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ABSTRACT

This report, prepared by a statewide Mathematics Advisory Committee, revises the framework in the Second Strands Report of 1972, expanding it to encompass kindergarten through grade 12. Strands for kindergarten through grade 8 are: arithmetic, numbers, and operations; geometry; measurement, problem solving/ applications; probability and statistics, relations and functions, and logical thinking. Goals of mathematics instruction and the early education program are discussed as well as specific program objectives for each strand. The strands for grades 9 through 12 are: arithmetic of real numbers, algebra, geometry, measurement, problem solving/applications, probability and statistics, relations and functions, logical thinking, and computers. Goals of instruction and objectives of the mathematics program are presented followed by general discussion of minicourses, remedial clinic programs, resource centers, programs for talented students, and college preparatory programs for technically and nontechnically oriented students. The appendices present: (1) program objectives by levels for each strand kindergarten through grade 8; (2) a time line for implementation of the metric system; and (3) criteria for evaluating instructional materials for kindergarten through grade 8. (JW)

Mathematics Framework

for California Public Schools

Kindergarten Through Grade Twelve

CALIFORNIA STATE DEPARTMENT OF EDUCATION
Wilson Riles—Superintendent of Public Instruction
Sacramento, 1975

Mathematics Framework for California Public Schools

Kindergarten Through Grade Twelve

Adopted by the California State Board of Education May 9, 1974

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Foreword

The "post-new math framework" might be an appropriate title for this document to signal its timing. With the demise of "new math," this framework presents improved mathematics to fill the void. The contents reflect the concerns of teachers rather than those of mathematicians.

The new framework identifies the child as the central figure in the educational scene, and that is as it should be. "The teacher assumes the role of a guide," say the writers of this document, a guide "who directs learners to explore, investigate, estimate, and solve everyday, realistic, pupil-oriented problems."

The "metric framework" might be another title ascribed to this document, because it establishes the International System of Units (SI) as the standard for measurement. With my endorsement and encouragement, the writers submitted and won this concession from the State Board of Education.

However, my preference for a title is the "basics framework," because the major concern of the writers is clearly the increased use of sound teaching techniques to enable California schoolchildren to learn basic mathematics. I wholeheartedly support this approach, and I hope for every teacher and student the excitement that comes with this way of teaching and learning.

Wilson Riles

Superintendent of Public Instruction

Acknowledgments

Ad Hoc Mathematics Framework Committee

Clyde L. Corcoran, *Chairman;* Teacher and Chairman, Mathematics Department, Whittier Union High School District

Shirley S. Myers, *Subcommittee Chairwoman;* Senior Mathematics Teacher, Temple City Unified School District; and Member, Curriculum Development and Supplemental Materials Commission

Carolyn H. Aho, District Mathematics Resource Teacher, San Francisco Unified School District

Lucile S. Borad, Retired Teacher, Hayward

Dwight O. Coblentz, Curriculum Coordinator, Mathematics/Computer-Assisted Instruction, Office of the San Diego County Superintendent of Schools

Gloria F. Cox, Vice-Principal, Garfield High School, Los Angeles Unified School District

Floyd Downs, Mathematics Teacher, San Mateo Union High School District

Rose Eleanor Ehret, SNJM, Associate Professor of Mathematics, Holy Names College, Oakland

Verda F. Holmberg, Elementary Teacher, Mt. Diablo Unified School District

Vera White, Teacher Assistant in Reading and Mathematics Development, Oakland Unified School District

Robert Tardif, Manager, Mathematics Task Force, California State Department of Education

Consultants to the Ad Hoc Mathematics Framework Committee

Arthur A. Hiatt, New Haven Unified School District

Robert L. McFarland, Office of the Alameda County Superintendent of Schools

Henry B. A. Palmer, Office of the Los Angeles County Superintendent of Schools

Judith Salem, Tamalpais Union High School District

Delmer G. Lansing, California State Department of Education

Fred Lorenzen, California State Department of Education

Joseph Hoffmann, California State Department of Education

Elaine H. Stowe, California State Department of Education

6

Preface

In 1963 the first framework for mathematics was published by the California State Department of Education. It was commonly referred to as "The Strands Report" because Part One of the framework outlined eight fundamental concepts or strands which "tied" the mathematics curriculum together in kindergarten through grade eight. Also considered in the framework was the dynamic character of good mathematics instruction; that is, pupils should be encouraged to guess, to experiment, to hypothesize, and to understand through active participation in the teaching-learning process.

The Second Strands Report (*Mathematics Framework for California Public Schools: Kindergarten Through Grade Eight*) was accepted by the State Board of Education in 1968. In this second report, the network of strands was designed as an integrated whole, and a satisfactory instructional program was described as one that would provide a balanced emphasis upon each of the strands.

The Statewide Mathematics Advisory Committee (SMAC), 1967–1970, which prepared the *The Second Strands Report,* was charged by the Curriculum Development and Supplemental Materials Commission and the State Board of Education to consider a suitable extension of the strands concept through grade twelve. Under the direction of its chairman, John L. Kelley, the advisory committee sponsored a conference of 45 participants, including mathematicians, scientists, secondary teachers of mathematics and science, and persons using mathematics in industry and computer technology. The present Ad Hoc Mathematics Framework Committee, whose members were appointed by the Curriculum Development and Supplemental Materials Commission in October, 1973, is indebted to SMAC for the work it accomplished. The framework committee gathered information from agencies, teachers, professional organizations, and concerned individuals; and it conducted meetings throughout the state in an attempt to ensure that a variety of opinions would be heard.

Because the mathematics program for individual high school students varies according to their interests, skills, and career objectives, the strands for the high school level were designed to respond to the flexibility of school programs. In this framework is contained what a total mathematics program can and should provide for high school students. Although the Department of Education plays no direct role in the selection of materials for mathematics programs in grades nine through twelve, the Department believes that this framework provides information useful to those responsible for mathematics programs at every level.

Those responsible for the selection of materials for mathematics programs in kindergarten through grade eight should find that the screening criteria contained in the framework are quite useful. The updated criteria reflect a number of concerns about the acquisition of basic mathematics skills that prevailed at the time the framework was revised. It is anticipated that the forthcoming statewide adoption of materials in mathematics will reflect the impact of this publication.

In the development of a school mathematics program for California, it seems pointless to refer to contemporary mathematics programs as the "new math." Our concern should be to make the very best in mathematics curriculum and instructional practices available to our students and teachers. We are also interested in informing the public that high-quality preservice and inservice teacher education programs are needed to prepare people to teach mathematics with the knowledge, skill, and enthusiasm necessary to serve our pupils with excellence.

The Ad Hoc Mathematics Framework Committee is hopeful that this publication will provide a set of creative guidelines for teachers, authors, and publishers to employ in the development of instructional materials. Further, we expect that the report will be useful to administrators and teachers in the development of instructional materials and comprehensive mathematics programs which have objectives consonant with the needs of pupils and society.

This publication is the result of the combined efforts of many interested and concerned individuals, and we express our thanks to them, especially to Clyde Corcoran, Chairman of the Ad Hoc Mathematics Framework Committee, and to the members of his committee, who are listed on page iv. We also thank the individual school districts, the offices of the county superintendent of schools, and the college that provided released time for their selected staff to complete the framework.

DONALD R. McKINLEY
Chief Deputy Superintendent of Public Instruction

WILLIAM E. WEBSTER
Deputy Superintendent for Programs

Contents

9

Introduction

The California State Board of Education, recognizing the need for a reappraisal of *The Second Strands Report: Mathematics Framework for California Public Schools* published in 1972, mandated the development of a new and more extensive framework. This 1975 framework will encompass the mathematics program from the kindergarten level through grade twelve.

