DOCUMENT RESUME

ED 129 631 SE 021 501

AUTHOR Damarin, Suzanne K.
TITLE Problem Solving: Polya's Heuristic Applied to Psychological Research.
PUB DATE [76]
NOTE 40p.

EDRS PRICE MF-$0.83 HC-$2.06 Plus Postage.
DESCRIPTORS *Educational Psychology; Higher Education; Learning; Mathematics; *Mathematics Education; *Problem Solving; Psychology; *Research Reviews (Publications)
IDENTIFIERS *Polya (George)

ABSTRACT

Using the "How to Solve It" list developed by Polya as a vehicle of comparison, research findings and key concepts from the psychological study of problem solving are applied to mathematical problem solving. Hypotheses concerning the interpretation of psychological phenomena for mathematical problem situations are explored. Several areas of needed research with respect to the solution of mathematical problems are discussed. Three elements of Polya's list are identified as having primary importance in the solution process. It is argued that psychological research does not support the usefulness of "devising a plan," but rather implies that problem solution is facilitated by the restructuring of data. (Author)

PROBLEM SOLVING: POLYA'S HEURISTIC APPLIED TO PSYCHOLOGICAL RESEARCH

Suzanne K. Damarin

The Ohio State University

ABSTRACT

Using the "How to Solve It" list developed by Polya as a vehicle of comparison, research findings and key concepts from the psychological study of problem solving are applied to mathematical problem solving. Hypotheses concerning the interpretation of psychological phenomena for mathematical problem situations are explored. Several areas of needed research with respect to the solution of mathematical problems are discussed. Three elements of Polya's list are identified as having primary importance in the solution process. It is argued that psychological research does not support the usefulness of "devising a plan," but rather implies that problem solution is facilited by the restructuring of data.

PROBLEM SOLVING: POLYA'S HEURISTIC APPLIED TO PSYCHOLOGICAL RESEARCH

The nature of the problem solving process and of optimal methods of training individuals to solve problems have been subjects of speculation, introspection, and controlled research among scholars as diverse as philosophers, mathematicians, experimental psychologists, and educationists for a long time. However, the specific questions and conclusions of theorists and researchers from any one of these groups seem not to have influenced greatly the work of those from other disciplines. There are many good reasons for this, ranging from fundamental differences in objectives through difficulty in interpreting data across fields.

The purpose of this paper is to integrate some of the findings of experimental psychologists, educational researchers, and mathematicians in order to test the theories of one group against the findings of the others. Although the laboratory findings of experimental psychology are generally not directly applicable to the mathematics classroom, this comparison and analysis reveals many potential relationships between these situations, as well as a multitude of hypotheses concerning the processes by which students succeed or fail in the solution of mathematical problems.

The basic vehicle for this comparison is Polya's heuristic as described in his book How to Solve It (1957). This approach to problem solving, developed by an esteemed mathematician and teacher of mathematics as an aid to students in solving problems, is ~~probably~~ the most influential volume on heuristic and its application to the mathematics classroom. Psychologists seem not to have paid a great deal of attention to this work, but as shall be shown below, the *a priori* plausibility of Polya's system as a *general* problem solving method, as well as its generality of statement, admit the possibility of classifying and examining experimental problems and the findings of psychologists according to this outline. Because psychological researches can be so classified, it is possible to use the empirical data from these studies to examine the general validity of the "How to Solve It" scheme, bringing the contributions of mathematicians and psychologists together in a single focus.

Polya's "How to solve it list"

Polya's listing of steps in problem solving includes four basic stages; each stage except the third includes several substeps. The short version of Polya's listing to be used here is as follows:

I. Understanding the problem

A. What is the unknown?

B. What are the data?

C. What is the condition?

II. Devising a plan

A. Think of a related problem

B. Think of a problem with a similar unknown

C. Can you restate the problem?

III. Carry out your plan

IV. Examine your results

The generality of Polya's system is evidenced by the vernacular (i.e., non-mathematical) form of the directives outlined above.

This generality allows the classification of information and events connected with the solution of a wide variety of specific problems. In table 1, six problems are outlined according to this system. The first three of these problems have been used frequently in classical psychological research on problem solving; the fourth problem is the Cryptarithm used by Newell and Simon (1972) in their work on human information processing. The final two problems are problems in algebraic-arithmetic reasoning which have also been used in studies of problem solving.

Insert Table 1 about here

The "two-string" problem is a classical example of an "insight problem." In it the would-be problem solver is faced with the task of tying together the ends of two strings which hang from the ceiling; the lengths of the strings and the distance between them prohibit the subject from simply securing one and tying it to the other. In order to solve the problem he must use an object in the room (frequently a wrench) to make a pendulum of one string.

