their partial equivalences and are classified, or are considered from the point of view of their differences, and are seriated. It is not possible to do both at once unless the qualities are abstracted (or eliminated), but then it is necessary to do both simultaneously.

The only way, then, to distinguish A_1, A_1', A_2', A_3', . . . is to seriate them: $A \rightarrow A \rightarrow A \rightarrow$, . . ., where $\rightarrow$ denotes the successor relation and A represents A_i' where all the qualities of the element of A_i' have been eliminated. Clearly, Piaget considers each A to be a unit-element, at once equivalent to, but distinct from all the others, where the equivalence arises through the elimination of qualities and the distinctiveness arises through the order of succession.

The notion of a unit is central in Piaget's system and is not deducible from the Grouping Structures, but rather is the result of the synthesis already alluded to. Once reversibility is achieved in seriation and classification, "groupings of operations become possible, and define the field of the child's qualitative logic" (Piaget, 1952, p. 155). Here operational seriation has as a necessary condition reversibility at the first level of reciprocity.

> A cardinal number is a class whose elements are conceived as 'units' that are equivalent, and yet distinct in that they can be seriated, and therefore ordered. Conversely, each ordinal number is a series whose terms, through following one another according to the relations of order that determine their

> respective positions, are also units that are equivalent and can therefore be grouped in a class. Finite numbers are therefore necessarily at the same time cardinal and ordinal . . . (Piaget, 1952, p. 157).

The development of classes and relations does not, as it may seem from the above quotations, precede the development of number in Piaget's theory, but those developments are simultaneous. Without knowledge of the quantifiers "a," "none," "some," and "all," which implicitly involve cardinal number, the child is not capable of cognition of hierarchical classifications. A genetic circularity consequently exists in the developmental theory of classes, relations, and numbers.

Quantity. It is now possible to discuss the notion of quantity as elaborated by Piaget (1952, p. 5). Strictly speaking, Grouping VIII, Multiplication of Relations, should be discussed prior to the discussion on quantity. Suffice it to say that Grouping VIII allows the child to consider two perceptual relations simultaneously (e.g., taller but narrower for two glasses of water).

In the subsequent discussion, quantity as viewed by Piaget is described; a replication study by Elkind is discussed; quantity is related to one-to-one correspondence; quantity as a scientific concept is contrasted with quantity in Piagetian theory; and the relationship of quantity and number is pointed out in Piagetian theory.

Quantity as viewed by Piaget. Whether it be continuous (i.e., liquid) or discontinuous (i.e, collections of objects) quantities, Piaget's logical analysis of quantity in children is the same. First, there is what it termed gross quantity. Piaget (1952) describes gross quantity as follows

> At the level of the first stage, quantity is . . . no more than the asymmetrical relations between qualities, i.e., comparisons of the type 'more' or 'less' contained in judgments such as 'it's higher,' 'not so wide,' etc. These relations depend on perception, and are not as yet relations in the true sense, since they cannot be coordinated one with another in additive or multiplicative operations. This co-ordination begins at the second stage and results in the notion of 'intensive' quantity, i.e., without units, but susceptable of logical coherence. As soon as intensive quantification exists, the child can grasp . . . extensive quantity. (p. 5)

An illustration of gross quantity was given where two containers of beads; one containing green beads (A_2) and one containing red beads (A_1) were placed before a child. The containers were of identical dimensions. The child was asked if there were the same amount of beads in the two containers, and if a necklace made from the green beads and red beads would be of the same length. The green beads (or reds) were then poured into a container taller but narrower than the two originals. Questions were then put to the child concerning the necklaces. Children who were capable only of gross quantity would think that the necklace of green beads would be either longer than the necklace of red beads or shorter, depending on which dimension he focussed. Such children were not able to coordinate the dimensions of the container.

Children capable of intensive quantity were capable of coordinating the two dimension of the container (higher but narrower). They could use this compensating coordination to explain why the number of beads doesn't change upon pouring from one container to another, if they knew that the numbers of beads were equal to begin with.

Psychologically, intensive quantity would not be sufficient for a child to compare, numerically, two circular arrangements of blocks of differing diameters but of equal number. One arrangement would be less dense bu of greater diameter (or circumference) than the other. But realizing this compensating relation would not guarantee that the two circular arrangements contain the same number of blocks. According to Piaget, it would be necessary that arithmetical units intervene.

Logical multiplication of relations and the intervention of the notion of the unit are the two conditions for quantity to be extensive quantity for the child. Logical multiplication of relations is a necessary (but not sufficient) intermediary between gross, one dimensional quantity and extensive quantity. In the case of two amounts of liquid in two full containers A and B, a child could make a decision about relative amounts of liquid in A and B through logical multiplication in the two cases where B is both taller and of greater diameter than A and where A and B have at least one constant dimension (height or diameter). In the case where both dimensions vary, no decision would be possible. In such a case, the notion of units would logically have to intervene before a comparison could be made. Piaget's claim is that, psychologically, if the child knows that the quantities are equal in some initial state, realizing that they are equal in a final state, where both dimensions of the cylindrical containers vary inversely, demands a conception of units (Piaget, 1952, p. 21). In the case of the red and green beads above, the unit is Piaget's arithmetical unit.

<u>Elkind's replication of quantity</u>. In his study replicating Piaget's experiments on quantity Elkind (1961a) gives the following summary:

> Eighty. . .children were divided into three Age Groups (4, 5, 6-7) and tested on the three Types of Material for three Types of Quantity in a systematic replication of Piaget's investigation of the development of quantitative thinking. Analysis of variance showed that success in comparing quantities varied significantly with Age, Type of Quantity,

Type of Material and two of the interactions. . . .

> The results were in close agreement with Piaget's finding that success in comparing quantity developed in three, age related, hierarchically ordered stages. . . . (pp. 45-46)

The types of material Elkind used were (1) wooden sticks 1/4" square by 1 1/4", (2) orange colored water, a tall narrow glass, and two drinking glasses, one a 16 ounce glass and one an 8 ounce glass, and (3) large wooden beads that would just fit into the tall narrow glass in (2) above. The types of quantity he compared were (1) gross quantity, (2) intensive quantity, and (3) extensive quantity.

In the study, gross quantities were easiest to compare, intensive were intermediate, and extensive were hardest. For the types of material, quantities involving liquids were hardest to compare, with no difference between sticks and beads. There was a significant interaction of age groups and the quantity compared. Comparisons involving gross quantities was easy for all three groups. However, comparison involving intensive quantities was quite difficult for the 4-year group and became increasingly easier for the two older groups. The same was true for comparisons involving extensive quantities, but these comparisons remained more difficult than the comparisons involving intensive quantities.

Since Piaget defines his stages in terms of the type of quantitative comparisons children are capable of making, it is clear from Elkind's study that a child may be able to make extensive quantity comparisons using materials of a given kind and thereby be classified at Stage 3, but changing the type of material could affect the type of quantitative comparison the child is capable of and thereby alter the stage.

classification. However, there is a definite statistical relationship between age groups and stages as exemplified by the interaction of age groups and quantity compared and high and significant correlations between types of material.

Quantity and one-to-one correspondence. Piaget (1968, pp. 36, 37) has identified two psychological types of one-to-one correspondence; qualitative one-to-one correspondence and numerical one-to-one correspondence. Qualitative correspondence is based on the qualities of the elements where an element of one class is made to correspond to some element of another class because of the qualities associated with the elements--e.g., color, shape, or size. Numerical correspondence is such that any element of one class is made to correspond to any element of the other class regardless of qualities of elements. "Each element counts as one, and its particular qualities have no importance. Each element becomes simply a unity, an arithmetic unity." (p. 37)

Another type of behavior associated with one-to-one correspondence tasks is optical correspondence (Piaget, 1968, p. 34). Essentially, this is where children make global evaluations. An example is in a task where the adult has, say, six red checkers aligned in a row and gives a child the black checkers and instructs him to put out the same number of black checkers as red checkers. An optical correspondence would be where the child aligns all the black checkers in a row adjacent to and the same length as the row of red checkers. Another optical correspondence is where a child places one black checker by one red checker but cannot conserve the correspondence established. If conservation is present, the correspondence is called operational.

Qualitative correspondence may be either optical or operational. A child making a correspondence between two collections based on the qualities of the elements may not be able to conserve the correspondence if the configuration of the elements is altered; in this case, the qualitative correspondence is optical and not operational. If the child is able to conserve the correspondence, this is an operational correspondence (i.e., the elements altered always have the possibility of being placed back in the original position). A numerical correspondence is essentially operational. Children pass through three stages regarding one-to-one correspondence. The first is global evaluation, or essentially no one-to-one correspondence (up to approximately five or six years of age). The second is optical qualitative correspondence and the third is operational or numerical correspondence. Piaget (1952) spells out the relationships between different types of quantitative comparisons and the different types of correspondences, i.e., "global evaluation corresponds to 'gross quantity,' qualitative correspondence to 'intensive quantity,' and numerical correspondence to 'extensive quantity' [p. 90]."

If two sets of objects are placed in rows in front of a child capable of qualitative correspondence (and hence of intensive quantification) and one of two sets is altered, then a proper judgment could arise in the case of:

(1) equal length and equal density of two sets;

(2) greater length and greater density of one of the sets;

(3) equal length and greater or less density, or greater or less length and equal density, of one of the sets;

but not in the case of:

(4) greater length and smaller density, or greater density and smaller length, since he must be able to deduce the proportionality of differences (Piaget, 1952, p. 91).

Quantity as a scientific concept and as a cognitive-development concept. Confusion exists concerning what Piaget means by intensive and extensive quantity and what intensive and extensive quantity means in a scientific sense. This section is an attempt to clarify that confusion.

A quantity can be viewed as a collection of elements for which criteria of comparison have been established (e.g., ordinal numbers). But it is well to view quantity in the general context of measurement. Measurement can be interpreted in terms of a function, where the domain of the function consists of a collection of objects (called bodies) with definite structure and the range (for the purpose of interest here) a subset of the real numbers. The structure in the domain is of particular interest. Through some empirical (or operational) procedure, the bodies of the domain can be ordered on the basis of some property (or dimension). The property is called intensive whenever there exists two physical relations < (order) and = (equivalence) such that given any two bodies B and

C, the trichotomy law holds, and the transitive property holds for >. It is important to realize the only way one can be sure that the law of trichotomy and the transitive property hold is through experiment. A property is extensive if it is intensive, if there exists a physical operation that is closed with respect to the property, if it is commutative and associative, and if it has the following properties: (1) if A = B and C = D, then A + C = B + D for all bodies A, B, C, and D, and (2) if A = B, then A + C > B for all C.

So, the domain of the function has definite structure, was stated without regard to number, and depends on whether the property is intensive or extensive as well as intensive (any extensive property is intensive, but not conversely). Once this structure has been identified it is possible, through assignment of some body as a unit body, to assign real numbers to bodies through a process called measurement (or application of the measurement function). The function thus defined must preserve the structure of the domain. For an intensive dimension, this means that (1) F(B) = F(C) if and only if B = C; (2) F(B) > F(C) if and only if B > C and, for an extensive dimension, F(B + C) = F(B) + F(C). Obviously, F depends on the unit selected so that F = k·G for another measurement function G defined on the same domain, where k is a positive real number. An example of a domain of bodies important for this study is the class of collections of physical objects, where the comparison between sets is based on one-to-one correspondence. If the unit selected is a single object, the measurement of a set is its count. The measurement function then assigns ordinal numbers to sets and preserves the additive structure (for a dimension to be extensive, it is sufficient for the bodies to be pair-wise disjoint).

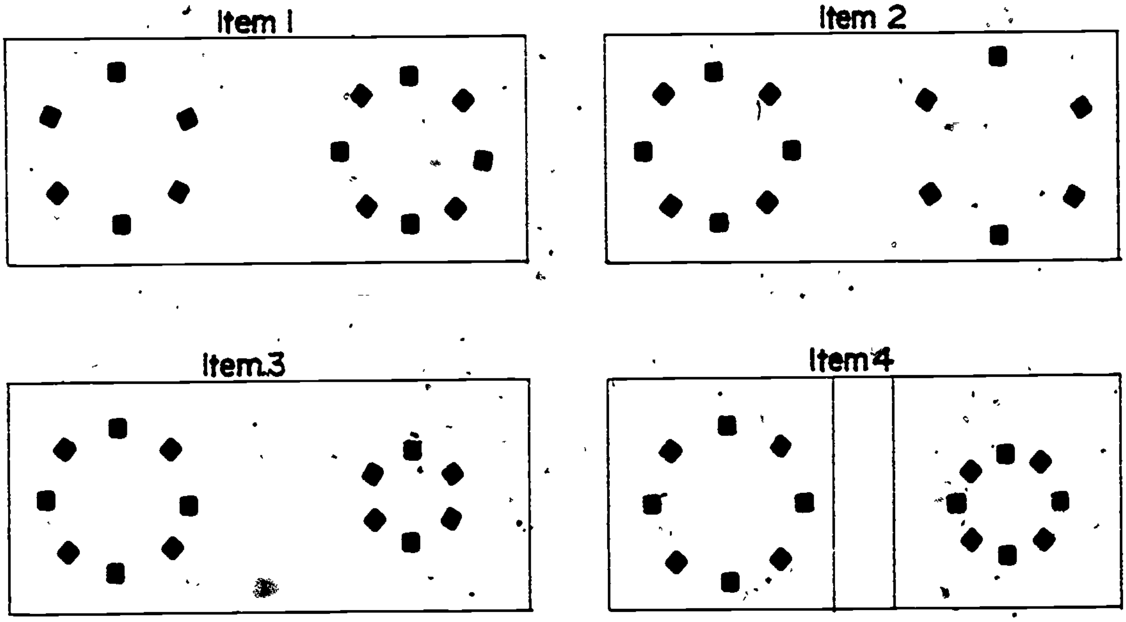
Contrast of Piaget's conception of intensive quantity and extensive quantity with the definitions given above are made with regard to the structure of the domain of the measurement function, with regard to units, and with regard to mathematical and cognitive structure. A child who is capable only of intensive quantity in a Piagetian sense would not have mastered the notion of units. However, units of measurement may intervene in an intensive quantity (e.g., density) in the scientific definition, but not in the Piagetian conception. When a child is capable of what Piaget calls "extensive quantity," units intervene. Apparently, a child capable of extensive quantity in the Piagetian framework would be likely to comprehend quantity, intensive and extensive, in the scientific sense. It should be noted, that it would be a restricted conception in the sense of a formal concept and in the sense of generality (i.e., a child would not necessarily be able to conceive of all different quantities such as real numbers or density). Surely, it sould seem for a child to comprehend an intensive or an extensive quantity in the scientific sense with units, he would of necessity have to be capable of extensive quantity in the Piagetian sense due to the intervention of units in the scientific definitions.

The reason a differentiation needs to be made between genetic structures concerning quantity and the scientific structures of quantity can be seen by example. Let B be a collection of collections of physical objects. If equivalence and order are defined on the basis of one-to-one correspondence in the usual manner, B together with the criteria for comparison which have been set up is a quantity. Do we have an extensive or an intensive quantity? It depends on whether or not the elements of B are or are not mutually exclusive, respectively. In either case, a unit is taken to be a singular object so that the collections are uniquely assigned numbers. A child's conception of gross, intensive, or extensive quantity in the Piagetian conception in no way depends on whether the collections of B are mutually exclusive. Rather, it depends on the cognitive operations of which the child is capable. If the child is capable of extensive quantity in the Piagetian sense, he ought to be capable of comprehending the structure of the measurement function under discussion, the unit of measurement, and the necessity of disjoint collections being used for addition of numbers.

Extensive quantity is identified with numerical one-to-one correspondence, both of which incorporate the notion of a unit. As was seen in the section Number in Piagetian Theory, the notion of a unit is essential to number and is arrived at by a synthesis of Grouping I and V, as is cardinal and ordinal number. Consequently, extensive quantity is paralleled by cardinal and ordinal number in Piagetian theory. Gross and intensive quantity correspond to stages in the development of number, which have not yet been discussed here.

Quantity and arithmetic. Two noteworthy studies have been conducted in which quantitative comparisons addition and subtraction, and manipulatable objects have been interrelated. In the studies (Steffe, 1966, Le Blanc, 1968), children for whom evidence was present that they were able to make extensive quantitative comparisons performed significantly better on tests of addition and subtraction problems than did children for whom no such evidence was present. Both of these studies were conducted toward the end of the school year using first grade children. Three four item tests were constructed, each of these being designed to measure the ability of children to make quantitative comparisons. Four geometrical arrangements were used, one for each test--circular, rectangular, and linear.

FIGURE 1



Note: Circular patterns have 4" and 7" diameters.

In item 1 of one of the test using circular arrangements (see Figure 1), if a child made a comparison based on the diameter of the two circular arrangements (making a gross comparison), he would no doubt give an incorrect response to the examiner's question, "Are there more blocks here or are there more blocks here or are there the same number of blocks here as here ("here" is identified by pointing)?" A gross comparison could also be made based on relative density alone, which would lead to a "correct" response. A child could also make an intensive judgment that one circle had more blocks because both circles were of the same diameter but one was more dense; this would also be a correct judgment. It was, therefore, possible for a child to respond correctly on this item without making an extensive quantitative comparison. The same can be said for Items 2 and 3. However, for a child to respond correctly on Item 4, an extensive comparison had to be made if one ascribes to the theoretical interrelationships of correspondence, quantitative comparisons, and logical multiplication. Certainly an intensive comparison was not possible since there were the same number of blocks in each circle, all equally spaced, so that the arc distance between the blocks was always in the same ratio to the diameter.

The two remaining tests were strictly analogous to the test using circular arrangements. In the two studies under review, there was no attempt to explicate experimentally the theoretical interrelationships mentioned immediately above nor are such attempts made in the present study. The assumption is made that for a child to respond correctly to items analogous to the last item of each test of quantitative comparisons under review, a process of "forward transformation" had to be initiated and the forward transformation involved quantitative comparisons, which in turn involved logical multiplication of relations.

The concept of forward transformation has been advanced by Beilin (1969).

Forward transformation is a more significant type of transformation than reverse transformation* since it is the basis of many kinds of problem solving. It is apparently more difficult to initiate, however, than backward or reverse transformation. Carrying out the forward transformation inevitabley means involving a compensation procedure with the dimensions of length and width and so the transformation is inextricably involved with logical multiplication. . .

Successful response in the quasi-conservation** task is much more difficult than in the classic conservation task. The difference, as we have suggested, highlights the role of the analytic-set which triggers an internal transformation process that gives rise to some kind of conflict among inferences. No conflict exists on the stimulus side of the equation per se. Conflict results only from the subject's disposition to analyze the data of his experience in such a way as to generate inferences which are in conflict because of their logical incompatability (i.e., "the objects cannot be both identical and nonidentical at the same time") [p. 435].

Of the 341 first-grade children tested for the "addition" study (Steffe, 1966), 128 were incorrect in at least one item of each test. Since Item 4 of each test was very difficult for the 128 children, these children may be viewed as being gross quantitative comparers. They were designated as Level 4. Three other levels of an ability to make quantitative comparisons were identified: Level I where all items of all tests were scored correctly; Level 2 where all items on exactly two tests were scored correctly; and Level 3 where all items of exactly one test were scored correctly. Analyses of variance indicated that statistical differences ($p. < .01$) existed among the mean performances of the four levels for addition and subtraction problems. It is important to note

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*Reverse transformation is a process initiated by a child, either physical or mental, where, e.g., a collection of objects are returned to their initial positions in a conservation of one-to-one correspondence problem.

**Quasi-conservation refers to a task where the objects are not moved physically, as in Test I.

that the test of quantitative comparisons was administered starting March 8 and the test of problems was completed on April 12, 1966.

Sullivan (1967), in his critical analysis of Piaget's theory as it relates to School Curriculum, has stated:

> A substantial correlation between number readiness (e.g., conservation of number) and the achievement of addition and subtraction can be interpreted in both directions. Simply, it raises the question of "which came first, the chicken or the egg;" that is, we do not clearly know whether learning of addition or subtraction enhances conservation or whether the opposite obtains (p. 21).

When the significant differences among the four levels of quantitative comparisons noted earlier are considered, it must be pointed out that the children in the studies received very little or no direct instruction on extensive quantitative comparisons as measured, but had received instruction on processing sums and differences. Children who showed little aptitude for making extensive quantitative comparisons involving forward transformation performed statistically less well on the problems than children who were successful in making extensive quantitative comparisons. Since the children in Level 4 did have a mean solution rate of approximately two out of every three addition problems and one out of every two subtraction problem, it cannot be said that where instruction on processing sums has been given, the ability to make extensive quantitative comparisons involving forward transformations is necessary for the solution of problems.

But the results of the two studies were as theory predicted. Further analyses showed children who did not make an extensive quantitative comparison and, in consequence, did not (probabilistically) initiate or unsuccessfully initiated an internal forward transformation, also performed poorly on the type of problems most demanding a forward transformation. For the problem structure a - b = x; where the problems were verbally presented without manipulatable objects, the mean score was only 24 percent. For the problem structure a + b = x under the same conditions, the mean score was 49 percent.

A later study conducted by Steffe and Johnson (1971) was designed to answer questions raised in the first two studies (Steffe, 1966; Le Blanc 1968). During November, 1967, 199 first grade children were given a test of quantitative comparisons. These 199 children were in eight classrooms housed in four different school buildings in a rural Georgia County. Between January 15 and 24, 1968, 192 of the previously tested 199 children were administered the Lorge-Thorndike Intelligence Test, Level 1, Form A. The test of quantitative comparisons (discussed later) contained 15 items. Evidence was strong that a child could make extensive quantitative comparisons if he scored at least 10 of the 15 items correct. If he scored seven or less evidence was considered weak for extensive quantitative comparisons. Of the 192 children, 127 with IQ scores in the range of 80-97 or 103-120 were used in the study. Four groups of children were then defined by crossing the two classification variables. During the month of May, 1969, a 48 item problem solving test was administered to 108 children remaining in the study at that time. Twelve problems for each of the following four problem structural types were presented to each child: a + b = n, a - b = n, a + n = b, and n + a = b. A treatment variable called Problem Conditions (presence or absence of manipulatable objects during problem solution) was used where children were randomly assigned to the two conditions. The following research hypotheses were of interest.

1. Children who are able to make extensive quantitative comparisons are able to solve arithmetical word problems with structural types a + b = n, a - n = b, a + n = b, and n + b = a better than children who are not able to make extensive quantitative comparisons.

2. Children who are not able to make extensive quantitative comparisons are able to solve arithmetical word problems with the four structural types in the presence of manipulatable objects significantly better than in the absence of manipulatable objects.

3. The problem structure a - b = n, is correlated higher with the problem structures a + n = b and n + a = b than with a + b = n.

In the analysis of the data, it was found that mean performances of children in the high and low categories of quantitative comparisons differed substantively on addition problems (a + b = n) in the case of no manipulatable objects present during problem solution (48 vs 75 percent). But mean performances across the two categories did not differ in the case of manipulable objects present for the same problem structure. The analogous mean performances for the problem structures a - b = n, a + n = b, and n + b = a did not differ within objects present or objects absent. However, mean performances on the structural types a - b = n, a + n = b, and n + b = a was between 46 and 54 percent, inclusive. The mean performance for the structural type a + b = n was approximately 75 percent. So, hypothesis (1) was rejected for all problem structural types except for a + b = n.

The presence of manipulatable objects was a strong variable for all problem types for all categories of children. Hypothesis (2) was not rejected and was extended to include extensive quantitative comparers.

The correlation of the problem structure a + b = n with the three others was in the interval [.45, .59] while the intercorrelations among these latter three structural types fell in the interval [.65, .79] with most greater than .70. These correlations do not contradict hypothesis (3).

Moreover, in view of the low mean scores for the subtraction problems and in view of the significance of quantitative comparisons in the case of the structural type a + b = n, instead of considering the ability to make forward and reverse transformations basic to an ability to solve arithmetical problems of the various structural types, it is now hypothesized that the ability to make forward and reverse transformations is basic to the <u>acquisition</u> of an ability to solve arithmetical word problems.

The test of quantitative comparisons used in the Steffe and Johnson (1971) study was developed in an earlier study (Harper and Steffe, 1968). Eight of the test items involved a forward transformation and seven a reverse transformation. Of the eight items involving a forward transformation, six involved a comparison of two equal sets, three of six objects per set, and three of eight objects per set. The geometrical configurations varied across these six items with configurations of (1) circles, (2) rectangles, (3) lines, and (4) triangles, since comparisons of two equal sets of objects are easier in a rectangular configuration than in a circular or a linear configuration (Steffe, 1966). The objects in two of the eight items involving a forward transformation were arranged in lines--one of six objects and one of eight objects. These items were included to provide some floor in the test. If two rows of objects have equal length but one has greater density, an intensive quantitative judgment would suffice for a correct comparison of the numbers of objects in the two sets. One of the two items was exactly of this nature. In the other item, the row of eight objects was shorter than the row of six objects. Actually, an intensive comaprison should be necessary for a correct response, but children who were capable only of gross comparisons should have responded correctly to the item if they focussed on density, which seems to be the most likely focus. The six items which had the same number of objects in both sets required the children to make an extensive quantitative comparison if they were to respond correctly.

Since it is the extensive quantitative comparison that makes possible a numerical correspondence, the child who made a correct comparison by using one-to-one correspondence was said to have established a numerical relation between the sets of objects. If a child made a correct comparison by counting, then, because the three stages in coordination of cardinal and ordinal numbers corresponds to the three stages in numerical correspondence, the child was said to have established a numerical relation between the two sets of objects.

The remaining seven items of the test involved objects which the child moved. Four of these items involved situations in which the child had to compare two sets of objects with the same number in each set. These items varied in many ways from the corresponding six in the first eight discussed above. One of the most striking differences was that, in the items with movable objects, the one-to-one correspondence was established by the children before they were asked to compare the two sets in their final state. A principal component analysis supported a contention that different abilities were required to distinguish between the items containing equal numbers of objects in the sets to be compared and the items containing unequal numbers of objects in the sets to be compared. It is important to note that these items varied across transformational types (forward and reverse). Other fluctuation of item difficulty was not a function of the transformational type as Beilin found (1969), but rather a function of the final geometrical configuration of the objects.

An interesting study has been reported (Mpiangu and Gentile, 1975) where an experimental test was made of the hypothesis that conservation of number is a necessary condition for learning other number concepts. The children used in the study were kindergarten students enrolled in two schools in surburban Buffalo, New York.

An eight item conservation of number test and a fifteen item arithmetic test were administered to the children as pretests. Any child who scored at least seven on the arithmetic pretest was discarded from the study. The children were then randomly assigned to experimental and control groups. The experimental group was given ten 20-minute arithmetic training session and the control group was given the 15-minute session playing a card game. The arithmetic concepts tested were: rote and rational counting; number recognition; relations (just before, just after, between); number synthesis and analysis.

The experimental group dramatically outperformed the control group on the posttest arithmetic test. When the post achievement test in arithmetic was regressed on the pretest conservation scores, no differences could be detected in the slopes of the regression lines. This lack of differences in the slope of the regression lines was taken by the experimenters as meaning that conservation of number is not a necessary requirement for learning arithmetic.

There are, of course, great differences in the studies reported by Steffe (1966); Le Blanc (1968); Steffe and Johnson (1971); and Mpiangu and Gentile (1975). The first three studies concentrated only on problem solving performances, whereas the latter study included basically order concepts. This difference in criterion variables is very important, as Brainerd (1976) has shown order concepts (transitivity of weight) to precede cardinal number concepts (his test was analogous to the extensive quantitative comparison test, static items) by as much as two years. His critical ages where order concepts were present and cardinal number concepts not spanned the age interval from 5 to 6 years. Consequently, it would not be expected that one would predict learning of one from the other during this age span. The situation is not as clear, however, for first grade children.

One should also consider that in essentially two weeks of arithmetic instruction, the experimental children went from a mean of 3.57 to a mean of 11.17 out of 15--from approximately 24 to 74 percent. When considering the scope of the learning tasks, the mean increase is quite substantial for such a short period of time. The children were required to count in both directions from any number between 0 and 11 and count by two's; find the name of a missing number in a given sequence (1-10); find numbers just before, just after, or between any two in the sequence (1-10); and find the correct answer and provide a correct justification to an item such as "three and two make how many?" Either the children were very able or else the criterion items were very close to the content taught. No delayed posttest was given, so it is not possible to ascertain the quality of the training in the sense of retention over time.

The four studies discussed in this section definitely raises a fundamental question needing resolution. This question is as follows:

> Are children who are capable of only gross quantitative comparisons able to acquire arithmetical knowledge to the same extent as children capable of extensive quantitative comparisons?

The question, as stated, is imprecise. It will, however, be made more precise in other sections of the report.

Quantity and set partition. In Part III of The Child's Conception of Number, Piaget (1952, p. 115) discusses the additive and multiplicative composition of number. In the discussion of the additive composition of numbers, the goal was to discover whether the child is capable of understanding that a whole remains constant irrespective of its parts. In the first problem, the child was told that he is to have four sweets at one time and four at another. The next day, he is to have the same number but, because he will be less hungry at the first then at the second, he will have only one sweet at the first time and all the others at the second. Beans were used to illustrate each statement, three beans being taken from one pile of four and put with the other four to represent the situation the second day. The child was asked to compare the other two [(4 + 4) and (1 + 7)] and to say whether he would eat the same number of sweets on both days. The second problem consisted of giving the child two unequal sets of counters and asking him to make them equal (apparently it was always possible to do so). In the third problem, the child was given some counters and was asked to divide them into two equal parts (again, it apparently was always possible to do so).

Three stages were identified regarding the three problems, where the stages were the same across problems. In the first stage, the children grasped neither the equality of the two arrangements (4 + 4) and (1 + 7) nor the permanence of the whole in spite of changes in the distribution of the parts, the latter being a characterization of the famous class-inclusion problem reported in the same volume (Piaget, Chapter 7, 1952). The last (and operational) stage was characterized by reversible operations. The middle stage is a transitional stage where the child can be led to a realization of the invariance of the whole, but does not discover it spontaneously. The same type of phenomena can be observed with regard to the remaining two problems.

Two aspects of the relationship between a set and its partitions are essential: The first is that a partition exhausts a collection, and the second is that any two partitions are not equal sets. The first is essential for the child to realize what is invariant relative to the second. Piaget's study of addition concentrated on these two aspects when he asked a child to recognize, for example, that 4 + 4 = 1 + 7. So, it would seem that <u>partitions of a collection</u>, as a concept, is developmental and highly related to quantity and number in Piagetian theory.

Class Inclusion

In the section Number in Piagetian theory, it was pointed out that Piaget views nested classification as being essential for number, and reciprocally, number as being essential for nested classifications. Piaget (1952) has stated that "class and number are mutually dependent, in that while number involves class, class in it's turn relies implicitly on number" (p. 184). The difficulty of understanding the serial inclusion associated with whole number was pointed out by Sinclair (1970, pp. 150-151). In an experiment designed by A. Morf (Greco and Morf, 1962 pp. 71 ff), a collection of 9 cubes is placed in front of the child. The experimenter had one block and added to it until a good deal more than 9 were present. The question put to the child was whether there was a time when the experimenter and child had the same number. The five- and sometimes the six-year olds were not at all sure. Class-inclusion, then, is to Piaget an integral aspect of a childs' numerical reasoning. On the other hand, numerical reasoning is an integral part of class inclusion.

Dodwell and Elkind have performed replications of Piaget's experiments on the ability of children to include partial classes within a total class, i.e., if $A \cup B = C$ ($A \cap B = \phi$), then $A \subset C$ or $B \subset C$. For his subjects, Elkind (1961a) selected twenty-five children from each of the grades kindergarten to third. The question asked of each child was, "Are there more boys (or girls depending upon the sex of the child being questioned) or more children in your class?" Other questions were also asked to gain assurance that the children understood the above question. On the basis of the responses, the children were placed in three stages; Stage 1 if either $C \subset A$ or $C \subset B$, (A = boys, B = girls, and C = children), Stage 2 if $C = A$ or $C = B$, and Stage 3 is either $C \supset A$ or $C \supset B$. Fifty percent of the five-year-olds, thirty-two percent of the six-year-olds, twelve percent of the seven-year-olds, and eight percent of the eight-year-olds were in Stage 1. Correspondingly, 48, 56, 76, and 92 percent respectively were in Stage 3. The four distributions of percents were statistically different.

Dodwell (1962) was interested in investigating the response to class inclusion questions and responses made on the tests of provoked and unprovoked correspondence discussed earlier. In the discussion of the results, he stated that the "ability to answer correctly questions which involve simultaneous consideration of the whole class and its (two) component subclasses, appears to develop to a large extent independently of understanding of the concept of cardinal numbers (as measured by the tests for provoked and unprovoked correspondence)" (p. 158).

The above studies are what may be called "one-shot" studies, that is, studies that test an indivudual at a point or points in time. The question immediately arises, then, if a child is on a given stage at a given point in time with reference to a particular situation and particular materials, will the same child be on the same stage at a different point in time, all other things constant? Dodwell (1961), using the tests devised in an earlier study, made a test-retest reliability study with intervals of one week and three months. He comments, "The short-term reliability of the test is highly satisfactory, and compares well with the reliabilities of many commercially available cognitive tests. The long-term reliability indicates considerable stability in the development of number concepts. . . ." (p. 30).

In this same study, Dodwell examined the data from his original sample of 250 children to detect differences due to sex and socio-economic status. He reports that differences were extremely small, insignificant, and did not favor either sex. To test for socio-economic status, the children were divided into three groups on the basis of their fathers' occupations: (1) professional, (2) clerical and semi-skilled, and (3) semi-skilled or unskilled trades. No differences were detected among the groups, but the higher socio-economic groups scored more favorably.

Class inclusion being unrelated to one-to-one correspondence does not prove conclusively that it is not an integral part of the child's conception of number in a serially inclusive sense, nor does it prove that it is not an integral part of whole number operations. The latter two problems remain to be studied more definitively.

Logically, addition and subtraction of whole numbers and Piaget's class-inclusion problem are inextricably intertwined. Little data are available, however, concerning acquisition of addition and subtraction and performance on the class inclusion problem. Sullivan (1967), in his critical appraisal of cognitive development theory to school curriculum, noted that "If a relationship was demonstrated. . .between the attainment of addition and subtraction and the wooden bead problem, it might just as well be interpreted that addition and subtraction is a necessary condition for class inclusion. . . (p. 2)" Sullivan unwittingly may be partially correct as, already noted, Piaget sees number as a synthesis of Groupings I (Primary Addition of Classes) and V (Addition of Asymmetrical Relations). Operational classification, however, awaits the development of number where the elements of the classes are considered as units. Consequently, Piaget's formulations lead to a genetic circularity among classes, relations, and number. Class inclusion is taken as the criterion of presence of Grouping I, so that, from a genetic point of view, there is no reason to attribute necessity to one or the other of class inclusion and addition and subtraction (as studied by Piaget) for the presence of the other. So, addition and subtraction may be necessary and sufficient for class inclusion.

Training studies. Beilin (1971) has given an extensive review of the literature pertaining to training children to perform logical operations. Class inclusion was included in his review. In fact, he found few data regarding the training of classification beyond that pertaining to class inclusion. The major goal of the training studies reviewed by Beilin was to determine whether class inclusion is symptomatic of an underlying mental organization pertaining to classification, Grouping I: Primary Addition of Classes, or whether it is mainly the result of experience.

The most noteworthy of the studies reviewed by Beilin is the one conducted by Kohnstamm (1968) due to the results and the subsequent controversy created by the study. Kohnstamm's approach was to use a total educational experience to teach children class inclusion. He used three instructional approaches, one a pure verbal method and the others a verbal method supplemented by pictures or by physical objects. In the purely verbal method, he asked questions such as "In the whole world, are there more dresses or more clothes?" In the case of incorrect answers the children were told they were incorrect and were given the correct answer as well as a reason for the answer.

In the second instructional approach, the purely verbal method was supplemented with pictures of different classes. The same feedback procedures were used. In the third instructional approach, the verbal method along with pictures was supplemented with Lego-blocks.

In the case of the purely verbal instructional approach, six of twenty five-year old children were observed to have learned how to solve the class inclusion problem. In the case of the second instructional method, eight of twenty five-year old children could solve the pictorial items as well as the verbal ones. In the case of the third instructional group, sixteen of twenty children could solve the picture items as well as the block items.

Kohnstamm's (1967) results clearly indicated that experience may be a primary factor in solving the class inclusion problem. But the Piagetians' took exception to his interpretation of the results of his experiment, claiming they were "figural structures" rather than operative structures. In response to Kohnstamm's work, Inhelder and Sinclair (1969) undertook a learning experiment in class inclusion with eleven children. When using Kohnstamm's criteria, they observed that nine of the 11 children succeeded in class inclusion. When more stringent criteria were established, only two of eleven succeeded. The more stringent criteria involved a valid explanation and correct response to a problem of a different form.

The response of Inhelder and Sinclair to the Kohnstamm experiment is very important because Grouping I would imply that a child who is operational with class inclusion has at his disposal a potential of elaborating a nested hierarchy of classes not restricted to a class and <u>one</u> of its subclasses. The class inclusion problem is merely a convenient way of tapping this potential. Children trained on a narrow front (with only two classes) may <u>act</u> as if they have the potential of elaborating a nested classification but may not, in fact, be capable of doing so. A similar situation in mathematics teaching is where a student is trained to prove the triangle inequality ($a + b \geq c$, where a, b, and c are the lengths of the sides of a triangle) and, given a triangle, 'knows' that the sum of the length of two sides always exceeds the length of the third, but thinks it is possible to construct a triangle out of a three inch segment, a four inch segment, and an eight inch segment.

Rather than dwell on the complete set of training studies surrounding class inclusion, Beilin's (1971) summary statements are cited.

> These studies of class inclusion point to the fact that training can lead to successful acquisition of this logical ability. . . .
>
> The question of operative achievement from instruction and training still appears not fully resolved. . . . (p. 105).

It must be pointed out that when Beilin states that training can lead to successful acquisition of class inclusion, he considers class inclusion as being simply one class included in another. The position being taken here is that the structure of the class inclusion relation (a partial ordering) must be taken into account for any claim of operativity to be made. It is not enough to train children on a particular problem or set of class inclusion problems, test them on the same problem or set of problems, and claim class inclusion has been internalized as a flexible, functional scheme.

Classification. In order to fully appreciate classification behavior of children, it is necessary to discuss classes per se. Generally, when objects are classified together, they share common properties. For example, quite dissimilar objects can be classified together under the heading, "fruit." What makes these objects "fruit" is what is common. Within the class of fruit, however, important differences exist -- oranges and apples are different. Given a universe of objects, three distinct kinds of properties exist (Inhelder and Piaget, 1964).

1. Properties specific to members of a given class (e.g., the properties which make items fruit) which distinguishes the class from other classes (from vegetables, meat, etc.).

2. Properties which are common to members of a given class and those of other classes to which it belongs (e.g., that which is common to fruit and vegetables).

3. Properties which differentiate members of a given class one from another (those which differentiate a pear from an apple, for example).

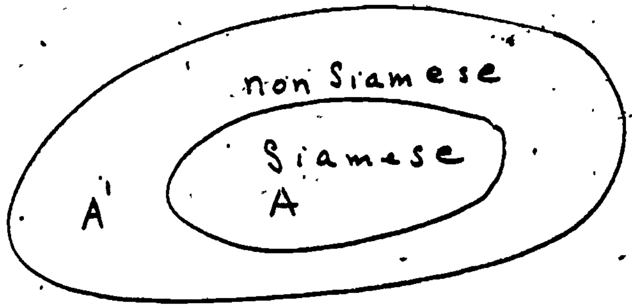
The intension of a class is the properties common to the elements, and the extension of a class is just the members of the class. The coordination of the intension and the extension of a class is what develops in children in stages.

Young children below about six years of age have been shown to employ primitive behavior in attempting to form classifications. The types of collections formed by these children have been called complexive collections or graphic collections (Inhelder and Piaget, 1964). For example, children were asked to classify a collection of geometric objects together, some triangular shapes, some square shapes, and some half ring shapes. At least three varieties of graphic collections were identified. First, some children constructed a number of subcollections, ignoring the rest of the material which was never classified. The subcollections had no common property--the child would change criteria of classification within a subcollection. Sometimes, subcollections were not formed but properties of individual items noted. Second, successive similarities between one object and the next were formed. While this is an improvement over the type of behavior noted in the first example, it is not true classification since no over-all criteria for classification were found for subcollections; subcollections were not differentiated, and part-whole relationships were not identified. Third, definite figures are made out of the objects--a "house" is made, then windows, etc. That is, the child makes no real attempt at classification, but instead plays with the objects, constructing whatever comes to his fancy.

The graphic collections described above have two features differentiating them from true classes. First, some collections are formed on the basis of the spatial arrangement of the objects. Second, no criteria for classification (no properties which tied all the elements together) were isolated by the children. These two aspects are simply another way of saying that intensive properties were not identified by the children--these children are at Stage I (pre-operational) as regards their classification behavior.

Stage II (or transitional) classification behavior is an advance over Stage I classification behavior, but it is not yet operational classification behavior. Stage II classification behavior can best be characterized by a recognition of intensive properties, with no complete coordination between the intension of a class and the extension of a class. Given a class of objects; children are able to separate the class of objects into subclasses. This means that they understand that all elements can be classified, each subclass contains elements of a specific kind or which possess a specific property, and two or more subclasses are constructed. Yet, the subclasses formed are not thought of as forming a hierarchy of classes. The class-inclusion relation is not mastered.

The class-inclusion relation being mastered means simply that, given a class A which is contained in a class B, the child understands all of the A are some of the B but all of the A do not constitute all of the B. For example, if A is the class of Siamese cats and B is the class of cats, then all Siamese cats are certainly cats, but they do not exhaust the cats. That is, there are cats that are not Siamese cats. So, all of the A do not consittute all of the B, but just some of the B. Children at the transitional stage of classification certainly realize that Siamese cats are indeed cats, and in fact are part of the set of cats. So, one would think they would understand class-inclusion. But they may not. It is critical they understand that there are cats other than Siamese cats or, in other words, that all cats are not Siamese. If A' denotes the set of nonSiamese cats, then $A \cup A' = B$ and $A = B - A'$.



To understand class-inclusion, the child must be able to engage in reversible thinking. To do so is to be able to conceive that the Siamese cats, together with the nonSiamese cats ($A \cup A'$), make up the cats (B); and that the cats, minus the nonSiamese cats, make up the Siamese cats ($A = B - A'$). In this reversible reasoning, the child has to be able to conceive of the total class of cats as being made up of the two subclasses at one and the same time. Stage II children, when focusing on the cats, lose sight of the subclasses, and when focusing on the subclasses, lose sight of the total collection. Typical responses of transitional children (Stage II) are given in the following situations. A picture is shown to the children on which there are, say, four Siamese cats and three cats which are not Siamese. The children are asked to compare the number of cats to the number of Siamese cats. When asked to do so the children will compare the Siamese cats to the other cats.

The children at Stage III (concrete operational) are capable of solving the class inclusion problem, and are much more flexible in their classification behavior than are Stage II children. Stage II children are able to build hierarchies of classes. For example, they are capable of conceptualizing such hierarchies as maltese terriers are part of the terriers, terriers are part of the dogs, dogs are part of the mammals, and etc. Stage III children are not only capable of building hierarchies of classes, but are able to change the criteria of classification and reclassify a set of elements in a new way. The child may consider new dogs in his classification and refine the classification to include many more classifications than those given. Two complementary processes exist that describe the Stage III flexibility in classification. One, given a classification, the child can go back and construct finer classifications or whole new classifications and not be tied to the one constructed. Two, a child can anticipate a classification before it is done.

In summary, the following three stages in children's classificatory behavior have been identified:

Stage one. (Preoperational) Given a collection of objects and told to "put everything together that goes together," a child at this stage forms what is known as "graphic collections." If he does anything, he constructs one or more spatial wholes. This is a child's first attempt to coordinate part-whole relations with those of equivalence and difference.

Stage two. (Transitional) At this stage, the constructed collections are no longer graphic collections. Trial and error plays a large role in construction of classifications and no over-all plan is present. Children cannot yet solve the class-inclusion problem but do understand that all elements need classifying, each subclass contains elements which possess a specific property, and two or more subclasses are constructed.

Stage three. (Concrete Operational) Children at this stage are able to coordinate the intension and extension of a class, as evidenced by the solution of the class inclusion problem. Children at this stage are capable of conceiving of hierarchial arrangements of classes, and are capable of imposing more than one classificational system on the same collection of elements, anticipating the new classificational system before carrying out the classification.

Class inclusion and arithmetic. Surprisingly, class inclusion has not been used to any great extent by mathematics educators in studies of children's acquisition of arithmetical content or in studies designed to assess effects of instruction in logical reasoning . . . especially classification. A study of the latter type was carried out by Johnson (1975). This study is critical for the utilization of class inclusion as a readiness variable in the present study, so it will be discussed in some detail.

The purpose of the study was to determine if specific instruction on classification would improve the ability of young children to (a) form classes, (b) establish selected equivalence and order relations; and, if so, would transfer occur to other class-related activities or the transitive property. The sample consisted of kindergarten and first-grade children with chronological ages in months in the intervals (64, 76) and (77, 89), respectively. The children were further categorized into IQ intervals of (80, 100) and (105, 125), as measured by the Otis-Lennon Mental Abilities test. Random assignment was used in forming an experimental and a control group.

The learning material was designed to provide children with experiences in forming classes, intersection and union of classes, the complement of a class, and relations between classes and class elements. The intensive properties of the classes could be abstracted through simple abstraction of physical properties (e.g. red) or else were functional properties (e.g., things to ride in). The first three sessions (I, II, III) were designed to provide experiences in forming classes. In the next three sessions (IV, V, and VI) work was done on the intersection and complement of the intersection of classes. The children were put in a conflictive situation where it was pointed out that an object could not be placed inside of two nonoverlapping hula hoops simultaneously. The two following sessions (VII and VIII) included activities concerning formation and union of classes. Sessions XII - XV contained activities designed to operationally define the relations "more than," "fewer than," and "as many as." The remaining sessions (IX - XI) involved practice material on formation of classes involving union, intersection, complementation, and nested classifications.

hould be pointed out that all of the necessary content was inclu the instructional session to enable the child to solve the class sion problem. All he had to do was structure the information. Essent everything through Stage II classification was include he instruction, as well as the set relations "more than," "fewer t nd "as many as" which occur in the question of the problem. The instr included recognition of intensive properties specific to a given cl alization that an object can possess more than one intensive pro intersection and union); recognition of properties that separate elements of a given class (complementation); and recognition of properties common to members of a given class and other classes to which it belongs (nested classification, intersection, union, and complementation). It was felt that the class inclusion relation must be structured by the child. If instruction is to be assimilated regarding classification, it must be broadly based, including intension and extension of classes. But the child must coordinate the intension and the extension. Specific training, such as Kohnstamm's only serves a narrow function in acquisition of Stage III classification behavior.

The posttests were separated into achievement tests and transfer tests. The achievement tests were a connective test (and, or, and not), a relations test, and an intersecting rings test. The transfer tests were a multiplication of classes and relations test, a class inclusion test, and a transitivity of relations test.

The data analysis showed that the treatment greatly affected achievement. Age did not yield significance, whereas a categorization variable (IQ) did yield significance for all achievement mean differences. The means for the connective achievement test were 71 percent for items based on content contained in the learning material, and 72 percent for items based on content not contained in the learning material in the case of the experimental group. For the control group, the analogous means were 42 percent and 33 percent. For the relations achievement test and the intersecting rings test, the means were 78 and 44 percent for the experimentals and 52 and 11 percent for the controls. These means indicate that the treatment was highly effective for its designed purpose.

In the case of the transfer tests, treatment was effective for the multiplication of classes and transitivity tests, but not for the class inclusion test. Again, age was not significant for any test, but intelligence was, especially for the class inclusion test, which had a grand mean of only 29 percent. The grand mean for the transitivity test was 68 percent and for the multiplication of classes and relations test, the means were 63 percent (3 x 3 matrices) and 55 percent (2 x 2 matrices) for the experimentals and 44 percent (3 x 3 matrices) and 39 (2 x 2 matrices) for the controls.

From the training sessions, it was evident that class inclusion was resistant to the instruction. The intersecting ring items showed improvement due to the training, but there was strong evidence that the control children viewed the intersecting rings as forming three subregions due to the most frequent response choice in a multiple choice format. Moreover, direct instruction was given on intersecting rings.

It appears, then, that while one can dramatically improve children's classification capabilities in the sense of recognition of intensive properties of a class, it is quite difficult to improve the coordination of the intension and extension by instruction on the intension and necessary subskills. Direct training is effective, as shown by Kohnstamm, but that training is shallow, as shown by Inhelder and Sinclair.

The conclusion drawn here is that class inclusion is resistant to training, if the goal of that training is to influence the structure of class inclusions as a relation. While this conclusion is stronger than the one made by Beilin, he did not have the advantage of Johnson's study in his review. Johnson (1975) goes on to extrapolate "When considering the results of the study . . . a serious problem is revealed in that children are being presented with concepts they are conceptually unable to handle. In a subtraction problem such as 9 - 5 = 4, if a child thinks that the difference is larger than the minuend he might just as well write something like 5 - 9 = 4" (p. 143).