This is the third mathematics curriculum framework developed for use by California public schools. *The Second Strands Report* provided an excellent basis for mathematics curriculum development in the state, but a continuous assessment of a mathematics framework is needed because of expanding information and knowledge, shifting emphasis in subject areas, and changing organizational patterns for instruction.

The principal assumption which underlies the thinking of this ad hoc framework writing committee is that the school mathematics program should be designed to educate each child to the child's optimum potential in mathematics.

The recommendations of the revised framework for kindergarten through grade eight reflect the following changes in emphasis:

1. An *increased emphasis* on the application of mathematical concepts to physical objects familiar to children
2. An *increased emphasis* on computational skills along with the development of the structural aspects of mathematics
3. An *increased emphasis* on ways to improve children's attitudes toward mathematics
4. An *increased emphasis* on metric units known as the International System of Units, which will be the basis for standard measurement instruction
5. An *increased emphasis* on the total concept of decimal numbers
6. An *increased emphasis* on application and problem-solving skills
7. A *decreased emphasis* on numeration systems other than the familiar decimal-based system
8. A *decreased emphasis* on the computation of fractional numbers in kindergarten through grade six
9. A *decreased emphasis* on set theory

The purpose of a framework is to provide a base from which schools, school districts, and offices of county superintendents of schools can develop adequate goals and objectives for their programs. In addition, the mathematics framework provides the basis for the development of criteria for the evaluation of instructional materials to be considered for adoption by the state of California. This mathematics framework contains a description of the major components of the school mathematics program, kindergarten through grade twelve. These components are:

- *Broad goals and objectives*
  General goals for learners
  Content and topic goals
  Program objectives
- *General content guidelines*
  The elementary strands
  The secondary strands
- *Methods and materials*
  Classroom climate
- *Suggestions for program evaluation*
- *Criteria for screening instructional materials*

The Climate and Environment for Learning Mathematics

The most effective and efficient climate and environment for learning provides for the following:

- Experience with objects from which the learner can develop concepts
- A means of communication that the learners can understand
- Opportunities for learners to become involved in activities
- Opportunities for the teacher to study the learner's habits of work and thought
- Motivation for learners to continually improve their proficiencies in mathematical skills and concepts

The kindergarten through grade twelve mathematics program provides for the following:

- A rich variety of opportunities for the learning of mathematical concepts
- The application of these mathematical concepts to socially useful mathematical problems
- The accumulation of mathematical maturity and proficiency for use in other disciplines
- A climate for learning

Learning is a group experience in that group behavior affects the learning process, as pupils do learn from one another. Mathematics becomes a vibrant, vital subject when points of view are argued, and for this reason interaction among pupils should be encouraged. As pupils build mathematics together, they develop special pride in learning activities, and their work gains momentum. Manipulative materials provide effective means for facilitating learning. These materials are often simple and can be pupil-made or collected. Manipulating may mean handling an object, comparing objects, viewing objects represented in a pictorial mode, or engaging in paper-and-pencil activities. The materials should provide a smooth transition from concrete learning experiences to the abstract.

A significant feature of a mathematics learning environment is the spirit of free and open investigation. The learning of mathematics is many-faceted. Pupils and their teacher must feel free to express and explore those facets that have particular meaning for them. The classroom environment is an important but often overlooked facet. It should be organized and equipped to appear as a laboratory for learning and should relate learning to past experiences while providing new experiences as needed. Well-equipped and organized classrooms allow pupils to accept the responsibility for their own learning and progress.

The learning climate in the classroom should provide an atmosphere of open communication between pupil and teacher. The teacher should encourage questions and accept problems from the pupils. The mathematics' instructional materials should be relevant to the pupil's interest and needs and should provide for pupil experimentation.

The establishment of a classroom climate, under the direction of the teacher, should be pupil oriented, self-directed, and non-threatening. Using definable instructional objectives, the teacher assumes the role of a guide who directs learners to explore, investigate, estimate, and solve everyday, realistic, pupil-oriented problems.

The ideal classroom climate fosters the spirit of "discovery." It provides a variety of ways for pupils to direct their own learning under mature, patient guidance of an experienced, curiosity encouraging teacher. Self-directed learning requires pupil involvement in creative learning experiences that are both pupil motivated and teacher motivated. The classroom climate should encourage pupils to solve problems in a variety of different ways and accept solutions in many different forms. All pupils should express creative thinking even when it differs from the pattern anticipated by the teacher or when it produces a different conclusion or result.

Instructional materials adopted by the state to implement the mathematics program should be sufficiently flexible to be able to be used with a variety of teaching methods and organizational plans. Whether or not ability grouping takes place, it is clear that in any classroom the rates of learning will vary, and the pacing of instruction should be planned accordingly. Perhaps of more significance, the pupils' modes of thinking will differ: some think best in concrete terms; others, in abstract formulations. The introduction of a new mathematical concept should be done in such a way as to appeal to each of these ways of thinking.

Mathematics Program Evaluation

Program evaluation is a sequence of activities that culminates in a judgment about the success or failure of a program. In evaluating a program at any level, one must provide a response to the question, "Did the program achieve its objective(s)?" The evaluation discussion which follows is designed to provide information useful at the classroom level.

Teachers conduct their classes so that pupils learn mathematics. In order to evaluate their own efforts, teachers use a variety of tools and techniques to assess the progress of each pupil. If pupils do not progress as expected, then the programs should be modified or expanded to accommodate the talents and the needs of those pupils. If the objectives of a mathematics program are reasonable and comprehensive, the quality of the program can be measured according to the accomplishment of those objectives.

Evaluation is a multipart process. Program objectives should be stated and based on an assessment of the needs of pupils. The target population then can be surveyed to ensure an accurate appraisal of its needs, and the program can be adjusted to reflect current conditions in the pupil population. When the instructional program is complete, the population can be assessed to determine the degree to which the program objectives were accomplished.

Matrix sampling is used in California. The procedure requires the development of a pool of test items that provide comprehensive coverage of the mathematics content. After a valid pool of items has been developed, the items are distributed randomly to a number of subtests so that subtests are of similar difficulty. The subtests are randomly assigned and administered to pupils in the examinee population. The underlying theoretical model permits the resultant data to be used to estimate the achievement characteristics of the examinee population as if every examinee in the population had responded to every test item on every subtest. At the school level, the examinee population might be all the sixth-grade pupils in the school. At the classroom level, the examinee population would include all the pupils in the class. This testing procedure has potential for use at the district level, at the school level, and even at the class level.

When the examinee population is small, it is necessary to administer a greater number of items to each examinee to preserve, at least partially, the integrity of the results. Models for the development of a matrix sampling plan are presently available. However, the procedure estimates group characteristics only and cannot be used to measure the achievement of individuals.

The progress of individual pupils in a class is vitally important at the class, and possibly the school, decision-making level. If one wishes to learn of the needs of a particular pupil, it is necessary to consult pupil personnel files, to conduct diagnostic testing and interviews, to observe the learning behavior of the pupil, to utilize achievement test results and teacher-made test results, and to consult parents regarding the status of their children. It is essential that teachers learn as much about their pupils as they can to better serve pupil needs. A needs assessment process sets up the pupil-level objectives which direct the teacher's behavior. Thus, teaching behavior can be directed both by total class achievement and by the achievement of individual pupils.