The "water jar" problems, first studied by Luchins and Luchins (1942, 1950), are familiar to many people as puzzles. In a typical problem the subject is asked to describe a procedure for measuring 5 quarts of water given three jars with capacities (a) 18 quarts, (b) 43 quarts, and (c) 10 quarts.

In the "Towers of Hanoi" a subject is presented with a board on which three posts are mounted; on one post is a stack of discs of different sizes, piled according to size. The problem is to transfer the entire stack to another post by moving discs one at a time without ever placing a disc on top of a smaller disc. This is to be accomplished in the least possible number of moves.

The cryptarithmetic problem:

```
  D O N A L D        D = 5
+ G E R A L D
  -----------
  R O B E R T
```

is discussed in great detail by Newell and Simon (1972) as an example of the information processing technique of problem volving. In theory this problem could be solved by linear algebra, but this solution method is quite complex and subjects do not use it. However, subjects do sometimes use algebraic notation and singletons or pairs of equations in the solution process.

The problem of the Donkey and the Mule was used by Buswell in his 1956 study of mathematical problem solving:

> A donkey and a mule were laden with wine. If 1 gallon were taken from the donkey and placed on the back of the mule, the mule would have double the load of the donkey. But if 1 gallon were taken from the mule and placed on the donkey, the two animals would have equal burdens. How many gallons of wine did each animal carry at first? (Buswell, 1956, p. 102)

This problem can be solved by simple algebra of one or two variables or by "reasoning." Buswell's study revealed that very few subjects (high school and college students) were able to solve this problem on the first attempt, and that more subjects used "logical" than "algebraic" methods in constructing correct (or incorrect) solutions.

The final problem listed in the table was used by Paige and Simon (1966) in their study of cognitive processes involved in solving "algebra word problems:"

> A car radiator contains exactly 1 liter of a 90 per cent alcohol-water mixture. What quantity of water will change the liter to an 80 per cent alcohol mixture? (Paige and Simon, 1966, p. 119)

This problem is ambiguous in that it can be interpreted to mean that water can be added to dilute all of the current solution or, alternatively, to mean that some of the 90 per cent solution is to be removed and replaced by water. Under either interpretation the problem is readily solved by algebra (one variable). The experimenters do not report any subjects solving or attempting to solve this problem by means other than algebraic, sometimes with accompanying figure.

These problems have been dissected according to "best fit" with Polya's list. Reading down Table 1 we are able to make some a priori comparisons among the problems and to relate the experimental findings with respect to some aspect of one problem with findings, hypotheses, or questions about the others.

Understanding the Problem

In Polya's analysis understanding a problem involves the determination of three categories of information: information concerning the definition of the "unknown," information in the form of data, and information concerning the relationships among the data and unknown which define the problem (conditions).

In all of the problems outlined above the definition of the unknown can be assumed to be well understood by the subject. In these problems the goals and unknowns are virtually identical, and experimenters in making clear the goals of the problem have also elucidated the unknown. (There do exist studies of situational problem solving in which the nature of the unknown is less clear to the subject, however.)

What are the data?

The definition of "data" used in this paper is intentionally broad; "data" is taken to include not only numerical information, but also the content of experimenter directions and the characteristics of any materials made available to the subject.

In psychologists' discussions of insight problems the construct "functional fixedness" refers to the failure of subjects to see an object ordinarily serving one purpose as relevant to the resolution of a situation in which it might serve another purpose. This phenomenon is classified here as related to the subjects' determination of available data. Under this view, subjects who fail to solve the two-string problem may do so because, although they have the datum "wrench" they do not have the datum "steel object weighing about two poun ds."

A related phenomenon seems to occur in some subjects' treatment of quantitative problems in that subjects appear to see some part of the data as "fixed" by one stated relationship, and, therefore, to relate this data improperly, if at all, to the remainder of the data in the problem. A striking example of this phenomenon appears in Paige and Simon's report of a subject's attempt to solve the alcohol-water problem; it shows a case in which addition is inflexibly used. This subject's model of the problem seemed to be "(initial mixture) + (water added) = (final mixture)"; her data then became ".90 + x = .80" which she wrote down. This subject was unable to solve the problem, apparently because she "fixed" the additive relationship of water and solution.