Very little work has been done attempting to show a causal relationship between class inclusion and addition and subtraction. Vitale (1976), in a study conducted to evaluate the Comprehensive School Mathematics program at the kindergarten and first-grade level, observed a correlation of .06 between class inclusion and subtraction computation and a correlation of only .28 between class inclusion and addition computation. These low correlations cannot be taken as showing no relation between class inclusion and subtraction and addition, because the addition and subtraction items were computation items presented using numerals. However, she also observed a correlation of only .09 between class inclusion and subtraction problem solving. As the subtraction problems had to be read by the children, possible effects of class inclusion may have been masked due to reading difficulties. Moreover, the study does not show possible effects of the lack of class inclusion on learning of addition, subtraction, and especially of the missing-addend problems. It does indicate that not as much relationship exists between class inclusion and whole number operations as Johnson implied.

The Achievement Variables

Quantitative comparisons and class inclusion are personalogical variables of a cognitive nature. They are based in Piaget's grouping structures but have a logical relationship to cardinal and ordinal number. But the extent to which they are readiness variables for learning different aspects of cardinal and ordinal number is yet to be determined.

Seven clusters of variables are defined in the subsequent presentation. Each one of these clusters is used in a multivariate analysis of variance where Quantity is used as a categorization variable--extensive quantitative comparers versus gross quantitative comparers. Through these analyses, a determination of Quantity as a readiness variable for acquisition of content of first grade mathematics can be accomplished. While the multivariate analyses cannot be used to prove deductively Quantity is a readiness variable, statistical differences can be used to gain support or rejection of hypotheses arrived at through logical analyses. The statistical differences would be especially compelling if the mathematical learning tasks of the children are controlled to include what are considered as mathematical learning tasks critical to acquisition of the content in question.

In the next section, those aspects of cardinal and ordinal number important for instruction of the children, for the definition of the achievement variables, and for ascertainment of a logical connection between the two readiness variables and the achievement variables, are presented.

Theoretical and Empirical Background of Cardinal and Ordinal Number

In his classic work, Set Theory, Hausdorff (1962, p. 29) leaves cardinal and ordinal number completely undefined and asserts that relations between cardinal and ordinal number are merely convenient ways to express relations between sets. Hausdorff (1962) commented that "this formal explanation says what the cardinal numbers are supposed to do, not what they are...we must leave the determination of the 'essence' of cardinal number to philosophy" (pp. 28-29). Although Hausdorff's point of view is consistent with modern postulational developments in mathematics, it does not lessen the importance of his work on cardinal (and ordinal) number for research on acquisition of mathematical knowledge. For the structures which characterize the mathematical knowledge the child is asked to acquire seldom, if ever, correspond exactly in form to structural aspects of the child's natural thought. It is truely the case that Hausdorff is not concerned with the nature of cardinal (and ordinal) number and leaves the determination of their "essence" to philosophy, and ultimately to psychology as well. Not only is there a difference in the way in which the objects called cardinal and ordinal numbers are viewed in mathematical structures as discussed by Hausdorff and in genetic structures as discussed by Piaget, but there are formal differences in the structures and these differences are profound.

In the following exposition, only "naive" set theory is dealt with. In this theory, such constructions as "the set of all cardinal numbers" lead to antinomies. For a theorem is provable which leads to an unbounded sequence of cardinal numbers--which means that for any set of cardinal numbers, there is still a greater one. Consequently, "the set of all cardinal numbers" is not conceivable even though it would appear to be so. In the axiomatic treatment of set theory, these obvious contradictions have been removed (Kelly, 1955, pp. 250-81). Since the theory does not allow for unlimited construction of sets, the objects {x: x is a cardinal number} and {x: x is an ordinal number} do not represent sets. A distinction is made between a class and a set, in that a class is undefined, whereas a set is a class which is a member of another class. That is, a class x is a set if and only if there is a class y so that x is a member of y. Using this special restriction, cardinal and ordinal numbers are defined to be sets of a special kind. Rather than follow this axiomatic treatment of the development of cardinal and ordinal number, the treatment by Hausdorff is adhered to because it is felt to be closer to modeling child thought.

Ordered systems. During subsequent discussion, occasion arises to employ general ordered systems. The basic concept of ordered systems is that of a partially ordered set. A ready example of a partially ordered set is the set of subsets P(X) of a given set X ordered by the set inclusion relation "$\subset$."

If P is a partially ordered set and E a subset of P, then an element x of P is called an upper bound for E if for every $e \in E$, $e < x$. An element x is the least upper bound for E if for any other upper bound $y \in P$, $x < y$. Analogous definitions can be given for lower bounds and the greatest lower bound of E. A lattice is a partially ordered set for which every two element subset {x,y} has a least upper bound and a greatest lower bound. Examples of lattices are P(X) ordered by set inclusion and the positive integers ordered by "a divides b." The least upper bound of any two sets A and B of P(X) is $A \cup B$ and the greatest lower bound is $A \cap B$; and the least upper bound of any two positive integers ordered by "divides" is their least common multiple and the greatest lower bound is their greatest common divisor.

A <u>chain</u> in a partially ordered set P is a subset C of P in which < is connected (that is, a subset C where if $x, y \in C$, $x < y$ or $y < x$). Any such subset C of P is partially ordered by < and is a lattice as well as a chain. The set of natural numbers ordered by < is an example of a chain.

<u>Cardinal Number</u>. Hausdorff (1962) assigns objects, called cardinal numbers, to sets in such a way that if object a corresponds to set A and object b corresponds to set B, a = b if and only if A is equivalent to B. It is important to note that the set A to which the cardinal number a is assigned may or may not be an ordered set. Two cardinal numbers may be compared* by comparing the sets to which they are assigned. $a \leq b$ means that $A \sim B_1$ where $B_1 \subset B$. It may be that $A = B_1$ in which case $A \subset B$.

The sum and product of cardinal numbers determine their arithmetic. "The sum a + b of two cardinal numbers is the cardinality of the set-theoretic sum $A \cup B$**, where A and B are any two disjoint sets having the cardinalities a and b respectively (Hausdorff p. 33)." This definition is

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*Subleties exist concerning comparison of any two cardinal numbers in that it is, in fact, true that the comparability of any two cardinal numbers relies on Zermelo's well-ordering theorem, which states that any set can be well-ordered. This theorem is necessary (in Hausdorff's development) to show that there do not exist two incomparable sets, i.e., that it is never the case that there exist no A_1 and no B_1 so that $A_1 \sim B$ and $B_1 \sim A$.

**"$\cup$" has been substituted for "+".

justified because if $A \sim C$ and $B \sim D$ where D and C are disjoint, then $C \cup D \sim A \cup B$, so that the cardinality of $C \cup D$ is equal to that of $A \cup B$.

The product of two cardinal numbers a and b is defined as follows. "The product ab of two cardinal numbers is the cardinality of the set theoretic product A x B, where A and B are two sets with cardinalities a and b respectively (Hausdorff, p. 35)." The product of a and b is invariant of the particular choice of the sets A and B just as was the sum except that in the sum, A and B had to be disjoint. That is, if $A \sim C$, and $B \sim D$, then $A \times B \sim C \times D$, so that the cardinality of $C \times D$ is equal to that of $A \times B$. The commutative, associative, and distributive laws hold for the processes just defined, and depend directly on the commutative, associative, and distributive laws for set operations.

Barnes (1963, p. 194) defines a cardinal number as an equivalence class of sets without regard to order. With regard to this definition, there are two uses of cardinal number -- a class usage and a member-of-a-class usage. The member-of-a-class usage refers to the practice of using a representative set (a standard set, but not necessarily so) from a particular equivalence class. For example, one might look at a horse and say, "that horse has four legs." "Four" in this case is used in the member-of-a-class sense. The class usage of "four" is implicit in the following statement: "Horses normally have four legs." In the first case, a particular collection was referred to, and in the second, a class of collections was referred to. Logically, set equivalence is a critical concept for the class usage of number. Set equivalence, of course, is based on one-to-one correspondence.

<u>A contrast of cardinal number in mathematics and number as defined by Piaget</u>. It would seem that the class usage of cardinal number would not

develop until numerical one-to-one correspondence is available. But here a fundamental difference exists between the mathematical development of cardinal number and Piaget's notion. That difference can be characterized by the notion of a unit. In the mathematical development, no analogue of Piaget's "arithmetical unity" exists except for elements of sets. "Set" is taken as an undefined object and relations and cardinal number are defined in terms of sets. Such a procedure is acceptable although Piaget (1971, p. 37) is of the opinion that to define cardinal number in such a way is to introduce "number" into the definition of number. His opinion is based on the different psychological types of one-to-one correspondence--operational one-to-one correspondence assumes number.

Whether Piaget's psychological analysis is correct should be discussed. Van Engen (1970, p. 40) commented that Piaget does not distinguish between a relation that may exist between two or more elements of a set and the elements of a set. Van Engen's claim is certainly true, because Piaget's arithmetical unity depends on order relations for its construction--order of some type is essential for the objects to be considered as distinct, but yet equivalent.

The question of whether order is essential for the development of cardinal number needs an answer. Logical identity is involved. Tarski (1954) has defined logical identity as follows: "x = y if, and only if, x and y have every property in common (p. 55)." Examples and nonexamples may clarify the concept. Set equality is an example of logical identity and set equalivance is a nonexample of logical identity. From the definition, one can conclude that (1) every thing is identical to itself; (2) if x = y then y = x; and (3) if x = y and y = z then x = z. Logical identity is therefore an equivalence relation and has an accompanying symmetrical difference relation "not identical to." This symmetrical difference relation seems to be quite important in classification because if objects are classified together, they share common properties, but they also are different one from the other even if this difference is no more than the fact that the objects are distinct. Suppose we have two physical objects with every physical property in common. Would they be indistinguishable? Certainly not, for two distinct physical objects are always nonidentical because they can never have every property in common -- spatial position is an example they can never have in common. In case of number, however, it is clear that Piaget is considering mental constructions and not physical objects. But the arithmetical units Piaget speaks of need not be arranged mentally in a linear order to be held distinct. Rather, they can be arranged mentally most any way by virtue of the fact they are distinguishable as objects (albeit mental constructions). They are not logically identical to one another because they are different objects -- they may have independent existence in the mind much the same way objects which look alike physically have independent existence.

The beginning of number for a child may be set equality, which is a logical identity. A collection of objects in one beaker is the same collection no matter whether it is thrown out the window of an airplane or poured into another beaker. A child may think there are more objects in one beaker than the other, but also know they are the same objects. Here "more" denotes a global evaluation having little to do with the objects themselves. It is in this sense that the member of a class meaning of cardinal number may arise before the class meaning.

The above argument was advanced to illustrate the possibility that cardinal number and ordinal number may be distinguishable in their development.

That does not mean logical identity is to be considered a necessary and sufficient condition for the psychological existence of cardinal number. Nothing may be farther from the actual case. It would be rather surprising, though, if a child had a concept of cardinal number but not logical identity.

If one does not consider "number" to be necessary for operatory one-to-one correspondence, how is one to account for the development of operational one-to-one correspondence? If a child sets up a qualitative one-to-one correspondence between two classes and one or both of the classes were rearranged, there would be no hope that without logical identity the correspondence would be maintained. Following Van Engen (1970, pp. 34-52), if a number (e.g., four) is regarded as a particular set in the member-of-a-class meaning, then logical identity is surely a logical prerequisite to number while one-to-one correspondence is not. It is quite feasible that a child learns member-of-a-class meaning of cardinal number before the class. After (or when) the member-of-a class meanings are established, (at least for small numbers) the child may then construct one-to-one correspondence concepts.

Ordinal number. Just as set equivalence is a basic notion for cardinal number, set similarity is a basic concept for ordinal numbers. For clarity, the order relations discussed below are asymmetric and transitive (strict partial orderings) as well as being connected. Two ordered sets are called similar if there exists a one-to-one correspondence between their elements that preserves the order in the two sets. In symbols, "A is similar to B" is denoted by "$A \cong B$." Set similarity is an equivalence relation just as is set equivalence. Hausdorff (1962, p. 51) assigns order types to ordered sets in such a way that similar sets, and only similar sets, have the same order type assigned. In symbols, r = s means $R \cong S$. If a set is well-ordered, then its order-type is called an ordinal number.

In general, the arithmetic of order types is not isomorphic to the arithmetic of cardinal numbers. For if A and B are disjoint ordered sets, then the set theoretic sum of A and B (A + B) is a new ordered set such that the order of the elements of A is retained, the order of the elements of B is retained, and every $a \in A$ precedes every $b \in B$. If a is the order type of A, b the order type of B, then a + b is the order type of A + B. That $a + b \neq b + a$ in general can be seen by the following example. Let A = {1,2,3.,...,n} and B = {n + 1, n + 2,...}. The order type of A is n, the order type of B is ω, and the order type of A + B is $n + \omega$, where ω is the order-type of the natural numbers. But the order type of B + A is $\omega + n$ which is not ω because B + A = {n+1, n+2, ..., 1,2,...,n} contains a last element (A + B does not). So $\omega + n \neq n + \omega$. Because n and ω are ordinal numbers and, in general, since the above example shows that the sum of two ordinal numbers is not commutative, the arithmetic or ordinal numbers is not isomorphic to the arithmetic of cardinal numbers. Nevertheless, two sets with the same ordinal number necessarily possess the same cardinal number. If A is a well-ordered set, A is a chain. An intuitive example of a chain important to subsequent discussion is as follows: Let A be a well-ordered set. Then A has a first element, say a_0; A - {a_0} has a first element, say a_1; A - {a_0, a_1} has a first element, say a_2; etc., so that A = {a_0, a_1, a_2, a_3, ...}. The notation used here is that the index of every element is the ordinal number of the set of elements preceding it. For a_3, "3" is the ordinal number of {a_0, a_1, a_2}, which is called a <u>segment</u> of A determined by "a_3." In more general terms, each element a of A determines some segment S where $S = \{x \in A: x < a\}$. If $Q = \{x \in A: x \notin P\}$, then $A = S + Q$. Note that $a \notin S$ because $<$ is irreflexive, so a is the first element of Q. A result of this definition is that a well-ordered set is never similar to one of its segments, which leads to the fact that for any two ordinal numbers a and b, either $a < b$, $b < a$, or else $a = b$. In particular, $a < b$ means that A is similar to a segment of B. Of course, if it were possible for B to be similar to one of its segments, then it would be true that $a = b$ as well as $a < b$.

As indicated above, the elements of a set A which is well ordered can be indexed by successive ordinal numbers. If A is a finite set, then $A = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$ and n is the ordinality of A where 0 is the ordinality of the empty set. Because any ordering of a finite set is a well-ordering, it is impossible to distinguish the orderings with reference to the ordinal number of the set; i.e., all orderings give the same ordinal number. Thereby, the ordinal and cardinal numbers of finite sets correspond, and it is possible to find the cardinal number of a set by a process of counting, that is, by indexing the elements of the set A by the ordinal numbers $\{0,1,2,\ldots, n-1\}$ by virtue of successive selection of single elements. (Select some a_0, then some a_1, etc., until the last one a_{n-1} is selected.) Then n is called the cardinal number of the set. This process is often referred to as counting. It is important to recognize that counting has its basis in ordinal number.

The notion of equvalence classes of finite sets is implicit in the above discussion because $\cong$ is an equivalence relation. This observation has led to the definition of an ordinal number as an equivalence class of well-ordered sets (Barnes, 1963, p. 194). The set $\{0,1,2,\ldots,n-1\}$ of cardinality (and ordinality) n can be considered as the standard set of an equivalence class of sets each of cardinality n. It must be explicitly pointed out that the arithmetics of cardinal numbers and ordinal numbers of finite sets are, in fact, isomorphic.

To view a cardinal number as a class of sets should be no more foreign than to view the objects of a finite field formed by the integers modulo a prime as classes of sets. Of course, to tell a five year old child that a number is an equivalence class of sets is absurd. The identification of a number as a set of objects, however, is a natural way to think about cardinal and ordinal number. In the well-known "empty hat" (Van Engen, 1970, pp. 38-39) approach to cardinal number, "0" is defined to the empty set, "1" is defined to be the set containing 0 as an element, etc. More formally, $0 = \emptyset$; $1 = \{0\}$; $2 = \{0,1\}$; $3 = \{0,1,2\}$; $4 = \{0,1,2,3\}$; ...; $n = \{0,1,2,\ldots,n-1\}$. Thus, "4" is the ordinal number of the segment $\{0,1,2,3\}$ and is identified with the segment itself. Because cardinal and ordinal numbers are indistinguishable in this context, it is also the cardinal number of the set.

Concretely, if A is a finite set to be counted, then by successive selection of elements, successive segments of set A are determined and a chain of ordered sets is formed. "One," in the selection of the first element has both cardinal and ordinal characteristics in that "one" tells how many elements have been selected and also that the first one has been selected. A subset of the collection A of one element has also been determined. "Two" in the selection of the next element also has both cardinal and ordinal characteristics in that "two" tells how many elements have been selected and also that the second one has been selected. The segment corresponding to "two" is an ordered set, is a subset of the collection A, and contains the set consisting of the first element. It is ordered by the relation "precedes," which is transitive and asymmetrical (and is thereby a strict partial ordering). If this counting process is continued until A is exhausted, then $A = \{a_1, a_2, \ldots, a_n\}$ has been well-ordered by the relation "precedes." A chain of sets has been established in that if $A_1 = \{a_1\}$, $A_2 = \{a_1, a_2\}$, etc., then $A_1 \subset A_2 \subset \ldots \subset A_n$. In this sense, one can say that one is included in two, two is included in three, etc. If A is counted in a different way, $A = \{a_1^*, a_2^*, a_3^*, \ldots, a_n^*\}$. It must be noted that while a_i^* may not be the same element as a_i, nevertheless a_i^* is the ith element and also the cardinal number of $A_i^* = \{a_1^*, a_2^*, \ldots, a_i^*\}$ where $i < n$. While A_i and A_i^* are similar (and therefore equivalent), they are not necessarily equal sets.

<u>Set similarity as a developmental concept</u>. The concept of set similarity has been shown to be developmental by Piaget (1952, p. 97). He differentiates between qualitative correspondence between two seriations and numerical correspondence between two series. The construction of a single series and that of finding a one-to-one correspondence between two series amounts to the same thing insofar as his behavioral analyses show. Children go through three stages with regard to set similarity--no conception of the possibility of seriation, or similarity; seriation or similarity based on perceptual processes; and then numerical correspondence between two series.

<u>Cardinal and ordinal number as developmental concepts</u>. Piaget's (1952 p. 157) definition of number is close to the concept of a well-ordered finite set. In his study of ordination and cardination, Piaget (1952, Cha. VI) employed three experimental situations, one involving seriation of sticks, one seriation of cards, and one seriation of hurdles and mats. In the seriation of sticks experiment, the child was asked to seriate ten sticks from shortest to longest and then was given nine more sticks and

was asked to insert these into the series already formed (the material was constructed in such a way that no two sticks were of the same length). He was then asked to count the sticks of the series after which sticks not counted (or sticks the child had trouble counting) were removed apparently along with one or two he did not have trouble counting. The experimenter then pointed to some stick remaining and asked how many steps a doll would climb when it reaches that point, how many steps would be behind the doll and how many it would have to climb in order to reach the top of the stairs formed by the sticks. The series was then disarranged and the same questions as before were put to the child who would have to reconstruct the series in order to answer the questions.

There is no question that aspects of ordinal number and cardinal number were involved in the above experiment. Any conclusion drawn with regard to number, however, by necessity is a function of a capability to construct a series of sticks based on the connected asymmetrical relation "longer than" having little to do with ordinal number. To demonstrate the point more concretely, an eight year old child was asked which of a collection of books on a table would be the third one. He answered, "What do you mean, any one could be third!". Piaget's experiment with the staircase, then, was more an experiment concerning similarity between a set of n sticks ordered by "shorter than" and the standard counting set {1,2,...,n} than it was an experiment concerning ordination and cardination. A similar analysis holds for the seriation of the cards experiment. While no analysis of the hurdles and mats experiment is given, suffice it to say that it too involves specific relations.

Piaget's (1952, Cha. 5) experiments with set similarity were also dependent upon particular relations. As such, it may be that the particular relations influenced the outcomes of the experiments. In the mathematical development, the connected, asymmetrical, transitive relation "precedes" is what is important--not "shorter than" for dolls, or "smaller than" for hats. While particular order relations determine order of precedence, precedence is only incidental and not primary in the ordering.

It should be clear that Piaget views a child's conception of number as both cardinal and ordinal. A child can make cardinal judgements and ordinal judgements, but when one is present the other is always possible.

Addition and subtraction of ordinal number. Brainerd's (1976) data not withstanding, there is not enough evidence available to make a decision whether cardinal and ordinal number develop as a unified construct or whether they develop somewhat independently. Piaget's data, of course, lead to the conclusion that they develop as a unified construct. If a child's number concept is as Piaget views it, addition for a child would be best modeled by addition of ordinal numbers. An example of ordinal number addition follows. If $\alpha = 5$ and $\beta = 3$, then $5 + 3$ is the ordinal number of the set $\{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}$, where $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$. To rename $5 + 3$, the child could count "one", "two", "three", "four", "five", "six", "seven", "eight", or could count "six", "seven", "eight", which represents a counting-on of B to A. In both cases, $5 + 3$ is renamed as 8.

Subtraction of ordinal numbers is possible in special cases. If α and β are ordinal numbers and $\alpha < \beta$, α and β determine a unique ordinal number ξ satisfying the equation $\alpha + \xi = \beta$ (Hausdorff, 1962, p. 74). ξ is of type $W(\beta) - W(\alpha)$ where $W(\beta) = \{\text{ordinal number} < \beta\}$. Clearly, if $\alpha<\beta$, $W(\alpha) \subset W(\beta)$. An example is if α is 7 and β is 9 then $W(\alpha) = \{0,1,2,3,4,5,6\}$; $W(\beta) = \{0,2,3,4,5,6,7,8\}$ and $W(\beta) - W(\alpha) = \{7,8\}$. ξ is a remainder in the following sense. If a is an element of a well ordered set P, $S = \{x \in A: x < a\}$ and $Q = \{y \in A: y \geq a\}$, then $P = S + Q$ and S is the <u>segment</u> and Q is the <u>remainder</u> determined by a. Essentially, then, ξ is the ordinal number associated with the remainder of W (β) determined by α. The solution ξ of $\alpha + \xi = \beta$ is denoted by $\beta - \alpha$ for finite α and β.

In the case of the equation $n + \alpha = \beta$ where $\alpha < \beta$, the solution is also represented by $\beta - \alpha$ for finite α and β. However, the solution is arrived at by the following process. $n + (\alpha - 1)$ is the predecessor of β; $n + (\alpha - 2)$ the predecessor of $n + (\alpha - 1)$; and so forth, until n is reached. Concretely, if $x + 5 = 11$ is the equation, one counts back from 11 to reach 6 (the solution) in the following way: "ten", "nine", "eight", "seven", "six"; so since six is the predecessor of $x + 1$, x must be six.

In the case of the equation $5 + x = 11$, the solution is found by counting the remainder, starting with the <u>first element</u> of the remainder and proceeding to the last. It should be clear that one could also start with the last element of the remainder and count backward to the first. In either case, a double counting process is necessary: Ten is one; nine is two; eight is three; seven is four; six is five; so the answer is six. Or; six is one: seven is two; eight is three nine is four; ten is five; eleven is six; so the answer is six. In the case of counting-back, rather than counting predecessors of elements in the remainder, one can count the elements themselves: eleven is one; ten is two; nine is three; eight is four; seven is five; so the answer is six.

The above counting processes associated with addition and subtraction of ordinal numbers are hereafter referred to as "counting on" and "counting-back" strategies. On the assumption that the child's concept of number is basically modelled by ordered finite sets, they are viewed as being central processes when children find sums or differences. Counting-on and counting-back do not necessarily involve tallying as in the above strategies for solving equations. Imagine a situation such as six blocks being in full view of a child and three not visible. If the child knows there are three not in view, he could <u>start</u> with three and count the remaining six on without tallying. Likewise, a task could be designed to tap counting back without it involving talling.

<u>Addition and subtraction problems and counting</u>. Counting is not only involved in addition and subtraction in the sense of counting-on and counting-back, but is involved in other ways for the child. There are three types of counting easily identifiable. The types are rote counting, point counting (or one-to-one correspondence counting), and rational (or mental) counting. The basis in mathematics for rote counting is the set of ordinal numbers $\{0, 1, 2, 3, \ldots n - 1\}$. Behaviorally, rote counting is the recitation of the symbol chain "one," "two," "three," . . . . The basis in mathematics for point counting is the one-to-one correspondence between a collection of n elements and the set of ordinal numbers $\{0, 1, 2, 3 \ldots n - 1\}$ represented by indexing elements: $A = \{a_1, a_2, a_3, \ldots a_n\}$. Behaviorally, successive elements of A are selected until they are exhausted. The basis in mathematics for rational counting is counting-on and counting-back. But it must be understood that counting-on and counting-back must be associated with mental representations of collections such as A immediately above. Behavioral aspects of rational counting are given below. The child who is a rote counter can recite an ordinal number sequence while pointing to a set of objects, but fails to index the elements correctly. That is, a rote counter miscounts a set of objects because of failure to tally for each object of the set once and only once.

The rote-counter would not be expected to solve addition and subtraction problems. Consider this general problem: "Here is a set with s things in it. Here is a set with q things in it. How many things do you have altogether?" When the items of sets S and Q are placed together (assuming $S \cap Q = \emptyset$) the child must determine the count correctly for $S \cup Q$, regardless of the configuration of the items, if # $(S \cup Q)$* is to be correctly determined. The rote counter will have difficulty finding # $(S \cup Q)$ for many configuration of the elements of S and Q. The rote counter has analogous problems with subtraction.

For a point counter, each item in a collection is perceived as an unique element and is tallied once and only once when determining the cardinality of the set. The child at this level of counting is heedful of objects in the set that are likely to be counted more than once. But the successive selection of elements of a collection does not, in the counter's mind, determine a chain of inclusive sets: one is included in two, etc. The count is a labeling process.

When presented an addition problem, the point counter solves it using a <u>counting all strategy</u>. Given two collections of things defined in the problem (S and Q), the child counts S, counts Q, then counts $P = S \cup Q$. This involves counting all of the objects to count P. When subtracting the objects of the set S from the set P, the child counts out elements of P, counts out elements of S, then counts all of the elements of P-S to determine # (P-S) = #Q = p-s. At this level of counting the basic addition and subtraction facts would not be available to th[illegible] unless explicit teaching had taken place.

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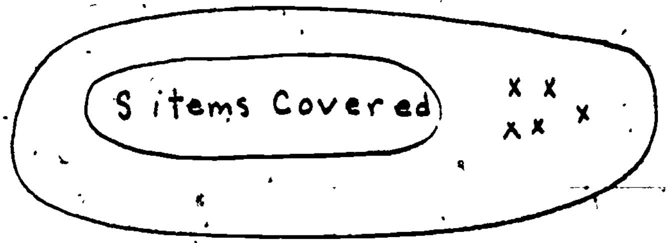
*# $(S \cup Q)$ means the number of $S \cup Q$.

Rational counting is stratified into several levels according to the complexity and scope of the problem solving behavior exhibited by the child. The levels, labeled as R-1, R-2, R-3 and R-4, are delineated below.

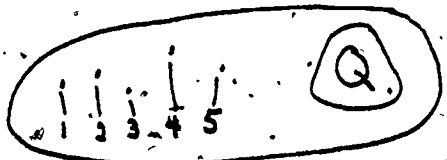
The first level of rational counting, R-1, is counting-on. The child point-counts correctly, but also can determine correctly the cardinality of the set $P=S \cup Q$ when S is covered or otherwise not subject to a point-count.



A collection of p objects with s objects covered.

The child holds s as both the cardinality of the covered subset S, and at the same time, mentally recovers the ordinal property of the s such that through mental awareness of s, he recalls the existence of the successors of s; $s + 1, s + 2, . . ., s + q = p$. The child at the (R-1) level can extract the ordinal sequence $q + 1, q + 2, . . ., n$ from his internalized simple verbal chain (1, 2, . . ., q, q + 1, . . ., n), for each integer q of that chain, when $q < n$.

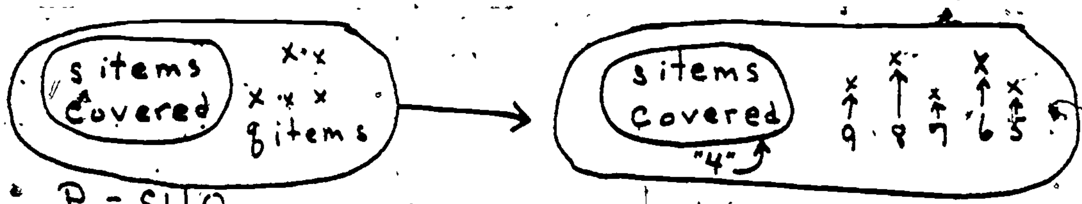
The child at Level R-2 can count-on in the manner described in R-1, as well as solve problems in the class presented below. The child can count S, "one," "two," "three," "four," "five," and then considering $Q \subset P$, and



#P is known: say 9
#S is known: 5.
Problem: find Q.

S ∪ Q = P, count on from five: "six," "seven," "eight," "nine" while at the same time tallying the count of Q.

The next two levels of rational counting are determined by counting back. It is possible to conceive of point counting back as well as point counting forward from "one" as described above. A child may start at say, "ten," and count back to "one" verbally or by labeling a collection "ten," . . ., "one," the latter being point counting back. As a precursor to rational counting back, the child can use a sophisticated point count back to solve a missing addend problem. Let # P = 9, Q visible,



#Q = 5, and S covered. The child can count back assigning the numbers "nine," "eight," "seven," "six," and "five," by a point-count, to the five visible objects in Q. The child can then construct symbolically the set S with tallies assigned by a continued count backward of the sequence "four," "three," "two," and "one." He then counts the tallies and determines #S.

The student at level R-3 can extract mentally, from the ordinal sequence p-1, p-2, . . ., c, . . ., 1, the fact that c in the sequence conveys the cardinality of the set of units available for the continued count-back. The fact is used to solve the missing addend problem by eliminating the need for a tally of the unknown number of objects when counting-back, and inherently, the point-count of the tally. When given a set P and a set Q ⊂ P, the student can find #S when the set is covered using only a counting-back procedure.

In the problem diagrammed above, let p = 9 and q = 5. The child at level R-3 point-counts the five items of the uncovered set Q using the "backward verbal chain" "9, 8, 7, 6, 5". At this point, the student understands that to continue the count "4, 3, 2, 1" is the reverse of the count "1, 2, 3, 4". Hence the #S is determined without additional counting.

The level R-4 can be used to solve problems where P = S ∪ Q, #P is known, #Q is known, but S and Q are both covered.

p known, q known, s unknown.

Again, for concreteness, let p = 9, q = 4 in the above diagram. The child at level R-4 can determine a tally of q = 4 while counting-back the predetermined number of tallies.

The tally is to q = 4: 1 1 1 1 (1 2 3 4)

The count-back is "9, 8, 7, 6" and is corresponded to the tally. The #S = s is then determined by the student's awareness that the end of the count-back produces s = 5.

The types of counting are summarized below.

Rote counting -- recitation of a simple verbal chain but with incorrect point counting

Point-counting -- correct use of one-to-one correspondence counting.

Rational counting --

R1 - rational counting-on

R2 - rational counting-on with tally

R3 - rational counting-back

R4 - rational counting-back with tally

The counting all strategy can be used to solve addition and subtraction problems as well as exercises such as 3 + 2 or 5 - 2. Children also use the counting-on strategy to solve the problem modeled by s + q = □. Rational counting-on, R-1, is used, since the child considers either one of the numbers as a starting point and the other number to represent a set of units in the verbal chain. For example, to solve 9 + 3 = □, the student might select <u>nine</u> as his starting point in his verbal chain and count on three units more in the chain: "ten," "eleven," "twelve." There is no need to count through the forward verbal sequence to the number nine since the child mentally extracts the cardinal property of "nineness." At the same time the child initiates a three unit count from the number <u>nine</u> in the verbal (mental) chain. The child does not need to keep a tally in the problem if he makes marks over "3" to begin with. Otherwise, a tally is needed.

The missing addend problem is solved with a rational counting-on strategy that utilizes a tally. The level of rational counting used is R-2. When given the missing addend problem, the missing addend is perceived as part of the total sum. Given the problem 3 + □ = 11, the student counts-on from three to eleven and symbolizes the units of the missing addend with a tally as he counts.

The child solves a problem like 9 - 5 = □, by starting at nine to count the units in the backward ordinal sequence. He counts back five units to the number five and mentally extracts the next number in the backward-ordinal sequence and names it as the solution to the problem. In this problem situation, the student is asked to solve the problem by counting back.

<u>The Achievement Tasks</u>

<u>Ordinality and cardinality tasks</u>. In Piaget's study of cardinal and

ordinal number, he was concerned with two basic problems. First, the child had to determine a cardinal number given an ordinal number and, second, the child had to determine an ordinal number given a cardinal number (Piaget, 1952, p. 149). These experiments have been discussed earlier in the section on Cardinal and Ordinal Number as Developmental Concepts. It was pointed out in that section that Piaget's experiments were more an experiment concerning similarity between a set of n sticks ordered by "shorter than" and the counting set {1, 2, 3..., n} than they were experiments concerning the two problems posed. Different tests for the coordination of cardination and ordination need to be constructed to eliminate that criticism of Piaget's work, while emphasizing the relation "precedes."

The following terminology is adopted for description of the testing formats. "P" denotes a finite well ordered set ordered by the relation "precedes". "S" is a segment of P determined by any element p of P and Q is the remainder. The minimum element of Q is denoted by q.

Two tasks were designed (see Appendix A.1). In the first task (A) a row of 12 counters was presented to the child. The first two questions of the task determine whether children can start with any counter, given an orientation of first and second, and determine a sucessor and a predecessor. The third question determines whether a child can start with an element q + 2 of Q and determine the cardinality of S, the segment determined by q, the minimum element of Q. The third question also determines whether the child can determine the cardinality of P given the cardinality of S. However, various response sequences are possible. For example, if a child was not able to determine the cardinality of S given the ordinal number q + 2, he was asked to determine the cardinality of P given q + 2. If successful in finding the cardinality of P, he was then asked again to find the cardinality of S.

The second task (B) was presented with S determined by the three objects covered. The basic intent of the task was to determine the cardinality of P given an element of Q (in this case q + 1). The first question was designed to orient the child to the direction of the relation "precedes". Again, different response sequences were possible. But whether the child found the cardinality of P or not, he was asked to find the cardinality of Q.

Logically, class inclusion and quantitative comparisons should be readiness variables for a child's acquisition of the ability to perform the two tasks above. Class inclusion is logically involved because in order to find the cardinal number of P given that S is covered and Q visible, the child needs to know that the objects of S, even though they are covered, make up part of the objects of P (or all of the objects together), and that there are more in P than in either S or Q. If S and Q are not united in the mind of the child, but remain separated, the most likely answer for the total number of objects is the number of visible objects, or the number in Q because there would be no conception of P. Class inclusion is also logically involved in finding the number in S given q + 2. The child seemingly would have to realize that the objects of S, even though they are covered, are part of the objects of P.

Quantitative comparisons should be a readiness variable for the two tasks because ordinal numbers and cardinal numbers satisfy the conditions for a quantity from the scientific point of view. As noted in the contrast of the Piagetian's notion of quantity and the scientific notion of quantity, it would seem that a child capable of extensive quantity in the Piagetian sense should be capable of comprehending extensive and intensive quantity in some cases in the scientific sense. Cardinal and ordinal number should fit those cases, because a singular object can be considered as a unit.

From the two tasks on cardinal and cardinal number, the following variables were identified.*

1. Successor of an element (task A, question 1a or c; task B, question 1a or b). Range: {0,1,2}

2. Predecessor of an element (task A, question 2a). Range: {0,1}

3. Number in S (#S) (task A, question 3a; taskB, question 3a). Range: {0,1,2

4. Number in P (#P) (task A, question 3b; task B, question 2a). Range: {0,1,2}

5. Number in S + Number in P (task A, questions 3a or d or f or h; task B, questions 3a or b; task A, questions 3b or c or g; task B, questions 2a or b or c). Range: {0,1,2,3,4 }

The sucessor and predecessor variables should be clear. However, the others need discussed.

1. Number in S. In task A, question 3a was the first time the child was asked to find the number in S. Likewise, in task B, question 3a was the first time the child was asked to find the number in S. In task B, there was a distinct possibility the child would use the information that there were eight in set P and five showing, and do a subtraction problem. Nevertheless, he would obtain the number in S from ordinal information, but not necessarily by counting back. He could count on to find the number in P, then use a subtraction fact.

2. Number in P. In task A, question 3b, the child could possibly use the information that there were seven in set S (question 3a) and then use addition. Again, however, he would use ordinal information to find the number in P. Task B, question 2a, was the first time the child was asked for the number in P in that task.

3. Number in S + Number in P. This variable was identified because it was felt that some indication had to be given for cases where the children found the number in S or the number in P after prompting. Given that the child missed one or more questions of the Number in S or of the Number in P questions, the experimenter attempted to give the children additional information which could help them find the Number in S or P. If a child was successful after prompting, it could be said that he had the notion of the interrelationship between ordinal and cardinal number, but did not initiate the counting process on his own. There is the possibility that the child appeared to find the number in S or the number in P after prompting, but did so as a result of rote counting. This possibility dictated that the #S + #P variable be differentiated from either of the

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*Domain of each variable is contained in parentheses and the range is stated in set notation.

#S or #P variable. The range of #S + #P is {0, 1, 2, 3, 4} due to the questioning sequences, but was not merely the sum of #S and #P variables. A child could score 0 on either of the latter two variables, but 4 on #S + #P variable.

Counting back, just before, just after, and between tasks. Counting has been described mathematically in the section on ordinal number. Because the cardinal and ordinal numbers of a finite set correspond, it is possible mathematically to index the elements of a set A by the ordinal numbers {0,1,2, ..., n - 1} by virture of successive selection of single elements (select some a_0, then some a_1, etc., until the last one a_{n-1} is selected). Then n is the ordinal (and cardinal) number of the set. Of course, counting begins with 1, so the above set is replaced by {1,2,3,...,n}, where n is associated with the last element selected and with the entire set rather than being associated with only the set A.

The tasks for Counting Back, Just Before, Just After, and Between are presented in Appendix A.2. From the tasks, the following variables were identified.

1. Counting Back. If a child could not count three discs and count back from 3, no further questions were asked. If a child successfully completed the task with discs, he was asked to do it with 8. If successful, he was asked to do it with 12. If successful, he was asked to count back from 15 without objects and without counting up to 15. If a child made errors with 8 discs, he was asked to do it with 4, but the variable was scored 0. If he made an error with 12, the variable was scored 1 and the task was terminated. If he made an error with 15, the variable was scored 2 and the task terminated. If he did each task correctly, the task was scored 3. Range: {0,1,2,3}

2. Just Before. In task B, the child had two chances to score 1 on the task. If not correct on the preliminary task, he was told the correct answer. Range: {0,1}.

3. Just After. In task C, the child had two chances to score 1 on the task. If not correct on the preliminary task, he was told the correct answer. Range: {0,1}.

4. Between. The child received 1 point for each number he found between two others. In case he could not find a number between 1 and 3, he was told the correct answer. Range: {0,1,2,3,4}.

It was expected that the variable Counting Back would not be related to class inclusion nor to quantitative comparisons. The reason for this expectation was that one-to-one correspondence counting or rote counting would be sufficient for completion of the tasks. The variables Just Before and Just After are in the same category--rote counting would be sufficient for correct solution. The variable Between, however, was expected to have quantitative comparisons as a readiness variable. The reason for the expectation is that for a child to realize that 10 is between 8 and 12, he had to realize that 10 is a successor of 8 as well as a predecessor of 12. Consequently, reversibility at Piaget's first level (R") would seem to be involved. This being the case, quantitative comparisons would be a readiness variable because reversibility and quantitative comparisons are manifestations of the same general scheme.

It is not as clear that class inclusion is a readiness variable for Between. One could make the argument that class inclusion is a readiness variable because, in a nested set sense, 8 is included in 10, which is in turn, included in 12. So, 8 is less than 10 and 10 is less than 12. Likewise for 9 and 11. Again, however, class inclusion may not be a readiness variable because ordering processes may be sufficient to answer the question--9 is after 8 and before 12, so it is between 8 and 12.

Verbal problems. Rational counting is not necessary to be able to solve verbally presented addition and subtraction problems modeled by the sentences $a + b = n$, and $a - b = n$ where $b \leq a$. One-to-one correspondence counting and a "counting all" strategies are sufficient. If the problem, "Mary has five marbles. Nancy has two. How many marbles do both girls have?" is presented to a six-year old, the child could count out five marbles from a marble source, two more, then count the selected marbles, starting from one and proceeding through seven. Of course, such a strategy does not incorporate all of the information available to the child, as both subcollections were counted twice. A counting-on strategy would be more efficient, but not necessary, in the solution.

In the case of a subtraction problem, say, "John has seven peppermint sticks. He eats three. How many does he have left?" a child could count seven objects from an object source, count three of the seven, and then count the ones remaining. This solution process is merely the reverse of the "count all" process described above. Consequently, it is referred to as a "count all" process for subtraction. The more sophisticated process of counting back modeled by subtraction of ordinal numbers would entail the child starting at seven and counting back three in either one of the two ways identified in the section Addition and Subtraction of Ordinal Numbers.

The missing addend problem is modelled by the sentence $a + n = b$. An example is "Joe has five pennies. His father gives him some more. Now he has nine. How many did Joe's father give him?" Of course, two solution strategies are possible, one involving rational counting and one involving a counting-all strategy for subtraction.

It would seem to be the case that gross quantitative comparers would not be capable of solving the missing addend problem through rational counting procedures. However, extensive quantitative comparers would have a greater incidence of solution through counting-on than gross quantitative comparers, but would not always be capable of initiating such solution procedures because rational counting level R-1 and strategies used to solve the missing addend problem are not isomorphic. The statement of the missing addend problem would undoubtedly militate against a counting-all subtractive solution.

Many children who are capable of only gross quantitative comparisons may have mastered counting-all procedures for addition and subtraction. Consequently, quantitative comparisons would not be as strong a readiness variable for addition and subtraction problems as for missing addend problems. Quantitative comparison should be, however, a strong readiness variable for rational counting solutions to addition problems and counting back solutions to subtraction problems. That is not to say, however, that quantitative comparison is not a readiness variable for learning to solve addition and subtraction problems. In the section, Quantitative Comparisons and Arithmetic, statistical differences were observed between performance on addition and subtraction problems between the group of gross and the group of extensive quantitative comparers.

Presence or absence of manipulatable objects during problem solution has been convincingly shown to be a significant variable (Steffe, 1966; LeBlanc, 1968; Steffe and Johnson, 1971). The reason the variable is significant may be because objects facilitate one-to-one correspondence counting. Two different sets of six problems per set were constructed. In the case of verbal problems with objects, the objects were always present where the objects used were objects named in the problem statement. Each problem was read to the child in total before he started to solve it. If necessary, the problem was reread to the child. The child was told he could use the objects to help him find the answer and was urged to use them if he made mistakes. The problems are presented in Appendix A.3.

In the verbal problems with no objects present during solution, the child was asked to write the number sentence for the problem. Records were made of whether the child considered the sentence as open or closed and whether he processed the information before or after he wrote the sentence. The problems are presented in Appendix A.4.

It would seem that class inclusion would be a readiness variable, especially for the missing addend problems and the subtraction problems. In cases where children solve the problems by a counting-all procedure, no relationship would be expected. But where children are faced with the necessity of counting on to solve a problem, one would expect relationships with class inclusion and processes used on those problems. The children were urged to read each problem if they could. If not, the experimenter read the problems to the children. The sentence for the first problem, if not written by the child upon request, was written by the experimenter. However, every means possible was used to urge the child to write the sentence. The variables are identified as follows.

Problems with objects present.

1. Addition. Range: {0,1,2}
2. Subtraction. Range: {0,1,2}
3. Missing Addend. Range: {0,1,2}

Problems with objects absent.

1. Addition. Range: {0,1,2}
2. Subtraction. Range: {0,1,2}
3. Missing Addend. Range: {0,1,2}

Set partition tasks. In the section, Quantity and set partitions, arguments were advanced that quantitative comparisons and class inclusion would be readiness variables for set partitions. Four tasks were constructed to test the ability of a child to form set partitions. The tasks were constructed to test the invariance of the number of objects in a collection regardless of how they are partitioned. They are presented in Appendix A.5.

In the first two tasks, the child counted to establish equivalence and in the last two tasks he was told they were equivalent--in the third a particular number was involved (100) and in the fourth only a relation. The intent of the tasks was to establish that the two collections had the same number in each, and then partition them into a different number of subcollections. If the child focused on the number of subcollections he would respond to the item incorrectly. He had to disregard the number of subcollections and judge them on the basis of the information before the partitioning. Two variables were identified.

1. Set Partitions with counting. Range: {0,1,3}
2. Set Partitions without counting. Range: {0,1,2}

Addition and subtraction of ordinal number tasks. It has been argued that the objects Piaget calls finite numbers (cardinal and ordinal) develop in the child as a synthesis of Grouping I and V. After presentation of cardinal and ordinal number in mathematics, it became clear that Piaget's notion of number is quite well modeled by a well-ordered finite set. But Piaget never extended his developmental theory beyond the objects he called finite numbers. Consequently, one cannot know the extent to which operations on cardinal and ordinal numbers are developmental. Obviously, without a conception of number, it would not be possible for operations to be there. But if the concept of number is present in the child's mind, does that also imply the operation? Or, are the operations a later acquisition, perhaps dependent upon school instruction?

Addition and subtraction of ordinal numbers would seem to have quantitative comparisons as a readiness variable as well as class inclusion. What is meant by addition and subtraction of ordinal numbers is the counting-on strategies and counting back strategies with tallying discussed in the section on the mathematics of addition and subtraction or ordinal numbers. The tasks are presented in Appendix A.6.

The tasks in the addition of ordinal number test are graduated in complexity. The first three tasks are warm up tasks and are presented verbally. Errors are freely corrected. Tasks 4 and 5 are counting-on tasks and are a test of ordinal number addition items modeled by the equation $\alpha + \beta = \xi$, where ξ is unknown. Tasks 6 and 7 are ordinal number addition tasks modeled by the equation $\alpha + \xi = \beta$, where ξ is unknown.

The first three tasks of the subtraction of ordinal number test are warm-up tasks presented verbally to the children where corrections of the children's errors were made freely. Tasks 4 and 6 are counting-back without tallying. Tasks 5 and 7 are counting-back tasks with tallying. All tasks are modeled by the equation $\xi + \alpha = \beta$, where ξ is unknown (or equivalently, by $\xi = \beta - \alpha$). Four variables were identified.

1. Rational Counting-on (Tasks 4 and 5, Addition of Ordinal Numbers): Range {0,1,2}
2. Addition of Ordinal Numbers (Tasks 6 and 7, Addition of Ordinal Numbers): Range {0,1,2}
3. Rational Counting-back (Tasks 4 and 6, Subtraction of Ordinal Numbers): Range {0,1,2}
4. Subtraction of Ordinal Numbers (Tasks 5 and 7, Subtraction of Ordinal Numbers): Range {0, 1, 2}.

<u>Mental arithmetic problems</u>. Tasks have been presented which could be legitimately called mental arithmetic. But none of the tasks were such that the problem was presented in a written numerals format where the child was discouraged from using his fingers or tally marks as aids. For a task to be a test of mental arithmetic, the child must not use physical, pictorial, or bodily aids in solving the problem. The task could be presented in more than one stimulus mode. The one selected here is written numbers. Whether quantitative comparisons or class inclusion are readiness variables for mental arithmetic is uncertain. The mental arithmetic test is presented in Appendix A.7.

Two scores were obtained for each problem, a product score (answer score) and a time score (the number of seconds it took to start one problem and then finish by writing the sum or difference obtained). The variables were simply addition and subtraction time and product scores.

Nested classification tasks. In the section Classification; a distinction was made between the intension of a class and the extension of a class. The intension of a class was identified as the properties which are common to the elements of a class and the extension the members of the class. Coordination of the intension and extension was identified as what develops in stages in children. In addition to a class inclusion test, it was decided that it was necessary to include a test which would give the child an opportunity to demonstrate, within a particular hierarchy of classes, the ability to:

1. Identify properties specific to members of a particular class which distinguishes the class from other classes;

2. Identify properties specific to members of a given class and to other classes to which it belongs;

3. Identify properties which may be specific to one or more members of a given class which distinguish them from other members.

The nested classification tasks were designed to tap Stage II classification behavior identified by Inhelder and Piaget (1969) as a minimal capability. An indirect technique was used. The two tasks were designed using the same material set, the first task including one instance of the inclusion relation and the second two instances.