A cyclical evaluation process that teachers could employ is presented in Figure 1. The cycle is entered by making a preliminary review of the accomplishments and talents of the pupils in the class. That needs assessment gives rise to a tentative set of objectives and a corresponding mathematics program. While the program operates, the teacher uses various tools and techniques to gather data on the condition of pupils in the class. The teacher also seeks parental input regarding the status of the children with respect to school activities. This interim information-gathering activity provides feedback about the progress of pupils and provides a quasi-scientific basis for making program adjustments to better accommodate pupil strengths and weaknesses. That continual needs assessment activity is the link between program development and program relevance.

As the time allotted to the program runs out, the teacher should complete the development of the final evaluation system. The testing instruments for assessing the achievement of individual pupils should be selected or developed with both the stated program objectives and the pupil-level objectives well in mind. In developing an item pool for matrix sampling applications (usually accomplished with the cooperation of other teachers), the program objectives should be used to

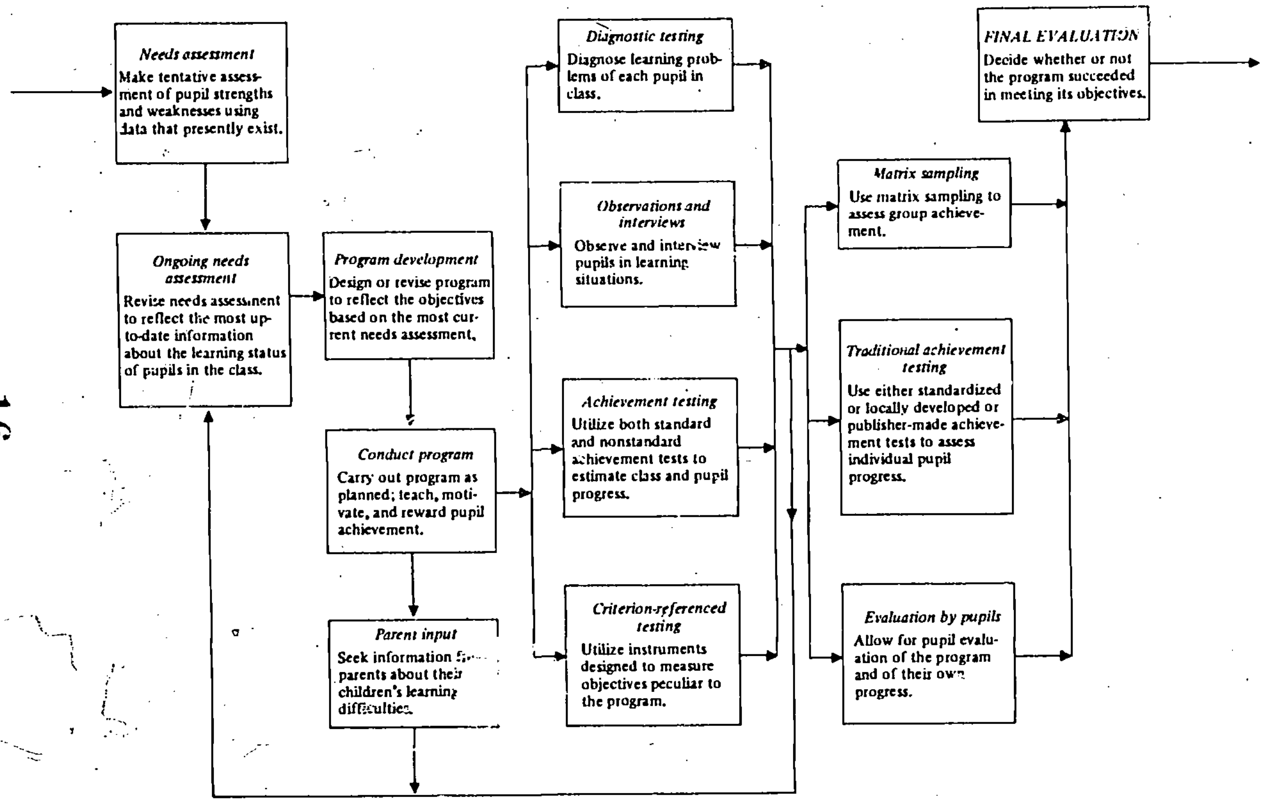

Fig. 1. The schematic of a plan for evaluating a mathematics program at the classroom level

ensure comprehensive coverage of the content domain, both topically and by level of development or cognition. When the final evaluation activities are carried out and the data on the total program impact are in, then it is necessary to judge how successful the program has been. Did it accomplish its objectives? Should it continue or should it be changed or dropped?

The procedures outlined in this section should help to remind the reader of the things that might be done in evaluating a program. However, the philosophy which governs evaluation is eminently more important than the tools alluded to earlier. At the class level, the teacher must deliver a program to pupils which fits the needs of those pupils, a program that is designed to help pupils learn the mathematics they must know to enrich their lives. The teacher must keep the program abreast of pupil needs. And, finally, the teacher must realize that the program is successful only if it serves to diminish the learning difficulties of the pupils in the class.

The Mathematics Program in Kindergarten Through Grade Eight

Mathematics programs should respond to the needs of children and the needs of a career-oriented society in presenting the content and structure of mathematics. While the study of mathematics for its own sake is possible in such a response, it is not likely to be the prevailing reason for mathematics study by children. When children become aware of the fact that the study of mathematics tends to open up certain career options, their inclination or enthusiasm for mathematics increases. Some children acquire such awareness slowly. It is recommended that programs be designed to permit pupils to study and learn mathematics as long as they attend school, regardless of their level of attainment in the subject matter or the lateness of their decision to engage in further study.

Goals of Mathematics Instruction

As a result of mathematics study, pupils should learn to function smoothly in their everyday encounters with mathematical situations. Their studies should allow them to advance to further study commensurate with their ability and desire to so do. The study of mathematics should also acquaint pupils with the richness of the design of mathematics to allow that element of beauty to become a part of their knowledge. It is expected that most pupils will not become mathematicians; however, no program should prevent such a career outcome.

A number of other goals for programs have had a pervasive influence on the preparation of the framework. They are listed below:

- Mathematics programs should progress from concrete experiences to abstract experiences for all learners, with substantial emphasis on those elements of the environment which are familiar and likely to kindle interest.
- Mathematics programs, to be maximally effective, must be implemented by the efforts of a sensitive, knowledgeable, and skilled teacher.