In a slightly different instance 11 of Buswell's 61 subjects read the data in the donkey and mule problem in a way that led them to choose

> "There can be only one gallon on each animal to begin with if the mule's load is doubled after moving one gallon from the donkey to the mule."

from among six "basic assumption" alternatives. These subjects seem to have (erroneously) fixed the doubling relationship; the effect of shifting wine from the mule to the donkey does not effect their thinking with regard to the first transfer at all.

Paige and Simon (1966), presenting subjects with self-contradictory problems, had several subjects who failed to find the contradictions; the protocols given for such subjects seem to show non-integrative analyses of the problems. These researchers hypothesize that in making operational definitions of variables these subjects identified all noun phrases containing certain key words. It seems clear from the protocols and the experimenters' analyses of them that subjects who failed to find inconsistencies did so because they "fixed" variables in accordance with certain salient relationships and failed to consider appropriately any other restrictions on these variables.

Two fundamental sets of questions about the nature of mathematical problem solving emerge from the observation of this relational fixedness. The first questions concern the nature and extent of such fixedness. Most mathematics teachers would agree that 20, the response obtained by fixing the relation "twice as old as" would be a popular response to the problem "Jane is twice as old as Bob; in five years Bob will be 10. How old will Jane be in 5 years?" Similar examples have been studied by Nesher and Teubal (1975). Yet little is known about the interaction of individuals with relations of this sort. Do some individuals tend to fix many relations, or do some relations tend to invite "fixing?"

The second family of questions arises from comparing results of experimental studies of functional fixedness with the findings of Paige and Simon (1966), Buswell (1956) and others to the effect that subjects do not always (or even usually) spontaneously detect incompleteness or inconsistency in data. Psychologists studying performance on insight problems have observed similar failure of subjects to spontaneously discover (or use) properties inherent in objects at their disposal for use in solving problems. However, Saugstad and Raaheim (1958) found that subjects who "had" the data that a newspaper can be used as a tube and a nail as a hook were able to solve an insight problem while subjects who did not list these as possible functions of the paper and nail were not. With this finding in mind we might ask whether Paige and Simon's subjects "had" the notions "extraneous root" or "insoluble equation"; more generally, does "having" certain bits of mathematical theory and knowledge help in solving algebraic problems?

Data Seeking

In several studies of mathematical problem solving behavior there is a question of the relevance of various data available to the subject (Buswell, 1956; Forehand, 1967; Rimoldi, et al., 1962, 1964; Gormly, 1971). Buswell's data, taken from a questionnaire in which subjects were asked to identify relevant and irrelevant facts included in problem statements, indicates that high school and college students are not very efficient in such identification (Means: 6.5 of 10 relevant and 5.3 of 10 irrelevant facts correctly labelled), and that college students perform no better than high school students on this task. Moreover, three additional pieces of information were needed to solve Buswell's problems; in the mean subjects were able to point out only 1.2 of these missing facts.

Rimoldi et al. (1962, 1964) and Gormly (1971) have studied subjects search for additional data by providing a mathematical problem followed by a series of relevant and irrelevant questions whose answers can be obtained by removing tabs from the problem sheet. Rimoldi and his psychometric group have been especially interested in training subjects to be more selective in the choice of information for consideration, and have shown that such training is feasible, while Gormly's interest has been chiefly in investigating the relationship of the personality trait "comprehensiveness" to the number of questions selected. The data of these authors clearly suggest that many subjects do not use strictly logical criteria in discriminating between relevant and irrelevant data for the solution of mathematical problems.

What Is the Condition?

The conditions explicit in the statements of the problem considered in Table 1 seem to be understood by the subjects. Newell and Simon's cryptarithmetic subject S3 requests (and receives) clarification of conditions at the outset of his protocol and later when he asks whether it is possible to have a digit preceding ROBERT. At times he seems to be clarifying the condition for himself.

Paige and Simon do not report that the ambiguity of the alcohol and water problem caused subjects any difficulty; apparently subjects made an interpretation of the condition and proceeded to work with that interpretation.

As observed above, some subjects, faced with contradictory data (Paige and Simon, 1966) did not consider the "possibility" condition implicit in the problems of the sort presented.

A related phenomenon seems to occur in experimental studies of the solution of non-mathematical problems; one example occurs in Wier's (1971) report on the problem "get as many marbles as you can." For this experiment Wier built an apparatus which dispensed marbles through one of three holes if the appropriate button were pushed at the right time. Marbles were dispensed by the experimenter randomly subject to the constraint that two-thirds of them went to hole number one. Children soon learned to stick with the first button in order to maximize their gains, but adults, apparently believing there had to be a non-random solution, did not.