The material set consisted of seven pieces of polygonal shaped objects, three round objects that were not buttons, four round nonwhite buttons, and 15 round white buttons. The polygonal shaped objects included three triangular shapes, two square shapes, and two parallelogram shapes. The round objects included an orange felt, a black checker, and a red disc. The nonwhite buttons included two blue, a yellow, and an orange. In each task, the child was asked to sort the items in particular ways. In order to ensure the child recognized the properties of the classes, specific items were given to the child to classify. After the child classified these items (with help, if necessary), the child was shown a box which contained an object. The child had to agree that something was in the box. The experimenter then placed it into its respective place,

The two tasks are presented in Appendix A.8.

Two basic questions were asked, one a question of possibility and one a question of fact. In task A, the first question was asked to determine if a child could differentiate properties specific to members of a given class from properties specific to members of another specific class. Hereafter, such properties are called type 1--the first type of properties listed in this section. Type 2 and type 3 properties are those properties in (2) and (3) of that same list. The first three questions were to test identification of type 1 properties--or relevant attributes. The fifth and sixth questions were of type 3. Questions 5a, b, and e were of type 1.

In task B, the first three and the sixth questions asked about type 1 properties. Questions 1a, b; 2a; 3a; 4a; and 5a were also of type 1 properties. Some of these questions contained aspects of type 2 properties (for example, question 2 tested a type 2 property). Question 6 was a type 1 question. Questions 4 and 5 tested type 3 properties.

Neither of task A nor task B explicitely asked the child to solve the class inclusion problem. A child could conceivably answer every question correctly and still not realize that the classes of objects were nested. This was by design. It would seem that neither class inclusion nor quantitative comparison would be readiness variables for the nested classification tasks.

A supplement to the nested classification tasks was included after the presentation of task B. This supplement was designed to test class inclusion within the context of the nested classes.

Two variables were identified based on the number of collections.

1. Nested classification A. Range: {0,1,2,3,4,5,6}
2. Nested classification B. Range: {0,1,2,3,4,5}

Loop inclusion tasks. The loop inclusion tasks were designed to test the capability of a child to view regions as being nested or intersecting. Johnson (1975) found that one reason children failed to solve the class inclusion problem was that they viewed nested regions defined by two boundaries as separating the regions into two separate regions. Moreover, as Piaget and Inhelder (1963) claim that the concept "inside" develops early on in childhood (as early as four years of age), little difficulty should be present for a child to comprehend the concept "inside" as it pertains to a single loop. Difficulties are introduced when two or more loops intersect or when they are nested. "Inside" was defined operationally in the present study by placing a stick vertically in a loop and showing that the loop could not be pulled through the stick. Quantity was not expected to be a readiness variable for loop inclusion tasks. These tasks are presented in Appendix A-9.

The Variables

The achievement variables used in the present study were partitioned for the purpose of data analysis. The partitioning is based on logical grounds. Investigation of the achievement variables per se is contained in the correlational study following the readiness study.

Cluster 1. The first cluster of variables was composed of some of the variables identified in the description of the ordinality and cardinality tasks.

1. Number in S (#S). Range: {0,1,2}
2. Number in P (#P). Range: {0,1,2}
3. Number in S + Number in P (#S + #P). Range: {0,1,2,3,4}

Cluster 2. The second cluster of variables was formed by the remaining variables identified in the ordinality and cardinality tasks and in the counting back, just before, just after, and between tasks.

1. Counting Back. Range: {0,1,2,3}
2. Just Before. Range: {0,1}
3. Just After. Range: {0,1}
4. Between. Range: {0,1,2,3,4}
5. Successor. Range: {0,1,2}
6. Predecessor. Range: {0,1}

Cluster 3. The third cluster of variables was formed from the verbal problems to be solved using no objects.

1. Addition. Range: {0,1,2}
2. Subtraction. Range: {0,1,2}
3. Missing Addend. Range: {0,1,2}

Cluster 4. The fourth cluster of variables was formed from the verbal problems to be solved with objects and the set partitions test.

1. Addition. Range: {0,1,2}
2. Subtraction. Range: {0,1,2}
3. Missing Addend. Range: {0,1,2}
4. Set partitions With Counting. Range: {0,1,2}
5. Set partitions Without Counting. Range: {0,1,2}.

Cluster 5. This cluster of variables was formed from the addition and subtraction of ordinal numbers tasks.

1. Rational Counting On. Range: {0,1,2}
2. Addition of Ordinal Numbers. Range: {0,1,2}
3. Rational Counting Back. Range: {0,1,2}
4. Subtraction of Ordinal Numbers: S and Q covered: Range: {0,1,2}

Cluster 6. This cluster of variables was formed from the mental arithmetic problems.

1. Addition Product Score. Range: {0,1,2}
2. Subtraction Product Score. Range: {0,1,2}
3. Addition Time Score. Range: {0,1,...,n}
4. Subtraction Time Score. Range: {1,2,...,n}

Cluster 7. This cluster of variables was formed from the nested classification tasks, the loop inclusion tasks, and a post administration of the class inclusion test.

1. Class Inclusion. Range: {0,1,2,3,4,5}
2. Loop Inclusion. Range: {0,1,2,3}
3. Nested Classification A. Range: {0,1,2,3,4,5,6}
4. Nested Classification B. Range: {0,1,2,3,4,5,}

CHAPTER II

Presentation of the Design

The Design of the Readiness Study

Sample

The first grade children in Huntington Street Elementary School and Roberston Lane Elementary school City, Southeast, were used as the initial population. All of these children were administered the SMSG first grade test selected by PMDC staff in September of 1974. The two scales used in the selection were SMSG Scale 204, Counting Members of a Given Set, and SMSG Scale 205, Equivalent Sets. Only those children for whom evidence was present that they could count to at least seven were included in the population.

Two readiness tests, quantitative comparisons and class inclusion, were administered individually to all of the children in the population. Children were judged to be either gross quantitative comparers, extensive quantitative comparers, or indeterminate. Children for whom evidence was present that they could not solve the class inclusion problem were then selected. The children were then randomly ordered within each group of gross and extensive quantitative comparers within each school. The first six of each of the two quantitative comparison groups were assigned to the experimental group and the second six to the control group, as diagrammed.

School	Huntington		Roberston	
Quantity \ Treatment	Experimental	Control	Experimental	Control
Extensive	6	6	6	6
Gross	6	6	6	6

During the course of the experiment two control children (one extensive and one gross comparer) moved from the district and were subsequently replaced by two extensive quantitative comparers.

In summary, the characteristics of the 48 children in the sample were as follows:

1. Each child could one-to-one correspondence count to at least seven
2. No child could solve the class inclusion problem.
3. Twenty four of the children were extensive quantitative comparers and 24 were gross quantitative comparers.

Tests

Description of criteria on the readiness tests. In the case of quantitative comparisons, evidence was considered strong if a child answered correctly at least five of eight questions with justification. A child was judged to be a gross quantitative comparer if judgements were made on the basis of perceptual cues and a majority of the answers were not correct. An "inconclusive" category was also present when clear judgements could not be made.

A criterion for the class inclusion test was not at issue because 88 of the 107 children scored the possible score of zero. Of the remaining children, seven scored one.

Administration of the achievement tasks. The test involved in Cluster 4 variables was administered individually to the 48 children in the sample as pretests during the first two weeks in October 1974. During February 1975, the tests in all of Clusters 1-7 and the quantitative comparisons test were administered.

During February, each child was interviewed in three different sittings of no more than 30 minutes per sitting. Each interview was audio-video recorded as well as hand recorded by the interviewer. During the first sitting, the class inclusion test, the loop inclusion test, and the nested classification test were administered, in that order. During the second sitting, counting back, just before, just after, and between tasks; task A of the ordinality-cardinality test, task B of the ordinality-cardinality test; quantitative comparisons test; verbal problems with objects; and the set partition test were individually administered, in that order. During the third sitting, the mental arithmetic test, the verbal problems without objects; the ordinal number addition test; the ordinal number subtraction test; and a test called the formalization test (which has not been described) were administered, in that order. The testers for the experimental group were Mr. Charles Lamb and Mr. James Hirstein. The testers for the control group were Mr. Curtis Spikes and Mr. Leslie Steffe.

Data sources. Each audio-video tape was viewed and all data were extracted from the tapes. The data were coded on record sheets which are presented in Appendix B. The record sheets are presented in the order that the tasks were administered to the children.

The first record sheet presented Appendix B is for class inclusion. The column on the left provides opportunity to code whether the children pointed out each subset and the containing set. The two possible answers are presented in the middle column for each task with the correct choice in all capital letters. The last column provides the opportunity to code the child's response to the "Why?" question. For an item to be correct, the child had to respond correctly to each of the two questions asked. The rationale was included as supporting evidence for the presence of class inclusion when available. Verbal justification was not necessary for an item to be scored correctly. Each item scored correctly on the second administration was given a score of 1. A score of 0 was assigned any other response pattern for an item. In case of the readiness test, strong evidence would be available for the presence of class inclusion if four out of five items were answered correctly with justification on at least one of those four items. Strong evidence would be available for the absence of class inclusion if a score of 0 or 1 was obtained. Anything else would be indeterminate.

The second report sheet presented is for loop inclusion. Task A included three directions to the child. In order to be given credit for doing task A correctly, a child had to place the stick correctly for all three directions. In such case, a 1 was assigned. In any other case, a 0 was assigned. The third direction was critical as the child had to realize that it is not possible to place a stick inside the green but not inside the red. Likewise, the child had to place the stick correctly for each of the three directions in task B to be awarded a score of 1. If not, a score of 0 was awarded. In task C, the child had to place the stick correctly for each of the two directions to be awarded a score of 1. Otherwise a score of 0 was awarded.

The third record sheet is a flow chart depicting the possible response paths a child could follow through the questions for nested classification Task A. There are five rows (corresponding to the boxes), one for each item. The insert in the upper right indicates whether the child classified the items in the warm-up task independently or whether help was needed. The solid lines represent the correct response path. On any item, (each box in the left column represents an item) a score of 1 was awarded if a child's response followed the solid line response path. A 0 was awarded otherwise.

The fourth record sheet is a flow chart depicting the possible response paths a child could follow through the questions for nested classification task B. There are six rows (corresponding to the boxes), one for each item. The insert in the upper right hand corner indicates whether the child classified the items in the warm-up tasks independently or whether help was needed. The insert in the lower right hand corner indicates the responses on the nested classification supplement. Again, the solid lines represent the correct response path. On any item, a score of 1 was awarded if the child's response followed the solid line. A 0 was awarded otherwise.

The fifth record sheet presented consists of flow charts depicting the possible response paths for the counting back test and just before and just after tests. P_8; P_4; and P_{12} indicate action sequences where the child counts back from 8, 4, and 12. A score of 1 was awarded if a child correctly performed P_8 but not P_4. A score of 2 was awarded if a child correctly performed P_8 and P_{12} but could not count back from 15. A score of three was awarded if all three were done. A score of 0 was awarded in any other case. B_{14}; B_{11}; A_{14}; and A_{11} indicate the responses given to questions concerning what number comes just before (or just after) 14 or 11. A score of 1 was awarded to the "just before" problem if either B_{14} or B_{11} were correct. A score or 0 was awarded in case each was incorrect. Likewise, a score of 1 or 0 was awarded to the "just after" question.

The sixth record sheet presented consists of a flow chart depicting possible response paths for the between tasks. A score of 1 was awarded for each correct response.

The seventh record sheet presented consists of a flow chart depicting possible response paths a child could follow through the questions for the cardinal and ordinal number Task A. The first box ($p \Rightarrow p + 1$) represents the child's response to question 1 asking for the successor of 9. If immediately correct, a score of 1 was awarded for Successor. If the child counted from the beginning or didn't know, a response was ascertained (box 1c) for the successor of 10 ($p \wedge p + 1 \Rightarrow p + 2$). In the case a child was immediately correct, answering "10" a score of 1 was awarded for Successor. In any other case, a score of 0 was awarded Successor. Box 2 ($p \Rightarrow p - 2$) represents the child's response to question 2 asking the child to name the seventh counter given the position of the ninth. If immediately correct, a score of 1 was awarded to Predecessor. In any other case, a score of 0 was awarded to Predecessor. Box 3 ($q + 2 \Rightarrow \#S$) represents the child's response to question 3 asking the child to find the cardinality of S given the position of $q + 2 = 10$. If correct, a justification was asked for (How do you know?). Box 3b ($q + 2 \Rightarrow \#P$) represents the child's response to question 3b in Task A asking the child to find the cardinality of P. Regardless of the response ($q + 2 \Rightarrow \#P$), the task was terminated. If the response ($q + 2 \Rightarrow \#S$) for question 3 was incorrect, various other question sequences were possible. Box 3c ($q + 2 \Rightarrow \#P$) represents the child's response to question 3c of Task A asking for the cardinality of P given $q + 2$. If correct, question 3 was repeated [Box 3d ($q + 2 \Rightarrow \#S$)]. In case of any response to question 3d, Task A was terminated. If the response in box 3c ($q + 2$

$\Rightarrow$ #P) is incorrect, various response paths were possible and should be self-evident. The #S variable was scored from the response in box 3. The #P variable was scored from the response in either box 3b or 3c. The #S + #P variable was scored from the response in either of box 3, box 3d, or box (3f or 3h); and from the response in either of box 3b, box 3c, or box 3g. It should be clear that responses from box 3d, or box 3f or 3h; and box 3g were facilitated by the experimenter.

The eighth record sheet presented is a flow chart depicting possible response paths a child could follow through the questions for the cardinal and ordinal number task B. The first box ($q + 1 \Rightarrow q + 2$) represents the child's response to question 1 asking for the successor of 5. If the response was correct, a 1 was awarded for Successor. If incorrect, question 1b was asked. If the response was correct a 1 was awarded for Successor. In any other case, a 0 was awarded for Successor. Box 2 ($q + 1 \Rightarrow$ #P) represents the child's response to question 2 asking the child for the cardinality of P (the total number of objects) given the position of $q + 1$. If the child's response was correct, he was sequenced through box 2a. If the child's response was incorrect, he was sequenced through the appropriate sequences; box 2b or box 2c. The #S variable was scored from the response in Box 3. The #P variable was scored from the response in Box 2. #S + #P variable was scored from the responses in Box 2, Box 2b, Box 2c, Box 3, or Boxes 3b.

The ninth record sheet presented is for the quantitative comparison test and the verbal problems with objects test. In order to be given credit for an item on the quantitative comparison test, a child had to answer the relational question correctly and have a response basis which indicated something other than a solution based on perceptual features for one or more of the eight items. One point was given for each correct item. The verbal problems were scored on a right-wrong basis for the product score. Whether the child used objects and any observable processes were recorded.

The tenth record sheet is for the partitions test. An item was scored as correct if a correct response was given to the relational question and a justification.

The eleventh record sheet is for recording responses to the mental arithmetic test and the verbal problems without objects test. Of the data recorded on the verbal problems without objects record sheet, only the column "answer" was used in the readiness analysis. The columns headed by E, S, E + S denote when the problem was read by the experimenter, the child or both. Other columns are self-explanatory. The items on both tests were scored on a 0-1 basis, 0 incorrect; 1 correct, in the case of the answer.

The twelfth record sheet is for recording the children's responses to the tests of addition and subtraction of cardinal and ordinal number. Of the data recorded, only the columns "answers" were considered for analysis in the readiness study. Some comments are necessary to interpret the other coding schemes. On the addition test, records were made of (1) whether the basis of the response was an immediately given fact, whether the child counted on, or counted all; (2) where the observed process were correctly employed (a child could make an executive error and still have the correct process); and (3) what answer the child produced based on a given process. A section was provided for comments. On the subtraction test, space was provided to record whether the child found the minimum element of Q, whether the correct process was employed for a given process, the answer, and the source of cardinality of S. The latter could be obtained in various ways.

Treatments

Description of the treatments. The children in the Control Group participated in their regular mathematics program, Elementary School Mathematics for Kindergarten through Grade 6 (Eicholz and Martin, 1971). The children in the experimental group participated in mathematics classes conducted by Leslie P. Steffe and W. Curtis Spikes. The 12 experimental children in Oglethorpe School were taught from 10:00AM to 11:00AM Monday, Tuesday, Thursday, and Friday and the 12 experimental children at Whitehead Road School were taught from 12:00PM to 1:00Pm on Monday, Tuesday, Wednesday, and Friday. Instruction began October 1, 1974 and ended January 17, 1975 for the experimental children.

The instruction in the experimental groups was highly individualized for each child in that very few sessions were held where group interaction or group demonstration was used. Because the instruction was individualized the children from the experimental group were pooled for data analysis.

The first instructional week was spent on classification where the terminology "and," "or," "not," "some," and "all" was introduced. The content of the classifications were dog, squirrel, and bird cutouts and balloons, toy soldiers, toy horses, and toy cowboys. A sample instructional session is given below.

Objectives: Given a collection of five dog cutouts, two squirrel cutouts, and three bird cutouts, the children should be able to:

1. Select all of the dogs, squirrels, or birds;
2. Select all of the animals that are not birds, squirrels, or dogs;
3. Select all of the animals that are not birds and not squirrels, etc;
4. Select some of the animals; and
5. State that the dogs are some of the animals, but not all of the animals, etc.

Activities:

a. Give the children the animal cutouts and have them select the dogs, the squirrels, and then birds.
b. Have the children select the dogs. Then ask, "Do you have some of the animals?"
c. Repeat (b) using the squirrels and birds.
d. Repeat (b) using combination of two of the obvious subsets.
e. Have the children select all animals that are not squirrels, birds, or dogs.
f. Repeat (e) using combinations of two subsets.
g. Ask the children to compare the number of squirrels and birds, squirrels and dogs, and birds and dogs. Then have them compare the animals and dogs, the animals and squirrels, and the animals and birds.

The second instructional week was spent on partitioning collections of objects. Three basic activities were designed. The first was designed using two subcollections with counting, the second three subcollections with counting, and the third more than three without counting. Samples of three activities are given below:

Objectives: Given a collection of objects, the child should be able to partition the collection into subcollections and realize that:

1. There are as many objects in the subcollections as in the original collections, and
2. The number of subcollections are compensated by the number in each subcollection.

Materials: Construction paper with two nonoverlapping rings drawn inside of another ring.

Activities:

a. Instruct each child to count out ten objects and stress that each has ten.
b. Have the children place five objects in each of the two rings, and ask, "How many here (pointing to the other)?" and "How many altogether?"
c. Have each child take one object from one of the rings and place it in the other and repeat questions in (b).
d. Continue additions and subtractions of one object between the rings until all combinations summing to 10 are covered.
e. Repeat the above activities using nine objects; eight objects; seven objects.

f. The experimenter and children each take five objects. The experimenter puts two in one ring and three in the other with the three covered so that the children cannot see the objects. The children are then asked to find out how many the experimenter has covered by using their objects. Repeat with other combinations.

g. Repeat (f) using other total number of objects.

h. Instruct each child to pour his popcorn into a glass. Ask the children to estimate the number of kernels of popcorn in their glass. Check the estimation less than 20 through counting. After a few estimations, tell the children that there are 100 kernels of popcorn in each glass because they were counted.

i. Form pairs of children and have one child of each pair pour his popcorn into five glasses and the other into twenty glasses. Then ask each pair who has more popcorn or if they both have the same number of kernels.

j. Repeat activity (b) changing the number of glasses around in each pair.

k. Have each child pour his popcorn into ten glasses. Ask the child if he has the same number of kernels of popcorn before pouring as after pouring.

1. Line 50 glasses in a row and have each child pour his popcorn into the glasses, some in each glass. Ask each child to compare the number of kernels before pouring to the number of kernels after pouring.

The third instructional week was spent on loop inclusions and intersections. Sample activities are given below.

Loop Inclusions

Objectives: Given a chain of rings,  , the child should be able to:

1. Place an object inside of exactly one ring, exactly two rings, etc.,
2. Ascertain that any object inside of a given ring is also inside of its containing rings, and
3. After objects are placed inside of each ring, find how many are inside of a given ring.

Materials: Three closed strings of different colors where the strings can be used to form concentric circles, and a pile of tile.

Activities:
a. Give each child one ring. Have the children put one of their hands inside of it. Take the ring and show the children that it will not come off, so their hand is inside of it.
b. Give the children two concentric rings. Have them put one of their hands inside of exactly one ring. Show the children that their hand is not inside of the innermost ring because it can be picked up and their arm is not

inside of it. However, for the outermost ring, their hand is "caught."

c. Using two concentric rings, have the children place one of their hands inside of exactly two rings. Assist children who have difficulty by showing them that neither ring will come off their hand.

d. Give the children three rings and have them place tile inside of exactly one ring, exactly two rings, and exactly three rings.

e. Place the five rings on the floor in the appropriate way. Instruct a child to step inside of exactly one ring. Discuss why the child is inside of exactly one ring using the operational definition given earlier.

Loop Intersections

Objective: Given two or three overlapping rings, a child should be able to identify the interior of exactly one or more rings.

Materials: A collection of rings made of different colored yarn.

Activities: a. Place two overlapping rings on the floor and have each child stand in different parts of the interior--inside of one, of the other and both. Discuss why the children stand where they do each time.

b. Give each child a checker, and have them place the checkers inside of exactly one ring (e.g., the blue ring only), and inside of exactly two rings (e.g., the blue ring and the red ring).

c. Place three overlapping rings on the floor, a blue, a red, and a green ring. Repeat (a) and (b) with appropriate modification.

The remaining instructional time was spent on addition and subtraction. The instruction was sequenced according to the learning instructional phases for addition and subtraction. It is here that the instruction was highly individualized. Consequently, it is very difficult to describe any one uniform instructional sequence. However, the learning-instructional phases for addition and subtraction are presented, after which activities are elaborated.

In the exploratory phase, for the children with rote-counting abilities, addition and subtraction problems were not attempted until they acquired point-counting abilities. This means that children who were rote-counters were given many concrete examples of point counting to bring their level of counting up to the level of point-counting. This was done in the context of counting all strategies for addition and subtraction. The counting-all strategy was used to solve addition and subtraction exercises at the exploratory phase. The children at this phase were given the problem of determining how many elements there were in two sets, S and Q, when all the elements of both were put together. The elements of S were counted out, the elements of Q were counted out and placed with the elements of S. The children then counted out all of the elements of $S \cup Q = P$. The students continued these types of activities with objects and with their fingers, and worked spontaneously from both verbal and written instructions for basic addition facts. This means that being told: "solve this problem: How much is six and four?"

and being given the symbolized statement--"6 + 4 = ____," elicited the same problem-solving behavior. In the case of using their fingers, the students counted out six fingers, counted out four fingers, and then counted each finger and determined that the answer was "ten." Concrete objects were abandoned by all of the children after about two weeks of instruction o addition and subtraction. Finger dexterity increased if the sums were ten or less.

All of the children in the experimental groups were introduced to the exploratory phase of addition and subtraction. The reason for this decision was that an attempt was made to let the children differentiate themselves in instruction to the abstraction-representation phase for addition and subtraction. It was expected that the children who were extensive quantitative comparers would enter the abstraction-representation phase more quickly than would the gross quantitative comparers. The abstraction and representation phase is described below.

In the abstraction-representation learning phase for addition, the children can use the counting-on strategy to solve the problem $s + q = \square$. Rational counting-on, R-1, is used, since the child considers either one of the numbers as a starting point and the other numbers to represent a set of units in the verbal chain. For example, to solve $9 + 3 = \square$, the student might select nine as his starting point in his verbal chain and count on three units more in the chain: "ten," "eleven," "twelve." There is no need to count through the forward verbal sequence to the number nine from one since the child extracts, mentally, the cardinal property of "nineness." At the same time the child initiates a three unit count from the number nine in the verbal (mental) chain. The child does not need to count each unit in the problem but does need to keep a tally of three units. The missing addend is solved with a rational counting-on strategy that also utilizes a tally, but in a different way. The level of rational counting labeled R-2, is needed here. When given the missing addend problem, the missing addend is perceived as part of the sum total. Given the problem to solve, 3 + □ = 11, the student counts-on from three to eleven and symbolizes the units of the missing addend with a tally as he counts. In finalizing the solution, the child point-counts the tally either simultaneously while counting on or after. The subtraction problem is solved with R-3 level counting--counting back without tally. The child solves a problem like 9 - 5 = □, by starting at nine to count the units in the backward ordinal sequence. He counts back five units to the number five and mentally extracts the next number in the backward-ordinal sequence and names it as the solution to the problem. In this problem situation, the child is asked to solve the problem by counting back.

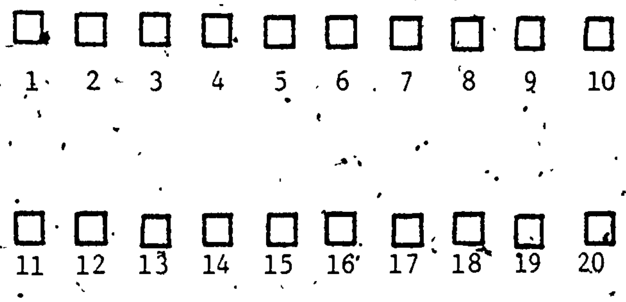
Instructions on counting on and counting back activities were given to each child. The first R-1 level counting activities were as follows. A card with three rings on it  was used. Objects were counted out while being placed into one of the rings. These objects were screened from view. Objects were counted out while being placed into the other ring. The children were then asked to find how many were in the big ring. Counting-all strategies could be used to solve the problem as well as R-1 counting. The goal of such activities was to have the children abstract, through counting activities, that the objects covered did not have to be recounted, but one could start with the number of objects covered and count-on, as described above in the abstraction-representation phase.

From the active involvement in counting physical objects, children were presented with exercise sheets with sums. They were encouraged to use tally marks with pencils in either counting all or R-1 counting as they were able.

The missing addend problem was first presented using a variation of R-1 counting behavior, transforming it to R-2 counting behavior. Instead of counting each collection and covering one, the children were told there were a certain number in the big ring, some under the cover, so how many were altogether. R-2 counting behavior was modeled by the teachers and by able children for those not able to display it.

Because some of the children had a great deal of difficulty with R-1 and R-2 counting, the solution to the missing addend problem presented in symbols (5 + □ = 7) was modeled using partitioning as a base. In the case of the example, seven objects were counted out, five of the seven were counted out, and then the two were counted out to go into the box. A child with counting-all strategies could execute the solution presented in that way. Efforts were then made to take the children that were able into solution by R-2 level counting.

Counting-back activities were also presented, first point counting-all and then rational counting back without tallying. Then counting-back activites were then incorporated into subtraction exercises such as 5 - 3 = □. The children were given a counting-back board as follows. They were shown that to process 5 - 3 on the board, they would



start at five and count off three, to find the answer "two." An attempt was made to emphasize that even though, say "6" appeared under a particular tile, it told how many tiles there were up to and including that tile. Structured materials were used due to the great difficulty child experienced in rational counting back.

All of the children were presented with counting-on and counting-back strategies associated with the three equations $\alpha + \beta = \xi$; $\alpha + \xi = \beta$; and $\beta - \alpha = \xi$, and α and β known and ξ unknown. The third learning-instructional phase was also dealt with in <u>instruction</u>. This learning-instructional phase is called the formalization-interpretation phase.

The formalization-interpretation learning phase for addition and subtraction is characterized by the interrelationships of addition and subtraction. The student in this final learning phase for addition and subtraction can relate problems of the type $9 - 5 = \square$ and $9 = \square + 5$. To become aware of the latter equation from the first one, the R-4 counting--counting-back with tally--must be employed and utilized in a special way. The student counts back five units <u>nine</u> to the number <u>five</u> with a tally (mental). He preserves the solution as four units of the nine and he preserves the five units counted-back, as part of the nine. The numbers <u>five</u> and <u>four</u> are parts of the number <u>nine</u>, The numbers four and five are considered as units.

So, the child realizes (with reconstructing the 5 units he counted back) that 5 units counted back on to four units results in the original 9 units. In this way, addition and subtraction are interelated. So, when a child finds the sum of 5 and 4, he also knows the difference of 9 and 4.

The opportunity was given each child in the treatment to enter this learning-instructional phase through written work. Families of equations were presented to the children for solution, such as $4 + 5 = \square$; $4 + \square = 9$; $\square + 5 = 9$; $9 - 4 = \square$; and $9 - 5 = \square$. The children were never told the interrelationships but were left to make the observations. The written work for each child was retained as children differed greatly in the amount of written work they could do.

Addition, subtraction, and missing addend problems were given to the children to solve during instruction on addition and subtraction. The problems were presented in written format. Children who could read the problems were encouraged to work independently. They were encouraged also to write a mathematical sentence for each problem they solved. The problems were read to the children who could not read. These children were also encouraged to write mathematical sentences for the problems they solved.

The children were allowed to use the hand-held calculator during the last four weeks of instruction. The role of the calculator was to check sums or differences.

The Research Hypotheses

The research hypotheses for the readiness study are stated for each cluster of variables. Rationale for the hypotheses are contained in the previous sections and are summarized whenever appropriate.

Cluster 1. The research hypotheses advanced for Cluster 1 variables are as follows:

1. Extensive quantitative comparers obtain cardinal information from ordinal information to a greater extent than gross quantitative comparers.

2. Extensive quantitative comparers who are taught counting strategies will be able to obtain cardinal information from ordinal information to a greater extent than extensive quantitative comparers who are not taught counting strategies.

3. Gross quantitative comparers are not able to obtain cardinal information from ordinal information regardless of being taught counting strategies.

Cluster 2. The research hypotheses advanced for Cluster 2 variables are as follows:

1. Extensive quantitative comparers and gross quantitative comparers do not perform differently on the variables Counting back, Just Before, Just After, Successor, and Predecessor.

2. Extensive quantitative comparers will outperform the gross quantitative comparers on the variable Between.

3. The children in the experimental and control groups do not perform differently on all variables in Cluster 2.

Cluster 3. The research hypotheses advanced for Cluster 3 variables are as follows:

1. Extensive quantitative comparers will solve verbally presented missing addend problems better than gross quantitative comparers. Differences will also exist on addition and subtraction problems, but not as acute as for the missing addend problems. Moreover, subtraction is more difficult for the gross quantitative comparers than addition.

2. The experimental group will out-perform the control group on all three problem types.

Cluster 4. The following research hypotheses are advanced for the pretest.

1. Gross quantitative comparers are not able to solve the missing addend problems nor the subtraction problems.

2. Extensive quantitative comparers can solve addition and subtraction problems and can, with moderate success, solve missing addend problems.

3. Gross quantitative comparers are not able to solve set partition problems but extensive quantitative comparers are able to solve these problems.

4. Extensive quantitative comparers will outperform gross quantitative comparers on all variables of the cluster.

In the case of the posttest, the following hypotheses were advanced.

1. The children in the experimental group will outperform the children in the control group on all variables except addition, where there will be no difference in performance. This hypothesis should be especially true for the gross quantitative comparers.

2. The extensive quantitative comparers will outperform the gross quantitative comparers in the control group on the variables missing addend, subtraction, set partitions with counting, and set partitions without counting.

Cluster 5. The following research hypotheses are advanced for Cluster 5 variables.

1. Extensive quantitative comparers are able to (a) rational count on, (b) rational count back, (c) solve ordinal number addition problems, and (d) solve ordinal number subtraction problems to a greater extent than gross quantitative comparers.

2. The experimental gross quantitative comparers will outperform the control gross quantitative comparers on the rational counting on and the ordinal number addition problems.

3. Rational counting on and addition of ordinal number problems are highly related.

4. Rational counting back and ordinal number subtraction problems are highly related.

Cluster 6. No research hypotheses are advanced for Cluster 6 variable.

Cluster 7. The research hypotheses advanced for Cluster 7 variables are as follows:

1. Quantity is not a readiness variable for any of the classification tasks.

2. The experimental group will outperform the control group on nested classification tasks and on the loop inclusion tasks.

Statistical Analyses

Item analyses. An item analysis was conducted for each test whenever appropriate. Program ANLITH, an item analysis computer program made available by the Educational Research Laboratory of the University of Georgia was used to conduct the item analysis. The program was initiated for use at the Educational Research Laboratory by Yi-Ming Hsu and was developed by Thomas Groneck and Thomas A. Tyler.

Item difficulty (p-values) are reported for each item. A p-value is a ratio of the number of correct responses to the total number of responses for an item. Test means, standard deviations, and Cronbach's Alpha reliability coefficient are reported for each test as well as the frequency distribution of total scores.

Analyses of variance. Multivariate analyses of variance were conducted for each cluster of variables and were used to test the research hypotheses. Program MUDAID, Multivariate, Univariate, and Discriminant Analysis of Irregular Data, was used for the analyses of research (Applebaum & Bargman, 1967). This program is available through the Educational Research Laboratory at the University of Georgia.

Quantity was used as a classification variable (Extensive vs. Gross) and treatment as an independent variable in all analyses of variance. Each analysis of variance then, was 2 x 2. A univariate analysis of variance and one or more discriminant functions (corresponding to the significant effects) are reported in cases of significant interactions or main effects in the 2 x 2 multivariate analyses. Correlation matrices of the dependent variables are also presented.

CHAPTER III

Presentation of the Results

Results of the Readiness Study

Item analyses

Item analyses are presented for tests of the readiness variables and for tests of some of the achievement variables. These analyses include a difficulty index for each item, a frequency distribution for each test, an internal consistency reliability coefficient for each test, test means, and test standard deviations.

Quantitative comparisons. The test of quantitative comparisons (Appendix A.11) was administered to 107 children as a pretest. Table 1 contains the difficulty indices for each item, and item characteristics. Items 1, 2, 3, and 6 were of comparable difficulty. These items either had

Table 1

Difficulty Indices and Item Characteristics for Quantitative Comparisons Pretest

Item	Difficulty	Item Characteristic
1	.70	Triangular arrangement; six red, six green
2	.74	Rectangular arrangement; six red, eight green
3	.73	Random arrangement; six red, six green
4	.57	Linear arrangement; six red, six green
5	.49	Linear arrangement; eight red, eight green
6	.72	Random arrangement; eight green, six red
7	.59	Circular arrangement; eight red, eight green
8	.54	Random arrangement; eight red, eight green

a configuration conducive to solution by visual inspection (triangular or rectangular), had two collections of six objects with a random arrangement (item 3), or contained a collection which apparently had more than the other (item 6). These items all demanded an extensive quantitative comparison for correct solution due to difficult geometrical configurations or eight objects in each collection to be compared. They were the critical items to separate the extensive quantitative comparers from the gross quantitative comparers.

The test mean was 5.01, standard deviation 2.58, and internal consistency reliability .84. The reliability of .84 supports the classification into extensive and gross categories. Further justification of the validity of the two quantitative categories is that, if a child scored at least 5 out of 8 correctly with justification for his answers, evidence was strong he would have made an extensive quantitative comparison. Evidence was strong because at least one of items 4, 5, 7, or 8 would by necessity have to be answered correctly with justification.

The distribution of total scores for the eight item test was as follows. Eleven children scored zero, five scored one, five scored two, seven scored three, eight scored four, ten scored five, twenty-one scored seven, and nineteen scored eight. The rather large frequencies for the scores five, six, seven, and eight can be attributed to items 1, 2, 3, and 6. In retrospect, those items did not necessarily measure extensive quantity.

The test of class inclusion (Appendix A.10) given to the total first-grade population was extremely difficult (88 out of 107 scored zero), so no psychometric analysis was needed. Evidence was available that only nine children had class inclusion.

Number in S and number in P. Table 2 contains the difficulty indices for the Number in S and Number in P tests (Appendix A.1). The first item on Number in S test was more difficult than the second. The first is probably more indicative of the difficulty of the #S items due to the

Table 2

Difficulty indices for #S and #P Tests

Test Item	Number in S	Number in P
1	.31	.46
2	.54	.44

fact that the second item was from the second ordinality task and the child had processed a considerable amount of information about the task before asked to find the number in S.

Table 3

Frequency Distributions, Means, Standard Deviations, and Reliabilities of the #S and #P Tests

Test	Frequency Distribution Total Score 0	1	2	Mean (Percent)	Standard Deviation	Reliability
#S	16	23	9	.85 (42)	.71	.15
#P	17	19	12	.90 (45)	.77	.33

The frequency distributions, means, standard deviations, and reliabilities for #S and #P tests are given in Table 3. None of the distributions appear to represent normally distributed variables. The reliabilities are extremely low and are a reflection of the rather large number of children scoring one out of the two items correctly. The items were not homogenous. This heterogeneity may be a result of the items being on different tasks and in different sequences in each task.

While the low reliabilities may be attributed to the fact that the tests contained only two items, the tests were administered individually by competent testers. Such individual administration should minimize errors of measurement. This argument strengthens the necessity for better task design for tests of #S and #P variables.

In the event differences for main effects are detected in the analyses of variance for #S or #P variables, they can be interpreted. The reason such interpretation is possible is that, given significant differences (say, for quantity) a preponderance of the children scoring zero would have to be in one category and a preponderence of the children scoring 1 or 2 would have to be in another category. For children scoring either zero or two, it is reasonable to conclude that they did not or did have the ability to obtain cardinal information from ordinal information, respectively. For children scoring one, however, difficulties of interpretation are present.

In the event differences are not detected in the analyses of variance for #S or #P variables, no interpretation should be made.

Problem solving without objects. Table 4 contains the difficulty indices for problem solving test without objects (Appendix A.4). The indices are surprisingly high for a problem solving test with no objects present.

Table 4

Difficulty Indices for Problem Solving Test Without Objects

Item	Difficulty	Item Type
1	.75	Addition
2	.81	Addition
3	.73	Subtraction
4	.69	Subtraction
5	.54	Missing Addend
6	.42	Missind Addend

The missing addend problems are more difficult than the four addition and subtraction items, as expected. The indices for the addition and subtraction items are quite comparable but greater than indices for the missing addend items, indicating that counting-all strategies were used during solution.

Table 5 contains the frequency distribution, mean, standard deviation, and reliability information for the three problem types. None of the distributions appear to represent normally distributed variables. The internal consistency reliabilities are quite substantial for Subtraction and Missing Addend but rather modest for Addition. Inspection of the frequency distributions show that the missing addend problems were

Table 5

Frequency Distributions, Means, Standard Deviations, and Reliabilities of the Addition, Subtraction and Missing Addend Tests Without Objects

Test	Frequency Distributions: Total Score 0	1	2	Mean (Percent)	Standard Deviation	Reliability
Addition	4	13	31	1.56 (78)	.64	.35
Subtraction	10	8	30	1.42 (71)	.81	.75
Missing Addend	22	6	20	.96 (48)	.93	.87

almost an all-or-nothing phenomenon. The subtraction problems were easier than the missing addend problems, but yet only 8 children scored one of the two correctly. The addition problems were quite easy for the children and all but four scored at least one of the two correctly. The analyses of variance should be interpreted with caution in the case of the addition problems if no significant differences exist. If differences do exist, a preponderance of children who scored 0 or 1 would have to be in a category together. So, here again, interpretation would have to be made with caution. There is no difficulty interpreting results in the analyses of variance for the two other problem types.

Problem solving with objects and partitions tests. The problem solving test with objects (Appendix A.3) and the partitions test (Appendix A.5) were administered as pretests. The problem solving test consisted of two addition items, two subtraction items, and two missing addend items. These three item-types were considered to be subtests. The partition test was made up of two items where the child counted and two items where he did not count. These item types were considered as subtests. Table 6 contains the difficulty indices for each item of each test.

Table 6

Difficulty Indices for Problem Solving and Partition Pretests

Item \ Test	Problem Solving		Partitions	
	Difficulty	Item Type	Difficulty	Item Type
1	.48	Addition	.52	With Count
2	.48	Subtraction	.60	With Count
3	.19	Missing Addend	.44	Without Count
4	.46	Subtraction	.29	Without Count
5	.56	Addition	—	—
6	.15	Missing Addend	—	—

It is apparent from Table 6 that the indices for addition and subtraction are approximately the same, but the missing addend problems were more difficult. Moreover, the indices for the two items with counting in the partition test are each considerably greater than the two items without counting, which did not include a particular number--only the relation "the same number."

Table 7 contains the frequency distributions for the two total tests and their subtests. None of the distributions appear to represent normally distributed variables. For items with counting and without counting, responses for the variable Partition nearly reflected an all-or-nothing phenomenon, since scores of one were relatively few in number.

Table 7

Frequency Distributions for Problem Solving and Partition Pretests

Test \ Score	0	1	2	3	4	5	6
Problem Solving Total	16	4	3	9	9	4	3
Addition	17	12	19	-	-	-	-
Subtraction	20	11	17	-	-	-	-
Missing Addend	36	8	4	-	-	-	-
Partition Total	15	4	14	3	12	-	-
With Count	17	8	23	-	-	-	-
Without Count	27	7	14	-	-	-	-

Because 36 children scored 0 on the missing addend items, that subtest did not contribute a great deal to the middle three scores in the total problem solving test distribution. As the difficulty indices for addition and subtraction were around .50, those items should have contributed heavily to the middle three and possibly upper three scores in the total test distribution. Consequently, one would expect the a distribution of nonezero scores to be nearly bell-shaped. The actual distribution of the total test met this expectation.

Table 8 contains the internal consistency reliability coefficients, means, and standard deviations. The reliabilities of the tests are substantial except for the variable Missing Addend, which was a very difficult test. The reliabilities support further analyses of the data

Table 8

Reliabilities, Means, and Standard Deviations of Problem Solving and Partitions Pretest

Test \ Statistic	Reliability	Mean (Percent)	Standard Deviation
Problem Solving	.82	2.31 (38)	2.00
Addition	.68	1.02 (52)	.86
Subtraction	.70	.94 (47)	.88
Missing Addend	.58	.33 (16)	.62
Partitions	.80	1.85 (46)	1.54
With Counting	.80	1.12 (56)	.90
Without Counting	.84	.73 (36)	.88

and allow those analyses to be interpreted with the confidence that the criterion measures are internally consistent. In fact, the reliabilities associated with the two-item subtests support analyses conducted using those subtests as dependent variables.

The problem solving with objects and partition tests were administered as posttests as well as pretests. Table 9 contains difficulty indices of the items of each test. Very substantial gains from pre-to-posttest were

Table 9

Difficulty Indices for Problem Solving and Partition Posttests

Item \ Test	Problem Solving		Partitions	
	Difficulty	Item Type	Difficulty	Item Type
1	.71	Addition	.77	With Count
2	.52	Subtraction	.73	With Count
3	.58	Missing Addend	.71	Without Count
4	.52	Subtraction	.60	Without Count
5	.75	Addition	-	-
6	.50	Missing Addend	-	-

made in scores on the addition and missing addend items and in all of the items of the partitions test. The difficulty indices for the subtraction and missing addend problems are now comparable and all are less than the indices for the addition items. The problem solving item difficulties in Table 9 are consistent with those observed by Steffe and Johnson (1971) but not consistent with those observed in Table 4 for the subtraction problems, a result to be explained in the section Analyses of Variance.

Table 10 contains the frequency distributions for the two total tests and their subtests. All frequency distributions changed from pre-to-posttest from the lesser to the greater scores (See Table 7). As subtraction was worked on in the experimental group, it is surprising that the frequency distribution was not altered in the same magnitude as the other distributions. Interpretation of the changes in the distributions is delayed until the section Analyses of Variance.

Table 10

Frequency Distributions for Problem Solving and Partition Posttests

Test \ Score	0	1	2	3	4	5	6
Problem Solving	6	5	7	4	6	3	17
Addition	7	12	29	-	-	-	-
Subtraction	21	4	23	-	-	-	-
Missing Addend	17	10	21	-	-	-	-
Partitions Total	8	3	6	4	27	-	-
With Count	8	8	32	-	-	-	-
Without Count	13	7	28	-	-	-	-

Table 11 contains the internal consistency reliability coefficients, means, and standard deviations. All of the reliability coefficients, except for the addition subtest, are substantial and again support analyses of variance using the subtests as criterion tests. The rather low reliability of the addition tests is to be expected because the test was

Table 11

Reliabilities, Means, and Standard Deviations of Problem Solving and Partition Posttests

Test \ Statistic	Reliability	Mean (Percent)	Standard Deviation
Problem Solving	.86	3.58 (60)	2.22
Addition	.54	1.46 (73)	.73
Subtraction	.91	1.04 (52)	.96
Missing Addend	.74	1.08 (54)	.89
Partition	.87	2.81 (70)	1.55
With Count	.72	1.50 (75)	.76
Without Count	.82	1.31 (65)	.87

relatively easy (mean score 73 percent). The standard deviations remain substantial and reflect the fact that children scored at each possible score on the criterion scale, with heavy loading at the extremes.

Addition and subtraction of ordinal numbers. Table 12 contains the diffioulty indices for the addition and subtraction of ordinal number tests (Appendix A.6). The rational counting on items, modeled by

$\alpha + \beta = \xi$, ξ unknown, were each fairly easy items. The missing addend problems, or ordinal number addition items modeled by $\alpha + \xi = \beta$, ξ unknown were also surprisingly easy. However, the ordinal number subtraction items were difficult, as were the counting-back items. Item difficulty is somewhat a function of the particular numbers involved.

Table 12

Difficulty Indices for Addition and Subtraction of Ordinal Number Tests

Test	Ordinal Number Test	Ordinal Subtraction Test
Item	Difficulty Type	Difficulty Type
1	.77 Counting-On $\alpha + \beta = \xi$	.54 Counting-Back $\xi + \alpha = \beta$ No tallying
2	.73 Counting-On $\alpha + \beta = \xi$	.31 Ordinal Subtraction $\xi + \alpha = \beta$ Tallying
3	.71 Ordinal Addition $\alpha + \xi = \beta$	.56 Counting-Back $\xi + \alpha = \beta$ No tallying
4	.56 Ordinal Addition $\alpha + \xi = \beta$	.19 Ordinal Subtraction $\xi + \alpha = \beta$ Tallying

Table 13 contains the frequency distributions, means, standard deviations and reliabilities for the total tests of addition and subtraction of ordinal numbers. The reliabilities associated with the two tests with equation forms $\alpha + \beta =$ and $\xi + \alpha = \beta$ with tallying are rather low. The former is easy and the latter difficult, each of which contributes to low reliabilities. The analyses of variance for these two tests should

Table 13

Frequency Distributions, Means, Standard Deviations and Reliabilities of Ordinal Number Addition and Subtraction Tests

Test	Frequency Distribution Total Score 0	1	2	Mean (Percent)	Deviation	Reliability
Counting-On	6	12	30	1.50 (75)	.71	.50
Ordinal Addition	14	7	27	1.27 (64)	.88	.84
Counting Back	20	15	13	.85 (42)	.82	.61
Ordinal Subtraction	20	20	8	.75 (38)	.72	.47

be definitely interpreted, but with some caution if no differences are detected in the analyses.

Mental arithmetic. Table 14 contains the difficulty indices for the mental arithmetic test (Appendix A.7). The difficulty indices for the time score represent an average time for each item. The subtraction exercises took longer, on an average, than did the addition exercises. Not only did they take longer, but they were more difficult.

Table 15 contains frequency distributions for the mental arithmetic test, product score. Table 16 contains the same data for the time score.

Table 14

Difficulty Indices for Mental Arithmetic Product and Time Scores

Item \ Test	Product Score	Time Score	Item Type
1	.86	14.62	Addition
2	.75	14.02	Addition
3	.67	21.34	Subtraction
4	.40	24.00	Subtraction

Table 15

Frequency Distributions for Mental Arithmetic Test:

Product Score

Test \ Score	0	1	2	3	4
Total	7	3	9	16	13
Addition	7	12	29	-	-
Subtraction	15	15	18	-	-

Table 16

Frequency Distribution for Mental Arithmetic Test: Time Score

Test \ Interval	1	2	3	4
Total	16-115* (36)**	115-214 (9)	214-313 (1)	313-412 (1)
Addition	0-18 (0)	18-117 (44)	117-216 (3)	---
Subtraction	0-99 (45)	99-198 (1)	198-297 (1)	---

*time range in seconds

**Number of students that completed the test within the time interval given.

Most of the students completed the total test within the interval of 16 to 115 seconds -- within approximately two minutes.

Table 17 contains the reliabilities, means, and standard deviations associated with each of the product score and the time score. In some cases, the reliabilities are extremely low. For the addition items product score, the reliability is only .15. The fact that the test was easy certainly contributes to this low reliability. No interpretation should be given to the analysis of variance on that measure. The reliabilities associated with the time score should be interpreted as a measure of the consistency of the time it took to do each problem. If, for example, it took consistently much longer to do one subtraction item than the other, a low reliability would be the result. But if it always took about the same time, a substantial reliability would show. Just because the addition items are much closer in difficulty than the subtraction items, one cannot say they took closer to the same time than did the subtraction items, for the reliability is less for the addition time scores than for the subtraction time scores.

Table 17

Reliabilities, Means, and Standard Deviations of the Mental Arithmetic Test: Product and Time Scores

Test \ Statistic	Reliability	Mean (Percent)	Standard Deviation
Total: Product	.19	2.65 (66)	1.50
Addition: Product	.15	1.60 (80)	1.16
Subtraction: Product	.65	1.06 (53)	.82
Total: Time	.77	73.98	55.29
Addition: Time	.49	28.64	29.12
Subtraction: Time	.71	45.34	31.83

<u>Class inclusion</u>, <u>loop inclusion</u>, <u>nested classification task A</u>, and <u>nested classification task B</u>.* Table 18 contains difficulty indices for the items of each test. The difficulty indices for the class inclusion items are quite close. Moreover, the correlations of each item with the total test corrected for overlap are .58, .77, .70, .87, and .82, for item 1, 2, 3, 4, and 5, respectively. These very substantial item-test correlations indicate each item functioned well as a discriminator.