- The program shall strive to have pupils learn to reason logically and independently and to develop a fondness and inclination for inquiry.
- The experiences of pupils in the mathematics program should equip them with the skill to think and communicate in mathematical terms.
- The program should result in continuous individual pupil growth in the skills of computation and measurement to ensure functional competency of pupils as citizens in a complex society.
- The experience of learners in a mathematics program should result in an understanding and appreciation for the fundamental concepts, structure, and usefulness of mathematics.
- Mathematics programs should be more activity oriented than theoretical, and mathematics programs should require pupils to engage in useful activities designed to generate enthusiastic learning and positive attitudes toward mathematics as a useful tool in their lives.
- The program design should be flexible and provide for a variety of teaching and learning styles. More specifi 'ly, the programs conducted should lead pupils to acquire the uowing:

1. A sound background in the concepts and skills of the real number system, including experiences with:
   a. Sets of numbers and basic operations defined on those sets
   b. Computational algorithms for the basic operations
   c. Properties of the basic operations defined on the sets of numbers
   d. Equality, inequality, and other relations
   e. Functions and other relations
   f. Mathematical sentences
   g. Decimal systems of numeration and place value
2. A background in the concepts of geometry, including experiences with:
   a. Simple geometric constructions
   b. Basic plane and solid geometric configurations
   c. Congruence and similarity
   d. Perpendicularity and parallelism
   e. Symmetry
   f. Circles and polygons
   g. Transformations
   h. Measurement of angles, perimeters, areas, and volumes
   i. Maps and scale drawings

j. Graphing and coordinate geometry in one and two dimensions

3. An appreciation of or an ability to apply mathematics, including experiences with:
   a. Measurement with standard units, including the use of decimals
   b. Estimation, comparison, and scientific notation
   c. Probability and statistics
   d. Discovery of mathematical relationships
   e. Simple deductive systems
   f. Telling time
   g. Strategies and tactics for problem solving
   h. Analysis of problems using mathematical models
   i. Methods of logical reasoning

The relative success of a program can be estimated in terms of how well it seems to meet the above goals. However, a formal evaluation will require assessment against a set of specific program objectives based on these goals and on the specific needs of the pupils and the community served. A program should have a formal evaluation component for judging strengths and weaknesses. With such a component, educators can make meaningful adjustments for program improvement and continuing student growth.

The Early Education Program

The interrelated ideas of mathematics become part of our human experiences at a preschool age. These early experiences provide intuitive background essential to the development of later mathematical content. Therefore, it is important that the instructional program in mathematics begin in the early education of the child.

In the beginning, the development of mathematical concepts for all children should be of an informal and exploratory nature; the goals and objectives set up in this framework provide for the establishment of a program of such a nature. For early mathematical experiences to be effective, guidelines need to be established which will provide a frame of reference for guiding adults involved in developing learning experiences for children. Incidental learning is a useful tool for developing mathematical concepts.

Activities should provide for the involvement of children with physical objects that are usually found in the environment. Activities may also be centered on materials designed to develop certain mathematical ideas. Children should be encouraged to compare, classify, and arrange objects according to shape, color, and size; to experiment with symmetry and balance; and to discover and create patterns. They should discover the comparative relations of "more," "fewer," and "as many as" through the activity of matching groups of objects. In these activities, the child develops understanding of mathematical concepts while learning to name and understand the number of a set, to count, and to develop positional relationships such as inside, outside, on, first, next, last, before, after, between, left, right, above, and below.

Throughout their activities, children should be encouraged to ask questions and talk about what they are doing, both with the teacher and among themselves. Children at this level are imitative and are interested in words. They are increasing their vocabulary rapidly. If the teacher introduces word and language patterns easily and naturally, children will begin to assimilate the words and patterns into their own speech and thoughts. The key words here are *easily* and *naturally*. Children's own ways of conveying their ideas must be accepted at the time; but, concurrently, they should have the opportunity to learn to express ideas with clarity and precision.

Strands for Kindergarten Through Grade Eight

The content of the mathematics programs for elementary levels, kindergarten through grade eight, is organized into seven major content areas (strands), while the secondary program has nine major content areas. For each strand in kindergarten through grade eight, the program objectives are listed by levels in Appendix A, as illustrated in Figure 2.

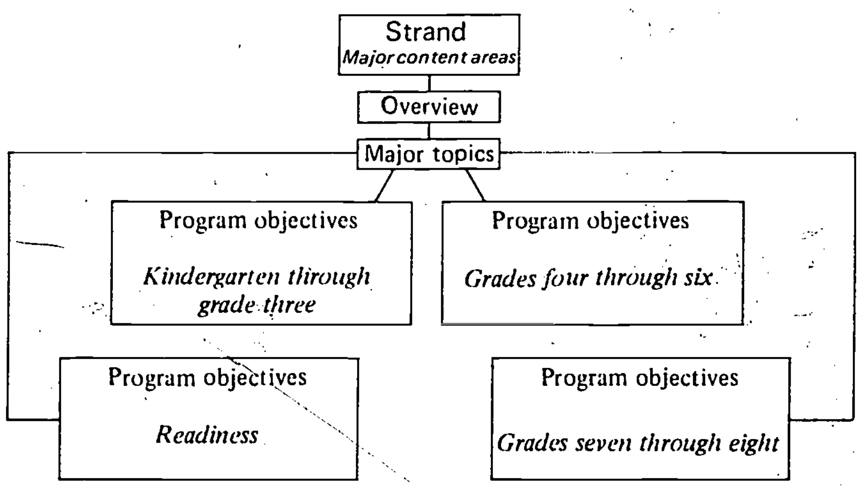

Fig. 2. Organization of mathematics program strands in kindergarten through grade eight

The program objectives identify skills and concepts to be presented by teachers in the following order of conceptualization:

1. Exploration
2. Identification
3. Recognition
4. Development
5. Acquisition/demonstration
6. Application/utilization
7. Maintenance

Arithmetic, Numbers, and Operations Strand

The development of the *arithmetic, numbers, and operations* strand constitutes the most important portion of a mathematics program in all levels in kindergarten through grade eight. This strand reflects the growing concerns of educators throughout California with regard to the urgent need for a reemphasis on children's learning and maintenance of computational skills. The study of the real number system begins with the counting numbers to be followed by the whole numbers, integers, rational numbers, and real numbers.

At the early childhood level, children are provided with exploratory counting and comparison experiences using concrete sets of objects. The basic addition and multiplication facts should be presented and mastered early in the mathematics program. Intuitive experiences with manipulative materials should be used to motivate the development of computational algorithms. At appropriate levels, these techniques should be tied to the ideas from which they derive their validity; namely, the properties of closure, commutativity, associativity, distributivity, identity elements, and inverse elements for each of the number systems studied. Place value is the fundamental principle for naming numbers. In this respect, decimal notation, as well as computational skills with decimal numbers, should receive early priority at the primary level, as described in the *measurement* strand.

Informal mathematical experiences are important at all levels of learning. Most learners need to relate the symbols of mathematics to objects and to images of events from their own experiences in order for the symbols to become meaningful. In developing the ability to work with abstract symbols, pupils should first work with physical models: (1) build with blocks and other materials; (2) handle objects of different shapes and sizes, noting characteristic features; (3) sort and classify objects; (4) fit objects inside others; (5) arrange objects in order of size; (6) experiment with a balance; (7) recognize positional relationships and symmetry; and (8) search for patterns.

Pupils may then progress through pictorial representations to the more abstract and symbolic representations of the same concepts.

A mathematics program should provide for the introduction of new terms and language patterns in close association with other learning activities. Assimilation of vocabulary and language patterns into a learner's own speech and thought is expected to develop gradually. When learners show evidence of familiarity with an idea, they should be provided with opportunities for many relevant activities. Gradually, attention can be given to new terms in reading vocabulary. Most learners should acquire an understanding of, and an ability to read, the standard terminology and language patterns. New terms and language patterns can take place through many opportunities in using the terminology rather than through memorization and parrotlike repetition.

The program-level objectives for the *arithmetic, numbers, and operations* strand are grouped under the following major topics:

1. Counting
2. Operations
3. Place value
4. Patterns
5. Nature of numbers
6. Properties

Activities that guide a learner to recognize and generalize the central unifying ideas in the real number system aid the learner in development of a methodology for systematic thinking. Continual diagnosis of a learner's growth, a planned program of maintenance, reinforcement of skills, and remedial instruction are essential ingredients of the *arithmetic, numbers, and operations* strand.