This phenomenon could well be related to the failure of Paige and Simon's subjects to find inconsistencies in their data; the latter may have felt that there "had to be" an algebraic solution, and therefore accepted their equations as solutions. For some subjects the mere statement of a problem may imply the condition "there is a solution in a standard algebraic form." Over a long period many writers on problem solving in the schools have criticized the problem sets typical of most textbooks and courses as nurturing this expectation (Dilworth, 1966: Henderson and Pingry, 1953; Thorndike, 1922; and a host of others).

Devising a Plan

The second major step in problem solving as outlined by Polya entails the planning of a solution method. Polya's advice in this connection is that the would-be problem solver search his memory for a related problem, restating the problem, if necessary, to find a relevant solution method.

The evidence of psychologists' subject protocols does not support the notion that planning of the sort suggested by Polya is a natural component of subject behavior during problem solving. The protocols published by Newell and Simon (1972) and Paige and Simon (1966) provide no evidence for this type of planning, and, in fact, the extensive protocol for subject S3 solving the cryptarithm D O N A L D + G E R A L D + R O B E R T shows considerable evidence of apparently random efforts to gain insights into parts of the problem. Nowhere in the protocol does S3 state an explicit plan for moving from one step of the solution to another.

Gagne and Smith (1962), on the other hand, have shown that subjects can be encouraged to improve performance by verbalizing--at least for the Towers of Hanoi--and that the development of a strategy does seem to enhance subjects' performance on more complex (i.e., more discs) problems of the same sort. These experimenters assigned subjects to four groups for training on this problem with 3 and 4 discs. Subjects in two of these groups were instructed to verbalize their solution processes, giving reasons for each move. Moreover, one verbalizing group and one non-verbalizing group were instructed to try to devise a general strategy. All groups were then given the five disc problem with the result that verbalizers used only 7.9 (strategy group) and 9.3 (non strategy) moves more than the minimum while nonverbalizers made 48.1 and 61.7 extra moves, respectively. It is clear from these data that the verbalizers had a considerable advantage; it seems that those subjects who worked on a general strategy may have had some slight advantage although these differences are not significant. All of the verbalizers were able to state a complete (6) or partial (8) verbal principle for solving this problem, whereas only 7 of the 14 nonverbalizers were able to state a partial principle (of these 7, 6 were in the strategy group.

Even if Gagne and Smith's data is accepted as demonstrating some advantage for planning solutions, these same investigators provide indirect evidence that detailed planning is not inherent in the solution processes of naive subjects. Many of the reasons given by the verbalizing Ss for moves made in the three- and four-disc problems were classified as "just to try it," "don't know why," or "the only possible move." Some subjects did verbalize moves in the direction of an immediate subgoal, e.g., "to free up a space" or "to get at a larger disc." Gagne and Smith do not report any evidence that subjects who mentioned subgoals had an advantage over subjects moving one piece at a time.

A problem in some senses similar to the Towers of Hanoi was used by John Hayes (1966) in his study of subgoals as a part of the solution process. Hayes' spy problems are based on the notion that not all spies in a spy-ring can communicate directly with each other. In these experiments subjects first memorize a list of eleven pairs of spies who may talk to each other, which they learn to a stiff criterion. They are then given tasks such as "Get a message from Shower to Horse," and the solution time is measured. In order to determine whether subjects were planning a strategy for delivering the message or simply moving it one step at a time, Hayes introduced subgoals by altering the problem statement. While some subjects were given a problem such as "Get a message from Joe to Cat," for others the problem statement was modified to include <u>necessary</u> steps in the solution process yielding, for example, "Get a message from Joe through Ape and Waterfall to Cat." The addition of this subgoal information had the effect of <u>increasing</u> the time required to solve the problem even though the only solution to the problem involved the information given in the subgoal condition (e.g., sending the message through Ape and Waterfall).

Thus Hayes has shown that experimenter suggested plans are far more helpful to subjects in the process of solving problems of this type. Moreover, since the stages selected for giving to the subject were necessary parts of the message delivery systems, these data suggest strongly that subjects did not plan their solutions in advance.

For his 1956 study of patterns in solution of the donkey and mule problem, Buswell first collected voice protocols from a group of pilot subjects. He then classified the steps used by pilot subjects into 81 categories. A second group of subjects were then asked to choose the steps of their solutions from among these 81. Finally the 81 were narrowed to 38 which were arranged in a tree-like structure. The final sample of 61 subjects were then given the problem, and were instructed to choose branches successively until they traversed the tree to a solution. Buswell's data shows no consistent pattern of solution. Nor do the data describing successive attempts of a single subject to solve the problem reveal any planning.

