DOCUMENT RESUME

ED 143 557 SE 023 041

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TITLE Studies in Mathematics, Volume XX. Mathematical Uses and Models in our Everyday World.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 72
NOTE 161p.; For related documents, see SE 023 028-040; Contains numerous light type

EDRS PRICE MF-$0.83 HC-$8.69 Plus Postage.
DESCRIPTORS Arithmetic; Elementary Grades; Elementary School Mathematics; *Instructional Materials; *Mathematical Applications; *Mathematical Models; Secondary Grades; *Secondary School Mathematics; *Textbooks
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This book is intended as a demonstration that a variety of interesting problems suitable for use in the school mathematics experience of everyperson can be fabricated from available sources. It is intended to be illustrative rather than exhaustive. The problems in the book are intended to be accessible to children by the middle school years. The expository sections may be difficult for middle school students. The underlying idea throughout the book is that of mathematical models. Chapters included in the book are: (1) Uses of Numbers for Description and Identification; (2) Uses of Pairs or Triples of Numbers; (3) The Role of Measures in Application; (4) Formulas as Mathematical Models; and (5) Examples of Problem Collection Themes. A selected bibliography concludes the book. (RH)

SCHOOL MATHEMATICS STUDY GROUP

Studies in Mathematics

VOLUME XX

Mathematical Uses and Models in Our Everyday World

By
Max S. Bell

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Preface

As mathematics has become more obviously essential in many of the inquiries and much of the work of the world there have been an increasing number of appeals for more attention to "teaching mathematics so as to be useful." The following are typical:

> Since mathematics has proved indispensable for the understanding and the technological control not only of the physical world but also of the social structure, we can no longer keep silent about teaching mathematics so as to be useful. In educational philosophies of the past, mathematics often figures as the paragon of a disinterested science. No doubt it still is, but we can no longer afford to stress this point if it keeps our attention off the widespread use of mathematics and the fact that mathematics is needed not by a few people but virtually by everybody. ("Why to Teach Mathematics So As to be Useful," H. Freudenthal, 1968)*

> The big unfinished business is to collect appropriate examples of honest applied mathematics for earlier levels of education. I know it is possible to bring real applications into the secondary, and even the elementary, school and to motivate and illustrate much mathematics by such examples. A major, probably international, effort is needed to collect a sufficient variety of examples to fit all our different school situations. ("On Some of the Problems of Teaching Applications of Mathematics," H. Pollak, 1968.)

This book is one person's attempt to respond to the need reflected by these appeals. It had its origins in work with the School Mathematics Study Group (SMSG) efforts since 1966 to produce a mathematics curriculum for about grades 7-10 setting forth principally mathematics judged to be essential to "everyman." (The results of that project are now available as SMSG's <u>Secondary School Mathematics</u> (SSM) in two versions, one for average and one for "slow" learners.) From its inception, all concerned with the project recognized that such a curriculum should illuminate the uses to which mathematics is put. But it soon became clear that it is one thing to say this should be done and quite another to find the appropriate materials to accomplish it. At some point I took it as something of a personal challenge to expand the supply of problem material that would support such intentions. This volume is the result. My work on this project began with these three beliefs:

1. That interesting numerical information and good explanations of elementary uses of mathematics in the world's work are available in ample supply and from readily accessible sources.

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*All books and articles referred to are listed at the end of this book (alphabetically by title) with complete bibliographic information given.

2. That from such sources interesting problems can be fabricated that use a variety of mathematical models, including many requiring only the mathematics available by the middle school years.

3. That such problems, once fabricated, can be effectively worked into curriculum materials and made an integral part of the school mathematics experience.

It seems to me that the first belief is well supported by the variety of sources referred to by problems in this book, and these by no means exhaust the possibilities. As to the second belief, turning this raw material into problems has proved to be often difficult, always time consuming, but possible. As to the third proposition, I still believe it, but this volume makes no contribution to its proof. Still, I hope that the existence of this and other such collections encourages others to work both on extension of the supply of problems and on the difficult task of making them an integral part of school mathematics.

I wish to acknowledge here the steady encouragement of E. G. Begle and his patience through a succession of missed deadlines. A draft version was very much improved by suggestions from these graduate students in mathematics education at the University of Chicago: Diane Detitta, Raphael Finkel, Susan Friedman, James Goehmann, Raymond Klein, David Porter, and Ronald Teeple. The patience and help of my wife and family is acknowledged, as is the assistance of Marilyn Brown and Rosalind Stephens in typing several versions of the manuscript. Finally, my debt to many other writers of books and articles will be obvious to any reader of these problems, and is acknowledged.

Max S. Bell
The University of Chicago
May, 1972

Introduction

This book is intended as a demonstration that a variety of interesting problems suitable for use in the school mathematics experience of everyman can be fabricated from readily available sources. It is intended to be illustrative rather than exhau ve; indeed, for every problem that appears here there are source materials in my files that would support the formulation of several more problems.

The organization of the book is somewhat peculiar. It does not follow the topical sequence of any known curriculum; nor is it organized in order of difficulty of problems; nor around specific areas of application of mathematics. Principally, it is organized around a few basic mathematical ideas that seem to me useful or essential for everyman yet neglected with respect to applications in most school materials. Hence it begins with a deeper than usual consideration of the uses made of single numbers, starting with counting then considering such uses as ordering, indexing, coding, and identification. The second chapter considers uses of pairs and triples of numbers, first in coordinate systems of various sorts, next with respect to ratios, and finally with some uses involving ordinary calculations. The third chapter tries to give some picture of the very widespread "measure" uses of numbers and some general sense of what is going on when one measures something. The fourth chapter takes up the use of measures in various formula models. Among other things, this chapter tries to make the point that similar mathematical models serve in a wide variety of situations. The Fifth Chapter gives three examples of themes that can support problem collections. The intention is to suggest that many more such themes could be explored to good effect.

There are many "problems" in the book that do not so much ask a question as invite the reader to formulate questions. There are also problems where the information is insufficient or of somewhat doubtful validity; in most cases this is deliberate. There are questions where there is not a definite answer called for or appropriate; this too is usually deliberate. That is, I don't see why school problems should invariably be tidy and unambiguous, when the problems presented by life are not.

Readers of manuscript versions of this book have asked about intended audience for and probable use of the book. I confess I do not have a precise answer to such inquiries. The problems themselves are intended to be accessible to youngsters by the middle school years, hence required mathematical knowledge has been kept to a modest level. The expository sections may be another story, since they are frequently quite condensed and it may be difficult for some middle school readers to cope with them. Also, some of the problems may well require that elusive "mathematical maturity" beyond the technical knowledge required. It is unlikely that an entire book of page after page of assorted problems would be seen as a suitable textbook for a school course (though I think this might be worth trying). The book should, however, serve as a useful supplement over several years to existing textbooks, most of which have few genuine applications. Working teachers can use the book both to expand their own knowledge of a range of applications and as a source book from which to assign problems or sections to their students. I hope that many youngsters use the book, with or without the intervention of their teachers. I should think it would be useful in the training of prospective or in-service teachers, both for elementary and secondary schools. Most important, I hope that the sorts of problems represented by this and similar collections will eventually be woven into the fabric of curriculum materials that define the school mathematics experience.

The underlying idea throughout this collection is, of course, that of mathematical models. There is a brief introduction to the processes involved in building mathematical models in the first chapter of the book but it is not possible for a student to acquire a thorough understanding of these processes with any single experience. Rather the ideas and processes must be encountered again and again over a long time span, perhaps with rather pointed reminders periodically about what is going on. Problems such as those in this collection provide a context in which such reminders and discussion can take place but are not sufficient by themselves.

There are many omissions from this collection, for a variety of reasons. Probability and statistics applications are missing largely because they will be attended to much better than I could do them by a joint American Statistical Association and National Council of Teachers of Mathematics project. Two books from that project should be available by the time this book appears, one each from Holden-Day and the Addison-Wesley Company. I regret not being able to include (because of printing difficulties) examples of the marvelously informative graphs and charts of the "Road Maps for Industry" series, which would provide good bases for many nice problems. (Teachers can receive these upon request to The Conference Board, 845 Third Avenue, New York, N.Y. 10022). It was not possible to include problems based on three dimension relief maps (land terain, population distributions, pollution distributions, etc.), and the two dimensional projections of these into contour lines, level lines, and other equal-characteristic curves. Such applications are illuminating, important, and very much neglected in the school experience, as are several other sorts of projections compressing three into two dimensions. Time pressures prevented inclusion of more problems based on social statistics and the controversies and decisions that face average citizens (armaments, inflation, pollution, government budgets, etc.). There are risks, of course, in the use of such materials, but they are the stuff of responsible citizenship.

These omissions are mentioned not by way of apology but as an indication of unfinished business. I believe the sampling of problems contained herein indicates that headway can be made in greatly expanding the supply of good applications for use at many school levels. Other recent and pending problem collections give the same positive indication. I hope with these and the efforts of many teachers and other mathematics educators, perhaps we can at last begin to solve one of the most persistent problems in mathematics education; that of "teaching mathematics so as to be useful."

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Chapter I

USES OF NUMBERS FOR DESCRIPTION AND INDEXING

1.1 Preliminary Remarks About Applications of Mathematics and Mathematical Models

> "The use of applied mathematics in its relation to a physical problem involves three stages: (1) a dive from the world of reality into the world of mathematics; (2) a swim in the world of mathematics; (3) a climb from the world of mathematics back into the world of reality, carrying a prediction in our teeth." (J. Synge, quoted in the American Mathematical Monthly, October 1961, page 799).

Starting thousands of years ago, man invented numbers, computation, and geometry to help understand and keep track of his world. For the past few hundreds of years mathematics has been seen as having "unreasonable effectiveness" in providing equations, formulas, and other mathematical models to help understand real world events related to astronomy and physical sciences. But only in the past few tens of years has mathematics been used extensively in many fields outside the physical sciences, especially since about 1950, when the first practical electronic computers were built. As a result, statements such as these are now commonplace:

For economics:

> The applied contributions of mathematical economics cover a wide range of areas. It has helped in the planning and analysis of ... the measures designed to eliminate recessions and inflations ... . It has helped to promote efficiency and reduce costs in the selection of portfolios of stocks and bonds, and to the planning of expanded industrial capacities and public transportation networks. In economic theory, it has helped us to investigate more deeply the process of economic growth and the mechanism of business cycles. In these and many other areas, the use of mathematics has become commonplace and has helped to extend the frontiers of research. ("Mathematics in Economic Analysis")*

For biology:

> There now exists, at least in outline, a systematic mathematical biology which, in the words of one of its pioneers, is "similar in its structure and aims (though not in content) to mathematical physics." ... Moreover, this mathematical biology has already greatly enriched the biological sciences ... ("On Mathematics and Biology")

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*Full information about all articles and books referred to in this book is in the Bibliography, listed alphabetically by title.

For business:

> The use of mathematical language ... is already desirable and will soon become inevitable. Without its help the further growth of business with its attendant complexity of organization will be retarded and perhaps halted. In the science of management, as in other sciences, mathematics has become a "condition of progress." (Mathematics in Management)

The process by which mathematics becomes useful to workers in these fields is indicated by the quotation at the beginning of this chapter. (Read it again now.) The "dive into the world of mathematics" typically results in what is called a "mathematical model" of the real world problem--some bit of mathematics such as a numerical expression, an equation, or a geometric diagram which expresses in abstract terms something about the real situation.

This same process of using abstract "mathematical models" to express something going on in the world also characterizes everyday uses of mathematics. Consider, for example, the mathematical abstractions used by most people every day of their lives--the counting numbers 1, 2, 3, ... These numbers appear to have been needed, and hence invented, early in human history and by virtually every human society that we know about. By examining the number words in various languages, anthropologists have found many number systems based on ten (probably the fingers on both hands), on five (the fingers on one hand), on four (the spaces between the fingers), on three (perhaps counting of knuckles on fingers), and some fairly sophisticated mathematical systems based on twenty (fingers and toes?). This dive into mathematics was first by way of spoken number words; written number systems were a later development. After this dive into mathematics, such things as "addition" and "multiplication" of counting numbers were invented to describe something about what happens when two sets of things are combined. That is, what man has done for centuries in applying numbers is not very different in basic spirit and method from the modern use of mathematical models to solve complicated problems in business, science, government, or social sciences.

Problem Set 1.1a

1. Give several situations where one could actually count things in the world of reality, and where the mathematical model would involve only the set of natural numbers and the operation of addition.

2. Give an example of a situation in the world of reality for which the mathematical model produced in the world of mathematics involves only the set of natural numbers and the relations =, <, >.

3. Give a situation using addition, where the numbers that represent things

in the real world are natural numbers, but where actual counting would probably not be possible.

4. Give an example of a situation in the world of reality for which the mathematical model would involve fairly small natural numbers and subtraction. (For example, I had five pennies and gave away three of them; how many did I have left?) Now think of a different kind of situation for which the mathematical model could be exactly the same. (You have five pennies and I have three; how many more do you have than I?)

5. Give an example of a situation for which the mathematical model would involve natural numbers and division. Now try to think of quite a different situation which would have the same mathematical model.

6. Give a situation which would be described by natural numbers, but where counting of individual objects probably did not lead to the numbers. (For example, how many miles from Chicago to New York? How many atoms in an ounce of Uranium? What is the present world population?)

7. Give a situation where the initial description is in terms of natural numbers, but where manipulation in the world of mathematics forces one to consider fractions.

8. Give a situation for which the appropriate mathematical description involves negative integers.

9. Give a situation from the world of reality for which the dive into mathematics would involve both geometry and arithmetic.

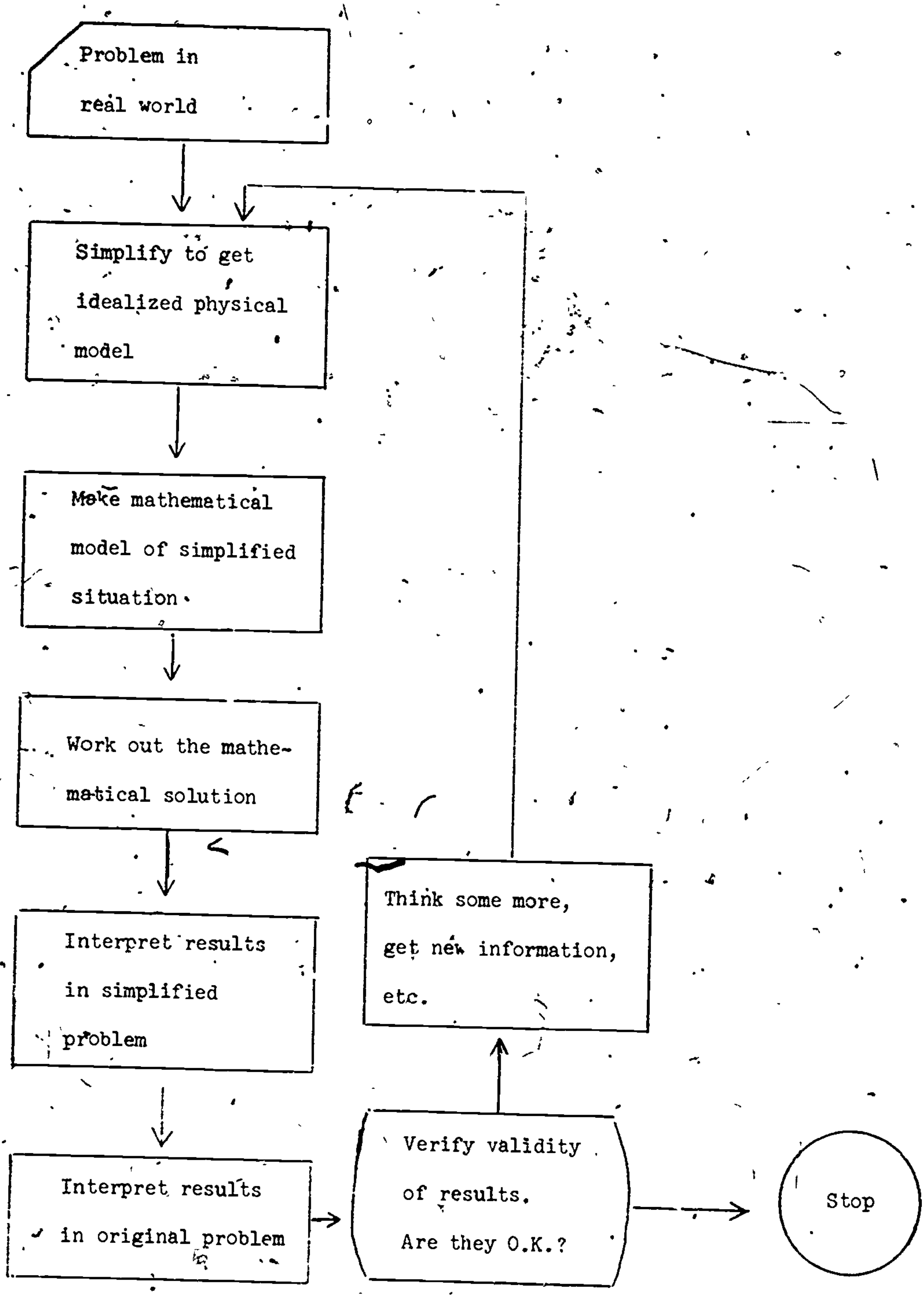
10. Give a situation where the appropriate mathematical model involves an equation.

"Abstract" mathematics is applied to solve problems about "real life" situations by using "mathematical models" in which the real life things are represented by mathematical things. This process can be illustrated by a "flow chart," such as computer programmers use in analyzing problems:

Problem Set 1.1b

1. For the next two days make a log of <u>every</u> use you see (outside your mathematics class) of numbers, calculations with numbers, or of other mathematical things. If some computation with numbers is involved, indicate whether it was done with the assistance of a machine. For about five of these uses, explain how each is a use of mathematical models.

2. Go through a daily newspaper or weekly newsmagazine (<u>Time</u>, <u>Newsweek</u>,



In simple, everyday, applications one does not, of course, go explicitly through all these steps.

U.S. News and World Report, etc.) and record or clip out every instance you find of some use of numbers or other mathematics. Go over your record (or collection of clippings) and see if you understand every such use; if not, try to do something about that. See if the information presented seems accurate and if the conclusions draw from it are defensible. Are there other interpretations reasonably possible from the same information? Identify the mathematical models that are used (arithmetic, percentage, probability, graphs, etc.) and try to identify stated or unstated assumptions that would need to be made in simplifying the real situation enough to fit a mathematical model to it.

3. Among your parents, relatives, parents of friends, etc., try to find someone who uses mathematics as a regular part of his (or her) job. Interview him and write a brief paragraph describing the uses of mathematics discussed in the interview, including the level of mathematical training needed (arithmetic, high school mathematics, college mathematics).

1.2 Uses of Single Numbers for Description

Usually our first and most frequent encounter with mathematical models is with numbers used to describe or quantify some situation. Often the numbers are whole numbers obtained by some counting process; frequently they express measures of something; sometimes a single number description is obtained from combining several other numbers. Here is an assortment of information given in numerical form from many sources and problems for you to do using the information given.

Problem Set 1.2

1. The population of the world in selected years since 1650 is given in the table below:

Estimated World Population

Year	World Total (Millions)
1650	470
1750	694
1850	1,091
1900	1,571
1930	2,070
1940	2,295
1950	2,517
1960	3,003
1969	3,561

a) The table is given in millions--write it out so it shows actual number of people, and read each number.

b) Draw a line graph of this information with years along the bottom and total population on the side.

c) Starting with 1650, <u>about</u> how long did it take for world population to double? Starting at this new point, about how long did it take for it to double again? Again? Continue answering that question up to 1969. Make an "educated guess" about how long it will take for the 1969 population of the world to double. In doing so, what assumptions would lead to a different prediction?

d) The 1918 influenza epidemic is said to rank with the "black death" plagues of earlier times as one of the most destructive disasters ever to sweep the earth. It is said that influenza took some twenty million lives in a few months. Assuming that the world population in 1918 was slightly less than that shown above for 1930, what death rate would this represent: one in a hundred? one in ten? or what?

e) World War I was still going on when the influenza epidemic struck. According to the 1970 <u>Information Please Almanac</u> (page 795), sixteen countries had together about 65 million mobilized armed forces for that war, of which about 8,500,000 were killed or died. What is the death rate of those mobilized? Which was the most destructive for those directly involved, the war or the influenza epidemic?

f) Austria-Hungary had 7,800,000 forces mobilized for World War I, and 1,200,000 were killed or died. What death rate does this represent? (Total casualties for Austria-Hungary, including the wounded and prisoners, came to 7,020,000.) The United States came into the war near its end; it had about 4,700,000 forces mobilized, with about 117,000 dead. What death rate does this represent? For the British Empire the figures were about 8,900,000 versus 900,000 killed; for France about 8,400,000 with 1,400,000 dead. What were the death rates?

g) <u>Information Please Almanac</u> and other good almanacs have similar casualty figures for World War II. Look up this information and make a judgment about the destructiveness of World War II as compared to World War I.

h) Look up in an almanac or other source death and injuries from automobiles in the United States (and, if you like, to other forms of transportation). Compare the "automobile plague" with some of those above.

2. The information that a citizen needs to process in order to make responsible and intelligent political/social decisions is frequently given in numerical form. Here, for example, is a sampling of items from the October 17, 1971 New York Times. For each item, suggest a problem or two suitable for this book.

   a) It is estimated that there are between one and two million illegal aliens now working and living in the United States. They continue to enter at a rate of at least 2,000 per day. In the fiscal year 1971, 420,126 illegal aliens were "captured" by the Immigration and Naturalization Service. The reason for coming is usually to escape poverty. For example, an 18 year old youngster was working in Mexico for $1.25 a day, came illegally to Chicago and earned $ 14.75 a day plus 2 meals, sending back $700 to his father in the year before he was caught.

   b) Last year 3490 cases of smuggling of Mexicans into the United States were reported. The usual fee from the border to Chicago or San Francisco is $250 per person, with groups ranging from 15 to 70 people.

   c) One estimate of the amount of money sent out of the United States by illegal aliens is 5 billion dollars per year.

   d) In Raleigh, N.C., rats have been found able to survive $2\frac{1}{2}$ to 6 times the normal killing does of a widely used rat poison. Apparently a genetic trait is involved. In Scotland about half the farms have rats, and 40% of them are resistent to this poison.

   e) Contrary to the tendency of Canadians to blame the U.S. for everything wrong, the Science Counsel of Canada recently reports that many of Canada's economic troubles result because Canada "lacked the spirit of adventure." For example, they own 94 billion in life insurance while the Americans that outnumber them 10 to 1 have 159 billion.

   f) A Gallup survey (of a sample population of 8,935 people) showed these results:

	1971			1940		
	Republican	Democrat	Independent	Republican	Democrat	Independent
Professional and Business workers	31%	34%	39%	47%	29%	24%
Clerical and Sales workers	25%	43%	32%	37%	41%	22%
Manual workers	19%	48%	33%	32%	50%	18%
	1971			1960		
Blacks	9%	72%	19%	23%	54%	23%

g) Three hundred stations carry Sesame Street, and an estimated two-thirds of the nation's 12 million three and four year olds watch the program, which has a budget of $13.8 million per year. There is some evidence that each year's new class of kindergarten kids knows more than the previous class, and some of this is attributed to the Sesame Street program.

h) A Census Bureau survey from September 24th to October 2nd showed that of 2,216 persons asked, only 33% said President Nixon's price freeze has stopped price increases.

i) Trains are fuller under Amtrak, but it is thought to be largely because there are fewer trains. The nationwide total of about 185 trains a day is only half the number that were operating before Amtrak. Amtrak is losing a lot of money, and may need $260 million operating subsidy over the next 2 years. (In 1969, before Amtrak, there were direct rail passenger losses of $200 million.)

j) In 1900, a New York to Chicago train took 18 hours using steam locomotives; Amtrak's time is now 18 hours 40 minutes on one route and 17 hours on another. The average speed is around 50 mph nationwide, compared to 70 mph for the New York to Washington Metroliner and 125 mph on the Japan Tokaido line and some lines in Germany.

k) Failure of engine mounts in 1965 through 1969 Chevrolet Nova, Chevelle and Camaro models (which represent 1 out of every 12 vehicles now on the road) have led to 500 complaints received by the National Highway Traffic Safety Administration. General Motors reported to the safety agency that it has replaced about 100,000 of the mounts; one report is that the replacement job costs about $30. [Note: G. M. later announced it would recall 6.5 million Chevrolet cars and trucks to replace motor mounts.]

ℓ) "Thus far George Meany has been batting 1.000 in his efforts to induce the administration to tailor the wage part of the control program to his design." (What exactly does "batting 1.000" mean? Is its meaning likely to be understood by most newspaper readers? Can you think of other sports statistics that can be used in a popular article and understood by most people?)

m) "Mr. Meany did not delay in letting it be known that he thought the goal [of 2 to 3% inflation by the end of 1972] was impractical. On the west coast...the dock union was unwilling even to discuss an offer calling for pay increases of 32.6% over 2 years."

n) The administration is trying to eliminate 1.5 million needy children --perhaps 400,000 in New York alone--from the free and reduced price component of the school lunch program for 7.3 million children. This summer a cut in the amount of subsidy per child was ordered but an angry Senate stopped it. The current move would cut down on the number of children by permitting subsidized lunches only for children from "poor" families (families with less than $3,940 income). But John R. Cramer claims the cut in cost of the summer proposal (more children but less money per child) would total $432 million while the second approach (more money but fewer children) also cuts costs by $432 million.

o) The total Black enrollment in all colleges was less than 800 in 1900; 7000 in 1920; about 100,000 in 1950; 200,000 in 1960 (130,000 or 65% in Black colleges); 234,000 in 1964 (120,000 or 51% in Black colleges); 434,000 in 1968 (156,000 or 36% in Black colleges); 470,000 in 1970 (160,000 or 34% in Black colleges).

3. The Guiness Book of World Records is packed with numerical facts and descriptions. For example, here are some items from the 1971-'2 edition; For each, answer any questions asked and make up at least one problem suitable for a book such as this.

a) On page 417, we learn that the greatest distance recorded for a sling-shot is 1,447 feet. Is that more or less than an ordinary city block ($\frac{1}{8}$ mile)?

b) We are told on page 235 that the new Sears Tower in Chicago, which will be 109 stories and 1451 feet tall, will become both the tallest building in the world (not counting TV antennas), and with 4,400,000 square feet (101 acres), also will have the most floor area. The Merchandise Mart in Chicago was biggest in floor area with a mere 4,023,400 square feet.

c) The Merchandise Mart cost $32 million in the early 1930's. How much is this per square foot of space, and how does that compare with building costs now? (As an index of 1970 building costs, the several buildings of the world trade center in New York will have a total area of rentable space of 10,018,000 square feet and will cost $575 million.) Read this problem again and see if you have an accurate basis on which to compare present building costs with those of 40 years ago.

d) A successful, popular singing artist can make a tremendous amount of money. For example, page 201 tells us Bing Crosby had sold 200 million records by 1960 (from 2600 singles and 125 albums) and 300 million by 1968. The Beatles are claimed to have had sales of 330 million records between February 1963 and January 1970. (For the week ending March 21, 1964, Beatles records were 1st, 2nd, 3rd, 4th, and 5th on the list of best selling U.S. singles, and 1st and 2nd on list of the best selling L.P.'s in the U.S.). Make an estimate of the total cost to buyers of the 330 million Beatles records sold by January 1970 with some reasonable allocation of these among albums and singles. Try to find out how something about how the income from record sales is distributed and estimate how much of this might go to the Beatles. How much might go to the recording company? To record distributors and retailers? How much of the performers share might go to managers and the like?

e) Page 169 gives a chemical compound that has a 1,913-letter name. About how many lines of print in this book would that be?

f) This is only a sample. One can open such a book to any place and find interesting numerical data. Check out this book or another record book and make up at least five more problems suitable for these problem sets.

1.3 Numbers Used for Indexing, Ordering, Identification, or Coding

We generally think of numbers in connection with "counts" or "measures." But numbers are also very widely used in indexing, or identifying people or things, or controlling sequences of events. Here are some examples and some questions to go with them.

Problem Set 1.3

1. a) Years ago telephone numbers had just as many digits as were necessary in a local situation. For example, a three digit number might do very nicely for a small town while a 6 or 7 digit number might be necessary for a large city. In large cities telephone numbers were frequently a combination of letters and numbers, for example, BU 1-2345; with the letters being abbreviations for regional names in the city. Recently telephone companies throughout the United States agreed to make all telephone numbers of 7 digits, doing away with letter designations and considerations of the size of town where a phone is located. What are some of the reasons that might have prompted such a decision?

   b) Actually telephone numbers are now identified by a 10 digit code, with the first 3 digits being a so-called "Area Code." Look at a map or listing of area codes in a telephone book and make some conjectures about the way in which they were assigned. If possible, have someone in your class check with a telephone company official about the correctness of your conjectures.

   c) For the 7 digits following the area code, what is the largest possible number of telephones that can be included within an area determined by one area code.

   d) Area codes always have second digit of 0 or 1, but no regular telephone numbers do. This is so that the machinery can detect when an area-code number is being dialed. What is the largest possible number of such area codes that telephone companies can assign? In theory, can more telephones be accomodated with such a three digit area code followed by a seven digit number or by simply assigning anyone who has a telephone a ten digit telephone number?

2. Get a U.S. map that shows the interstate highway systems and the older national highway system. Study the numbers of interstate highways and determine if any particular principles in assigning numbers apply. For example, is there any significance to even and odd numbers? To three digit versus two digit numbers? To the relative size of the number? Now study the numbers assigned to the older U.S. highway network and answer similar questions. When you've finished, try to verify from

some source the correctness of your conclusions.

3. The "Dewey Decimal System" for library cataloguing has whole numbers (hundreds) for broad categories (e.g., 500 is science and mathematics), and tens and one for more specific categories. As the cataloguing becomes more complex and more categories are needed, decimals are used (e.g., 510.19). The Library of Congress system is basically similar, but uses code letters along with numbers and allows for more categories, hence is often used for very large collections.

   a) Find out what system of cataloguing is used by your school library. Find out how fine the divisions are. For example, is any book in your library identified by a decimal number?

   b) How many books are in the largest library near you? (Don't forget to consider college or university libraries, which frequently have very large collections.) What cataloguing system is used?

4. The mathematician Philip Davis tells this story:

> "On a recent trip to New York I got off at Pennsylvania Station. I walked to the taxi platform, but the train had been crowded and soon dozens of people poured out of the station for cabs. Some of them waved and yelled; some stepped in front of the cabs rushing down the incline; some, who had porters in their employ, seemed to be getting preferential treatment. I waited on the curb, convinced that my gentle ways and the merits of my case would ultimately attract a driver. However, when they did, he was snatched out from under me. I gave up, took the subway, cursing the railroad, the cabbies, and people in general and hoping they would all get stuck for hours in the crosstown traffic.
>
> Several weeks later, on a trip to Philadelphia, I got off at the 30th St. Station. I walked to the taxi platform. A sign advised me to take a number. A dispatcher loaded the cabs in numerical order, and I was soon on my way to the hotel. It was rapid, it was pleasant, it was civilized. And this is a fine, though exceedingly simple, way in which mathematics may affect social affairs. ...The numbers have an order, and can be used to simulate a queue [line of people] without the inconvenience and indignity of actually forming a queue. The numbers are a catalyst that can help turn raving madmen into polite humans." ("The Criterion Makers: Mathematics and Social Policy.")

   a) List some places you have been lately where the natural order of the whole numbers and the "take a number" system have been used to simulate a queue to make order of service more fair.

   b) Could the set of fractions be conveniently used in this way? Why or why not?

c) Try to think of some other places where the natural order of whole numbers play a role. (Games, sequencing work, putting together model kits, etc.)

d) How do such uses of whole numbers differ from counting uses?

e) Can you imagine a situation where strict adherence to numerical order might not be warranted in dispensing some service?

5. Charge account cards are very common, and the numbers on the cards enable efficient processing of the cards by machine. There are sometimes annoying errors, however, mostly in clerical recording of the numbers. Since machines will "read" whatever number is recorded, there should be some way of removing cards with incorrectly copied numbers from machine control so that a human operator can look at other information on the card to correct the error.

In fact, a system has been devised to do this. It depends on adding an extra digit called a "check digit" to the end of the regular charge account number. Here is a system used by a book publisher. All charge account numbers are five digits long plus a check digit which is assigned by taking the sum of twice the last digit, three times the next to last digit, etc., and then finding the remainder after division of this sum by eleven. This remainder is the check digit. For example in the case of number 35796 the check digit is found as follows: 12 + 27 + 28 + 25 + 18 = 110; 110 ÷ 11 = 10 remainder 0; so 0 is the check digit and the complete number is 35796 * 0.

a) Verify that a transposition error (such as writing 28143 for 21843) will be detected by this routine.

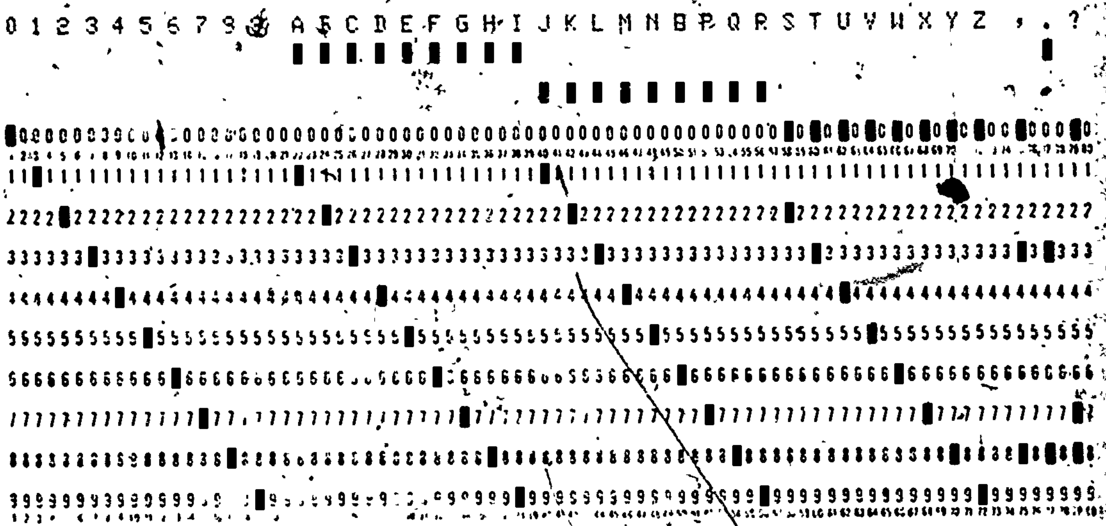
b) Verify that leaving out a digit (e.g., 1843 for 21843) or repeating a digit (e.g., 218843 for 21843) will be detected. (The computer would probably read 1843 as 01843; it might also have another routine programmed into it to reject invoices with serial numbers of the wrong length, such as 1843 or 188043.)