*See Appendices A.10, A.9, and A.8.

Table 18

Difficulty Indices for Class Inclusion, Loop Inclusion and Nested Classification Tests

Item \ Test	Class Inclusion	Loop Inclusion	Nested Class Task A	Nested Class Task B
1	.38	.36	.66	.81
2	.26	.36	.53	.38
3	.26	.66	.55	.79
4	.32	—	.30	.64
5	.36	—	.74	.43
6	—	—	—	.81

The difficulty indices of the items in the three other tests fluctuated a great deal. In the case of the loop inclusion test, the third item was relatively easy compared to the first two. When considering the item context (intersecting rings), it appears to be measuring something quite different than the two others. This claim is supported by the low correlation (connected for overlap) of .35 between the third item and the total test. In retrospect, the third item could be answered correctly even though a child looked at two intersecting rings as forming three separate regions. In the case of items 1 and 2, the probability was great that a child had to view the nested rings as being nested in an inclusive sense.

In the nested classification task A, the most difficult item involved a sequence of four questions involving attributes irrelevant in the nested classification. Whether the number of questions, the irrelevant attributes, or both contributed to the difficulty is not clear. The two most difficult items of task B did not involve irrelevant attributes, but rather classification of the buttons. Apparently, children in some cases thought that the object in the box could be a white button.

Table 19 contains the frequency distributions and Table 20 the reliabilities, means, and standard deviations of the tests. It is apparent that the class inclusion test is quite difficult with approximately one half of the children scoring zero. Fifteen of the children displayed scores of at least three, which indicated that these children learned how to solve the class inclusion problem during the time from September to February.

Table 19

Frequency Distributions of the Class Inclusion, Loop Inclusion, and Nested Classification Tests.

Test	0	1	2	3	4	5	6
Class Inclusion	23	8	1	3	7	5	_
Loop Inclusion	12	16	8	11	_	_	_
Nested Class: A	2	7	13	11	5	9	_
Nested Class: B	4	1	5	8	10	8	11

The class inclusion test is highly internally consistent, but the remaining three are only moderately reliable given that they were constructed to measure a single capability. As the last item of the loop inclusion test is faulty, it undoubtedly contributed to the low reliability.

Table 20

Reliabilities, Means, and Standard Deviations of the Class Inclusions, Loop Inclusion, and Nested Classification Tests

Test \ Statistic	Reliability	Mean (Percent)	Standard Deviation
Class Inclusion	.90	1.57 (31)	1.94
Loop Inclusion	.65	1.38 (46)	1.10
Nested Class: A	.55	2.79 (56)	1.44
Nested Class: B	.76	3.85 (64)	1.80

Whether the nested classification tests are good measures is, at this point, an open question. They do not possess particularly good psychometric properties given the way they were constructed.

Analyses of Variance

Cluster 1. The variables included in Cluster 1 were Number in S (#S), Number in P (#P), and Number in S + Number in P (#S + #P). These variables are defined in the section The Achievement Tasks. The research hypotheses to be tested in this section are that (1) extensive quantitative comparers obtain cardinal information from ordinal information to a greater extent than gross quantitative comparers, (2) extensive quantitative comparers who are taught counting strategies obtain cardinal information from ordinal information to a greater extent than the extensive quantitative comparers who are not taught counting strategies, and (3) gross quantitative comparers are not able to obtain cardinal information from ordinal information regardless of being taught counting strategies.

The multivariate F for interaction of Quantity and Treatment ($F_{3,42} = .62$) was not significant. The multivariate F for Quantity was significant ($F_{3,42} = 3.14$, $p < .05$). The multivariate F for Treatment was not significant ($F_{3,42} = .39$). Table 21 contains the raw weights of the discriminant function for Quantity and the total group correlations of the original variables with the discriminant function. Because Quantity was significant ($p < .05$) the univariate F-ratios are presented in Table 22. The variables showing significance were #P and #S + #P for Quantity. The fact that #P and #S + #P were significant for Quantity in the univariate tests corresponds quite well to the fact they contributed most (correlations, Table 21) to the separation of the extensive and gross quantity groups. The cell means for the #S variables are presented in Table 23, for the #P variable in

Table 21

Weights of the Discriminant Function and Correlation of Original Variables with the Discriminant Function for Quantity: Cluster 1.

Statistic \ Variables	#S	#P	#S + #P
Weights	-.045	.036	.055
Correlation	.34	.71	.92

Table 22

Quantity Versus Treatment Univariate Analysis of Variance: Cluster 1.

	Number in S		Number in P		Number in S + Number in P	
Source of Variation	Mean Square	F	Mean Square	F	Mean Square	F
Quantity (Q)	.59	1.14	2.16	5.01**	13.42	8.33**
Treatment (T)	.03	<1	.01	<1	1.04	<1
QXT	.45	<1	.96	1.85	.90	<1
Error	.52		.52		1.61	

**(p<.01)

Table 24 and for the #S + #P in Table 25. The first hypothesis tested was supported in the multivariate analyses and was supported for the #P and the #S + #P variables in the univariate analyses. Apparently, the extensive quantitative comparers were able to utilize the hints in the

Table 23

Cell Means for #S

Quantity \ Treatment	Control	Experimental
Extensive	42%	54%
Gross	41%	33%

Table 24

Cell Means for #P

Quantity \ Treatment	Control (Percent)	Experimental (Percent)
Extensive	50%	63%
Gross	41%	25%

Table 25

Cell Means for #S + #P

Quantity \ Treatment	Control (Percent)	Experimental (Percent)
Extensive	67%	81%
Gross	48%	48%

cardinal-ordinal tasks to a greater extent than were the gross quantitative comparers. This finding is quite significant. The extensive quantitative comparers, especially those in the experimental group, seemed quite capable of solving problems of the nature presented. Solution strategies necessary for the tasks were apparently available to the extensive quantitative comparers and were easily activated.

Apparently, the task design for #S produced too much conflict for extensive quantitative comparers to activate relevant strategies to the same extent as in the two other variables. This opinion is based on the results of the #P and #S + #P variables. Consequently, the results for the #S variable are viewed as inconclusive, neither supporting nor refuting the first hypothesis of this section. A test of the hypothesis for #S awaits better and more reliable task design.

In case of the #P variable, the extensive quantitative comparers outperformed the gross quantitative comparers, especially in the experimental group. An interaction between quantity and treatment is suggested by the means in Table 24, but was not significant statistically. One can say that children who are extensive quantitative comparers can obtain cardinal information from ordinal information better than gross quantitative comparers as long as that information can be obtained from counting forward rather than backward. The effect of Quantity was not as strong for #P as it should have been theoretically. But it must be remembered that the reliability for #P variable was low. The first hypothesis was supported by the data from the #P variable, but one should not place strong confidence in the results. A more conclusive test awaits better task design. Even though the interaction of Quantity and Treatment was not significant for any of the three variables, the second hypothesis seemed supported by the results in Tables 23, 24, and 25. The results are suggestive enough that the hypothesis should be tested again. The third hypothesis appears to be supported, although not strongly.

The correlations of the variables are presented in Table 26. The correlation of .54 between #S and #P is significnat at $p < .01$. This modest correlation between #S and #P is further evidence that improved task design is necessary for the two variables. The two remaining correlations are spurious due to definition of the variable #S + #P.

Table 26

Correlations Among Variables in Cluster 1

	#S	#P
#S	--	
#P	.54**	--
#S + #P	.61**	.61**

**(p < .01)

Cluster 2. The variables included in Cluster 2 were Counting Back, Just Before, Just After, Between, Successor, and Predecessor. These variables are defined in the section The Achievement Tasks. The research hypotheses to be tested in this section are that (1) extensive quantitative comparers and gross quantitative comparers do not perform differently on the variables Counting Back, Just Before, Just After, Successor, and Predecessor, (2) that the extensive quantitative comparers outperform the gross quantitative comparers on the variable Between; and that (3) the children in the experimental and control groups do not perform differently on all the variables in Cluster 2.

The multivariate F for interaction of Quantity and Treatment ($F_{6,39} = 1.46$) was not significant. The multivariate F for Quantity was significant ($F_{6,39} = 2.57$, $p < .05$). The multivariate F for Treatment ($F_{6,39} = .94$) was not significant. The raw weights of discriminant function for Quantity and the total group correlations of the original variables with the discriminant function are presented in Table 27. The variables Counting Back and Between contribute most to the separation of the extensive and gross quantity groups. These two variables are also significant in the univariate analyses presented in Table 28.

Table 27

Weights of the Discriminant Function and Correlation of Original Variables with the Discriminant Function for Quantity: Cluster 2

Statistic \ Variables	Counting Back	Just Before	Just After	Between	Predecessor	Successor
Weights	.052	-.113	.020	.048	-.039	.043
Correlation	.69	-.05	.38	.63	.21	.37

Table 28

Quantity Versus Treatment Univariate Analyses of Variance: Cluster 2

Variable	Counting Back		Just Before		Just After		Between		Successor		Predecessor	
Source of Variation	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F
Quantity (Q)	10.51	8.31**	.01	<1	.23	2.54	15.33	6.96**	.88	2.43	.20	<1
Treatment (T)	1.35	1.07	.02	<1	.21	2.54	.17	<1	.00	<1	.00	<1
Q X T	.63	<1	.244	1.52	.00	<1	4.59	2.08	.13	<1	.00	<1
Error	1.36		.160		.09		2.20		.36		.26	

$^{**}(p < .01)$

Table 29

Cell Means for Counting Back and Between

Variable	Counting Back		Between	
Quantity \ Treatment	Con	Exp	Con	Exp
Extensive	72%	83%	67%	79%
Gross	64%	44%	54%	35%

Table 29 contains the cell means for the two variables for which Quantity was significant in the univariate analyses. In case of the variable Counting Back, the gross quantitative comparers had a mean score of only about 54%, which indicates some difficulty with point counting back for these children. In that the treatment had no positive effects, one can expect teaching gross quantitative comparers to count back to be somewhat ineffective if the teaching is not sustained and repeated over time. Gross quantitative comparers have a difficult time determining the numbers between two given numbers, as shown in Table 29. Again, the concept was resistant to instruction on counting-on strategies given for these children.

The first hypothesis tested in this section is not supported in a multivariate sense. To locate precise differences, the univariate analyses were run. It was found that hypothesis was not supported for Counting Back, but is supported in the case of the remaining variables. The second hypothesis was supported. Acquisition of the concept Between appears related to quantitative comparisons as hypothesized. The third hypothesis was supported.

The correlations among the variables are presented in Table 30. The critical correlation is r = .30 to be significant for p < .05, and r = .35 to be significant for p < .01. The correlations are modest at best given that the variables are conceptually related.

Table 30

Corrlation Among Variables in Cluster 2

	1	2	3	4	5
1. Counting Back	-				
2. Just Before	.19	-			
3. Just After	.31*	.36**			
4. Between	.29	.49**	.44**	-	
5. Successor	.18	-.03	.03	-.03	-
6. Predecessor	.52**	.34*	.15	.34*	.00

*(p < .05). **(p < .01)

Cluster 3. The variables included in Cluster 3 were Addition, Subtraction, and Missing Addend problems to be solved in the absence of physical objects. The research hypotheses tested in this section are that (1) extensive quantitative comparers will solve verbally presented missing addend problems better than gross quantitative comparers. Differences will also exist on addition and subtraction problems, but not as acute as for the missing addend problems. Moreover, subtraction will be more difficult for the gross quantitative comparers than addition, and (2) the experimental group will out-perform the control group on all three problem types.

The multivariate F for interaction of Quantity and Treatment ($F_{3,42} = 1.23$) was not significant. The multivariate F for Quantity was significant ($F_{3,42} = 6.89$, $p < .01$). The multivariate F for treatment ($F_{3,42} = 1.03$) was not significant. Table 31 contains the raw weights of the discriminant function for Quantity, and the total group correlations of the original variables with the discriminant function.

Table 31

Weights of the Discriminant Function and Correlation of Original Variables with the Discriminant Function for Quantity: Cluster 3

Statistic \ Variable	Addition	Subtraction	Missing Addend
Weights	.013	-.003	.131
Correlations	.40	.51	.99

The missing addend problems contribute a great deal to the separation of the extensive and gross quantity groups. The subtraction problems are next, and then the addition problems, which contribute relatively little. To further understand the variables, univariate F-ratios and cell means are presented as Table 32, and Table 33, respectively.

Table 32

Quantity versus Treatment Univariate Analysis of Variance: Cluster 3

Variable	Addition		Subtraction		Missing Addend	
Source of Variation	Mean Square	F	Mean Square	F	Mean Square	F
Quantity (Q)	1.76	3.49√	3.26	5.54*	13.33	21.51**
Treatment (T)	.40	<1	1.89	3.20[a]	.34	<1
Q X Q	1.78	3.52√	.90	1.52	.04	<1
Error	.50		.60		.62	

*(p < .05) **(p < .01) √(p < .07) [a](p < .09)

Table 33

Cell Means for Cluster 3 Variables

Variables	Addition		Subtraction		Missing Addend	
Treatment / Quantity	Con	Exp	Con	Exp	Con	Exp
Extensive	88%	79%	81%	88%	69%	75%
Gross	50%	79%	41%	75%	14%	25%

The F-ratio for Quantity is significant for Missing Addend. In the case of Missing Addend the mean for the extensive quantity group is 72 percent and the mean for the gross quantity group is 20 percent — a striking difference.

It appears as if Quantity and Treatment should have interacted for Addition and Subtraction. In the case of the control group, the differences in the means for the extensive and gross quantity groups were 38 and 40 percent for Addition and Subtraction, respectively. The same differences were 0 and 13 percent for the experimental group. These interactions are not significant statistically.

Because of relatively large within-cell variances and the fact that only one degree of freedom was available for the numerator of the F-ratio, strong between group differences had to exist before they were statistically significant. Consequently, if a main or interaction effect was significant statistically it was certain to be significant educationally. Moreover, some between group differences could be concluded as educationally significant when not statistically significant. The interaction effects in the analyses of variance for the addition and subtraction tests fall in this category. The differences in the means for the control group are of a magnitude that they would be significant if differences in experimental groups were of the same magnitude. In fact Quantity was significant for Subtraction. At any rate, the interactions of Quantity and Treatment are considered as educationally significant for Addition and Subtraction, and are explanable in terms of the treatment.

The experimental children were encouraged to use their fingers in doing addition and subtraction problems using a counting-all strategy. The gross quantitative comparers apparently learned to execute the strategy about as well as extensive quantitative comparers in case of the experimental group. However, due to the counting-on necessary for the missing addend problems, counting-all procedures were not appropriate. The gross quantity children apparently had a great deal of difficulty learning counting-on procedures even though such procedures were taught.

The correlations among the variables in Cluster 3 are presented as Table 34. The correlation between Addition and Missing Addend is modest. A more extensive discussion of the correlations is offered in the next section when the analysis for the variables in Cluster 4 is presented.

Table 34

Correlations Among Variables in Cluster 3

	1	2
1. Addition	-	
2. Subtraction	.55**	-
3. Missing Addend	.33*	.49**

*(p < .05) **(p < .01)

The first hypothesis is suppported in a multivariate sense as well as a univariate sense. The predicted differences were observed for addition, subtraction, and missing addend problems. The second hypothesis was not supported. However, the gross quantitative comparers who were in the experimental group outperformed the gross quantitative comparers in the control group in the case of addition and subtraction.

Cluster 4. The variables included in Cluster 4 were Addition, Subtraction, and Missing Addend Problems to be solved in the presence of objects and Partitions With Counting and Partitions Without Counting.

This cluster of variables was administered to the children before the treatments began and after the treatments were over. Two sets of data are then presented--pretest data and posttest data. The pretest data are presented first.

The research hypotheses for the pretest data for Cluster 4 are that (1) the gross quantitative comparers are not able to solve the missing addend problems nor the subtraction problems, (2) the extensive quantitative comparers can solve addition and subtraction problems and can, with moderate success, solve missing addend problems, (3) the gross quantitative comparers are not able to solve set partitions problems but extensive quantitative comparers are able to solve these problems, and (4) extensive quantitative comparers will out perform gross quantitative comperers on all operation variables.

The interaction of Quantity and Treatment was not significant. The multivariate F for Quantity was significant ($F_{5,36} = 9.84$, $p < .01$). The multivariate F for treatment was not significant.

Table 35 contains the raw weights of the discriminant function for Quantity and the total group correlations of the original variables with the discriminant function. Partition problems do not contribute a great deal to the separation of the two groups involved as shown by the correlations. All other variables do contribute, with subtraction and missing addend problems contributing quite heavily.

Table 36 contains the univariate analyses for all of the variables. Quantity was highly significant for Addition, Subtraction, and Missing Addend; but was not signigicant for either Partitions With Counting or Partitions Without Counting. In order to inspect the cell means Table 37 is presented. Any differences due to treatment groups was strictly

Table 35

Weights of the Discriminant Function and Correlation of Original Variables with Discriminant Function for Quantity: Cluster 4 Pretest

Statistic \ Variable	Addition	Subtraction	Missing Addend	Partition Count	Partition No Count
Weight	.051	.134	.174	.019	-.054
Correlation	.53	.84	.70	.27	.009

Table 36

Quantity versus Treatment Univariate analysis of Variance: Cluster 4 Pretest

riable	Addition		Subtraction		Missing Addend		Partitions With Count		Partitions No Count	
urce of riation	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F
antity (Q)	7.50	12.46**	12.89	30.56**	6.38	21.19**	2.41	3.27	.00	<1
eatment (T)	.68	1.12	1.96	4.64*	.112	<1	.83	1.13	1.07	1.25
T	.00	<1	.12	<1	.00	<1	.67	<1	.46	<1
ror	.60		.42		.30		.74		.85	

*(p < .05) **(p < .01)

Table 37

Cell Means for Cluster 4 Variables: Pretest

Variable	Addition		Subtraction		Missing Addend		Partitions With Count		Partitions No Count	
Treatment / Quality	Con	Exp	Con	Exp	Con	Exp	Con	Exp	Con	Exp
Extensive	88%	75%	94%	66%	44%	38%	75%	75%	38%	42%
Gross	46%	33%	33%	17%	4%	0%	38%	63%	25%	50%

due to chance fluctuations in sample selection. The subtraction problems appeared to be easier for the control children than for the experimental children.

The first hypothesis was strongly supported for the missing addend problems but only weakly supported for the subtraction problems, as a mean score of 25 percent was obtained by the 24 children who were gross quantitative comparers on the subtraction problems. The extensive quantitative comparers, however, had a mean score of 80 percent on the subtraction problems and a mean score of 82 percent on the addition problems, but only a mean score of 41 percent on the missing addend problems. Consequently, hypothesis (2) is strongly supported.

Hypothesis 3, surprisingly, was not supported by the data. No differences were found between the extensive and gross groups on either of Partitions With Counting or Partitions Without Counting. Moreover, the correlations between the partition tests and the other three variables are low as shown in Table 38. Partitions with and without counting do not correlate with subtraction or missing addend problems. The partition tests do correlate significantly with addition problems, but

Table 38

Correlations Among Variables in Cluster 4: Pretest

	1	2	3	4
1. Addition	-			
2. Subtraction	.58**	-		
3. Missing Addend	.78**	.29	-	
4. Partitions With Count	.36*	.29	.12	-
5. Partitions No Count	.31*	.10	.23	.46**

*(p < .05) **(p < .01)

the correlations are barely significant. The addition problems correlate substantially with subtraction and missing addend problems, which, in turn, do not significantly correlate.

Hypothesis 4 was supported in a multivariate sense. The univariate analyses showed that the hypothesis was supported.

The research hypotheses for the posttest data are (1) the children in the experimental group will outperform the children in the control group on all variables except addition, where there will be no differences in performance. This hypothesis should be true especially for the gross quantitative comparers. (2) the extensive quantitative comparers will outperform the gross quantitative comparers in the control group on the variables Missing Addend, Subtraction, Set Partition With Counting, and Set Partitions Without Counting.

The multivariate F for the interaction of Quantity and Treatment was not significant. The multivariate F for Quantity was significant ($F_{5,40} = 8.24$, $p < .01$). The multivariate F for Treatment was not significant ($F_{5,40} = 2.37$) for $p < .05$, but was significant for $p < .06$. Table 39 contains the weights of the discriminant function for Quantity and the correlations of the original variables with the discriminant function.

Table 39

Weights of the Discriminant Function and Correlation of the Discriminant Function for Quantity: Cluster 4 Posttest

Statistic \ Variable	Add	Sub	Missing Addend	Partitions With Count	Partitions No Count
Quantity	.027	.024	.219	-.033	.004
Correlation	.60	.51	.98	.32	.38

Table 40 contains the results of the univariate analyses. Every variable was significant for Quantity. These significant F-ratios are a reflection of the correlations in Table 39 in that each variable contributed to the separation of the extensive quantity and gross quantity groups with the operations variables contributing more than the partition variables. Table 41 is a table of cell means for the two factors across

Table 40

Quantity versus Treatment Univariate Analyses of Variance:

Cluster 4 Posttest Variables

ariable	Addition		Subtraction		Missing Addend		Partition With Count		Partition No Count	
ource of ariation	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F
uantity (Q)	6.46	6.06**	8.79	12.00**	18.29	43.24**	2.49	4.78*	4.35	6.45*
reatment (T)	3.12	7.77**	1.38	1.88	.47	1.13	.00	<1	.05	<1
XT	.11	<1	.14	<1	.00	<1	.51	<1	2.29	3.40
rror	.40		.732		.42		.52		.67	

*(p < .05) **(p < .01)

the variables, and Table 42 is a correlation table for the variables. Hypothesis 1 was not supported for any of the variables. In fact, in the case of addition, the control children outperformed the experimental children. Hypothesis 2 was supported and can be extended to include addition.

It is now possible to make statements from the perspective of the pretest and posttest cluster 4 variables. On the pretest, the control children uniformly outperformed the experimental children on the subtraction problems (significantly) and the addition problems (nonsignificantly). On the posttest, the control children uniformly outperformed the experimental children on the addition problems (significantly) and

Table 41

Interaction Table for Cluster 4 Variables: Posttest

Variable	Addition		Subtraction		Missing Addend		Partition With Count		Partition No Count	
Treatment / Quantity	Con	Exp	Con	Exp	Con	Exp	Con	Exp	Con	Exp
Extensive	100%	79%	84%	62%	88%	79%	92%	83%	88%	71%
Gross	68%	38%	36%	25%	27%	17%	59%	71%	36%	62%

Table 42

Correlation Among Cluster 4 Variables: Posttest

	1	2	3	4
1. Addition	-			
2. Subtraction	.37*	-		
3. Missing Addend	.48**	.41**	-	
4. Partition With Count	.14	.31*	.44**	-
5. Partition No Count	.43**	.47**	.38*	.71**

*(p < .05) **(p < .01)

the subtraction (nonsignificantly). By inspection of the means, one can say that the experimental children did not improve a great deal from pre-to-post test on the addition and subtraction problems for Cluster 4 variables. One must remember, however, that objects were present during solution. For the addition and subtraction problems with no objects present, the experimental children uniformly performed quite well (see Table 33). The control children performed no better on the addition and subtraction problems without objects than with objects. The fact that the experimental children performed better (the gross quantitative comparers) on addition and subtraction problems without objects than with objects is not consistent with research reviewed (Steffe, 1966; LeBlanc, 1968; Steffe and Johnson, 1971); nor is it consistent with the results for the control children. The result is explainable in terms of the treatment. The experimental children were encouraged to use their fingers in computational work utilizing counting-all strategies for addition and subtraction. For these children, the counting-all strategies were personal and easily activated. Comparing the results of Tables 32 and 41 for the gross quantitative comparers, it is easy to see that the presence of manipulatable objects interfered with the solution strategies they had been taught. In the treatment, those children resisted using manipulatable objects, much prefering to use their fingers as aid to solution.

The experimental children also used the hand calculator to perform computations. But the results presented for the mental arithmetic test refute the possible interpretation that the experimental children had learned their facts and used them in solving problems without objects.

The missing addend problem remained quite difficult for the gross quantitative comparers for problems with objects present and was difficult for problems with no objects present. This difficulty is attributable to the solution strategies necessary for solution of the problem. Further discussion of the results for the missing addend problem is delayed until results of Cluster 5 variables are presented.

The results for the set partition problems are a curiosity. On the pretest, they were not correlated with subtraction or missing addend problems and were correlated only moderately with addition problems. These results cannot be attributed to test difficulty -- that some tests were too easy or too difficult for the sample. Moreover, as shown in Table 8 in the section Item Analyses, the internal consistency reliability of Cluster 4 variables were substantial on the pretests, except for the difficult missing addend problems. One can only conclude that, on the pretest, the two set partition tests functioned relatively independently of the other tests. This result was entirely unexpected and was not as theory might predict.

On the posttest, the extensive quantitative comparers had acquired the facility to solve the set partition problems to a greater extent than had the gross quantitative comparers. Consequently, one may say that Quantity was a readiness variable for learning to solve the set partition problems. Moreover, the test retained its substantial reliability on the posttest as shown in Table 11 in the section Item Analyses, so that the results can be accepted with confidence that the test functioned quite well.

On the posttest, there was a convergence of performance on Cluster 4 variables in that all were significantly correlated (Table 42) except Addition and Partitions With Counting. Partitions Without Counting correlated fairly well with addition, subtraction, and missing addend problems. This result is considered as reflection of beginning emergence of number facility on the part of the gross quantitative comparers.

It should be clear, however, that set partitions is not a precursor to addition or subtraction nor is it true that addition or subtraction is a precursor to set partitions. The capabilities emerge in the same age range, but as unrelated phenomena.

The relative magnitude of the correlations between Addition and Subtraction depends on the test conditions and also depends on the time of the year that tests are administered. On the pretest for problem solving with objects present, Addition correlated substantially with Subtraction and Missing Addend (Table 38), but Addition was not as strongly correlated with Subtraction and Missing Addend on the posttest (but correlated significantly) as on the pretest. On the test for problem solving without objects, Addition and Subtraction were correlated to a greater extent than Addition and Missing Addend (Table 34). This result was the opposite of the results obtained from the objects present pre-or-posttest. Moreover Subtraction was substantially correlated with Missing Addend for the test of problem solving without objects.

The above mixture of results of correlations among Addition, Subtraction, and Missing Addend variables indicates that underlying solution processes are not necessarily reflected in product scores. One would think that in a "natural state," children's success in solving subtraction and missing addend problems would be highly related due to counting strategies. But after "counting-all" strategies had been taught, solution of addition and subtraction problems should be highly related, but solution of missing addend problems would not be related to solution of addition or subtraction problems because of the almost certain necessity of "counting on" to solve the missing addend problem. The correlations in Table 34 and Table

42 do not support this reasoning.

Cluster 5. The variables included in Cluster 5 were Rational Counting On, Addition of Ordinal Numbers, Rational Counting Back, and Subtraction of Ordinal Numbers. The research hypotheses to be tested in this section are that (1) extensive quantitative comparers are able to (a) rational count-on, (b) rational count-back, (c) solve ordinal number addition problems, and (d) solve ordinal number subtraction problems to a greater extent than gross quantitative comparers; (2) the experimental gross quantitative comparers will outperform the control gross quantitative comparers on the rational counting-on and the ordinal number addition problems; (3) rational counting-on and addition of ordinal number problems are highly related; and (4) rational counting-back and ordinal number subtraction problems are highly related.

The multivariate F for interaction of Quantity and Treatment was not significant, ($F_{4,41} = .93$). The multivariate F for Quantity was significant ($F_{4,41} = 3.50$, $p < .05$). The multivariate F for Treatment, ($F_{4,41} = 1.00$), was not significant. Table 43 contains the weights of the discriminant function and correlations of the original variables with the discriminant function for Quantit

Table 43

Weights of the Discriminant Function and Correlational of Original Variables With Discriminant Function for Quantity: Cluster 5

Statistic \ Variable	Rational Count-On	Ordinal Addition	Rational Count-Back	Ordinal Subtraction
Weights	.119	-.078	.018	.105
Correlation	.74	.47	.62	.81

Table 44 contains the univariate analyses and Table 45 contains the cell means for all the variables in Cluster 5. The univariate analyses are consistent with the results in Tables 43. The significant F-ratios are consistent with the substantial correlations of the original variables with the discriminant function.

Table 44

Quantity versus Treatment Univariate Analyses of Variance: Cluster 5

Variable	Rational Count-On		Ordinal Addition		Rational Count-Back		Ordinal Subtraction	
Source of Variation	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F
Quantity (Q)	3.62	8.19**	2.42	3.36√	3.62	5.71*	7.22	9.77**
Treatment (T)	.43	<1	1.15	1.61	.12	<1	.14	<1
QXT	.57	1.28	2.43	3.39√	.22	<1	.45	<1
Error	.44		.72		.64		.46	

**(p < .01) *(p < .05) √(p < .08)

Table 45

Interaction Table for Cluster 5 Variables

Variable	Rational Count-On		Ordinal Addition		Rational Count-Back		Ordinal Subtraction	
Quality \ Treatment	Con	Exp	Con	Exp	Con	Exp	Con	Exp
Extensive	88%	87%	77%	71%	65%	58%	58%	54%
Gross	50%	71%	32%	71%	23%	21%	9%	25%

The interaction effect of Quantity and Treatment was significant at $p < .08$ for the variable Ordinal Addition. Inspection of Table 45 shows that the experimental gross quantitative group outperformed the analogous control group 71 to 32 per cent. An analogous result appeared for Rational Counting-On, but was not as strong as for Ordinal Addition. These results are educationally significant as counting-on strategies were emphasized in the experimental group. Apparently, the instruction was effective for the gross quantitative group. In fact, the gross quantitative comparers in the experimental group solved exercises like 6 + 3 = □ by counting-on three from six during the last week of instruction. No appreciable differences existed in the mean of the control and experimental groups for the two other variables. In the case of Rational Counting-Back and Ordinal Subtraction, the experimental treatment was not effective, suggesting more resistance to instruction for Rational Counting-Back than Rational Counting-On.

Table 46 contains the correlations among the variables of Cluster 5. Rational Counting-On is highly correlated with Ordinal Addition and

Table 46

Correlations Among Variables of Cluster 5

	1	2	3
1. Rational Count On	-		
2. Ordinal Addition	.72**	-	
3. Rational Count Back	.46**	.63**	-
4. Ordinal Subtraction	.47**	.66**	.68**

**($p < .01$)

Rational Counting-Back is highly correlated with Ordinal Subtraction. These results are as theory predicts.

Hypothesis (1) is supported in a multivariate and univariate sense. The variable Quantity appears to be a readiness variable for all variables of Cluster 5. Hypothesis (2) is considered as being supported for ordinal number addition problems even though statistically nonsignificant results were obtained for the interaction of Quantity and Treatment for Ordinal Number Addition. The results are strong enough to be considered as educationally significant. Hypothesis (3) and (4) are also supported by the correlations in Table 46.

Apparently, there was a training effect for rational counting-on strategies with tallying (ordinal number addition). The training effect did not transfer to the missing addend problems with no objects or with objects (Table 33 and Table 41). The latter two problem types were different than the ordinal number addition problems. The ordinal number addition problems had the objects of the segment visible and the remainder covered, so the child could use the visible objects to count initially. The missing addend problem with objects present during solution had 10 objects present but the sum was less than 10. Extra objects were then available for use. The missing addend problem with no objects did not have objects present, but rather referred to objects familiar to the children. Apparently, the visible objects in the ordinal number addition test activated available solution strategies. It must be pointed out, however, that the ordinal number addition test was just like the instructional tasks. The experimental gross quantity children's Counting-On behavior was apparently specific to the tasks given in instruction, but it is encouraging to note the improvement obtained. The fact it was not generalized across tasks is suggestive of a "process in the making."

Cluster 6. The variables included in Cluster 6 were Addition Product Score, Subtraction Product Score, Addition Time Score, and Subtraction Time Score. No hypotheses were advanced because there was no predictive theory from which hypotheses could be generated.

The multivariate interaction and main effects for Quantity and Treatment were not significant (Q: $F_{4,41} = 2.07$; T: $F_{4,41} < 1$). Even though Quantity was not significant in the multivariate analysis for

Table 47

Quantity versus Treatment Univariate Analysis of Variance: Cluster 6

Variable	Addition Product		Subtraction Product		Addition Time		Subtraction Time	
Source of Variation	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F
Quantity (Q)	2.57	5.30*	2.84	4.44*	1494	4.45*	958	2.74
Treatment (T)	.13	<1	.26	<1	494	1.28	696	1.99
QXT	.03	<1	.29	<1	756	1.66	624	1.79
Error	.48		.64		322		349	

*(p < .05)

$p < .05$, it was significant for $p < .10$. As three univariate analyses showed Quantity significant ($p < .05$) and Quantity was significant in the multivariate analysis ($p < .10$), the univariate analyses are presented in Table 47. Quantity was significant for addition product and time scores and for Subtraction product scores. Cell means are presented in Table 48.

Table 48

Interaction Table for Cluster 6 Variables

Variable	Addition Product		Subtraction Product		Addition Time		Subtraction Time	
Quantity \ Treatment	Con	Exp	Con	Exp	Con	Exp	Con	Exp
Extensive	85%	87%	73%	58%	10.4	11.6	19.6	18.9
Gross	59%	67%	41%	42%	29.6	14.8	35.8	20.7

There was a tendency for the subtraction exercises to be more difficult than the addition exercises and take longer to process. Because subtraction was presented in the experimental group, they could have outperformed the control group on the two subtraction variables. No such differences were observed in the analyses.

Quantity apparently is a fairly weak readiness variable for learning to mentally process addition and subtraction exercises and the time it takes to do them. A most plausible reason Quantity is not a strong readiness variable for Cluster 6 variables is that addition and subtraction exercises can be solved using counting-all strategies.

Table 49 contains the correlations of Cluster 6 variables. Low, and sometimes significant, negative correlations exist among the time and product variables. This result weakly supports popularly held beliefs that children who score addition and subtraction exercises correctly will

Table 49

Correlations Among Cluster 6 Variables

	1	2	3
1. Addition product	--		
2. Subtraction product	.38*	--	
3. Addition Time	-.28	-.30*	--
4. Subtraction Time	-.33*	-.26	.75**

take less time, on the whole, than children who score them incorrectly. The correlation of .75 between Addition Time and Subtraction Time is substantial.

Cluster 7. The variables included in Cluster 7 are Class Inclusion, Loop Inclusion, Nested Classification Task A, and Nested Classification Task B. The research hypotheses to be tested are as follows: (1) Quantity is not a readiness variable for any of the classification tasks, and (2) the experimental group will outperform the control group on the nested classification tasks and on the loop inclusion tasks.

The multivariate main and interaction effects for Quantity and Treatment were not significant (Q: $F_{4,41} = 2.29$; T: $F_{4,41} = .819$; QXT: $F_{4,41} = .819$). However, $F_{4,41} = 2.29$ is significant, $p < .10$. Consequently, univariate analyses are presented. Table 50 contains the univariate analyses for the variables and Table 51 contains the cell means for the variables.

No differences were attributable to Treatment or to the interaction of Treatment and Quantity. This result is somewhat surprising due

Table 50

Quantity versus Treatment Univariate Analyses: Cluster 7

Variable	Class Inclusion		Loop Inclusion		Nested Class A		Nested Class B	
Source of Variation	Mean Square	F	Mean Square	F	Mean Square	F	Mean Square	F
Quantity (Q)	5.19	1.33	7.09	6.39*	6.02	2.74	15.18	3.60
Treatment (T)	1.45	<1	1.25	1.13	.82	<1	.52	<1
QXT	2.91	<1	.88	<1	2.38	1.05	.02	<1
Error	3.91		1.11		2.27		3.96	

*($p < .05$)

Table 51

Cell Means for Cluster 7 Variables

Variable	Class Inclusion		Loop Inclusion		Nested Class A		Nested Class B	
Quality \ Treatment	Con	Exp	Con	Exp	Con	Exp	Con	Exp
Extensive	43%	27%	59%	61%	60%	63%	65%	69%
Gross	20%	23%	24%	44%	55%	40%	47%	51%

to the rather substantial amount of instruction given on loop inclusion and classification in general in the experimental group. Apparently, the children did not profit from the instruction. These results are not completely consistent with those of Johnson (1975), even though his tests were quite different than the four in Cluster 7. In that no differences existed between the experimentals and controls on Class Inclusion is consistent with the results of Johnson (1975). Moreover, the fact that Treatment is not significant for Loop Inclusion is consistent with the results of Johnson (1975), because to do items 1 and 2 on that test, the child had to go beyond the physical knowledge of the operational definition and reason logically. The third item could be done using the physical knowledge of the operational definition and the separation of the subregions formed by the intersecting loops. Even so, the first two items on the test did demand more than physical knowledge for solution and were a test of whether the comprehension of loop inclusion could be improved through the operational definitions given. The results are negative.

The results of the nested classification tasks are not completely consistent with the results of Johnson (1975). A child could score high on either of the nested classification tasks and do so through using the results of physical knowledge. The properties of the classes were all physical properties, so they should have been easily recognized by the children (round, polygonal shaped, white, button, nonwhite). The conclusion from the Johnson study was that physical knowledge pertaining to classification should be easily acquired by first grade children. While the results of the nested classification tasks do not contradict Johnson's results due to task difference, it is true that the children had a great deal of difficulty applying their physical knowledge in a problem setting (finding out what was in the box) which demanded the children use their physical knowledge. The problem was a discrimination problem that did not demand the use of class inclusion.

Table 52 contains the correlations of the variables in Cluster 6. All of the correlations are low except for the correlation of .50 between Task A and Task B, nested classification. Even this correlation, however, is lower than expected due to the nature of the tests. It was expected that class inclusion and loop inclusion would correlate rather substantially.

Table 52

Correlation Among Cluster 7 Variables

	1	2	3
1. Class Inclusion	-		
2. Loop Inclusion	.28	-	
3. Nested Classification A	.19	.29	-
4. Nested Classification B	.14	.14	.50**

**($p < .01$)

In that they did not, one should not use one to test transfer effects of instruction of the other in future studies.

Hypothesis 1 was not strongly supported in a multivariate sense. Quantity was significant for Loop Inclusion, so hypothesis 1 was not supported for this variable. Hypothesis 2 was not supported for either the multivariate or univariate analysis of va[illegible]

Correlations Among the Variables

The variable clusters were formed through logical analysis of task structure. Apparently, the logical analysis led to credible results except in all but a few cases as the within-cluster correlations generally were substantial. But because it was not of major interest to do a structural analysis of children's thought processes, factor analyses were not performed. Consequently, the presentation in this section is limited to correlations of the most interest. In order to obtain the correlations reported below, the 29 dependent variables were considered as a 29 element vector in a two-by-two multivariate analysis of variance. This procedure was used so that effects due to Quantity and Treatment would be eliminated statistically from the correlations.

Variables apparently requiring rational counting in solution. Some variables apparently requiring rational counting in solution may have been solved by processes other than rational counting. However, in all but a few cases, as shown in Table 53, significant correlations exist among variables apparently demanding rational counting for solution. The exceptions that exist are due to the across cluster correlations involving Cluster 1 variables (#S, #P, and #S + #P). The correlations within Cluster 1 variables were significant, the most important one being the correlation of .54 between #S and #P. The two others are spurious due to the definition of #S + #P. It is somewhat surprising that the correlation between #S and #P was significant due to the low reliabilities of the tests used to measure the variables. Of the remaining 21 correlations involving Cluster 1 variables, nine were not significant, seven were in the range .30 to .39, and the remaining five were in the range .43 to .50. All of the remaining correlations in Table 53 were significant.

Table 53

Correlations Among Variables Apparently Requiring Rational Counting in Solution*

	1	2	3	4	5	6	7	8	9
1. #S	--								
2. #P	.54	--							
3. #S + #P	.61	.61	--						
4. Missing Addend With Objects	.50	.39	.39	--					
5. Missing Addend Without Objects	.43	.16	.31	.44	--				
6. Between	.45	.48	.46	.53	.43	--			
7. Rational Counting-On	.16	.15	.30	.39	.35	.32	--		
8. Ordinal Addition	.32	.21	.21	.46	.56	.38	.72	--	
9. Rational Counting-Back	.23	.30	.27	.49	.30	.42	.46	.63	
10. Ordinal Subtraction	.33	.10	.16	.32	.44	.32	.47	.66	.68

* $|r| \geq .30$ Significant $p < .05$

Variables requiring at most point counting in solution. Table 54 contains the correlations among variables requiring at most point counting. Sixty of 105 correlations reported in Table 54 were not significant. Eight of the 26 correlations involving the two time variables were significant, but only marginally--significant correlations were observed for Just Before, Addition No Objects, Subtraction Objects, and Subtraction Product Score. The results weakly support the popularly held belief that children who score addition and subtraction facts correctly will take less time in computation and children who do addition and subtraction problems correctly will take less time in computation. But speed in computation is not negatively correlated universally with tasks requiring at most point counting.

Of the 26 correlations involving the two Partitions variables, 18 were not significant. Of the eight that were significant, four of them were with addition or subtraction problems. However, none of these four were greater than .47. All of the correlations involving Successor were not significant and all of the correlations involving Just After except two (Counting Back and Just Before) were not significant.

Table 2

Correlations Among Variables Requiring at Most Point Counting in Solution*

	1	2	3	4	5	6	7	8	9	10	11	12	13
1. Counting Back													
2. Just Before	.19	--											
3. Just After	.31	.37	--										
4. Successor	.18	-.03	.03	--									
5. Predecessor	.52	.34	.15	0	--								
6. Addition No Objects	.48	.31	.25	.07	.32	--							
7. Subtraction No Objects	.39	.48	.25	.10	.43	.55	--						
8. Addition Objects	.38	.14	-.02	0	.30	.39	.42	--					
9. Subtraction Objects	.39	.32	.20	0	.53	.37	.33	.37	--				
10. Partitions With Objects	.41	.21	.08	.04	.32	.22	.21	.14	.31	--			
11. Partitions Without Count	.32	.15	-.09	.01	.48	.22	.33	.43	.47	.71	--		
12. Addition Product Score	.19	-.20	.06	-.03	.03	.44	.30	.25	.31	.12	.07	--	
13. Subtraction Product Score	.19	.19	.07	.01	.23	.57	.46	.32	.26	.03	.12	.38	--
14. Addition Time Score	-.23	-.33	-.03	-.05	-.03	-.42	.29	-.08	-.39	-.28	-.34	-.23	-.30
15. Subtraction Time Score	-.17	-.30	-.07	.26	-.28	-.38	-.25	-.02	-.24	-.08	-.12	-.33	-.26

* |r| ≥ .30 significant $p < .05$

Variables apparently requiring rational counting vs. variables requiring at most point counting. Table 55 contains 150 correlations, 67 of which were not significant. The correlations give some indication of the relationship between rote and point counting and rational counting. The relationship is stronger than one might expect due to the different numbers of children who could point count but not rational count. However, there is a nesting characteristic between point counters and rational counters and likewise between point counters and rote counters by definition. Consequently, the correlations which are significant are a reflection of the nested character of the three major types of counting identified. In fact, the correlations in Table 55 appear to be more substantial than those of Table 54.

Correlations between classification and numerical variables. Table 56 contains the correlations between the classification and numerical variables. The variables Between, Just Before, and Number in S were the most consistently correlated with the classification variables. Class Inclusion, however, was significantly correlated with Just Before and Ordinal Subtraction. All of the significant correlations, however, were only marginally significant except one, which was .46.

As the tests given in this study were given in January and February of 1975, the question arises concerning the correlations between classification variables and numerical variables at the beginning of instruction in mathematics in the first and second grades. The low correlations reported here may be a result of instruction in school mathematics in the first grade. Moreover, it is possible that as children progress into second grade, the correlations between classification variables and numerical variables would increase due to the fact that class inclusion is easier for older children.

Table 55

Correlations Between Variables Apparently Requiring Rational Counting vs. Variables Requiring at most Point Counting*

Point \ Rational	Number in S	Number in P	Number In S + Number in P	Missing Addend With Objects	Missing Addend Without Objects	Between	Rational Counting-On	Ordinal Addition	Rational Counting-Back	Ordinal Subtraction
Counting Back	.25	.29	.42	.36	.29	.29	.29	.27	.19	.25
Just Before	.32	.13	.12	.20	.47	.49	.19	.31	.33	.30
Just After	.10	.20	-.10	.08	.07	.44	.02	.17	.04	.05
Successor	.09	-.07	.07	-.06	-.01	-.04	.17	.23	.01	.04
Predecessor	.36	.35	.36	.36	.32	.34	.20	.26	.29	.29
Addition No Objects	.28	.30	.38	.49	.33	.37	.54	.60	.47	.48
Subtraction No Objects	.50	-.39	.46	.44	.49	.39	.34	.41	.29	.28
Addition Objects	.37	.41	.32	.48	.27	.35	.32	.41	.47	.46
Subtraction Objects	.39	.44	.37	.41	.16	.40	.37	.35	.50	.42
Partitions With Count	.30	.40	.31	.44	.23	.44	.31	.13	.14	.32
Partitions Without Count	.46	.50	.35	.43	.21	.46	.36	.27	.23	.13
Addition-Product Score	.06	.14	.16	.22	-.02	-.02	.47	.40	.27	.41
Subtraction Product Score	.34	.24	.25	.39	.41	.15	.37	.47	.42	.43
Addition Time Score	-.12	-.23	-.24	-.39	-.32	-.25	-.47	-.40	-.42	-.26
Subtraction Time Score	-.09	-.03	-.15	-.31	-.18	-.15	-.29	-.15	-.31	-.22

* $|r| \geq .30$ significant $p < .05$

Table 56

Correlations Between Classification and Numerical Variables*

Classification \ Numeral	Counting Back	Just Before	Just After	Between	Successor	Predecessor	Addition With Objects	Subtraction With Objects	Partitions With Count	Partitions Without Count	Addition Without Objects	Subtraction Without Objects	Number in S	Number in P	Counting-On	Ordinal Addition	Counting-Back	Ordinal Subtraction	Missing Addend With Objects	Missing Addend Without Objects	Number in S + Number in P
Class Inclusion	-.13	.33	.24	.10	.03	.06	-.07	.26	.06	.05	.27	.19	.27	0	.20	.25	.16	.30	.08	0	-.12
Loop Inclusion	.04	.26	.09	.35	-.09	.17	.14	.25	.17	.24	.07	.08	.19	.18	-.13	.20	.24	.12	.16	.08	.16
Nested Class A	.13	.21	.06	.28	.05	.27	.21	-.11	.03	.14	.27	.25	.32	.29	.19	.34	.18	.22	.27	.34	.11
Nested Class B	.23	.11	.13	.46	.05	.13	.21	.12	.12	.14	.31	.18	.36	.47	.12	.26	.15	.20	.23	.25	.25

* $|r| \geq .30$ significant $p < .05$

In the fall of 1975, tests constructed by PMDC staff (see Apendix A. 12 and A.13) were administered to 132 entering first grade children and 97 entering second grade children in City, Southeast. Each of the first and second grade tests contained two items constructed to measure class inclusion. The first grade test also contained five other subscales: Elementary Counting (9 items); Advanced Counting (4 items); Problem Solving (6 items); Set Equivalence (8 items); and Before-After-Between (10 items). The Cronback Alpha reliabilities were as follows: Class Inclusion (.61); Simple Counting (.79); Advanced Counting (.59); Problem Solving (.67); Set Relations (.79); and Ordering Numbers (.86). Table 57 contains test-test correlations for the first grade test. It is apparent that the correlations of Class Inclusion with the five other subscales are negligible.

Table 57

Test-Test Correlations: PMDC First Grade Test*

	1	2	3	4	5
1. Elementary Counting	-				
2. Advanced Counting	.47				
3. Problem Solving	.57	.46			
4. Set Equivalence	.65	.39	.58		
5. Ordering Numbers	.72	.53	.65	.74	
6. Class Inclusion	.11	.07	.07	.19	.24

* $r \geq .23$ signifcant $p < .01$; $r \geq .19$ significant $p < .05$

The second grade test contained eight subscales other than class inclusion. The subscales, the number of items in each subscale, and the reliability of each subscale are as follows: Elementary Counting (7 items, .47); Advanced Counting (5 items, .78); Patterns (2 items, .70); Place Value (8 items, .95); Equivalent Name (6 items, .89); Ordering Numbers (4 items, .55); Addition-Subtraction (4 items, .58); Missing Addend (4 items, .59); Class Inclusion (2 items, .67). Table 58 contains test-test correlations for the subtests of the second grade tests. The correlation of class inclusion with the eight other subscales were low, the greatest being .37.