In a sense Buswell's first choice point, "Will you solve this by algebra or logic?" forces subjects to commit themselves to a plan. However, subjects who failed in their first attempts frequently changed from one mode to another for the next attempt. In only 7 of the 104 comparisons of nth with (n + 1)st attempts of a single subject is there consistency through three or more stages of problem solution, suggesting that subjects were not really planning strategies.

The evidence cited above seems to indicate that subjects developing their own solutions to problems with mathematical or relational content do not spontaneously develop long range solution plans, but rather that they move from one step to another.

Polya, of course, does not say that subjects _do_ plan a solution method, but rather that they _should_. The evidence cited above does not tell us definitely whether this is good advice or not. The results of Forehand suggest that it might be, while those of Hayes suggest that it is not.

Finding a Related Problem

The second aspect of Polya's advice to problem solvers who are seeking a _method_ of solution is that they search for a similar problem with which they are familiar, or restate the problem in more familiar terms. The literature reviewed here provides some evidence that subjects do use such similarities in their solution processes. For example, subjects in Paige and Simon's study of solving algebraic word problems exhibit the use of structural similarity of problems when they spontaneously draw diagrams, or translate the verbally stated problem into an algebraic statement.

Suppes, Loftus, and Jerman (1969) in their study of the performance of fifth graders on arithmetic problems found that the single most important variable effecting problem difficulty was the nature of the previous problem; problems were easier when preceded by problems of a similar type than when preceded by dissimilar ones.

Buswell's generalization problem, on the other hand, provides an example in which the appropriate relationships among problems was not observed. In Buswell's experiment subjects were presented with addition problems such as 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 for which they were to discover a general rule for rapid addition. Subjects were given clues until they reached such a generalization. Subsequently subjects were given the problem:

How many x marks are in the pyramid below? There are 9 in the center column.

```
        x
       xxx
      xxxxx
     xxxxxxx
    xxxxxxxxx
   xxxxxxxxxxx
  xxxxxxxxxxxxx
 xxxxxxxxxxxxxxx
xxxxxxxxxxxxxxxxx
```

Buswell reports that only 23 of the 60 subjects who made a correct generalization for the arithmetic problems were able to apply it to this figural problem. In this case the majority failed to discover a similarity to problems they had just finished working on, if indeed they were "trying to find similar problems." The similarity of this problem with those previously solved was apparently too weak for transfer to operate.

The well-documented results of the Luchins' (1942, 1950) water jar problems provide evidence of a different sort. Subjects who were given a series of eight problems involving jars A, B, and C, all having solution B - 2A - C, were unable to find the solution A - C for a subsequent problem. In this case the similarity of the test problem with previous problems was too strong for subjects to overcome; if subjects were indeed looking for similar problems they found them quickly and to their detriment.

The question of what constitutes a useful similar problem for subjects in a mathematical problem solving situation is thus one which should be investigated empirically.

Technical Aspects (Polya's Carrying Out the Plan)

There is a strong tendency among mathematics teachers to distinguish between "setting up a problem" and "solving equations:" very likely this distinction is related to Polya's definition of a third stage in problem solving as "Carrying out the Plan." As we have seen above, it is not clear that subjects actually develop a plan to carry out!

Nonetheless the issue of the relationship of solving equations and/or performing computations to the ability of subjects to handle algebraic problems is of some importance. More than one subject in Paige and Simon's study of algebraic problems apparently felt he had solved the problem once he had established a linear equation descriptive of the data given, even though the equation had no solution consistent with the physical situation (owing to inconsistencies in the data)! The beliefs of these subjects that the equation was equivalent to the solution of the problem seems to have caused their failures to discover the inconsistencies.

In situations where no such inconsistencies exist the relationship between rote computational ability and "creative" problem solving is an unresolved issue with some evidence supporting the view that there is a high correlation between these abilities and other evidence supporting their independence.

Suppes, et al. (1969) observed that their problem solving subjects seemed to prefer certain modes of computation over other equivalent modes [e.g., the form a - (b + c) is more popular than the equivalent (a - b) - c], and suggested that the area of computational preferences and habits should be studied.

Buswell's (1956) monograph includes a description of a study of the order in which computations were performed. Subjects were presented with three problems, each involving the multiplication of four numbers; these problems were to be solved on a specially designed form from which the experimenter could determine the order in which the computations were made. Bu ll's fairly elaborate analysis of computational patterns yield no significant relationships with the order in which the numbers were presented in the problem statement, nor with the magnitude of the numbers.