Counting. Activities in which learners can compare the number of objects in different sets, without resorting to counting, lead to counting concepts. In these activities, pupils compare the number of objects in two given sets by pairing the members and then discovering the size relationship. Counting requires matching the members of a set of objects on a one-to-one basis with the members of the set of counting numbers. More sophisticated experience in counting can be obtained by grouping sets into ones, tens, hundreds, and so on. Learners should have experiences with many types of counting activities, which include experiences with equal sets, equivalent sets, finite sets, and infinite sets.

Operations. The learners should be provided with experiences that will enable them to acquire proficiency in computational skills. The possession of such skills should help the pupil to develop self-confidence in the ability to deal with numbers, the basic operations of numbers, and the applications of numbers. The developmental stages for these concepts range from an intuitive development of the definitions of the basic operations by joining and separating sets to the higher-level ability of developing and using the algorithms for the operations on the systems of whole numbers, integers, and rational numbers in decimal and fractional form.

Place value. The study of place value develops an understanding of the decimal place-value notation. The two major principles of a place value of numeration are base and position. The learner's concept of the decimal place-value numeration system has its beginning in prekindergarten and kindergarten experience when the learner first names numbers. These concepts should be refined and extended throughout the mathematics curriculum. Some of these concepts developed as the learner gains understanding of decimal place-value notation are the following:

1. A decimal system requires ten symbols.
2. The order of numbers.
3. Positional notation to indicate value.

Consideration of systems of numeration that utilize the principle of nonplace-value system can lead to an appreciation of the advantages of a place-value system and an awareness of the historical development of numeration systems. However, numeration systems other than the decimal should be a minor part of any mathematics program.

Patterns. The study of patterns is valuable to the pupil in the study of number systems (their operations and properties). The study of patterns assists the learner in the discovery and development of generalizations, providing not only practice in using the basic facts but experience in working with large numbers. Mathematics has been described as the study of patterns. Important applications of mathematics are a result of the search for trends or patterns among data derived from experiments or from the solutions of problems. The discovery of new ideas through the study of numerical relationships that display unusual patterns should be a regular part of the school mathematics program.

Nature of numbers. The study of the nature of numbers leads to an understanding of the real number system. The nature of numbers encompasses the following:

1. The chief characteristics of numbers, such as whether they are prime or composite, whether they are even or odd, what their factors and multiples are, and what their relation is to other numbers (greater than, less than, relatively prime to)
2. How numbers are used in daily living, such as in counting, measuring, and computing; and how numbers appear in nature, such as in plants, flowers, and shells

Properties. The learner should develop an intuitive understanding and appreciation of the properties of the basic operations and of their applications to everyday problems. The study of properties of the basic operations should include some level of conceptual understanding of commutativity, associativity, identity elements, inverse elements, distributivity, and closure. The transitive property for equality and order relations should be presented. Number sentences are particularly useful in guiding learners to discover patterns for the properties of operations. The same properties are later applied to the solutions of mathematical equations and inequalities.

Geometry Strand

The mathematics program for kindergarten through the various levels of the curriculum should provide for the development of a strong, intuitive grasp of basic geometric concepts such as point, line, plane, and three-dimensional space. Experiences in geometry should relate to familiar objects, since so much of the world is of a geometric nature. Opportunities should be provided at all levels to use manipulative materials for investigation, exploration, and discovery; the opportunities should consist of a wide assortment of informal geometric experiences, including the use of instruments, models, and simple arguments. Thus, a geometry program should provide the foundation for later, formal study.

Teacher awareness of geometry in the environment will enhance the total curriculum and lead to opportunities to incorporate geometry with other disciplines. Examples can be drawn from art forms of all cultures, from industry, and from nature.

From the outset, deliberate effort should be made by the teacher to use appropriate and correct terminology in the development of the geometric concepts. Yet, language should not become a barrier or deterrent to the exploration of and experimentation with geometric ideas. Vocabulary building should include the language of sets in a natural way.

The program-level objectives for the geometry strand are grouped under the following major topics:

1. Geometric figures
2. Reasoning–logical thinking
3. Coordinate geometry
4. Measuring geometric figures

Geometric figures. The identification of geometric figures should begin by handling physical models such as triangular, rectangular, and circular objects. Sorting or grouping objects according to shape and size in the early learning level makes pupils aware of *similarities* and *differences.* Later, classification of geometric figures should be refined by specifying additional properties. Pupils should be familiar with figures such as triangles, rectangles, cubes, spheres, and pyramids.

Two basic concepts of geometry are similarity and congruence. Similarity can be thought of as a transformation which preserves shape but not necessarily size. Congruence is a transformation that preserves size as well as shape. While early experiences with similarity and congruence can be accomplished through sorting and matching, later experiences can include tracing and paper-folding activities and the use of measuring instruments. Many learning activities lead to identifying specific conditions that will ensure these relationships.

Pupil experiences should include experiences with the four transformations of reflection, rotation, translation, and dilation (i.e., scale drawing).

Reasoning–logical thinking. The elementary geometry program in kindergarten through grade eight is a program of "informal" geometry. The word informal refers not to casual manner of presentation or emphasis but to the absence of a formally developed subject using an axiomatic approach. A goal of the geometry program from kindergarten through grade eight should be to provide the foundation for later formal study. When appropriate, the teacher may present short deductive and inductive arguments.

Coordinate geometry. First experiences with concepts of coordinate geometry should be informal and preferably of a physical nature. Arrangements in rows and columns and movement in specified directions are appropriate activities. Children then can plot points in the first quadrant and can graph data recorded in experimental situations. Successive experiences should involve all four quadrants, leading to the ability to graph simple linear and quadratic equations.

Measuring geometric figures. Through realistic situations, the concepts of length, perimeter, area, volume, and angle measurement should be developed. Experiences such as pacing the perimeter of a rectangle (e.g., classroom, hallway, or school yard), tiling plane surfaces with regular shapes, and constructing rectangular solids with building blocks assist pupils to discover patterns leading to general statements or formulas.

"Measuring geometric figures" is an obvious intersection of the two strands *measurement* and *arithmetic, numbers, and operations* with the strand *geometry*. The application of the concepts of measurement provides a wealth of problem situations that frequently demonstrate the practical value of geometry.

Measurement Strand

Often one is unaware of the extensive use made of the process of measurement in daily living because measurement so permeates one's experience. In fact, measuring is a key process in many of the applications of mathematics and serves as a connecting link between mathematics and the evironment. Nonetheless, measurement skill is an acquired skill that is best learned through the *act* of measuring.

In most recent textbook series, only a single chapter or unit at each grade level has dealt specifically with measurement concepts. Additionally, incidental teaching of measurement occurred in problem-solving and application activities in other sections of the instructional materials. Such presentations of measurement often had little to contribute to the overall mathematical development of the learner, except for some possible computation practice and memorization of facts needed to convert within a measurement system.

With the introduction of a metric system, a stronger feeling for measurement can be developed easily because of the way metrics ties directly into our decimal system of numeration. A metric standard will foster a better understanding of measurement concepts because the decimal (tens) nature of metrics is related to the base ten place-value system.

The *measurement* strand is not merely an outline for transition to a different measurement standard but rather an outline for improving the presentation of measurement. Pupils must be given extensive opportunities to use measurement tools and to acquire skills useful in adult life. The transition to metrics provides a convenient opportunity to improve measurement instruction along with the adoption of a universal and less complex standard for measurement. (See Appendix B for a time line for the transition to metrics.)