Table 58

Test-Test Correlations: PMDC Second Grade Test*

	1	2	3	4	5	6	7	8
1. Elementary Counting	-							
2. Advanced Counting	.34							
3. Patterns	.16	.37						
4. Place Value	.33	.80	.41					
5. Equivalent Names	.25	.49	.14	.34				
6. Ordering Numbers	.38	.58	.40	.57	.31			
7. Add-Sub	.39	.31	.27	.29	.21	.50		
8. Missing Addend	.24	.48	.50	.61	.19	.47	.45	
9. Class Inclusion	.13	.25	.15	.21	.12	.23	.37	.31

* $r \geq .25$ significant $p < .01$; $r \geq .20$ significant $p < .05$

The fact that class inclusion was not significant for the variable Quantity, did not correlate significantly with any variable of clusters 1, 3, 4, and 5 except for Ordinal Subtraction (this correlation was only marginally significant), and, on the PMDC Fall 1975 tests, had negligible or significant but low correlations with all other subscales, indicates that class inclusion is not related to numerical variables in a product sense. It appears reasonable, based on the data in Tables 56-58, to strongly conjecture that classification variables are only weakly related to numerical variables. It could be argued, however, that in order to solve missing addend problems or solve the ordinal addition and subtraction problems, class inclusion would of necessity intervene in the numerical reasoning. This observation does not weaken the conjecture because (1) the conjecture was made with regard to product socres rather than process scores and (2) it is conceivable that processes other than those involved in class inclusion produced the numerical product score in Clusters 1, 3, 4, and 5.

CHAPTER IV

Discussion of the Results

Quantitative Comparisons as a Readiness Variable for Learning First Grade Arithmetical Content

In the past, counting was not explicitly considered in studies (Steffe, 1966; LeBlanc, 1968; Steffe & Johnson, 1971; Mpiangu & Gentile, 1975) of Quantity as a readiness variable for learning first grade arithmetical content. Moreover, only a restricted collection of variables were considered in any one study, so that conflicting results are present across studies. In the current study, a wide variety of variables were included so that information on Quantity as a readiness variable could be obtained for the variables on a constant sample.

Variables Apparently Requiring Rational Counting in Solution

Theoretically, Quantity as a classification variable should be significant for any achievement variable apparently requiring rational counting (rational counting-on with or without tally and rational counting-back with or without tally) in solution. These achievement variables and their level of significance for Quantity are presented in Table 1. The variables are consistently statistically significant for Quantity except for #S, which had an associated internal consistency reliability of only .15. Missing Addend With and Without Objects was strongly significant (mean scores 84 vs. 20 and 72 vs. 20 percent, respectively for extensive vs. gross quantitative groups).

Further empirical confirmation that Quantity is a readiness variable for aspects of arithmetic apparently requiring rational counting are the consistently high scores of the extensive quantitative comparison groups across the variables in Table 2. It made little difference whether an extensive quantitative

Table 1

Achievement Variables Necessitating Rational Counting:
Level of Significance for Quantity

Variable	Number in S (#S)	Number in P (#P)	#S + #P	Missing Addend With Objects	Missing Addend Without Objects
Level of Significance	n.s.	.01	.01	.01	.01

Variable	Between	Rational Counting-On	Ordinal Addition	Rational Counting Back	Ordinal Subtraction
Level of Significance	.01	.01	√	.05	.01

√ Interaction of Quantity and Treatment significant

comparison group was the experimental or control group. This fact is displayed in Table 2. It must be remembered that the control group received little or no instruction on counting strategies, whereas the experimental group received explicit instruction on counting strategies. The fact that the extensive control

Table 2

Mean Scores on Achievement Variables
Necessitating Rational Counting: Percents

Group		#S	#P	#S + #P	Missing Addend With Objects	Missing Addend Without Objects
Extensive	E	54	63	81	79	75
	C	42	50	67	88	69
Gross	E	33	25	48	17	25
	C	41	41	48	21	14

Group		Between	Rational Counting-On	Ordinal Addition	Rational Counting-Back	Ordinal Subtraction
Extensive	E	79	87	71	58	54
	C	67	88	77	65	58
Gross	E	35	71	71	21	25
	C	54	50	32	31	9

children used counting strategies across variables strongly suggests those children possessed a counting scheme in the sense of Piaget's schemes.

The instruction on counting strategies given to the experimental gross quantitative comparison group resulted in improved group mean scores in the case of only two variables--Rational Counting-On and Ordinal Addition. The tasks used to measure these two variables were contained in the experimental treatment. Since Missing Addend was analogous to Ordinal Addition in logical structure, it offered an excellent test of transfer of the learned processes present for Rational Counting-On and Ordinal Addition especially since all children were given opportunity to solve missing addend problems (orally presented) in the experimental treatment. No such transfer took place as was revealed by the mean scores for Missing Addend With and Without Objects in Table 2. The improvement in rational counting for the gross quantitative comparers was specific, then, to the tasks on which the children received explicit training. They did not initiate learned counting strategies in novel tasks, whereas the extensive quantitative comparers performed uniformly well across all tasks involving rational counting.*

Beilin's (1969) concept of forward transformation is very useful in exploration of the lack of transfer of learned counting strategies on the part of the gross quantitative comparers. Apparently rational counting-on with and without tally was possible for the experimental gross quantitative comparers, but the lack of initiation in novel but closely related tasks suggested that the counting strategies were processes "in the making." They had not attained the status of mental operations but rather were learned algorithms or procedures for solving certain tasks. It is in this sense that Quantity is a readiness

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*Counting-Back, Ordinal Subtraction and #S were more difficult than the others.

variable for learning rational counting.

The above results differ from those observed in the study by Steffe & Johnson (1971). In that study, Quantity was not significant for the missing addend problems in the presence or absence of objects. The fact that Quantity was significant for Missing Addend in Table 1 strongly suggests the mathematical experiences the children engage in are instrumental in acquisition of mental operations associated with rational counting. Because teachers do not explicitly teach rational counting, it is apparently enough that extensive quantitative comparers be exposed to problems which stimulate rational counting. The textbook series used by the teachers in the Steffe & Johnson (1971) study (Morton, Grey, & Rosskopf, 1965) did not emphasize missing addend problems whereas the textbook used in this study did.

The ability to make extensive quantitative comparisons signals the presence of a synthesis of Groupings I and V if Piaget's theory is to be believed. But as Piaget's theory does not account for rational counting, exact relationhips are not to be expected between Quantity and Missing Addend. This expectation is empirically verified by Cluster 4 pretest variables as the mean score for the total extensive quantitative comparison group was 41 percent for the missing addend problems. This rather modest mean score is comparable to the means observed by Steffe & Johnson (1971) for extensive quantitative comparers. That the gross quantitative comparison group in this study performed at about the same level as the extensive quantitative comparison group in the Steffe & Johnson (1971) study can be explained by the different times of test administration--October vs. May of the first grade.

The data strongly suggest that counting is not developmental in the sense of the grouping structures but rather the emergence of the grouping structures allows children's culturally induced rote and point counting capabilities to take on numerical meaning not possible prior to the emergence of the grouping structures. Mathematical instruction apparently serves to solidify the counting schemes and to raise them to the level of rational counting-on and rational counting-back for the extensive quantitative comparers. For the gross quantitative comparers, mathematical instruction alone apparently is not sufficient for acquisition of rational counting-on or rational counting-back at the level of mental operations. For the latter group, the role of mathematical instruction is still not understood for the acquisition of rational counting-on and rational counting-back.

Cardinal information from ordinal information. The extensive quantitative comparers could obtain cardinal information from ordinal information better than the gross quantitative comparers in the case of counting forward (#P) but not in the case of counting backward (#S). Moreover, the extensive quantitative comparers could utilize the hints in the cardinal-ordinal number tasks (#S ± #P) to a greater extent than the gross quantitative comparers. In symbolic notation, if P is a finite set ordered by "precedes," S a segment of P and Q the remainder, and r some element of Q, but not the minimal element; then extensive quantitative children when told the position of r could count forward to determine the number in P better than the gross quantitative children. In the tasks, S was screened from view and Q was visible. Moreover, upon prompting by giving one (or more) adjacent (or successively adjacent) positions to r, the extensive quantitative comparers could utilize the prompts and solve the tasks to a greater extent than the gross quantitative comparers. There was no evidence that the extensive quantitative comparers could count backwards better than the gross quantitative comparers to determine the number in S.

Other than the low reliabilities of the measures of the variables #S and #P, an explanation exists for the reason there were not greater differences between the extensive and gross quantitative comparers. Because the children were told the position of the 10th element in Task A and the fifth element in Task B (see Appendix A.2), it would be possible for the children to employ point counting behavior to find the number in P. Moreover, as all of the children could point count to at least seven (all quite byond seven at the time the tasks were administered) the possibility the children utilized point counting is very strong. It has also been observed that children who cannot rational count-on or rational count-back can, given a particular number name, orally count-on or count-back from that number. The basis for this observation is the measure for Cluster 5 variables in the preliminary items (see Appendix A.6). The child being told that a particular object was 10th or 5th certainly could have elicited rote counting-back or rote counting-on. That some gross quantitative comparers correctly found the number in S could be a result of knowing three comes before four and seven comes before eight on a rote counting basis. Conflict must be introduced into the task design in such a way to separate the false positives (children who scored the item correctly but who could not rational count-back) from the true positives. One way would be to add objects to S and require the children to (1) find the new number of S and (2) find the position of some r of Q.

Between. The variable Between not only demanded that the child had synthesized the before and after relations, but also that he be able to synthesize internalized counting with the relations. The child was asked to find a number between 8 and 12, then another, then another, then another. The questioning sequence assumes the child gave only one, then one more, then one more. The last question was asked to be assured the child would not give a number between 8 and 12 after exhuasting the possibilities. Of course, the questioning sequence was altered to fit the child's responses. The child was then asked to give a number between 8 and 6. One point was given for each correct number cited. The way the item was scored allowed children to obtain a nonzero score on the basis of guessing or merely employing the "after" or "before" relation along with the counting sequence. Moreover, a child could give too many numbers and still receive a nonzero score. Even so, Quantity was significant (extensive vs. gross quantitative comparison group means 73 vs. 44 percent) for Between. With more stringent scoring procedures, one would expect even greater differences. Improved scoring procedures should penalize children for giving too many numbers as well as not enough. The reason children were not penalized for the former was that it is possible for children to be induced to write a number as being between two others because of the demands placed on him by the questions. If a child thinks he has them all and another is asked for, he may give one just to please the experimenter. Consequently, caution must be exercised in task design to minimize the possibility of false incorrect responses.

Point Counting in Sums and Differences

Achievement variables which require only point counting in solution processes may be significant for Quantity if the solution processes require more than immediate application of an algorithm. Addition and subtraction problems with objects and without objects and addition and subtraction product scores required only point counting in the solution process. These variables and their level of significance for Quantity are presented in Table 3.

Table 3

Achievement Variables Necessitating Point Counting: Level of Significance for Quantity

Variables	Addition With Objects	Subtraction With Objects	Addition Without Objects	Subtraction Without Objects
Level of Significance	.01	.01	√	a

Variables	Addition Product Score	Subtraction Product Score
Level of Significance	.05	.05

√Interaction of Quantity and Treatment significant $p < .07$.
[a]Interaction of Quantity and Treatment significant $p < .09$.

The extensive quantitative comparers invariably initiated a point counting procedure in the solution of items on the test of quantitative comparisons. They point counted regardless of whether the item could be solved by a gross quantitative comparison or demanded what Piaget would call an extensive quantitative comparison. The fact that the gross quantitative comparers did not initiate point counting techniques (or initiated them, but relied on perceptual cues for the answer) in the test of quantitative comparisons was strong enough to carry over to the problem-solving contexts especially for the control gross quantitative comparison group.

Treatment was significant in the case of Addition With Objects in favor of the control group. However, without objects, the experimental gross quantitative comparers outperformed the control gross quantitative comparers for both Addition and Subtraction (Table 4). These results can be attributed directly to the treatment. The children in the experimental groups were encouraged to use their fingers to aid computation and soon abandoned using physical objects to perform sums and differences. The physical objects in fact impeded them in performing counting-all procedures in finding sums. Evidently being a gross quantitative comparer does not impede a child from learning counting-all procedures in finding sums and differences if the child is allowed to use his fingers.

The above results differ from those of past studies (Steffe, 1966; LeBlanc, 1968; Steffe & Johnson, 1971). In those studies, presence of manipulatable objects facilitated problem solution. The above results strongly suggest that children's superior performance in the presence of manipulatable objects, in finding sums or differences is a result of a universal educational practice of children's being encouraged to use objects or pictures of objects but discouraged in using fingers as an aid to calculation. The hypothesis is strong that the use of fingers in performing arithmetical calculations is critical in the formation of <u>mental operations</u> associated with counting, finding sums, and finding differences. Physical objects are external to the child and offer little in the way of sensory impressions. Children often finger count by touching their lips, nose, cheek, or other facial parts, thus gaining sensory impressions of units in time sequences.

The fact that the control extensive quantitative comparers outperformed the control gross quantitative comparers on every variable in Table 4 agrees with the studies cited above except for subtraction in the case of Steffe & Johnson (1971). In that study, Quantity was not significant for subtraction. However, mean scores were low, suggesting that the children had not been given experiences with finding differences.

Table 4

Mean Scores on Achievement Variables Necessitating Point Counting: Percents

Group		Addition With Objects	Subtraction With Objects	Addition Without Objects	Subtraction Without Objects	Addition Product Score	Subtraction Product Score
Extensive	E	79	62	79	88	87	58
	C	100	84	88	81	85	73
Gross	E	38	25	79	75	67	42
	C	68	36	50	41	59	41

Successor, Predecessor, Just Before, Just After and Counting-Back

The variables Successor, Predecessor, and Counting-Back involved at most point counting capabilities but did not require rational counting. Of these variables only Counting-Back was significant for Quantity--77 vs. 54 percent for the extensive vs. gross quantitative comparison groups, respectively. That Successor and Predecessor were not significant for quantity is not surprising (they were very easy items for all children). Essentially, rote counting capabilities would be sufficient for task solution for these two variables, just as was true for Just Before and Just After (also not significant for Quantity). That Quantity was significant for Counting-Back is consistent with the operational definitions of extensive and gross quantity.

The variables immediately above overlap with those studied by Mpiangu & Gentile (1975). They required children to count in both directions between 0 and 11; count by twos; find the name of a missing number in a given sequence (1-10); find numbers just before, just after, or between others in the sequence (1-10); and find the correct answer and provide justification to an item such as "These two make how many?" Mpiangu & Gentile (1975) found that number conservation is not necessary for learning the above content. The fact that Successor, Predecessor, Just Before, and Just After were not significant for Quantity is consistent with Mpiangu & Gentile's (1975) results. Moreover, the statistical significance for Counting-Back does not contradict their findings because the gross quantitative comparers had a mean score of 54 percent. Undoubtedly, both the extensive and gross quantitative comparers gained facility in point counting. The addition and subtraction product scores displayed in Table 4 also are consistent with Mpiangu & Gentile's (1975) findings. However, exception must be taken to their conclusion that conservation of number is not a necessary requirement for learning arithmetic in that it is overly simplistic, as shown by data in this study. When rote counting or point counting is all that is required of the task, gross quantitative comparers are quite capable of being trained to perform the task as the knowledge is in the main what Piaget (1964) has called physical knowledge. The training in rational counting, however, was narrowly acquired by the experimental gross quantitative comparers. As already noted, it is doubtful that the learned counting behavior was at the level of mental operations. It is in this sense that Quantity serves as a readiness variable for learning arithmetic rather than prohibiting acquisition of the simplest arithmetic skills. Curriculum designers who rely on learning hierarchies can be confident of task specific training but not horizontal or vertical transfer in the case of gross quantitative comparers. Even so, it should not be concluded that task specific training is necessarily undesirable. Only further research of a longitudinal nature will answer the question of the utility of training gross quantitative comparers to count-on and count-back.

If any future investigators include Just Before or Just After as variables, they should design the tasks to include five to six items per measure and use a less-than-10, 10-20 and 20-99 trichotomy to insure a range of responses. But at the present time, the best evidence available indicates that Quantity is not a readiness variables for the two relations.

Successor and Predecessor fall into the same category as Just Before and Just After. Improved task design is necessary if the variables are to be used in future experiments.

Counting-back. The variable Counting-back deserves special mention. The test (Appendix A.2) was designed sequentially in that if a child could not point count-back from 8, he would not be asked to point count-back from 12, and if he could not point count-back from 12, he would not be asked to rote count-back from 15. Of course, in the case of success, the child would proceed through the task. The sequential nature of the test was predicated on the assumption that children can point count-back from a digit before they can either point count or rote count-back from a two digit number because of the familiarity of children with the digits 1-9. The fact that Counting-Back was significant certainly resides in the sequential nature of the test. The extensive quantitative comparers had a mean score of 77 percent vs. a mean score of 54 percent for the gross quantitative comparers. These means along with the fact that 2/3 of the gross quantitative comparers could not rote count back from 15 confirms that the basis of the significance of the variable is the capability to rote count-back from 15. Critical tests of Quantity as a readiness variable for learning counting-back skills would be the familiarity with two digit numbers or the capacity to acquire such familiarity. Such tests were not made in this study.

Partitions

As noted in the section on Quantity and set partition, Piaget demonstrated that a child's "additive composition" of number developed in stages synonomous with those associated with cardinal and ordinal number. Consequently, the variable Partition should be related statistically to Quantity. Partition was assessed at two different times--prior to the treatment and after the treatment. On both pretest and posttest, the internal consistency reliabilities were quite substantial for Partitions With Count and Partitions Without Count (all at least .80 except one, which was .72). As shown in the analyses of Cluster 4 variables, Partitions With Count and Partitions Without Count contributed negligibly to the separation of the gross and extensive quantitative comparison groups in the discriminant function in case of the pretest and only nominally on the posttest. Quantity was not significant for Partitions With Count nor Without Count on the pretest. On the posttest Quantity was significant ($p < .05$) for Partitions With Count and Without Count with a strong suggestion of a Treatment by Quantity interaction. Mean scores are presented in Table 5. The significance for Quantity on the posttest must be discounted because the same general pattern of scores were present on both test adminstrations. The one notable exception is that the extensive quantitative comparers improved on the measure for Partitions Without Count to a greater extent that did the gross quantitative comparers. It must be remembered that the experimental group received instruction on set partitions whereas the control group did not, having followed the mathematical curriculum as exemplified by the textbook.

Table 5

Mean Scores for Partitions: Percents

Test		Pretest		Posttest	
Group \ Variable		Partitions With Count	Partitions Without Count	Partitions With Count	Partitions Without Count
Extensive	E	75	42	83	71
	C	75	38	92	88
Gross	E	63	50	71	62
	C	38	25	59	36

The above results are at variance with the results of Piaget reviewed in the section <u>Quantity and set partition</u>. Evidence is very weak that Partitions and Quantity are related either before formal instruction in the first grade or after such instruction.

Classification

Even though there was a general trend for Quantity to be significant for the four classification variables, only Loop Inclusion was statistically significant. In view of the substantial internal consistency reliabilities reported in Table 20 of the Item Analysis section for the four classification variables, the results can be interpreted with confidence the tests were measuring what they were designed to measure. In the case of Class Inclusion (reliability .90) the results are consistent with Dodwell's (1962) results that class inclusion and provoked and unprovoked correspondence develop independently.

The nonsignificance of Quantity for the nested classification tasks is not suprising in view of the fact that they were designed to test Stage 2 classification behavior. The significance of Quantity for Loop Inclusion was due mainly to the low mean score (24%) of the control gross quantitative comparison group. The mean score (44%) for the experimental gross quantitative comparison group can be attributed to the treatment. There does seem to be a relationship between Quantity and Loop Inclusion, although weak.

Class Inclusion as a Readiness Variable for Learning First Grade Arithmetical Content

No statistical tests were possible on Class Inclusion as a categorization variable. Only 12 children out of 107 scored at least two on the class inclusion test in late September, 1974. Consequently, only children for whom evidence was present they could not solve the class inclusion problem were admitted to the study. There was no hope that Class Inclusion would be related to Quantity on the pretest because of the poor performance on the class inclusion test. But because every child in the sample could not solve the class inclusion problem, judgments could be made on the necessity of class inclusion for learning first grade arithmetical content as measured by the posttests. These judgments have to be based on heuristical argument. Evidence would be present for Class Inclusion as a readiness variable for some other variable in any of the following cases.

Case 1. If Treatment were not significant and a strong correlation existed between Class Inclusion and the other variable on the posttest.

Case 2. If Treatment were significant and a strong correlation existed between Class Inclusion and the other variable on the posttest.

Case 3. If each child who scored well on the other variable could also solve the class inclusion problem.

Evidence would be absent for Class Inclusion as a readiness variable for some other variable in any of the following cases.

Case 1. If Treatment were not significant and a negligible correlation existed between Class Inclusion and the other variable on the posttest, given comparable group mean scores.

Case 2. If Treatment were signifcant and a negligible correlation existed between Class Inclusion and the other variable on the posttest, given comparable mean scores.

Case 3. If the mean scores for the other variable significantly exceeded the mean score for class inclusion.

Loop Inclusion, Partition, and Nested Classification

The class inclusion test was administered as a posttest to ascertain possible improvement in class inclusion scores. The treatment contained class inclusion activities, partitioning activities, and loop inclusion activities, so a treatment effect could have been possible for classification variables. However, treatment was not significant for any classification variable, indicating that classification per se is resistant to training, especially class inclusion. The result that class inclusion was resistant to training is consistent with the conclusion made in the review of literature in the section Class inclusion and arithmetic. There, it was concluded that if the goal of classification activities is to influence class inclusion as a scheme (or a relational structure), then it would be resistant to training. The results are at variance with those of Kohnstamm (1968) as he observed dramatic improvement in class inclusion through training. The results are in agreement with those of Inhelder & Sinclair (1969) and Johnson (1975). Both Inhelder & Sinclair and Johnson used problems of a different form in the testing than were used in the training just as was the case in this study. Apparently, this may have been enough to wash out any possible training effects in this experiment. Even though only two-stage problems were used, they contained content different than that used in instruction, except for one of the four items. But the difficulty of that item (item 4) was not discernibly different than the other four items (see Table 18, Item analyses). A more plausible explanation for the lack of training effects than change in item form is that there was a three months time interval between the classification experiences and the classification tests. But if class inclusion had been induced by the treatment as a flexible functioning scheme, the three months delay should not have been important, as one characteristic of operativity is that loss of operational concepts does not occur due to forgetting. Consequently, the treatment was not effective especially for inducing class inclusion.

Only 32 percent of the total sample answered correctly at least 3 out of a possible 5 items on the class inclusion posttest. As Quantity was not significant for Class Inclusion, one cannot predict gain in class inclusion proficiency from Quantity. Moreover, a lack of demonstrated knowledge of class inclusion at the beginning of the experiment did not deter that 32 percent from gaining class inclusion. But specific experiences in classification, as noted above, did not enhance acquisition. Apparently, there are more important factors than specific training in acquisition of class inclusion.

Because Treatment was essentially not significant for Partitions With Counting, Partitions Without Counting, and Loop Inclusion after substantial instruction took place, one may be tempted to conjecture that lack of class inclusion had a dampening effect on training of these variables. But as variables other than Class Inclusion could account for lack of treatment effects, posttest correlations must be considered in the heuristical analysis. The correlation between Loop Inclusion (mean, 46 percent) and Class Inclusion (mean, 31 percent) was only .28 and not statistically significant. So the Loop Inclusion and Class Inclusion tests functioned independently. One would expect that in the absence of a treatment effect the two variables would improve synchronously over time if they were both part of general schemes of classification, one logical and one infralogical. The correlation of .28 does not support the thesis that Class Inclusion is a readiness variable for acquisition of loop inclusion. Moreover, the mean scores on the loop inclusion test do not support the thesis that the infralogical operation would follow the logical operation in development, as item difficulties were comparable except for the last item of the loop inclusion test which was high, indicating an easy item.

The correlations between Class Inclusion and Partitions With Counting and Class Inclusion and Partitions Without Counting were only .06 and .05 respectively. The mean scores on the three tests were 13, 75, and 65 percent for Class Inclusion, Partitions With Counting, and Partitions Without Counting. On the pretests, the respective means were 0, 56, and 36 percent. The relatively high means of the set partitions tests and the essentially zero correlations should discourage any further conjecture that Class Inclusion and Partitions are related variables, one being a readiness variable for the other.

Treatment was not significant in the case of either of the nested classification tests and the correlations between Class Inclusion and the two Nested Classification variables were only .19 and .14. But Class Inclusion should not be expected to be a readiness variable for Nested Classification due to the way Nested Classification was tested. Only Stage 2 classification behavior was required for task performances on the Nested Classification tests.

Rational Counting

In the case of the tasks designed to assess the ability of children to obtain cardinal information from ordinal information, it was argued in the section The Achievement Tasks that Class Inclusion should be a readiness variable for acquisition of the ability. As every ordered set P can be thought of as the union of a segment S and remainder Q (P = S + Q), if a child is told the position of some element of Q, say q, and asked to find the number of elements of S (or P), class inclusion is logically part of the task. The child, it would seem, must regard S as being part of P, and S and Q together comprising P. As the tests of the #S and #P variables are of low reliability, only conjectual statements may be made.

Nine children correctly answered both of the items for the #S variable and twelve correctly answered both of the items for the #P variable. Correlations between Class Inclusion and #S, #P, and #S + #P were .27, 0, and -.12 respectively, none of which were significant. Consequently, some children acquired capability to obtain cardinal information from ordinal information independently of their ability to solve the class inclusion problem. These results, however, are tenuous due to low internal consistency reliabilities for the test used to measure the variables.

Cluster 5 variables were of greater interest than were Cluster 1 variables because the measures were more reliable and rational counting strategies were clearly used in task solution. For the same reason that was given for Cluster 1 variables, Class Inclusion should be a readiness varible for Counting-On, Ordinal Addition, Counting-Back, and Ordinal Subtraction. The internal consistency reliabilities for these measures were .50, .84, .61, and .47, respectively. The means were 75, 64, 42, and 38 percent, respectively. The reliabilities are substantial enough to interpret the data, albeit with some caution for Ordinal Subtraction. The analyses of variance showed Treatment as significant for the gross quantitative comparers in the case of Rational Counting-On and Ordinal Addition. The means for the experimental group in the case of these two variables were 71 percent for both. However, the posttest correlations of Class Inclusion with the four variables (as ordered above) of .20, .25, .16, and .30 (considered in conjunction with the mean scores) does not lead to the conclusion that Class Inclusion is a readiness variable for acquisition of rational counting strategies.

Addition, subtraction, and missing addend. Even though counting-all procedures were sufficient for solution of the subtraction problems, it is somewhat surprising that the extensive quantitative comparers had a mean score of 80 percent on the pretest. The gross quantitative comparers had a mean score of 25 percent. These mean scores destroy any illusions that Class Inclusion is a readiness variable for solution of simple subtraction problems early in the first grade year. Even though the extensive quantitative comparison group had a mean score of 42 percent for the missing addend problems, on the pretest, it seemed possible that Class Inclusion still would be a readiness variable for acquisition of an ability to solve such problems as the gross quantitative comparers had a mean score of only 2 percent. On the posttest, the analogous mean scores were 85 and 22 percent. But the posttest correlation between Class Inclusion and Missing Addend with Objects was .08. In the face of such a low correlation, one would not conjecture that Class Inclusion is a readiness variable for acquisition of solution to missing addend problems in the presence of objects.

Because Treatment was effective for Addition and Subtraction Without Objects, it could have been the case that Class Inclusion was a readiness variable for solution to orally presented addition (mean 78 percent) and subtraction (mean 71 percent) problems. If so, those children who solved such problems should be able to do the Class Inclusion problems. However, posttest correlations were only .27 and .19 between Class Inclusion and Addition and Subtraction Without Objects, respectively. There was a 0 correlation between Class Inclusion and the Missing Addend Without Objects (mean 48 percent).

The above results are consistent with those of Dodwell (1962) and Vitale (1975). Dodwell observed that class inclusion and provoked and unprovoked correspondence apparently develop independently and Vitale observed small correlations between class inclusion and addition and subtraction computation.

Class Inclusion as a Correlate of Arithmetic Achievement

Not being satisfied with the results of Class Inclusion as a readiness variable for acquisition of rational counting, problem solving, loop inclusions and interrelation of cardinal and ordinal number, class inclusion items were included as part of entering first- and second-grade mathematics tests constructed by PMDC staff (see Appendix A.12 and A.13). Class Inclusion correlated nonsignificantly with all first-grade number scales (Table 57 in Correlations Among the Variables). In case of the entering second-grade test Class Inclusion correlated low but significantly with Advanced Counting, Place Value, Ordering Numbers, Addition-Subtraction, and Missing Addend.

Based on the above data, it is reasonable to strongly conjecture that Class Inclusion as measured is only weakly related to numerical variables. It is not tenable to regard Class Inclusion as measured as a readiness variable for achieving arithmetical content tested in Clusters 1, 3, 4, 5, and 7 and tested in the first- and second-grade PMDC tests. Where does this leave Piaget's theory that number in children is a synthesis of classification and seriation?

The item content for the Class Inclusion items was of a pictorial nature and perceptually distracting. Moreover, as Johnson (1975) has shown, children regard intersecting ring items as separating the occupied region into disjoint rather than overlapping subregions. A similar phenomenon may be operating with regard to pictorial items--children may regard the problem one of comparing two subclasses due to the overwhelming perceptual features of the stimulus configurations. As a result, the class inclusion test may be conservative in that too many false negatives occur (children who have the potential of solving the class inclusion problem but fail). This conjecture is plausible due to the fact that class inclusion should intervene by necessity in solution to Cluster 5 variables and missing addend problems. However, if such is the case, then class inclusion would be present each time a child solved a numerical reasoning problem logically requiring class inclusion. In such cases, there would be no need to assess class inclusion because it would be synonomous with such numerical reasoning. In any case, then, it is questionable whether class inclusion needs to be considered as a readiness variable for learning arithmetical content, unless new measures are developed which are related, but do explain, numerical reasoning.

Piaget's theory may be still intact as it concerns number. A critical test would entail a class inclusion measure that did not depend on pictorial items containing distracting features but yet would be different than tests implied by the question "Which are there more of in the whole world, children or boys?" The problem associated with pictorial tests have been pointed out. The problems with the latter test are numerous, one of the most obvious being the necessity of attempting to imagine all of the children in the world. A universe of objects must be selected that is comprehendable by the child but yet does not contain perceptually distracting features. It has been conclusively shown in this study that such distracting features have nothing to do with pure numerical reasoning.

The Treatment

Content was included in the treatment not normally included in the mathematics programs for grade one. The features of the treatment were the inclusion of classification activities, set partitioning activities, counting activities by levels, the learning instructional phases for addition and subtraction, problem solving activities for addition and subtraction, the hand-held calculator for drill on basic facts, and the individual nature of the mathematics instruction. The treatment was included in the study to control the mathematical experience of the children in the experimental group. There was little interest in accelerating the learning of a particular topic per se. But there was a great deal of interest in determining the effects of particular mathematical experiences on different groups of quantitative comparers and on various closely related mathematical topics.

In short, the emphasis was placed on understanding the role of mathematical instruction on the development of mathematical concepts for identifiably distinct groups of learners.

It must be emphasized that the achievement variables are performance variables and consequently do not measure the important aspect of how the children progressed through instruction. Various comments will be made throughout this section concerning observations made during instruction. These comments are a result of daily observations of the children as they progressed through instruction and are offered to shed further light on various results. The 29 variables identified do give a good picture of a cross sectional nature of where the children stood at the time of testing.

Classification and Partitioning

There were no significant differences due to treatment for any of the four classification variables (Class Inclusion, Loop Inclusion, Nested Classification A, or Nested Classification B) nor did treatment interact with Quantity for any of the four variables. Moreover, treatment was not significant for either Partition With Counting or Partition Without Counting nor did Treatment and Quantity interact for either of the latter variables. In the case of Partition Without Counting, it appeared that Treatment and Quantity should have interacted but the mean scores were a reflection of how children began on the pretest for set partitions (see Tables 37 and 42 in the section Analyses of Variance).

Three instructional weeks were devoted to classification, set partitioning and loop inclusions. The total instructional time amounted to 12 instructional days of approximately 45 minutes per day. The children enjoyed the instructional activities and seemed at the time of instruction to profit. The basis of the instruction was operational definitions. For example, in the classificational activities, the terms "and," "or," "not," "some," and "all" were clarified for the children through their actions on animal cutouts and toys. The children were required to follow directions such as "select some of the animals." Corrections were made in the case of incorrect performance. In this way, the terms were defined operationally. The children had little trouble in learning the operational meanings of the terms. Class inclusion activities were also emphasized utilizing the terminology developed in an attempt to train the children to focus on all the animals when comparing all the animals with some of the animals. It was felt that children may in many cases focus on some of the animals (e.g., dogs) in comparison of all the animals (e.g., dogs and cats) with some of the animals (cats). At the time, clarification of the terminology seemed to help most of the children in solving class inclusion problems. But it was difficult at the time to know whether the children were being trained to respond to the verbal cues all and some, knowing all is more than some. The results of the class inclusion test (Tables 50 and 51 in the section Analyses of Variance) in the posttests support the contention that no real improvement was the case for class inclusion problems. In fact, the control children had a greater mean score than the experimental children (41 vs 25 percent, respectively). Moreover, the extensive quantitative comparers in the experimental group did not do any better than did the gross quantitative comparers in the experimental group on the class inclusion posttest. Had Quantity been a readiness variable for learning class inclusion, the extensive experimental group should have gained a great deal from the instruction on class inclusion.

There are good theoretical reasons for hypothesizing that the experimental extensive quantitative comparers would in fact acquire the facility to solve the class inclusion problem. In the section Number in Piagetian Theory, it is illustrated how hierarchical classifications are involved in children's conception of number. Number intervenes into classificational hierarchies through the quantifiers "a," "some," "none," and "all," which must carry numerical meaning in Piaget's analysis of the interrelations of classification, relations, and number for children to conceive of hierarchical classifications. Children who are extensive quantifiers have, in theory, the notion of a unit essential to extensive quantification in Piaget's theory (see the section Quantity as viewed by Piaget). The unit is also essential to the child's conception of number as outlined in the section Number in Piagetian Theory. Consequently, those children who were extensive quantitative comparers who failed to solve the class inclusion problem should have done so for reasons other than not possessing the conception of a unit, failing to attach numerical meaning to the quantifiers "a," "none," "some," and "all," or not being able to conceive of hierarchical classificational systems. Possible reasons for failing to solve the class inclusion problem for these children are the dominance of the perceptual configuration of the tasks or not understanding the verbal direction. The instruction in the treatment was organized to eliminate these two possible reasons for failure of extensive quantitative comparers to solve the class inclusion problem. It was not expected that the gross quantitative comparers would acquire the facility due to lack of Grouping I capabilities--not being able to conceive of hierarchical classifications. The extensive and gross quantitative comparers in the control group did not receive instruction on classificational systems.

The lack of a statistical (or educational) significant Quantity by Treatment interaction for Class Inclusion (Table 50 in the section <u>Analyses of Variance</u>) strengthens the conclusion made earlier in this chapter that Class Inclusion is resistant to training. In that discussion, extensive vs gross quantity was not highlighted. It can now be concluded that class inclusion is resistant to training regardless of whether the children are extensive quantitative comparers or gross quantitative comparers. The lack of interaction of Quantity and Treatment also strengthens the conclusion that class inclusion need not be considered in future studies as a readiness variable for learning first grade arithmetical content unless dramatically different measures for class inclusion are devised. Under the hypothesis of hierarchical classificational schemes being an integral aspect of number and therefore extensive quantity, the instruction given in the treatment on classification should have been assimilated into operational schemes available for classification. The lack of the aforementioned statistical interaction throws into question the premise that the extensive quantitiative comparers possessed hierarchical schemes of classification which were not activitated on the pretest of class inclusion. The premise that number precedes hierarchical classification certainly deserves serious consideration. The question of merger is also interesting.

The loop inclusion tasks were designed to measure the application of Grouping I to spatial content--as a measure of of Piaget's infralogical operations (Sinclair, 1971). It is generally accepted that infralogical operations develop later than logical operations so it was hypothesized that Quantity would not be a readiness variable for learning loop inclusion. But due to the experience given to the children on loop inclusions, it was hypothesized that the experimental group would outperform the control group. The operational definition given for an object to be inside a loop (a simple closed curve) was that if the loop had to be taken over the top of the object to be pulled away. This operational definition was particularly effective when the object was a stick placed on end inside the loop, or if a child were standing inside one or more loops.

Quantity was significant for Loop Inclusion with a suggestion of the experimental gross quantitative comparison group performing better than the analogous control group (mean scores 44 vs 24 percent, respectively). The mean scores for the two extensive quantitative comparison groups were approximately equal. However, it cannot be claimed that the experiences given to the gross quantitative comparers in the treatment caused the 20 percent difference as it could be just as well attributed to chance. The difference is just not great enough to warrant any suggestion that the treatment was effective for the gross quantitative comparers. That Quantity was significant is somewhat surprising in view of the fact that it was not significant for Class Inclusion. The means for the two groups were 60 and 30 percent, respectively, for the extensive vs gross quantitative comparers. This difference does given some encouragement that suitable measures can be found for class inclusion which would be at least statistically related to Quantity.

In the test for Loop Inclusion, the "inside" was defined operationally for all children. It is apparently nonproductive to spend more than one or two days on showing children the operational definition of "inside" for simple closed curves because the level of achievement was comparable in the case of the experimental and control groups.

The instruction on set partitioning was included because at the time set partitioning was considered to be instrumental in establishing meaning for addition and subtraction and eventually numeration. The instructional activities were designed in such a way that for a particular collection of objects, all two-subset and three-subset partitions would be considered. For example, given a collection of seven objects, they would be partitioned into subcollections of 6 and 1, 5 and 2, 4 and 3, 3 and 4, and 2 and 5, and 1 and 6 objects successively, by moving an object from one subcollection to the other. In each case, the chidren were focused on the constancy of seven and the changing numbers in the subcollections. Partitioning activities were also presented to the children using approximately 100 kernels of popcorn and four or more glasses into which the popcorn was poured. No counting was included in the latter type of activities. During the course of instruction, children seemed to be generally successful with the activities. The partitioning activities are analogous to Piaget's additive composition of number.

In the section Quantity and set partition, it was pointed out that what Piaget calls the additive composition of number develops in three stages paralleled by gross, intensive, and extensive quantity. Even though the pretest data on set partitions did not relate to Quantity as expected (no differences existed due to Quantity where the mean total score was 44 percent for the gross quantitiatve comparers and 58 percent for the extensive quantitative comparers), it was felt that the instruction on set partitions would lead to improvement for the extensive quantitative comparers at least for set partitions which did not include counting. All of the children improved on Set Partitions Without Count as well as on Set Partitions With Count but there were no educationally significant treatment by Quantity interactions. Because the control children improved at least as dramatically as did the experimental children (the control children did not receive direct instruction on set partitions), and because Set Partitions are at best only weakly related to other numerical variables in the study, there seems to be little reason in the future to include direct instruction on set partitioning in first grade instructional programs. There is no evidence that the children in this study abstracted the meaning of addition or subtraction through partitioning activities.

Counting by Levels and Learning Instructional Phases for Addition and Subtraction

As the instruction was individualized for each child in the treatment group, no one instructional sequence may be described. It was the case, however, that each child was presented counting activities which progressed through rote counting, point counting, and rational counting. The instruction for addition and subtraction progressed through the learning instructional phases exploratory, abstraction-representation, and formalization-interpretation. The children were programmed through the learning-instructional phases at different rates and did different amounts of work. With few exceptions, the extensive quantitative comparers progressed through the abstraction-representation phase and associated counting activities more rapidly than did the gross quantitative comparers. Even though each child was given the opportunity to progress through the formalization-interpretation phase, only eight of the 48 children in the total sample actually did. It is important to note that tests were given for the formalization-interpretation phase even though they are not reported in this monograph.

At the culmination of the learning activities, all of the children were using rational counting-on to process exercises such as 4 + 5 = □ . It is interesting to note what seemed to be critical instruction for children who were at most point counters to progress to that level. The instructional procedure used was to direct the children to make marks on their paper to represent the two addends and then gradually lead them into a realization that only marks for one of the two addends would be necessary if one would start counting from the other addend. An analogous procedure was used with finger calculation. The children were then encouraged to not mark or use fingers, but to count the smaller addend on to the larger (in the case of unequal addends) mentally. After the children had mastered the procedure, they seemed very impressed with their powerfulness in calculating sums, now being able to find sums such as 15 + 4, 25 + 3, etc. Such sums were found even though the children did not know numeration.

Initially, each child was given experience in rote and point counting activities. All of the children learned to point count and write the numerals to at least 50. Point counting-back activities were also given, first starting with 10 and progressing through 20 or greater, depending on the child. The children, some with great difficulty, learned to point count back from 20. Addition and subtraction activities were integrated with the counting activities where children used the counting-all procedures with objects to process sums and differences of the basic fact variety ($a + b \leq 10$). The children who were extensive quantitative comparers soon tired of using objects and wanted to use finger calculation. Thereafter, it soon became apparent that all of the children wanted to abandon the physical materials in favor of finger calculation. They were allowed to do so. The extensive quantitative comparers (with the exception of one child) easily learned to process sums such as 4 + 3 by counting-on three to four--"five," "six," "seven"--either through using finger calculation or mental calculation. The gross quantitative comparers, however (with the exception of two children, one of which was one of the best students) used counting-all procedures with finger calculation and did not internalize the counting process until direct instruction was given. It is important to note that trials (on an individual basis) during instruction were provided these children to give them the opportunity to change counting strategies from counting-all to counting-on while processing sums such as 4 + 3. The trials were used as checks to insure that children were not held to counting-all procedures when in fact they could use more efficient counting strategies. It was not until the last week of instruction that the gross quantitative comparers (with the exceptions noted) were able to progress on to counting-on activities (after approximately six weeks of instruction using counting-all strategies with physical objects and finger calculation). Work with the hand-held calculator and problem solving were interspersed within the same six weeks, so six weeks should not be considered as a required time. But it does give indication of the extreme difficulty children have of acquiring counting-on without tallying if it is not within their cognitive competence.

The above procedures of instruction--integrating rational counting with finding sums--may only lead to what one may call algorithms for finding sums. The induced counting behavior may not have been counting schemes. In fact, the evidence is strong that gross quantitative comparers did not generalize the counting-on without tallying procedures taught across tasks as noted in the discussion of quantitative comparisons as a readiness variable for learning first grade airthmetical content. On the addition problems without objects, (Table 33 in the section Analyses of Variance) and the counting-on test (Table 45 in the section Analyses of Variance), the gross quantitative comparers in the experimental group performed quite well. But on the addition problems with objects (Table 41 in the section Analyses of Variance), the experimental gross quantitative comparers performed quite poorly. But it is important to note the instructional procedures were effective over a rather narrow range of problems and gave the gross quantitative comparers a sense of intellectual competence (as observed in instruction) in performing arithmetical exercises.

The effects of instruction on counting-on with tallying and the missing addend problems were also interesting. The instruction was synthesized so the children were not aware that two different goals were being accomplished with the same activities--the capability to count-on with tallying and the capability to solve the missing addend problems. The missing addend problem was initially presented using a counting-all strategy. For example, to solve $4 + \square = 7$, the children were instructed to take seven objects, count out four and the ones remaining would be the answer. Invariably, children who did not possess counting-on with tallying confused the procedure with previously learned counting-all procedures for processing sums. That is, to process sums such as represented by the sentence $3 + 8 = \square$, the children would count out eight objects, count three and the five remaining represented the result of the algorithm. It was necessary to explicitly point out the different appearance of the two types of sentences for these children. Through successive examples, the gross quantitative comparers did discriminate between the two sentence types and apply the correct algorithm. The same learning problem, however, did not occur for the children who were able to count-on with tally. They conceptualized the sentence $4 + \square = 7$, as four and how many is seven--five, six, seven--so it is three. Consequently, no problems in discriminating solution procedures existed for these children for the sentence types represented by the sentences $3 + 5 = \square$, and $3 + \square = 9$.

The counting all procedure for solving the sentence type 3 + □ = 8 seemed to interfere with the more natural counting-on strategy available to some of the children. After being shown the counting-all procedure, such children seemed to view it as the preferred solution process and were very reluctan[t] to employ counting-on with tallying. It should be recognized that counting-on with tallying requires more mental effort than does the counting-all procedure which may be the cause for some children's great reluctance to use the more sophisticated counting strategy. But it also should be recognized that adults presented the counting-all procedure which may have given it a status of being the preferred adult solution.

The counting-all procedure for solving missing addend sentences was used initially, of course, so that the gross quantitative comparers would have a procedure for solving the problems which (it was hoped) could be transformed into a counting-on procedure. In the transformation, an analysis of the counting-all procedure was attempted in the following manner. After a child had solved, say, 3 + □ = 7, by counting out seven, taking three, and then counting the remaining ones to obtain four, they were instructed to refocus their attention on the three, then count-on the four obtaining seven. This analysis move was not effective for some children as they could not count-on without tallying, which was a minimal requirement to conceptualize what was being analyzed. Direct instruction was also given to tie the missing addend sentence to rational counting-on with tallying. Problems were presented where some of a collection of objects were screened from a child's view. The children were then asked to find how many were screened. They had counted all of the objects to find the number in the total collection before some of them were screened. The unsuccessful children were allowed to "peek" behind the screen and count the objects there. These procedures were associated with missing addend sentences, e.g., 4 + □ = 7, in the obvious ways after the physical problem was solved. Encoding of the physical and mental actions seemed extremely difficult for children who were not able to count-on with tally. These children seemed "lost" in instruction.

The posttest data on the missing addend problems and the ordinal addition problems showed that the gross quantitative comparers in the experimental group were quite capable of solving ordinal addition problems (mean 71 percent) but were particularly inept at solving missing addend problems with objects (mean 17 percent) and without objects (mean 25 percent). It was in fact surprising that the experimental gross quantitative comparers performed so well on the ordinal addition problems (see Table 45 in the section Analyses of Variance) because during the treatment they seemed particularly inept at doing so. They apparently used trained procedures within a problem context familiar to them. It was particularly pleasing to note that the extensive quantitative comparers in the experimental group performed quite comparably to these in the control group on the missing addend problems and ordinal addition problems. The experimental extensive quantitative comparers, when forced to do so, did utilize counting-on with tallying in problem contexts not solvable by counting-all procedures.

Based on experience in instruction with children not capable of counting-on with tally or without tally, it is recommended that teachers not present missing addend problems to these children until counting-on schemes are acquired either through development or instruction. While such children can learn to solve such missing addend problems through counting-all procedures, the solution process is algorithmic and conceptualization of the problem is lacking. In the case of children capable of counting-on with tally, the missing addend problem should be presented with solution process that of counting-on. These children, in their own time, should produce more efficient solution procedures. It is strongly urged that the child's counting capabilities be the determiner of whether the missing addend problem is presented or not.

Children who are capable of counting-on, even if it is only without tallying, should be presented with addition through counting-on procedures rather than counting-all procedures. The counting-on procedures should lead to knowledge of basic facts more quickly. Moreover, the children can be exposed to more sophisticated sums (such as 43 + 4 or 56 + 5) and thereby gain a sense of competence not possible through counting-all procedures. Essentially, the exploratory phases of addition and subtraction can be done very minimally with these children. While counting-all procedures should not be forbidden (especially for differences with minuend less than or equal to ten), they should not be emphasized.

Conceptually, counting-back is to differences as counting-on is to sums. While differences may be found by counting-on with tallying, there is not presently available data which shows a child is capable of conceptualizing differences in terms of counting-on if counting-back and counting-on are not synthesized (formalization-interpretation phase), one being associated with differences and one with sums. In the instructional activities, counting back with and without tallying seemed especially difficult for most of the children. Presentation of the activities seemed to cause dissonance, with children refusing to participate mentally. While the extensive quantitative comparers fared much better than the gross quantitative comparers, the instruction on counting-back seemed to be not well received by the children. But because of its importance to differences, instructional procedures need created and tested before definitive recommendations are made concerning the introduction of counting-back with and without tallying.

Problem Solving Activities for Addition and Subtraction

Addition, subtraction, and missing addend problems were presented to the children in oral and written contexts. These problems were an integral part of the instruction utilizing the learning-instructional phases for addition and subtraction. Consequently, only features of the problems not discussed heretofore are presented. Children who could not yet read were given problems to solve in an oral presentation. Children who were able read the problems. One main goal of the problem solving activities was to teach the children to write mathematical sentences for the problems. In the main, the children were not capable of determining the defining relationships in the problem, writing an associated open sentence, solving the sentence, and then interpreting the solution back in terms of the problem. Rather they solved the problems mentally (if they in fact solved them), and then wrote a closed mathematical sentence to symbolize what they had done. This procedure was manifest in the posttest of addition, subtraction, and missing addend problems without objects. The children were asked to write the associated sentences in doing the problems. Observations were made concerning whether the children first processed the information and then wrote the sentence, or vice-versa. For the addition problems there were 88 attempts to write a mathematical sentence by the children. In 80 of these 88 attempts, the children first processed the information and then wrote the associated sentence. For the subtraction problems, in 71 of 82 attempts to write a mathematical sentence, the children first processed the information and then wrote the sentence. For the missing addend problems, the analogous numbers were 67 out of 78. In total, then, there were 248 attempts to write a mathematical sentence for a given problem. In 218 out of these 248 attempts, the children first processed the information and then wrote the mathematical sentence.