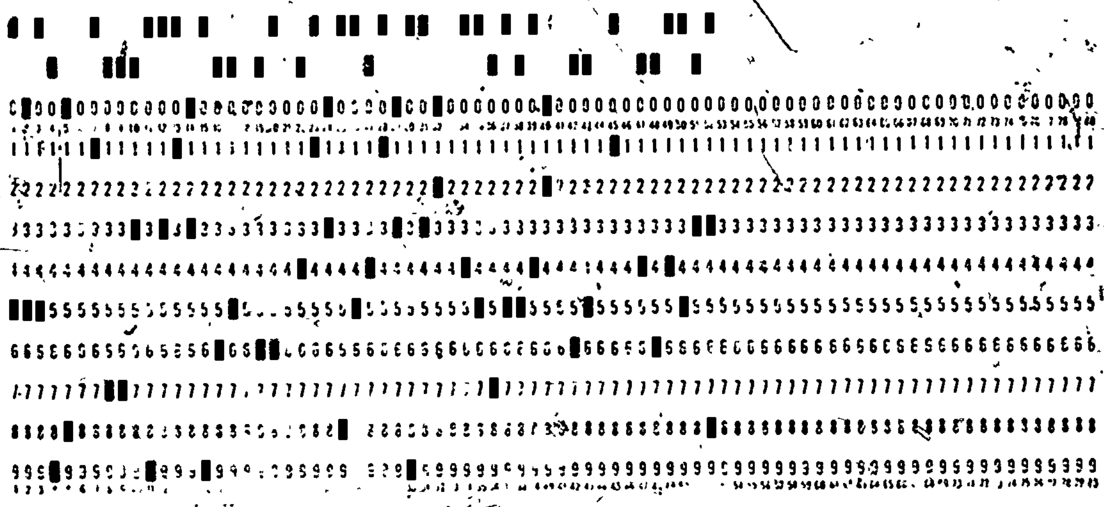
c) Think of some other errors that might be made in copying numbers and see if the check digit system would detect them.

d) Why might the inventors of this system have used 11 as a divisor in making the check digit? Why not use 5; or 9; or 10? Would it make any difference what number was used?

e) See if any of the credit cards used by people you know have check digits. The system used is probably different than outlined here--try to figure it out.

6. A very widespread use of code numbers these days is on the tabulating cards used with computers. In the first of the two pictures of such tabulating cards, the holes show how the numbers 0 through nine are punched into such cards, and how each letter of the alphabet is punched. See if you can "read" the message punched into the card below it. The next time you see such a card, see if you can read the information on it from the punched holes only.





1.4 Large and Small Numbers; Order of Magnitude

In today's world we frequently deal with information that involves very large numbers; for example, the billions of dollars in the United States budget and the millions of dollars in local, city, and state budgets. On the other hand, we are also frequently required to deal with information about very small quantities; for example, parts-per-million in various air and water pollution indexes.

Exercises 1.4a

For many purposes it is helpful just to realize <u>about</u> how big a number must be to describe a situation adequately. Consider the following table and put a check mark in the column that you think gets you "in the right ball park." Then check with classmates or friends and resolve disagreements in your judgments, perhaps by consulting almanacs or encyclopedias.

	Less than a billionth	Billionths	Millionths	Thousandths	Hundredths	Tenths	Ones	Tens	Hundreds	Thousands	Ten Thousands	Hundred Thousands	Millions	Ten Millions	Hundred Millions	Billions	More than Billions
Average height of man (meters). (Remember that a meter is about the same as a yard)																	
Average small bird (meters)																	
Insect (meters)																	
Bacteria (meters)																	
Virus (meters)																	
Molecule (meters)																	
Atom (meters)																	
Annual tons of garbage in U.S.																	
Motor vehicles per U.S. family																	
Pounds of food per U.S. person per day																	

	Less than a billionth	Billionths	Millionths	Thousandths	Hundredths	Tenths	Ones	Tens	Hundreds	Thousands	Ten Thousands	Hundred Thousands	Millions	Ten Millions	Hundred Millions	Billions	More than Billions
Gallons of water used per U.S. family per year																	
Books sold in U.S. each year																	
Reader's Digest circulation																	
Words per min. on typewriter																	
Words per min. by high speed printer attached to a computer																	
Time (seconds) for 3 digit × 3 digit multiplication by desk calculator																	
Time (seconds) for 3 digit × 3 digit multiplication by computer																	
Cost of new high school bldg. for 1,000 students (dollars)																	
U.S. Govn't annual budget (dollars)																	
Total annual wages in U.S. per year (dollars)																	
Annual Chicago school budget (dollars)																	
Annual Atlanta, Ga. school budget (dollars)																	
Cost of a new car (dollars)																	
Cost per month of running a new car, including insurance, depreciation, etc. (dollars)																	

	Less than a billionth	Billionths	Millionths	Thousandths	Hundredths	Tenths	Ones	Tens	Hundreds	Thousands	Ten Thousands	Hundred Thousands	Millions	Ten Millions	Hundred Millions	Billions	More than Billions
Distance from San Francisco to New York (miles)																	
Distance from El Paso to nearest Canadian border (miles)																	
Diameter of the earth (miles)																	
Circumference of the earth (miles)																	
Distance to the moon (miles)																	
Distance to the sun (miles)																	
Distance to the nearest star other than the sun (miles)																	
Diameter of known universe (miles)																	
Miles per gallon for automobile																	
Number of public school teachers in U.S.																	
Walking speed (mph)																	
Automobile speed (mph)																	
Jet airplane speed (mph)																	
Moon rocket speed (mph)																	

	Less than a billionth	Billionths	Millionths	Thousandths	Hundredths	Tenths	Ones	Tens	Hundreds	Thousands	Ten Thousands	Hundred Thousands	Millions	Ten Millions	Hundred Millions	Billions	More than Billions
Atlanta, Georgia population																	
Chicago, Ill. population																	
Children age 6-18 in U.S.																	
U.S. population																	
World population																	
Annual net world population increase																	

Problem Set 1.4b

1. In some scientific work an instrument called a diffraction grating is used. This is a slightly concave mirror with 30,000 equally spaced lines to the inch cut into it by a diamond needle so that when a beam of light strikes it, the light is diffracted into over 100,000 distinguishable colors. Try to estimate the thickness of the paper on which this page is printed, and the number of such lines which could be included in a space equivalent to the thickness of the page.

2. A recent (and somewhat controversial) book speculates that the ultimate source of life on earth is the energy supplied to the earth from the sun over billions of years. Indeed, the sun does put out an enormous amount of energy. One book (Mathematics in Everyday Things) estimates that the sun continuously generates the equivalent of 5.2×10^{23} horsepower of work. About one two-billionth $\left(\frac{1}{2 \times 10^9} = \frac{1}{2} \times 10^{-9} = .5 \times 10^{-9} = 5 \times 10^{-10}\right)$ of this is received by the earth. If this is so, about how much horsepower is continually being intercepted from the sun by the earth? An average automobile engine may be rated at about 200 horsepower; how many such engines would it take to equal the sun energy intercepted by the

earth? (The book doesn't say how much of this energy is absorbed by the earth's atmosphere and how much actually reaches the earth's surface.)

3. According to 1001 Questions Answered About Trees, the U.S. Christmas tree market in 1959 was about 40 million trees, with about 94% coming from forest lands and the rest from tree plantations. Make an estimate of the average cost of a Christmas tree and assuming things haven't changed very much since 1959 estimate about how much Americans spend on Christmas trees every year. How does this compare with annual spending in the U.S. for something else that you are interested in?

4. The article Intellectual Implications of the Computer Revolution, makes the point that when technology achieves an order of magnitude (tenfold or factor of 10) increases in efficiency, it usually changes the way one thinks about things. For example, going from the horse and buggy days of 5 to 10 miles per hour to an automobile of about 60 miles per hour (a one order of magnitude change in speed) has changed the entire face of the land and how we think about transportation. Similarly, the one order of magnitude jump from automobile speed to jet airplane speed has changed many of our concepts of the world. The article says that computers have improved in speed by at least six orders of magnitude (ten used as a factor six times, or 10^6, or a millionfold increase) and remarks: "In order to understand the factor of a million, consider the following two situations: first, you have only one dollar, and second, you have one million dollars. You can readily see that in the two different situations there are fundamental differences in the view you adopt of yourself and of the possibilities that are open to you." With the use of such high speed electronic computers the cost per unit of computing has decreased by something more than a thousand three orders of magnitude) over calculating done by hand on desk calculators, and there are far fewer mistakes. "It is as if suddenly automobiles now cost 2 to 4 dollars, houses 20 to 60 dollars." List some other situations where changes of one or several orders of magnitude makes a great difference in the way things are thought about.

5. The origins of the earth and universe are subject to much speculation and inquiry. One theory holds that the universe began as a tremendously concentrated "cosmic egg" with an initial temperature of over 10 billion degrees Celsius, (1×10^{10} C°) or 18 billion degrees Fahrenheit (1.8×10^{10} F°). This cosmic egg exploded and will expand until it loses its energy, then contract again into another cosmic egg at which point the

cycle will be repeated. One estimate that goes with this bit of speculation is that the period of oscillation is about 80 billion years. How many life spans of a man is that? Try and find some estimate of the possible maximum age of the earth and see where we might be in such an oscillating period.

6. A light bulb uses about 100 watts of power. The power in music and other sounds can also be measured in watts. In an ordinary conversation a man produces about one 10-millionth of a watt (.0000001 w or 1×10^{-7} w). How many people talking would it take to produce the power used by 100 watt bulb?

7. The author recently (October 1971) went to a rally to hear a speech given by one of the candidates for nomination for President in 1972. Among other things, he said that Illinois pays 14 billion dollars per year in federal taxes, of which 9 billion dollars goes through the Pentagon (i.e., for military and defense expenditures). Of this $9 billion, about $3 billion in a recent year went to southeast Asia. Guess one of the issues of this candidate. The 1970 census found 11,114,000 people in Illinois. Assuming these figures are correct, about how much per person do residents of Illinois pay annually in federal taxes? To the Pentagon? Via the Pentagon to southeast Asia? What are some possible uses you could make of such information? What assumptions might underlie this candidate's use of this information? Pick something familiar to you and relate it to these figures. For example, how many automobiles can be bought for the money the Pentagon spends? How many schools? How many housing units? Etc. (The point here is not necessarily that the money should be spent in some other way, but just to find ways of getting familiar examples to make more real just how much money is involved.) Much of the business of informed citizenship these days involves understanding such numbers and putting them in perspective. At the time you read this, are there things in the news involving large expenditures of money?

Chapter II

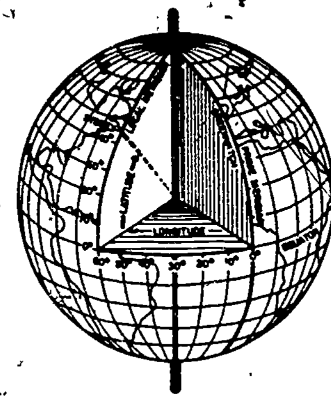
USES OF PAIRS AND TRIPLES OF NUMBERS

In chapter one we talked about the use of mathematical models and about ways of using single numbers for numerical descriptions, for counting or measuring, for ordering, for coding, for identification numbers, and in other ways. We discussed the "order of magnitude" of numbers and "scientific notation" for very large and for very small numbers. In this chapter we will be dealing primarily with numbers used in pairs or triples in a variety of ways.

2.1 Coordinate Systems

On a long automobile trip a highway can be idealized as a "number line" with the mileages along that number line serving to locate oneself on the journey. Generally speaking, however, man wants his location not merely on a line but on a surface--on the earth, within a country, within a city. For this he usually needs a <u>pair</u> of numbers. Similarly, if he needed to locate "Room #306" within a building, he might realize that the number 306 probably does might not mean the three-hundred-sixth room but rather third floor, sixth room. That is, a pair of numbers is concealed in the 306.

To begin with, let us consider the problem of precisely describing locations on the surface of the earth. As in all problems of location we need to have standard reference lines. In the case of the earth, one of these is the earth's equator. (For this discussion and the problem set that goes with it, it will be helpful to have a standard world globe before you.) Locations north and south of the equator are on circles parallel to the equator. Each such circle is assigned a number between 0 and 90 north or 0 and 90 south. (What is the radius of the "circles" numbered 90S or 90N?) The number assigned is the measure of an angle in a plane perpendicular to the equator and through the center of the earth, with the vertex of the angle at the center of the earth and rays intercepting the equator and the parallel circle north or south of the equator. (See the diagram on the following page, showing the 30th parallel North.) As with other angle measures, if more precise measurement is required, the circle is assigned a measure in degrees and minutes, or even degrees, minutes and seconds. (For problems in this book, however, round off to the nearest degree.) This part of your location on the globe is often



called the latitude. (According to Mapping geographers refer to it as "phlatitude," because they label it with the Greek letter Phi (ϕ). This may help you remember the meaning, since the letter P goes with parallel, and latitudes refer to "parallels.")

Problem Set 2.1a

1. Find the city you live in on a world globe and find out what parallel it lies on.

2. The Canadian-United States border runs roughly along what parallel? Excluding Alaska, what is the latitude of the northernmost point in the United States? Including Alaska, what is the latitude of the northernmost point? Roughly speaking, where does our southern border run, i.e., along what parallel? What is the phlatitude of the southernmost point in the United States? Is Hawaii further south than most other places in the United States? Where are such American dependencies as Puerto Rico? Is all of Mexico south of the southernmost point in the United States? Where is Havana, Cuba with respect to the southernmost point in the United States?

3. As the world moves, the stars in the sky seem to move, but there are certain "fixed" stars. The most famous of these is the North Star (Polaris). It is said to be the case that the latitude of a location can be found by measuring the altitude in degrees of the North Star above the horizon. Try to figure out why this should be so (draw a diagram) and try to verify it by first finding the latitude of your city on the world globe and then comparing it with the angle in degrees between your horizon and the North Star. (For very accurate measurements of this sort navigators use an instrument called a sextant. You may want to see if one is available and find out how to use it, but there are probably other ways to get a rough measure of the angle in question.)

4. After World War II the "38th Parallel" was defined as the boundary between North Korea and South Korea. (This boundary was changed somewhat by the Korean War.) Look at your world globe and make a list of other important boundaries that seem to be defined in terms of parallels of latitude.

Knowing only the latitude is not much help in locating yourself, because that only puts you someplace on a very large circle parallel to the equator. (Unless your latitude is near 90°; in that case, what would you be wearing?) To locate yourself in the east-west direction consider the "great circles" through poles of the earth. (A great circle is the intersection of the earth's surface with a plane through the center of the earth.) Meridian or longitude lines are semicircles from North Pole to South Pole that lie on great circles through these poles. To attach numbers to these meridian lines, one of them is arbitrarily designated the "prime meridian" and all others are assigned numbers from 0 to 180 East or 0 to 180 West of this prime meridian. Prime meridians has been defined in many ways throughout history and you could make it the meridian passing through place you are right now but on most maps it is taken as the one through the Admirality Observatory in Greenwich, England.

Problem Set 2.1b

1. Keeping in mind that there are 24 hours in a day and that the earth is rotating counterclockwise when looking down on the north pole, why is it that the meridian lines marked on many maps are spaced 15 degrees apart at the equator. These particular meridian lines (called central meridians or time meridians) run approximately through the middle of "time zones" around the world so that when it's high noon at the Greenwich Prime Meridian, it is 1 o'clock at the 15° meridian east, 2 o'clock at the 30° meridian east and so on around the world. Time zones are about 15° wide with irregular boundaries determined for national and regional convenience; for example, it would be a nuisance if a given city were divided into two time zones. Look at a map and see what time zone you are in and how many degrees its time meridian is from Greenwich. Look at the boundaries of your time zone and figure out why they run as they do.

2. The international dateline is, roughly speaking, the meridian numbered 180 east or west, halfway around the world from the Greenwich Prime Meridian. From the information just given, figure out why this should be so and why it is one day later on one side of the dateline than on the other.

3. Find out the average meridian (or longitude) of your city. (Longitude is designated by geographers with the Greek letter lamda, written λ.) That is, find out how many degrees around from the Prime Meridian you are.

4. Latitude can be found by finding the angle of the North Star above the horizon; longitude can be found by knowing what time it is over the Prime Meridian when it is exactly "high noon" by the sun over your location. Hence up until a few years ago accurate navigation of ships and airplanes depended on having very accurate timepieces (chronometers) set to the time at the Prime Meridian. One then simply had an instrument to determine when it was exactly noon (sun exactly overhead), read the chronometer, and from that figured out how many degrees one's present location was from the Prime Meridian. Set your watch to "Greenwhich mean time", then take a reading at noon tomorrow (by the sun, not the clock) and see if this works out properly.

5. In addition to latitude and longitude it is sometimes important to know your altitude. What is the (average) altitude of your city? What are the highest and lowest places in your state? In the United States? (In each case try to give approximate latitude, longitude, and altitude of high and low places.) What is the lowest place on earth, including the deepest place in the ocean? What is the highest place place on earth? What is the lowest place if you don't go under water; that is, what is the lowest dry-land place on earth?

6. With latitude, longitude, and altitude, there are three coordinates to locate a point in our ordinary 3-dimensional space. If you were directing two airplanes in such a way as to avoid a collision, you would need to know in addition to latitude, longitude, and altitude the time each airplane is at a given place; that is, four coordinates are needed. (Authors of popular science articles and science fiction stories thus sometimes characterize time as being "the fourth dimension.") Considered this way, the number of dimensions is simply the number of independent pieces of information needed to describe a situation. Can you think of situations where other pieces of information would help in locating yourself?

7. When locating ourselves in everyday life we frequently use a number of separate pieces of information. For example, if somebody wanted to locate you in school, they might need to know in addition to the city you are in such information as the street address, room number, desk in the

room, and the period or time. How many separate pieces of information are given in the following set of directions? A certain school in Chicago is located at 1362 East 59th Street; during second period a student is in room 306, at the second desk in the first row in the room. Could the location of this student be described precisely with fewer pieces of information? Could latitude, longitude, altitude, and time exactly locate this student if it were possible to precisely determine them?

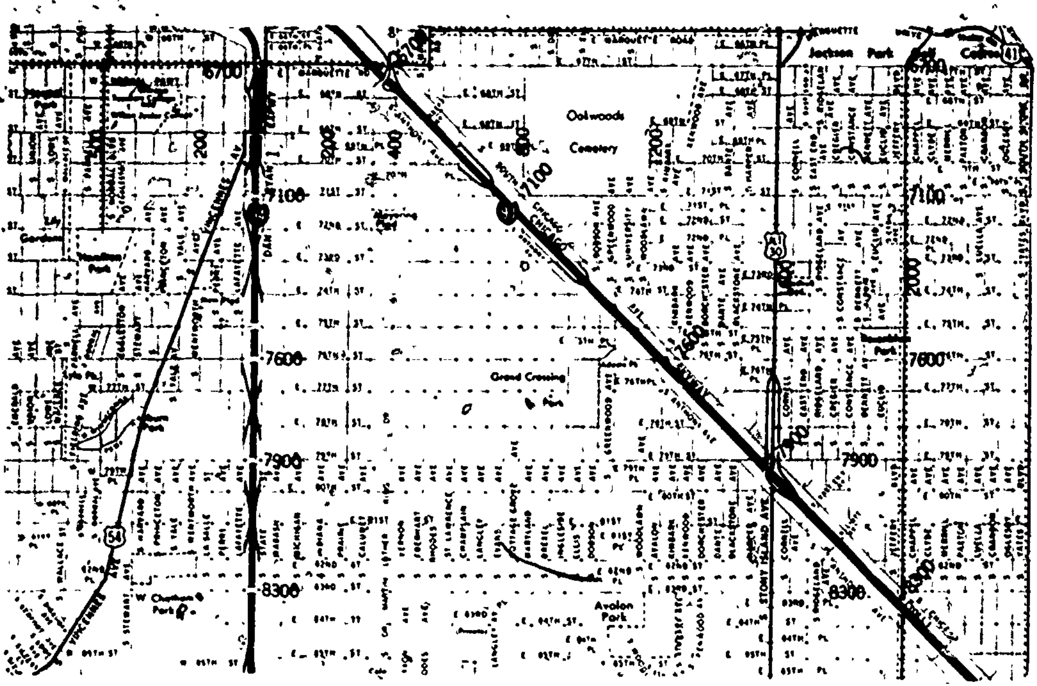
8. In the case of latitude, the standard reference line is the equator. In the case of longitude the standard reference is usually the Greenwich Prime Meridian. What is the reference for expressing altitude and how is it determined?

9. To avoid serious disputes it is important that real estate property be described very precisely; take a moment and think of situations where serious disputes could arise if the boundary lines of a piece of property were unclear. Here is a description of an actual residential building lot in Chicago, Illinois:

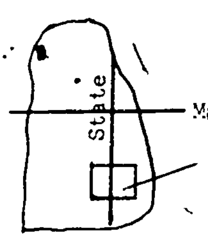
> Lot 14 in Block 5 in Ashland, a Subdivision of the North three quarters and the North 33 feet of the South quarter, both of the East half of the North East quarter (except the North 167 feet thereof) in Section 18, Township 38 North, Range 14, East of the Third Principal Meridian in Cook County, Illinois.

The description mentions "the Third Principle Meridian" as a reference line and also "Range 14" as a reference line. Find out (perhaps from someone in real estate) what this description means and how you could locate the property in question. If possible, get a description of the property on which your own home or school building is located and figure out what it means. (If you get stuck, the book Mapping, will give you information on how the United States has been surveyed and boundary lines established for Sections, Townships, and so on. This book is full of fascinating geographical information, and will also supply some interesting details about navigation, "earth coordinates", and map making.)

On the following page there is a map of a part of the south side of Chicago, Illinois. Like many of the "newer" cities in the United States, Chicago is laid out on a rectangular grid so that two pieces of information (distance north-south, distance east-west) give unambiguous locations and also serve to show the relation of one location to another. Since many of the streets are named rather than numbered, one also needs a key to know which location coordinate goes with the name; a list of some of these is given under the map. The reference line running east and west is Madison Street; the one running



Anthony (400E 6801S-SE)
Ashland (1600W)
Avalon (1234E)
Baldwin (1900E)
Bennett (1911E)
Birkhoff (634W)
Blackstone (1437E)
Calumet (325E)
Champlain (625E)
Chappel (2038E)
Clyde (2100E)
Constance (1836E)
Cornell (1632E)
Cottage Grove (800E)
Crandon (2300E)
Cregier (1800E)
Cyril Ct (1934E)
Dante (1433E)
Dobson Avé (1026E)
Dorchester (1400E)
Drexel Av (900E)
East End (1700E)
Eberhart (500E)
Eggleston (430W)
Emerald (732W)
Euclid (1934E)
Fielding (518W)
Gilber Ct (700W)
Givins Ct (721W)
Greenwood (1100E)
Harper (1500E)
Harvard (319W)
Indiana (200E)
Jeffery (2000E)
Kenwood (1342E)
Kimbark (1300E)
King Dr (400E)
Lafayette (26W)
Langley (700E)
LaSalle (150W)
Luella (2224S)
Marquette Rd (6600S 6700S)
Marquette Rd S (500E)
Merrill Av (2125E)
Michigan (100E)
Normal Av (500W)
Parnell (534W)
Prairie (300E)
Princeton (300W)
Rhodes (532E)
St. Lawrence (600E)
South Chicago (400E 6700S-SE)
South Shore Dr (1734E 6700S-SE)
State St (1E W)
Stewart (400W)
Stony Island (1600E)
Union (700W)
Vernon (420E)
Vincennes (700E 3500S-SW)
Wallace (600W)



Madison

Enlarged map above

north and south is State Street. Hence the address 1362 East 71st Street describes the location a little over 13 blocks east of State Street and 71 blocks south of Madison Street (since in Chicago all the numbered streets are south of Madison.) Some of the older cities in the United States have much less tidy addressing systems; below we'll indicate some of them and also some other rather nice systems.

Problem Set 2.1c

1. Notice that South Chicago Ave. runs diagonally across part of the map. If (and only if) you have had enough algebra to know about slopes of lines and equations of lines, figure out what the slope of South Chicago Street is if Madison is the X axis, State Street is the Y axis, and north and east respectively are regarded as positive directions. (Remember that unless indicated otherwise the top of a map is regarded as the northerly direction.)

2. For each of ten Chicago addresses of your invention (but on the map) give a pair of numbers that would pinpoint the location in question. You will probably need to use E-W and N-S codes along with the numbers. If a coordinate system were set up as in problem 1 above, could all of the addresses be expressed unambiguously without E-W and N-S codes? How?

3. Two additional things about the Chicago addressing system can be noted: even numbered addresses are all on one side of any street, odd numbered addresses on the other side and every increase of 100 in the numbering system means $\frac{1}{8}$th of a mile. The numbered (East-West) streets on the south side of Chicago are $\frac{1}{8}$th mile apart but for North-South streets there is no such regularity. (In some parts of the city there are twelve North-South streets per mile; in other parts sixteen; in some places neither of these--as you can see from the map.) Main traffic streets are usually a half-mile apart. Find out how your own city or town (or the largest city nearest you) is laid out for addressing purposes and see if the even-odd or eight blocks to a mile conventions are followed.

4. Addresses on New York's Manhattan Island are not as neatly arranged as in Chicago. Each of the North-South streets must be regarded as a separate number line with its own system of coordinates; hence, "45 Fifth Avenue" may not be anywhere near "45 Fourth Avenue," nor is it likely to have anything to do with numbered streets such as Forty-fifth Street. Try to get a map and other information about Manhattan addresses and find out if there is some systematic way of finding various locations from

their addresses.

5. Detroit has a number of concentric circle main streets around the central city core--"One Mile Road," "Two Mile Road," "Three Mile Road," and so on. Other main streets go out from the center of the city like the spokes of a wheel. Given such a design for a city, how could one invent an unambiguous addressing system? Try to find out if the system you devised does in fact resemble that used by Detroit. (There is a mathematical coordinate system called "polar coordinates" that resembles the Detroit map. Find out something about it or ask your teacher about it.)

6. To get a letter delivered promptly in Washington, D.C., you must include NW, SW, NE, or SE after the street address. Why do you suppose this is? Verify your conjecture.

7. Many of the older cities of the world have address systems that are very confusing indeed, at least to a stranger. For example, along a street in Paris, France, there may be a number of small segments each of which has a different name, with each named segment having an independent numbering system along it. It gets even worse in some places. For example, it is reported that in some towns buildings are numbered according to the order in which they were built, so that a new building might have a very high number right next to a building with a very low number. Try to find out about the address systems in, say, Tokyo, Japan, or any other city that interests you.

2.2 Ratios and Percents

When using pairs and triples of numbers as "coordinates" to locate something, each number retains its identity as a separate piece of information. But frequently two or more numbers are combined into a single number, and the individual numbers lose their identities. For example, you have often combined two or more numbers into single numbers using ordinary arithmetic operations such as addition and multiplication.

In many cases the most useful way to consolidate two pieces of information into a single piece of information is to express a "rate" or a "ratio" by combining the two numbers into either a fraction or a quotient. For example, 1001 Questions About Trees tells us that it used to be thought that in making paper from wood, only the wood fibers from soft wood were suitable because they are much longer than the fibers from hard wood (about one tenth of an inch long for soft wood and less than one twenty-fifth of an inch long for certain hard woods). Later it was found that what mattered was not the length of the fiber, but the ratio of the length to the diameter. Since hard wood fibers are both thin and short, a length-to-diameter comparison by ratio shows that they are just as suitable for making strong paper as are longer and thicker soft wood fibers.

In the papermaking example given above the ratio is formed using two measures given in the same units. Frequently ratios are formed using measure units that are not the same. For example, consider average automobile speeds expressed in miles per hour. Suppose someone tells you that he made a trip at an average speed of 50 miles per hour. This information might have come from a trip of 100 miles made in two hours or a trip of 25 miles made in a half-hour; in either case, information using two different units is compared. Also, the original information given by pairs of numbers is lost when the average speed is expressed by a single number--50 miles per hour.

Problem Set 2.2

1. People are frequently interested in how many miles per gallon (mpg) they get from their automobiles. "Fifteen miles per gallon" could be gotten from any of the following measures: 300 miles, 20 gallons; 420 miles, 28 gallons; 330 miles, 22 gallons. Find out the "mpg" for your family car. What might affect the mpg?

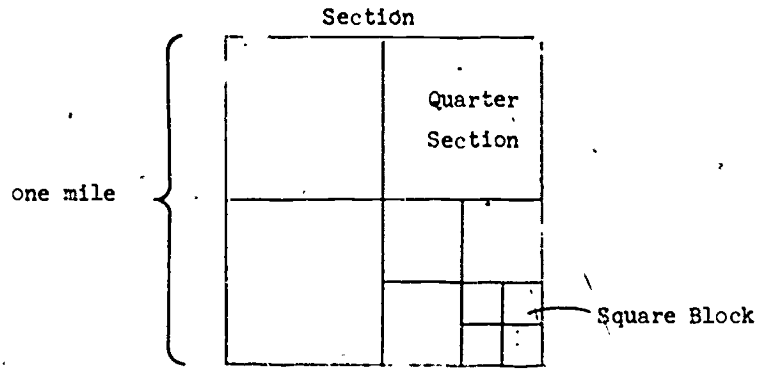
2. Still other common examples of ratio are found in government statistics such as per capita (that is, per person) income or per capita tax revenues. In this connection recall problem number seven in Problem Set 1.4b about U.S. Income Taxes from Illinois and how they are allocated to certain uses; how much per citizen; and so on. Find some other such government statistics and make up several problems suitable for this section.

3. We often express information in terms of a comparison to something else that might be regarded as important. For example, in <u>1001 Questions Answered About Trees</u> we are told that in 1945 damage to "saw timber" (timber made into lumber for construction) by insects and diseases was $3\frac{1}{2}$ billion board feet per year, and that this loss was more than 3 times the loss from fires. By 1959 (when the book was published) the insect and disease damage was around 7 billion board feet per year and this was said to be about 9 times the fire loss. What was the approximate fire loss in 1945? What was the approximate fire loss around 1959? Is the fire loss decreasing or increasing? How could you have answered this last question without actually figuring the fire losses given for 1945

and 1959?

4. <u>1001 Questions Answered About Trees</u> tells us that a bamboo tree resembles certain grasses in its structure (the scientific name of bamboo, <u>dendrocalalamus</u>, means "tree reed") and that one can almost see a thriving bamboo tree grow; as much as 18 inches in a single day. What is the average growth per hour of a bamboo tree growing at that rate?

5. Much of the land in the United States is surveyed and divided into mile-square "sections"; these in turn are divided into quarter sections, then quarter-quarter sections, then in square blocks (see diagram).



Hence there would be eight blocks along a one mile side of a section. (See if this is the case for your city.) If there are 8 blocks per mile, how many feet per block are there (5280 feet per mile)? How many yards per block? About how many football field lengths or 100 yard dash track lengths per block? If you run hard for a block, have you done a 100 yard dash or a 220 yard dash or a 440 yard dash or what? Notice how many times the word "per" is used in the few sentences above; this word frequently indicates ratio or rate expressions.

6. Athletic fields frequently have a running track around the outside of a football field; about how long is such a running track likely to be and how many times around it will make a mile?

7. One common use of ratios is to express "scale" on such things as building plans and maps. For example, a frequently used ratio in architectural drawings is one eighth inch to one foot. Using this scale, for how large a building could you draw a plan on a page of this book leaving reasonable margins and space for title and explanations? What sort of scale would be appropriate if you were going to draw a floor plan of your mathematics classroom on a page of this book? What scale would be appro-

priate to draw one entire floor of your school on this page?

8. The book Mapping tells us that the scale of a map is often expressed as a "representative fraction" which tells exactly how much the map is "shrunk" with respect to the actual ground it represents. For example, a representative fraction of 1/5,280 would tell us that one foot of the map would represent 1 mile of actual distance (a mile is 5,280 feet). If, as is fairly common, a map you are looking at has one inch "equal" to a mile, what would the representative fraction be; that is, how many inches are there in a mile? At a scale of one inch per mile how long would a map have to be to represent the United States excluding Alaska and Hawaii coast-to-coast (about 3000 miles)? How wide? If your school has a flat roof, is it large enough to accomodate such a map?

9. Mapping tells us that the purpose of a map is often defined by its scale. For example, maps with representative fractions of 1:1,000,000 and smaller are usually atlas maps giving general views of the earth's surface, shapes of continents, boundaries of countries, and so on. Maps between 1:1,000,000 and 1:10,000 allow more detail about a given region. Maps of a larger scale than 1:10,000 are usually used to show real estate divisions and surveys. We are told that the nickname for maps with 1:1,000,000 scale is "a million map." Such a million map may be part of a large map-making project, "The International Map of the World," which many nations throughout the world are cooperating on. In this scale about how many miles are there to an inch?

10. Suppose that you are dealing with a world globe that is one foot in diameter; this will make it about 38 inches around the equator (why?). The earth itself is about 8000 miles in diameter, hence about 25,000 miles around the equator. So about what distance does one inch on such a globe represent? Now, using the number of inches in a mile, change this to a representative fraction. (Since the 12 inch diameter globe would have a circumference of a little less than 38 inches and the equatorial circumference is very nearly 24,902 miles, it would be a little bit silly to give answers to the above questions in very exact numbers.)

11. We have the impressions that the earth is very irregular with very deep oceans and very high mountain peaks. This is indeed so, yet it has been said that if the earth were shrunk to the size of a billiard ball then it would be fine to play billiards with! That is, relative to earth's enormous size, the irregularities don't amount to much. To see why this would be so, remember that the earth has about an 8000 mile diameter.

Figure out about how many miles high the highest mountain is, then figure out what part of the 8000 mile diameter this would be. Similarly, find out how deep the deepest point in the ocean is in miles and compare it to the 8000 mile diameter.

12. The world is not perfectly round; it bulges somewhat around the equator. See if you can find out how much this bulge is relative to its size. Would irregularities of this magnitude seriously affect the "trueness" of a billiard ball?

13. Even a fairly large world globe or map would not show the mountains and ocean depths clearly if the scale used vertically to show depths and heights were the same as that used in the rest of the map to show planar distances. That is, any contour map has the vertical scale grossly distorted with respect to the other scales used. If you can find a relief map somewhere (perhaps in a library) verify this by measuring the height of a mountain on the map, then computing what height would be represented if the vertical scale were the same as that used on the rest of the map. For example, if the highest mountain peak on a relief "million map" measures one inch, what height would this represent if the vertical scale were also 1:1,000,000?

14. The book Mapping tells us that ratios are also used in defining standards of accuracy for surveying. "First order of accuracy" is one unit error allowed in 25,000 units; "second order of accuracy" is one unit in 10,000 units; "third order of accuracy" is one unit in 5,000 units. What would be the allowable error in miles for each of these three standards of accuracy for a direct line survey coast-to-coast for the U.S.?