Earliest experiences should center on physical activities requiring arbitrary units (e.g., width of a hand, capacity of bottles, and clapping of hands) to develop concepts of measuring distance, capacity, or the passage of time. In learning to measure, pupils should begin to use simple measurement tools to measure quantitative attributes of familiar objects. As learning progresses, pupils should be provided experience in carrying out more complex measurements.

The *measurement* strand intersects with the strands of *geometry* and *arithmetic, numbers, and operations*. The use of rulers and protractors to measure geometric figures will lend added insight to the concepts in geometry. When the units used for measuring are metric units, this activity promotes an understanding of decimal notation. Computation involving measurements will provide practice in the operations of arithmetic. The *measurement* strand also provides opportunities for numerous interdisciplinary learning situations that relate mathematics to subjects such as social studies, geography, science, industrial arts, and home economics. Measurement also includes the topics of time and temperature.

The program-level objectives for measurement are grouped under the following topic headings:

1. Arbitrary units of measure
2. Standard units of measure
3. The approximate nature of measurement
4. Estimation in measurement

Arbitrary units of measure. In the introductory stage of learning measurement skills, pupils first become familiar with the properties of the objects to be measured. Next, they learn to make discriminations among those properties. They then learn to compare objects according to the quantitative properties the objects possess in such terms as "is equal to," "is less than," or "is greater than." At this point, arbitrary units of measure are selected or devised to allow the comparison of common properties of objects. Pupils should understand that the arbitrary units selected by others may differ. Experience with arbitrary units should lead pupils to discover the merits of selecting more widely accepted units of measure and to establish the need for standard units.

Standard units of measure. Measurements expressed in standard units result in measurement statements that can be universally understood. The International System of Units (SI) should be the system of standard units taught in the schools of California. Conversion, involving computation from metric to U.S. Customary units and from U.S. Customary units to metric units, must be avoided. However, informal comparisons of metric units with comparable U.S. Customary units may be profitably used during the transitional period and for historical discussions.

The techniques of measurement learned while using arbitrary units are the same as those used with standard units. The metric system, long in use by most of the countries of the world, is a decimal system of standard units. Thus, early instruction with numbers and operations using decimal notation should precede instruction in the use of metric units. The precision required and the complexity of the ideas represented dictate the level at which different units and associated measurement terminology are introduced into the instructional program.

The approximate nature of measurement. The exact measure of a line segment is called its length, just as the exact measure of a surface is called its area. The physical act of measuring a segment with a ruler or a tape measure produces, at best, an approximate measure, due to the limitations of the ability to read a measuring instrument and of the precision of the measuring instrument. The process of "rounding off," familiar to students as a part of operations with numbers, becomes meaningful when applied to the measuring procedure.

In general, pupils should learn to understand that in making and recording measurements they are dealing in approximations. Ordinarily, one tries to obtain as accurate an approximation of the measurement as possible, although frequently a good estimate may serve the purpose.

Estimation in measurement. The ability to estimate effectively is a skill that has great practical value. Frequently, an offhand estimate will provide a ready check for the result of a calculation and will act as a deterrent to continued calculations with incorrect measurements. Because a good estimate is an "educated guess," skill at estimating develops through many measuring experiences.

In early grade levels, estimates with measurements should be encouraged through comparisons with already accepted measures, whether arbitrary or standard. Later, pupils should develop an intuitive grasp of and familiarity with the standard units so as to be able to make reasonable estimates through direct observation, using visual or other appropriate senses.

Problem Solving/Applications Strand

One major goal of a mathematics program is for pupils to develop the ability to formulate and solve problems and the ability to apply these problem-solving skills in practical situations. In applying mathematics, we are concerned with situations that arise inside as well as outside the domain of mathematics. Application of mathematics requires one to (1) formulate problems that are suggested by given situations; (2) construct, if possible, adequate mathematical models of these formulated problems; (3) find the solution of these models; and (4) interpret these solutions back in the original situations. Problem solving requires one to select strategies for the analysis of a given problem and to use certain mathematical skills and techniques identified in the analysis to solve the problem. Clearly, the ideas of problem solving and mathematical applications are interrelated.

Concrete mathematical applications selected from a wide range of sources should be systematically included in a mathematics program, along with the development of problem-solving strategies and skills. Constant exposure to concrete mathematical applications enables pupils to use concepts, techniques, and skills they have already developed to attack and solve useful problems. This exposure to interesting and useful problems can motivate students to develop new and more significant mathematical skills and techniques.

The strand *problem solving/applications* should be consistently interwoven throughout the mathematics program. Each of the other six strands provides tools for the development of problem-solving strategies and skills. The other strands also contribute toward the development of techniques for expressing and relating mathematical concepts that arise both inside and outside the domain of mathematics.

The ideas of *problem solving/applications* are so important that a mathematics program should include periodic study of formulation techniques, problem-analysis strategies, and problem-solving techniques. However, the strategic principles of problem solving should not be presented as a specific format that must be followed nor as a step-by-step procedure to which all solutions must conform. The creative solution of a problem is more valuable than a burdensome routine. *Creative thought or insight should not be stifled by having to conform to unnecessary formalism.*

The program-level objectives for the *problem solving/applications* strand are grouped under the following major topics:

1. Problem formulation
2. Problem-analysis strategies and tactics
3. Constructing mathematical models of problems
4. Finding the solution
5. Interpreting the solution

Problem formulation. Problem formulation should be an outgrowth of pupil experiences that arise in the context of some interesting event—often a phenomenon arising in everyday life, in the social sciences, in the life sciences, in the physical sciences, in the humanities, or in mathematical recreations. The situations selected for study should be so meaningful that the pupil will honestly expect to experience the situation or will have some assurance that other people actually do experience the situation. As pupils work with concrete situations, they consciously or unconsciously pose problems that seem to need solution, or they ask questions such as "why does this work?" The ability to formulate meaningful problems has as much value in the marketplace as does the ability to solve problems.

Each attempt on the part of a pupil to formulate a problem should be nurtured and encouraged. To this end, it is recommended that mathematics programs include a significant number of concrete situations that require pupils to explore, analyze, and investigate. Some of these situations should lead to problems that are open-ended in the sense that they invite conjectures.

Problem-analysis strategies and tactics. A mathematics program should systematically assist pupils in devising strategies for analyzing problems that lead to some success in solving the problems. The first step in any strategy is to make sure the problem is understood. Regardless of the origin of the problem, the solvers must understand the problem so well that they can restate it in their own words. The solver should be able to pinpoint the purpose of the problem, to indicate the unkowns, and to identify the given data. Several tactics are available to help the pupil at this point:

1. Guess some answers, try them out, and observe the results of the different guesses.
2. Construct a diagram, a graph, a table, a picture, or a geometric representation of the situation, and observe the relationship between the various parts of the problem.
3. Construct a physical model of the situation, or use physical materials to simulate the features of the problem.
4. Search for and identify underlying functional relationships in the problem.
5. Compare the problem or parts of the problem with similar or simpler problems that are more easily understood.

The development of problem-analysis strategies and tactics should start with the pupil's first mathematical experiences and accompany the development of basic mathematical concepts and skills. The use of a variety of analysis tactics should become the habitually accepted thing to do.

Constructing mathematical models of problems. Mathematics does not literally deal directly with the raw physical situation but only with a refined model of the situation. Mathematics does not divide ten apples by five children. Rather, mathematics provides an operation which divides the number ten by the number five; the answer, two, is interpreted as meaning that each child will have two apples. The distinction between the model and its origin is crucial, especially in more complex situations in which the model does not fit the situation so exactly. To illustrate this, one assumes in the preceding example that the ten apples are all at least edible.