More recently Burns and Yonnally (1964) have completed a similar study and claim that problems in which the data are presented in the "natural" order or the "order in which it is used" are easier than others. However, in the sample problem provided by these researchers fractions must be used when computations are performed in some orders, but not others, and this could explain the differences in difficulty.

In both of the studies cited above the problems used involved only multiplication and division. Since the order of operations does not effect the answer to such problems, the importance of these studies *per se* is questionable. However, they do suggest that similar studies in which the order of operations is important should be done.

Along this line Kennedy, Eliot, and Krulee (1970), while investigating error patterns of problem solvers, found that low ability subjects were somewhat more likely to use data in the order in which it is presented than high ability subjects; this result was only marginally significant, however, and the phenomenon needs further study.

A related issue which seems to have received very little attention is the question of whether and how the complexity of the numbers involved in an algebraic word problem effects the nature of the solution process (distinguished from the computational process). This author hypothesizes that subjects' performance on a problem such as the donkey and the mule problem would be greatly effected if it were restated so that 3/7 gallon or .9423 liters were moved from one beast to another, and that the differences in subject behavior would be observable throughout the problem solving process, and not limited to computational phases of it.

Examination of the Solution Obtained

An important aspect of mathematical problem solving is the final judgment of the solver that he has arrived at a complete and correct solution. Hadamard (1945) speaks eloquently of the relation of this aspect to "good" mathematical problem solving:

> ". . . in our domain, we do not need to ponder on errors. Good mathematicians, when they make them, which is not infrequenty, soon perceive and correct them. As for me (and mine is the case of many mathematicians), I make many more of them than my students do; only I always correct them so that no trace of them remains in the

> final result. The reason for that is that whenever an error has been made, insight--that same scientific sensibility we have spoken of--warns me that my calculations do not look as they ought to." (p. 49)

The psychological literature bearing directly on this point is not as large as in the other areas discussed above. For insight problems the process of examining the solution is generally trivial--either one has or has not succeeded in tying together two strings! In minimization of steps problems such as the Towers of Hanoi, on the other hand, the complexity of a proposed solution ($2^n - 1$ steps if correct) is such that the subject can hardly be expected to remember it, let alone examine it for errors.

The question of whether subjects spontaneously examine their solutions to problems for accuracy would seem to be moot. Newell and Simon's S3 did verify his final solution to DONALD + GERALD = ROBERT; earlier in the protocol he expressed the intention to examine his prior work: "Now I'm going back to see if I've made some obvious fallacy." He was precluded from doing so by the experimenter's remark, "You haven't made any obvious fallacies." On the other hand Paige and Simon's subjects fail to examine their solutions spontaneously--and thus do not detect the fallacies involved in some of the problems. When led by the experimenter to reconsider these problems the subjects cited were able to analyze the problems and detect the inconsistencies involved.

Buswell, investigating whether subjects could estimate reasonable solutions to problems in order to "check" computationally derived answers, found great deviations from the anticipated estimates. Many subjects computed the actual figure rather than estimating and 14 percent of these (high school) subjects were unable to cope with the problem at all. It is therefore doubtful that Hadamard's insight would warn these subjects when their "calculations do not look as they ought to."

The Structuring of Problem Situations

Searching for a related problem is but one possible method of acting on and reforming a similar problem; while the psychological literature on problem solving does not support the notion that seeking similar problems is either popular or especially fruitful as a method of approaching problems, this literature does reveal that successful problem solvers restructure problem data during the solution process.

The importance of understanding the structure of a problem and altering the problem in successive stages through structurally consistent steps was first proposed by the Gestalt psychologists in the early forties (Duncker Wertheimer, Hartmann). According to the Gestalt analysis a problem solving subject moves from one state of affairs to another which is structurally similar. When a subject fails to solve a problem it is because he has an inadequate view of the situation; he may be lacking in the breadth of his view of the problem situations, or conversely, viewing the structure of the situation too broadly.

More recent work tends to support the view that problem solving involves structuring and restructuring the information given. Forehand (1967) studied organization of information by problem solvers in a task involving matching pairs of individuals. Subjects in this study were given problems followed by a series of facts in scrambled order; these facts were to be recorded and used in the solution of the problem. The records of facts made by subjects were analyzed to determine whether data were organized in accordance with 0, 1, 2, or 3 of the available variables. All subjects sorted on at least one variable, and analysis of variance showed that success in solving the problems was significantly related to the number of variables used in organizing the data.