These data are important in that they elucidate the role of the mathematical sentence in the solution of arithmetical problems for young children. The children were quite capable of symbolizing their mental activity but did not represent the problem condition in written symbols and then work with the representation. Rather, any representation of the problems was internal. The mathematical sentence did not carry the power of representation of defining relations, but was rather only a manifestation of mental activity engaged in by the child.

The Hand-Held Calculator

The hand-held calculator was used each instructional day during the last four weeks of instruction. Each child in the experimental group was given a calculator and was allowed to use it during the entire class period, but was not required to use it. Children had little difficulty with the mechanical aspects of the calculator, quite readily learning to enter sums and differences. The role of the calculator in the classroom was to check answers arrived at through other means. The children enjoyed the calculators enormously during the time they used them. There was little evidence, however, that the calculators improved speed or accuracy of computation because Treatment was not significant for the addition and subtraction product or time scores (see Table 47 in the section Analyses of Variance).

At times, some of the children wished to do calculations on the calculator just to get them done. These sessions were very ineffective from the point of view of the children remembering basic facts. They seemed to be not interested in the answers, just writing them down.

The calculators seemed to be particularly ineffective for children who could only use count-all procedures to process sums and differences. Such children displayed little memory for basic facts, each sum or difference being unrelated to other sums or differences already found. While the calculators were an effective motivational device in instruction, they did not help the children remember basic facts.

Correlations Among Selected Variables

Variables Apparently Requiring Rational Counting In Solution

The minimal correlations between Cluster 1 and Cluster 5 variables (See Table 53 in the section Correlations Among the Variables) was disconcerting if they represent valid correlations. The #S and #P. variables were constructed to measure the childs ability to interrelate cardinal and ordinal number. The tasks were based both in mathematics and in developmental psychology. Piaget (1952) has strongly asserted that "Finite numbers are . . .necessarily at the same time cardinal and ordinal, since it is of the nature of number to be both a system of classes and of asymmetrical relations blended into one operational whole" (p. 157). In the review (see the section Cardinal and ordinal number as developmental concepts) of the tasks and theory supporting Piaget's assertion, it was noted that particular relations--longer than, shorter than, etc., for dolls and sticks--may have influenced the outcomes of the experiments Piaget performed. It is the relation "precedes" which in general determines order of precedence. Position is also a critical concept in ordinal number. The position a particular element occupies is entirely dependent upon the particular way in which the elements are ordered. In the tasks in Appendix

A.1, the child had to determine the cardinal number of certain segments and of the whole collection from being given the position of a particular object. The task design was an attempt to eliminate the criticism of Piaget's tasks that particular relations may unduly influence the outcome of the tasks.

The tasks were based also in mathematics in that if some finite set P is represented as $\{a_1, a_2, a_3, \ldots a_n\}$, any particular element, say a_r, $1 < r < n$, determines a segment $S = \{a_1, a_2, \ldots a_{r-1}\}$ and a remainder $Q = \{a_r, a_{r+1}, \ldots, a_n\}$. The tasks were presented in such a way that the order was determined by the row of objects and the segment $S = \{a_1, a_2, \ldots, a_{r-1}\}$ was determined by the cover. Given the position of a_{r+1} or a_{r+2}, the child had to give the cardinality of S and of P. The numbers selected for P (12 and 8) were small enough to be within the experience of the children. Piaget's theory predicts that children who are in Stage II with respect to number should solve the task, especially in the cases where hints were given.

The rational counting-on, ordinal addition, rational counting-back, and ordinal subtraction tasks were based on the same structural analysis of number as were the #S and #P tasks. The important differences resided in the facts that (1) no order of the elements was implied by the physical arrangement of the objects except for the physical determination of the segment and remainder through covering the objects, and (2) the cardinal number of the segment, remainder, or the total set was always given rather than the position of some element.

If the minimal correlations between Cluster 1 and Cluster 5 variables represent valid correlations, the concept of position as it relates to other aspects of children's conception of cardinal and ordinal number will have to be elucidated through further experimentation. However, given the low internal consistency reliabilities of the #S and #P tests, improved task design for those variables must be accomplished before any conclusions are drawn regarding a child's concept of position as it relates to other numerical variables.

The eight correlations between the two missing addend problems and the variables of Cluster 5 were all significant but modest. The greatest correlation was between Ordinal Addition and Missing Addend Without Objects. The modest correlations can be attributed to the extraneous variables present in missing addend problems. The children had to translate the orally presented problems from natural language into a numerical procedure. In the case of objects present, the child had to ignore the fact there were more objects present for use than were needed--a difficult task for many children as they never bothered to count all of the objects, but rather counted out the first given number and then counted the remainder for the answer. In the face of such extraneous variables, that the Missing Addend variables correlated as well as they did with variables of Cluster 5 supports the contention that rational counting procedures are critical for comprehension and solution of missing addend problems.

The significant correlations for the variable Between with all other variables in Table 53 and the correlation of Ordering Numbers with Advanced Counting in Tables 57 and 58 in the section <u>Correlations Among the Variables</u> supports the contention that knowledge of Between demands rational counting as a prerequisite.

Variables Apparently Requiring at Most Point Counting in Solution

Set partitions. Set partition is part of the mathematics of addition and subtraction of cardinal and ordinal number. On the pretest (See Table 38 in the section Analyses of Variance), Partitions with Count and Partitions Without Count correlated negligibly with Subtraction and only marginally with Addition. The analogous correlations on the posttest (See Table 42 in the section Analyses of Variance) were greater and were all significant except Partitions With Count and Addition. Moreover, Partitions With Count and Partitions Without Count correlated negligibly with Addition and Subtraction with no objects (See Table 54 in the section Correlations Among the Variables). The correlations of the two tests of set partitions with the addition and subtraction time product scores were essentially zero. Both set partition variables did correlate significantly with Counting-Back and Predecessor but the correlations were less than .50. Apparently, then, set partitions is not a critical apsect of cognitive functioning on arithmetical tasks requiring point counting for task performance.

This assertion is strengthened by inspection of the distribution of total scores for the variables. Partitions With Count and Addition Without and With Objects had quite similar distributions of total scores (see Table 5 and Table 10 in the section Item Analyses). The distributions of total scores for Partition Without Count and Subtraction With and Without Objects also were similar. In the former case, the correlations were .14 and .22, respectively. In the latter case, the correlations were .47 and .33, respectively. The only correlation of the four which shows strength of association was the correlation of .47 between Partitions Without Count and Subtraction With Objects. With the exception of this correlation of .47 and the possible exception the correlation of .44 between Partitions Without Count and Addition With objects, (two variables also with similar frequency distribution), correlations involving the set partition variables with other variables with similar frequency distributions were marginal or nonsignificant. It was therefore possible for children to succeed (or not to succeed) on set partition items but not succeed (or succeed) on tests based on point counting.

If one argues that set partitioning is an integral aspect of the <u>meaning</u> of addition or subtraction of cardinal numbers, the correlations of set partition variables with Cluster 5 variables and with the missing addend problems in the posttest of Cluster 4 variables should be seriously considered in the argument. Children who were not capable of set partitioning should not have been able to find sums or differences using point counting strategies because they would not be capable of applying the strategies. Children who were capable of set partitioning may or may not be able to find sums or differences if the argument is accepted as valid. In the face of the small correlations, the argument does not seem plausible.

Osborne (1967), in a study of subtraction through partitions, conjectured that "If the child is not perceptually or cognitively ready to conserve the whole upon sub-division, then he cannot acquire the concept of subtraction via a group manipulative approach" (p. 107). Because of the relative independent functioning of Partitions in this study, Osborne's conjecture is not supported. In fact, because the experimental group engaged in partitioning with associated number facts, the evidence is negative concerning Osborne's conjecture.

Another conjecture made by Osborne (1966) was that "Given an instructional approach to subtraction, if the child thinks in terms of manipulation of groups, then the child will understand subtraction better than if he thinks in terms of one-by-one manipulation" (p. 107). The evidence is also against this conjecture because of the small correlation between the two partitions variables and the various variables of a subtractive nature.

Addition and subtraction. The addition and subtraction product scores were correlated negligibly with all variables in Table 54 in the section Correlations Among the Variables except the addition and subtraction problems without objects. Although these correlations were modest, they are logical in that in both cases, mental or finger calculation had to take place. The mental calculation could involve knowledge of number facts. The correlations are comparable with a correlation of .46 reported by Steffe (1966) between a number facts test and an addition and subtraction problem solving test without objects. The correlations between addition and subtraction with objects and the two product scores are somewhat less than the correlation of .41 between comparable tests reported by Steffe (1966). The correlations in this study are more consistent with the conjectures advanced by Steffe (1966, p. 43) that the presence of objects in the solution of addition and subtraction problems would lower the correlation between an addition facts test and an addition and subtraction problem solving test. However, finger calculation serves a functional role in both number facts tests and addition and subtraction problem tests. Due to children's great reliance on finger calculation, it destroys the role of knowledge of number facts as an explanation of performance on addition and subtraction tests. Consequently, although correlations may be somewhat greater between "number facts" tests and orally presented arithmetic addition and subtraction problems without objects than it is between the former and orally presented addition and subtraction problems with objects, they are not enough greater to strongly suggest that objects are critical for formation of mental operations associated with addition and subtraction. If objects were critical in formation of mental operations associated with addition and subtraction, one would expect a negligible correlation between addition and subtraction problems presented in the presence of objects and number facts tests. The presence of objects would enable the children to do the problems independently of knowledge of number facts which would manifest in an essentially zero correlation. On the other hand, one would expect addition and subtraction problems presented to children without objects to be related substantially to number facts tests if for no other reason than mental calculation would seem to be necessary for solution. But in the face of the correlations, the intervening variable of finger calculation destroys the line of reasoning and also destroys the illusion that physical objects are critical for early learning of arithmetic. This conclusion is supported by the fact that all six correlations among the four addition and subtraction problem solving tests were significant and of approximately equal range (.33 to .55). It was the case also that correlations between the problems with objects and problems without objects (.39, .42, .37, .33) were not a great deal different than within objects (.37) and within no objects (.55).

Counting back, just before, just after, successor, predecessor. The five variables under consideration had only four significant intercorrelations out of ten. Counting Back and Predecessor correlated .52. This correlation as well as the correlation of .34 between Predecessor and Just Before is manifestation of the fact a child had to count back from nine to name the seventh element with a point count in the test for Predecessor. That Successor was not correlated with any other variable in Table 54 in the section Correlations Among the Variables, is somewhat surprising. These essentially zero correlations lead to the conjecture that children's ability to start at a number and count-on in a rote fashion does not lead to arithmetical competence of any kind and should not be taken as being essential in learning arithmetical content.

The variable Predecessor correlated significantly with the problem solving variables as well as with Just After, Counting-Back, and Just Before. Counting-Back also correlated significantly with the problem solving tests. But, Just Before and Just After did not correlate significantly with the problem solving variables except for one case (Subtraction With No Objects and Just Before). These results signal the commonality of solution process among variables requiring point counting. Just Before and Just After did not require point counting--only rote counting.

Variables Apparently Requiring Rational Counting vs Variables Requiring at Most Point Counting

Set partitions. The two set partition variables correlated greater with variables apparently requiring rational counting for solution than with variables requiring at most point counting for solution. However, the correlations of the two set partitions variables with variables apparently requiring rational counting for solution are not easily explained in that some are significant and some are not significant. Those correlations not significant are for the variables Missing Addend Without Objects, Ordinal Addition, Rational Counting-Back and Ordinal Subtraction (see Table 55 in the section Correlations Among the Variables). Those correlations which are significant are for the variables Number in S, Number in P, Number in S + Number in P, Missing Addend With Objects, Between, Rational Counting-On, and Ordinal Subtraction (Significant for Partitions With Counting).

The significant vs. nonsignificant dichotomy cannot be explained by whether the initial equivalence in the partition test was established by the child through point counting. In fact, the correlations for Partition Without Counting generally exceeded the correlations for Partition With Counting and both generally had significant or nonsignificant associated correlations. It would seem plausible that a physical objects present vs physical objects absent dichotomy could explain the difference in the significant vs nonsignificant correlations because the test for Partitions included physical objects. In the case of nonsignificant correlations, the children had to answer questions concerning objects screened from view even though some objects could be seen (except in case of Ordinal Subtraction). For these variables, no image of screened objects would be available to the children through direct perception. But in the case of the significant correlations, direct perception of objects, was not the case either except for one variable--Missing Addend with Objects. Consequently the physical objects present vs physical objects absent dichotomy is not a tenable explanation for the dichotomy significant vs nonsignificant correlations.

A rational counting-on vs rational counting-back dichotomy does not explain the significant vs nonsignificant dichotomy for the correlations. Consequently, due to the rather marginal nature of the significant correlations under consideration (none were greater than .50), it is concluded that the underlying basis for the significant vs nonsignificant dichotomy for the correlations has no discernable explanation and may be attributed to chance fluctuation of the sample. Partitions, then, is only weakly related to (1) arithmetical operations (addition and subtraction) in the case where rational counting-on or rational counting-back is required for solution, (2) rational counting-on, (3) rational counting-back, (4) the ability to obtain cardinal information from ordinal information, and (5) knowledge of betweeness for numbers up to 12.

In view of the frequency distribution of total scores (Table 10 in the section Item Analyses) for Partitions With Count, one would expect that the variable would not be correlated with Number in S (frequency distribution, Table 3 in the section Item Analysis), Number in P (frequency distribution Table 3), Missing Addend Without Objects (frequency distribution, Table 5), Counting-back (frequency distribution, Table 13) or Ordinal Subtraction (frequency distribution, Table 13). One would expect, however, that the variable could be correlated with Counting-On and Ordinal Addition (frequency distribution, Table 13). The correlation of only .31 and .13 for the latter two variables only strengthens the above conclusion that Partition variables are weakly related to the variables apparently requiring rational counting-on in solution.

Analogous inspection of frequency distributions for Partition Without Count and other variables under consideration would lead to the expectation of significant correlations in the case of Missing Addend With Objects (actual correlation .43), Ordinal Addition (actual correlation .27), and Missing Addend Without Objects (actual correlation .21). These three correlations again strengthen the above conclusion. When the frequency distributions were such that it would be possible for significant correlation between Partition variables and other variables of this section, the correlations were minimal.

Addition and subtraction problems. Only five of 40 correlations involving addition and subtraction problems were not significant. Two of the five were between addition and subtraction problems with objects and missing addend problems without objects. Two others were between subtraction problems with no objects and rational counting-back problems and ordinal subtraction problems. Solution procedures in both cases certainly may have contributed to the negligible correlations. For addition and subtraction problems with objects, children could use counting-all procedures but for missing addend problems without objects, children counted-on with tally either using their fingers or mentally. For subtraction problems without objects, children generally used counting-all procedures with their fingers, but for rational counting-back and ordinal subtraction, children had to go through a backward ordinal sequence either without tallying (counting-back) or with tallying (ordinal subtraction). These procedures were quite different and may be used to explain why the variables involved did not correlate to a greater extent than they in fact did.

Inspection of the frequency distributions for subtraction problems with objects (Table 10 in the section Item Analyses) and missing addend problems without objects (Table 5 in the section Item Analyses), would lead one to expect a significant correlation due to similarity of the distributions. That the correlation was not significant strongly supports the process analysis given above. The other variables with nonsignificant correlations had dissimilar frequency distributions, which certainly does not contradict the fact that children use widely varying solution procedures in solving problems.

The correlation of .60 between ordinal addition problems and addition problems with no objects is quite surprising in view of the dissimilarity of the frequency distributions. Rational counting-on problems and addition problems with no objects correlated .54 and had quite similar frequency distributions. But rational counting-on problems and ordinal addition problems correlated .72. Consequently, evidence is strong that children who solved the addition problems without objects did so in the main by counting-on either on their fingers or mentally. Or at least they were capable of doing so. This contention is further supported by (1) the correlation of .47 and .48 between addition problems without objects and rational counting-back problems and ordinal subtraction problems, respectively, especially in the face of the great dissimilarity of frequency distributions between the former and the latter two problem types, and (2) the correlation of .49 between the addition problems without objects and missing addend problems with objects.

The remaining significant correlations in the main reflect statistical relationships rather than analogous solution procedures. However, counting types are nested by definition--so children who can perform ordinal addition tasks, for example, can also point count and thereby use counting-all procedures in solution. However, the correlations are dampened by the fact that children who apply appropriate solution procedures arrive at incorrect answers through mechanical errors and by children who can utilize counting-all procedures but not rational counting procedures in solution.

Addition and subtraction product and time scores. The addition product scores only correlated significantly with variables obtained from the ordinal number addition and subtraction tests. The subtraction product scores also correlated significantly with the variables obtained from this test. However, the subtraction product scores also correlated significantly with missing addend problems with and without objects. In the mental arithmetic tests, children were admonished not to use their fingers nor make marks on the paper. Apparently, the admonition was effective enough that the subtraction exercises were thrown into the realm of mental arithmetic for some children, which explains the significant correlation for the subtraction product scores noted above. The admonition was effective also for the addition exercises to be thrown into the realm of mental arithmetic. But it was a fact that the addition exercises were easier than the subtraction exercises, which explains the nonsignificant correlation of the addition product score with the missing addend problems with and without objects (see Tables 5, 10, and 15 in the section Item Analyses for distributions).

The correlations of the addition and subtraction product scores with variables apparently requiring rational counting do not contradict the contention that addition facts should be considered as abstractions from mental operations associated with rational counting-on or ordinal addition. The case for subtraction is not as clearcut. However, the correlations certainly do not contradict the contention that subtraction facts should be based on at least an integration of rational counting-back with rational counting-on. In any case, teachers who drill children on addition or subtraction facts in the absence of strong rational counting capabilities (at least counting associated with ordinal addition) run a great risk of frustrating the child.

The negative correlations between the addition and subtraction time scores and the variables Missing Addend With and Without Objects, Rational Counting-on, Ordinal Addition, Rational Counting-Back, and Ordinal Subtraction supports the relationship observed for the product scores. A weak indication was present that children who obtained correct answers on the test items for the variables just noted, tended to work faster than the children who did not. But the association is weak and should not be considered as vitally important in planning arithmetic instruction.

Counting back, just before, just after, successor, and predecessor. Of the 50 correlations involving the five variables in the paragraph heading above, only 15 were significant. Six of the 15 involved Predecessor. Counting Back was correlated significantly with two variables (#S + #P, .42 and Missing Addend With Objects, .36); and Just After was correlated significantly with only one variable (Between, .44). Just Before was correlated significantly with six variables. Obviously, then, Just Before and Predecessor are the only two variables related consistently with variables apparently requiring rational counting in solution. But in general the correlations were not strong except for the correlations between Just Before and Between (.49), and Just After and Between (.44). These two correlations support the logical relationship which exists among the three variables. For a child to find a number between two others, conceiving of numbers just before and just after the two given numbers and being aware of which is which, is extremely important for being successful.

The significant correlations for the variable Predecessor were only marginally significant. But they may reflect on underlying conceptualizing ability on the part of the child.

Some Problems Needing Further Study

Ginsburg (1976, p. 147) has given a useful characterization of children's knowledge of arithmetic in terms of three cognitive systems. System 1, informal in nature, develops outside of the formal school setting and involves perception and thought used to deal with quantitative problems. Counting is not part of System 1. System 2 involves counting but is still informal in that it develops outside of the context of schooling. It has a cultural component since it depends on social transmission of counting. System 3 is formal in that it deals with arithmetic taught in school. Ginsburg (1976) conjectures that a great deal of interaction takes place between System 2 and System 3. "Probably the great majority of young children interpret arithmetic as counting <u>regardless of how they are taught</u>...they probably use counting as the basic method for dealing with arithmetic" (p. 148). While Ginsburg did not identify counting typologies used in this study, he is essentially correct in his observation that mathematics educators have <u>assumed</u> counting in their mathematics programs for early childhood. Counting has been viewed as being acquired by children through experiences outside of the mathematics curriculum. Serious attention has not been given to counting in school mathematics texts for early childhood and its role in the formation of mathematical concepts and principles.

Freudenthal (1973) has pointed out the importance of counting to arithmetic: "We stressed the didactical priority of the counting number... The child should learn to add by counting further, to subtract by counting backwards, to articulate the counting by tens, to multiply by counting with other intervals but 1, and so on" (p. 242). The data of this study clearly confirm that one cannot be arbitrary concerning how a particular child <u>should</u> learn to add, subtract, etc. It is clearly important that one be assured counting schemes are available to the child before addition or subtraction are done as Freudenthal suggests. But if addition and subtraction are connected in the mind of the child through counting-on and counting-back, a great savings transfer could occur in the learning of subtraction. Studies need to be designed to determine if such transfer occurs.

Counting has been used to identify learning-instruction phases in addition and subtraction. But data other than that presented in this monograph are necessary to establish psychological credibility of these phases. In other words, can a child, for example, operate cognitively at the formalization-interpretation phase but yet not have synthesized counting-on and counting-back? Is that synthesis a critical cognitive function for the interrelation of addition and subtraction? Experiments also need done to determine if there exist transitional characteristics from one counting typology to another and from one learning instructional phase to another. Such transitional characteristics, if they exist, could be critical indicators of instructional procedures.

Because children possess different counting capabilities, counting must be moved from Ginsburg's System 2 to System 3. It should be a function of school instruction to develop counting capabilities and to develop their use in the learning of concepts and principles in school mathematics. It is the role of research to study development of counting in children and how they use counting in other aspects of mathematics. Other than addition and subtraction, numeration is given as an example of how counting may be used by children to learn important school mathematics concepts.

It should not be surprising that learning instructional phases for numeration are logically identifiable and are closely allied with learning instructional phases for addition and subtraction. New elements, of course, are introduced into definitions of the phases.

The exploratory phase for numeration corresponds to the capability to point count and to the exploratory phase for addition and subtraction. Here, children may be expected to count out collections of a given number (e.g., count out collections of two tens and five from a pile of thirty-two objects). However, the collections are looked upon by the child as being just that--piles of objects having no particular significance in the sense of being lasting in the mind of the child. They may not be looked at as representing two tens and five but rather cease to exist in any way upon being physically destroyed. There is no representation of "ten" in the child's mind (here, we are not speaking necessarily of an image but a unit consisting of a plurality).

The child who is capable of counting-on without tallying is capable of counting a set P (where #P = 15) by counting a set S of ten, holding that in mind as an entity, and counting "one ten and one," "one ten and two," "one ten and three," "one ten and four," "one ten and five." So, here, $P = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\} \cup \{q_{10+1}, q_{10+2}, q_{10+3}, q_{10+4}, q_{10+5}\}$. The element q_{10+1} means that the child holds S in mind as an entity and conceives of q_{10+1} as representing one ten (S) and one (the element q_{10+1}): The assumption here is that for a child to conceive of 15, it is necessary for him to have the potential of constructing 15 as one ten and five more in the way described by the counting-on process. Obviously, under the assumption, 10 + 5 must be a conception of 15. Then if 15 is to mean one ten and 5 more, a counting-all strategy alone is not sufficient to allow the child to connect the ten and the 5 more into one number, 15.

Counting-on without tallying, then, is assumed as critical for the abstraction-representation phase for numeration. But that is not enough. Counting-on with tallying is also assumed as necessary. If a child has a collection of, say, nineteen, and knows it, if he counts out ten, he should be able to count-on from ten keeping track of how many he has counted-on, so he doesn't go past 19. That is to say, counting-on with tallying is conceived of as essential for a child to find the number of tens in a given number. It would seem counting-on with tallying is a minimal condition for knowing how to find the number of tens in a given number at some level other than counting out piles of tens.

Another example is when a child counts a collection he <u>doesn't know the number of</u>, say a pile of 25 objects. The child can count:

*	*	*	*	*	*	*	*	*	*
1	2	3	4	5	6	7	8	9	10

1	2	3	4	5	6	7	8	9	10
*	*	*	*	*	*	*	*	*	*
11	12	13	14	15	16	17	18	19	20

1	2	3	4	5
*	*	*	*	*
21	22	23	24	25

The numeral under the stars represent the child's count and means he takes one, says "one," takes another, says "two," etc., until 10 is reached. He then takes another, says "eleven" and tallies "one," takes another, "twelve" and tallies "two," etc. The above procedure represents a <u>mental</u> count of a pile of objects. When the child is done, he <u>knows</u> he counted out two piles of ten. One may object and say children do not do the above. Perhaps true, but in the case where a child is asked to find the number of tens in 36, he should to so on a basis other than merely being trained to say "3." The <u>assumption</u> is that for the question to have numerical meaning for the child, he will have to <u>construct</u> the collections of ten through counting and tallying, or be able to do so. Such a procedure involves the above represented tallying procedure. The tallies, of course, may be fingers!

Numeration activities for a child in the exploratory phase would be counting out piles of ten from a pile of objects, counting the piles and then the ones remaining, and associating a numerical "ab" (a and b are digits) with the procedure. Visuals, such as a bundle of ten could be used, but numerals would have little conceptual (or numerical) meaning, but would have figurative meaning. The abstraction-representation phase implies the child has internalized counting-on strategies available. Numeration now has the potential of carrying numerical information and is much richer in its meaning.

In the abstraction-representation phase, the child can be presented with more abstract content concerning numeration than children in the exploratory phase. Here, children are capable of learning the concept of place-value, of learning the numerals, their names, and of conceptualizing one hundred as ten tens. They essentially view a numeral such as "56" as being five units each of plurality ten and one unit of plurality six, because they are capable of mentally constructing (through rational counting-on with tallying) the various units. They also conceive of the five units of ten and the one unit of six as making up a total unit of 56. This discussion brings to mind set partition. But set partition is now viewed, at this age level, as being made possible because of rational counting with tallying. But, for a child to have achieved the abstraction-representation phase with regard to numeration he should know, for example that it would take more two's to make twelve than three's to make twelve. Such capability is taken as manifestation of the above described conception[35] of "56."

The formalization phase for numeration presupposes the formalization phase for addition and subtraction. The flexibility of thought implied by the formalization phase for addition and subtraction is, of course, that once a child starts at some point in the ordinal sequence and counts-on k, having p + k, he knows, without actually counting-back, that if he would count-back, k, p would be the result. The number 36 means thirty and six more, so a child should know that 36 less 6 would be 30 because 36 is thirty and six more.

At the formalization phase, a child has to order numbers. So to order 30 and 36, the child should know the connecting link both ways: $30 + 6 = 36$; or $30 = 36 - 6$. Ordering 48 and 55 is not so easy. The child should be able to mentally manipulate two digit numbers in such a way that he knows 48 to 50 is 2, and from 50 to 55 is 5, so from 48 to 55 is 7. He would also know, then, that from 55 to 48 is 7. Another example is 39 and 71. The child should be able to go from either one to the other through rational counting-on with tallying or rational counting-back with tallying. For formalization of numeration, any two numbers should be connected by the child being able to find the distance between them, or equivalently, by solving $a + \square = b$ or $\square = b - a$, and solving one, knowing the other. That is, if a child figures out it is 42 from 39 to 71, he should know it is also 42 from 71 to 39.

Formalization of numeration <u>does</u> <u>not</u> involve two-digit addition and subtraction in the sense of algorithm work. However, it does involve the sense of order. That is, $a < b$ whenever $b - a = c > 0$ (or equivalently, if there exists c where $a + c = b$). The point is, the child must not only order the numbers, but <u>find</u> <u>c</u> mentally. The concept of <u>place-value</u> constructed at the abstraction-representation level should mediate, at some point in time, the process in finding c. For example, from 39 to 71, 49 is ten, 59 is twenty, 69 is thirty, thirty-one, thirty-two. So, 32 is the answer. Formalization also implies that verbal number names and the written numerals are coordinated and each has two connotations—a place value connotation and a position in the number sequence.

The work with standard algorithms for addition and subtraction may be looked at as <u>following</u> acquisition of the formalization-interpretation phase for numeration. The algorithms may be, to the child, just efficient procedures for processing sums and differences. Learning-instructional phases must be also developed for multiplication and division and validated.

It should be clear that a great deal of work remains in the construction and validation of models for learning and instruction of particular concepts. Measurement and particular aspects of space and geometry are critical as are addition, subtraction, multiplication, and division of whole numbers and fractions. These models should be of the nature outlined by Beilin (1976) to be maximally useful to school programs.

Other than the problems outlined above, it continues to be of importance to continue to study the influence of variables in Ginsburg's System 1 on the acquisition of knowledge in his System 3. One study of immediate importance is to determine the influence of Quantitative Comparisons on acquisition of numeration concepts.

References

Applebaum, M., & Bargmann, R. E. A fortran II program for MUDAID: Multivariate, univariate, and discriminant analysis of irregula data. Technical Report NONR 1834(39). Urbana: University of Illinois Press, 1967.

Barnes, W. E. Introduction to abstract algebra. Boston: D. C. Heath, 1963.

Beilin, H. Stimulus and cognitive transformation in conservation. In D. Elkind, & J. H. Flavell (Eds.), Studies in cognitive development: Essays in honor of Jean Piaget. New York: Oxford University Press, 1969.

Beilin, H. The training and acquisition of logical operations. In M. F. Rosskopf, L. P. Steffe, & S. Taback (Eds.), Piagetian cognitive-development research and mathematical education. Washington: D.C.: NCTM, 1971.

Beth, E. W., & Piaget, J. Mathematical epistemology and psychology. (Translated from the French by W. Mays.) Dordrecht, Holland: D. Reidel, 1966.

Brainerd, C. J. Analysis and synthesis of research on children's ordinal and cardinal number concepts. In R. A. Lesh (Ed.), & D. A. Bradbard (Tech. Ed.), Number and measurement: Papers from a research workshop. Ohio State University: ERIC/SMEAC, 1976.

Dodwell, P. C. Children's understanding of number concepts: Characteristics of an individual and of a group test. Canadian Journal of Psychology, 1961, 15, 29-36.

Dodwell, P. C. Relations between the understanding of the logic of classes and of cardinal number in children. Canadian Journal of Psychology, 1962, 16, 152-160.

Eicholz, R. E., & Martin, E. Jr. Elementary school mathematics. Reading: Addison-Wesley, 1971.

Elkind, D. The development of the additive composition of classes in the child: Piaget replication study III. The Journal of Genetic Psychology, 1961a, 99, 51-57.

Elkind, D. The development of quantitative thinking: A systematic replication of Piaget's studies. The Journal of Genetic Psychology, 1961b, 98, 37-46.

Freudenthal, H. Mathematics as an educational task. Dordrecht-Holland: D. Reidel, 1973.

Greco, P., & Morf, A. Structures numériques élémentaries. Paris: Presses Universitaires de France, 1962.

Ginsburg, H. Learning difficulties in children's arithmetic: A clinical cognitive approach. In A. R. Osborne (Ed.) & D. A. Bradbard (Tech. Ed.), Models for learning mathematics: Papers from a research workshop. Ohio State University: ERIC/SMEAC, 1976.

Harper, H. E., & Steffe, L. P. The effects of selected experiences on the ability of kindergarten and first-grade children to conserve numerousness. Research and Development Center for Cognitive Learning, Technical Report No. 38, 1968.

Hausdorff, F. Set theory (2nd English Edition). New York: Chelsea Publishing Company, 1962.

Inhelder, B., & Piaget, J. The early growth of logic in the child. (Translated from the French by E. A. Lunzer & D. Papert.) London: Routledge and Kegan Paul, 1964.

Inhelder, B., & Sinclair, H. Learning cognitive structures. In P. J. Mussen, J. Langer, & M. Covington, Trends and issues in developmental psychology. New York: Holt, Rinehart & Winston, 1969.

Johnson, D. C. Learning of selected parts of a Boolean algebra by young children. In L. P. Steffe (Ed.), Research on mathematical thinking of young children: Six empirical studies. Reston: NCTM, 1975

Kohnstamm, G. A. Piaget's analysis of class inclusion: Right or wrong? (Translated by M. Peeck-O'Toole & G. Uildriks-Bone), the Hague: Mouton & Co., 1967.

Kelly, J. L. General topology. New York: D. Van Nostrand, 1955.

LeBlanc, J.F. The performance of first grade children in four levels of conservation of numerousness and three IQ groups when solving arithmetic subtraction problems. (Doctoral dissertation, University of Wisconsin), Ann Arbor, Michigan: University Microfilm, 1968, No. 68-7111.

Morton, R. L., Gray, M., & Rosskopf, M. F. Modern arithmetic through discovery, book I. New Jersey: Silver Burdett, 1965

Mpiangu, B. D., & Gentile, J. R. Is conservation of number a necessary condition for mathematical understanding? Journal for Research in Mathematics Education, 1975, 6, 179-192.

Osborne, A. R. The effects of two instructional approaches on the understanding of subtraction by grade two pupils. (Doctoral dissertation, The University of Michigan) Ann Arbor, Michigan: University Microfilms, 1967, No. 67-08321.

Piaget, J. The child's conception of number. (Translated by C. Gattegno & F. M. Hodgson), London, Routledge & Kegan Paul, Ltd., 1952.

Piaget, J. Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds), Piaget rediscovered: A report of the conference on cognitive studies and curriculum development. Cornell University, 1964.

Piaget, J. Genetic epistemology. (Translated by Eleanor Duckworth.) New York: Columbia University Press, 1970.

Piaget, J., & Inhelder, B. The child's conception of space (Translated by F. J. Langdon & J. L. Lunzer). London: Routledge & Kegan Paul, 1963.

Sinclair, H. Number and measurement. In M. F. Rosskopf, L. P. Steffe, & S. Taback (Eds.), Piagetian cognitive development research and mathematical education. Washington, D.C.: 1971.

Steffe, L. P. The performance of first grade children in four levels of conservation of numerousness and three IQ groups when solving arithmetic addition problems. Technical Report No. 14, Research and Development Center for Learning and Re-Education. Madison: University of Wisconsin, 1966.

Steffe, L. P. An application of Piaget-cognitive development research in mathematical education research. In R. Lesh (Ed.), Cognitive psychology and the mathematics laboratory. Ohio State University: ERIC/SMEAC, 1973.

Tarski, A. Introduction to logic. New York: Oxford University Press, 1954.

Van Engen, H. Epistemology, research and instruction. In M. F. Rosskopf, L. P. Steffe, & S. Taback (Eds.), Piagetian cognitive-development research and mathematical education. Washington, D.C.: NCTM, 1971.

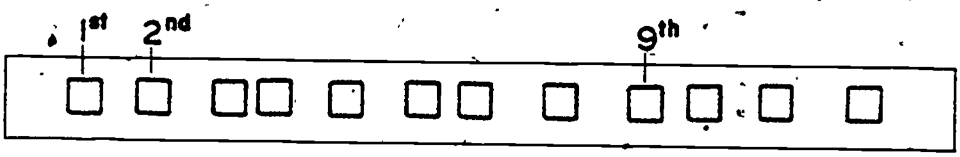
Vitale, M. A. The effects of the comprehensive school mathematics program on kindergarten and first grade children's ability to perform class inclusion, seriation, and transitivity tasks. (Doctoral dissertation, University of Georgia) Ann Arbor, Mich: University Microfilms, 1976, No. 76-2271.

Appendix A.

ACHIEVEMENT TASKS

Appendix A.1: Ordinality and Cardinality Tasks.

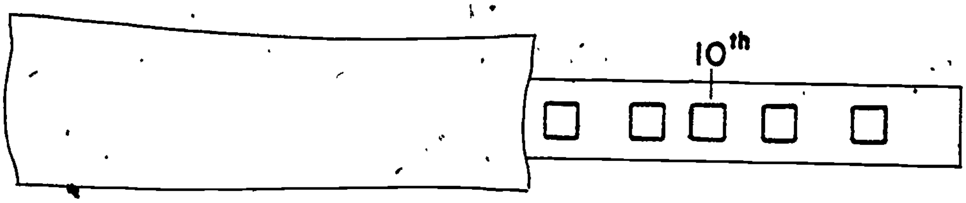
Task A. (12 counters in a row)



HERE ARE SOME COUNTERS IN A ROW. IF WE START COUNTING FROM THIS END (S's left), THIS ONE IS FIRST (point), THIS ONE IS SECOND (point), THIS ONE IS THIRD (point).

1. THIS ONE IS NINTH (point). WHICH ONE IS THIS? (point to tenth)
   a. [ ] correct immediately (go to #2)
   b. [ ] correct but counts from the beginning
   c. [ ] incorrect
      THIS ONE IS NINTH (point), THIS ONE IS TENTH (point). WHICH ONE IS THIS? (point to eleventh)
         [ ] correct immediately
         [ ] correct but counts from the beginning
         [ ] incorrect

2. THIS ONE IS NINTH (point). WHICH ONE IS THIS? (point to seventh)
   a. [ ] correct immediately
   b. [ ] correct but counts from the beginning
   c. [ ] incorrect



3. (cover seven with cloth) - THIS ONE IS TENTH (point). HOW MANY ARE COVERED?

a. [ ] correct - HOW DO YOU KNOW THAT?

b. [ ] HOW MANY ARE THERE IN ALL

c. [ ] incorrect - THIS ONE IS TENTH (point), HOW MANY ARE THERE IN ALL?

d. [ ] correct - RIGHT, AND HOW MANY ARE COVERED?

e. [ ] five - FEEL THE FIRST ONE. WHICH IS NEXT? (feel second)

f. [ ] correct - HOW MANY ARE COVERED?

[ ] correct

[ ] incorrect

[ ] incorrect - STOP

g. [ ] incorrect (not 5) - THIS IS TENTH (point), THIS IS ELEVENTH (point), THIS IS TWELVTH (point). HOW MANY ARE THERE IN ALL?

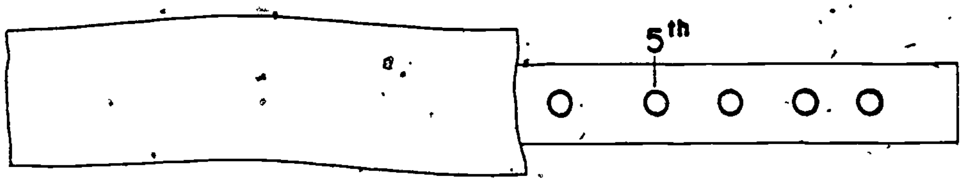
h. [ ] correct - HOW MANY ARE COVERED?

[ ] correct

[ ] incorrect

[ ] incorrect - STOP

Task B.



HERE ARE SOME COUNTERS IN A ROW. SOME OF THEM ARE COVERED. FEEL THE FIRST ONE HERE.

1. THIS ONE IS FIFTH (point). WHICH ONE IS THIS? (point to sixth)
   a. [ ] correct - go to #2
   b. [ ] incorrect - THIS ONE IS FIFTH (point), THIS ONE IS SIXTH (point). WHICH ONE IS THIS? (point to seventh)
      [ ] correct
      [ ] incorrect
2. THIS ONE IS FIFTH (point). HOW MANY ARE THERE IN ALL?
   a. [ ] correct - HOW DO YOU DO THAT?
   b. [ ] five- REMEMBER, THERE ARE SOME UNDER THE COVER. FEEL THE FIRST ONE. THIS ONE IS FIFTH (point). HOW MANY ARE THERE IN ALL?
      [ ] correct
      [ ] incorrect
   c. [ ] incorrect (not 5) - THIS ONE IS FIFTH (point), THIS ONE IS SIXTH (point), THIS ONE IS SEVENTH (point). WHICH ONE IS THIS (point to eigth)
      [ ] correct - HOW MANY ARE THERE IN ALL?
         [ ] correct
         [ ] incorrect
      [ ] incorrect - FIFTH (point), SIXTH (point), SEVENTH (point), EIGHTH (point).
         HOW MANY ARE THERE IN ALL?
         [ ] correct
         [ ] incorrect

3. THIS IS THE FIFTH ONE (point). HOW MANY ARE COVERED?

   a. [ ] correct - done

   b. [ ] incorrect - THIS ONE IS FIFTH (point). WHICH ONE IS THIS (point to fourth)

      [ ] correct - HOW MANY ARE COVERED

         [ ] correct immediate

         [·] correct, trial and error

         [ ] incorrect

      [ ] incorrect - FIFTH (point), FOURTH (point). HOW MANY ARE COVERED?

         [ ] correct

         [ ] incorrect

Appendix A.2: Test of Counting Back; Just Before; Just After; Between.

COUNTING BACK

WE ARE GOING TO PLAY A GAME. IT GOES LIKE THIS: (Count out 5 discs, from the child's left to right, then count backward from the fifth disc).

NOW YOU PLAY THE GAME (Give the child 3 discs).

A. YOU PLAY THE GAME (Give the child 8 discs).

[ ] correct. PLAY IT WITH THESE (12 discs).

[ ] correct COUNT BACKWARD FROM 15

[ ] wrong. PLAY WITH THESE (4 discs).

JUST BEFORE - JUST AFTER

TELL ME THE NUMBER THAT COMES JUST BEFORE 3

TELL ME THE NUMBER THAT COMES JUST AFTER 3

B. TELL ME THE NUMBER THAT COMES JUST BEFORE 14

[ ] correct. Stop

[ ] incorrect. JUST BEFORE 11

C. TELL ME THE NUMBER THAT COMES JUST AFTER 14

[ ] correct. Stop

[ ] incorrect. JUST AFTER 11

BETWEEN

CAN YOU GIVE ME A NUMBER THAT GOES BETWEEN 1 AND 3? (Show child card with 1, 2, 3, 4, 5 on it).

REMOVE CARD: STOP AT FIRST WRONG ANSWER

C. CAN YOU GIVE ME ANOTHER NUMBER THAT GOES BETWEEN 8 AND 12? [ ]

ANOTHER? [ ]

ANOTHER? [ ]

CAN YOU GIVE ME A NUMBER BETWEEN 8 AND 6? [ ]

Appendix A.3: Verbal Problems With Objects

1. BILL HAS 3 MARBLES. TOM GIVES HIM 5 MORE. HOW MANY MARBLES DOES BILL HAVE NOW?

2. THERE ARE 7 APPLES IN A BASKET. SALLY TOOK 5 OUT TO MAKE A PIE. HOW MANY APPLES ARE LEFT IN THE BASKET?

3. MIKE HAS 5 BLOCKS. HE FOUND SOME MORE. NOW HE HAS 8 BLOCKS. HOW MANY DID HE FIND?

4. THERE ARE 8 BUTTONS IN A BAG. JANE TOOK 2 BUTTONS OUT OF THE BAG TO SEW ON A DRESS. HOW MANY BUTTONS ARE LEFT IN THE BAG?

5. RON HAS 4 TOY CARS. MARY GIVES HIM 3 MORE TOY CARS. HOW MANY TOY CARS DOES RON HAVE NOW?

6. LORI HAS 3 JACKS IN HER HAND. SHE PICKED UP SOME MORE AND NOW HAS 7 IN HER HAND. HOW MANY DID SHE PICK UP?

Appendix A.4: Verbal Problems With No Objects

1. KEVIN HAS 4 CRAYONS. JERRY GIVES HIM 3 MORE CRAYONS. HOW MANY CRAYONS DOES KEVIN HAVE NOW?

2. THERE ARE EIGHT MARBLES IN A BAG. JANE TOOK 2 MARBLES OUT TO PLAY WITH. HOW MANY MARBLES ARE LEFT IN THE BAG?

3. SALLY HAS 5 PENNIES. HER MOTHER GIVES HER 5 MORE. HOW MANY DOES SALLY HAVE NOW?

4. TOM HAS 5 COMIC BOOKS. HE GOT SOME MORE FOR HIS BIRTHDAY. NOW HE HAS 8 COMIC BOOKS. HOW MANY MORE DID HE GET FOR HIS BIRTHDAY?

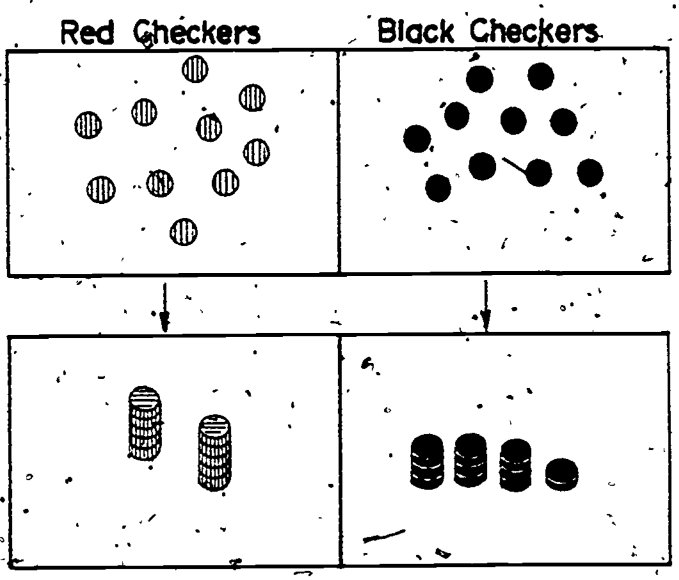
5. THERE ARE 7 NUTS IN A DISH. TOM TAKES 5 NUTS OUT TO EAT. HOW MANY NUTS ARE LEFT IN THE DISH?

6. MIKE HAS 3 CATS. HIS MOTHER GAVE HIM SOME MORE. HE NOW HAS 7. HOW MANY DID HIS MOTHER GIVE HIM?

Appendix A:5: Set Partitions Tasks

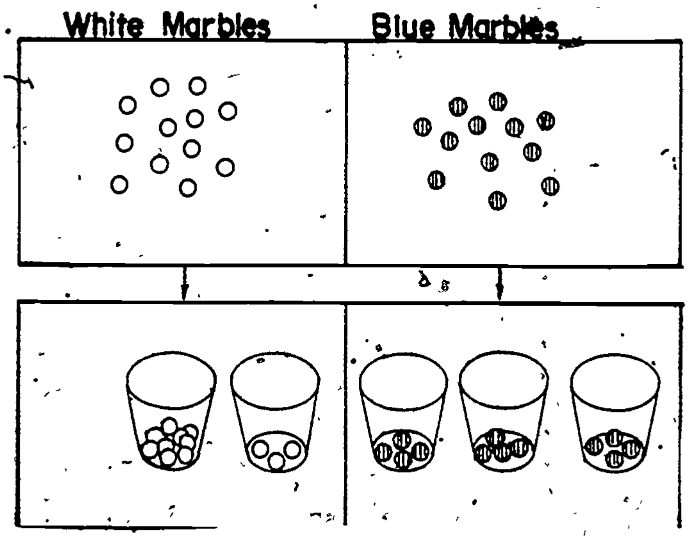
1. HOW MANY RED CHECKERS ARE HERE (child counts)?
HOW MANY BLACK CHECKERS ARE HERE (child counts)?
(Get s to count and agree that there are 10 of each. Then stack the blacks in stacks of 3, 3, 3, and 1 and the reds in stacks of 5 and 5).
TELL ME IF THERE ARE MORE BLACK CHECKERS, OR MORE RED ONES, OR IF THEY ARE THE SAME.

WHY?



2. HOW MANY WHITE MARBLES ARE HERE?
HOW MANY BLUE MARBLES ARE HERE?
(Get the child to count and agree there are 12 of each. Then put into transparent glasses, white 9 - 3 and blue 4 - 4 - 4).
TELL ME IF THERE ARE MORE WHITE MARBLES, OR MORE BLUE ONES, OF IF THEY ARE THE SAME.

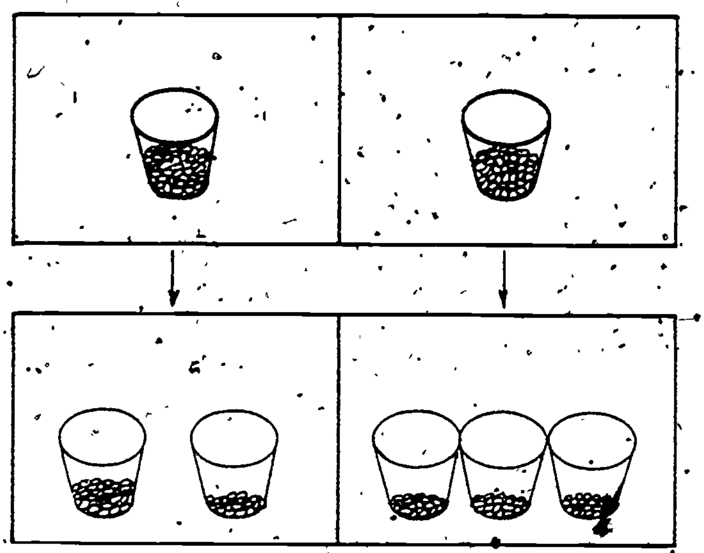
WHY?



3. The child was presented with two transparent glasses of beans filled to the same level, and told there were 100 beans in each. If necessary, adjustments were made so the child would agree there were the same number in both cups. The beans were then poured into transparent glasses, one into two glasses (most-few) and the other into three glasses evenly.

TELL ME IF THERE ARE MORE BEANS IN THESE CUPS (motion over the two glasses), OR MORE IN THESE CUPS (motion over the three glasses), OR IF THEY ARE THE SAME.