15. Mapping speaks of a Cape Kennedy satellite tracking job in 1960 that required placement of cameras on a survey with an accuracy of one unit in 400,000; the U.S. Coast and Geological Survey actually delivered one in one million accuracy. With this latter standard of accuracy, how many feet error might there be in a coast-to-coast survey of the U.S.?

16. With the use of artificial satellites, it is possible to measure the distance between the Atlantic and Pacific coasts with an error of only 10 to 15 feet; what standard of accuracy does this represent?

17. "Specific gravity" is a measure of how much heavier or lighter something is than an equal volume of water. For example, the specific gravity of mercury is 13.596 because a cubic centimeter of mercury has a mass of 13.596 grams, while a cubic centimeter of water has a mass of 1 gram.

The units of measure cancel in the ratio, leaving a number with no units attached to it. Given the fact that water weighs 62.43 pounds per cubic foot and mercury weighs about 848.80 pounds per cubic foot, form the ratio and do the division, showing that in either system of units the specific gravity of mercury is still about 13.6.

18. If something sinks in water, it has a specific gravity greater than one; if it floats, less than one. Do you see why this is so, given the definition of specific gravity? Figure out the specific gravity of these common American hardwoods: hickory, beach, and black locust may weigh 56 pounds per cubic foot (of seasoned wood). On the other hand, willow and poplar may weigh only 20 pounds per cubic foot, while butternut, basswood and tulip tree wood weigh about 25 pounds per cubic foot.

19. Some soft woods weigh nearly as much as the heavier hard woods. What is the specific gravity of Western Larch weighing 53 pounds per cubic foot of seasoned wood?

20. In the West Indies, Florida, and the Yucatan there is a black ironwood tree which weighs up to 88 pounds per cubic foot. What is its specific gravity? Does it float?

21. Make a rough calculation of your volume (perhaps by thinking of yourself as a box or cylinder), weigh yourself, then figure out what your specific gravity is. Devise several different ways of estimating your volume and indicate what simplifying assumptions, etc. go into each of these models for the estimation.

22. Form the appropriate ratio and compare the average speeds (yards per second, etc.) for each of the following records (for humans) given in <u>Guinness Book of World Records</u>: 9.1 seconds for 100 yards, 19.5 seconds for 220 yards, 44.7 seconds for 440 yards, 1 minute 44.9 seconds for 880 yards, 3 minutes 51.1 seconds for 1 mile, and 46 minutes 37.8 seconds for 10 miles. For which distance is the greatest average speed attained? Does this surprise you?

23. A handy reference point for comparing speeds is that one mile per minute (60 miles per hour) is the same as 88 feet per second. Show that this is true. Use this information, ratios, and appropriate conversions to compare each of the following animal speeds with man's maximum speeds as given in the previous problem. These are from <u>Animal Facts and Fallacies</u>: cheetah 70 mph; antelope 60 mph for two miles and 36 mph for 27 miles; deer or race horse 45 to 50 mph; greyhound 35 mph; buffalo 30 mph; elephant 25 mph.

24. The record high jump for man is about $7\frac{1}{2}$ feet and the record broad jump about 29 feet. Animals comparable to the size of man, such as a deer, impala, or kangaroo can jump 8 to 10 feet high but a jerboa from Africa whose body is about 5 inches long can jump 3 to 4 feet high. How high could a man jump if he could do as well relative to his size? (In Chapter V we will discuss why man can't jump that high.)

25. Considering their size, many animals can do better than man in horizontal jumping. For example, an impala can jump 30 to 40 feet, a large jack rabbit about 25 feet, a kangaroo 25 to 30 feet, while a jerboa can jump 12 to 15 feet; and a cricket frog only an inch long can jump 3 feet. Ask and answer a couple of interesting questions based on these data.

26. Using the information above, make up and answer a question about ratio of horizontal jump distances to vertical jump distances for man and several other animals.

Note: In the following problems, as in many past ones, you should use good sense and reasonable estimating procedures rather than trying for fussy and exact answers. The information about animal sizes is from <u>Animal Facts and Fallacies</u>.

27. What is the ratio of the adult weight to the birth weight of an average human being? Of adult length to birth length?

28. The birth of young bear cubs is during the winter sleep of bears; they sometimes weigh no more than 6 to 8 ounces, while an adult bear may weigh about 200 pounds. What is the ratio of adult weight to birth weight?

29. For both kangaroos and oppossums the young are born only a couple of weeks after conception. Then with strongly developed front legs they climb through the fur of the female into a pouch from which they emerge fully developed about 70 days later. (Such animals are called marsupials; the pouch is a marsupium.) A newly-born kangaroo is only about an inch long; weighs a couple of ounces and bears <u>no</u> resemblence to its parents. What is the approximate ratio in kangaroos of adult weight to birth weight and adult length to birth length? This is said to be a greater size difference than for any other mammal.

30. On the other hand (still speaking of mammals), a newborn whale at birth is often nearly half as long as his mother. A female sperm whale only 32 feet long was found containing a calf 14 feet 8 inches long. A Blue Whale 80 feet long had a baby 25 feet long weighing 16,000 pounds. If you can find out about how much an 80 foot blue whale would weigh, figure out the

ratio of adult weight to birth weight for this animal. The Random House Unabridged Dictionary says that this whale, also called the sulfer-bottomed whale, is the largest mammal that has ever lived.

31. It is said that ocean sun fish weigh more than 1200 pounds and are over 8 feet long, yet lay eggs that hatch into fish only $\frac{1}{10}$ of an inch in length. What is the ratio of adult length to birth (hatch) length? It has been said that the young sun fish must increase its weight approximately 60,000,000 times before it gains maximum size; if this is so, then what is the approximate weight of the newly hatched fish?

32. Musical notes blend together perfectly if one frequency is exactly twice another. If three frequencies are in a ratio of 4 to 5 to 6 this combination of notes is called a "major triad." The musical scales which are used in the western world in pianos and the like are precisely formulated from certain ratios. Find out something about the relationship of various ratios to the construction of musical scales. (One source is <u>Science, Numbers, and I</u>, pages 124 through 135.)

33. In the metric system of measure, water plays a role in several places. For example, a cubic centimeter of water weighs one gram, hence 1000 cubic centimeters weighs a kilogram, and the volume of a liter is defined to be 1000 cubic centimeters. The Celsius temperature scale has zero as the freezing point of water and 100 as the boiling point of water; hence water is a liquid between $0^{\circ}C$ to $100^{\circ}C$. The prefix "mili" means "thousanth." Is one mililiter (abbreviated ml.) the same as one cubic centimeter? How many grams per mililiter of water? (Note to teachers: the smallest unit in a Cuisenaire rod set is one cubic centimeter; it might be useful to get a handful of those to show the students what a cubic centimeter or mililiter volume looks like and hence how much water is in a gram. Also, because the weight of water was found to be slightly different than what it was thought to be when metric units were invented, the equivalence of a cubic centimeter to a gram is only approximately correct, though accurate to a fair number of decimal places.)

34. A unit of heat in common use in the metric system is a "calorie." This is the amount of heat it would take to raise the temperature of one gram of water by one degree Celsius (actually from $14.5^{\circ}C$ to $15.5^{\circ}C$.) It turns out that this isn't much heat, so to have a convenient number to work with most applications use "kilocalories." (Sometimes, but not always, this "calorie" is capitalized: 1 Calorie = 1000 calories.) For example, when a diet chart says that an ounce of butter contains say 200 "calories" it really means 200 kilocalories. Get a calorie chart like

those used for dieting. Look at the calories per unit of various foods and try in your own mind to relate that information to the heat definition of calories. (For example it is clear that more heat energy can be gotten from an ounce of fat than from an ounce of spinach.) Incidentally, an ounce is about 30 grams--more precisely, 28.23 grams.

35. Weather and Health tells us that the most important mechanism for man's surviving in extreme heat is sweating and evaporation of the sweat from the skin. Each of us has about 2,000,000 sweat glands. For a short period of time we can survive with a water loss of about 2 quarts per hour provided this is replaced by drinking an equivalent amount of fluid. This corresponds to a heat loss of about 700 kilocalories per square yard of body surface per hour. Over a 24 hour period a loss through sweat at the rate of about one pint per hour can be tolerated, with adequate fluid replacement. Estimate the number of square yards of body surface of the average human adult. From this, estimate how many kilocalories of heat a person releases by sweating the two quarts per hour; by sweating the one pint per hour. Remembering that "a pint's a pound the world around" (for water), what percent of the body weight of a person weighing 150 pounds would the two quarts per hour fluid loss through sweating represent if not replaced? The pint per hour loss?

Comment: As you probably know, a "percent" is just a standard and convenient way of expressing ratios where the comparison is always to 100 units. For example, 27 percent means exactly what you expect it to mean, namely 27 units per 100 units. Percents, of course, are in very widespread use in everyday life applications.

36. From daily newspapers, advertisements, and other experience over several days, list at least 10 different uses of percents.

37. In Weather and Health we are told that a loss of about 2% of body weight through sweat makes a person thirsty. In your own case how much water loss would that be (that is, what is 2% of your own weight)? Using "A pint is a pound" what volume of water is this?

38. If the water loss in sweat is 5% of the total body weight, there is a very intense thirst and the body temperature and pulse rate rise rapidly. How much water loss by sweat would that be in your own case?

39. Sweat water losses of over 7% of body weight can lead to circulatory failure and possible death. Compute what sort of water loss that would be for you. (These figures apply to those who are adults or near adults;

small children and babies have much less tolerance.)

40. The body is much less tolerant of certain other kinds of losses from the body. For example, sweating also removes salt from the body. A loss of one gram of salt for every 4 pounds of body weight normally results in fatigue while the loss of 1.5 grams per 4 pounds of body weight may result in a drop of (systolic) blood pressure and possible shock. In your own case, how many grams of salt loss through sweating would result in fatigue and how much in the more serious effects? (There are about 30 grams in one ounce.)

41. With some estimate for the percentage of salt in perspiration, try to estimate how much loss of fluid through sweating would lead to a dangerous loss of salt. Use either your own weight or that of an average male (about 150 pounds) as a reference. (Maybe one of your science teachers can help you find out how much salt there is in perspiration.)

42 On the TV show Emergency, the paramedics handling an injury or heart attack are nearly always told to "Start I.V. ... ." (I.V. means "intravenous.") An I.V. puts fluid (usually saline--salty) directly into the blood stream. What might this have to do with the above discussion of dangerous fluid and salt losses?

43. Weather and Health tells us that 78% of our air at the earth's surface (by volume) is nitrogen (N_2) and that 21% is oxygen (O_2). Most animal life processes depend on oxygen. It is thought that very early in the life of the earth there may have been little or no oxygen in the air and that the presence of plants on the earth brought oxygen into the atmosphere. (As you may know, plants use carbon dioxide and give off oxygen as a waste product, while animals use oxygen and throw off carbon dioxide as a waste product.) Ozone has its highest concentration at an altitude of about 15 miles and is very important to life but when it forms near the earth's surface in smog it's highly irritating to our mucous membranes. Atomic weights are defined as very small units by which a hydrogen atom weighs about 1 and a carbon atom weighs 12. With this unit an atom of oxygen weighs about 16 (hence an O_2 molecule weighs about 32) and an atom of nitrogen weighs about 14 (so an N_2 molecule weighs about 28). With that information figure out the proportions of nitrogen and oxygen in air by weight instead of by volume, ignoring the 1% of other constituents of air.

44. What would be the atomic weight of an atom of H_2O (water)?

45. Oxygen and nitrogen are valuable gasses that are frequently packaged in tanks and sold for various uses. It is clear from their abundance in air that it might very well be economically profitable to separate ordinary air into its components and thus extract the nitrogen and oxygen for bottling and sale. Indeed this is the case. Is it likely that this would be an economical way to get a supply of carbon dioxide? Carbon dioxide is used in making soft drinks (it gives such drinks the fizz) and in manufacturing "dry ice." Try to find out how carbon dioxide is manufactured for commercial use. It is possible to make fruit juice or flavored water into carbonated soft drinks by dropping in a piece of dry ice and letting it "melt" into carbon dioxide gas, at the same time cooling the drink. Find out at what temperature the change from solid "dry ice" to carbon dioxide gas takes place. (In using dry ice in this or any other way, certain cautions about its handling are in order--find out what they are <u>before</u> you do anything with dry ice.)

46. Maple sugar is made mostly from mature maple trees perhaps 100 years old, perhaps 70 feet tall with diameters at the base of 2 to 4 feet. The sap collected from the trees is about 97% water and 3% maple sugar (mostly succrose). According to <u>1001 Questions Answered About Trees</u>, the average yield of maple sugar per tree is about 2 or 3 pounds per season, with very fine trees in an exceptional year yielding perhaps 5 or 6 pounds of sugar. About how many gallons of sap must be collected from a tree for a 3 pound yield of sugar? (Hint: First get the answer in pounds of sap and then using "a pint a pound" convert it to gallons of sap.)

47. A newspaper report on sugarin' says that it takes a barrel of sap to produce a gallon of maple syrup. Is this consistent with the above information?

48. Maple syrup is produced when most of the water is boiled out of the sap. Further cooking then yields the sugar. The yield of syrup is just slightly higher than the yield of sugar because there is very little water in the syrup form of the sugar. Supposing a tree could yield 4 pints (pounds) of syrup in a season, how many trees would you need to produce one gallon of syrup per season? At one Indiana maple grove, genuine maple syrup sells for $10.50 per gallon. At this rate, how many trees would you need for a reasonable income?

49. In a grove of sugar maples visited by the author the average distance from tree to tree is about 25 feet. For such a grove, find out about how many acres of ground it would take to accomodate the number of mature

trees you got as your answer to the previous problem.

50. The source cited talks only about maple sugar and is not very clear about what percent of the sap can be turned into syrup. See if you can get some more accurate information on this point. One indication might be that the same farm that sold pure maple syrup at $10.50 per gallon and $2.25 per pint, sold maple sugar at $1.25 per half-pound. From this, what would you suppose about the relative yields of syrup and sugar?

51. Here is some information from 1001 Questions Answered About Trees about the commercial uses of timber from our forests and from "tree farms": saw logs, primarily for various kinds of construction, use up 68%; pulpwood and pitprops (the latter used in mining) about 23%; other industrial uses about 8%. Of the pulpwood, 95% goes into paper and paperboard, the rest into plastics, cellophane, and rayon. Of the 95% that goes to paper and paperboard 51% is paperboard and 49% paper. Half the paperboard is used for corrugated cardboard; about 30% for boxes of cardboard other than of corrugated cardboard; 10% for building board of various sorts; and 10% for all other uses. The 8% that goes into "other industrial uses" mostly goes into poles, piling, and posts. For example, three million telephone poles are used each year, and pilings for docks, piers, highway and railway bridges takes an immense number of trees. Masts of boats take many trees but many fewer than they used to. Using this information along with whatever other information that you can dig up, ask at least 5 interesting questions. For example, what percent of the total use of wood goes to paper, assuming that most of that listed as "pulpwood and pitprops" is in fact pulpwood?

52. The same source estimates that fences in the United States use 600,000,000 posts a year along 15,000.000 miles of fencing. Consider the United States as roughly a rectangle about 2500 miles long and 1000 miles wide and see if 15,000,000 miles of fencing seems reasonable. (For example, if all over the whole country every square mile was fenced how many miles of fencing would this amount to?) If the estimates given are more or less correct, about how many posts per mile of fencing are used each year?

53. P. T. Barnum (1810-1891) is supposed to have said that "There is a sucker born every minute." How many per year is this? Assuming that he is talking about the United States and assuming that the rate has not changed, what percent of the 3,718,000 live births in the United States in 1970 (census bureau statistics) were born suckers? Do you think Barnum made

a reasonable estimate? What percent of the people that you know would you consider gullible enough to fall for a not too obvious fraud or deception? (Who was P. T. Barnum anyway? Do you suppose the remark was made in his professional capacity?)

54. The 1902 Sears Roebuck catalog has recently been reprinted. If you can find it (perhaps in your library), make some price comparisons with a recent catalog. What proportion of items in 1902 Sears Roebuck Catalogue would still be ordered if listed in the most recent catalog (for actual use, not as an antique or curiosity)? (You will probably want to formulate some way of sampling the catalog rather than doing every page, or else share the work with others.)

2.3 Numerical Information Combined by Calculation into New Information

We have seen that pairs or triples of numbers play a role in various coordinate systems, in a number of ratio situations and in tables and graphs. It is also very common for two, three, or more numbers to be combined together by ordinary arithmetic operations to solve everyday life problems. This section will contain a variety of these. You should be on the lookout for different categories of uses; e.g., use in occupations, in one's daily lives, in information acquisition for particular interests, in making social and political decisions, and so on.

Problem Set 2.3

1. The Boeing 747 is the largest commercial airliner flying as of 1971. According to the information given by the pilot of a recent trip between Chicago and Los Angeles, the plane carries a maximum of 450 passengers; it has a four-person flight crew and a fourteen-person passenger service crew (including two security people to prevent highjacking and other incidents); it costs about 1000 dollars for fuel for the trip from Chicago to Los Angeles, with 300 gallons of fuel used just for the take-off. The fare one way from Chicago to Los Angeles is about $110. Considering just the wages of the crew and other on-board employees plus the cost for fuel, make an estimate of the cost of flying the plane from Chicago to Los Angeles. How many passengers would it have to carry in order to "break even?" (I'm not sure whether the Chicago-Los Angeles trip counts as a full day or a half day of work for crew members; you will need to get information on this and on pay rates for such employees.)

2. Try to list other services directly chargeable to a flight such as the

747 flight from Chicago to Los Angeles; for example, ticket agents, baggage handlers, food service getting the meals ready and into the airplane, refueling and other servicing of the airplane, and so on. It is reasonable to suppose that each person so involved costs the airline from $5 to $10 per hour for his services, including fringe benefits and use of the various kinds of equipment. Add this to the cost of the flight and find a new "break even" passenger load level.

3. A Boeing 747 costs about twenty-three million dollars, which is often borrowed by the airline at about a 6% annual interest rate. How much interest is this per year? How much per day? If the 747 makes 3 such trips per day, allocate the daily interest costs and see how many passengers are necessary per Chicago-Los Angeles flight to pay this cost.

4. A 747 has a wing span of about 196 feet and a length of about 231 feet. Compare these figures to the length of the airplane first flown by the Wright brothers and how far it flew on the first manned, power flight.

5. The 747's cruising speed is about 595 miles per hour. The still experimental supersonic transports TU 144 (Soviet) and Concorde (French-English) are scheduled to cruise at about 1500 miles per hour and carry about 120 passengers. Strictly on the basis of flight time and passengers carried (not including ground time and assuming full loads in all cases) which would be the most efficient passenger carriers, a 747 or one of the supersonic jets? What might "most efficient" mean in this case?

6. "Since a high proportion of automobile drivers in fatal crashes (44% to 60%) have been found to have enough alcohol in their blood to significantly impair their driving ability...it seems that if a significant proportion of motorists could be prevented from driving while intoxicated the rate of fatal auto accidents might be reduced considerably. In Great Britain,...[with] an enforcement program utilizing the breath analyzer... fatalities in auto accidents during the following year decreased by 15%, serious injuries decreased by 11%, and total auto accidents decreased by 10%." (From "Technological 'Shortcuts' to Social Change.")

Consider this statement and the table given just below and answer the questions following the table.

MOTOR VEHICLE ACCIDENTS

NUMBER AND DEATHS BY TYPE OF ACCIDENT

ITEM	1950	1960	1970 (Preliminary)
Motor vehicle accidents	8,300,000	10,400,000	16,400,000
Accidents per 10,000 vehicles	1,888	1,397	1,471
Traffic deaths total	34,763	38,137	55,300
From noncollision accidents	10,600	11,900	15,700*
From collision accidents:			
With other motor vehicles	11,650	14,800	23,200
With pedestrians	9,000	77,850	9,900
With other vehicles or objects	3,490	3,610	6,500*
Traffic death rates:			
Per 100,000 population	23.0	21.2	27.1
Per 10,000 motor vehicles	7.1	5.1	5.0
Per 100 million vehicle miles	7.6	5.3	5.0

*Not comparable with earlier years.
(Adapted from The American Almanac (for 1972), page 540; original data from the National Safety Council.)

a. How might one account for the wide range of estimates (44% to 60%) about percent of intoxicated drivers in fatal crashes? Using the low and the high figures give the range of number of auto accident deaths for 1970 that might be blamed on intoxicated drivers. ("Fatal crashes probably means collisions with vehicles or fixed objects so you may need to exclude "collisions" with pedestrians. What proportion of the total auto deaths come from "crashes" rather than pedestrian or non-collision accidents?)

b. The first year experience of Great Britain as reported above didn't quite hold up; in the second year there was a 10% reduction in fatalities, 9% in injuries, 10% in total accidents. If it were possible by controlling drunk drivers to reduce the death, injury, and accident rate by 10% in automobile "crashes" how many lives might be saved annually in the United States?

c. A U.S. Department of Transportation study indicates that the average "economic loss per fatality" in automobile accidents (including loss of future earnings of those killed) is about $90,000 now and will probably rise to about $110,000. Taking $10,000 as the estimated average loss, make an estimate of the total dollar loss from automobile fatalities in 1970. Now make an estimate of the total dollar loss from those deaths in which intoxicated drivers are presumably involved.

d. In Chapter 1 we asked some questions about death rates in various "plagues"--influenza, "black death," and wars, for example. Look at the traffic death rates given in the table; convert these to whatever standard you were using in the earlier problems; and comment on the seriousness of the "plague" of automobile deaths.

e. Notice from the table that while the traffic death rate per 100,000 population has increased from 1950 to 1970, the rate per 10,000 motor vehicles and per 100,000,000 vehicle miles has actually decreased. What inferences could you draw from that?

f. Note that the rates per 10,000 motor vehicles and per 100,000,000 vehicle miles are almost the same all of the way across the table. Can you make any inferences from that fact?

g. Look in the table at the "accidents per 10,000 vehicles" in 1970 and estimate the chance that a given vehicle in a given year would have an accident. Ask one of your parents how much the "collision" provision in their automobile insurance costs them and how much it might cost to repair the automobile if it were involved in an accident. Keeping in mind that insurance rates are based on just such information about total population and not merely about your parents (who may be safer drivers than most) comment on whether the insurance premium your parents pay is "fair." (Remember that insurance rates are often relatively higher in some places with high accident risks--large cities, for example.)

h. In some of the problems above you may have found it convenient to use "powers of ten notation" in doing your calculation. For example, in problem c. above you could have expressed the average loss as $\$1 \times 10^5$ ($100,000), the number of deaths as 5.5×10^4 (55,000) and hence the annual loss as $\$5.5 \times 10^9$ ($5.5 billion or $5,500,000,000). This is especially useful when you need to make reasonable estimates using very large or very small figures. Using the table below about the U.S. budget and statistics such as the number of people in the United States or in your city, ask and answer some interesting questions about the relative allocation of funds by the Federal government, increases in the budget from 1950 to 1971, costs per citizen of various budget items, and so on. (The table is adapted from The American Almanac, 1972.)

FEDERAL BUDGET

RECEIPTS AND OUTLAYS FOR 1960, 1965, 1971

In millions of dollars for years ending June 30.

	1960	1965	1971 (est.)
Total Receipts	92,492	116,833	194,193
Source of Receipts			
Individual income taxes	40,741	48,792	86,300
Corporation income taxes	21,404	25,461	30,100
Social insurance trust funds	14,683		48,973
Excise taxes	11,676	14,570	16,800
Estate and gift taxes	1,606	2,716	3,730
Custom duties	1,105	1,442	2,490
Miscellaneous receipts	1,187	1,594	3,800
Total Outlays	92,223	118,430	212,755
Function of Outlays			
National defense	45,908	49,578	76,443
International affairs and finance	3,054	4,340	3,586
Space research and technology	401	5,001	3,308
Agriculture and agricultural resources	3,322	4,807	5,262
Natural resources	1,019	2,003	2,636
Commerce and transportation	4,774	7,364	11,442
Community development and housing	971	298	3,858
Education and manpower	1,286	2,533	8,300
Health	756	1,730	14,923
Income security	17,977	25,453	55,546
Veterans benefits and services	5,426	5,722	9,969
Interest	8,299	10,357	19,433
General government	1,327	2,210	4,381
Pay increase and contingencies	(x)	(x)	800

7. Sometimes calculations with exponents are useful directly. For example, a power raised to a power such as $(10^2)^3$ means $10^2 \times 10^2 \times 10^2$ or 10^6; in general, in such cases one multiplies exponents. Here are some examples based on information from Mathematics in Everyday Things about a transatlantic communications cable that stretches about 2,250 miles from Newfoundland to Scotland.

a. Distributed approximately evenly along the cable's length are 51 electronic amplifiers which keep the messages at a level high enough to be detected at the other end. About how far apart are the amplifiers?

b. Each amplifier increases the signals by a factor of about a million (10^6). Hence the total amplification to compensate for losses would be a million multiplied by itself 51 times, which would mean $(10^6)^{51}$. What would be the total amplification be, as ten to some power? (This is, of course, an enormous number. It has been said that the number of atoms in the entire observable universe probably does not exceed 10^{85}.)

c. Try to find other instances where very large numbers are involved in a mathematical model, a description of a situation, or as the result of calculations. Similarly, find some very small numbers that result from calculations or descriptions. (For starters you might try the Guiness Book of World Records or The Lore of Large Numbers, both of which are probably in your school library.)

8. The Odd Book of Data notes that small amounts of wear from friction can have large effects; for example, just a pound or two of material removed from a large truck by wear can consign the truck to the scrap heap.

a. Suppose a truck weighing five tons is made unusable by loss of one pound of metal by wear on its moving parts. What percent of the weight is this one pound loss?

b. Suppose the truck has eight cylinders in its engine, each with a diameter of four inches and a length of 5 inches. What is the total surface area of the cylinders? Suppose the surface of each cylinder has .020 inches removed from it by wear (about as much as the thickness of five pages of this book). About what volume of metal is lost? If the engine block is made of cast steel weighing about 490 pounds per cubic foot (.28 pounds per cubic inch), what is the weight of the metal lost? (.020 inches of cylinder wear would be more than enough to make an engine need overhaul, and in some cases would make an engine unusable.) One way to get a picture of what is going on here is to consider that a number $2\frac{1}{2}$ can of food (pumpkin, tomatoes) just about represents the wearing surface of a cylinder in a fairly large engine. That being so, you can estimate about how many can thicknesses are equivalent to the wear in this problem and get an estimate of metal loss in that way.

c. Bearing surfaces and gears are particularly important to the functioning of cars, trucks, and other machinery. Pick some piece of machinery or equipment (perhaps a motor in a household appliance) and try to identify the places where a small loss of material by friction would make a large difference in how well things function.

d. Suppose in draining the oil from the engine crankcase or from the transmission or differential of his car your father found that a small handful of metal chips or a small quantity of metal dust had accumulated. Even if it were just a few ounces, do you suppose he would be worried? Suppose the accumulation were 2 ounces; what part of the weight of your father's car would that be? What percent? Do you suppose he would take comfort from the facts you would present him showing that the lost metal was only a very small part of the car?

9. On page 79 of the American Almanac we learn that of U.S. residents 14 years of age or older about 45% of the men and 30% of the women presently smoke cigarettes; about 33% of the men and 60% of the women have never smoked; about 20% of the men and 8% of the women have quit smoking. (For about 2% their status is "unknown.") The total population covered is about 60 million men and about 68.5 million women. Find out from this information how many smokers, former smokers, and nonsmokers there are in these populations.

10. On page 308 of the American Almanac we learn that in 1968 about 9.8 billion dollars were spent on tobacco; about how much is this per smoker per year?

11. It is variously estimated that $\frac{1}{20}$ to $\frac{1}{10}$ of those who drink alcoholic beverages in the United States are "alcoholics" or "problem drinkers." Why is there such a range in the estimates? What would be the problems in making such estimates? See if you can find in some source enough information to make an estimate yourself.

12. A Chicago newspaper reports as follows: "Of the nearly three million industrially employed persons in the Chicago metropolitan area, we estimate that more than 171,000 are alcoholics, said the president of the Kemper insurance group." What percentage does that represent? The same article says that the national average is about 5.3% of the industrial force and urges that management help rather than fire such employees, if only because "it costs more than $1,000 to get an employee processed and on the payroll." Make up and answer some questions based on this information.

13. The speed of light in a vacuum is said to be, by the most accurate current available measurement, about 186,272 miles per second. How many miles per hour is this, approximately? Distances to stars are measured in light years, where one light year is the distance light can travel in one year. About how far would this be? (Here's a place where rounding off and using powers of ten notation would probably be very helpful in getting a good estimate.)

14. Traveling at the speed of light, about how many times could one go around the world (say at the equator) in one second?

15. The speed of sound through air is said to be about 1,100 <u>feet</u> per second. What part of a mile (approximately) is 1,100 feet and hence, what is the speed of sound in miles per second? In miles per hour? Traveling at the speed of sound, how long would it take to go around the world at the equator?

16. One of the fastest experimental aircrafts goes at a speed of about 2,500 miles per hour. At that speed about how long would it take to go around the world at the equator?

17. Manned space capsules can travel at about 18,000 miles per hour. How long would it take at that speed to go around the world, assuming you were traveling at the earth's surface and at the equator?

18. Since the radius of the earth is about 4,000 miles (diameter, then is about 8,000 miles), the circumference at the equator is about 25,000 miles. Suppose you were in a manned satellite in an orbit 100 miles above the earth's surface. How much would that increase the distance you would travel in going "around the world"? Using your new figure, how long would it take the satellite to go around the world?

19. It is astonishing to reflect that until the invention of the automobile late in the last century man's maximum traveling speed was probably under

30 miles per hour, and his usual speed, say by horse and buggy, was probably 5 to 10 miles per hour. What consequences might that have had in the early history of this country in the living patterns and tastes of people, the placement of towns and cities, and so on? Each subsequent invention (automobile, jet airplane, rocket) represents about an order of magnitude (tenfold) change in the speed that one could go. What were the consequences of each such advance in changing our world? Do order of magnitude changes such as these merely change how fast one can get from one place to another or are the changes more far-reaching?

20. Here is a problem formulated while helping with a tedious task last summer: General Foods sells 2 brands of pectin for use in making jams and jellies from fresh fruit (the pectin makes the jam or jelly "jell.") Suppose Certo sells for $.40 per bottle; Sure Jell for $.20 per box. Here are their respective recipes for Blueberry Jam:

Certo Recipe

Crush fruit. Add 2 tablespoons lemon juice to $4\frac{1}{2}$ cups crushed fruit. Add 7 cups (3 pounds) sugar and 1 bottle Certo. Cook, etc. Yield: 12 medium jelly glasses.

Sure Jell Recipe

Crush fully ripe fruit. Add 2 tablespoons lemon juice to four cups (2 pounds) of crushed fruit, add 1 box Sure Jell, and bring to hard boil. Add four cups ($1\frac{3}{4}$ pounds) sugar. Cook, etc. Yield: 8 medium jelly glasses.)

Considering the higher cost and higher yield of the Certo recipe, decide which product a housewife should use if the Blueberries cost her $.20 per pound and sugar is $1.20 for 10 pounds. Should a difference in the cost of fruit change her decision? If so, what is the break even point i.e., the price of fruit at which it would make no difference which recipe is used (in cost per unit of jam). What about the price of sugar? If fruit were scarce, which recipe would you prefer? (We assume here that the taste and quality of the end products are equivalent but this may not be so.)

21. A travel brochure tells me that Caracas, Venezuela, is about 3,000 feet above sea level on the slope of a mountain that rises abruptly from the waterline to 9,[illegible] feet. "As the crow flies," the distance from Caracas to the waterline is 1[illegible] miles. If there were a road of about constant slope from the waterline up to the city in a straight line, what would

the slope of the road be? Express the slope in number of feet rise per 100 feet, which is easily converted to percent slope. As a matter of fact, the actual distance by road is about 50 miles; if the road has a constant slope (which it probably does not), how steep a road is this? (You may be interested to know that the maximum grade for a railroad is about a 2% slope; automobile roads are frequently steeper than this, but seldom as much as 10%. For a grade as steep as 10% an automobile would have to shift out of high gear into a lower gear.)

Death from Heroin

Early this spring, Joseph W. Spelman, chief medical examiner of the city of Philadelphia, addressed medical colleagues on the topic of sudden death from heroin. To a shocked audience he showed photograph after photograph of victims with needles remaining in their veins who had died after self-administration of drugs.

In New York City, among the estimated 100,000 heroin addicts, more than 900 fatalities due to drugs occurred in 1969. In that city, for the age group 15 to 35, drug abuse is now the leading cause of death. According to Michael M. Baden, deputy chief medical examiner, the majority of fatalities are due to an acute reaction to the intravenous injection of a mixture containing heroin. The mechanisms involved in the deaths are not clearly established: overdosage has been suggested by some investigators; others speak of an allergic reaction. A survey of practices attending the production, distribution, and usage of heroin leaves one amazed that the death rate is not higher. The method of illicitly extracting morphine from opium is crude. The impure morphine is subsequently acetylated to heroin in secret laboratories, mainly in France. Purity of the product is of the order of 90 percent. Subsequently, the heroin passes through a complex distribution system and is adulterated repeatedly in unsterile conditions with a variety of additives, including quinine, mannitol, and other white powders.

The Office of the Chief Medical Examiner of New York City analyzed 132 street samples of drugs, all of which supposedly contained heroin. They found that 12 samples contained no heroin at all, and among the remainder the concentration of the drug ranged from less than 1 to 77 percent. Variability in the amount of the drug could be responsible for many fatalities. A user accustomed to a low concentration is likely to die from an injection of almost pure heroin.

Hard core addicts subject themselves to more than 1000 intravenous injections each year, and they are thus exposed repeatedly to possible antigens in the crude heroin or in its adulterants. In addition, the repeated use of unsterile drugs, unsterile equipment, and unsterile technique leads inevitably to human wreckage. In a description of the major medical complications of heroin addiction,* Donald B. Louria and his colleagues have identified the most common medical problem as liver damage arising from hepatitis. Other organs that are particularly subject to attack include the heart and lungs. Infection of the heart, though not so frequent as hepatitis, is more often fatal.

Drug abuse, which was once predominantly a disease of Harlem, is now a plague that is spreading to the suburbs. Drug use has been glamorized, while descriptions of the dreadful consequences have been muted. Parents and educators must inform the young of the corpses and of the physical wreckage. Despite warnings, adventurous youth will sample the illicit--and many will be hooked. The number of addicts is already estimated at 200,000, and the annual cost of their drugs at $5 billion. With so much at stake in lives and in money, the nation should increase its efforts to curtail drug abuse and to find better ways to rehabilitate addicts. Two relatively new methods seem promising. One is the use of methadone.** A second approach is a psychiatric one, which emphasizes attitudinal changes and utilizes ex-addicts to give emotional support to addicts who wish to stop. Determined and imaginative effort might well disclose even better methods. This nation should provide the necessary funds to move vigorously against a spreading plague.