In a study along similar lines Schwartz (1971) investigated the behavior of problem solvers given "who done it" problems involving affirmative and negative statements. Subjects were encouraged to write down everything they did. Examination of these records revealed that for positively stated problems those subjects who organized data in matrix-like structures were most likely to solve the problems. In a follow-up to this study Schwartz and Fattaleh (1972) gave subjects "who done its" with the data presented in various modes (sentences, matrices, networks). Contrary to their hypothesis that subjects would fare better with problems presented in the matrix mode, these authors found that success was influenced most by the ability (or inclination) of the subject to restructure the data, regardless of the form in which it was presented.

The main burden of these research findings is thus consistent with the Gestaltist position. The job of the problem solver is apparently not to seek (and perhaps be misled by) similar problems, nor to work at itemized solution plans, but to comprehend the internal unity and structure of the problem.

Summary and Conclusions

While the research findings cited above suggest many interesting hypotheses which should be pursued in both laboratory and educational research settings, their major implication for mathematics education lies in their pinpointing of three parts of Polya's solution scheme as more important to the solution procedures of subjects than the others. These three, "What are the data?," "What is the condition?" and "Can you restate the problem?" are all concerned with the understanding and analysis of the problem as a unit, rather than with its similarities to other problems or its place in a hierarchy of specific and general problems. These questions are related to each other in that the restatement of a problem depends upon the organization (or reorganization) of the data in a manner consistent with the condition.

The findings of psychologists suggest, moreover, that it is difficult to separate the subjects' treatment of data and condition. In the analysis above we have hypothesized two mechanisms which tend to enhance this identification, especially in subjects who fail to solve problems:—"relational fixedness" which can be thought of as the tendency to treat some salient relationships as absolute and unchanging; and the tendency of subjects to alter problem statements by the addition of conditions (and related "data") which seem to be determined by their past experience. These phenomena are closely linked and seem to occur because problem solvers have difficulty in overcoming various types of mental associations.

The documented importance of restating or restructuring problem situations rests, then, on the singular effectiveness of this strategy in eliminating non-essential associations. While the dictum "Think of a similar problem" may invite the subject to build on irrelevant associations, or even invent new ones, the admonition to restate the problem demands that he carefully analyze the constituent parts and the relationships among them in order to derive a new and usable statement.

REFERENCES

Burns, Paul C., Yonally, James L. "Does the order of presentation of numerical data in multistep arithmetic problems affect their difficulty?" <u>School Science and Mathematics</u>, 1964, 64, 264-270.

Buswell, Guy T. <u>Patterns of Thinking in Solving Problems</u>, University of California Press, Berkeley: 1956.

The Conference Board of the Mathematical Sciences, <u>The Role of Axiomatics and Problem Solving in Mathematics</u>, Washington, D.C.: 1966.

Dilworth, R. P., The Role of problems in Mathematical education, <u>The Role of Axiomatics and Problem Solving in Mathematics</u>, Conference Board of the Mathematical Sciences, Washington, D.C.: 1966.

Duncker, K "On problem solving," <u>Psychological Monographs</u>, 1945, Vol. 58, whole no. 270.

Forehand, Garlie A., <u>Individual Differences in Problem Solving Processes of College Students</u>, Carnegie Mellon University, Pittsburgh, Pa., 1967.

Gagne, Robert M., Smith, Ernest C., Jr., "A study of the effects of verbalization on problem solving," <u>Journal of Experimental Psychology</u>, 1962, 63, 12-18.

Gormly, John, "Cognitive Structure: Functional unity in verbal and mathematical performance," Journal of Personality, 1971, 39, 70-78.

Hadamard, Jacques, The Psychology of Invention in the Mathematical Field, Princeton University Press, 1945.

Hartmann, George W., Gestalt Psychology and Mathematical Insight, 1937, reprinted in Teaching Mathematics: Psychological Foundations, F. Joe Crosswhite, et al., eds., Worthington, Ohio, 1973.

Hayes, John R., Memory, goals, and problem solving, in Problem Solving: Research, Method and Theory, B. Kleinmuntz, ed., New York, 1966.

Henderson, Kenneth B., Pingry, Robert E. , Problem Solving in Mathematics, in The Learning of Mathematics, Its Theory and Practice, Twentifirst Yearbook, National Council of Teachers of Mathematics, Washington, D.C., 1953.

Kennedy, George, Eliot, John, Krulee, Gilbert. Error patterns in problem solving formulations, Psychology in the Schools, 1970, 7, 93-99.

Luchins, A. S., Mechanization in problem solving; the effect of Einstellung, Psychological Monographs, 1942, 54, No. 28.