The form of the mathematical model is usually one or more written sentences using mathematical symbols. Other models may, at times, be more appropriate; e.g., a picture of sets and a geometric figure. Basically, a mathematical model of a problem is any representation which permits manipulation by mathematical principles. A mathematical model tries to copy some of the characteristics of a given situation. To be successful, a model should accomplish the following:

1. Include as many of the main characteristics of the given situation as practical.
2. Be designed so that the included characteristics of the given situation are related in the model as they are in real life.
3. Be simple enough so that the mathematical problems that are suggested by the model can be solved readily.

A mathematics program should provide pupils with experience in discussing and constructing mathematical models of given situations. The reverse process is equally important: Given a mathematical model, the pupil will construct a real situation for it.

Finding the solution. The solution of a problem requires a wide variety of technical skills. Basic computational skills and an understanding of number properties are essential to finding solutions. Pupils also need the skills related to solving equations and inequalities, to graphing, to constructing geometric figures, and to analyzing tabular data. A mathematics program should include a substantial number of ready-to-solve problems that are designed specifically to develop and reinforce these technical skills and concepts.

In most problem situations, the results should be anticipated by estimating in advance. Estimating should be introduced early to all pupils as a standard operating procedure. Sometimes a solution when compared with an estimate may reflect a significant oversight, and then the major concerns should be: How did you go about it? Is the model adequate or valid? Was the solution process completed correctly? or Were the assumptions made too broad or restrictive?

Interpreting the solution. A mathematics program should systematically include experiences in the interpretation of the solutions obtained. The problem and its solution should be reviewed to judge the validity of the model and the accuracy of the mathematical manipulations. Discussion of a solution should resolve the following questions:

1. Was the problem solved?
2. Would another model work?
3. Can the model be improved?
4. Can the model be extended to solve related problems?

Probability and Statistics Strand

People today are overwhelmed with data from the mass media. They need to understand, interpret, and analyze these data in order to make decisions that affect everyday life. Therefore, experiences in collecting, organizing, and interpreting data should be included in a school mathematics program. These experiences should begin in kindergarten and should be a part of the instructional program at each succeeding level through the eighth grade and beyond.

Statistics is the art and science of collecting data, organizing data, interpreting data, and making inferences from data. One deals with some degree of uncertainty when trying to make these inferences. It is at the stage of decision making that one applies the concepts of probability so as to select alternative courses of action which are likely to produce desired results.

The program-level objectives for the *probability and statistics* strand are grouped under the following major topics:

1. Collection, organization, and representation of data
2. Interpretation of data
3. Counting techniques
4. Probability

Collection, organization, and representation of data. Collecting data should be the outgrowth of experiences involving observations by the pupil. The classroom, as well as the world, provides the pupil with an abundant source of data.

Organizing data is an art that the pupil must learn. The information in a table, graph, or chart must be presented in such a way that it fits the purpose for which the data were orginally gathered. In this topical area, the emphasis is on the construction and interpretation of the various graphs and tables needed to organize data.

Pupils working in the *probability and statistics* strand should have opportunities to see the relationship of mathematics applied to other areas of the curriculum.

Interpretation of data. Graphic devices are useful for offering a quick visual summary of a large collection of measurements or facts. If further interpretation or comparison of data is required, then measure of central tendency or scatter is needed (e.g., range, percentiles, mean, median, mode, and standard deviation).

Counting techniques. Before the more formal aspects of probability theory are presented, the pupils should develop a feeling for techniques of counting events. Counting procedures include tree diagrams, combinations, permutations, and sample space.

Probability. It is possible to introduce some of the beginning concepts of probability at the elementary level; however, most of the concepts in probability should be presented at the high school level.

Mathematical models of many scientific and economic problems exist within probability theory. The ability to assign numerical values to ideas that involve uncertainty is one of the concerns of probability. Probability theory is necessary to carry the interpretation of data to the point of making statistical inferences or "wise decisions" in the face of uncertainty.

The following ideas are considered appropriate for pupils in the elementary schools: sample space, definition of probability, probability of an event, independent events, probability of certainty, probability of nonoccurrence, $P(A \cap B)$, $P(A \cup B)$, and complementary events.[1]

Relations and Functions Strand

Mathematics offers a way of organizing and understanding most observations of the world about us, both in and out of school. One justification for including mathematics in the school curriculum seems to reside in the exploration of the notion of patterns and relationships. This approach to mathematics enables a child to discover and describe something of the shape and pattern of the universe. From the day children enter school, teachers should organize experiences that will encourage the children to think, seek, and discover ideas for themselves, to look for patterns and relationships, and to form generalizations. As these relationships are seen and discussed, concepts become clearer, and fundamental principles emerge that have value in unifying the study of mathematics to follow. Mathematics *is* the story of relationships.

[1] $P(A \cap B)$ represents the probability of A *and* B occurring. $P(A \cup B)$ represents the probability of A *or* B occurring.

The program-level objectives for the *relations and functions* strand are grouped under the following major topics:

1. Patterns
2. Relations
3. Functions
4. Graphs

Patterns. In preschool activities, the term "patterns" refers to painting, drawing, woodworking, making collages, and constructing models. The language used while engaged on a particular piece of work and the child's and the teacher's observations on various aspects of the work will help to heighten the awareness of size, shape, pattern, and relative position of objects.

At the primary level, patterns activities provide visual representations for discussing symmetry, repetition, counting, ordering, and pattern discovery. The child should develop an appreciation for the use of patterns to predict and make conjectures about future events. From experiences with patterns of two elements, the child should also become familiar with the notion of ordered pairs. In another application of patterns, the child can learn to recognize physical or pictorial representations of fractions.

Relations. Most of mathematics is concerned with relations. The young child learns early to relate certain objects or sounds with other objects. For example, a child associates other children with their parents. Intuitively, a child recognizes without formal articulation certain associations between pairs of objects or names of objects. Thus, the child learns to form ordered pairs, such as name and object, in the development of his or her language.

Sets of related pairs of objects are studied throughout the mathematics program. The process of forming pairs should be introduced early in the mathematics program. In beginning arithmetic, pupils learn to associate a set of objects with a number. First, they learn to count by pointing to the objects in sequence and pairing the objects with the set of ordinals. Then the pupils find that counting is a way of determining what number is to be associated with a certain collection of objects. Thus, counting determines certain related pairs; namely, *(set, number).*

Another example of related pairs can be observed in the relationship "greater than," which may be thought of in the form *(number, greater number).* This is an example of a relation in which a single number is related to many other numbers.

As pupils develop skill in collecting empirical data, they should begin to search for meaningful relations in the data. For example, a pupil could discover the relation between the number of diagonals that can be drawn and the number of sides of a polygon.

Functions. One type of relation is of particular importance to science and mathematics: Each member of a set is related to one and only one member of a second set. For example, for each child there is just one natural mother. Such a relation is called a functional relation or, simply, a function. Thus, the pairs of related objects that form a functional relation have the property that just one related pair exists with a given first member. Some functional notation should be used systematically by the end of the eighth grade. Several notational schemes may be suggested as shown in Figure 3, and different notational schemes should be used on occasion since different notations are suggestive of different aspects of the function concept.

The function concept includes mathematical operations. Elementary pupils encounter functions when learning the number facts. Multiplication by five, for example, identifies a function defined on the set of numbers.