22. For this and the next problem refer to the accompanying editorial reprinted from Science. ("Death from Heroin," Abelson, P. H., Science, Volume 168 12 June 1970) 1298. Copyright 1970 by the American Association for the Advancement of Science; reprinted by permission.)

    If the figures given in the editorial are correct, what is the annual average cost per addict for drugs? What is the death rate among addicts in New York City? Ask some other questions based on the information given in the editorial.

23. According to the American Almanac the number of active drug addicts at the end of 1969 that had been reported to the U.S. Bureau of Narcotics and Dangerous Drugs by police or hospitals was 64,915, including 15,000 newly reported within the previous year. How do you account for the difference in these statistics and those given in the Science editorial?

*D. B. Louria, T. Hensle, J. Rose, Ann. Intern. Med. 67, 1 (1967).
**J. Walsh, "Methadone and heroin addiction: rehabilitation without a 'cure'," Science 168, 684 (1970).

Chapter III

THE ROLE OF "MEASURE" IN APPLICATIONS

3.1 Measures Everywhere

"Measure" (verb) is a very general concept and "measures" (noun) play a role in many different situations. To measure something is to assign a number (its measure) to it. An example is when we assign a weight to each head of lettuce in a store. We are measuring the lettuce; its measure is its weight. Usually we make certain standard requirements of measures; for example, they are usually non-negative numbers (weight is usually not negative) and we usually want the measure of two objects together to be the sum of the measures of each one alone (two heads of lettuce weighing 2.1 and 1.7 pounds should together weigh 3.8 pounds). Frequently we get measures directly with some sort of instrument, such as a weighing scale for the lettuce or a tape measure to assign a length and width to a room. Some measures are derived from other measures, as when we find the area of a room by multiplying together its length and width measures. Often we use functions and formulas to produce measures indirectly from other measures that can be easily or directly obtained; for example, the formula $S = 16t^2$ tells us <u>how far</u> something has fallen by measuring a <u>time</u>.

There are more uses of measure than we usually think of. For example, while sitting in a recent conference I began a list of measurements that may have been considered in making the hotel conference room available. A wise planner would have begun by getting a measure of how necessary and how profitable that type of facility would be before beginning to build it. The measurements that then went into architectural planning, construction bids, and the actual construction were very numerous. The room was air conditioned; many measures would be required to make this extra comfort available. Furnishing the room involves measures of various profitable uses of the room, economical maintenance, and, of course, measures involved in manufacturing and installing the furnishings themselves. Once the room is completed and furnished, there are several ways to express its measures: floor area, volume, seating capacity for people listening to a speaker, seating capacity for banquet use, seating capacity for a group using a conference table, rates to be charged for various uses during various seasons of the year, and so on. Close examination of almost any situation will yield a similarly varied list of measure considerations.

Problem Set 3.1

1. Consider yourself and your present school situation and list as many applicable measures as you can. Start with direct measures that apply to you (e.g., height, weight, clothing sizes, etc.). Next, what measures are in your counseling file at school? Now move out to measures applicable to the people in your class and to the whole student body. Next, think about the initial planning of the school buildings you meet in, their construction, and their maintenance. Next consider the measures that apply to the financing of your education, and to possible ways of measuring the "yield" on this investment in your education.

2. Once you have such a list, look it over and note the items in the list for which it is useful to think in terms of "maximizing" or "minimizing" something.

3. Look at a box of dry breakfast cereal and do some "cereal box arithmetic." For example, these boxes usually have statements of the nutritional content of the cereal--a certain amount of minerals and vitamins or a certain percentage of the "minimum daily requirement" (MDR) in each serving. Try to find out how MDR's are defined. Determine how "a serving" is defined; measure it out and see if this is a normal serving in your household. See if the nutritional content is given for the cereal itself or the cereal with milk, sugar, and fruit--obviously it would make a difference. (I have the impression that such things used to be stated for cereal with milk, but that regulations now require nutritional content of the cereal alone.) Current government regulations now require that when the contents of foods are listed, the ingredient that forms the highest proportion must be listed first; if the ingredients in this cereal are listed, check out what it is composed of most, second most, and so on. If you have several kinds of cereals, do some price-per-unit comparisons among them, such as cost per ounce of cereal, or cost per mg (milligram) of vitamin B_6. If there are premium offers on the box (for example, toys for boxtops or for boxtops and money), see if accurate information about the size of the things offered is given. I recall one offer which showed plastic basketballs and footballs which looked large in the picture, but the small print revealed they were only a couple of inches in diameter, hence probably not of much use as toys.

4. It is frequently the case that measures are defined in such a way that they can be added; that is, when two things are combined the total measure is the sum of the measures of the things that are combined. A way of expressing that is to say that $m_1 + m_2 = m_{total}$. List some everyday measure situations where things are combined or put together and see in which ones this "additive" property is true and in which ones it is not.

5. Here is a situation where $m_1 + m_2 = m_{total}$ does not seem to hold: "To convert Star brand instant powdered milk into one quart of milk, add $1\frac{1}{3}$ cups of dry milk to one quart of water." List some other everyday situations where measures "don't add up."

6. This same package of powdered milk gives us alternate instructions: "By weight, use 3.2 ounces of powder to one quart of water." Weights such as this really should "add up," hence if a quart of water weighs 32 ounces (a pint weighs about a pound) what should this "quart" of milk weigh? Does that mean that we will really have more than a quart or does a quart of milk weigh that much more than a quart of water?

7. In order to assign numbers to objects or to situations, one must settle on some sort of unit in which to express the measure. For example, what units might be used in measuring the thickness of a page in your book or the size of a small object on your desk? In what units would you measure your desk? The length or width of your classroom? The distance you walked to school this morning? The distance traveled in an automobile trip? Distance to the moon? Distance to a star? Diameter of the universe? Make up a similar problem that starts with measure in very small units and ends up with quite large units.

8. Once a unit is decided on, anything at a given moment should have an exact measure; for example, a pencil can't be both six inches long and seven inches long at the same time. In practice, however, it is not possible to say exactly what the measure is; for example, two people might give 6.14 inches and 6.15 inches as measures of the same pencil. Think about and discuss with others in the class or with friends or family what some of the practical barriers are to the "unique" assignment of numbers to things. Which of these are due to human errors? For which does accuracy depend on the instrument used for measurement? Are there any theoretical reasons why it would be impossible to assign a "unique number" to each object? (If you are familiar with the distinction between "real" and "rational" numbers, what would be the problem in assigning a number if the "actual" measure were an irrational number? Can you

think of situations where the measure would be an irrational number? Ignore this parenthetical question if you haven't yet covered such material in your mathematical experience.)

9. Can you list any situations where the very act of measuring might change the measure? (If you are interested in science or philosophy, you might find Heisenberg's "uncertainty principle" interesting in this connection.)

3.2 Measures and Conversions for Length

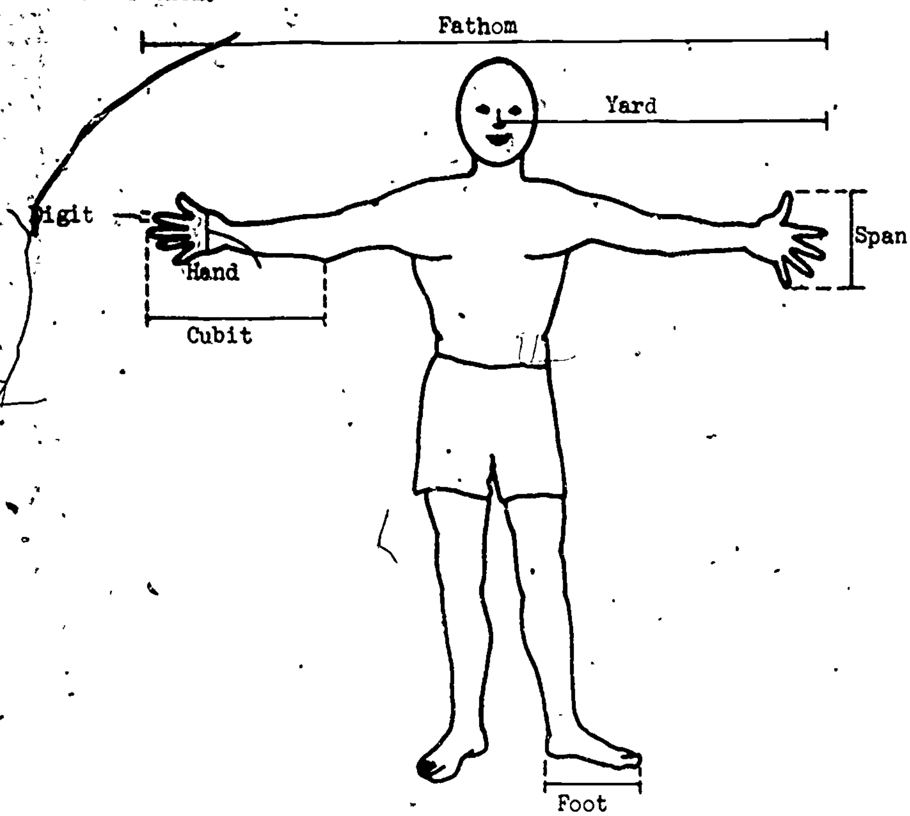
Measure of length probably appeared fairly early in human history and was probably based on things familiar to the people involved. Just as many early numeral systems are based on 10, probably because man has ten fingers, many early measure systems were based on body parts. This is the origin of such words as "foot," "digit," "span," and "hand," each of which names measuring units. Eventually, of course, these measures had to be standardized in some way and conversions within a given country's measure system established; hence, for example, in the English system a foot was defined as 12 inches and a yard defined as 3 feet. In days when communications between groups of people was not nearly so good as it is now, every group of people isolated by distance or mountains or other barriers from other groups might very well have a different measure system. Also, in modern times, new and more convenient systems have been invented and are used side by side with the traditional systems. Hence, in addition to conversions within a given system, it is necessary to establish standard conversions between different systems. It is convenient to think of these conversions as "conversion functions"; with one measure as "input" and another as "output." For example, $f = 3Y$ defines the conversion from yards to feet.

Problem Set 3.2

(Much of the information in this set comes from Realm of Measure.)

1. The origin of some of our common measures gives us a good way to approximate various things. For example, the picture on the following page shows some common measures based on dimensions of the human body. The yard as the distance from the tip of the nose to the tips of the fingers was often used in stores for measuring cloth. With a string or towel or length of cloth or something, see how long this is with a number of people, both classmates your own age, children younger than you, and adults. Draw some conclusions about how often such a way of measuring cloth could come pretty close to giving a standard yard among people who

are fully grown. While you are at it, draw some conclusions about growth patterns; for example, is the "yard" of an 8 year old about half as long as that of a 16 year old? How about a "yard" for a 1 or 2 year old child? Ask some other interesting questions along this line and try to answer them.



2. The span from the end of the thumb to the end of the little finger in an outstretched hand is a very useful way to measure things. Standardize your own span by finding out about how long it is, then use it in measuring some things. Do some comparisons of your span with that of other people. (According to the Random House Dictionary, a span is usually taken to be 9 inches.)

3. The cubit (from elbow to finger tips) is about $\frac{1}{2}$ yard--the dictionary says it has been variously standardized from 17 to 21 inches. Find some references to "cubit" in the Bible or other works of literature and convert the cubit reference into terms more familiar to you. Find out what your own "cubit" is.

4. A fathom is from finger tip to finger tip of the outstretched arms and is said to be derived from an Anglo-Saxon word meaning "embrace." Are

there other uses of the word "fathom" that are independent of measure, but do suggest "embrace"? The depth of water is often measured in fathoms, perhaps because a "leadsman" on a boat or ship in the days before electronic depth finders would drop a weight on the end of a rope until it hit bottom and then count the number of finger tip to finger tip measures as he gathered in the line. A whole vocabulary was built up around taking depths for navigational purposes in shallow waters ("by the deep, six"; "mark three"; "mark twain"). Try to find a reference that tells you something about this vocabulary and what it meant to the sailors in the days before modern instruments.

5. A fathom has been standardized at 6 feet (2 yards); why would you expect this, given the diagram of body measurements above? Look on a map that shows water depth (say for a coastal region). Find out if the depths are given in fathoms or some other measure, and if they are given in fathoms, find the maximum depth in feet, first having formulated a "conversion function" to make the conversion from fathoms to feet.

6. Nowadays the depth of water is measured by a "fathometer" which sends out a pulse of energy, then records how long it takes an echo from this pulse to return from the sea floor and be received by this instrument. What are the essential parts that such an instrument must have? What things must be known to convert this echo-time measurement into a depth (length) measurement? Might the instrument require a different "calibration" in salt water than in fresh water. Might the temperature of the water make a difference.

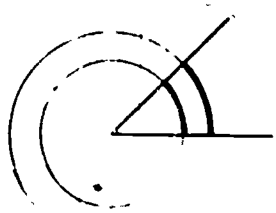
7. Units of length too long to be measured conveniently with the body tend to vary widely from country to country, at least as far as traditional units go. For example, where we use a mile (5280 feet), the Russians use a "verst" (which is about 3,500 feet). Verify that a verst is about $\frac{2}{3}$ of a mile and that a mile is about $1\frac{1}{2}$ versts. Try to find some other measures of longer distances and their origin. Find out what other Russian units of length go with the verst. What is the origin and use of a "league" (as in Twenty Thousand Leagues Under the Sea)?

8. Some special uses dictate special measures; for example, horses are said to be so many "hands" high (the hand is standardized now as 4 inches). Estimate about how many hands high you think an average horse is at its shoulder, then check your estimate against some information about horses.

9. We said earlier that in many city surveying systems, a block is $\frac{1}{8}$ of a

mile. (Horse racing uses the "furlong," which is also $\frac{1}{8}$ of a mile.) Surveyors use a somewhat decimalized system by defining a "chain" as $\frac{1}{10}$ of a block. Then what part of a mile is a chain? About how many feet in a chain? How many yards? The system is further decimalized by defining a "link" so that there are 100 links in a chain. So about how long is a link?

10. An arc in a circle is sometimes spoken of as having a degree measure. As shown in the diagram, a number of different actual arc lengths could all have the same degree measure. A more accurate subdivision is made with "minutes of arc," with 60 minutes in each degree. How many minutes of arc are there in a "quadrant" of a circle; that is, in an arc defined by a 90 degree angle?



11. Earlier in this book there are a number of problems having to do with the latitude of a location on the earth's surface. This was defined as the degree measure of the angle from the center of the earth between the location and a point on the equator directly north or south of the location. One attempt to standardize a length, by using a fact of the world itself, is to define a "nautical mile" as the arc length equivalent to one minute of arc; which (as you just computed) is $\frac{1}{5400}$ of a quadrant of the earth's surface. The length of such a quadrant arc varies in length, though not by very much, depending on where on earth it is located. The standard reference usually used is the one going through Paris, France. Along that line, according to the Realm of Measure, a minute of arc is equivalent to about 6076.097 feet and this length has been defined as the International Nautical Mile.

a. The measure given above is given to thousandths of a foot; why do we still say "about" in the sentence giving the measurement?

b. According to the unabridged Random House Dictionary, the International Nautical Mile is defined to be 6076.1033 feet. I'm not sure of the reason for the discrepancy with Realm of Measure; consult some other standard reference and see if you can resolve it. What is the difference (in inches) between the two values?

c. Assume for the moment that the dictionary is "correct"; what is the percent error in that case in the figure given by the Realm of Measure? (That is, compare the discrepancy of .0.03 feet to the "true value" of 6076.1033 and express the resulting ratio as a percent.)

d. The British chose to round off 6076.1033 and define the "Admiralty Mile" as 6080 feet. What is the percent "error" for that figure compared to the "true value" for a nautical mile given by the dictionary?

12. Both ship and air navigation are done using nautical miles. Why do you suppose the connection between minutes of arc and length in nautical miles would be useful in navigation?

13. Take the latitude of a central location in the city you live in; then for some other location that you might like to visit and find out what its latitude is. Now, using differences between coordinates, find out how far in nautical miles you are north or south of the other place in question. The similar problem for longitudes and distance is more complicated; can you see why? Is there an easy way of getting an arc length measure of the direct connection along a great circle between you and the other place in question, and hence the shortest distance in nautical miles. Some globes have instruments for this included with them; if the globe you are using does not, you could make a sort of protractor that would just fit the globe and give you the measurement you want.

14. If the places you've chosen are not too far away from each other on the surface of the globe, you could get a pretty close estimate of the direct line distance from the north-south distance and east-west distance and by using the Pythagorean theorem. Try this with problem 13 just above and see how close it gets to the actual distance. Name two cities of the world where it would be obviously inappropriate to use the Pythagorean theorem and the north-south and east-west distances to find the distance between the cities. You know, of course, that the Pythagorean theorem is a theorem of plane geometry, and hence doesn't really apply on the surface of a sphere. It does give "close enough" results in most situations.

15. What other theorems of plane geometry would not apply on a surface of a sphere? For example, are two sphere surface "lines" that are perpendicular to the same line "parallel"? Try it for two lines perpendicular to the equator.

16. A country's number system or traditional measure system is very persis-

tent over time and is only changed with great difficulty. For example, the United States is nearly alone among major countries in the world in not using the metric system in business, although it is universally used in science. The metric system itself was imposed only in a time of great upheaval and with considerable difficulty during and after the French Revolution. The committee appointed by the French Revolution intended to make the meter a ten-millionth of a quadrant, measured from the equator to the North Pole through Paris, hence this distance would be 10,000,000 meters. Unfortunately, with more sophisticated measuring techniques, this quadrant was later found to be 10,002,288.3 meters. By the time this was found out, too much had been done with the meter to change its length and so the earth was no longer used as the standard of measure.

a. Rounding off the above figures to a discrepancy of about 2000 parts in 10 million parts, what percent discrepancy is there between the intended and true measure?

b. Why in the next to last sentence of the paragraph above should we have said "about 10,002,288.3 meters"?

c. In 18[illegible]5 the International Bureau of Weights and Measures prepared a standard meter bar out of platinum-iridium alloy, which is kept at Severs, France, under the most exact and careful conditions possible. Why would such a standard be important? What would be the difficulties in maintaining such a standard?

d. The British have a standard yard manufactured of bronze in 1844. This was intended to be exactly $\frac{3,600}{3,937}$ of a standard meter. What is the connection between that ratio and the "conversion factor" which tells us that a meter is 39.37 inches? From a shrinkage of the bronze or some other reason, a recent measure makes the British Imperial Yard equal to $\frac{3,600,000}{3,937,004}$ meters. The Realm of Measure says that this amounts to a difference of about $\frac{1}{30}$ of an inch in a mile; check this.

e. The International Inch is said to be equal to exactly 2.5400 centimeters. What is the purpose of the extra zeros? According to the Realm of Measure this makes the standard yard equal to $\frac{3,600,000}{3,937,008}$ meters. Can you figure out from the information given here how the denominator was obtained?

f. There have been some moves recently to return to a standard of length that depends on some invariant world fact. Several suggestions have been based on the fact that each color of light has a certain wave length and chemical elements give off characteristic colors when heated. Can you see any advantages in defining length by such a standard?

g. The wave lengths associated with colors are very short; the wave length of cadmium red light has been measured as equal to 0.0000006438086 meters. About how many such wave lengths are in a standard meter. If you are interested, check with your science teacher or in some reference book and see what the most recent thinking on this question has been. You might also want to look into what sorts of measurement instruments are used to get such exact measures of length as that just given for the length of a single wave length of cadmium red light.

17. If the entire earth including the International Standard Meter Bar, were to shrink or expand in the same way, would there by any way for us to detect that this was happening?

18. We have already mentioned that a fath meter measures distance by timing echos. Radar works in much the same way, but not with actual sound waves and echos. Find out something about how radar works.

19. In the old days of ship navigation, navigators relied on very accurate clocks and very accurate readings, from such instruments as a sextant to find their location with respect to the sun or certain stars at a given time. They then used books of tables to help them turn these measurements into exact locations. You might be interested in getting such a book in the library (it is called an ephemeris) and seeing how this sort of navigation is done. As late as the 1940's similar techniques were applied by airplane navigators. Why would these methods probably be unsuitable now for commercial air transportation, supersonic flights, military aircraft, and space travel. You might try to find out something about the network of electronic navigational aids that characterize air and sea transportation, radars, and the system of tracking stations used in our space flight efforts.

7.5 Measurement of Weight and Volume

The human units of measures of length were probably invented first based on familiar things such as length of foot, length of arm, and so on. It is likely that the next measures developed by mankind were measures of weight, and perhaps soon after that, related measures of volume. In both cases, many different systems based on locally familiar things developed; for example, India's traditional system of weight and volume measures is based on very different standards from that of the United States, and there are differences even between Great Britain and the United States. The only truly international standard of weight and volume is the metric system. This system dominates science and much of the commerce throughout the world.

There is often confusion between "mass" and "weight." The distinction may become clearer if you consider an astronaut who weighs in with his equipment on earth at a certain amount; becomes "weightless" at some point in his trip to the moon, so that he and anything not tied down can easily float in the air of the space capsule; then upon landing on the moon he has weight again, but much less weight than he had on earth. Clearly, even though "weights" change, something about the astronaut and his equipment remains constant through all this, and that something we call the "mass." In everyday affairs we deal only with the "weight" which results when a given mass is acted upon by the gravitational attraction between it and the earth we live on. The gravitational attraction of the moon to a given mass is much less because the moon is much smaller.

Throughout this section, we will usually be speaking of the weight that you would read on a scale on the earth. That is, even though, strictly speaking, "grams" and "kilograms" are metric system measures of masses--and hence are constant no matter where they are measured--in practice we treat them as measures of weight and find what they are by using scales on earth. In a later section when we are dealing with some scientific formulas, we'll have to take up the distinction between mass and weight again, but for the moment we'll stick to everyday usage.

Problem Set 3.3

1. The use to which measures are put often determine their character. For example, in the Europe of the Middle Ages there were many traditional systems, none compatible with the others, but with the rise of trading among countries it became necessary to standardize and agree on certain weights. The Realm of Measure discusses this in some detail and the next few problems are based on information from that book. The first "pound" to be widely accepted was the Troy pound; perhaps its name comes from the name of the city Troyes, France. The Troy pound is divided into 12 Troy ounces; each Troy ounce equivalent to 20 pennyweights, and

each pennyweight to 24 grains. These measures were used for precious objects where a small error might have an important effect. A British penny in the Middle Ages contained 24 grains of silver and the English pound had 240 pennies. So at one time the English unit of money, the pound, was literally a Troy pound of silver. Look up the present day price of silver and determine how much a Troy pound of silver would be worth now. How does this compare with the present value of the English pound? Is the [illegible] still used in dealing with precious metals?

2. Another link between the system of weights during the middle ages and the modern day system is the designation of gold as being "14 carat," "18 carat," etc. According to Realm of Measure, there is an old coin called a "mark" that had a mass of 24 carats, with a carat equal to half a pennyweight, or 12 grains. But without strict standards it would be easy to cheat by, for example, making a 24 carat coin half gold and half copper: that is, containing only 12 carats of gold. Hence, it became the custom to call such a coin as that just described a 12 carat coin and strict penalties were exacted for violating standards. Nowadays, any object regardless of its total weight, is marked 14 carat if it is $\frac{14}{24}$ gold. Find out what the present price of gold is on the world market and then compute what an ounce of 14 carat gold and of 18 carat gold would be worth. (Does "ounce" here mean the Troy ounce or our ordinary ounce.)

3. What we would call a pharmacist was once called an "apothecary." This profession also deals with very small quantities of materials, so it has traditionally used a system of weights based on the Troy pound, but somewhat differently divided. By talking to a pharmacist or using an encyclopedia or dictionary, explore this system of measure and find out what is meant by [illegible] ounce; dram; scruple; grain. Do pharmacists still use this system or have they switched over to the metric system?

4. The system of weights in common use in the United States and some other English speaking countries is based on the avoirdupois pound (from the medieval French [illegible] "goods of weight"). Each such pound is divided into 16 ounces; each ounce into 16 drams; each dram into $27\frac{11}{32}$ grains. How many grains are there in each ounce? How many drams in a pound?

5. [illegible] the relationships between avoirdupois, apothecary, and Troy measures. [illegible] the "conversion

factors" that allow you to change from Troy to avoirdupois pounds and vice-versa.

6. As suggested by the derivation from "goods of weight," the avoirdupois units were originally used when large quantities needed measuring. In American usage the hundredweight is (as you would expect) one hundred pounds and 20 hundredweights (2000 pounds) is a ton. In Great Britain another traditional unit is mixed in--the stone, which has an obvious derivation and which is standardized in modern times as 14 pounds. If you weigh yourself on a British weighing machine, your weight may be given in stones. What is your present weight in stones? Curiously enough, the British define a "quarter" as two stones and a "hundredweight" as four quarters, so how many pounds are there actually in a "hundredweight" in the British system? A British ton is still 20 British "hundredweights"; so how many pounds in a British ton? American tons are often called short tons; British tons, long tons. Try to find some information in a reference book or newspaper story or someplace where quantity is given in long or short tons. "Net ton" is another way of referring to the short ton; what do you suppose the corresponding term for the long ton is? Try to find some information which specifies "net ton."

7. In a rather curious way some units of weight have come to mean something quite different from weight in our nuclear age. For example, what is meant by a "20 kiloton" nuclear explosion? What is meant by a "megaton" bomb? How did such units of weight become attached as measures of explosive power. Is there some other measure of explosive power and if so, what is its relationship to the "kiloton"? Though depressing, such measures seem to be a standard feature of our present, uncertain world. Here are some problems having to do with such a world:

   a. Is it possible to convert either kilotons or some other measure of explosive force into destructive power? Try to find some printed discussion which does so, and make up a few problems based on this information.

   b. Many questions of public policy over the past couple of decades (how long is a decade?) have been concerned with the nuclear bomb balance of power between the two major nuclear powers--the USSR and the USA. See if you can find reliable estimates of the current nuclear stock piles in these two countries in such terms as "megatons."

c. It has been claimed that both the USA and the USSR have stockpiled enough nuclear weapons to destroy each other. Find some discussion of this and analyse it, paying particular attention to the "measures" that are used in whatever arguments are put forth.

d. Along with the destructive potential of nuclear power, there have been, of course, many peaceful uses, especially in the production of electrical power. Find some discussion of this and compare the measurement words used when discussion use of atomic energy for generating electrical power or providing power plants for ships) with the units used when discussion nuclear bombs.

8. To return to happier and smaller measures, diamonds and other precious stones are measured by unit of weight called the "carat" but this is a different carat from that used in the measure of precious metals. "Questions for Gems" (National Geographic, December 1971) is an interesting discussion of precious gems, with pictures of some fine stones in the Smithsonian Institute collection in Washington. The following problems are based on the information contained in that article:

   a. "Carat" has meant a number of different things throughout history, but currently is standardized in such a way that there are 142 carats in one avoirdupois ounce. What is the "conversion factor" to convert carats used for the measure of diamonds into the carats used in the measure of gold and other precious metals?

   b. Diamonds used in engagement rings range from $\frac{1}{[illegible]}$ carat to 1 carat for most rings, and larger for very expensive rings. The quality of the diamond itself and type of "[illegible]" used in cutting it has a lot to do with its price. By spending a few minutes in your local jewelry store or [illegible] a mail order catalogue, get some idea of the retail price of diamonds as they appear in engagement rings. Try to make some estimate of how much of the cost of a ring is in the setting and how much is in the diamond itself and then make an estimate of the range of retail price per carat for diamonds in engagement rings. [illegible] larger diamonds cost more or less per carat than small diamonds?

   c. The diamond mines near Pretoria, South Africa, have been the source of some of the most famous large diamonds in history. In 1905 a rough diamond of [illegible],1[illegible] carats was mined there. What is the weight of this in pounds and ounces. This stone was purchased by Transvaal for $[illegible], [illegible] and given to King Edward VII of England. What was the price per carat of the rough stone?

d. That $750,000 would be worth much more today in terms of purchasing power. Try to get some information on what it would be worth now.

e. The rough stone mentioned above was cut into two very large stones, the 530.2 c Star of India which heads the British Royal Septre and a 317.4 carat stone which appears in the British Imperial State Crown. These stones are "priceless" in the sense that they would not be sold no matter how much was offered, but, using the figures you derived for retail price per carat of diamonds and engagement rings, what would each of these stones be worth if converted efficiently into engagement rings? How much was left of the original rough stone to be cut into other stones and as waste that couldn't be made into usable stones?

f. The Star of India just mentioned is, roughly speaking, in the shape of an egg about $1\frac{1}{2}$ inch wide and 2 inches long. Estimate its volume, and then estimate the weight per unit of volume. Is this more or less than for water? Can you make an estimate of the "specific gravity" of diamonds? If so, try to check out your estimate with some figures from some other source.

g. Diamonds of South Africa are mined from tubular (roughly cylindrical) veins of a material called Kimberlite which range from a few feet in diameter to hundreds of feet in diameter and in many different lengths and depths. In modern mining for diamonds the Kimberlite is scooped up and run through a rock crusher which crushes it to pieces roughly $1\frac{1}{2}$ inches in diameter. Then these pieces are examined for their diamond content. The ratio of actual diamond to waste material is about 1 to 14,000,000. How many pounds of Kimberlite are scopped up in order to mine a 1 carat diamond? (Remember that there are 142 carats per avoirdupois ounce.) How many tons? (From now on unless we say otherwise, tons means American, net, short tons and all of our measures of weight refer to the avoirdupois pound.)

h. In spite of the efficiency of the rock crushers, in 19-4 a 426.5 c rough diamond, larger than $1\frac{1}{2}$ inches in one dimension slipped through the crushers. From this the 128.25 c Niarchos diamond was cut; this diamond is worth about $2,000,000. Estimate the price per carat. How does this compare with your earlier estimates for diamonds in engagement rings? If the price per carat here is quite different from that for diamonds in retail trade in engagement rings, how would you account for the difference? If it were possible to assume that the Star of India could have a value set on it that is proportionate to its weight compared to the Niarchos diamond, what would be the value of the Star of India. What are some of the hazards of making such comparisons.

i. The marketing of diamonds is rigidly controlled since their value depends in part upon their scarcity, but smuggling of diamonds does take place. It is estimated that . . . c of Sierra Leone diamonds are smuggled out by way of Liberia each year. Make some estimate of the value of . . . of diamonds in the smuggling trade. What are some of the hazards of this kind of estimation and what additional information would you want in order to make a more accurate estimate?

j. There are a number of synthetic imitations of diamonds on the market, ranging from synthetic white saphir worth about $5 per carat to a recent counterfeit called YAG (yttrium alumi garnet) at $50 per carat. How does the price of YAG per carat compare with your earlier estimate of price per carat of diamond for engagement rings? YAG is said to be so close in appearance to genuine diamonds that it is very difficult to tell the difference; it can be distinguished from diamonds by immersion in mineral oil: the diamond retains its luster, but YAG does not. You should be warned that it is hazardous to buy "bargain" gems of any kind outside legitimate and ethical trading channels. Besides the possibility of mistakes, what are some of the other hazards in this.

k. Manufacture and sale of "counterfeit" gems is of course a legitimate business as long as they are clearly identified. Assuming the price given for counterfeits include the cutting and faceting, how much would it cost you to get a counterfeit duplicate of the Star of India. How would the price of a counterfeit Niarchos diamond compare with the real article

l. Real diamonds have been produced artificially by General Electric. The small diamond chips used for cutting tools and other commercial uses thus produced are economically competitive with industrial diamonds from natural sources. But diamonds as large as $\frac{1}{4}$ carat of gem stone quality are too costly to manufacture, and very large stones are still impossible to produce artificially (as of late 1971). It is said to be the case that over a million pounds pressure and 2500° Fahrenheit are needed in the artificial manufacture of diamonds. This is said to correspond to the pressures in the earth 150 miles deep. Verify this if you can. Using this information and whatever else you can, find out about manufactured diamonds and make up some problems that could be added to this book.

m. Diamonds are useful in industry because they are just about the hardest think known, which means that they can be used to scratch or cut nearly anything else. The statement that they are the hardest thing known (or nearly so) suggests that some measure of "hardness" exists. Find out what system of measure applies to that. Compare the hardness of diamonds to such things as glass, steel, gold, and so on.

9. The Realm of Measure speculates that units of length probably appeared first in history, then units of weight (or mass), then units of volume. As with the other measures, there are many traditional systems for measuring volume, each based on some container that was in common use in a given culture. Why would it eventually become necessary to standardize units of volume?

10. The standardization of units of volume seems initially to have been based on units of weight; hence, the British Imperial Gallon holds 10 pounds of water. An American gallon is about $\frac{5}{6}$ of the Imperial Gallon and holds 8.337 pounds of water. How many pints are in a gallon? How much does an American pint of water weigh? What is the percent error (for water) in the old saying "a pint's a pound"? A liquid pint contains 16 fluid ounces. Does a "fluid ounce" measure more or less fluid than an "avoirdupois ounce"? Figure out the conversion factor between fluid ounces and avoirdupois ounces.

11. It would be more consistent mathematically to standardize our units of volume on the basis of linear units such as inches and feet rather than on the basis of weights. Picture for yourself or construct a volume of 1 cubic foot and make a guess or some [illegible]er estimate of how many gallons it would contain. Now, check that guess against the fact that a cubic foot of water weighs about 62.5 pounds.

12. According to the Realm of Measure a Biblical unit of weight called a "talent" was supposed to have been roughly equivalent to the weight of water contained in what we would call a cubic foot. Actually the Greek talent weighed only 57 pounds, a little less than our cubic foot of

water, while the Hebrew talent of the Bible was about 94 pounds; can you relate it to anything that might give a plausible reason?

13. A talent of silver came to be a unit of money, namely that amount of silver. Given the current price of silver, what would a Hebrew talent be worth in U.S. dollars? The Hebrew <u>shekel</u> (still a slang term for money) was about $\frac{1}{3000}$ of a Hebrew talent; what would this be equivalent to in U.S. dollars?

14. Below (from the <u>Realm of Measure</u>) is a table of liquid volumes that were once in very common use. In what industries might some of these measures still play an important part? Try to find some references in literature or history which use some of these terms. Make up some problems suitable for this book which use conversions from the table. "Minus" were likely to have been used only by apothecaries; a minim is about equal to 1 drop of liquid. Is "a drop" a fairly standardized measure of volume, or would it depend a lot on the material being measured and the dropper used? How many gallons in a tun and, if these are American gallons, about what would be the weight of a tun of water? Does this last suggest anything to you about the possible derivation of the word "ton"?