Luchins, A. S., and Luchins, E. H. New experimental attempts at preventing mechanization in problem solving, Journal of General Psychology, 1950, 42, 279-297.

Nesher, Perla and Teubal, Eva. Verbal cues as an interfering factor in verbal problem solving. Educational Studies in Mathematics, 1975, 6, 451-51.

Newell, Allen, Simon, Herbert A., Human Problem Solving, Englewood Cliffs, New Jersey.

Paige, Jeffrey M., Simon, Herbert A., Cognitive Processes in solving algebra word problems, in Problem Solving: Research, Method and Theory, B. Kleinmuntz, ed., New York, 1966.

Polya, George, How to Solve It, Princeton University Press, 1945.

__________, Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving (two volumes), New York, 1962.

__________, On teacher problem solving. In The Role of Axiomatics and Problem Solving in Mathematics, Conference Board of the Mathematical Sciences, Washington, D.C., 1966.

Rimoldi, H. J. A., Fogliata, Hermelinda M., Haley, John V., Reyes, Isabel O., Erdmann, James B., Zacharia, Rose Mary, Training in Problem Solving, Loyola Psychometric Laboratory, Chicago, Illinois, 1962.

Rimoldi, H. J. A., Fogliata, H. M., Erdmann, J. B., Donnolley, M. B. Problem Solving in High School and College Students, Cooperative Research Project No. 2199, Loyola Psychometric Laboratory, Chicago, Illinois, 1964.

Saugstad, P., Raaheim, K., Problem solving, past experiences, and availability of functions. In Thinking and Reasoning, P. C. Wason and P. N. Johnson-Laird, eds., Middlesex, England, 1968.

Schwartz, Steven, Modes of representation and problem solving: well evolved is half solved, Journal of Experimental Psychology, 1971, 91, 347-350.

__________ and Fattaleh, Daniel L. Representation in deductive problem solving, the matrix. Journal of Experimental Psychology, 1972, 95, 343-348.

Suppes, Patrick, Loftus, Elizabeth F., Jerman, Max. Problem solving on a computer-based teletype. Educational Studies in Mathematics, 1969, 2, 1-15.

Thorndike, Edward L., The psychology of problem solving, The Mathematics Teacher, 1922, 15, 212-227, 253-264.

Weir, Morton, Developmental changes in problem solving strategies, Psychological Review, 1964, 71, 473-490.

Wertheimer, Max, Productive Thinking (Enlarged Edition, Michael Wertheimer, ed.), New York, 1959.

Table 1. Analysis of six problems used i

Problem	Understanding the Problem			Devising a
	What is the unknown?	What are the data?	What is the condition?	Think of a related problem
Two string	Method of getting two strings together	Experimenter directives, distance to be overcome, properties of objects in the room, hints	Strings to be tied while hanging from ceiling	"Real life" problems involving pendulums, plumb bobs, etc.
Water Jar Problems	Method of obtaining specified amount of water	Capacities of containers, rules for measurement	Infinite source, precise measurement	Previous water jar problems
Towers of Hanoi	Most efficient legal method of moving stack	Beginning state, terminal state, definition of legal move	Discs can be moved only by legal moves, number of moves to be minimized	Relate to problems with smaller stacks, other games and puzzles
Cryptarithm DONALD +GERALD ROBERT	Digits represented by the ten letters	D=5, additive relationship, positional relationships (decimal notation)	Each letter represents a different digit	Similar puzzles
Donkey and Mule	Number of gallons on donkey, number of gallons on mule	Ratios of loads under two transformations	Total load is constant, transfer would produce given ratios	---Mixture Problems, Other problems from High School Algebra
Water/ Alcohol	Amount of water to be added	Initial solution, size of tank, nature of final solution	AMBIGUOUS water to be added, or solution to to be replaced to yield 80% solution	---Algebra, Physics, Chemistry and other science

Plan: Think of a problem with similar unknown	Plan: Can you restate the problem	Carry out your plan	Examine your results
"Real life" problems of making things fit, stretching, extending	???	Psychomotor task	Trivial
"Real life" problems involving measurement of quantities differing from measures available	Possible diophantine equation	Solve equation	Check the solution
Any minimization of steps problems	Establish subgoals?	Complex execution of sequential pattern of moves	Analysis of sequence (learning during solution process, errors) Heavy memory load
???	Digit-by-digit	???	Perform addition, check for double usage
	Transform data; algebraic, logical	Solve equations Select and test possible solutions	Do obtained values satisfy condition?
	Picture? Translate data, equation(s).	Solve equations	Do obtained values satisfy the condition? Are they physically feasible?