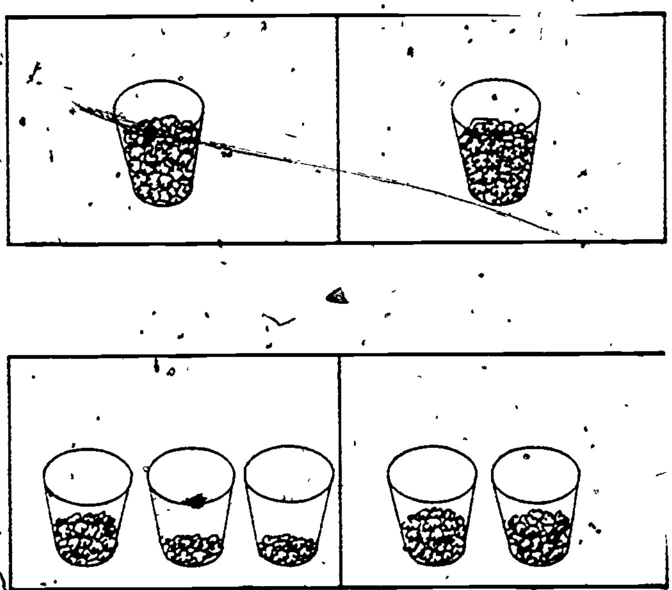
WHY?



4. The child was presented with two transparent glasses of kernels of popcorn filled to the same level, and <u>told</u> there were the same number of kernels in each. The popcorn was then poured into two transparent glasses, one into two glasses evenly and one into three glasses (most-few-few).

TELL ME IF THERE ARE MORE KERNEL'S OF POPCORN IN THESE CUPS (motion over the two galsses), OR MORE IN THESE CUPS (motion over the three glasses), OR IF THEY ARE THE SAME.

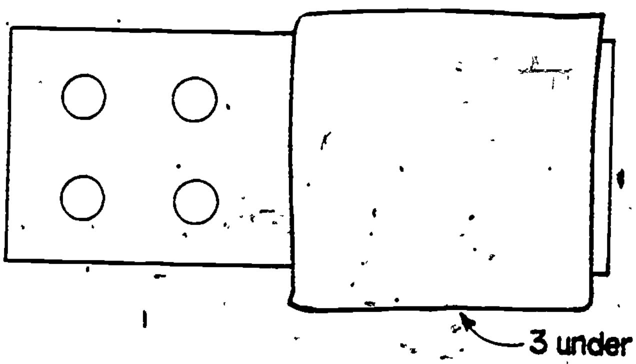
WHY?



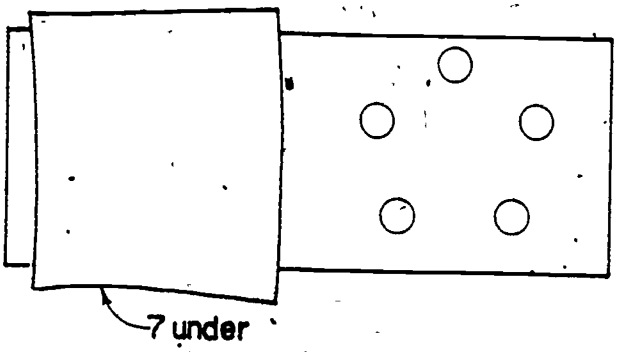
Appendix A.6: Addition and Subtraction of Ordinal Numbers Tasks

Addition of Ordinal Numbers.

1. START AT FOUR AND COUNT ON THREE MORE NUMBERS FROM FOUR (If unsuccessful, demonstrate).
2. START AT SEVEN AND COUNT ON FOUR MORE NUMBERS FROM SEVEN (If unsuccessful, demonstrate).
3. START AT TWELVE AND COUNT ON THREE MORE NUMBERS FROM TWELVE (If unsuccessful, demonstrate).
4. Three checkers covered with a cloth are presented to the child. Four visible checkers arranged randomly are also presented to the child.
   E: THERE ARE THREE CHECKERS UNDER THE CLOTH. COUNT ON TO FIND HOW MANY CHECKERS ARE THERE ON THE CARD?

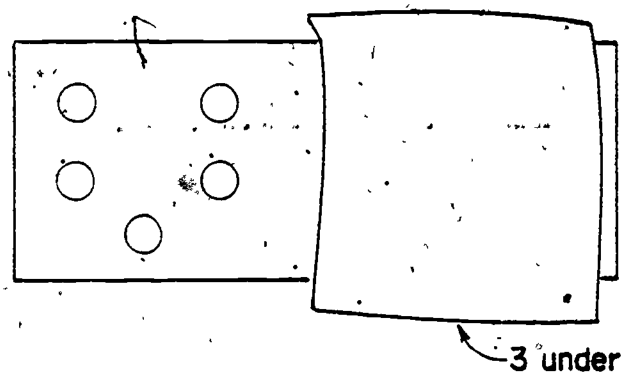


5. The same as (4) except seven checkers were under the cloth and five checkers were visible.

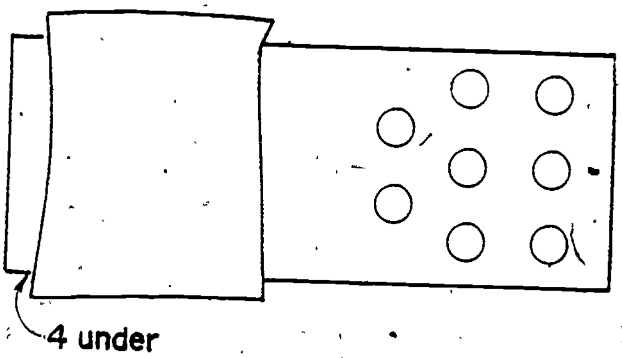


6. Three checkers covered with a cloth are presented to the child. Five visible checkers arranged randomly are also presented to the child.

E. HERE ARE FIVE CHECKERS. THERE ARE SOME MORE UNDER THE CLOTH. THERE ARE EIGHT CHECKERS IN ALL ON THE CARD. COUNT ON TO FIND HOW MANY CHECKERS ARE UNDER THE CLOTH.



7. The same as (6) except there are 12 checkers in all, eight visible.



Subtraction of Ordinal Numbers

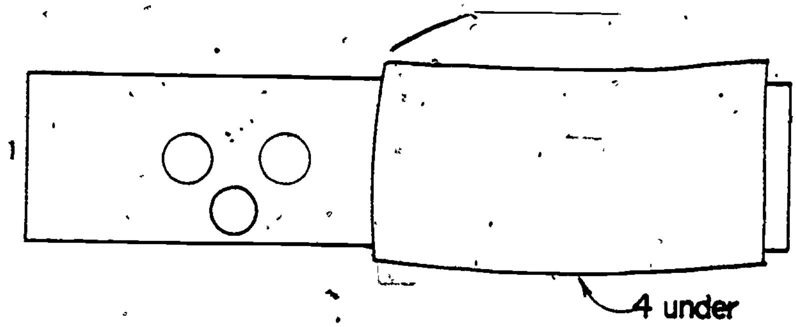
1. START AT FOUR AND COUNT BACK THREE NUMBERS (If unsuccessful, demonstrate).

2. START AT SEVEN AND COUNT BACK THREE NUMBERS (If unsuccessful, demonstrate).

3. START AT TWELVE AND COUNT BACK FOUR NUMBERS. (If unsuccessful, demonstrate).

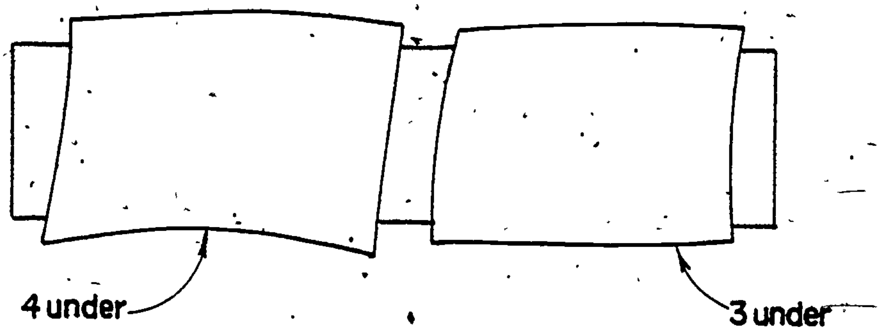
4. Four checkers covered with a cloth are presented to the child. Three visible checkers arranged randomly are also presented to the child.

E. THERE ARE SOME CHECKERS UNDER THE CLOTH. I COUNTED THEM ALL ON THE CARD AND THERE ARE SEVEN. COUNT BACK, STARTING AT SEVEN, TO FIND OUT HOW MANY ARE UNDER THE CLOTH.

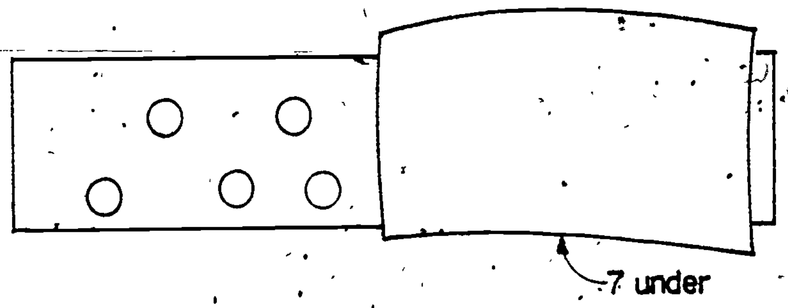


5. Seven checkers, four under one cloth and three under another cloth, are presented to the child.

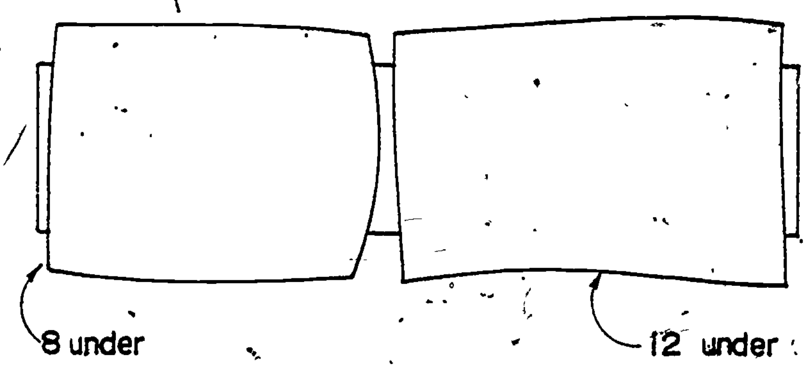
E. THERE ARE SEVEN CHECKERS ON THE CARD UNDER THESE CLOTHS. THERE ARE FOUR CHECKERS UNDER THIS CLOTH (point). COUNT BACK, STARTING AT SEVEN, TO FIND OUT HOW MANY ARE UNDER THIS OTHER CLOTH (point).



6. The same as (4), except there are seven checkers covered and five visible.



7. The same as (5), except there are four checkers covered under one cloth and eight under the other. The child is asked to count back from 12 fo find how many are under the cloth with four covered.



Appendix A.7: Mental Arithmetic Test

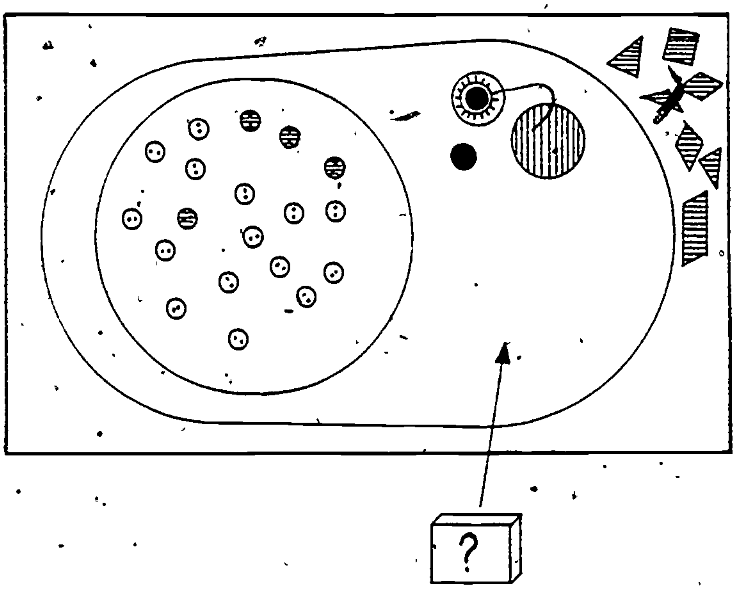
E. HERE ARE SOME ADDITION AND SUBTRACTION PROBLEMS. I WOULD LIKE YOU TO SOLVE THEM. DO NOT USE YOUR FINGERS TO HELP YOU, OR MAKE MARKS ON YOUR PAPER.

1. 5 + 3 = ______
2. 9 + 2 = ______
3. 8 - 2 = ______
4. 11 - 3 = ______

Appendix A.8: Nested Classification Tasks.

Task A: NESTED CLASSIFICATION

HERE ARE BUNCH OF THINGS. PUT ALL THE ROUND THINGS INSIDE THIS (big) STRING. (Put small string inside) PUT ALL THE BUTTONS INSIDE THIS (small) STRING.



Warm-up tasks (correct child's mistakes)

(green felt square) PLACE THIS WHERE IT GOES.

(brown round button) PLACE THIS WHERE IT GOES.

(green wooden disc) PLACE THIS WHERE IT GOES.

(black square button) PLACE THIS WHERE IT GOES.

Questions

THERE IS SOMETHING IN THIS BOX THAT GOES WITH THESE. (point and place box with round things not buttons).

1. COULD IT BE A SQUARE?

   [ ] no   [ ] can't tell

   a. [ ] yes   IS IT A SQUARE   [ ]no   [ ]yes   [ ]can't tell

2. COULD IT BE A BUTTON?

   [ ] no   [ ] can't tell

   a.   yes   IS IT A BUTTON?   [ ]no   [ ]yes   [ ] can't tell

3. COULD IT BE ROUND?

   [ ] no   [ ] can't tell

   a. [ ] yes   [ ]no   [ ]yes   [ ]can't tell

4. COULD IT BE BLUE?

   [ ]no-STOP

   [ ]yes   [ ] can't tell

   a. COULD IT BE A BLUE BUTTON?

      [ ]yes - STOP

      [ ]no

   b. COULD IT BE A BLUE CIRCLE?

      [ ]no - STOP

      [ ]yes

   c. COULD IT BE A BLUE SQUARE?

      [ ]no

      [ ]yes

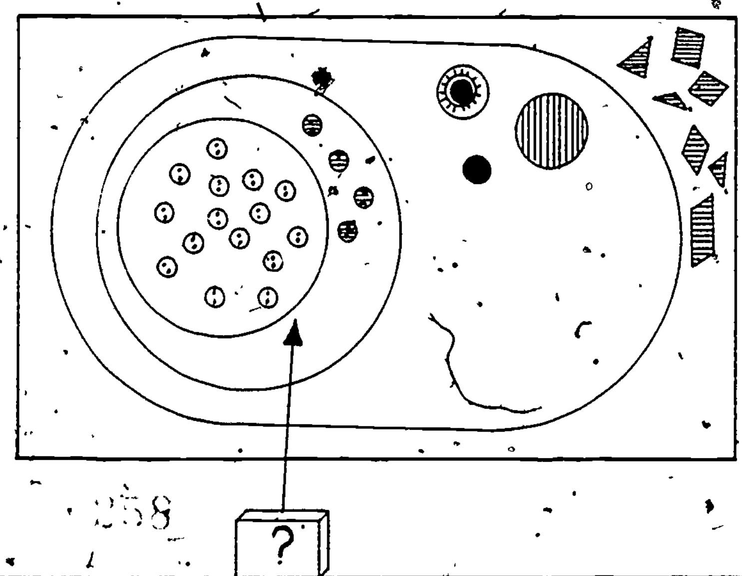
5. COULD IT BE A CHECKER?

   [ ]no   [ ]can't tell

   [ ]yes   DOES IT HAVE TO BE A CHECKER?   [ ]no   [ ]yes

Task B: NESTED CLASSIFICATION

(Place new loop inside buttons) PLACE ALL THE WHITE BUTTONS INSIDE THIS LOOP.



Warm-up tasks (correct child's mistakes)

(brown square tile) PLACE THIS WHERE IT GOES.

(blue wooden disc) PLACE THIS WHERE IT GOES.

(white square tile) PLACE THIS WHERE IT GOES.

(white felt circle) PLACE THIS WHERE IT GOES.

Questions

THERE IS SOMETHING IN THIS BOX THAT GOES WITH THESE. (point and place box with nonwhite buttons.)

1. COULD IT BE ROUND?

   a [ ] no IS IT A BUTTON? [ ] no [ ] yes [ ]can't tell

   [ ] yes IS IT ROUND? [ ] no [ ] yes [ ]can't tell

   [ ] can't tell

2. COULD IT BE A WHITE BUTTON?

   [ ] no [ ] can't tell

   [ ] yes IS IT A WHITE BUTTON? [ ] no [ ] yes

3. COULD IT BE A SQUARE?

   [ ]no [ ]can't tell

   a [ ]IS IT A SQUARE? [ ]no [ ]yes

4. COULD IT BE RED?

   [ ]no [ ]can't tell

   a [ ]yes COULD IT BE A RED CHECKER? [ ]no [ ]yes

5. COULD IT BE WHITE?

   no can't tell

   a yes IS IT WHITE? [ ] no [ ] yes

6. COULD IT BE A BUTTON?

   [ ]no [ ]can't tell

   a [ ]yes IS IT A BUTTON? [ ]no [ ]yes

NESTED CLASSIFICATION SUPPLEMENT

Check out

POINT TO ALL THE BUTTONS.

POINT TO ALL THE WHITE BUTTONS.

POINT TO ALL THE ROUND THINGS.

Questions

WHICH ARE THERE MORE OF, BUTTONS OR WHITE BUTTONS? WHY?

WHICH ARE THERE MORE OF, ROUND THINGS OR WHITE BUTTONS? WHY?

(STOP if both of the previous questions are correct.)

Otherwise -- (pretend the buttons are candy)

WHICH WOULD YOU RATHER HAVE, ALL THE CANDY OR ALL THE WHITE CANDY? WHY?

If correct - WHICH ARE THERE MORE OF, BUTTONS OR WHITE BUTTONS? WHY?

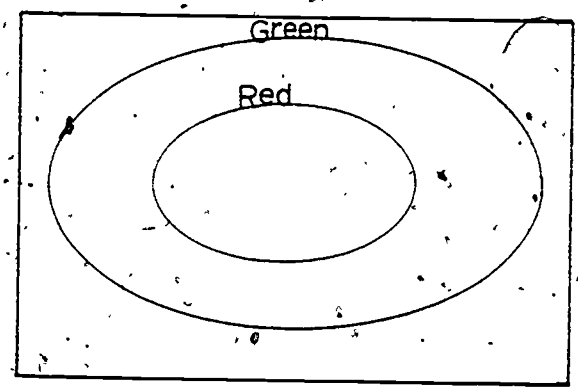
Appendix A.9: Loop Inclusion Tasks

Warm-up

(one loop, stick inside) THIS STICK IS INSIDE BECAUSE I CAN'T PULL THE LOOP OFF. (attempt to pull)

(two loops, orange inside blue, stick inside blue, outside orange) THIS STICK IS INSIDE THE BLUE LOOP BECAUSE I CAN'T PULL IT OFF. (attempt) THIS STICK IS NOT INSIDE THE ORANGE LOOP BECAUSE I CAN PULL IT OFF. (pull it off)

Task A



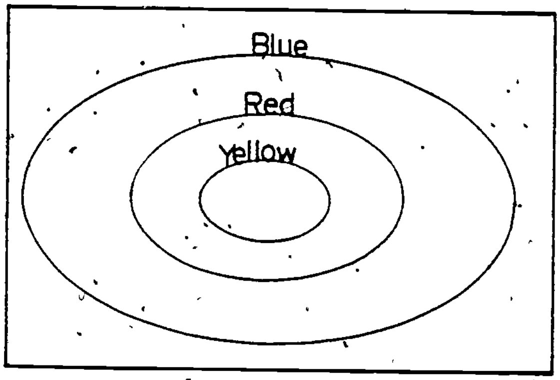
PUT THE STICK INSIDE THE RED LOOP BUT NOT THE GREEN ONE.

PUT THE STICK INSIDE THE RED LOOP AND THE GREEN ONE.

COULD YOU PUT THE STICK INSIDE THE GREEN LOOP BUT NOT INSIDE THE RED LOOP?

[ ]no [ ]yes SHOW ME HOW.

Task B

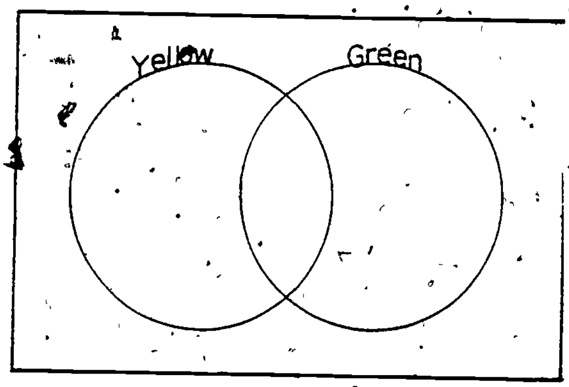


PUT THE STICK INSIDE THE BLUE RING ONLY.

PUT THE STICK INSIDE ALL THREE RINGS.

PUT THE STICK INSIDE EXACTLY TWO RINGS.

Task C

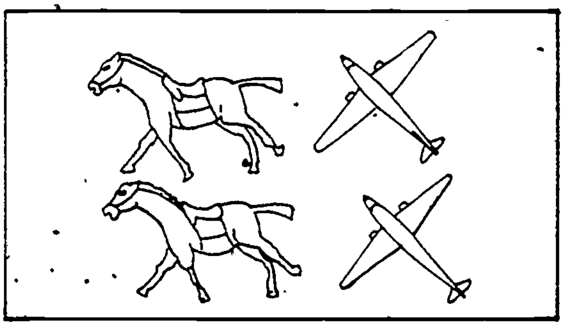


PUT THE STICK INSIDE THE YELLOW LOOP BUT NOT THE GREEN ONE.

PUT THE STICK INSIDE THE YELLOW LOOP AND THE GREEN ONE.

Appendix A.10: Class inclusion

Item 1.

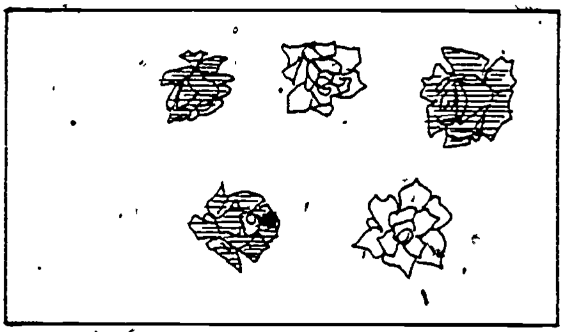


POINT TO THE AIRPLANES. POINT TO THE HORSES. POINT TO THE TOYS.

WHICH ARE THERE MORE OF, TOYS OR AIRPLANES?

WHICH ARE THERE MORE OF, AIRPLANES OR TOYS?

Item 2.

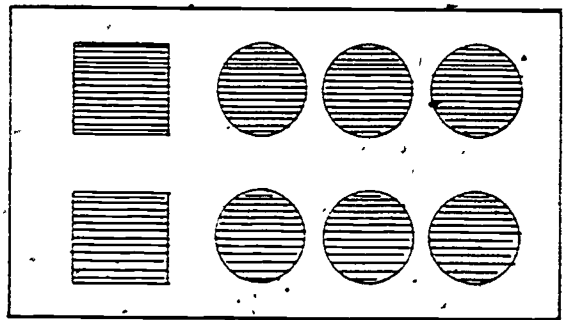


POINT TO THE FLOWERS, POINT TO THE WHITE FLOWERS, POINT TO THE RED FLOWERS.

WHICH ARE THERE MORE OF, RED FLOWERS OR FLOWERS?

WHICH ARE THERE MORE OF, FLOWERS OR RED FLOWERS?

Item 3.



POINT TO THE RED SHAPES.

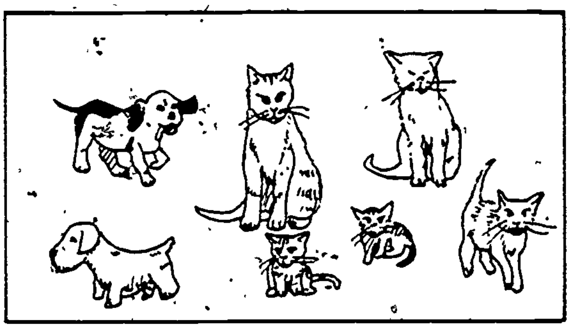
POINT TO THE SQUARE SHAPES.

POINT TO THE ROUND SHAPES.

WHICH ARE THERE MORE OF, ROUND SHAPES OR RED SHAPES?

WHICH ARE THERE MORE OF, RED SHAPES OR ROUND SHAPES?

Item 4.



POINT TO THE DOGS.

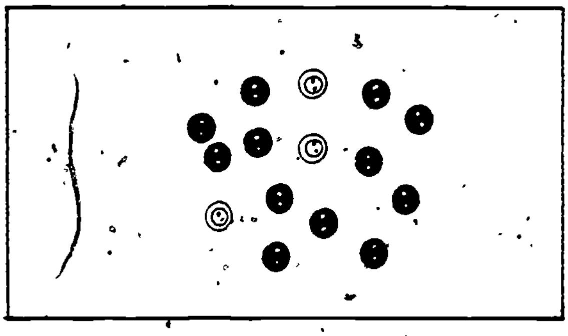
POINT TO THE CATS.

POINT TO THE ANIMALS.

WHICH ARE THERE MORE OF, ANIMALS OR CATS?

WHICH ARE THERE MORE OF, CATS OR ANIMALS?

Item 5.



POINT TO THE BUTTONS.

POINT TO THE WHITE BUTTONS.

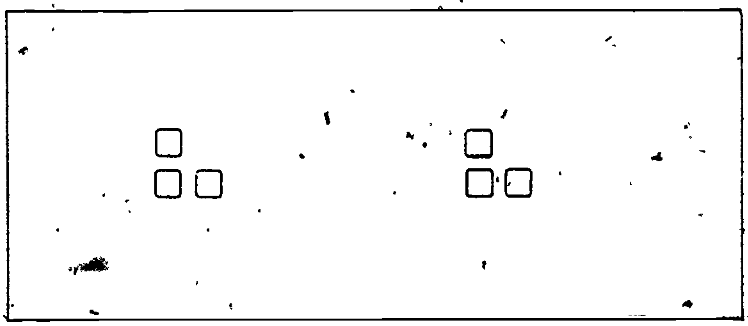
POINT TO THE BLACK BUTTONS.

WHICH ARE THERE MORE OF; BLACK BUTTONS OR BUTTONS?

WHICH ARE THERE MORE OF, BUTTONS OR BLACK BUTTONS?

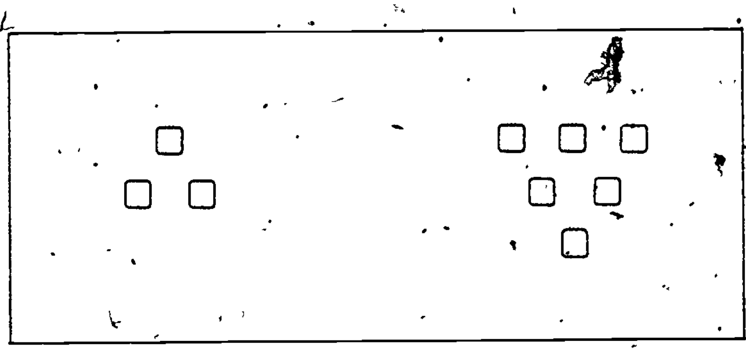
APPENDIX A.11. Quantitative Comparisons

Item W-1. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



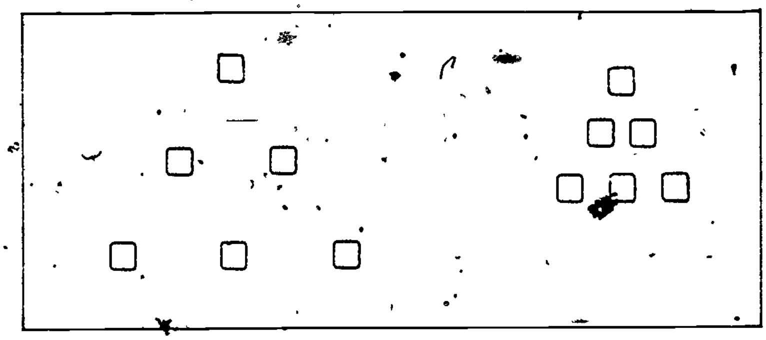
GREEN RED

Item W-2. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



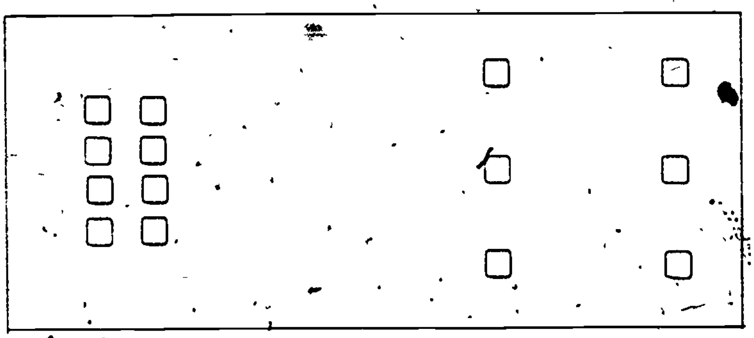
RED GREEN

Item 1. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



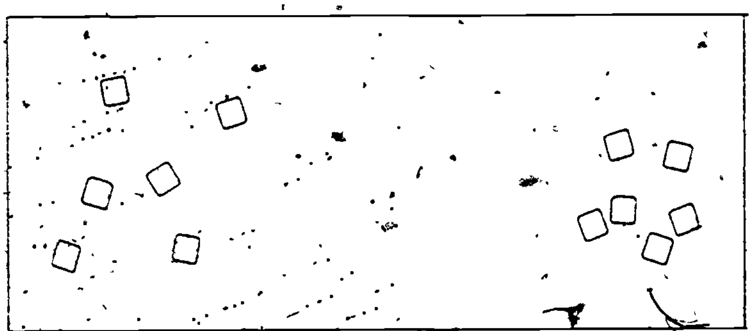
RED GREEN

Item 2. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



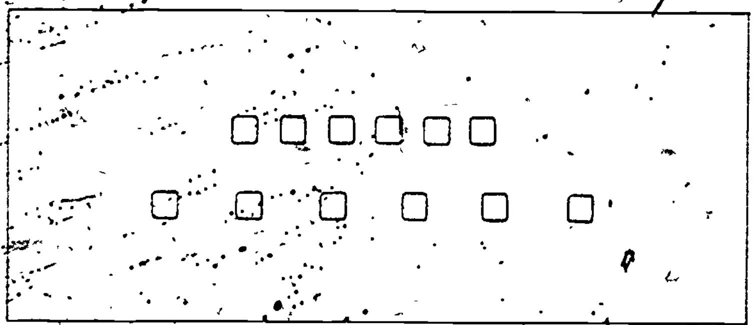
GREEN RED

Item 3. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



RED GREEN

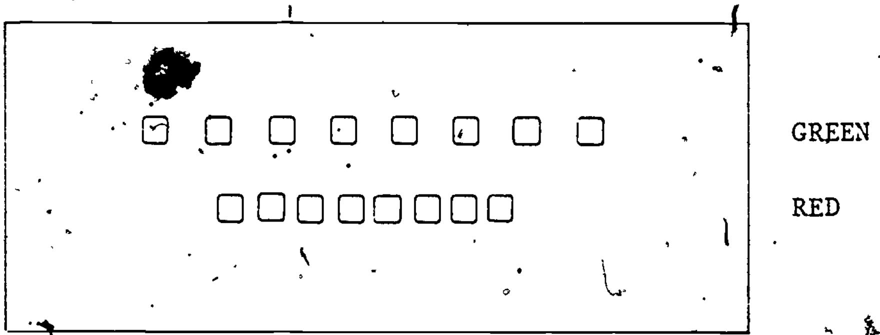
Item 4. TELL ME IF THERE ARE MORE RED ONES OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



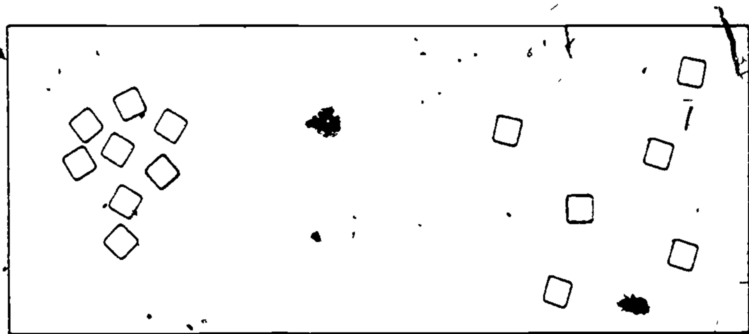
GREEN

RED

Item 5. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?

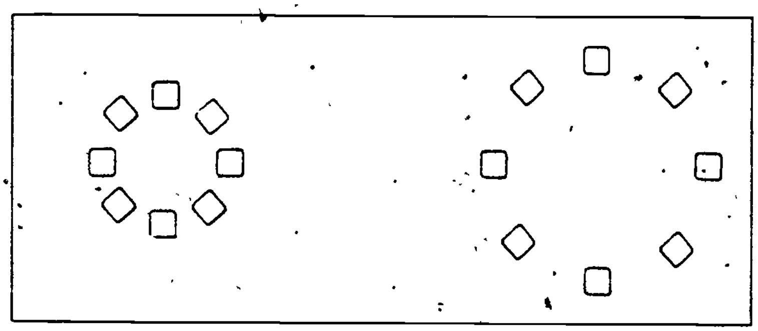


Item 6. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



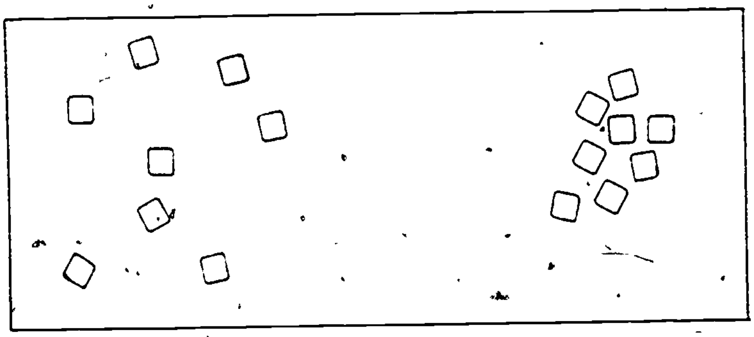
GREEN RED

Item 7. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



RED GREEN

Item 8. TELL ME IF THERE ARE MORE RED ONES, OR MORE GREEN ONES, OR IF THEY ARE THE SAME. WHY?



GREEN RED

Appendix A.12. PMDC Tests

PMDC ARITHMETIC TEST, GRADE 1

STUDENT'S NAME ____________

STUDENT PROFILE SHEET

STUDENT'S ID NUMBER ____________

Area	Item Response Number	TASK	YES	NO	COMMENTS
Elementary Counting	1 or 2	Count from 1 to 35			
	3	Construct a set, using beans, corresponding to a written numeral (6)			
	10 or 11	Count from 6 to 15			
	14	Count a picture set of horses (3)			
	15	Count a picture set of cows (7)			
	16 or 17	Count a picture set of animals (10)			
	35	Construct a set with 3 members in response to oral directions			
	36	Construct a set with 4 members in response to oral directions			
	46 or 47	Count back from 6 to 1			
Advanced Counting	23 or 24	Count from 90 to 112			
	29 or 30	Count by tens from 10 to 130			
	40 or 41	Count by twos from 2 to 20			
	44 or 45	Count by tens to determine the number of crayons in four boxes, each with 10 crayons			
Set Equivalence	9	Establish the number equivalence (5) of two picture sets without explicit directions to count the sets or to establish 1-1 matching between the sets			
	12	Construct a set with more members than a given pictured set (7)			
	20 or 22	Determine whether two sets have the same number (9) of members after the two sets were constructed by 1-1 matching			
	26	Determine the number of members in a set having established that it is equivalent to a set with 7 members			
	28	Construct a set with less members than a given pictured set (7)			
	42	Construct a set with one more member than a given pictured set (7)			
	48	Construct a set with one less member than a given pictured set (7)			
	52	Construct a set with the same number (7) of members as a given pictured set			
Ordering Numbers	4	Tell the number which comes just after a given number (3)			
	5	Tell the number which comes just after a given number (8)			
	6	Tell the number which comes just after a given number (13)			
	32	Tell the number which comes just before a given number (5)			
	33	Tell the number which comes just before a given number (3)			
	34	Tell the number which comes just before a given number (14)			
	49	Tell the number which comes between two numbers (3 and 5)			
	50	Tell the number which comes between two numbers (7 and 9)			
	51	Tell the number which comes between two numbers (14 and 16)			
	53	Tell the number which comes between two numbers (6 and 4)			
Addition Subtraction	13	Solve an addition problem-solving exercise (sum 5), oral directions			
	27	Solve a subtraction problem-solving exercise (minuend 7), oral directions			
	31	Find the number of a picture set (C) where one subset is explicitly shown (A) and a second subset is covered (B). N (B) is given			
	37	Given two disjoint sets (with 3 and 4 elements), determine how many altogether without joining the sets			
	39	Determine the number (7) of a set which was formed by joining two disjoint sets with 3 and 4 members			
	43	Solve a missing addend problem-solving exercise (sum 6), oral directions			
Class Inclusion	8	Answer a class inclusion question, without explicit direction, to count the sets (numbers 10 or less)			
	18	Answer a class inclusion question, after having counted the members in each set (numbers 10 or less)			

PMDC ARITHMETIC TEST, GRADE 2

STUDENT'S NAME ____________________

STUDENT PROFILE SHEET

STUDENT'S ID NUMBER ____________________

	Item Response Number	TASK	YES	NO	COMMENTS
Elementary Counting	1	Count a picture set of dots (13)			
	6 or 7	Count from 6 to 15			
	9	Count a picture set of horses (3)			
	10	Count a picture set of cows (7)			
	11 or 12	Count a picture set of animals (10)			
	14 or 15	Count from 35 to 46			
	21 or 22	Count back from 6 to 1			
Advanced Counting	29 or 30	Count back from 44 to 25			
	32 or 33	Count by tens from 10 to 130			
	39	Determine the number of a set represented by 6 bundles of ten straws			
	45	Determine the number of a set represented by 5 red chips, each red chip stands for ten			
	51 or 52	Count by tens from 26 to 126			
Place Value	40	Write the numeral for a set represented by 3 bundles of ten straws and 7 single straws			
	41	Tell the number of a set represented by 3 bundles of ten straws and 7 single straws			
	42	Construct a set using bundles of ten straws and single straws corresponding to a written numeral (34)			
	43	Construct a set using bundles of ten straws and single straws, with a given number of members (45), in response to oral directions			
	46	Write the numeral for a set represented by 5 red chips [each stands for 10] and 3 white chips [each stands for 1]			
	47	Tell the number of a set represented by 5 red chips [each stands for 10] and 3 white chips [each stands for 1]			
	48	Construct a set using red chips (10 each) and white chips (1 each) corresponding to a written numeral (37)			
	49	Construct a set using red chips (10 each) and white chips (1 each) to represent a given number (52), in response to oral directions			
Addition Subtraction	5	Use counters (beans) to solve an addition problem, sum 6			
	8	Use counters (beans) to solve a subtraction problem, minuend 7			
	23	Use counters (beans) to solve a addition problem, 2-digit (18) plus 1-digit (5)			
	28	Use counters (beans) to solve a subtraction problem, 2-digit (23) minus 1-digit (7)			
Missing Addends	20	Solve a written missing addend problem, sum 9			
	38	Solve a written missing addend problem, sum 27			
	44	Solve a written missing addend problem, involving multiples of 10 only			
	50	Solve a written missing addend problem, answer a multiple of ten (30)			
Patterns	34 & 35	Solve (without computation) an addition problem by using a related equation, 2-digit sum (24)			
	36 & 37	Solve (without computation) an addition problem by using a related equation, 3-digit sum (105)			
Ordering Numbers	2	Order four numbers (2, 3, 5 and 9) from smallest to largest			
	16	Tell which of two numbers (8 and 12) is more			
	24	Tell which of two numbers (19 and 31) is more			
	31	Tell which of two numbers (7 and 4) is less			
Equivalent Names	17	Identify names for the same number (6+3 and 5+4)			
	18	Identify names for the same number (4+1 and 3+2)			
	19	Identify names for the same number (6-1 and 3+2)			
	25	Identify names for the same number (5-2 and 4-1)			
	26	Identify names for the same number (10-5 and 7-2)			
	27	Identify names for the same number (4+1 and 7-2)			
[illegible]	4	Answer a class inclusion question, without explicit directions to count the members of sets (numbers 10 or less)			
	13	Answer a class inclusion question, after having counted the members in each set (numbers 10 or less)			

APPENDIX B

RECORD SHEETS

1. Class Inclusion Record Sheet

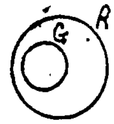
Item 1		
( ) airplanes	TOYS or airplanes	WHY?
( ) horses		
( ) toys	airplanes or TOYS	
Item 2		
( ) flowers	red flowers or FLOWERS	WHY?
( ) white flowers		
( ) red flowers	FLOWERS or red flowers	
Item 3		
( ) red shapes	round shapes or RED SHAPES	WHY?
( ) square shapes		
( ) round shapes	RED SHAPES or round shapes	
Item 4		
( ) dogs	ANIMALS or cats	WHY?
( ) cats		
( ) animals	cars or ANIMALS	
Item 5		
( ) buttons	black buttons or BUTTONS	WHY?
( ) white buttons		
( ) black buttons	BUTTONS or black buttons	

Comments:

2. Loop Inclusion Record Sheet

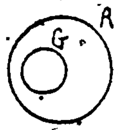
TASK A:

Inside red, not green.



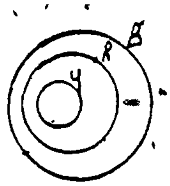
Inside red and green.

Inside green, not red.

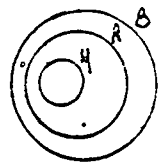


TASK B:

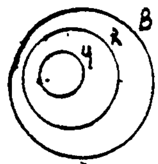
Inside blue only



Inside all three.



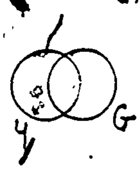
Inside exactly two.



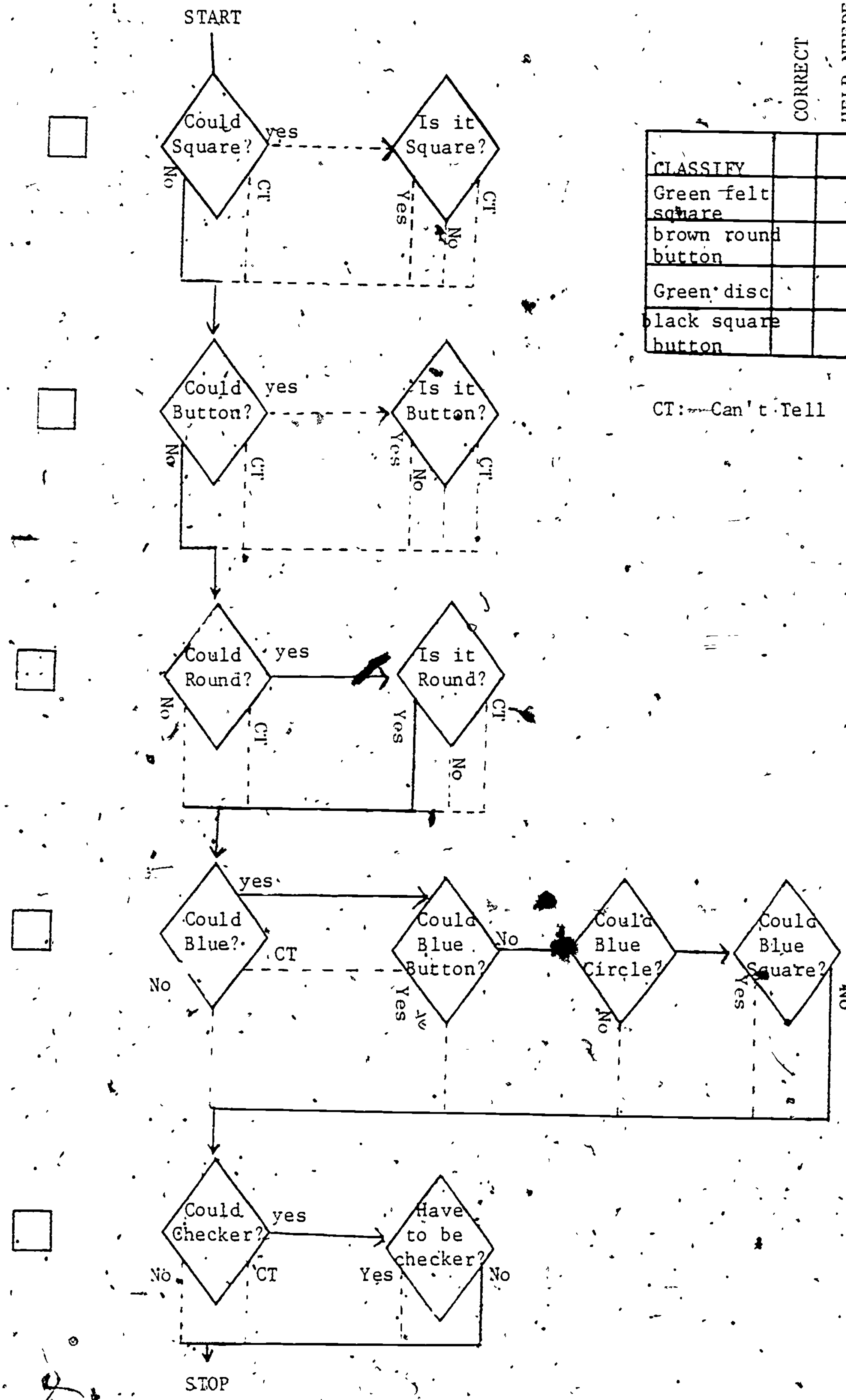
TASK C:

Inside yellow, not green.

Inside yellow and green.



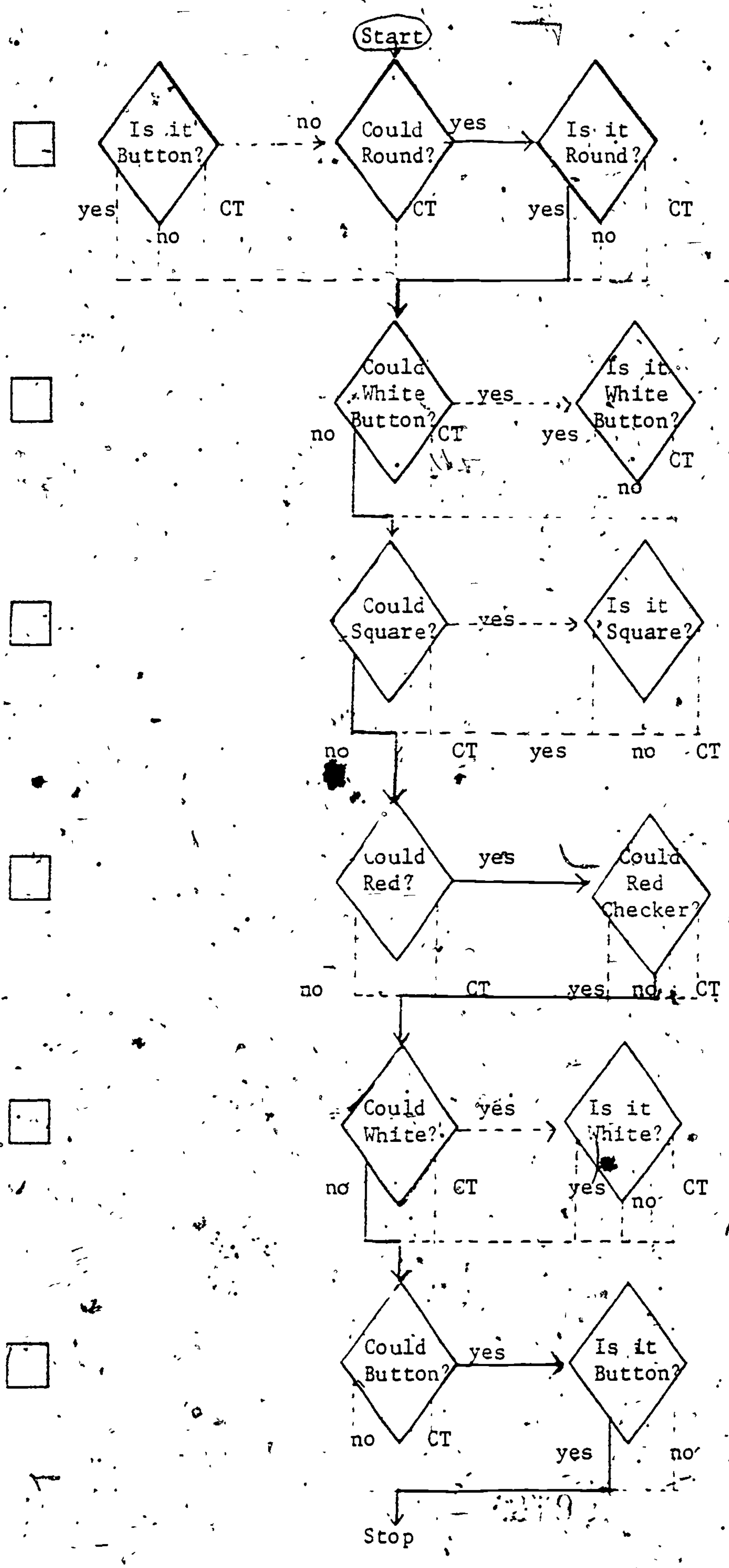
3. Nested Classification Record Sheet (Task A)



CLASSIFY	CORRECT	HELP NEEDED
Green felt square		
brown round button		
Green disc		
black square button		

CT: Can't Tell

4. Nested Classification Record Sheet (Task B).

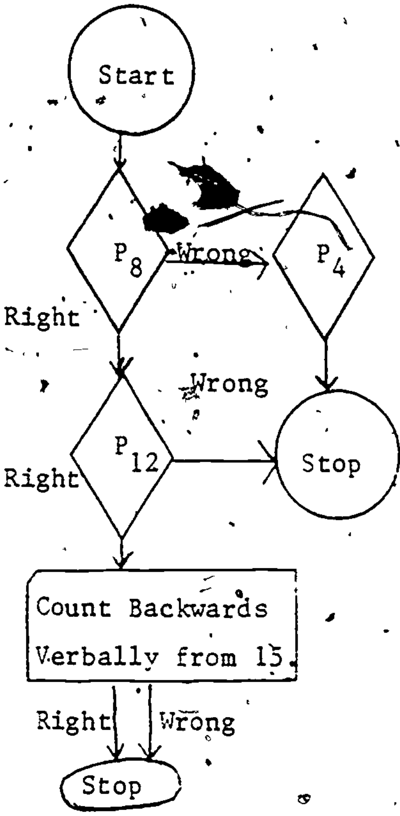


CLASSIFY	Correct	Needed Help
Brown Sq. Tile		
Blue Disc		
White Sq. Tile		
White Felt Circle		

Class Inclusion Suppl.