1 tun	=	2 pipes
1 pipe	=	2 hogsheads
1 hogshead	=	2 barrels
1 barrel	=	$3\frac{1}{2}$ firkins
1 firkin	=	9 gallons
1 gallon	=	4 liquid quarts
1 liquid quart	=	2 liquid pints
1 liquid pint	=	4 gills
1 gil	=	4 fluid ounces
1 fluid ounce	=	8 fluid drams
1 fluid dram	=	60 minims

15. What volume or weight is represented by "a shot" of liquor? (This is a common quantity used in mixed drinks.) Referring to our earlier discussion of the probable role of drunk drivers in fatal accidents, or some other information, can you make some estimate of how many drinks, each containing 1 shot of whiskey, might lead to a blood alcohol level that would constitute the legal definition of "drunk"? (This level is often set at 0.10%.) List some of the variables that make such an estimate somewhat difficult. One of these might be how long since the drink was consumed. (<u>The Odd Book of Data</u> says a healthy adult can "burn" about

0.34 fluid ounces of alcohol per hour.) Another might be how potent the whiskey is. In this connection, what sort of a measure is "proof" as applied to alcoholic beverages?

16. It is interesting to observe how frequently the number 12 figures in various traditional measures. List the places in the measures discussed so far where 12 (or some multiple such as 24 or 60) appears as a conversion factor or in some other form. Why would 12 arise so frequently? It is also interesting to observe how often powers of two (2, 4, 8, 16, etc.) appear. Why do you suppose this would be?

17. Traditional measures of volume and weight tended to depend on the use to which they were put, and there is a good deal of confusion among them. For example, nearly all of the traditional measures used in America differ somewhat from those used in Great Britain. In America there is still another system of quarts and pints used for "dry measure" that is slightly different from the liquid measure system given above. The conversion unit is this: 1 dry pint is equal to 1.164 liquid pints. Why would the same conversion factor be used in converting from liquid quarts into dry quarts?

18. One peck is eight dry quarts and one bushel is four pecks. Assuming that most things measured by bushels (wheat, tomatoes, other fruits and vegetables) weigh about the same for the same volume as water does, about what would you expect a bushel of tomatoes to weigh? A bushel of wheat? Is this assumption valid "for practical purposes"? Check on this with some reference giving actual weight of a bushel of wheat and bushels of other things.

19. There is a popular song from your parents' high school days that begins "I Love You a Bushel and a Peck... ." How is that for finding a measure of the unmeasureable? Do you know of other song lyrics or poems where measures--or at least measure words--appear?

20. Another traditional volume measure with a special purpose that you may not be familiar with is the "cord" of wood, which is a pile 8 feet long by 4 feet wide by 4 feet high. According to 1001 Questions Answered About Trees, seasoned wood can weigh from 20 pounds per cubic foot of willow or poplar to 50 pounds for many other hardwoods. With these figures, what is the range of weight of a cord of wood? If you were offered some wood, such as oak for your fireplace, at a cost either $20 for a cord or $15 for a ton, which would be the better buy?

3.4 More About the Metric System of Measures

The metric system of measures is more rational and easier to use than our traditional systems in nearly every way. We have already talked about the original definition of a meter as one ten-millionth of the distance from the equator to the North Pole along the earth's surface (and the subsequent discovery that this was slightly in error) and the fact that a meter is a little longer than a yard. In both the metric and traditional measures, area is measured in such units as square feet, square yards, square inches, square meters, centimeters squared, and so on. Again, there are in both traditional and metric volume measures such as cubic inches, cubic yards, cubic centimeters, and so on. But most of the traditional measures of volume have little relationship to this progression from linear to squared to cubic units; for example, there are no easy links between cubic inches or cubic feet and such things as quarts, bushels, or gallons. Likewise, except for the rough rule "a pint's a pound" there are few links between volumes and weights. In the metric system there are such links. The basic unit of volume, the liter, is the volume of a cube with edges ten centimeters (also called one decimeter) long, and there is also a tidy link to the weight of water.

Problem Set 3.4

1. How many cubic centimeters will there be in a cube with 10 centimeters on each side: i.e., in a liter?

2. As a convenient basic unit of weight, the inventors of the metric system took the weight of one cubic centimeter volume of water as the basic unit of weight. This unit was called a "gram." Hence, how much would you expect a liter of water to weigh? (More exact determination of the mass of water after this neat system was invented revealed minute errors. Since these errors are only on the order of .003%, they are not of concern except in the most delicate scientific measurements.)

3. A liter is very close in volume to an American liquid quart. The conversion function is that 1 liter is about the same as 1.06 American liquid quarts. You would expect, then, that a kilogram would weigh a little more than 2 pounds: actually the conversion function is that a kilogram is about 2.2 pounds. Make an estimate of how many grams there are in an ounce and check your estimate in some reference source.

4. There seems little question that the metric system will become the world standard of measure in business and industry, just as it is already the world standard of measure in science. Indeed, the United States Secretary

of Commerce recently proposed that the United States convert from traditional measures to metric measures over about a 10 year period. Because of the world-wide flow of goods, this will certainly put the United States in a better competitive position in world trade, but the changeover of manufacturing machinery will be enormously expensive. Moreover, it will present many difficulties to the average citizen accustomed to thinking in terms of our traditional measures. Here are some exercises which may show that it is not as hard as you may think to make the conversions, at least for everyday estimates of quantities.

a. In a letter to the magazine Science some time ago, it was suggested that transition to the metric system could be eased for "everyman" by temporarily renaming the metric units with the English system units that they are closest to. For example, liter would become a "metric quart"; a meter would be called a "metric yard"; 2.5 centimeters would be called a "metric inch." Other possibilities are to call $\frac{1}{2}$ kilogram (or 500 grams) a "metric pound" and 1.5 kilometers a "metric mile." Do you agree that each of these would be a pretty good rule of thumb for shifting to the Metric System? About what percent error is there in considering a liter to be a "quart"? About what is the % error in the other approximations? Where would the greatest errors occur? Would these be serious in everyday affairs?

b. An easy rule-of-thumb conversion between kilometers and miles is that a kilometer is about .6 miles. Would we use the same conversion factor to convert to miles per hour from kilometers per hour? If you are traveling in Mexico (which uses the metric system) and see a sign which means "speed limit 100" what should be the maximum speed showing on the speedometer of your American car? (Here and elsewhere in this set of problems, use your common sense in deciding on how precise to make your answers; in most cases a good approximation will do and it would seldom be appropriate to have your answers expressed to several decimal places.)

c. Many American car speedometers read from zero to 120; what would be the corresponding range on a speedometer calibrated in kilometers per hour? If you were driving a foreign car with a speedometer marked in kilometers per hour, what is the maximum reading on that speedometer that would keep you within a 60 mile per hour speed limit on an American road?

d. It is easy to imagine an automobile trip that begins in Canada, goes through the United States, and ends up in Mexico. Canada has miles must like ours, a dollar that is roughly the same as ours, but uses the British Imperial Gallon, which is about 1.2 American gallons. In Mexico, a peso is $.08 U.S. (or 12.5 pesos per American dollar), distance is measured in kilometers, and gasoline is sold by the liter. Suppose that you are an American making such a trip and you have a car that usually gets 15 miles per gallon on regular gasoline costing about 40 cents per gallon. Make the appropriate conversions to miles per Imperial Gallon and cost per Imperial Gallon and kilometers per liter and expected cost per liter to use as standards in judging your car's performance and costs on your journey in Canada and Mexico.

e. The Mexican government has a monopoly on the sale of gasoline, and all over Mexico one can get Supermex, which is 80 octane, Gasolmex, which is 90 octane and about equivalent to an American "regular" grade, and Pemex 100, which is 100 octane and about equal to an American premium grade gasoline. The respective prices are usually about 0.80 peso per liter for Supermex, 1.00 peso for Gasolmex, and 1.30 peso for Pemex 100. As in the United States, prices vary from place to place. How would these prices compare with prices for gasoline in gasoline stations near you. (By the way, what does "octane" mean as a measure of gasoline?)

f. Make up some more problems similar to b. through e. above that would be suitable for this book.

5. You have probably seen birth announcements which make some fuss about how much the baby weighed when it was born and how long it was. Suppose you get such an announcement from a cousin in France which announces that the weight of the baby was 4000 and its length was 60. What units are probably being used. Estimate how long and heavy the baby is in our traditional measures.

6. For very large measures of weight, the metric unit is a "megagram" (1 million grams). The Realm of Measure tells us that in most languages this unit is called a "tonne" and in English called a "metric ton." How many pounds are there in a metric ton and how does that compare with our standard "short ton" and the British "long ton"?

7. The common unit of land area in the United States is the acre, which according to several sources was originally equal to the amount of land

that a man could plow with a yoke of oxen in about half a day. It was eventually necessary to standardize this and, according to the Realm of Measure, it was standardized by English kings as a rectangle of land 40 rods long and 4 rods wide (a "rod" is $5\frac{1}{2}$ yards long). If you figure out how many rods there are in a mile, you see that the definition of acre makes it possible to fit an even number of acres in a square mile, which is lucky for land surveyers who use traditional measures.

a. How many acres are there in a square mile?

b. How many square yards are there in an acre?

c. What conversion factors would be used in a function changing square meters to square yards?

d. In the metric system, land is measured by a square 100 meters on each side; this unit of area is called a "hectare." How many square meters in a hectare? This is equivalent to how many square yards? How many acres does it take to make a hectare? Devise a convenient rule of thumb for converting from acres to hectares and from hectares to acres.

8. Have you ever wondered how much lava is produced by an erupting volcano? Writing in Science, January 14, 1972, D. A. Swanson concluded that active volcanos on the island of Hawaii over the past 20 years produced lava at an average rate of about 9 million cubic meters per month (9×10^6 cubic meters). ("Magma Supply Rate at Kilauea Volcano")

a. Try to get some idea of how much material that 9×10^6 cubic meters is. (For example, you might think of a meter as about a yard and see how many square blocks or square miles would be covered, say to the depth of 1 foot, by one month's lava flow or how many buildings of a certain size would be filled every day or every week or every month.)

b. Swanson drew his conclusions from the following data. The 1952 Halemaumau eruption produced 49×10^6 cubic meters (m^3) of lava in 4.5 months; the 1967-68 Halemaumau eruption produced 84×10^6 m^3 of lava in 7.25 months. Swanson averages each of these to find a fairly constant rate, then reduces the total volume by 15% to compensate for the fact that hot gasses in lava make it full of holes and channels and he is interested in the amount of solid lava. Using these figures, check to see if the final estimate of 9×10^6 m^3 per month is approximately valid.

c. Swanson suggests that this same amount of "magma" flows constantly into the Hawaiian volcanic system and is stored underground. This conclusion is based on the fact that when a volcano is actively erupting at the rates given above, the surface of the island is not deformed, while, when there is no eruption in progress, the land surfaces are deformed as if something were being stored underground. Make a mathematical model of the big island of Hawaii as a cone having the height of one of the higher mountains (say Maunaloa) and a suitable base (work from a map of the island). Estimate the volume of the cone and estimate how long it would take to build such a land mass with lava flowing at the constant rate suggested by Swanson. (It is very likely that Hawaii and many of the other Pacific islands were built from volcanic activity starting at the ocean floor.)

3.5. Some Miscellaneous Measures, Including Energy, Temperature, and Time

In a modern civilization the sources of and the uses of "energy" are necessarily of great concern. We need electrical energy to run light bulbs and appliances; we need fuel for automobiles and factories and many other things; and so on. To have a standard reference measure for energy we often convert discussions about any kind of energy into heat energy measured in calories. As we discussed in an earlier problem set, one calorie is defined to be that amount of heat needed to raise the temperature of one gram (or one cubic centimeter) of water by one degree Celsius (more precisely from 14.5° C to 15.5° C). As we also discussed earlier, the calorie used in discussions of diet and in most everyday discussions is in reality the kilocalorie (1000 calories), which clearly is the amount of heat needed to raise the temperature of 1000 grams (or one liter) of water by one degree Celsius. In countries where the Fahrenheit scale is used, another unit of energy is often used, namely, the amount of heat required to raise one pound of water by one degree Fahrenheit (to be more precise, from 59.5° F to 60.5° F); this is called the British Thermal Unit (BTU). On the other hand, when one is speaking of the expenditure of energy, one often uses something called the "foot-pound" which is the energy used in raising a one-pound weight one foot up against Earth's gravity. Here are some conversion units from the Realm of Measure to use in the problems that follow:

1 BTU	=	252.0 calories	=	0.252 kilocalorie
1 kilocalorie	=	3.97 BTU		
1 calorie	=	3.082 foot-pounds		

1 kilocalorie = 3,082 foot-pounds

1 BTU = 778 foot-pounds

Problem Set 3.5a

1. The food we eat supplies both the building materials for our body and the energy that is burned in our daily activity. Generally speaking, that part of food that goes to energy is measured in kilocalories (remember that in diet books and most everyday discussions the kilocalorie is called a calorie), and generally speaking the excess over what we actually burn up in our activity is stored in the body as fat. There are hundreds of diet books on the market; one of these tells us that a 130 pound man needs about 2700 kilocalories per day, and the needs go up by about 80 kilocalories for every 5 pounds increase in weight beyond that. For women the chart begins at about 2000 kilocalories for a 110 pound woman, going up at the rate of about 90 kilocalories for every 5 pound increase in weight above that. A note accompanying this information suggests that this is for a fairly active man or woman. For someone doing extra-heavy physical labor, the figures should be increased by about 15% and for a very calm and inactive life, decreased by about 8%. On a piece of graph paper construct a line showing the function thus defined for an average, fairly active man, assuming that this information is approximately correct and that it can be extended over a range from 90 pounds to 180 pounds. Then add with dotted lines or different colors the appropriate graphs for a person doing extra-heavy physical labor and for an inactive person. On another sheet of graph paper make a similar graph for women. Locate yourself on the appropriate graph.

2. From a diet book or elsewhere, find a table of the calorie (actually kilocalorie) content of a number of common foods. For a couple of days keep track of what you eat and then compute your calorie intake. Compare this calorie intake with your place on the graph from the problem above. If there are very wide discrepancies, consider why this may be (including the possibility that the information given in the problem above may be wrong or it may not apply to your particular situation).

3. It is said that each nation seems to have its own peculiar preoccupations; from popular literature and everyday conversation one might conclude that one of the American hobbies is concern about diets and weight. Books on a "new" method of dieting or magazine articles on someone's new theory are being widely discussed almost all the time. Are you aware whether or not this is true right now or fairly recently? Basically, it would

seem that changes in body weight would depend only on these two factors: the amount of energy used by the body and the amount of energy supplied to it by food or drink. Using the figures given in the table at the beginning of this problem set, verify that a kilocalorie of energy is equivalent to lifting about a ton and one half, one foot or three hundred pounds about 10 feet or 150 pounds about 20 feet. Consider an exercise where you jump up and down about one foot off the ground. About how many jump-ups would it take you to expend a kilocalorie of energy? Considering that most flights of stairs rise about 10 feet from bottom to top, how many flights of stairs would you have to go up to expend a kilocalorie of energy?

4. A pound of fat is said to have about 3500 kilocalories of reserve energy. How many flights of stairs would you have to climb in order to burn up a pound of fat stored in your body? (Of course exercise has many functions other than using up energy or fat, so don't let the results of this problem discourage you from exercising.)

5. According to one diet book, a one-ounce chocolate candy bar might have about 100 to 125 kilocalories. Find the net weight of some candy bar that you like to eat and figure out about how many calories it has. Then do two things: first, figure out how many jump-ups or flights of stairs it would take to use up this much energy; second, figure out how many such candy bars would be equal to the energy stored in a pound of fat.

6. With a diet book or other information on calorie content or exercise, make up some interesting problems that would be suitable for this book.

Problem Set 3.5b

1. Time is a very important thing to human beings, and probably always has been. We measure our days by the rising and setting of the sun. In times past, months were measured by the lunar (moon) cycle, from new moon to new moon, and years by the sun, from spring to spring. (Such archeological sites as Stonehenge, in England, are thought to have been used to determine the exact first day of spring each year.) Even so, it has been very difficult over the ages to devise an accurate calendar to keep track of the passage of years. Do some reading on this subject and say something about at least three other calendars used in modern countries and how they compare with our standard calendar. (For example, see an encyclopedia or <u>The Exact Sciences in Antiquity</u> by Otto Neugebauer.)

2. Find out how our own calendar evolved and what corrections have been necessary in it during the "Christian Era."

3. Investigate some other question about calendars that is of interest to you.

4. The measure of moment-to-moment time is accomplished nowadays by clocks and watches, with standard times available to us to "set" them by. What are some of the mechanisms that have been used to keep track of time by minutes, hours, seconds? How accurate are the most precise timepieces now available to us and in common use? How accurate and how expensive are the timepieces used to meet requirements for extremely precise measurements of time? What timing mechanisms are used nowadays by the National Bureau of Standards, the government agency with responsibility for providing "absolute" time standards?

5. According to Realm of Measure, using the rotation of the earth as a time standard is inadequate because the earth is slowing down slightly, and so the length of the day is very slowly increasing; by about one second every 100,000 years. By how much has the length of the day increased today as compared to ten years ago?

6. How could one detect that the earth is slowing down?

7. Mathematics in Everyday Things says that the length of the day is getting bigger by about 0.0016 seconds each century. Just to show that you can't believe everything you read, this source goes on to say "that the solar clock has run slow about $3\frac{1}{2}$ hours during the past 2,000 years." Can you see that this is obviously way out of line? How long would it take at the rate of .0016 seconds per century to accumulate $3\frac{1}{2}$ hours? At that rate how much slowing down would actually take place in 2,000 years? How long would it take to accumulate $3\frac{1}{2}$ seconds. Is the figure of .0016 seconds per century, approximately the same figure you are given in the previous problem: 1 second every 100,000 years? Why do you think this book made this error? Can you figure out what they meant to say or should have said?

8. The measure of time on your watch is really found by defining the position of the two hands on the clock face. Name some other ways by which time has been measured by an indirect reading depending on angle measure, length of something, volume of something, or some other measure. Is it possible to get a "direct" measure of the time or the passage of time? In what ways is measuring "time" different from measuring such things as

"length," "volume," "area"?

Problem Set 3.5c

1. Some of the other measures we've mentioned above depend not on direct measure of the thing in question, but the measure of something else instead; for example, energy measured in calories has something to do with temperature and volume of water: temperature itself is measured by the length of a column on the thermometer or as a rotation on another sort of thermometer. Make a list of those measures that seem to be directly accessible by some means and another list of those that use direct measures in order to derive another measure.

2. The traditional measure of temperature in the USA is in "Fahrenheit" degrees. Water freezes at 32° F and boils at 212° F. In all science and in many countries a centigrade or Celsius measure is used. The freezing point of water is 0° and the boiling point 100° C. Figure out a conversion function for taking Fahrenheit to centigrade degrees; the critical points will be 212 to 100 and 32 to 0. For reference points, figure out the Celsius equivalents of human body temperature (about 98.6° F) and room temperature [illegible].

3. Notice that both these systems allow negative values. Although most measures like distance and weight only have positive values, a few, like temperature and time, sometimes have negative values. It happens, however, that there is a "lowest possible" temperature. It is at -273.16° C. What is the lowest Fahrenheit temperature? There is a third scale of temperature which takes this into account. It is called the Kelvin or absolute scale, where 0° K = -273.16° C; 273.16° K = 0° C. Figure out the Kelvin equivalents of human body temperature and room temperature.

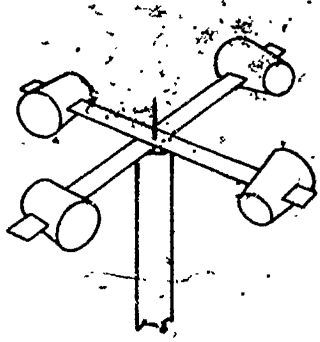
4. Measures are further complicated by the fact that in many cases one uses not a single measure, but some combination of measure; for example, speed is measured in miles per hour, and batting averages are measured in the number of hits per number of times at bat. Make a list of at least 10 such "compound measures" in ordinary use.

5. A shopper in a grocery store confronts a bewildering variety of brands, qualities, sizes, and prices; it is not uncommon for the "giant economy size" of a given brand of a product to cost more per unit than the "large" size. Spend an hour in a grocery store doing price comparisons on a cost per unit basis, first among various brands, and then among sizes within

brands. Some consumer organizations have advocated requiring stores to post their prices on a price per unit basis for easy comparison; do you think this is a good idea? Why or why not? How much of an extra burden would it put on a store manager and his employees? (For your exercise in price comparisons, you might want to use a slide rule. In many cases an approximation by rounding off quantities and prices involved is good enough information. While you're at it, notice that most canned goods have weights given both in ounces and in grams; use this to solidity your feelings about the relationship between ounces and grams. In some cases it is easier to get the price per gram rather than price per ounce.)

6. In America speed is usually measured either in miles per hour or in feet per second. It is nice to have a rule-of-thumb conversion between the two forms. Convert 60 m.p.h. to feet per second. Use this to convert the speed of a 100 yard dash in 9 seconds or of a four-minute mile to miles per hour.

7. 150 Science Experiments Step-by-Step gives instructions for constructing an "anemometer" for measuring wind speed by pushing the ends of two 12 inch long pieces of cardboard, crossed at right angles, through slits in four paper cups. This is then placed to rotate freely on a nail on top of a stick. Paint one cup a different color so rotations can be counted. The rule of thumb is that one should count the number of revolutions in 30 seconds, then divide by 5 to get the wind speed in miles per hour. Can you figure out why this should give a reasonably correct result? Should variations within reasonable limits of the length of the arms to which the cups are attached make any difference in this? Why or why not?



8. 150 Science Experiments Step-by-Step says that weather is often determined by the movement of air in the upper sky as well as by the direction of the wind at ground level. It suggests that in order to find out what this direction is one can mark off a circular mirror with the various directions N, NE, E, SE, etc. ; mark a dot in the center of the circular mirror; then watch in the mirror a cloud passing over the center spot until it goes over the edge of the mirror. By marking the point at which it goes over the mirror and looking at the point diametrically opposite

from it, one can then find out the direction from which the wind pushing the cloud comes. Try this if you are interested.

9. In your text books you have probably done problems of "indirect measure" which somewhere along the line involved measuring angles. Earlier in this book we talked about the role of angles in determining "latitude" and "longitude" coordinates on the face of the earth. Try to make a list of 5 or so other kinds of situations where angle measure plays some role.

10. Barometric pressure is a measure that is important in forecasting changes in weather. Find out what this means and how it is measured and what its uses are in weather forecasting.

11. A "measure" connected with weather is often given in the radio weather reports when they say that there is a 10% chance of rain or a 70% chance of snow. How does this "measure" differ from the other measures we've been talking about above? What do such statements mean?

12. There are other measures given in weather reports such as relative humidity and in some cases a "comfort index." Find out how such things as relative humidity are measured and what they mean in terms of human comfort. Why the "relative" in relative humidity?

13. An automobile is a very complicated piece of machinery requiring very precise specifications and measurements for many of its parts. For your family car, or any other car you choose, look up the specifications and find out what they mean and how they are measured. In particular, note the places where angles are measured. Also, note the precision with which the measurements are given.

3.6 Measures of Power

An earlier problem talked about calories as a measure of energy, and in particular you noted that a 1¼-pound man would have to go up about 2 flights of stairs to use 1 kilocalorie of energy. But the same amount of energy would be used up whether the man walked slowly or ran, although we know that there is considerable difference in how you feel after a slow walk and after running. Measures that take into account the rate at which energy is used are called measures of power: in everyday usage the most common examples are horsepower as applied to automobile engines or electric motors; and watts or kilowatts as applied to electricity. The key issue is not merely how much energy is supplied or used, but how much energy is used in a given unit of time, for example, per second, per minute, or per hour. When we speak of a man who uses in his normal activities 2,500 kilocalories per 24 hour day, we are clearly speaking of energy over a certain period of time and it is appropriate to think of this as power.

Problem Set 3.6

1. The Realm of Measure tells us that a man using 2,500 calories in a 24 hour day supplies about as much heat to his surroundings as does a 125 watt light bulb burning continuously. Assuming that information is correct, answer these questions:

   a. Most electrical appliances are rated in watts. About how many average sized people does it take to equal the heat equivalent to 1,000 watt iron, waffle iron, or toaster? Check several appliances such as these at home and see how many watts of power they consume in producing heat.

   b. A 1250 watt electric heater with a fan will keep an average size well insulated room comfortably warm at temperatures down to freezing outside the room. Do you suppose that thirteen people continuously in the room would keep it as warm as such a heater? If not, why do you suppose not?

   c. A large crowd of people crowded into a room does produce a considerable amount of heat. An auditorium warm enough to be comfortable when a crowd enters may actually need air conditioning even in winter to keep it from becoming uncomfortably warm when it is filled with people. This is also true for a crowded room at a party. Ask some interesting questions suitable for this book related to such issues.

2. The whole question of heating buildings is interesting and involves a good many numerical and mathematical calculations. Inquire of your local electric company and gas company and see how many watts of electrical power are needed to heat a room or a house and what the equivalent measures are when heat is supplied by gas fired furnaces heating either air or water to be distributed through the house.

3. Heat from electricity and from "fossil fuel," such as coal, oil, and natural gas, are in competition in most cities for the heating market. By talking to representatives of utility companies in your city or town, find out what the claims are for each and how much they claim each costs.

4. Take a survey of the appliances and motors and other electrical equipment in your home and try to find out how much power each uses when it is in

operation. Then, for several times during the day, make an estimate of how much total electrical power is being used in your house at that time. Are there periods in the day when a lot more power is being used than at other times?

5. Power is sold by utility companies in terms of kilowatt hours; where kilowatt means 1000 watts and kilowatt hour means, for example, 1000 watts used for 1 hour or 100 watts used for 10 hours. Using the estimates in the previous problem, try to estimate how many kilowatt hours are used in your home everyday. Then try to estimate how much is used each month. Check your estimate with a recent electric bill, and if you are way off, try to figure out why there is disagreement between your estimate and what is actually recorded by the electric company.

6. Power is measured as it comes into your home by the electric meter, which consists of a small motor which turns faster or slower, depending on how much power is being used in your home. This motor, in turn, operates dials recording the actual amount of power used. Have a look at your electric meter, and if it has a glass cover on it, watch the motor operating to turn the dials. Try to find out how to read the meter.

7. Do you suppose in a city such as yours that more electrical power is used by houses and apartments and other places people live or by business and industries? Given the information you just developed on how much power your own home uses, try to make some rough estimate of how much power your entire city uses and check this out with somebody at your local utility company.

8. According to The Physics of Music, "horsepower" as a unit of power was defined when mine operators began to replace horses with steam engines; it was the rate at which an average horse could do work over an average working day. It is now defined as the power required to lift 550 pounds one foot per second. A watt is about $\frac{1}{746}$ of a horsepower; or in other words, 746 watts are about one horsepower. So how far will a watt raise a weight of one pound in a second? If you jump up one foot each second, how many watts of power are you using?

9. Conversion of energy, and hence power, from one form to another is done many ways. The gasoline in our automobile tank ultimately becomes power applied to the rear wheels of the automobile; electricity becomes heat in a toaster or, through motors, the operating force for many kinds of machinery. List some more ways in which energy in some form is converted

into another form.

10. The conversion process of power (or energy) from one form to another is usually notoriously inefficient. For example, I have in my shop an electric motor which uses about 300 watts to produce $\frac{1}{4}$ horsepower. By computing the ratio of power out of the motor to the power that goes into the motor, attach a "percent efficiency" measure to this motor. (If the motor runs for a while it feels warm; this heat accounts for much of the lost power.)

11. Look into the relative efficiency of various kinds of motors; automobile engines, steam engines, and turbines, for example. Can you see why the search still goes on for more efficient electric or steam powered automobiles? Most automobiles have water circulating around the engine and being cooled in the radiator; what does this have to do with the "efficiency" of a gasoline powered engine?

12. The Physics of Music notes that sound is usually produced by some sort of mechanism that causes the air to vibrate; this vibration is sensed by the ears. Power considerations come into this in two ways: first in the power needed to make the sound-producing mechanism vibrate and second in the power actually radiated as sound. For example, a large organ may require a ten kilowatt engine to supply air to its pipes, but the actual sound radiated would probably be less than 15 watts. What is the efficiency of such an organ? A pianist playing furiously may use power at a rate of 200 watts with only about 0.4 watts radiated as sound from the piano. What is the efficiency in this case?

13. This same source notes that the human voice is about 1% efficient and that a bass singing voice radiates about 0.03 watts; how much energy does a singer put into such a sound? Average speech radiates about 0.00002 watts of radiated power; how much energy is put into ordinary speech if the 1% efficiency figure is valid?

14. The Physics of Music (on page 34) gives the measured power radiated by various musical instruments. Here is part of that information:

Source	Power in watts
orchestra of 75 performers (maximum)	70
bass drum (maximum)	25
snare drum (maximum)	12
trombone (maximum)	6
piano (maximum)	
bass saxophone (maximum)	0.3
bass tuba (maximum)	0.2
orchestra of 75 performers (<u>average</u>)	0.09
flute (maximum)	0.06
bass voice (maximum)	0.03
alto voice (maximum)	0.001
<u>average</u> speech	0.000024
violin at its <u>softest</u> passage	0.0000038

a. What is the ratio of the power of the full orchestra at its maximum loudness to the power radiated by a violin at its softest passage?

b. How many people would it take in an average conversational speech to radiate the power of a 100 watt light bulb?

c. Although there are enormous differences in the sound power radiations given in the table above, we do <u>not</u> perceive, for example, the bass saxophone as being 10 times as loud as the bass voice; and the bass drum as being nearly 10 times as loud as the bass saxophone and 100 times as loud as the bass voice. Instead, every time the sound radiation is multiplied by a factor of ten, our perception of the sound goes up by about a single step. Hence, we perceive the bass saxophone as being about twice as loud as the bass voice; the bass drum as being about 3 times as loud. This is an example where perception of the intensity of a stimulus is proportional not to the stimulus itself but to a so-called <u>logarithm</u> of the stimulus. This is also more or less true for such other sensations as pressure and sight. This is very nicely explained in <u>The Physics of Music</u> on pages 34-41, which also explains measures such as "decibels" used by sound engineers. Thus the sensation one gets from various sources depends upon the "order of magnitude" of power; one way to make the comparisons is to convert to "scientific notation" and then look at the exponent that goes with the ten. Use scientific notation to make comparisons among the ways we may perceive various sound listed in the table above.

15. Here are some tidbits of information from the chapter on energy in the Odd Book of Data:

a. Man's present average daily energy consumption per capita is equivalent to about 5.2 kilograms (11.4 pounds) of coal. This includes the equivalent of 0.46 kilograms of coal (2500 kilocalories) as food and 4.80 kilograms of coal (33000 kilocalories) for power.

b. 0.2 kilograms (0.45 pounds) of coal (about a handful) is equivalent to about 0.16 liters (0.04 gallons) of oil (about a glassful).

c. Sugar has about half as many calories as coal.

d. An electric power station serving a city of about a half million people would need a capacity of about 250 megawatts (million watts).

e. At 33% efficiency this 250 Megawatt generating plant can be run by 10 kilograms of coal for $\frac{1}{3}$ seconds; 10 kilograms of Uranium 235 (fission) for 13 days; 10 kilograms of hydrogen (fusion) for three months; 10 kilograms of matter via direct conversion to energy for 50 years.

With such information, the Odd Book of Data does a number of very interesting comparisons and produces a number of interesting facts. Try your hand at inventing some such exercise, resulting in either "odd data" or problems such as would be suitable for this book.

3.7 Approximations and Rules of Thumb

It frequently happens in everyday life that you want to find out the approximate measure of something without having measuring instruments at hand. Here are some suggestions to get you started on finding ways to do such things without instruments. In each case you are to try to extend the ways of finding approximate measures and check out the ones you are given and the ones you invent to see how accurate they are and how well you are able to use them in practical situations.

Problem Set 3.7

1. A handy way to measure rooms and buildings and plots of land is by a "pace." Most people can calibrate themselves so they can step off about 1 yard with each pace. If less or more than this is more comfortable to you, okay, but practice until you know what your pace is and can use it. The body measures given in the diagram in an earlier section are also very useful; from the tip of the nose to the end of the finger tip, about

a yard; between two knuckles, about an inch; the span of your hand, whatever it is; and so on. Lines in sidewalks are frequently spaced 5 feet apart. The height of the ceiling in most houses is about 8 feet. A brick is about 8 inches long; concrete blocks are often 1 foot long; floor tiles are usually either 9 inches or 1 foot on each side; ceiling tiles are usually 1 foot on a side. (Ceiling and floor tiles, of course, give you the means to get both linear and area measures.) The standard door height in residential construction is usually 6 feet 8 inches, and in commercial construction, often 7 feet. A compact car is about 15 feet long, larger cars range up to a maximum of about 20 feet long. Most cars are about 6 feet wide. A city block is often $\frac{1}{8}$ mile. In most houses the wooden "studs" on which the walls are nailed are spaced 16 inches apart, so if you can find one stud by some means or other, you can usually find the others by measuring from that one. Find some other everyday instances of standard measures and figure out some uses for them.

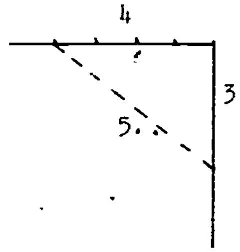
2. One can frequently find an otherwise inaccessible length by measuring angles and some length easy to get at, then making a scale drawing, or applying similar triangles or trigonometry. Do several such indirect measure problems; for example, the height of your school or its flagpole.

3. According to Mathematics in Everyday Things, if you close one eye, hold your arm out straight from you and parallel to the ground, and tilt your hand up so the palm is away from you, your eye will subtend an angle of about 8° in sighting first on one side of your hand at the second knuckle, then on the other side. Calibrate yourself and see if it really is about 8° by standing in the corner of a room and seeing how many "hand width angles" it takes to measure the right angle. Use this to estimate angles in doing several indirect measure problems as in the problem just above.

4. Engineers and artillary men define a "mil" as an angle which subtends one yard at a distance of 1000 yards (or one of any unit at a distance of 1000 units). According to Mathematics in Everyday Things if you stretch out your arm and sight across first one side of a finger, then the other side, an index finger usually measures about 40 mil; second finger, 40 mil; third finger, 35 mil; and the little finger 30 mil; with all four fingers, about 145 mil. Hence to measure the height of, say, your school building, you could walk [illegible] yards away from it, see how many mils from top to bottom of the building are measured off by your fingers at the end of your outstretched arm, and do the appropriate calculations (in this case $\frac{1}{10}$ of a yard for every mil--why?). Try this.

5. When the Lilliputians measure Gulliver for clothes in Gulliver's Travels, they measure his thumb and then apply a rule that "twice around the thumb is once around the wrist, and so on to the neck and waist. . ." Check this rule on yourself and on some friends.

6. If you are laying out a building or making a picture frame or any other application where you need a right angle, you can get one by measuring 4 units (inches or feet or yards or whatever) on one side of what you are building, 3 units down another side, and then adjusting the 2 sides until it is 5 units from the 4 unit mark diagonally across to the 3 unit mark. (You can also use 6, 8, 10 or any other multiple of 3, 4, 5.) Since it is very common to need angles in constructing buildings this technique is used quite often in the building trades. Why does it work?