Buttons	white buttons
Round Things	white buttons
All Candy	all white candy
Buttons	white buttons

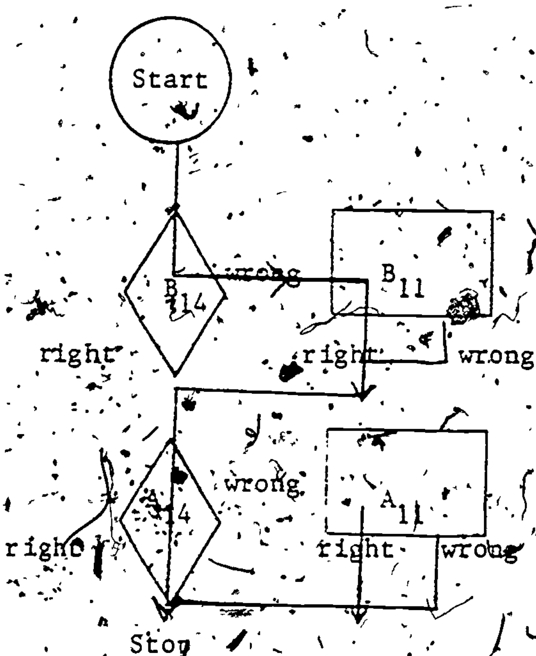
5. Counting Back and Just Before - Just After Record Sheet.



P_8	P_{12}	verbal

1=correct
0=incorrect
x=omit

Comments:

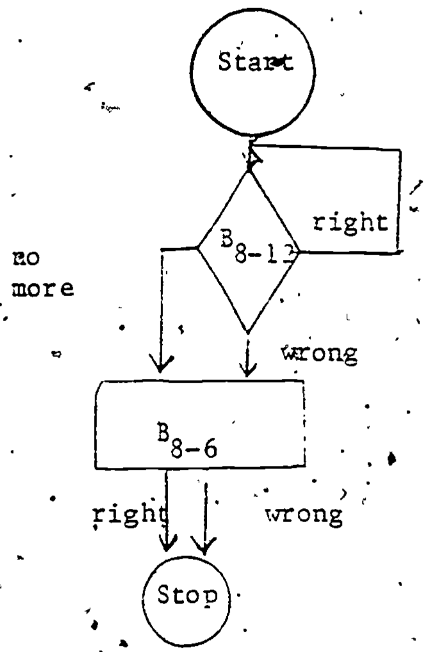


	14	11
before		
after		

1=correct
0-H =high
0-L =low
x=omit

Comments:

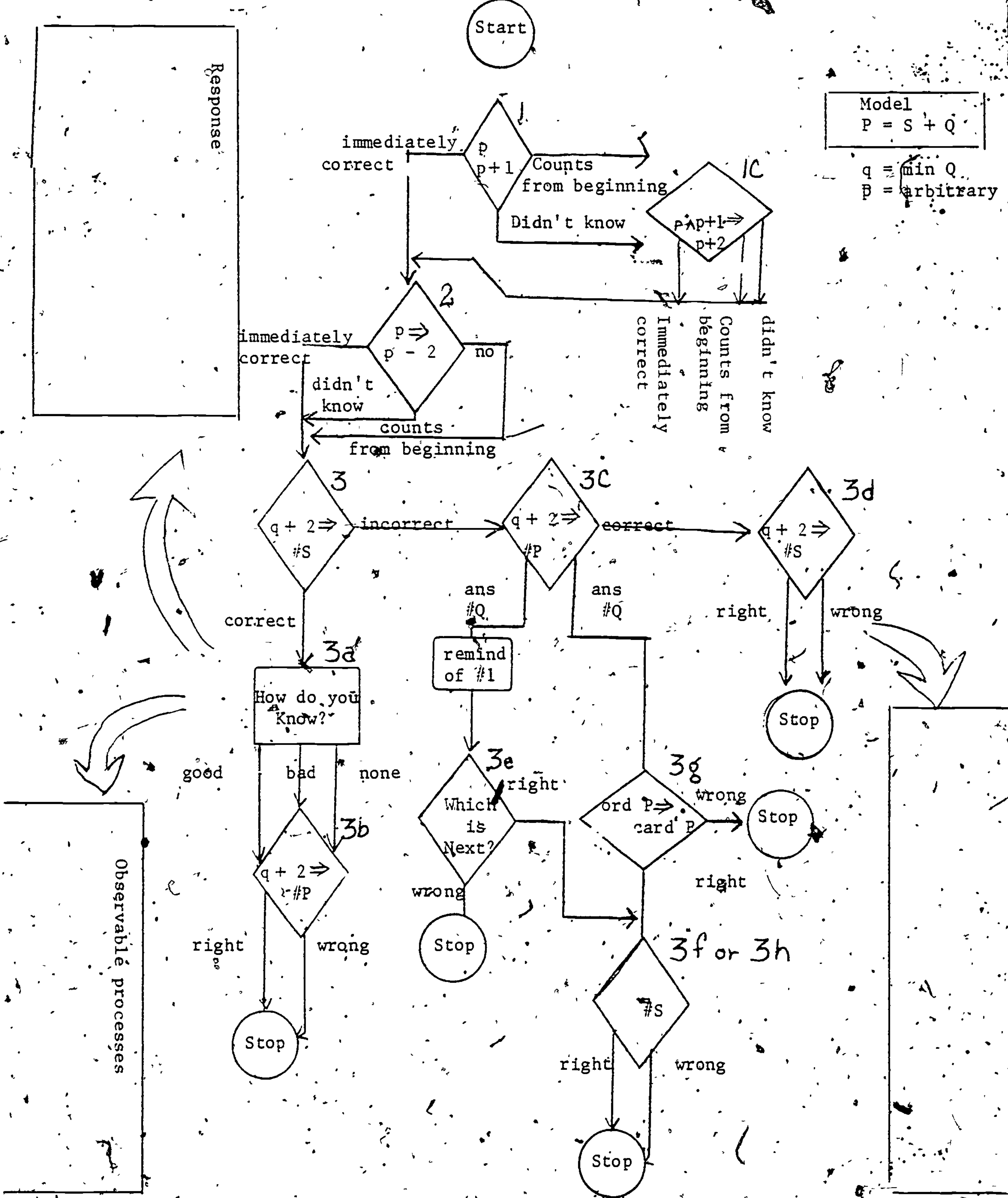
6. Between Record Sheet.



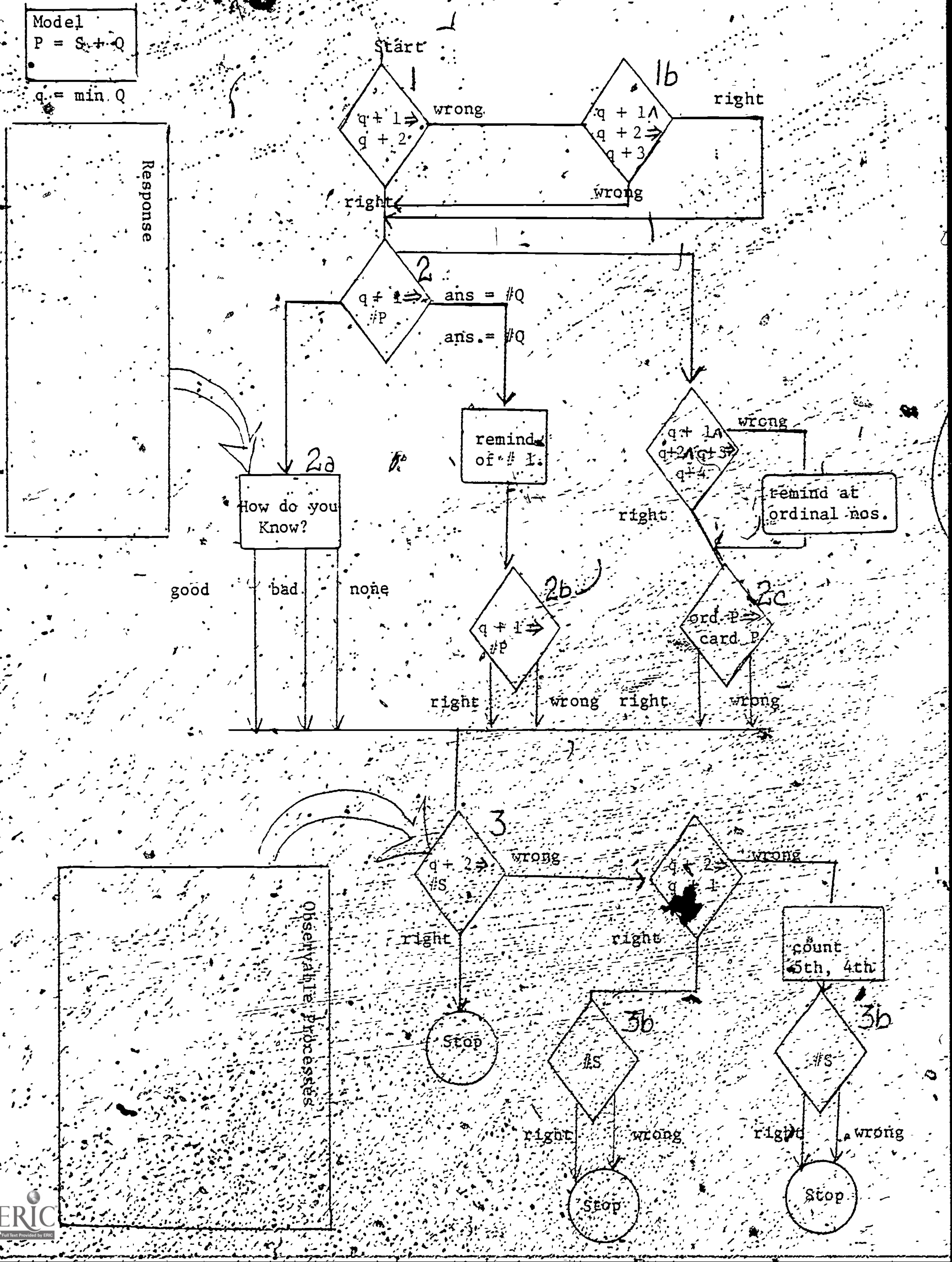
B_{8-12}		B_{8-6}
# right answers	exit	exit

1=correct
0-H=ans high
0-L=ans low

7. Cardinal-Ordinal Number Task A Record Sheet.



8. Cardinal-Ordinal Number Task B Record Sheet.



9. Quantitative Comparisons and Verbal Problems With Objects Record Sheet

ITEM	more red	more green	same	counted to ans	counted after why	WHY?
1	(more red)	more green	SAME	____	____	
2	(more red)	MORE GREEN	same	____	____	
3	(more red)	more green	SAME	____	____	
4	(more red)	more green	SAME	____	____	
5	more red	MORE GREEN	same	____	____	
6	(more red)	MORE GREEN	same	____	____	
7	more red	(more green)	SAME	____	____	
8	more red	(more green)	SAME	____	____	

[parens = larger display] [caps = correct ans]

Verbal Problems

no.	answer	used objects	Observable processes [none or describe]
1	8		
2	2		
3	3		
4	6		
5	7		
6	4		

10. Partitions Test Record Sheet.

[1] more red | more black | same

WHY?

[2] more blue | more white | same

WHY?

[3] more in 2 | more in 3 | same

WHY?

[4] more in 2 | more in 3 | same

WHY?

11. Mental Arithmetic and Verbal Problems Without Objects Record Sheet.

Record Sheet
Tape 3

Mental Arithmetic

response	time (sec)	Observable Processes [none or describe]
5 + 3		
9 + 2		
8 - 2		
11 - 3		

Verbal Problems

NO.	read by E,S,E+S (1)	first P,W (2)	eqn role C,O (3)	answer	equation	Observable Processes [none or describe]
1				7	4 + 3 = 7	
2				6	8 - 2 = 6	
3				8	3 + 5 = 8	
4				3	5 + 3 = 8	
5				2	7 - 5 = 2	
6				4	3 + 3 = 7	

(1) $S \leq 4$ wds $\Rightarrow$ E // $E \leq 4$ wds $\Rightarrow$ S // $E > 4$ & $S > 4$ wds $\Rightarrow$ E + S

(2) P = process information // W = write equation

(3) C = closed sentence // O = open sentence

12. Addition and Subtraction of Ordinal Numbers Record Sheet

ADDITION

write [1]
what's [2]
said [3]

		Observable Processes P = S + Q					Comments
		immed fact	count on (Q)		count all (P)		
	ans	ANS	CP	ANS	CP	ANS	
[4]	7	P		P		P 8	
[5]	12	P		P		P 10	
[6]	3	Q		Q		Q	
[7]	4	Q		Q		Q	

SUBTRACTION

write [1]
what's [2]
said [3]

		Observable Processes P = S + Q							source of cardinality of S		
		immed fact	count back (Q)			count all (P)					
	ans	ANS (S)	CP	minQ	ANS (S)	CP	minQ	ANS (S)	minQ	tally	other
[4]	4	S			S			S			
[5]	3	S			S			S			
[6]	7	S			S			S			
[7]	8	S			S			S			

CP = used correct process

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ABSTRACT

An exploratory investigation designed to gain insights into children's mathematical formulation of observed actions upon objects is presented. Eight episodes in which first and second graders were asked to interpret, in terms of number sentences, a sequence of actions with unifix cubes are also presented. Results of analysis of the videotaped episodes are presented and discussed in relation to children's concepts of equality. (MS)

PMDC Technical Report
No. 14

First and Second Grade Children's Interpretation of Actions upon Objects

Eugene D. Nichols

pmdc

1976

Financial support for the Project for the Mathematical Development of Children has been provided by the National Science Foundation: Grant No. PES 74-18106-A03.

PREFACE

This publication shares with the interested individuals the results of an exploratory investigation designed to gain insights into the children's mathematical formulation of observed actions upon objects. It is hoped that the reader interested in research on young children's mathematical thinking will find this publication a source of ideas for further exploration of this area.

A special gratitude is expressed to two doctoral students in Mathematics Education at the Florida State University: Patricia Campbell, for assisting the author with the managerial aspects of the interviews, and Max Gerling, for videotaping the interviews.

Thanks are due to the Project administrative assistant, Janelle Hardy, for coordinating the technical aspects of the preparation of the report, and to Joe Schmerler for the typing.

FOREWORD

Ed Begle recently remarked that curricular efforts during the 1960's taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better. The mathematician will, quite likely, agree with both parts of this statement. The layman, the parent, and the elementary school teacher, however, uqestion the thesis that the "new math" was really better than the "old math." At best, the fruits of the mathematics curriculum "revolution" were not sweet. Many judge them to be bitter.

While some viewed the curricular changes of the 1960's to be "revolutionary," others disagreed. Thomas C. O'Brien of Southern Illinois University at Edwardsville recently wrote, "We have not made any fundamental change in school mathematics."[1] He cites Allendoerfer who suggested that a curriculum which heeds the ways in which young children learn mathematics is needed. Such a curriculum would be based on the understanding of children's thinking and learning. It is one thing, however, to recognize that a conceptual model for mathematics curriculum is sound and necessary and to ask that the child's thinking and learning processes be heeded; it is quite another to translate these ideas into a curriculum which can be used effectively by the ordinary elementary school teacher working in the ordinary elementary school classroom.

Moreover, to propose that children's thinking processes should serve as a basis for curriculum development is to presuppose that curriculum makers agree on what these processes are. Such is not the case, but even if it were, curriculum makers do not agree on the implications which the understanding of these thinking processes would have for curriculum development.

In the real world of today's elementary school classroom, where not much hope for drastic changes for the better can be foreseen, it appears that in order to build a realistic, yet sound basis for the mathematics curriculum, children's mathematical thinking must be studied intensively in their usual school habitat. Given an opportunity to think freely, children clearly display certain patterns of thought as they deal with ordinary mathematical situations encountered daily in their classroom. A videotaped record of the outward manifestations of a child's thinking, uninfluenced by any teaching on the part of the interviewer, provides a rich source for conjectures as to what this thinking is, what mental structures the child has developed, and how the child uses these structures when dealing with the ordinary concepts of arithmetic. In addition, an intensive analysis of this videotape generates some conjectures as to the possible sources of what adults view as children's "misconceptions" and about how the school environment (the teacher and the materials) "fights" the child's natural thought processes.

The Project for the Mathematical Development of Children (PMDC)[2] set out

[1] "Why Teach Mathematics?" The Elementary School Journal 73 (Feb. 1973), 258-268.

[2] PMDC is supported by the National Science Foundation, Grant No. PES 74-18103-A03.

to create a more extensive and reliable basis on which to build mathematics curriculum. Accordingly, the emphasis in the first phase is to try to understand the children's intellectual pursuits, specifically their attempts to acquire some basic mathematical skills and concepts.

The PMDC, in its initial phase, works with children in grades 1 and 2. These grades seem to comprise the crucial years for the development of bases for the future learning of mathematics, since key mathematical concepts begin to form at these grade levels. The children's mathematical development is studied by means of:

1. One-to-one videotaped interviews subsequently analyzed by various individuals.

2. Teaching experiments in which specific variables are observed in a group teaching setting with five to fourteen children.

3. Intensive observations of children in their regular classroom setting.

4. Studies designed to investigate intensively the effect of a particular variable or medium on communicating mathematics to young children.

5. Formal testing, both group and one-to-one, designed to provide further insights into young children's mathematical knowledge.

The PMDC staff and the Advisory Board wish to report the Project's activities and findings to all who are interested in mathematical education. One means for accomplishing this is the PMDC publication program.

Many individuals contributed to the activities of PMDC. Its Advisory Board members are: Edward Begle, Edgar Edwards, Walter Dick, Renee Henry, John LeBlanc, Gerald Rising, Charles Smock, Stephen Willoughby, and Lauren Woodby. The principal investigators are: Merlyn Behr, Tom Denmark, Stanley Erlwanger, Janice Flake, Larry Hatfield, William McKillip, Eugene D. Nichols, Leonard Pikaart, Leslie Steffe, and the Evaluator, Ray Carry. A special recognition for this publication is given to the PMDC Publications Committee consisting of Merlyn Behr (Chairman), Thomas Cooney, and Tom Denmark.

Eugene D. Nichols,
Director of PMDC

THE EXPERIMENT

As part of several types of research activities of the Project for the Mathematical Development of Children, a clinical study of first and second grade children was carried out at an elementary school of about 1,000 children in the southeast. The purpose of the study was to find out how children interpret, in terms of number sentences, certain actions performed on physical objects. The objects used were single unifix cubes. To obtain uniformity of stimuli, a sequence of actions on the cubes was recorded on a videotape. The author performed the actions upon the cubes and subsequently used the tape individually with children.

The sequence of events in interviewing each child individually was as follows:

<u>Step 1.</u> After the child wrote his/her name on a sheet of paper, the experimenter said:

> How about writing a number sentence for me--any number sentence you like?

If the child wrote something that was not considered a number sentence (examples appear later in the text), the experimenter said:

> How about now writing something that has a plus or a minus and an equals sign?

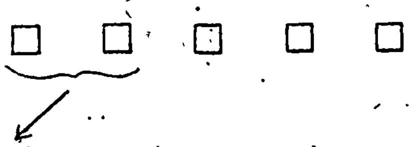
<u>Step 2.</u> Next the experimenter said:

> Now I am going to talk to you on TV. I'll tell you to do something. You watch and do it, OK?

<u>Step 3.</u> The eight action episodes were shown to the child on a 20-inch screen. After each episode, the tape was stopped, the child wrote a sentence, and then the next episode was shown.

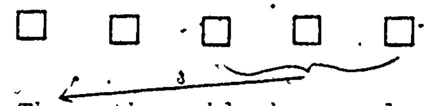
Each of the eight episodes was presented in the same mode. The first episode is fully described below along with the instructions in the order in which they were presented. These instructions were also repeated in each episode.

<u>Episode 1.</u> Five unifix cubes are placed on a table as follows:



The experimenter points to each cube in silence, giving the child an opportunity to count the cubes. Then he says, "Watch carefully." Two blocks on the left (child's view) are pushed off the table (a strip of cardboard is used to assure that the blocks fall off simultaneously). Then the experimenter says, "Write a number sentence that tells what I did." The resulting configuration, after the blocks have been pushed off, remains visible on the screen for from three to five seconds, then is phased out. The child writes a sentence and is asked to read it. Then the next episode is presented in the same sequence.

Episode 2.

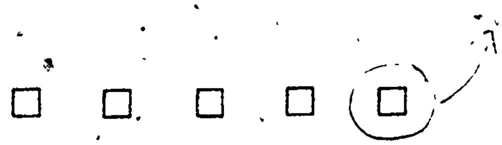


These three blocks are dropped from the table simultaneously.

Episode 3.

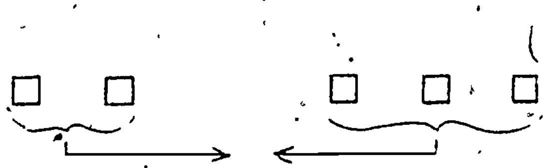
The experimenter picks up the two blocks with the right hand and removes then from the child's view.

Episode 4.



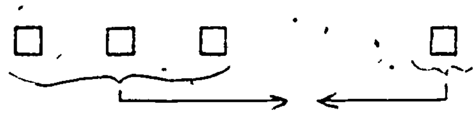
The experimenter picks up the one block with the left hand and removes it from the view of the child.

Episode 5.



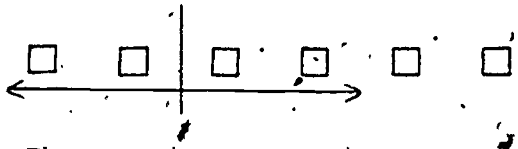
The experimenter pushes simultaneously the two and the three blocks together (two strips of cardboard are used for this purpose), so that one pile of blocks is formed.

Episode 6.



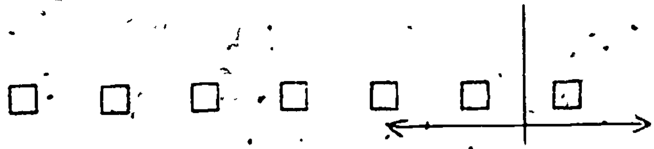
The experimenter pushes the three blocks and the one block together as in Episode 5.

Episode 7.



The experimenter, using two strips of cardboard, simultaneously pushes apart the two and the four blocks, so that two sets of blocks are obtained at the opposite ends of the table.

Episode 8.



The experimenter pushes the six and the one blocks apart, as in Episode 7.

THE RESULTS

As previously mentioned, it was necessary to ascertain the children had some referent for the phrase number sentence, thus the directions,

> How about writing a number sentence for me--any number sentence you like?

was given first. In response to these directions, the following are some examples of what children wrote.

First graders	Second graders
1 2 3 4 5 6 7 8 9 10 11 12	I like 2
11	I am nine years old
1 + 2 + 4 - 5	I had 5 pieces
	A boy is big
	1 2 3 4 5 6 7 8 9 10 ~~11 12 13~~
	
	I am 6 years old
	5

It is interesting to note that the responses the first graders wrote to the request for a number sentence fall into these categories:

(1) a sequence of numbers, or

(2) a single number, or

(3) a phrase containing addition and subtraction.

In the examples above it can be seen that second graders are more flexible in interpreting a "number sentence." This interpretation embraces English sentences which refer to numbers as well as size.

Following the second set of instructions, How about now writing something that has a plus or a minus and an equal sign?

all children wrote number sentences.

The following is a summary of the results for 22 first graders (beginning of March) and 25 second graders (middle of October).

Episode 1. Five blocks on the table, two pushed off

Sentences written by children	Frequency: First graders	Frequency: Second graders
5 - 2 = 3	12 (6 horizontal, 6 vertical)	14 (12h, 2v)
5 - 3 = 2	0	3 (2h, 1v)
6 - 3 = 3	2 (h)	1 (h)
3	1	2
other	7	5

Episode 2. Five blocks on the table, three pushed off

Sentences written by children	First graders	Second graders
5 - 3 = 2	12 (6h, 6v)	21 (17h, 4v)
2	1	2
5 - 2 = 3	1 (v)	1 (h)
other	8	1

Episode 3. Five blocks on the table, two picked up

Sentences written by children	First graders	Second graders
5 - 2 = 3	12 (7h, 5v)	21 (17h, 4v)
3	1	2
other	9	2

Episode 4. Five blocks on the table, one picked up

Sentences written by children	First graders	Second graders
5 - 1 = 4	11 (6h, 5v)	19 (16h, 3v)
4	1	2
other	10	4

Episode 5. Two and three blocks pushed together

2 + 3 = 5	3 (2h, 1v)	9 (8h, 1v)
3 + 2 = 5	2 (h)	8 (7h, 1v)
5 - 0 = 5	4 (3h, 1v)	1 (h)
5	2	3
5 - 5 = 0	2 (1h, 1v)	0
other	9	4

Episode 6. Three blocks and one block pushed together

3 + 1 = 4	3 (2h, 1v)	12 (h)
4 - 0 = 4	4 (3h, 1v)	2 (1h, 1v)
4	2	2
1 + 3 = 4	1 (h)	1 (v)
other	12	8

Episode 7. Six blocks, four and two separated

6 - 2 = 4	9 (5h, 4v)	6 (h)
6 - 6 = 0	2 (1h, 1v)	4 (h)
6 - 0 = 6	3 (h)	2 (h)
6 - 4 = 2	0	2
6	0	2
0	2	0
other	6	9

Episode 8. Seven blocks, one and six separated

7 - 1 = 6	6 (4h, 2v)	11 (h)
7 - 6 = 1	1 (v)	4 (3h, 1v)
7 - 7 = 0	2 (1h, 1v)	2 (h)
7 - 0 = 7	0	2
7	2	0

0	2	0
other	9	6

DISCUSSION

In selecting the first six episodes the PMDC staff postulated "key" responses. They were as follows:

1. 5 - 2 = 3
2. 5 - 3 = 2
3. 5 - 2 = 3
4. 5 - 1 = 4
5. 2 + 3 = 5 or 3 + 2 = 5
6. 3 + 1 = 4 or 1 + 3 = 4

Accepting these as "correct responses," the percents of "success" are as follows:

Episode	First graders	Second graders
1	55%	56%
2	55%	84%
3	55%	84%
4	50%	76%
5	23%	68%
6	18%	52%

With the exception of the first episode, the second graders have given the expected response much more frequently than the first graders. It would probably be safe to ascribe this difference to the effect of the longer period of teaching, during which the predominant emphasis was on addition and subtraction.

It is interesting to note the differences in preferences for the horizontal over the vertical form of writing sentences. For the expected responses, the following are the percents of children who used the horizontal form (the "keyed" response is taken to be 100%).

Episode	First graders	Second graders
1	50%	48%
2	50%	81%
3	58%	81%
4	55%	84%

5	80%	88%
6	75%	93%

The second graders' greater preference for the horizontal form (except for Episode 1) can probably also be attributed to instruction; at that particular school the horizontal form was used more frequently than the vertical form.

The construction of Episodes 7 and 8 was motivated by the investigations of children's concept of equality, discussed in other PMDC publications[3]. The crucial observation made in those investigations was that first and second graders reject the equality form a = b + c as being "wrong" and "backward." The author attempted to construct a dynamic situation with manipulatives which might suggest to children this sentence form. The obvious manipulation seemed to be a motion separating simultaneously a set of objects into two subsets. From the following results, it is seen that the intended interpretation did not take place. It seems that the sentence form a + b = c or a - b = c is so strongly imbedded in children's thinking that they employ these forms to the exclusion of others in interpreting actions upon objects.

The following results were obtained for the last two episodes.

Episode 7. Six blocks, four and two separated

	First graders	Second graders
6 - 2 = 4	41%	24%
6 - 6 = 0	9%	16%
6 - 0 = 6	14%	7%
6 - 4 = 2	0%	12%
other	36%	40%

Episode 8. Seven blocks, one and six separated

	First graders	Second graders
7 - 1 = 6	27%	44%
7 - 7 = 0	9%	8%
7 - 0 = 7	14%	0%
7 - 6 = 1	5%	16%
other	45%	32%

[3]Behr, M., S. Erlwanger, and E. Nichols. How Children View Equality Sentences (PMDC Technical Report No. 3); and T. Denmark, E. Barco, and J. Voran. Final Report--A Teaching Experiment on Equality. Tallahassee, Florida: Florida State University, 1976.

The complete abstinence from writing the form a = b + c should be investigated further. Although children reject it as "wrong" and "backward," one might construct an experiment in which children could be enticed into pretending that a sentence like 6 = 4 + 2 is alright and then asked to tell a story about real objects which would fit this sentence. It would be important to search for models which seem sensible to children and which promote the concept of equality as an equivalence relation, rather than as an operator. A study carried out by Coleen Frazer[4] points out that even college students do not possess an operational concept of the symmetric property of equality. The ability of an individual to accept, with great ease, the symmetric and possibly other properties of equality, does not necessarily mean that this individual is able to work with equal success with the two symmetric forms.

This exploratory experiment suggested that children begin very early in their school days to formulate mental constructs about the very crucial concept of equality and this particular construct, possibly extremely inadequate, might persist throughout the later years.

Our informal observation of second graders whose teacher taught the children to use the phrase "is the same as" for the symbol "=" suggested that this phrase, rather than "is equal to" might be more conducive to children's mental construct of equality as a relation.

If one accepts the thesis that young children should indeed perceive mathematics as an "action" subject and that the primary goal should be to teach these children how to <u>do</u> mathematics and, furthermore, if one would want the symbolism to be isomorphic to students' thinking about the actions suggested by the symbols, then the conventional use of the equality symbol is inadequate. More than that, this use is contrary to children's perceptions. The symbol, which would be consistent with children's perception of mathematical operations would have to be a non-symmetric, one-way symbol. For example, the symbol $\longrightarrow$ in $(4 + 3) \longrightarrow 7$ would more closely correspond to how first and second graders think about addition. It would suggest that <u>adding</u> 4 and 3 results in 7. The same symbols in $7 \longrightarrow (4 + 3)$ should then possibly suggest <u>separating</u> 7 into 4 and 3. The latter situation, however, raises the question about the use of the addition symbol: is it really analogous to the operation, expressed in $(4 + 3) \longrightarrow 7$, as the child perceives it? Perhaps separation of 7 into 4 and 3 would be more adequately expressed by $7 \longrightarrow (4, 3)$ and corresponding actions on objects performed in such a way that $7 \longrightarrow (4, 3)$ would be different from $7 \longrightarrow (3, 4)$.

This investigation suggests that the sentences $(3 + 4) \longrightarrow 7$ and $7 \longrightarrow (3 + 4)$ portray non-symmetric situations, as children perceive them, thus suggesting that the equality symbol, intended to have the symmetric property, is not the most appropriate one to use.

The matter of equality and the basic operations is central to the elementary school mathematics curriculum and beyond. The investigation

[4]Frazer, C. D. "Abilities of College Students to Involve Symmetry of Equality With Applications of Mathematical Generalizations," Florida State University, Tallahassee, Florida, 1976.

described in this paper is only a beginning of the kind of research that should continue. The main goal of the research should be to understand how children, as a result of their early experiences with mathematics, come to formulate mental constructs which possibly dominate their thinking for a long time.

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DESCRIPTORS *Developing Nations; Educational Change; *Educational Needs; Instruction; *Instructional Aids; Learning; Mathematical Models; *Mathematics Education; *Number Concepts; Problem Solving
IDENTIFIERS Brazil; *Calculators

ABSTRACT

This paper notes three objections to the use of hand-held calculators in schools: they would (1) block reasoning, (2) make individuals machine-dependent, and (3) broaden the gap between developed and underdeveloped nations. Each is addressed, with specific examples used to refute them. The belief is strongly expressed that calculators can aid in adjusting social imbalances between "have" and "have not" groups and nations. Projects in Brazil in which calculators are being used are cited. The use of calculators in modeling real problems is also discussed. (MS)

ISSUES ARISING ON THE USE OF HAND-HELD CALCULATORS IN SCHOOLS

by Ubiratan D'Ambrosio

Most of the objections to the use of hand-held calculators (HHC) in schools may be grouped into three main issues :

1st: HHC will block reasoning and will make individuals mentally slow;

2nd: the use of HHC will make individuals dependent on the machine, and the absence of it will be a handicap for daily needs;

3rd: HHC will broaden the gap between rich and poor, developed and underdeveloped nations.

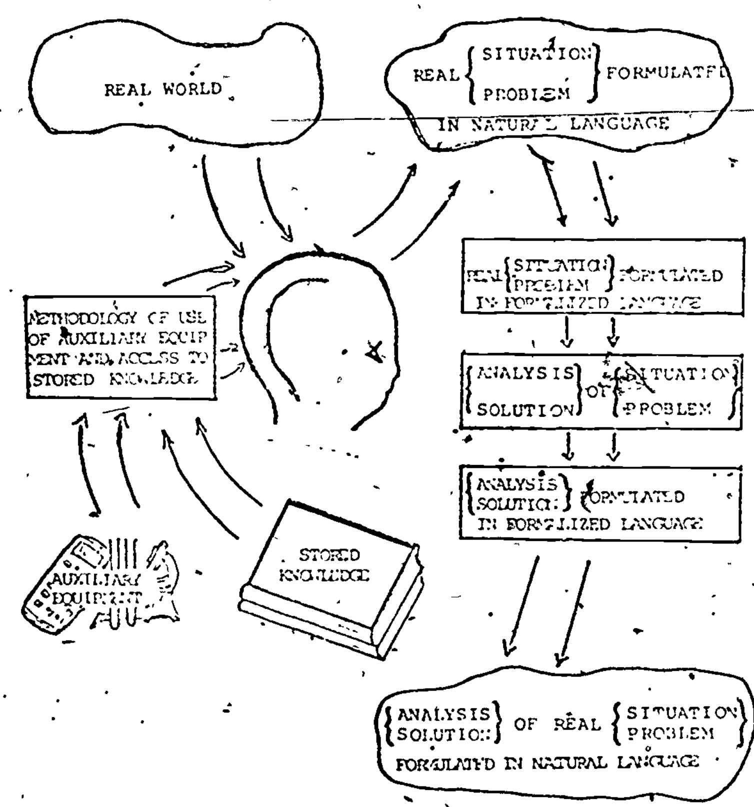
This talk will be addressed to questions derived from the three issues above. No doubt, there are fundamental questions which may be inserted in the very important branch of WHY's in education. A further question, obviously depending on the one just raised, is HOW. We will touch only briefly the question of "HOW" to use HHC. The WHY question deeply relies on philosophical considerations relating to the overall goals and objectives of mathematical education, and its underlying philosophy is present in the paper. We refer to [1] or to [2] , for an expanded version. The second question, related to "HOW" to use HHC, is the subject of much ongoing research and will obviously have a dynamic character, depending on the adopted philosophy of education, in particular of mathematical education, on accepted societal goals, and on technological advance. Anyhow, we will give a few examples on specific uses of HHC, as well as on some on-going projects, and also reference sources.

Let us address initially to the first and second objections, which are closely related. A Brazilian colleague of mine once said that "if a child forgets its calculator at home, it will forgets its head". As a preliminary, I must say that this is not the conception I have of the power and potential of a child's head. What, indeed can be said of the power and functioning of a child's mind? Not going into an almost endless discussion of the process of reasoning and creativity, we can briefly say that HHC reproduces, in a very unsophisticated and crude way, some of the basic operative functions of the brain.

In several instances, inventions which are, in a sense, similar to that of HHC, have caused impact and reaction. We might give the word to Plato, in his Phaedrus , which describes a conversation between the young King Theuth and the good old King Thamus, of Egypt, on the subject of the invention of writing. The old King dennounced it as a danger for civilization, saying that children and young people, who used to apply themselves to learn and retain whatever was taught them, would now cease to exercise their memories and consequently would be less diligent and capable.

Similar citations could be mentioned. Probably, the most striking is the rationale of G.W. Leibniz, about his calculating machine. In fact, the operational concept in mathematics is a recent fact, direct consequence of the invention of arabic algarisms, which by no means represent the essence of mathematics, and its inclusion in a course of general studies is even more recent, which was amply discussed in [2]. For long, manipulation of operations had been a merely mechanical ability, done with the aid of instruments or fingers and hands, recently replaced by the mechanical use of Arabic algarisms and positional notation. In other words, this is a mere mechanization of the structure which is the basis of positional numeric systems. Indeed, quantitative considerations, which carry the meaning of precise counting only up to lower two-digit numbers, are always an attribute of qualitative analysis. Linguistic considerations are illustrative of this. We mention, in particular, the recent works on the Inca "quipus", carried on by Marcia and Robert Ascher [3], which imply a strong attribute aspect to quantitative aspects of a discourse.

We regard the process of mentalization of reality in the following simplified scheme:



We agree with René Thom's description of mathematics as a finer language than natural language to describe reality.

As mentioned before, and anthropological research reinforces this view; numbers appear with the precision of units only in the lower range. Exemplifying, no one, with the exception of children doing exercises and exames in arithmetics, and of machines, ever uses numbers like "1,432,173". When someone needs these numbers, which occur in very specific branches of activities, they are dealt with by mechanical means, be it with the recent electronic equipment or with the mechanical heavy machines of the turn of the century. Meanwhile, the needed and important capability of "wise and experienced" men, which call for good quantitative evaluation, has been entirely subdued by the false importance of calculations precise to the unit! It would be useless to repeat examples of school failure in mathematics which could be avoided by a minimal amount of quantitative common sense. This quantitative common sense has been almost impossible to achieve due to overburdened emphasis on merely mechanical abilities, which undoubtedly do not represent the potential of the human mind. The overall and generalized use of HHC will probably make man less concerned with details of "precision to the unit" and more concerned with global quantitative evaluation. Of course, in some - a few - instances, precision to the unit may be desirable. Then, a HHC or even a larger machine will be needed and properly used.

This brings us to the second issue. The dependence on the machine is, indeed, a false issue. The argument between Kings Theuth and Thamus could be reproduced in practically every moment when a new invention is put into practice. It is remarkable the fact that Arabic algarisms were forbidden by legal edicts in Florence in the close of the XIII century, and I myself remember that when ballpoint pens were introduced, we were not allowed to use them, otherwise our handwriting would be spoiled to the point that we would be unable to write in an intelligent way. Probably, when wrist watches were invented, people would object, saying that "the moment your watches breaks, you will be unable to distinguish sunrise from sunset". The old always rejects the new. This rejection is probably the most active force against the absolutely needed dynamical character which should prevail in the educational process.

In fact, we brought to discussion a good comparative example for the issue of precise counting. Although in some daily practices, precise timing is needed, and appropriate chronological devices are used, for most of our routine activities unprecise and even intuitive time measurement are satisfactory. No one would dare to say he is sleeping in daytime for the reason his watch is not working! As we said before, the overall and generalized use of HHC will have the effect of changing the forces from "precision to the unit" to global quantitative evaluation. A HHC will always be available to someone, and with the same ease that we borrow a pen from a colleague when we forget ours, or we ask a passerby in the street "What time is it?", we will be able to remedy the situation of not having a HHC at hand when need is felt. It is remarkable to notice the drop in the price of HHC. They are cheaper than books: indeed the cost of a low cost model of HHC goes in the largest part to commercialization.

We now come to discuss the very important issue of what influence will HHC have in social unbalance, which prevail in most countries, and which seems to resist educational efforts. And also about the urgent need of bridging the gap between developed and underdeveloped countries.

In both cases, the local social unbalance or the global world desequilibrium between "haves" and "have nots" can be challenged only by eliminating the striking differences between available basic equipment and abilities. This is the rationale followed by training programs; this was the rationale behind school systems set up by the declining aristocracy to meet the challenge of the professional guilds, and this is the rationale for underdeveloped countries investing most of their human and material resources in education. For more dis -

cussion on this we refer to [2]. Indeed, the objective of all these programs is to prepare generations to compete with adequate abilities and tools. This competition, understood in its broad and global aspect, is the ultimate goal of an evolving society, in its full conception. Be it a lower class family with hopes of their children having better professional opportunities, be it an underdeveloped nation trying to deal and trade with developed nations in more dignifying circunstances. In both cases, it is necessary that the challenger be fully prepared to deal with the established structure, and if not in possession of the full equipment, certainly knowing how effectively powerful this equipment is. By rejecting sophistication in education with the argument of "this is costly" or "we are not yet ready for this", socially unprivileged classes or less developed nations risk perpetuating through the educational system a "status quo" which they must change. The oppressive power of "an absent electronic brain" is much more effective, then the debts which might result from learning that there is not such a thing!

Probably, the young boy in a Peruvian village who, after much effort, learned how to do arithmetic with paper and pencil, goes for a job in a city, and sees the boss pressing a few buttons and getting the results out of that "electronic brain", will experience the same sensation as his ancester warriors who met the gigantic armoured complexes of "Spaniards on horses". They were regarded as single entities!

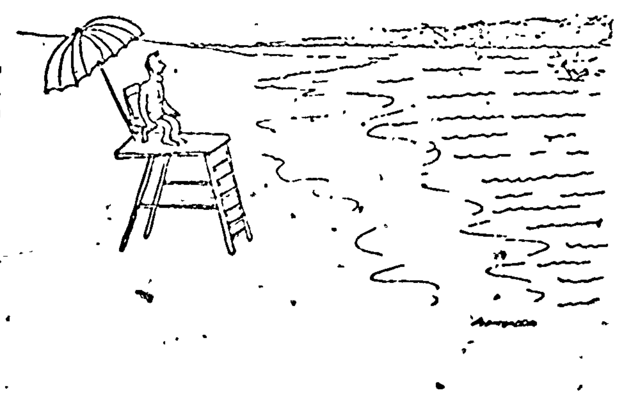
To introduce HHC in a school system is much more ~~a matter~~ of attitude, and the understandable and expected reaction to its use must be faced. This can be minimized if HHC is allowed to come into use, rather than forced upon a school system. Several strategies may be adapted. A rather successful one is to bring them into playing an important role in teacher training, through modelling courses. See, for example [4] and [5]. HHC must be brought into a useful device for daily practices. Once the teacher is "liberated" from prejudices and fear of HHC "spoiling minds", the acceptance of the instrument as a companion in daily practices will be conveyed to children. At the same time, evidence of rather immediate advantageous results of the use of HHC in schools may be an important factor. Research projects on HHC, conducted in schools, have the advantage of showing such results. Rather than the results of the research in iself, the main goal is to bring awareness to the use of HHC. In the Institute of Mathematics, Statistics and Computer Science of the State University of Campinas, we conduct a chain of research projects, in classes of 30 to 40 students, in various levels of schooling.

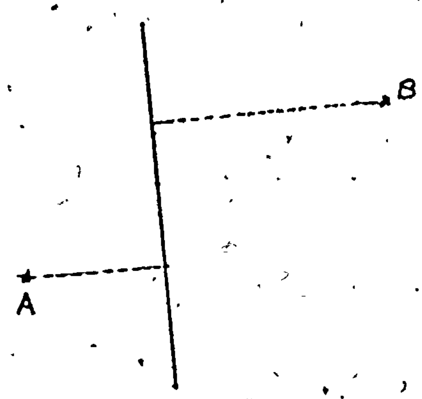
One typical project is being conducted in a lower income private school in the city of Campinas. A class-room of 45 students at the 7th year of Primary School (14 years old) is divided into groups of 15 students. Group A has no access to machines, Group B has limited access to machines (during class period) and Group C has total access to the machines (taking them home). Classes are conducted in the usual way, with the normal program. No change in the attitude of the instructor nor in the choice of curricular material, examples or home work. Three tests are given with intervals of one week, with the same kind of exercises and problems. The attitude of the student is observed and compared, in the course of the experi - ments..

Although the programatic material is not prepared for the utilization of HHC resources, the general attitude of the student should be affected, in the sense of giving him better and more adequate methodology for the utilization of what we have called "auxiliary equipment". This easiness is dealing with equipment in the analysis of real world problems or situations is the goal we hope to attain with the use of HHC.

In dealing with Mathematical Modelling, the use of HHC may bridge the gap between Mathematics and Applications, at an early stage. Concepts of the Calculus, like limits, derivatives and approximations find, in the use of HHC a natural vehicle for rather immediate applications. Model building, previously largely restrained to finite mathematics, has the possibility of reaching, through numerical manipulations, continuous phenomena. A few elementary examples are discussed in [4] and [5].

Furthermore, plausible reasoning, as presented by G.Polya, can be conveniently adapted to bring up a full understanding of hypothesis forming, in the very essence of the axiomatics method. In fact, borrowing from an example amply discussed in [5], we may present, through HHC, the full "mentalization" of analysis and solution of real problem situations. Starting with the example of a life-saver which has to reach a swimmer in trouble in a beach resort, we are able to build up the entire process of formulating hypotheses





which are approximations of a real situation. To bring the situation from the picture on the left to the diagram on the right represents the very essence of modelling. After this, the use of HHC shows, by simple computations involving solution of right triangles, through Pythagoras' theorem, that the solution of the problem of minimal path is indeed a broken line. The location of the point to "enter into the water" is found through a nesting interval technique. This example puts into effective combination, both the very deep conceptual approach of axiomatics, regarded as a modelling of natural phenomena in the purest line of thought of Euclid, Newton, and others, and the use of numerical methods as a tool for the analysis of the same natural phenomenum modelled into a mathematical problem.

A larger number of examples like the one just described, always close to the reality in which the educational process is taking place, will enable HHC to find its meaningful use in mathematical education. When we say reality we imply cultural and sociological framework, built up in an appropriate motivation for the age group in which the educational experience is taking place. And when we say meaningful use in mathematical education, we mean HHC used not merely as a tool for doing less painfully meaningless computations, but as a companion which can make possible through quantitative manipulation the mathematical analysis of natural phenomena, thus enhancing mathematics in its proper place among the sciences.

Summing up, the use of HHC will allow for the never before experienced power of employing numbers in modelling real problems situations and reaching their understanding.

REFERENCES

[1] : Ubiratan D'Ambrosio : Overall Goals and Objectives of Mathematical Education, chapter IX, New Trends in Mathematical Education IV, UNESCO-ICMI, Paris, 1977.

[2] : Ubiratan D'Ambrosio : Overall Goals and Objectives of Mathematical Education, Instituto de Matemática, Estatística e Ciência de Computação da Universidade Estadual de Campinas, São Paulo, 1976.

[3] : Marcia Ascher and Robert Ascher : The Quipu as a Visible Language, Visible Language, IX 4 Autumn 1975, pp. 329-356.

[4] : Ubiratan D'Ambrosio : Aproximación de funciones por funciones lineales: Noción de derivada, p. 141-157, Seminario-taller de modulos, UNESCO, Montevideo, 1977.

[5] : Ubiratan D'Ambrosio: Um exemplo sobre a estratégia de currículo dinâmico e integrado no ensino da matemática, Boletin do GCEM (Belo Horizonte) nº 2, 1977 (in print).

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