7. The volume of an object can very often be calculated by making an approximate mathematical model of its shape, and using standard formulas on the model to compute the volume. The volume of very irregular objects can be found (if they are small enough) by dunking them in water and measuring the amount of water displaced. For example, one could put a container, filled to the brim with water inside a larger container, then completely immerse the object, then measure the amount of water that spilled out of the small container into the large container. In this case one might very well want to have some way of converting pints, quarts, or cups of water into more standard volume measures, such as cubic inches. Devise a rule of thumb which helps you convert from pints into cubic inches and from gallons to cubic feet.

8. Even the volume of very large irregular objects can be found by water displacement if you have an accurate scale model of it. In this case, however, you must remember that the ratio of volumes is as the cube of the ratio of linear dimension, so that the volume of a model built to $\frac{1}{4}$ scale will be only $\frac{1}{64}$ of the volume of the original object. Why should this be? Check it out by immersing a medium sized doll proportioned about like the human body in water; measuring the amount of water displaced; and then doing the appropriate calculation using the height of the doll compared to your own height. You might want to carry this one step further by checking to see if this estimated volume more or less checks out with your weight. (Since you float in water, your weight is

less than "a pint a pound," but this approximation is still rather close.)

9. On a typewriter, "elite" type has 12 letters to the inch; "pica" type has ten letters to the inch; and typed copy usually has about six lines to the inch. Average word length is usually about five letters, or six spaces including the space between words. (Check that out by counting word~~s, letters, and spaces in the~~ three lines of print above.) Allowing one inch margins all around on standard typing paper, estimate about how many double spaced types pages will be required for a 1000 word theme.

10. The speed that sound travels through air is affected by several factors, including temperature, but, roughly speaking, it is about 1000 feet per second. The speed of light, on the other hand, is about 186,000 miles per second. With this information, devise a rule of thumb for telling about how far away a lightning flash is by estimating the time between when you see it and when you hear the thunder it produces.

11. Frequently, a close enough approximation "for practical purposes" can be found just by estimating the power of ten or order of magnitude involved; for example, expenditures of billions of dollars on behalf of millions of people. The Odd Book of Data has some very nice approximations of this kind all through it; here is one: The human eardrum has a surface area of approximately 1 square centimeter so that only a very tiny fraction of power coming from a source of sound strikes our eardrum. (For a mathematical model, think of the sound source as the center of a series of expanding concentric spheres, with a one centimeter square on the surface of these spheres. The surface of the sphere represents the way the sound power is distributed at various distances from its source and the centimeter square that part of the sound power that affects our ear.) Our ears are incredibly sensitive, being able to perceive sound waves with a power of only 10^{-16} watts of power actually impinging on the eardrum. The Odd Book of Data says that if our ears were only slightly more sensitive we could hear the background noise of collisions of molecules in the air. As noted earlier, about a ten-fold increase in the power of the sound source increases by one step our perception of loudness. Starting at the threshhold of hearing (10^{-16} watts) and visiting 10 locations, each of which would represent about a ten-fold increase in the power of the sound, you might go to the following places:

   a. a meadow with grass faintly rustled in a gentle breeze

   b. the center of a deserted city park

c. a street close to traffic

d. a large city square with background traffic hum

e. about 50 yards from a busy highway

f. inside a large department store

g. a sidewalk outside the store right next to a street in the midst of rush hour traffic

h. right next to a railroad viaduct

i. near a powerful motorcycle

j. across the street from a pneumatic drill

This last level of noise could damage the eardrum if we were exposed to it for a long period of time or regularly everyday, but one would need to make four more such ten-fold jumps before we would reach the "thresh-hold of pain," that is, the level at which we would experience real pain and likely damage. This is about the level of sound next to a modern jet airplane; which is why ground personnel at airports wear sound-proof earmuffs. Investigate what the relative noise level is at various places you might visit or work in. Find out what is meant by the "damage-risk limit"; that is, the noise level which leads to damage to ears exposed to it continuously. Compare various noisy places in the everyday living or working environment that may excede this level. For example, find out if recent statements that the noise level at a rock concert excedes the "damage-risk limit" have some validity or not.

12. A brand new nickel weighs 5.0 grams; a brand new penny weighs 3.1 grams. As you have already computed, an avoirdupois ounce is about the same as 28.53 grams. Suppose you wanted to use new nickels and pennies in some combination to "calibrate" a weighing scale. What combination of nickels and pennies would weigh close to one-half ounce? One ounce? Two ounces? About how many nickels to weigh one pound? How many pennies would weigh about one pound?

3.8 Measures in Building and Printing Trades

Arithmetic, rudimentary geometry, trigonometry, and sometimes even fancier mathematical tools are in common use in the work of the world. Here is some information about various trades and the computations that apply to them. The sizes used are pretty standard and don't change much, but the prices may be wrong for the area you are in, or they may be out of date. What you are to do is to study the information given in each of the numbered items below and then to ask some questions that a worker in that field might want answered or a home owner might want to answer. You may then want to go on and answer the questions yourself. Appropriate questions might revolve around the amount of material needed (allow something for waste--perhaps 10%), or the estimated cost of doing a certain job. In some cases the weight of materials and the amount that can be lifted by workmen is important. Much of the information given here is from Applied Mathematics.

Problem Set 3.8

1. The walls of the room that you are in probably consist of either "lath" covered by plaster, or sheets of material which amount to plaster sandwiches covered by paper. There are two kinds of lath in common use, gypsum lath and metal lath. Gypsum lath is usually $\frac{3}{8}$ inches thick in pieces 16" × 48", with 6 pieces per bundle. Cost per bundle is about \$1.60. One uses about 16 nails per piece in nailing it on.

2. Standard metal lath is usually 27" × 96" and comes in a 2.5 pound per square yard weight for walls and 3.4 pounds per square yard weight for ceilings. It comes in bundles of 10 pieces covering a 180 square feet at prices of \$40.00 to \$67.50 per bundle. It is attached with fourpenny nails about every six inches on each stud and a skilled workman can cover about 100 square yards (with no complications) in an 8 hour day at wages (1972) of \$5 to \$8 per hour.

3. Plastering is done in 3 coats on metal lath to a thickness of $\frac{1}{2}$" to 1" and often in 2 coats on gypsum lath with a usual thickness of $\frac{3}{8}$".

4. Common brick is $3\frac{3}{4}$" × $2\frac{1}{4}$" × 8". The amount of wall space that can be covered by bricks depends on how thick the mortar bond is between the rows of bricks and at the ends of the bricks; obviously, with a $\frac{1}{2}$" thick mortar bond you will need fewer bricks than with a $\frac{1}{4}$" mortar bond. (A $\frac{1}{4}$" mortar bond means $\frac{1}{4}$" all around each brick; hence, the joint between bricks is $\frac{1}{2}$" thick.) This makes quite a difference; for example, with a mortar joint of $\frac{1}{4}$" all around each brick, one needs about 698 bricks to cover 100 sq. ft., but with a mortar joint of $\frac{1}{2}$" one only needs about 616 bricks; in both cases for a wall one brick thick. (You might try to figure out how the numbers 698 and 616 were arrived at and if they are correct.)

5. Roofing is frequently done with "three tab" strip shingles that are each

36" × 12", but they overlap on the roof, and only 4 of the 12 inches are actually exposed to the weather. With that overlap, how many layers of shingle are there at any point on a finished roof? Each such shingle is secured by 6 nails. The unit used in roofing is the "square" which means 100 square feet of finished roof. In selling strip shi ies a square of shingles means enough shingles to finish a square of roof. The roofing felt often used underneath the shingles comes in rolls that are 3 feet wide by 144 feet long, costing about $11 per roll and secured with about 2 pounds of nails per square. A square of felt weighs 15 pounds. Nails cost about $20.00 per 100 pounds. The cost of "three tab" strip shingles made of asphalt is about $8 per square; a square weighs about 210 pounds. A carpenter or roofer can install a square of roofing in about 2 hours on an ordinary roof. Carpenters' wages are similar to lather wages given in item 2.

6. Wall paper comes in single rolls that are 18" wide and 24 feet long and in double rolls that are 18" wide 48 feet long. The cost ranges from $.75 to $20.00 per single roll.

7. In figuring the cost of a painting job, one must of course consider the area to be painted, the coverage of paint, the cost of paint, and the painter's wages. For figuring area, you're on your own. The coverage of paint is frequently given on the label; on wood some common coverages are 450 square feet per gallon for the first coat and 500 to 600 square feet per gallon for second and third coats. Generally speaking, one can pretend that there are no windows and doors (in order to compensate for waste) unless a large proportion of the area to be covered consists of windows, doors, and other things that are not to be painted.

8. The paper you are working on is probably $8\frac{1}{2}$" × 11" and you may have worked on 3 × 5 or 4 × 6 or 5 × 8 cards taking notes for an assignment. Books tend to come in several standard sizes. The reason that the same paper sizes come up in so many places is that they are cut from standard size larger sheets. A common problem for a printer, paper retailer, or wholesaler is to figure out the most efficient way to print and cut papers with the least waste. See if you can recognize in the information that follows how some common sizes of paper may have been cut.

   a. Although there are many kinds of paper, the following cover the majority of basic sizes: Newsprint 25" × 38"; book paper 25" × 38"; cardboard $25\frac{1}{2}$" × $30\frac{1}{2}$"; blotting paper 19" × 24"; bond 17" × 22" and paper for covers 20" × 26". Paper has a grain running in one direction and it is easier to fold or tear the paper with the grain than against the grain. (Verify this.)

b. Standard specifications are given in basic sizes with available weights per ream. (A ream is 500 sheets.) For example, this might be the listing:

Basis 17 × 22	16	20	24

This means the basic size for that paper is 17" × 22" and it comes at 16, 20, or 24 pounds per ream of 17" × 22" sheets. What would be the actual weight per ream of $8\frac{1}{2} \times 11$ paper cut out from 20 pound 17 × 22 stock? The underlining gives the dimension along which the grain runs.

c. For composing and laying out type we use a system approved in the year 1886. A point is very nearly $\frac{1}{72}$ of an inch (exactly 0.13837 inches). Twelve points is one pica; six points is $\frac{1}{2}$ pica, or one nonpareil. The common type sizes in terms of points are 6, 8, 10, 12, 14, 18, 24, 30, 36, 42, 48, 60, and 72. Figure out how high (in inches) each of the common type faces is. Which one is nearest to the size of type used in this book? What about newspaper print? What about newspaper headlines of various sizes?

d. If you were laying out a page to be set up in type, you would need know the dimensions of the space you wanted covered and of the type you were going to use in order to estimate certain things about the layout. For spacing between words and between lines, the printer uses spacers called "leads" which are defined by how thick they are, measured in points. The unit of area in printing is the "em," which is the area taken up by the capital letter "M" in whatever type size is being used; this is because the M is very nearly a square. The area actually covered by print depends on whether it is "type set solid" or with leads between lines.

Chapter IV

FORMULAS AS MATHEMATICAL MODELS

As a preface to the first section of this book we talked about how one uses mathematics in everyday affairs by formulating mathematical models. Just as a reminder of what is involved here, consider carefully the following quotation:

> What is a Mathematical Model? The scientist makes observations and experiments and then tries to construct a theory to fit his results. This theory often takes the form of a mathematical model, say an equation, which enables the scientist to summarize his results and to predict the outcome of new observations and experiments. Such a model is a substitute for reality and is simple to work with. Usually it represents an idealization in the sense that the model does not fit the facts with absolute precision. The fit should be close enough...that description and prediction are accurate enough for practical purposes. ("Mathematical Models of Growth and Decay")

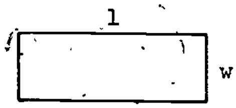
You will remember from our earlier discussion that in using a mathematical model you must also be aware of what "simplifying assumptions" were made in setting it up and that any results that you get from using such a model must be tested back in the world of reality.

4.1 Area and Volume Formulas

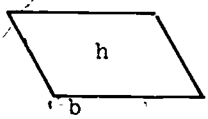
Most of the models we have dealt with so far in this book have involved more or less direct use of numbers in arithmetic calculations, or with measures, or with coordinates. We have already seen the many ways in which "measure" enters into our everyday lives and helps us investigate and describe the world we live in. In using mathematical models to investigate our world, a frequent result is a "formula" using several variables to express the relationships among measured quantities. For example, we seldom measure area directly by counting a certain number of square units; usually we measure certain linear dimensions and then use multiplication to get the area, as when we measure the length and the width of a rectangular room and then use the formula $A = lw$ to find its area. The usefulness of using formulas is even more apparent in the case of the areas of circles, where even if one wanted to go to the trouble of marking off square units and counting them, it would be difficult to accurately approximate the area because of the circular boundaries. It is easy, however, to measure the radius (r) of a circle and use the formula $A = \pi r^2$ to find the area. Even more difficult to find "directly," but easy to calculate using a formula model, is the surface area of a sphere; one measures the radius of the sphere and then uses the formula $A = 4\pi r^2$.

Problem Set 4.1

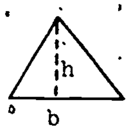
1. Here are some formulas for area:



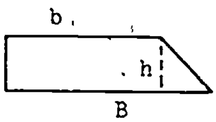
Rectangle: $A = lw$



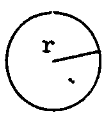
Parallelogram: $A = bh$



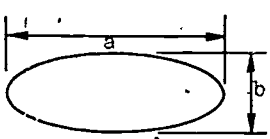
Triangle: $A = \frac{1}{2} bh$



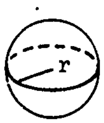
Trapezoid: $A = \frac{1}{2}(B + b)h$



Circle: $A = \pi r^2$



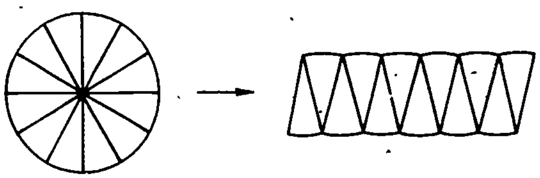
Ellipse: $A = \frac{1}{4}\pi r^2$



Surface of Sphere: $A = 4\pi r^2$

Surface of Right Circular Cone (not including base): $A = \pi rl$

a. Starting with a rectangle as being the basic kind of area figure, make sketches and see if you can figure out how the other common area formulas are derived, for example, by cutting up a parallelogram to make it into a rectangle and observing that any triangle is just half of some parallelogram (or half of a rectangle if it is a right triangle). Can you somehow cut up and reassemble a trapezoid and try to fit it into the scheme? Can you see from the diagram below how it would be possible to cut up a circle into very many pie-shaped pieces that could be fitted together to be approximately a rectangle with half the circumference ($\frac{1}{2}$ of $2\pi r$) for the base and the radius for the height and hence have area of about $\pi r \times r$ or πr^2?



Can you see how by unrolling a cylinder into a rectangle it would have the circumference of the circular base as one dimension and the height of the cylinder as the other? Try this last by rolling up a piece of paper into a cylinder and then unrolling it.

b. Use the area formulas that are given to approximate the surface area of your own body by modeling your head as a sphere; arms and legs and body as cylinders.

c. Use the area formulas given to find the area of a number of things around you; try to find some fairly irregular shaped things that can be approximated by dividing them up into a series of shapes that you can find the area of.

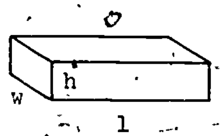
d. The fact that "A," "b," "h," and other letters appear in a number of different formulas above illustrates one of the hazards of using formulas. There are only a limited number of letters in the English alphabet available to us (twenty-six capital and twenty-six lower case symbols), so they get used over and over again in formula models, and they don't always mean the same thing. Try to find some other formulas in common use which use the letters "A," "b," "h," "r," "l," "w" in ways that are not closely related to their use in the formulas above.

e. I find it surprising that the surface of a sphere is exactly 4 times the area of a circle with the same radius. Do you? Can you think of any reason why this should be?

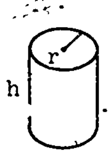
f. Start with a circular disk of paper and make a right circular cone, then unroll it and see if you can figure out from the flat shape why it should have the area formula that it does.

Comment: Notice that in all of the areas above, you end up either multiplying two linear dimensions together (in the case of the trapazoid you first have to find the average length of two bases) or by squaring one of the dimensions. As a matter of fact, area calculations nearly always come down to multiplication of two linear dimensions or squaring of a single dimension. (It may happen that to get a surface area several such products are <u>added</u>. Store this bit of information away for use in a later section of this book.)

2. Here are some formulas for volume:



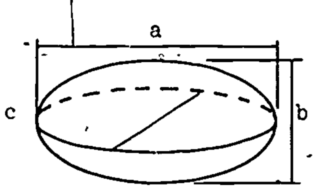
Rectangular "Box": $V = lwh$



Cylinder: $V = \pi r^2 h$

Circular Cone: $V = \frac{1}{3}\pi r^2 h$

Sphere: $V = \frac{4}{3}\pi r^3$



Ellipsoid: $V = \frac{4}{3} abc$
(a, b, c the semiaxes)

a. Model your own volume as you did your own surface area above by considering your head as a sphere and your various body parts as a series of connected cylinders. With this figure for your volume and the figure for your weight, figure out your density. Compare this with the density of water. (Water weighs about 62.4 pounds per cubic foot.) Except for the air spaces in your lungs, your body is fairly solidly filled up; do you suppose your density would still be such that you would float in water if it were not for the air spaces provided by your lungs? Find the approximate volume of your lungs, and re-compute your density with this space deleted.

b. Observe again that letters are used here in a variety of ways. Again, find the places where the letters in the formulas above are used in other formulas which have little or nothing to do with volume or area.

Comment: Observe that in the formulas for volume we always seem to be multiplying together three linear dimensions; squaring one linear dimension, then multiplying it by another linear dimension; or cubing one linear dimension. In fact, volume is characterized by being the product of three linear dimensions. Store this information away for use in later sections.

3. Here are 2 more formulas by which you can get the area of a triangle indirectly: the first by merely measuring the lengths of the sides of the triangle; the second (and less practical) by knowing the lengths of the sides of a triangle and the radius of a circle through the 3 vertices of the triangle (is there always such a circle?).

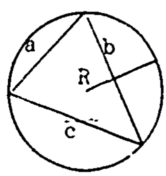
<u>Heron's Formulas</u>

$A = \sqrt{S(S-a)(S-b)(S-c)}$

where $S = \frac{1}{2}(a + b + c)$ and a, b, and c are lengths of the sides.

$A = \frac{abc}{4R}$

where R is the radius of the circumscribed circle.



Draw some triangles, preferably on graph paper, and verify that the formulas work by finding the area with these formulas and also with the standard formula. If you don't know how to find the center of a circle that goes through three points, you may want to check out the second formula by first drawing the circle, and then drawing a triangle by selecting three points on the circle. See if you can find some practical situations involving triangular shapes; make the measurements, and find the area by the first of the two formulas. If you have had enough algebra and geometry, try to figure out where Heron's formulas came from; that is, derive them using other facts of geometry and algebraic manipulation. Observe that the first formula involves multiplying together four lengths and then taking the square root of the product, so in effect, our general rule that area involves "squaring" one length or multiplication of two lengths, is not violated. Comment on whether the second formula fits this general picture of area as basically involving multiplication of two lengths.

4. An interesting extension of Heron's formula to the case of a quadrilateral inscribed in a circle is given below:

$$A = \sqrt{(S-a)(S-b)(S-c)(S-d)}$$

$$\text{where } S = \frac{1}{2}(a + b + c + d)$$

Draw such a quadrilateral on graph paper and verify that the formula will work. Observe that if a triangle were considered to be a "quadrilateral" with one of the "four sides" of zero length, then this formula would exactly give you the formula in problem 3. How else could you find the area of a (convex) quadrilateral that is not a rectangle or a parallelogram or a trapezoid? Would an appropriate "Heron's formula" work for a five-sided figure inscribed in a circle?

5. It is very common to measure quantities "indirectly." For example, temperature is often measured by measuring a length, namely, the height of mercury or some other liquid in an enclosed tube. Could you tell "directly" whether the outside temperature were 65° F (fahrenheit) or 60° F, or if your body temperature were 99° F instead of the "normal" body temperature of 98.6° F? Find and list as many other measures as you can that are determined not "directly" but by measuring a length, volume, deflection of a needle a certain number of degrees, or by some other means. If anybody you know has an automobile that still has a full scale instrument panel instead of the type that car buffs refer to as "idiot lights" (lights which flash on when some disaster is impending, such as "out of oil," but do not give continuous readings of such things as oil pressure and temperature), see how many things on that panel are indirect measures of something quite different from what is actually being read. If cars interest you, try to find out what the mechanism is that converts what is going on in the engine into a measure on a dial.

6. We can often assess the relationships among variables in a formula by assigning "constant" values to one or more variables and then observing what happens as other variables change in value. For example, in $A = \ell w$ (area of a rectangle), if the length (base) is assigned a constant value, then it is easy to see that the area gets larger as the width (or height) increases and gets smaller as the width decreases. Investigate the relationships among the variables in the other formulas above, then store this information away as one way of coming to terms with the formulas in the sections that will follow.

4.2 Variables, Constants, Careful Substitutions in Formulas

In using formulas it is very important to make sure you know what each letter (variable) in the formula means and what the appropriate replacements for it are. Hence, when a formula is given, each of the variables should be clearly defined and the units that go with the measurements that replace each variable should be specified. Here are some problems illustrating this point.

Problem Set 4.2a

1. A method for finding the day of the week for any date based on our current calendar follows:

Given: d = the day of the month of the given date
m = the number of the month in the year, with January regarded as the thirteenth month and February the fourteenth month of the previous year (for example, 2-13-1972 = 14-13-1971)
y = the year

Compute: $W = d + 2m + [\frac{3(m+1)}{5}] + y + [\frac{y}{4}] - [\frac{y}{100}] + [\frac{y}{400}] + 2$

(Each expression in brackets means the integer part of the quotient; for example, $[\frac{15}{4}] = 3$.)

Then the remainder, when W is divided by 7, is the day of the week of any given date, with Sunday the first day of the week, and Saturday the seventh or zero day (since $7 = 0$, mod 7). (From A Source Book of Applications, page 103.) Figure out the day of the week you were born on, and the day of the week your birthday will be on three years from now. There are many places to go astray in this formula, but if you work it out carefully, exactly according to the definitions of the variables, you will find that it always works for dates given by our current calendar. The occurence of leap years affects which day of the week a given date will fall on; can you see what in the formula takes care of that? Try to figure out what is the contribution of each of the various parts of this formula to the problem at hand.

2. Here is part of a simple econometric model of the United States based on data from before World War II and developed by Professor L. R. Klein:

   a. $C = 16.8 + .02P + .23P_1 + .8(W_1 + W_2)$

   This equation relates consumption (in billions of dollars) with profits this year (P) and the year before (P_1) and with wages in the private economy (W_1) and wages paid by the government (W_2).

   b. $I = 17.8 + .23P + .55P_1 - .15K_1$

   This equation relates investment with profit this year and the year before and with the stock of capital at the end of the previous year (K_1).

   c. $W_1 = 1.6 + .42X + .16X_1 + .13(t - 1931)$

   This equation relates wages paid by private employers with production this year (X), production the previous year (X_1), and the actual year stated numerically (t).

   d. $X = C + I + G$

   This equation states that the production of a given year is divided between consumption, investment, and the portion used by government. (Harry Schwartz, "America the Mathematized," New York Times, Nov. 7, 1969.)

Comments and questions on the above model: Notice that the formulas are not independent; that is, some of the variables that appear in the first formula also appear in the second and succeeding formulas. All the variables except "t" apparently have billions of dollars as the appropriate unit for replacement. Notice that many of the variables have a "coefficient" preceding them; for example, in the first one the coefficient of "P" is .02, the coefficient of "P_1" is .23 and so on. The coefficient gives a sort of "weighting" to each of the variables and gives you some idea of their relative importance. Notice that many of the formulas begin with a constant figure not attached to any variables. This suggests that even if the variables in the formula had the value of zero, there would still be something going on in the economy. The coefficient of "(t - 1931)" in formula C suggests that there has been a 13% increase in wages for every year since 1931. Try to interpret some of the other components of the formulas above. It may be a little hard to manage, but you might try to get appropriate information for a recent year on profits, wages, and the like, and check out whether the model works pretty well.

5. Mathematics in Everyday Things tells us that, for psychological reasons, it has become popular for merchants to inflate list prices in order to be able to offer large "discounts." Suppose that:

p = the fraction of the actual selling price which the dealer will keep as profit

d = the fraction of the list price which the dealer will deduct as the discount

C = the wholesale cost to the dealer

S = the actual selling price

L = the list price to yield S after discount d

The merchant then uses a double-markup procedure. His first markup is computed on the basis of the gross profit on actual selling price. He first adds the profit he wishes to make to the wholesale cost C to get the actual selling price S. Since p was defined above as the fraction of S which will be profit, the amount of profit will be $p \times S$. Therefore, we have the formula $pS + C = S$, which can be changed to the form $S = \frac{C}{1-p}$ by simple algebraic manipulations.

The merchant then calculates the list price L so that when the discount is made, it will leave the actual selling price S that was calculated as in the preceeding paragraph. Since d is the fraction of L which will be discounted, dL is the amount of the discount. Therefore, we have the formula $L - dL = S$, which can be changed to $L = \frac{S}{1-d}$; combining this result with the first formula we get $L = \frac{C}{(1-p)(1-d)}$. This looks complicated but it is easy to apply. Figure out what price tag you would put on a transistor radio costing you \$5 wholesale on which you want to make a profit of 30% on actual selling price and that you want to advertise as being sold at "40% DISCOUNT!" It is reported that some manufacturers print many different "suggested retail" price tags for the same article, thus enabling the retailer to use the one most in line with his own discount policy.

In the econometric problem above we observed that the "coefficient" attached to a variable often indicates the importance of that variable in the model. Another way of judging the influence of the given variable in a formula model is to observe where it comes in the structure of the formula and hence how much influence changing its value has. For example, in the formula

$P = 2i$ versus the formula $P = i^2$, replacement of i by 10 yields a value for P of 20 in $P = 2i$, but of 100 in $P = i^2$. Similarly, i has more important influence in the formula $P = 2i$ than in the formula $P = 2 + i$. The next problem set gives some more subtle instances of how the structure of a formula can determine the influence of a given variable:

Problem Set 4.2b

1. Einstein's relativity theories have some ra er curious and paradoxical consequences. One of them is the speculation that if a man were to take a trip in a spaceship traveling at near the speed of light (about 186,000 miles per second) he would travel for many years in space, and on return to earth, his friends would have grown old by earth time, but he could appear to have aged very little. The closer he could come to traveling at the speed of light, the less he would appear to have aged. If we take t as elapsed time relative to such a spaceship, T as earth time, c as speed of light (about 186,000 miles per second), and v as the speed (velocity) at which the spaceship travels, then the relationship between the spaceship time experienced by the space traveler and earth time is approximately given by the following formula: (We are ignoring the periods of acceleration to speed v and deceleration in slowing down to land.)

$$t = T\sqrt{1 - (\frac{v}{c})^2}$$

   a. In the structure of this formula it is clearly the ratio of the velocity you are traveling at to the speed of light that is the important thing. You must be sure that the two speeds are in the same units; for example, both in terms of miles per second. How many miles per second would ordinary automobile speeds of 60 miles per hour be? If that value were substituted into the formula, would it make t different from T in any but a very small amount?

   b. The speed of a space capsule such as we use in traveling to the moon reaches about 18,000 miles per hour. How many miles per second is that? Does such a speed have much effect in the formula; that is, do our moon tronauts come back from their trips much younger than if they had stayed on earth?

   c. Any ratio such as $\frac{v}{c}$ can be expressed as a decimal or as a percent; for example, we could express it as the percent of the speed of light. Suppose we could travel at 1% (.01) of the speed of light--

a mere 1,860 miles every second. Then the formula would be $t = T\sqrt{1 - (.01)^2}$. Would traveling at this speed through space let you stay much "younger" than people who stayed on earth?

d. The closest star (other than the sun) is Proxima Centauri, which is about 4 light years away. (How far is a light year?) Suppose of 2 newborn babies, one was put on a spaceship headed for Proxima Centauri at 99% of the speed of light. If the spaceship headed back immediately after arrival at the star, the round trip would take about 8 years, since it is traveling nearly at the speed of light and Proxima Centauri is 4 light years away. Using T as 8 and $\frac{v}{c}$ as .99, we need the value of t. In other words, what would be the apparent age of the spaceship baby? Are you convinced by now that given the structure of this mathematical model of time relationships in relativity theory, the variable v has little effect unless it is very large?

(Problems above were suggested by "On the Clock Paradox in the Relativity Theory.")

4.3 The Many Uses of Formulas with Same Structure: A = BC

Consider the formulas A = lw for the area of a rectangle and D = rt for the distance traveled by something traveling at a given rate for a certain length of time; for example, 50 miles per hour for 2 hours will take you 100 miles. From these examples it is clear that formulas can be mathematical models for very different things and yet be essentially alike in all respects except possibly the letters used in writing them down. (A technical term for "essentially alike in structure" is the word "isomorphic.") All of the formulas given in the next problem set will be "isomorphic" to each other and to the above two formulas. Another fact we wish you to observe in the following problem set is that there is a sort of "algebra of units" that can be used on measurement units as they appear in formulas. For example, in the formula $A = \ell w$, if ℓ and w are both in feet then the answer is in square feet, which could be regarded, just as in algebra, as being feet × feet. Similarly, since miles per hour is a ratio, miles/hour, then in the formula D = rt, one could write (miles/hour) × hours. The hours will "cancel," since one is in the denominator and the other a multiplier, and we will be left with simply miles, which is a proper unit for distance. Such an algebra of words is frequently appropriate either for determining the correct unit for an answer or for checking up on yourself to see if you are doing the right thing. If the algebra of words gives a ridiculous unit, then you are probably multiplying or dividing by the wrong thing. On the other hand, some of the measures used in science (especially electricity) are defined in ways that make it difficult to sort out this algebra of words, although if one were to trace the words back to their basic definitions, the algebra always works out. In the work that follows you should keep trying to do this manipulative algebra of units to see where it does and doesn't give nice results; being good at such a word algebra is frequently very useful.

Problem Set 4.3a

1. $D = vt$ is the formula for distance given the velocity (speed) and time. (You often see this in algebra books as $D = rt$, where r means "rate.")

   a. Problems in this area are very numerous. If you know even a little bit of algebra, you can find any one of distance, velocity, or time if you know what the other 2 are. Make up a few plausible problems which can be solved using the above formula.

   b. In this mathematical model some simplifying assumptions have obviously been made. For example, we speak of an automobile traveling at 50 miles per hour for 2 miles, but it is extremely unlikely that the rate is exactly 50 miles per hour all of the time, so we must be speaking of an <u>average</u> velocity. Are there other simplifying assumptions that are made in applying such a formula?

   c. The formula $D = vt$ gives, for example, the distance in miles traveled from the starting place in t hours at v miles per hour. We might want to consider a case where we have a "head start," that is, where we are already a certain distance d_0 from a given place and want the total distance. In that case the formula $D = vt + d_0$ might be more appropriate. Does this make sense to you? For example, if one were on a cross country automobile trip west from Chicago and starting out on the second day from Omaha, what would be the appropriate d_0 to use for keeping track of distance <u>from Chicago</u> at a given miles per hour for a given time during the second day?

2. $V = at$ gives the velocity after a certain lapse of time t and at a given acceleration a. For example, automobile specifications and results for certain kinds of speed contests are sometimes given in the time it takes to reach a certain speed. Here are some acceleration times from the handbook on the Mercedes Benz model 180A:

0 - 30 mph	5.6 sec.
0 - 60 mph	18.1 sec.
0 - 70 mph	28.2 sec.
40 - 60 mph	10.2 sec.

a. For each of the above find the average acceleration. Clearly this depends upon change in velocity over a given period of time. (The units that result might be $\frac{\text{miles}}{\text{hour second}}$, which can be read as "miles per hour per second." Does that make sense to you?) If you are interested in automobiles, find out if this represents pretty good acceleration or if it is relatively slow as automobiles go. What would a really impressive acceleration from 0 to 30 miles per hour be? From 0 to 60 miles per hour?

b. If the formula were taken literally, a car that was not accelerating, but simply going along at a steady speed, would have no velocity, for the acceleration would be 0 and 0 times anything would be 0. Hence, the more useful formula would be of the form $v = v_o + at$, where v_o is the velocity at which the car is already traveling. Here there may be complications in the measure units because velocity is given in miles per hour, but acceleration usually takes place over seconds or minutes in an automobile. Suppose someone were cruising along at 40 miles per hour and accelerated at the rate of 50 miles per hour per minute for a period of 30 seconds. How fast would he then be going?

3. In $W = rt$ we might be talking about the wages of a man for t hours at r dollars per hour. What might a formula such as $W = rt_1 + 1.5rt_2$ mean?

4. $C = np$ could be total cost of a certain number of articles at a price p per article. Think up a few problems like this and work them out. The formula $C = knp$ might reflect the cost during a sale or at a certain discount; for example, if there is a 10% reduction on the price of everything in a store, then one would expect that $C = .9np$ would apply to whatever one bought. Make up a few problems that appropriately use formulas of the type $C = np$ and of the type $C = knp$.

5. $e = ir$ is the so-called "Ohm's Law" and is much used in electrical circuits. Here e is the electrical pressure given in volts, i is the current given in amperes (abbreviated as "amps"), and r is the resistance given in ohms, to the flow of electricity. A useful analogy in thinking

about what is going on in electrical circuits is to think of water in a hose, with e corresponding to the water pressure, i corresponding to the amount of water actually flowing, and r to the friction and other resistance to flow. Think about how that would work and see if the interrelationship between the variables seems to fit the analogy. For example, suppose that for a constant flow of water the size of the hose is reduced and hence the resistance to the flow increased; this would increase the water pressure or the force with which water is ejected from the hose. Similarly, if the amount of water flowing through a hose were increased without the size or anything else about the hose changing, this would very likely increase the pressure or the force with which the water is ejected from the hose. (In these problems on electricity, a word algebra with the traditional units (volts, amperes, ohms) will <u>not</u> work out. All these units can be expressed in more basic units so a word algebra could work, but we won't do so here.)

a. Standard house current is usually between 110 and 120 volts; let us take it as 120 volts for convenience. Fuses or circuit breakers are put into house wiring circuits to keep the wires from becoming overloaded. What is the minimum resistance (in the form of irons, toasters, etc.) that could be handled by a circuit with a 20 amp fuse before the fuse would blow out and interrupt the flow of electricity?

b. Most modern automobiles have 12 volt batteries and generators (or alternators) that can supply up to 30 or 40 amperes of current. How little resistance can such a automobile electrical system handle?

c. Do you see why "short circuits" (near zero resistance) can cause trouble in houses and cars?

d. The amount of current that an electrical wire can safely carry without overheating and possibly causing a fire is pretty much a function of its size. From an electrician (or perhaps from a shop teacher in your school) get some information about how much current common sizes of house wiring will safely carry (two common sizes are number 12 wire and number 14 wire) and see what this has to do with the sizes of fuses or circuit breakers that are put into electrical circuits in your house. If is, of course, unsafe to put in such a large fuse that it is possible to overload the size of wire that is in that particular electrical circuit.

6. Most uses of electricity depend upon the amount of power delivered, and

for that you use the formula $P = ei$, where power is in watts, e is again the pressure in volts, and i is again the current in amperes.

a. Considering again that most house wiring supplies 120 volts and that a frequent circuit breaker size in a house is 15 amps, what is the maximum power load in watts which that circuit would carry? Most appliances that produce heat, such as toasters and irons, have power ratings of 1000 to 1500 watts; most lights 100 to 200 watts. Can you see why it is usually recommended to have one or more separate circuits just for the outlets in the kitchen? Compare the power demands of the appliances your parents have in the kitchen with what might have been the case, say, 30 years ago. You might ask your parents to make this comparison.

b. Look at the name plates on several appliances and find out what their power requirements are. Then have a look at the fuse box or circuit breaker panel for your house and see how many amperes of current are allowed for the various circuits. Make some estimate of what appliances can be hooked up in the various circuits without blowing the fuses or the circuit breakers. Appliances are the real problem; how many 100 watt bulbs would have to be burning at once in order to draw the same power as does your toaster or waffle iron?

7. $F = ma$ says that force is the product of the mass of an object and the acceleration of it. This just reflects your ordinary experience with, for example, an automobile that is accelerating and pushing you back with some force against the seat or decelerating (negative acceleration), where you are pitched forward as if there were some force pushing you. Similarly, when an elevator starts or stops, you feel pressed against the floor or almost lifted off the floor, but when it is traveling at a steady speed, you feel essentially nothing.

a. According to the formula, a constant acceleration but an increase in mass gives an increase in force. Interpret that in terms of your experience.

b. Similarly, for the same mass, increased acceleration would give increased force. Can you from your experience think of situations where a very small object, hence, a small mass, exerts considerable force because of rapid acceleration or deceleration?

c. With respect to the last question, look into the way that "atom smashers" work--very tiny masses are involved and enormous velocities resulting in enormous deceleration when they hit their targets.

d. Interpret the differences in the following two events. An automobile traveling at 40 miles per hour, hitting something solid like the foundation of a bridge, experiences considerable ~~force, to say the~~ least. An automobile travelling at 40 mph hitting the rear of another automobile, the brakes of which are not on (so that the stopping is extended over a longer distance and hence a longer period of time) experiences considerably less force.

e. Automobile safety experts have said that there would be considerably less damage in automobile accidents if shock-absorbing bumpers were installed on all cars; these essentially allow the bumper to move a certain distance while travel of the automobile is arrested. What does this have to do with the forces that will be inflicted on the automobile?

f. As far as the forces involved are concerned, is there any difference between acceleration and decceleration?

8. We remarked earlier in this booklet that the weight of people or of objects is different on earth than on the moon and even different on different places on earth; for example, one weighs slightly less on the top of a very high mountain than at sea level. Yet, as we have observed earlier, there is something about people and objects that does not change no matter where they are located. That something is called mass. Weight is a force, and the relationship between weight and mass is a special case of the formula just given, namely $F = ma$. In this case the formula is $W = mg$, where W is the weight and g is acceleration given any object by forces of gravity. An average value for the acceleration of gravity on the surface of the earth is about 32 feet per second every second (or in terms of a word algebra, 32 ft./sec.^{2}). But in ordinary everyday life we nearly always report masses as if they were weights. In the system of units we use (feet the unit for length, seconds the unit for time, pounds the unit of weight or force), the correct unit of mass is called the "slug." From the formula $W = mg$, we see that one slug of mass weighs about 32 pounds on the earth's surface.

a. How much do you weigh? Using g as 32 ft/sec^2, what is your mass in slugs?

b. The force of gravity on the ~~surface~~ of the moon is about $\frac{1}{6}$ as great as on the earth, and hence, the acceleration of gravity, g, is also about $\frac{1}{6}$. Given the figure you just obtained for your mass, and using $\frac{1}{6}$ of 32 ft/sec^2 for the acceleration of gravity, how much would you "weigh" on the moon?

c. The acceleration of gravity, g, on Mars is about 12.5 ft/sec^2; how much would you weigh on Mars?

d. The differences in weight of the previous couple of problems would only be observed if you were weighing yourself on an ordinary spring resistance scale such as most bathroom scales are. If you were weighing yourself on a balance scale such as is used in most doctors' offices, where a weight is moved along a bar and compared with your weight, your results would be the same on the moon as on the earth. Why?

e. The m in the formula $F = ma$ means mass. Suppose a 4000 pound automobile were involved in a collision that reduced its velocity from 40 miles per hour to 0 miles per hour in 5 seconds. Figure the mass of the automobile and the deceleration involved and then find out how much force is involved in the collision.

Frequently in using formulas it is useful to change from one form of the formula to another. For example, with $e = ir$, the equivalent version that gives the current directly from the voltage and resistance is $i = \frac{e}{r}$. If you haven't had enough algebra to see why these two formulas are just two different versions of the same thing, then skip the next problem set. Otherwise, continue.

Problem Set 4.3b

1. For each of the formulas of the form $A = BC$ in Problem Set 4.4a, imagine a situation where you would want some alternate form with another variable isolated. For example, in $D = vt$ you may want to concentrate on velocity instead of on distance, and so you would want the form $v = \frac{D}{t}$.
2. One frequently combines two formulas into a single formula. For example, the power formula in problem 6 in Problem Set 4.3a above is $P = ei$; e is given by problem 5 in that same problem set as $e = ir$; hence, this substitution could be made: $P = ei = (ir)(i) = i^2r$. This gives power in terms of current and resistance to current. In a radio circuit, if the current is .003 amps (3 milliamps) through a 20,000 ohm resister, what power is involved? (Such calculations are important in electronics.)
3. It ought to be possible in a similar way to combine the formulas $d = vt$

and $v = at$, but the situation is more complicated. The v in $v = at$ represents how fast you are going at the very end of the acceleration; you certainly haven't been going that fast all along. It would make more sense to say that the distance you have gone is the same as if you'd been traveling at the average velocity for that time period. Since the velocity has increased steadily, the average is obtain adding your initial velocity (0 in this case) to your final velocity (at), then dividing by 2. That is, $v_{avg} = \frac{(0 + at)}{2} = \frac{1}{2}at$. Then the distance you've gone is given by $d = v_{avg}t = (\frac{1}{2}at)\,t = \frac{1}{2}at^2$. For a freely falling body, acceleration is the acceleration of gravity, which near the surface of the earth is about 32 ft/sec^2, hence for such a freely falling body we get the familiar formula $d = \frac{1}{2}(32)\,t^2 = 16\,t^2$. How far would a fairly heavy object, (i.e., not subject to air resistance and other things ignored by this model) fall in 5 seconds if dropped from a sufficient height? How fast would it be going at the end of the 5 seconds? How far would the same object fall in 5 seconds if acted upon by moon's gravity? How fast would it be going?

4.4 A Potpouri of Formulas

Formulas often have the useful property of enabling us to see relationships that, in the real situation, are obscured by too many distractions. For example, a formula may tell us whether the relationship between two variables is a linear one, (the distance one goes in an automobile at 50 miles per hour is linearly dependent on time: $d = 50t$) or a quadratic one (the distance in feet something falls in 5 seconds is approximately given by: $d = 16t^2$). The formula model may also reveal inverse relationships; that is, relationships such that if one of two measures increases the other must decrease. One such case is when the product of two variables is a constant, as in $vt = 100$; another is when a variable is in a denominator, as in $v = \frac{100}{t}$. (Both formulas give the relationship between time and velocity in going 100 miles.)

A number of formulas are listed below, mainly to indicate how widespread is the usefulness of such models. The ones listed here are representative of thousands of such mathematical models in dozens of fields. You are not expected to understand how the formulas were arrived at. However, for a number of formulas of your own choice or assigned by your teacher you should respond to each of the following:

A. What is related to what, and how? (linear, quadratic, cubic, inverse, inverse square, square root, etc.)

B. What measurements are necessary to supply numerical replacements for each of the variables?

C. Wherever possible, check out the "word algebra" of the units of measure in the formula.

D. Many of the formulas have a "K-factor" which adjusts the formula model to particular situations. When such a "constant" is in a formula, consider what "adjustments" it may be intended to make. (For example, in a formula for radioactive decay, k would be different for uranium than for carbon; in the formula for amount of expansion of something when heated, k would be different for copper than for iron.)

E. What would be some reasonable inputs? Some reasonable outputs? Make at least one substitution where you replace all but one of the variables with numbers and get an output for the remaining one.

F. What experience, experiments, etc., might have led to the formulation of this model?

G. What are some of the simplifying assumptions inherent in the model? That is, what things about the actual situation are being ignored? Under what circumstances, if any, would the simplifying assumptions not be valid? (For example the "falling body formula" $S = 16t^2$ ignores air resistance and many other things, yet in most cases tells very accurately how far something in free fall will travel in a given time. But if air resistance were truly unimportan , sky-diving with parachutes would not be a very popular sport.)

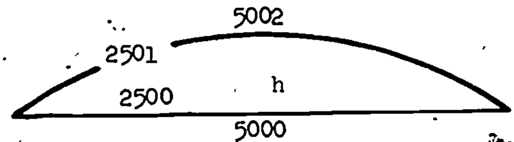
Problem Set 4.4

1. One big problem that engineers face in building bridges, roads, skyscrapers, and other structures is that most materials expand with an increase of temperature and contract with a decrease in temperature, so structures must be made expandable or flexible to keep from breaking when that happens. (Many thermostats, thermometers, and other control devices make good use of the expansion of metals with heat.) For this reason, in building highways it is necessary to provide for expansion joints every once in a while or the highway will expand and buckle and break on a very hot day. According to one source, here is a formula such as engineers might use in figuring out how much expansion to allow for:

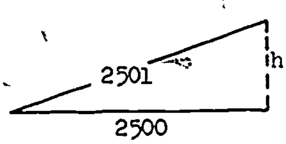
$$I = k\ell(T - t)$$

where I is the expansion (given in the same units as ℓ) at temperature T (degrees Fahrenheit--F^0), t the temperature at which the highway was built (F^0), and ℓ is the length of highway being considered. According to one source, k = .000012 is a reasonable value for some (2 lane) highways. How much would one mile of roadway expand at $T = 110^0$ F if t were 70^0 F?

2. Suppose a 5,000 foot long railroad rail is solidly anchored at both ends (with no expansion joints) and expands in length by 2 feet on a very hot day, buckling up as shown:



First, guess how high the distance h in the diagram would be. Next, consider the picture below as a rough model; use the Pythagorean theorem to find h, and compare the result with your guess:



$$2501^2 = h^2 + 2500^2$$

Would you expect this approximation for h to be more or less than the "true" value? ("On the Teaching of Mathematics so as to be Useful" by M. Klamkin says that this result is within 10% of the more correct answer obtained with much more complicated mathematical models such as elliptic functions or circular arcs and trigonometry. If you model this situation as an arc of a circle, can you find a way to find h, given the "arc length" of 5002 and "chord length" of 5000?)

3. The average population density in Black Africa is only about 25 per square mile of total area, but 750 per square mile in cultivated areas. Much of the agriculture in that area takes the form of clearing a piece of land, farming it for 1, 2, or 3 years, then abandoning it and letting it lie fallow to build up the soil again while shifting to another piece of ground for the next several crops. In looking into the question of how many more people the land could support without changing the traditional method of farming, one geographer formulated the following model: $D = \frac{AC}{B}$, where D is the density per acre; A is the proportion of cultivatable land;

B the length of rotation (period of cultivation plus fallow period); C the number of inhabitants per acre of land cleared each year. For example, if A = .8 (80%), B = 8 (perhaps 2 years harvest and 6 years "rest" for the land), and C = 4, what is the potential density per acre? At 640 acres to the square mile, what is the potential population density per square mile of total area, assuming these are reasonable numbers to use? ("Models of Agricultural Activity.") This author concludes that population in some parts of Africa could increase by a factor of ten without changing traditional methods of farming.)

4. If one takes the populations of all cities in a given area and ranks them in decreasing order of population size, one source says that the size of each of the cities is related to the largest by this "rank size" rule: $P_r = \frac{P_1}{r}$, where P_1 is the population of the largest city in the area, and P_r the population of the r^{th} ranked city. That is, the 4th ranked city has about $\frac{1}{4}$ the population of the largest city. Check this out with cities in your own area and see if it works out. ("Models of Urban Geography and Settlement Location," pages 326-329. Sometimes the r is given another exponent: $P_r = \frac{P_1}{r^n}$, with n any positive number. For example, if r^2 were used, the 1, 2, 3 ranked cities would have populations related as 1, $\frac{1}{4}$, $\frac{1}{9}$, which might fit some situations better.)

5. If you have ever filled a can full of water and then punched holes up along the side of the can, you observe that the water shoots out further near the bottom of the can than at the top; that is, the velocity of the water appears to depend upon how deep the water is, as we would expect. Here is a formula that is said to be a model for this phenomena:

$$v = kh$$

where v is the velocity of water at depth h coming through a sharp edged frictionless hole. What are some of the things that might determine the size of the constant k in this formula? What sort of simple experiments might you devise to test out whether this formula really works or not?

6. If you have ever had the experience of either standing still while a high-speed car comes by you blowing its horn or being in another car as one comes towards you blowing its horn, you notice that the pitch of the sound seems higher coming towards you than going away from you. The phenomenon was first explained in 1842 by Christian Doppler. Here is the formula for the "Doppler Effect":

$$f = \frac{F\,(C \pm V_o)}{C \pm V_s}$$

($+ V_o$ and $- V_s$ if source is approaching observer; $- V_o$ and $+ V_s$ if source is receding from observer.) Where f is the apparent or observed frequency (cycles per second--cps); F the actual frequency (cps); V_o the velocity of observer (ft/sec); V_s the velocity of sound source (ft/sec) and C the maximum velocity of waves (about 1120 ft/sec for sound waves).

Suppose you were in an automobile going at 30 miles per hour and a man in a car was coming toward you going 60 miles per hour blowing his horn, which has a 600 cps sound. Figure out how many cps the apparent sound will have as you approach and then what the apparent cps will be as you are going away from each other after passing. If you know, or can find out, anything about musical sounds, comment on what the net effect of this would be as far as musical pitches of the sounds go.

The Doppler Effect is thought also to account for the so-called "red shift" observed by astronomers; except that here we are dealing with frequencies of various colors and with the speed of light. From observing a shift in color from some nebulae, astronomers have speculated that some nebulae beyond our own galaxy are moving away from us at speeds approaching the speed of light. (Adapted from Mathematics in Everyday Things, pages 16, 17, and 110.)

7. Weather and Health tells us that a body loses heat partly by radiation (about 42%), partly by convection (26%), partly by evaporation of perspiration (18%), and partly through breathing (about 14%). Most of the heat loss from breathing is through evaporation of water from our breath, which in turn depends upon the temperature and humidity. Here is a formula that expresses this particular type of heat loss:

   $E_b = 5.4\ (62 - e_a)$ where E_b is heat loss per minute by evaporation through breathing (calories) and e_a is the prevailing "vapor pressure" of the air (millibars).

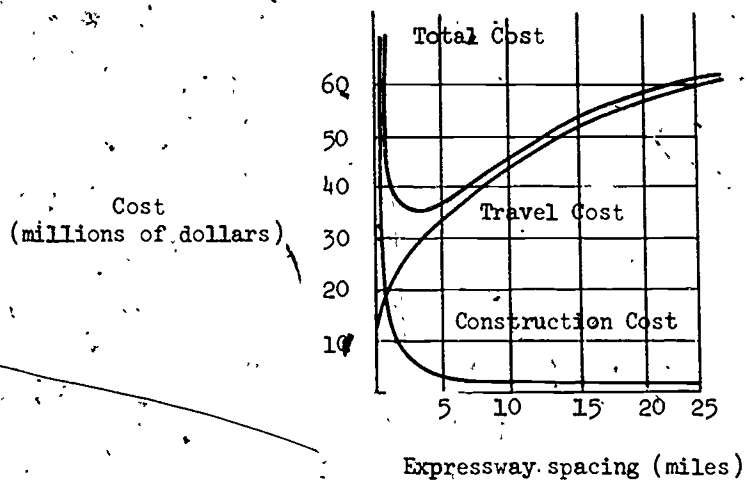
   The same source tells us that at a temperature of about 50° F and relative humidity of 50%, the vapor pressure is about 6.14 millibars. Under those conditions, what is the heat loss in calories per minute by evaporation through breathing? In that case, how many minutes breathing would amount to a kilocalorie heat loss? (Remember that kilocalories are the "calories" in diet books and popular literature.) How much heat loss from this source per hour? Per day? Considering the average calorie consumption of an average man (discussed in diet books and earlier in this booklet), what part of our calorie intake each day is dissipated through this particular source of heat loss?

8. Figuring out how many expressways to put in a city as well as othe. aspects of transportation policies is a very complicated business. It costs less to drive your car a mile on an expressway than on a crowded city street and the time saved is also worth something. On the other hand, expressways are very expensive to build. The Chicago Area Transportation Study (volume 3, 1962 p. 42) gives the following as a mathematical model for expressway spacing within a city, (that is, miles between expressways), which tries to take a number of factors into account:

$$Z = 2.24 \frac{C}{DKP_s (W_a - W_e)}$$

where Z is the best expressway spacing in miles; C is the average annual capital cost per mile of expressway; D is the "trip density" of the region for which the expressways are proposed (in vehicle equivalent trip destinations per square mile); K is a constant for converting travel cost differences between expressways and arterial streets to an annual dollar value; P_s is the proportion of all trips in that region that would use an expressway for part of their journey; W_a and W_e are the average cost per mile of travel on arterial streets and expressways. (Arterial streets are the city streets that carry the heaviest traffic.)

To make it easier for the readers of the report to figure out what is going on here, the report represents this information in a graph, with the assumptions that expressways cost about $8 million per mile to build; arterial streets are spaced about every half mile (as they are in Chicago); and an average of 20,000 daily trip destinations per square mile in the area:



The curves on this graph model show that travel costs are relatively low but construction costs are very high if expressways are spaced very close together, whereas, if expressways are very far apart, their construction cost is small but the total travel costs for motorists are relatively high. The total cost curve is found by adding the two curves; for example, add construction costs for freeway spacing of 5 miles to travel costs at that spacing to get total cost at that spacing.

a. According to the assumptions of these planners as shown in the graph, what is about the best expressway spacing for Chicago?

b. What different assumptions might be made that would change the results?

c. Are there assumptions that the planners have not made that they perhaps should make?

9. Clocks are sometimes governed by pendulums, and it is well known by now that the time a pendulum takes to make a full swing depends only on the length of the arm and not on the weight attached to the arm. (Do an experiment to verify this.) Here is a formula that is the model for this phenomenon:

$$T = 2\pi \sqrt{\frac{l}{g}}$$

where l is the length (in feet, meters, etc.); r is the acceleration from gravity (32.2 ft/sec or about 10 meters/sec.^{2} on earth); and T is the period for one swing of the pendulum (seconds).

a. How long a pendulum should you have in order for the period to be exactly one second?

b. For a child's swing hung from a frame or from a tree limb about 15 feet long, what will be the period of the swing?

c. Verify that the period for a given swing will be the same whether the swings go in short arcs or long arcs.

d. Because the acceleration of gravity is in the formula, you would expect things to behave differently on the moon than on the earth; how long should a pendulum arm be for it to have a period of one second on the moon? How would you adjust a pendulum clock to keep "correct" time on the moon?

10. Most cook books say that time for cooking a roast is proportional to the weight of the roast; some make corrections for smaller roasts. According to one mathematician the time is proportional to the $\frac{2}{3}$ power of the weight; that is:

$$t = kw^{2/3}$$ where t is time, w is weight.

(What this notation means, in case you haven't studied it yet, is that you first square the weight and then take the cube root of this squared value. w to the $\frac{3}{2}$ power ($w^{3/2}$) would mean cube the number and take the square root of the result.) The time can be in minutes or hours depending on how you fix the constant, k. Also, k might be different for different kinds of meat and for meat cooked to "rare," "medium," or "well done." I have the impression that if you want the time in hours, the range of k would be from about .4 to .7; look up some recommended cooking times in a cookbook and check this out. (From "On Cooking a Roast.")

11. Formulas very often play a role in skilled trades and in manufacturing operations. Here are a few examples:

a. Grinding wheels, pulleys, etc. are often attached to motors, and it is important to know how fast they are traveling at their rim. This obviously depends on how fast the motor is turning and how big the pulley or other disc is that is attached to the motor. The formula is:

$S = \pi DR$ where S is the surface speed in feet per minute (that is, the speed of a point at the rim); D is the diameter in feet; and R is the revolutions per minute that the motor is turning the disc (rpm).

A common speed for an electric motor is 1750 rpm; what is the surface speed of a grinding wheel with radius 8 inches? Similarly, the speed at which the blade of a bandsaw moves depends on the rim speed driving pulley; suppose an 18 inch diameter pulley is being turned at 800 rpm; how fast is the band saw blade going?

b. If one needs to find the length of the belt that goes around two pulleys, it's not so bad if the pulleys are the same size because the length is just the half circumference of each pulley plus twice the distance between them. (Draw a diagram and verify this.) If the pulleys are different sizes, things are a little more complicated because the belt touches more or less than half the rim of each pulley. Here is the formula that applies:

$$L = 2C + \frac{\pi}{2}(D + d)$$

where L is the total length of belt required; C is the distance between the centers of the 2 pulleys; and D and d are the diameters of the 2 pulleys. (L, C, D, and d must be in the same units.)

Make up a problem and answer it using this formula.

c. Gear arrangements are often used in machinery either to control speed or to control the amount of power delivered. If one gear has 30 teeth and a smaller gear has 15, every turn of the larger gear will turn the smaller one twice. More generally:

TR = tr where T and t are the number of teeth; R and r are the number of revolutions for the larger and smaller gears respectively.

Try to find some simple machines with the gears visible and verify this for yourself; for example, a hand operated drill, watch gears or clock gears, lawnmower gears, etc.

d. Screws and bolts are obviously used in a great many things. The number of threads per inch (n) and the so-called pitch of the thread (P) are related by the following formula:

$$P = \frac{1}{n}$$

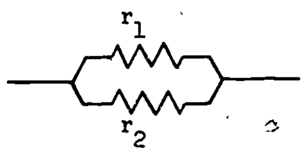
Pitch is the distance between peaks on 2 threads; verify to your own satisfaction that this formula makes sense. There are a number of different kinds of threads, as you may have found out if you have ever tried to put an ordinary machine threaded nut on a "Stove bolt";

it just doesn't work even if the bolt diameters are the same. Recent moves have been made to standardize the fairly confusing situation with respect to threads of nuts and bolts. If you are interested, look up some of this information, perhaps in Applied Mathematics, which is the source of the formulas in this problem. (Similar formulas appear in a great many trade manuals and books on mathematics for various vocational courses.)

12. We have said before that things that seem very different are frequently described mathematically by equations, formulas, or what have you that look very much alike. The 5 formulas that follow are examples of this, since some very different things are described by "isomorphic" expressions. As usual, you are not expected to understand all of the ins and outs of every one of the formulas below, but you should try to come to terms with as many of them as you can, verifying that they at least make sense and substituting some values for variables and working out results. For example, we use the first formula to find the total resistance to electricity flowing through a pair of resistors wired as shown. We know that the total resistance should be less than that of each of the resistors because there are more paths for the electricity to take; if you check out some values in the formula you will find that this is indeed the case.

a. Total resistance (ohms) in a parallel circuit:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

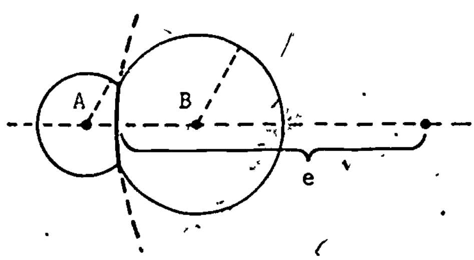


b. Lens: f is the focal length of a lens; d is the distance of object from lens; t is the distance of image from lens.

$$\frac{1}{f} = \frac{1}{d} + \frac{1}{t}$$

c. e is the radius of surface of intersection of two soap bubbles with radii A, B.

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{e}$$



d. S = Synodic period of a planet; i.e., the number of days it takes a planet to gain (lose) a whole lap around the sun with respect to the earth, P = Sideral period of planet (days to complete a circuit about the sun), E = Sideral period of the earth.

$$\frac{1}{S} = \frac{1}{R} - \frac{1}{E}$$

13. There are certain numbers that appear again and again in the investigations of science; π is one of these and the constant e is another, where e is an irrational (endless, non-repeating) number which begins $e = 2.171828. . . .$ (If you know how to deal with "factorial" notation, you may find it interesting to know that e can be approximated to any desired number of decimal places by the following expression:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + . . .$$

where $2! = 2 \times 1$, $3! = 3 \times 2 \times 1$, $5! = 5 \times 4 \times 3 \times 2 \times 1$, etc.).

Again, you are not expected to know enough to understand fully each of these formulas, but try to come to terms with as many as you can and follow up in detail on a few that especially interest you. Remember that the "k-factors" adjust the formulas for specific materials or situations.

a. $N = N_0e^{-kt}$.

Radioactive decay where N_0 is the initial number of atoms; N is the number of atoms after t hours. (A similar formula would apply in radioactive carbon decay measures used to date very old archeological remains.)

b. $N = N_0e^{kt}$

Population Growth of bacteria with ample food and space; N_0 is the initial population; t is the time; N is the population after time t.

c. $T = T_a + (T_b - T_a)e^{-kt}$

Cooling of heated object, where T_a is the temperature of air; T_b is the temperature of object when it starts to cool; T is the resulting temperature in t minutes.

d. $I = I_0e^{-kd}$

Intensity I of light passing through d centimeters of liquid, where

I_0 is the initial intensity of the light.

e. Every aspect of vision appears to be altered under water. These modifications in the appearance of the scene have a physical basis, for radiant energy is profoundly changed when it travels through water rather than air. The fundamental equation describing the transmission of energy is the same for air and water:

$$P = P_o e^{-kd}$$

where P is the radiant power reaching the distance d without loss, P_o is the radiant power at the initial point, e is the base of the natural logarithmic system, and k is the "extinction" or "attenuation" coefficient, and d is the distance.

The numerical value of k, however, is generally larger by a factor of 1000 or more for water than for air. The relatively large size of k means a rapid diminishing of light with increasing water depth. For example, if 90 percent of the initial light energy is transmitted through 1 meter of water, 81 percent will be transmitted through 2 meters and only 37 percent through 10 meters. (Suggested by "Underwater Vision.")

14. Most of you will sometime in your life have tied something fairly heavy to the end of a string and swung it above your head and observed that it develops quite a pull on the string. If you try it now, you'll observe that if you maintain a constant rate of turning that the pull on the string will increase as you let the string get longer. This is because in order to maintain the same rate of turning the object has to travel further in a larger circle than in a smaller circle, and hence, its velocity at the rim is higher. This force, called centripetal force if you are talking about the outward pull, is given by this formula:

$$F = \frac{mv^2}{r}$$

where F is the centripetal force, m is the mass (not weight) of the moving object, v is the velocity at the rim, and r is the radius of the circle.

(Remember that if you are dealing with an 8 ounce weight, you must first convert that to mass by using the relationship w = mg; if you are doing the calculation on earth, g is about 32 ft/sec^2.) Make up some problems and solve them using this formula. You could regard yourself as such a weight being spun in a circle as the earth rotates. What would your surface velocity be if you were at the equator? What would be the centrifugal force acting on you at the equator? Have you ever wondered why you don't fly off into space, as a rock would if you let go of the string? The next problem deals with this.

15. One of the genuine milestones in human discovery was Newton's discovery that a force of gravitational attraction exists between any two objects no matter what their size is, anywhere in our universe. This force increases with increasing mass, but decreases as distance between the two objects increases. Here is the formula that states Newton's law of universal gravitation:

$$F = \frac{gm_1m_2}{d^2}$$

where g 6.67 $\times 10^{-8}$ with metric units (grams, meters); F is the gravitational attraction; m_1 and m_2 are the masses (not weights) of two objects; d is the distance between their centers.

Since the mass of the earth is enormous, the attraction between anything (including you) on earth and the earth itself is quite strong. Therefore, we don't fly off into space from the centrifugal force of the spin of the earth, nor do we fall off when the earth turns "upside down." (Besides, our attraction to the earth defines "down.") Newton applied this great insight to far more than just the attraction of things to the earth; for example, he made it account for orbits about the sun of planets in our solar system (these orbits had already been described, but not explained, by Kepler). This law also accounts for the moon's orbit about the earth. Planets orbit about the sun because the sun has mass enormously greater than any of the planets, so the planetary orbits are determined by the centripetal force of their spinning around the sun balanced off against their gravitational attraction to the sun. The earth "captures" the moon in a similar way, but the pull is mutual, as evidenced by the fact that tides on earth are caused in part by the pull of the moon on the earth. You might look up some information on what the mass of the earth is and calculate what the gravitational attraction between you and the earth is; in this case the distance is effectively the distance between the center of the earth and you at the rim. You might also imagine yourselves orbiting the earth 1000 miles out in space and see to what extent this lessens your gravitational attraction to the earth. Make up some problems such as this and work them out.

16. A number of models in geography that deal with interactions between human population centers have the form:

$$I = \frac{k P_1 P_2}{d^q}$$

where P_1 and P_2 are the sizes of the two populations, I is an index of the interactions between them, and d is the distance between them.

The exponent q is frequently taken as 2, as in the gravity model above, but varies depending on the amount of "friction" distance puts on the interaction. For example, for two cities separated by a range of mountains, or an ocean, the effect of distance on certain interactions is different than for two cities with no geographical obstructions between them. Examples would be the number of telephone calls, banking transactions, and freight traffic between the two populations. Think of some other interaction situations where such a model might apply. ("Models of Spatial Patterns of Human Geography.")

17. Suppose that "Absence makes the heart grow fonder." Construct a formula expressing this, perhaps with "absence" measured in time: perhaps in distance. If you don't believe this proverb, make up another sort of model that you think gives a relation between "fondness" and "absence." You might also want to include the intensity of the feelings of the people involved.

Chapter V

THREE EXAMPLES OF PROBLEM COLLECTION THEMES

5.1 Biological Consequences of Linear, Square, Cubic Relationships

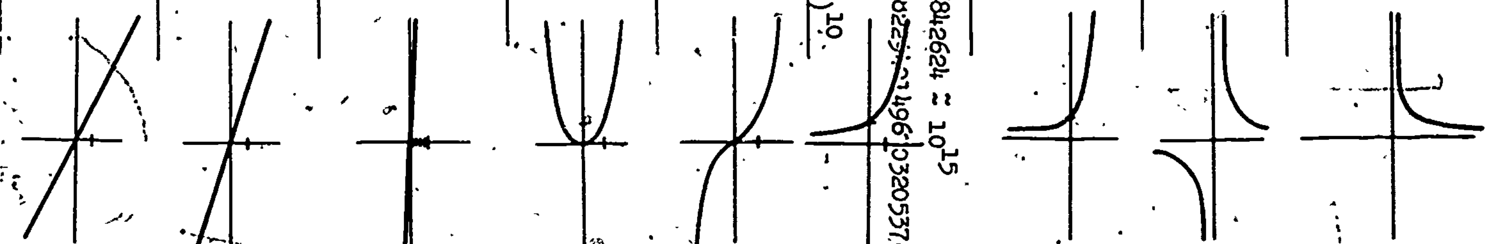
Every mathematics teacher knows that no matter what he does, some students will sometimes commit standard "freshman errors." They may cancel willy nilly, even when inappropriate (for example $\frac{1 + x}{2 + x} = \frac{1}{2}$), even though a simple trial substitution would show this to be absurd (for example, if the x in $\frac{1 + x}{2 + x}$ is replaced by 10 the value is $\frac{11}{12}$, and if x is replaced by 100, the value is $\frac{101}{102}$). Another common "freshman error" is to treat 2x, x^2, and 2^x as if they were all the same; or 3x and x^3 as if they were the same. But one only has to replace x by some numbers to see that 2x, x^2, and 2^x are very different things; for example, if x is replaced by 10, 2x is 20, x^2 is 100; and 2^x is 1024. Similarly, if x is replaced by 10, 3x is 30 but x^3 is 1,000. In mathematics, functions of the form $x \to 2x$ and $x \to 3x$ are said to be "linear"; functions of the form $x \to x^2$ are said to be "quadratic"; functions of the form $x \to x^3$ are called "cubic"; and functions of the form $x \to 2^x$ (or $x \to 10^x$ or $x \to e^x$--where e = 2.718...) are called exponential." As you may already have seen, such functions very often serve as mathematical models of real-world happenings; for example, quadratic relationships characterize area; cubic relationships characterize volume.

Problem Set 5.1a

1. On the following page is a partly completed table with headings corresponding to linear, quadratic, cubic, exponential, inverse, and inverse square relationships. Finish filling in the table, in this book if you own it, or on another sheet of paper ruled and labelled as on the facing page. (Some of the values of 2^x are filled in for you; see if you can fill in others from the ones given.) As you fill in the table watch where "order of magnitude" (factor of ten) differences develop. (We have observed several times in this book that such tenfold differences frequently make a fundamental difference in how we look at things; for example, walking speeds versus automobile speeds vs. jet airplane speeds; or \$100 versus \$10,000.) To help in this, express answers beyond 100 in "scientific notation"; for example, $2^{10} = 1024 = 1.024 \times 10^3$.

Growth of Numbers with Different Functions

x	Linear			Quadratic Areas	Cubic Volume	Exponential Population Growth		Inverse	Inverse Square Distance effects
x	d: $x \to 2x$	t: $x \to 3x$	m: $x \to 100x$	q: $x \to x^2$	c: $x \to x^3$	P: $x \to 2^x$	f: $x \to 10^x$	i: $x \to \frac{1}{x}$	s: $x \to \frac{1}{x^2}$
0	0	0	0	0	0	1	1	no value	no value
1									
2									
3									
4	8	12	400	16	64	16	10,000	$\frac{1}{4}$	$\frac{1}{16}$
5									
10						1024	10^{10}	0.1	0.01
20						1,048,576	10^{20}	0.05	.0025
50						$2^{50} = 1125899906842624 \approx 10^{15}$			
100	200 = (2×10^2)	300 = (3×10^3)	10,000 = (10^4)	10,000 = (10^4)	1,000,000 = (10^6)	$2^{100} = 1267650600228229401496703205376$	10^{100}	0.01 = (10^{-2})	0.0001 = (10^{-4})
1,000						$2^{1000} = (2^{100})^{10}$			
10,000						$2^{10000} = (2^{10000})^{10}$			



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2. Any function of the form F: $x \to kx$ is "linear." Even if the constant k gets relatively large (as 100x in [illegible] table), the linear relationships are eventually "swamped" by the quadratic, cubic, and exponential relationships. This would be true no matter how big k is (k stays the same throughout any discussion, of course; you don't suddenly decide to make it bigger). For example, suppose k is 1,000,000; so F: $x \to 1{,}000{,}000x$. Beyond what value of x is the output of q: $x \to x^2$ greater than for F: $x \to 1{,}000{,}000x$? Beyond what value of x is the output of c: $x \to x^3$ bigger than for F: $x \to 1{,}000{,}000x$?

3. Similarly, no matter what the value of k is in a quadratic function Q: $x \to kx^2$, the cubic relationship will eventually swamp it, as will the exponential relationship. Verify this by showing that even for Q: $x \to 100{,}00x^2$ there are values of x beyond which the outputs of c: $x \to x^3$ are bigger, and similarly for P: $x \to 2^x$.

4. Notice that in a linear relationship, numbers grow at a steady rate, but in quadratic, cubic, and exponential relationships, the rate of growth accelerates. For which of quadratic, cubic, or exponential functions does the rate of growth accelerate the fastest? For which do the order of magnitude jumps become the greatest in the table you are working with?

5. Look at the positive parts of the graphs; does the increasing acceleration show up on the graphs? How do the graphs reflect what you see going on in the table?

6. Notice that in the graphs for exponential and inverse models, there are no zero outputs; i.e., there is no replacement for x that gives a zero output for these functions. Why? That is, think about this.

7. Inverse relationships are common in formulating mathematical models. For example, if one were to make a table of time versus velocity (speed) for travelling 100 miles, faster speeds would mean less time; slower speeds more time. Think of some other inverse relationships. (Note: Inverse functions are not the only functions where "less" results in "more.")

8. In some other mathematical models that reveal inverse relationships, one of the variables has especially powerful effects; these often turn out to be "inverse square" function models. For example, sound decreases as the square of the distance from the source; so does gravitational attraction (see Newton's law of universal gravitation in a previous section). Try to find other situations where something decreases as the square of distance; or as the square of something else.

9. Does the table you just filled in convince you that "inverse square" models attenuate (diminish) much faster than does the other inverse function you worked with? Once again, this property can be temporarily concealed by multiplying by a constant, (k). However, no matter how we choose the constant k (as long as k is a positive number), $\frac{1}{x^2}$ will eventually get smaller than $\frac{1}{kx}$ as x is replaced by larger numbers. Verify this for I: $x \rightarrow \frac{1}{1000x}$ versus s: $x \rightarrow \frac{1}{x^2}$.

A number of biologists have written articles for "the intelligent layman" pointing out, for example, that the giants of fairy tales are a biological impossibility. The reason is that strength of legs and other bones depends on the cross sectional area of bones, while weight of the body, depends on volume. Since mathematical models for area are characteristically quadratic relationships, and for volume cubic relationships, giants would soon become so heavy that they couldn't walk without breaking their legs. That is, cubic relationships eventually swamp quadratic ones. The problems that follow talk about some more such biological consequences of linear growth, growth in area, and growth in volume. They are suggested by two fine articles: "On Being the Right Size," by J. B. S. Haldane and "On Magnitude," by D. W. Thompson. Both these articles are in Volume II of The World of Mathematics, which is almost certainly in your school library. You may want to read these articles either before or after going through this problem set.

In order to make sense of these problems you must somehow convince yourself, if you are not already convinced, that linear changes in the length or height of people, animals, etc., do take place; that whenever they do, changes of surface or of cross sectional area will depend on quadratic functions (maybe with K factors for adjustments); and that volumes will depend on cubic functions of the length. You must also believe that no matter what the particular form of the function or what constants are involved, cubic changes swamp quadratic ones and quadratic changes swamp linear ones in the long run.

Problem Set 5.2

1. Here is a quotation from Gulliver's Travels: "His Majesty's Ministers finding that Gulliver's stature exceeded theirs 12 to 1, concluded from the similarity of their bodies that his must contain at least 1728 of theirs and must needs be rationed accordingly." If Gulliver was about 6 feet tall then how tall were the Lilliputians? Since 12^3 is 1728 the Lilliputians were apparently aware that volume ratios depend on the cube of linear dimensions. Hence they provided Gulliver with a double hogs-

head for his daily half pint of ale. Given that a British barrel is 27 gallons and a double hogshead is 4 barrels, is this about the right proportion? That is, how many half pints in a double hogshead? Figure out some other equivalents in Gulliver's rations; for example, how would Lilliputian pumpkins or cattle look to him on his plate?

2. Haldane speaks of an illustrated Pilgrim's Progress that showed the giants Pope, Pagan, and Despair about 10 times as high as Pilgrim. He also says that bones will break under about 10 times normal stress. How much more volume would the giants have than Pilgrim, according to the relationships we are using? If everything were strictly similar, by what factor would the cross-sectional area of bones increase? How many times more has volume (and hence weight) increased than has the cross-sectional bone area? Is it likely that these giants could move around very well?

3. Thompson has this to say: "Understanding the correlation between weight and length in any particular species of animal, in other words a determination of K in $W = KL^3$ enables us...to weigh the animal with a measuring rod." Considering your own weight and height figure out what this K should be. Compare your value of K with others in the class. If you remember or have some record of your weight and height at some other time in your life, see if this same K gives approximately correct results for these other periods. If your parents kept a record of your birth weight and birth length see if the same K fits that data. If you have any pets whose weight and length you can measure, compute the K and try to figure out some way to verify whether the model works well for that animal.

4. The effects of the disparity between growth in volume and growth in area means that larger animals require larger and heavier bones out of proportion to their increase in length. Thompson tells us that the bones of a mouse or wren are about 8 percent of their total weight; of a goose or a dog about 14 percent; of a man about 18 percent. On the other hand, whales and porpoises (which are also mammals) have about the same proportion of bone weight even though a whale is much larger than a porpoise, presumably because they are floating in water and the effects of gravity on their bodies are irrelevant (hence bone structure need not be different). If Thompson's figure is correct, about how much do your bones weigh?

5. Dogs, of course, come in many different sizes; do you observe that the bone structure apparently changes in larger breeds? Does a large dog (say, a St. Bernard) weigh about as much as a comparably sized human? Our theory says that for a man about three times as long as a dog the

main structural bones should be about three times as long and have about nine times the cross-sectional area. If possible, check this out; perhaps your biology teacher can help you. The upper leg bone would be a good one to make some comparisons on.

6. The gazelle, the hippopotamus, and the giraffe all belong to the same order of animals, yet are constructed very differently. Consider these differences by linking them with what we are talking about here.

7. Just as important as the cross-section area of bones is the surface area of animals. Surface area is likewise a function of the square of the length while volume remains a function of the cube of the length. A mouse is about 4 inches long (without the tail), while a man is about 68 inches high (a factor of 17). Therefore the man's surface area will be bigger by about a factor of 300 (since $17^2 = 289$), but his volume and weight by a factor of about 5000 (what is 17^3?). Since air resistance of a falling body is proportional to the surface area (this is why parachutes work), a mouse can safely drop much further in free-fall through air than can a man. Now consider some insects and their relative weight versus their surface area; can you see why gravity has so little effect on them? Haldane claims that a mouse can be dropped down a thousand foot mine shaft and walk away; that a rat can do the same falling about a hundred feet or so. Explain this in terms of the relative size of mice and rats. What is the maximum fall that a man can walk away from unhurt and what might this depend on? Make some estimate of how far an elephant, horse, or cat could fall safely. Justify your estimate.

8. Gravity is less of a problem for a mouse than a man and no problem at all to an insect, but the effects of being wetted are something else again. Since man is large in relation to his surface area, when he steps out of a shower or a bathtub only enough water clings to him to make him 1% heavier. For your own weight, what weight of water would this be and what volume? According to our calculation above, a man is about five thousand times as heavy as a mouse, so how much does a mouse weigh? (Try to verify that this is about right.) Haldane says a wet mouse would have about its own weight of water clinging to it. Does that seem correct to you? For insects, relatively unaffected by gravity, getting wet is a disaster, as you may realize if you've ever seen a fly try to get out of a pool of water. What are some of the adaptations in insects to get drinks from pools of water without being caught in the water's "surface tension"? ("Surface tension" lets you fill a glass of water slightly

above the top of the glass without spilling; the molecular attractions act almost like a film on the surface.)

9. Thompson says that for animals as small as a bacillus "Brownian movement" takes precedence over gravity, surface tension, and other factors. Try to find out what Brownian movement is and explain what it has to do with a bacillus. Some bacillii are as small as 1 micron in length, that is, one millionth of a meter. Since a man is about 1.75 meters this means that man's length is about $1\frac{3}{4}$ million times that of a bacillus. By what factor, then, might man's volume (and hence his weight) exceed that of such a bacillus?

10. Another consequence of increased surface area relative to volume is heat loss at the surface. You may recall that earlier we stated that Weather and Health tells us that about 68 percent of man's body heat is lost in convection and radiation at the surface; with another 18% by perspiration from the surface and 14% by breathing. Therefore, the heat that the body disperses, and hence the food needed as fuel, is proportional to the surface area of the skin. Thompson says that man consumes about one-fiftieth ($\frac{1}{50}$) of his own weight in food daily, while a mouse eats about half its own weight in a day. About how much food is this for each and does it fit with the man-mouse ratios given earlier (for length about 17, for area about 17^2, for volume about 17^3)?

11. Here, from Thompson, are some other figures of weight versus calorie consumption; try to fit this information into the linear, area, volume theory that we have been talking about and see if you can account for any marked departures from what you expect (e.g., whales may not fit the model very well--why?).

	Weight (Kilo)	Calories per Kilo
Guinea pig	0.7	223
Rabbit	2	58
Man	70	33
Horse	600	22
Elephant	4,000	13
Whale	150,000	about 1.7

12. Thompson suggests that the furious rate at which a mouse has to gather and consume food reflects itself in a generally faster pace of living;

that is, it breeds faster and dies much sooner. Check this out against what you know about the life expectancy and gestation period (time it takes from conception to birth of the young animals) of animals both large and small. There are very few warm blooded animals smaller than a mouse; is it possible that volume versus surface area considerations put absolute limits on how small an animal can be? It would seem that this terrific need for food to overcome heat losses would make it difficult for small animals to live in very cold climates; check this out and see if any small warm blooded animals live year around in Arctic regions. Are there small birds and insects living in Arctic regions during winter seasons?

13. Other consequences of surface area come from the fact that we absorb oxygen directly from the air and the fact that absorption from intestine walls is the main way of supplying food to the body. A simple worm absorbs oxygen directly through the skin and food from its straight gut. But a factor of ten increases in length of an animal means, as we have seen, a thousand-fold increase in food and oxygen needs while absorption areas will have only increased by a factor of 100 unless there are some special adaptations. Find out from somewhere what man's lung surface area is and length of his small intestine. From the latter make several approximations to the surface area of the small intestine. If biology interests you at all, delve into a little bit of comparative anatomy of small simple animals versus larger, more complicated, ones and see what adaptations are made in various surface areas as animals get larger and larger.

14. According to Thompson, engineers are well aware of the fact that having built a bridge across a river it is <u>not</u> safe to use the same design in building a geometrically similar bridge twice as long. Given the discussion above, what would be some of the reasons for this?

15. The human eye has about 500,000 rods and cones as receptors for light and is in every way a very remarkable instrument. Each rod or cone is about 1 micron in diameter; they can't be much smaller than this because the range of wave lengths for visible light is from about .4 microns to .7 microns. Hence, smaller eyes simply have fewer rods and cones and hence, less visual acuity and color perception. With that information think about the following problems. Very small animals, such as mice, necessarily have eyes smaller than human eyes; what does this likely mean for the sort of vision they have? What does this mean for their

eye size in relationship to their total body size compared to humans? The human eye is already pretty good; is it likely that much larger animals would also need much larger eyes? Check this out for elephant's eyes versus man's eyes.

16. There is not really very much difference in how high a flea, a man, and a grasshopper can jump. Why should this be?

17. Some time ago I saw an advertisement for a telephone company, supposedly illustrating that the number of their subscribers had doubled in a given time period. In fact, if you measure the phones in the picture, the larger one is just twice as high (and twice as wide) as the smaller one; yet the picture still gives a very misleading impression. Why is this? Have you seen picture graphs that mislead in this way? In general, what do you expect the area relationships to be between two objects that are geometrically similar (same shape, different size)? What about volume relationships?

18. It is not unusual to find articles that speak about the consequences of volume versus area in biology and then by analogy argue that in some non-biological situations similar "scale effects" apply. For example, in his closing, Haldane has this to say: "To the biologist the problem of socialism appears largely as a problem of size...while nationalization of certain industries is an obvious possibility in the largest of states, I find it no easier to picture a completely socialized...United States than an elephant turning somersaults or a hippopotamus jumping a hedge." In a similiar vein, an article entitled "Weight-Watching at the University: The Consequences of Growth" outlines the "scale effects" in biology that we have been talking about above, then goes on to argue by analogy that universities can easily become too big. For such an analogy to be plausible it should be shown that there are linear versus quadratic, quadratic versus cubic, or linear versus cubic effects in the way some important variables fit together. Think about Haldane's statement of socialism, the probable content of the article on universities, or some other problem that you think such arguments might be applied to, and see if you can think of relationships between things in these situations that would be similar to those in the biological situations.

19. Animal Facts and Fallacies tells us that an ostrich egg measuring 6 to 7 inches in length and 5 to 6 inches in diameter is equivalent in volume to 12 to 18 hen's eggs. Find the length or the diameter of a hen's egg and see if this 12 to 18 is a plausible figure for the volume proportion.

This same source says that the eggs of the extinct "elephant bird" measured 13 inches in length and $9\frac{1}{2}$ inches in diameter. How much more than an ostrich egg and how much more than a hen's egg should this egg hold? (Animal Facts and Fallacies says 6 times and 150 times respectively.)

20. Weather and Health tells us that there appears to be a linear relationship between outside temperature and calorie requirements of people. From data gathered during World War II, it seems that every time the outside temperature goes down one degree Celsius (or about 2° F), people use 30 (diet) calories (actually kilocalories) more per day. At normal temperatures of about 21° C (about 68° F) a working man might use 3000 calories. Draw a graph using this information starting at a 21° C, 3000 calories point, and extending to -20° C. (According to this source, part of the increased calorie intake goes to produce a fat layer; one centimeter thickness of fat is said to be about equivalent to the insulation given by an ordinary suit.)

5.2 Gravity and Other Forces

When children learn that they live on a globe that rotates, they frequently wonder why people don't fall off as the world turns upside down. They don't because the huge mass of the earth acting on our mass exerts an attraction given by $F = Gm_1m_2/d^2$, where $G = 1.068 \times 10^{-9}$ in the English system and $G = 6.67 \times 10^{-11}$ in the metric system. This attraction also defines "down" for us.

Having had the experience of swinging a rock on a string around their head and having it fly away when they let go, some children wonder why people don't fly off this spinning world. The centripetal force that makes the rock fly off when released is given by $F = mv^2/r$, with m the mass of the rock, r the radius of the circle it spins in, and v the rock's velocity. But we don't fly off and we don't fall off, because of gravitational attraction to the earth.

It used to be said that whatever goes up must come down, but this seems today to be a rather old fashioned notion. We regularly put things up (say to the moon) which don't come down unless we put a lot of effort into bringing them down. We also put things in orbit so they go neither up nor down nor do they stay in the same place. Centripetal force certainly plays a role here, balancing the pull of gravity.

Even when you learn that you exert some tangible force against the earth as you stand on it or on a chair when you sit, you still must observe that the force varies under certain conditions. For example, if you get in a fast elevator going up you seem pressed more heavily towards the floor as it begins and almost rise off the floor as it stops. Similarly, in a car or airplane as you accelerate you are pressed back against the seat and as you decelerate you are thrown forward from the seat. The force acting here is given by the basic formula $F = ma$, where F is the force, m is the mass and a the acceleration. This formula simply confirms your experience that such forces increase both with acceleration and with increasing mass.

It is said to be the case that a man weighs less on the moon than on the earth, and in space on his way to the moon "weightlessness" becomes a problem. Yet his actual substance has not changed. The basic unchanging measure of this substance is mass. Weight is related to mass by taking account of the acceleration gravity gives a mass (on earth, this is 32 feet per second per second.) Weight is a force, and as a force $F = ma$ applies in the form $W = mg$ (or the equivalent $m = \frac{W}{g}$) where W is the weight, m the mass, g the acceleration due to gravity. The mass versus weight distinction must be reckoned with whenever physics formulas call for mass, as they usually do.

Two uniform fairly heavy objects dropped near the earth will fall the same distance in a given time no matter how much they weigh; or in other words, no matter what their mass. (This is said to be true of any two objects in a vacuum and it's pretty much true of any in open air except for objects so light that air resistance and wind currents becomes a problem.) The objects obviously "accelerate" as they fall because at the instant they are dropped they have a zero rate of speed but when they hit the ground they have acquired some speed. Furthermore, they go further during the second second than the first second, still further during the third second, and so on. Here we see in action the force of gravity and the acceleration it gives to falling objects. The general formula is $S = \frac{1}{2}at^2$, and on earth $a = g$ and g is about 32.2 feet per second per second or about 9.8 meters/sec^2.

The above phenoma are all related to one another and all play important roles in our lives. They also have much to do with why planets travel in the orbits they do, why the moon has the relation it does to the earth, and the science and technology that goes into space flight. The appropriate mathematical models are quite simple. In this section we will deal with some of these models and their consequences. The basic things operating in most of these problems are variations on the forces spoken of above. From these basic formulas, many other relationships can be derived. When some special variant on one of these formulas is needed, or if a new formula is required, it will be supplied along with the problem. More important than formulas, however, is your own use of common sense and intuition to get a feeling for what is going on.

Problem Set 5.2

The problems marked with asterisks are adapted or taken directly (by permission) from M. H. Ahrendt, The Mathematics of Space Exploration (copyright 1965 by Holt, Rinehart and Winston). This is a beautiful and highly recommended book with much information and many nice problems, some of them based on realistic NASA data. Appropriate formulas are often given in parenthesis after a problem, but if you can work the problem without using the given formulas, that is fine too.

1.* An automobile traveling 45 miles per hour comes to a complete stop 10 seconds after the brakes are applied. Find the average deceleration in feet per second per second. Find the distance required for stopping. ($S = \frac{1}{2}at^2$, a = acceleration = change in velocity/change in time).

2.* A person weighing 160 pounds stands on a spring scale in an elevator. When the elevator starts up, the scale momentarily reads 182 pounds, and when the elevator slows to a stop, the scale for an instant reads 130 pounds. Find the acceleration and the deceleration of the elevator. ($\text{mass} = \frac{\text{weight}}{g} = \frac{w}{32}$; $F = ma$; so $F = \frac{wa}{g}$)

3.* If the elevator above were to descend with an acceleration of 32 feet per second per second, what would the scale read during the period of acceleration?

4.* A Boeing intercontinental jet airplane has a take-off weight of 295,000 pounds and four turbojet engines with a thrust force of 18,000 pounds each. Neglecting friction and flight controls, what would be its maximum possible take-off acceleration? (How would you get $a = \frac{fg}{w}$ from $f = ma = \frac{w}{g}(a)$? The weight is w; the thrust force is f.)

5.* Ranger 7, a spaceprobe to the Moon, was orbited by NASA on July 28, 1964. It was designed to crash land on the Moon and to take and transmit to Earth pictures of the lunar surface during the last 17 minutes of its flight. It was found after launch that it was flying a little too fast and would pass in front of the Moon unless slowed down. Therefore, when Ranger was 101,044 miles out in space, a course correction was applied by radio signal to slow the 806 pound craft through the firing of a 50 pound thrust retrorocket. This reverse thrust slowed Ranger's speed by

67 miles per hour to 3927 miles per hour. What actual firing time in seconds was needed to decrease the velocity 67 miles per hour? (Ranger 7 took 4,316 pictures of the lunar surface, the best ones showing a resolution 2000 times as good as had been possible with Earth-based telescopes. Ranger hit the Moon within 10 miles of the selected aim point.)

6.* A string has a breaking strength of two pounds. It is attached to an object weighing $\frac{1}{4}$ pound (4 oz.), which a boy is whirling at a constant angular velocity of two revolutions per second, while gradually letting the string out to increase the radius. At what radius will the string break? (The centripetal force formula given in the introduction depends on the velocity of the object at the end of the string as well as its mass--which you can get from the given weight. If the angular velocity, or rate of turning, remains the same, the velocity will increase with increasing radius. Part of your problem here is to figure out what the radius has to do with the velocity at the rim, given a constant rate of two revolutions per second. You then need to find the radius at which the velocity will give a centripetal force exceeding the strength of the string.

7.* Some automobile manufacturers rate cars in terms of the number of seconds required to begin at a standing start and get the automobile to a speed of 60 miles per hour. If the automobile can reach 60 miles per hour in 10 seconds, it experiences an average gain in velocity (an average acceleration) of six miles per hour during each second. Find the gain in velocity in feet per second per second. What fraction of a "g" is this acceleration? (As often happens in mathematics, here we have another use of the letter "g". A "one g force" is a force that would produce an acceleration of about 32 feet per second per second; that is, the same acceleration earth's gravity force produces. A pull of one _g_ gives the mass of a body its normal earth weight; a two _g_ force doubles that weight, and so on.)

8.* During the flight of John H. Glenn, Jr. in Friendship 7, a Mercury spacecraft, the "g force" built up from one g at the beginning of launch to 6.7 g's when the booster engine in the first stage cut off at the end of 2 minutes 10 seconds of powered flight. If Glenn normally weighs 160 pounds, what would his weight be at 6.7 g's? At what rate in feet per second per second was the spacecraft accelerating at cut-off in stage one? During the burning of the second stage, lasting 2 minutes

52 seconds, the "g force" went from 1.4 g's to 7.7 g's. What was the acceleration at cut-off in stage two?

9. It is convenient in solving many problems to be able quickly to convert feet per second to miles per hour, and vice versa. If you multiply n feet per second by 3600 (the number of seconds in an hour), you will obtain the number of feet per hour. If you then divide by 5280 (the number of feet in a mile), you will obtain miles per hour. Simplify the resulting ratio to obtain a convenient multiplier for quickly changing feet per second to miles per hour. What would the multiplier be for changing miles per hour to feet per second?

10.* If the atmosphere exerts a pressure on earth (at sea level) of about 14.7 pounds per square inch, what is the air pressure (weight of the air) in tons upon one square foot of Earth's surface at sea level?

11.* Here are some interesting data from The Mathematics of Space Exploration. Ask some interesting questions and answer them using this data; such questions as these, for example:

a. In August 1971, Mars will be about 350,000,000 miles from the earth. Locate the neutral gravity point between them at that distance, ignoring the effects of other planets.

b. What percent of the total mass of our Solar System is concentrated in the Sun?

c. Which planet most closely resembles Earth in diameter, mass, density, period of revolution, surface gravity, and escape velocity?

d. Rank the planets first by distance from the Sun and next by period of revolution. Are there differences in the rankings and, if so, can you account for them?

Body	Diameter (miles)	Relative Mass* (Earth = 1)	Density (water=1)	Period of Revolution	Surface Gravity (Earth = 1)
Earth	7,920	1	5.52	365 days	1
Jupiter	88,640	317	1.34	11.9 years	2.64
Mars	4,200	.11	3.96	1.88 years	.39
Mercury	3,100	.04	3.8	88 days	.26
Moon	2,160	.012	3.33	$27\frac{1}{3}$ days	.17
Neptune	31,000	17.2	1.58	165 years	1.12
Pluto	?	.8	?	248 years	?
Saturn	74,500	95	.71	29.5 years	1.07
Sun	864,000	330,000	1.41	-----------	.28
Uranus	32,000	14.7	1.27	84 years	.91
Venus	7,700	.81	4.86	255 days	.86

*The mass of the Earth, in metric units, is about 6×10^{30} grams.

Body	Average Distance from Sun (millions of miles)	Eccentricity of Orbit	Surface Escape Velocity (miles per second)
Earth	93	.017	6.9
Jupiter	483	.048	37
Mars	141.5	.093	3.2
Mercury	36	.206	2.2
Moon	---	.054	1.5
Neptune	2,793	.009	14
Pluto	3,670	.248	?
Saturn	886	.056	22
Sun	0	--	387
Uranus	1,782	.047	13
Venus	67	.007	6.4

12.* If one were to take one unit of mass (for example, one slug) and use it with Newton's law of universal gravitation, the force exerted by earth's mass, M, would be as follows: (radius of the earth = 3960 miles)

$$F_c = \frac{G \times M \times 1}{3960^2}$$

Since the mass M of the earth is considered as concentrated at its center, the distance between the unit mass and M is the radius of the earth. The moon has .012 times the mass of the earth. Hence a calculation of the force of gravity on the unit mass on the moon would be:

$$F_m = \frac{G \times (.012M)(1)}{1080^2}$$

since the radius of the moon is 1080 miles. The ratio of the forces would be $\frac{F_m}{F_e} = \frac{G(.012M)}{(1080)^2} \cdot \frac{(3960)^2}{GM}$ which, if you do the calculations, tells you that moon's gravity pull is about $\frac{1}{6}$ of that on earth. How much would you weigh on a spring scale on the moon? (If you used a balance scale, there would be no difference--why?)

13.* By computation similar to that done above for the Moon, show that the surface gravity Mars is about .39 times the surface gravity on Earth and that the acceleration produced by gravity at the surface of Mars is about 12.6 feet per second per second.

14.* If man carried a load weighing 100 pounds on Earth up a flight of stairs through a vertical distance of 20 feet, he would do 100 × 20 or 2,000 foot-pounds of work. How much would he do in carrying the same object up similar stairs on the Moon? On Mars? On Saturn? On Venus?

15.* On October 24, 1970, Christos Papanicolaou set a world pole vault record of 18 feet $\frac{1}{4}$ inches. How high should he be able to pole vault on the Moon? On Jupiter? (Remember that his center of gravity is already at about 4 feet, so he is lifting his weight "only" about 14 feet.)

16.* The table above gives the distance to the Sun as 93,000,000 miles, while the mass of the Sun is about 330,000 times the mass of the Earth. Use this data to find the location of the neutral gravitational point between the Earth and Sun; ignoring other planets.

17.* Some of the matter in distant galaxies is said to be so dense that on Earth it would weigh 40 tons per cubic inch. Suppose that a man were able to obtain enough of this material so that a stone for the engagement ring of his fiance could be cut from it. Let us make the setting a cube which measures $\frac{1}{20}$ inch on one edge. Would she be able to wear the ring? What would the setting weigh?

18.* Weightlessness is a nuisance in space and this would be especially the case in a space station where people were living and working. One plan to overcome this is to make a space station roughly in the shape of a wheel, then by having it rotate like a wheel create a centripetal (or centrifugal) force that would act as an artificial gravity. "Down" would be toward the outer edge of the wheel. From the earlier discussion and for such centripetal force (introduction and problem 6 above) try and figure out what rate of spin would be needed for a space station with radius 300 feet in order to create an artificial gravity about half that of earth's gravity. The Mathematics of Space Exploration gives a formula for this: $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ where N is number of revolutions per second, a is the desired acceleration of artificial gravity, and r is the distance in feet from the center of rotation (hub) of the station to some point in the station. Check your calculation with this formula and try to figure out how it was derived from the formula for finding centripetal force.

19.* For extended periods of space travel, one might be limited by the amount of fuel that could be carried on board the space craft. Nuclear energy would extend the range, and other solutions have also been proposed. For example, The Mathematics of Space Exploration tells us that an experimental solar engine has been described which uses mirrors to gather heat from the Sun which is used to expand hydrogen gas. The heated gas will emerge as a jet giving a thrust of about 2 pounds. If the solar engine begins to work after a spacecraft weighing 40 tons has been placed in a parabolic escape orbit beginning 500 ~~miles~~ above the surface of the Earth, what is the total velocity (v) attained and distance covered after 80 hours? After 80 days? (Convert 40 tons to a mass, then use $f = ma$ and $v = at$.) This same source tells of a Solar sail which produces an acceleration of 0.01g. What velocity can it give the above craft in 80 hours? In 80 days? (How would a Solar sail operate?)

5.3 Mathematical Thinking in Everyday Situations

In recent years mathematics has been applied to a very wide range of problems, often with considerable success. Once you know something about the mathematical models process, it is not difficult to imagine how mathematical methods could, at least in theory, be applied to various real world situations. The following exercises are illustrative of the possibilities. Consider each of them and then answer the following questions: (Problems #2, 3, 5, and 6 below are suggested in "Problems of Teaching Applications of Mathematics.")

A. What simplifying assumptions might be helpful in making the problem manageable?

B. What data would need to be gathered, and how, before a beginning could be made on the problem?

C. Suggest ways in which the problem might be investigated and what part mathematics might play in the investigation.

D. If you can, suggest another problem or two in the general area referred to by the exercise, the investigation of which might also be aided by mathematics.

<u>Problem Set</u>

1. There is much concern these days about our environment. One of the things that most clutters up our environment is the disposal of waste products. Take for example the food and other wastes in an average household. One choice is to deposit the garbage in a garbage can, but then the air is polluted by the garbage truck, and also by the city incinerator. Another choice is for the housewife to wash waste food into the sewage system by way of a garbage disposal that grinds up the food, but this uses a lot of water in washing down the ground up garbage as well as putting an additional burden on the sewage purification facilities. Assuming these are the only two choices, which should be preferred if our criterion is least damage to the environment? What other choices might be considered, and how evaluated? Suppose the main criterion is minimum cost, including both direct and indirect actual costs of disposal. How would this change the consideration of the problem?

2. What is the largest rigid object that you can get through a door or around a corner? Consider first a flat object, like a piece of plywood, then consider three dimensional objects. (According to one source, no <u>general</u> solution to the corner problem exists, but you still should be able to attack, say, the problem of getting a couch up to an apartment.)

3. In moving cars through a long tunnel, how fast should you have the cars move? If they go very slowly it is safe to have them follow each other closely, which wastes very little space, but few cars get through in a

given time. On the other hand if they go very fast the spacing between them must be large for safety so there is a lot of waste space.

4. Long lines to the check out counter of a grocery store are a nuisance for all concerned but keeping too many clerks on duty can keep the store from making a profit. If lines are consistently too long, however, the store is likely to lose customers and hence profits. Formulate some problems in this area and suggest some ways of going about their solution.

5. For maximum parallel parking capacity, should streets be marked with parking spaces or left unmarked? If they are marked there is the possibility that large spaces still too small for a car will be left.

6. A restaurant wants to have an adequate supply of clean cloth napkins available each day. It can send napkins to a regular laundry which does the work cheap but takes a week to get them back. It can send the napkins to an express laundry which only takes one day but charges more. It can just buy and discard napkins and not bother to launder them. How can the restaurant assure itself of an adequate supply of napkins at minimum cost?

7. What are the actual costs of owning a car? (Don't forget depreciation, interest costs on borrowed money, and the fact that any money that you have invested in such as a car could be earning interest in a bank or as an investment.)

8. What are the actual costs of owning a home? How do they compare with rental of comparable space and amenities? Given a desire to minimize expenses, should one rent or buy a residence? What criteria and assumptions might operate in making the choice?

9. Suppose the flow of people to and from downtown area has increased to the point where both highways and public transportation are overstrained. How should the planner decide what to do about the situation?

10. An important national election is coming up and the television networks want to report results and predict the eventual outcome as quickly as possible. How should they proceed?

11. Is it better for a business to build its own office space or to lease facilities.

12. Suppose that in a given town, it is claimed that the school system is "doing a lousy job." How should a panel convened to evaluate the system go about its job; i.e., what would throw some light on the question?

13. Some have complained that welfare costs, public housing, attempts to train unskilled workers, etc., are too great a drain on the taxpayer. Others claim that the eventual actual dollar cost of <u>not</u> attending to these matters is more in the long run, to say nothing of social costs. What could be done to separate facts from fancies in this area?

14. Formulate and deal with a question similar to number thirteen above with respect to public school expenditures.

15. Because of the peculiarities of our electoral college system, it is possible for a President of the United States to be elected with less than a majority of the popular vote even if there are only two candidates. What is the minimum percentage of the total popular vote that could elect a President? With certain reasonable simplifying assumptions, such as only two candidates, and information on the population and number of electors in each state, this problem can be worked out with only arithmetic and simple algebra. The answer is a surprisingly small percent. Would you suppose that the minimum would be obtained by considering the voters from a collection of the least populous states, the most populous, or some combination? (See "The Minimum Fraction of the Popular Vote That Can Elect a President of the U.S.")

16. How many elementary arithmetic operations are performed in multiplying 932 by 47? Suppose a test had ten problems, each requiring four elementary operations and suppose a student does elementary operations with an error rate of 10%. If each of the ten problems counts ten points, with no partial credit, what is the range of possible scores for this student on this test? (Suggestions for further problems: a) Do a similar analysis of expected scores given the number of single operations and error rates in algebraic manipulations of various sorts. b) Is the increase in number of single operations in a given calculation a linear function of the number of digits in the numbers, or does it escalate faster than the size of numbers involved? c) A question similar to b) for algebraic manipulations.) (Read: "The Importance of Elementary Operations.")

17. Some years ago a rash of articles appeared with titles similar to "One Hundred Eminent Mathematicians." How would one go about ranking the eminence, importance, or influence of people in a given field over a given time span?

18. The number π, being irrational, is an infinite non-repeating decimal.

Years ago, someone spent a lifetime computing it to about 700 decimal places, but made an error. Recently this feat was duplicated in a matter of minutes, with no errors, using a computer. Why should anyone want π computed to so many decimal places?

19. The history of science is in part a history of improved instruments, especially as regards measuring and observation devices. Are there any limits other than technological ones on accuracy of measure or how small a thing can be observed?

20. Our society is a pretty complicated one. How does one get accurate indices of:

    a. The cost of living
    b. Inflation
    c. Recessions
    d. Unemployment
    e. Population growth rate
    f. Shifts in population
    g. Distribution of incomes
    h. Availability of services (medical, legal, etc.)
    i. Crime rates

21. What are the relationships between "inflation" and "the cost of living?" Are they just two names for the relative purchasing power of the dollar?

22. The 1902 Sears-Roebuck catalog has recently been reprinted (New York, Crown Publishers, Inc., 1969). Perhaps your library has it. Devise a scheme, and carry it out, if possible, for using just the data from this catalog and a current catalog to assess changes in, for example, some aspect of the "cost of living" since the beginning of the century, then check your conclusions with some standard index of cost of living.

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