Intuitive experiences will enable pupils to develop the concept of a function as a set of ordered pairs in which no two pairs have the same first element. Pupils should further realize that functions can be identified by statements, formulas or equations, tabulated data, and graphs. The pupils will then be on the way toward understanding a mathematical idea that has many applications. The mathematics program should offer the pupil familiarity with the functional notations given here and should enable the pupil to plot linear and quadratic functions, as well as functions with jumps such as the greatest integer function.

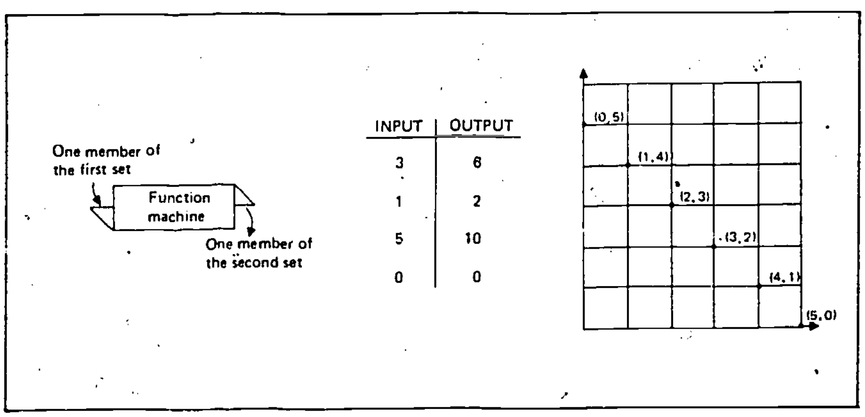

Fig. 3. Examples of notational schemes for functions

Graphs. Learning to present information in a graphical mode is essential. Graphs provide pupils with an organized method of recording and communicating their observations. Observations can be recorded as soon as a child has reached the collecting stage; it is while collecting and sorting data that the pupil begins to make comparisons and to form relations.

Use of a pictorial description of pairs of objects should begin early in the mathematics program. Plotting could be initiated with games such as tic-tac-toe; a class could graph the height of a plant on successive days of a month; or the class could record temperatures for each day in a month. The pupil's experience with such graphing reinforces the concept of the number plane, presents a picture for linear relations, and provides an excellent way of intuitively developing an understanding of the concept of grasping relations.

Graphing is also invaluable in the applications of mathematics. A class may record the length of a spring or rubber band as weights of successive sizes are suspended; or, at a more advanced stage, the period of a simple pendulum could be recorded while the length of the string is varied. In the study of measurement, a class may record the number of grams (or any standard unit) of water required to fill cylindrical jars of various diameters to a fixed depth. In the study of geometry, a class may plot circumference as related to diameter for different circles.

The mathematics program in kindergarten through grade eight should include the study of the coordinate plane. The association between each point in the plane with an ordered pair of numbers is basic to the mathematics that connects geometry and algebra. This association is also basic to the understanding of maps and, more generally, of scale drawings.

Logical Thinking Strand

The beginning approach to logical thinking is informal in grades one through three; beyond grade three, the requisites are more precise. Experiences with different kinds of sentences (using *and, or, not, if . . . then, all,* and *some*) and with some fundamental patterns of reasoning should be provided. These sentences should be verbal sentences, as well as mathematical, so that the experiences with the sentences will aid the pupils in seeing the importance of logic in relation to patterns of thought in ordinary life as well as in mathematics.

The elementary mathematics program should help pupils learn to organize ideas and to understand what they learn. Though much of informal logical thinking and deductive reasoning is a matter of common sense, the use of standard logical techniques can help pupils to organize the thought processes involved. Children should be able to decide whether a particular mathematical construct fits a definition and to recognize a specific application of a general principle.

In fact, logical thinking is both a desired outcome of mathematics programs and a capability essential for learning mathematics. If pupils are to progress steadily in mathematics, they must learn to recognize patterns. To recognize patterns, they must think logically enough to make discriminations and to find order in those discriminations. Hence, logical thinking can be considered to embrace two topical divisions: (1) patterns in mathematics; and (2) formal and informal reasoning.

Patterns in mathematics. The close connection between the ability to recognize patterns and the ability to think logically should be utilized at all levels; it will prove invaluable to the pupil in the study of number systems and operations. Patterns exist in most life situations, in nature, in history, in music, and so forth. In mathematics the pupil can be taught to utilize not only these patterns but also those existing in numbers and geometric figures in order to gain an understanding and appreciation of the beauty, logic, and order of the world.

Formal and informal reasoning. Any program for kindergarten and the early grades should provide many opportunities for children to explore and manipulate concrete objects, to identify likenesses and differences, to classify and categorize objects by their characteristic features, and to state generalizations.

At the lower elementary level, the use of many definitions of mathematical terms should be nontechnical in nature. The pupils should become acquainted with terms such as *all, some, and, or, if . . . then,* and *not* in such a way that they will be able to understand the meaning of these terms in a mathematical context.

Venn diagrams and a variety of mathematical sentences should be used consistently throughout the program and in every strand where appropriate. Starting with kindergarten, the language of sets should be used as needed to gain clarity, precision, and conciseness in mathematical communication and to aid in the reasoning process.

The Mathematics Program in Grades Nine Through Twelve

One important ingredient for a successful mathematics program in grades nine through twelve is the establishment of confidence on the part of learners. A program with "built-in" success can improve the student's self-image and can assist the student in gaining confidence in mathematical capabilities. A positive and open classroom climate is essential in the attainment of this goal.

If students are involved in the learning process through activities that emphasize discovery, inquiry, or experimentation, then the students can be provided with meaningful success experiences. Successful learning experiences encourage the student's further involvement and can also set the stage for increased motivation to learn mathematics. Such an approach in grades nine through twelve can start a positive learning cycle for many students in mathematics.

Diagnostic techniques should be used with students who have learning difficulties to identify the areas of difficulty, and then prescriptive teaching can be designed to meet the needs of those students. One danger in this strategy is that this technique may not be motivating to many students. Pointing out to students their past failures and asking them to eliminate their shortcomings often result in an immediate "turnoff." Instead, the introduction of new techniques and materials that emphasize a more positive approach can capture the interest of students and promote a better learning environment.

Continuous and flexible inservice training programs must be organized and funded on a state, regional, and local level to develop and maintain the mathematics program described in this framework. Up-to-date and responsive preservice programs must also be under constant development and evaluation. The success of the mathematics program in grades nine through twelve depends in large part on the mathematical competence of the teachers.

The improved preparation of teachers is certainly an important prerequisite of any improved mathematics program. While many teachers may need updating in content areas such as probability, statistics, transformational geometry, linear algebra, metric system, or computer mathematics, it appears equally important that teachers be ·epared to teach average and below-average students who have career or school goals that require mathematics. Teachers need to be able to guide students in learning activities that emphasize discovery and inquiry. Relevant and systematically organized inservice training programs and realistic preservice programs in mathematics can provide the background and support that teachers need to implement a strong mathematics program for all students.

The magnitude of the role that mathematics teachers play in the counseling of students into mathematics classes must increase substantially. Additionally, to communicate clearly and correctly the many changes in the needs and requirements in job training and college programs, mathematics teachers must accept and be given greater responsibility in the placement of students in mathematics courses in grades nine through twelve. Mathematics teachers universally regard the placement of students in classes for which the students are emotionally, mentally, or technically unprepared as one of the major causes of deterioration of the learning evironment, not only for the misplaced students but also for their classmates. Mathematics teachers should be given the time, information, and support necessary to act as informed advisers to students in relation to the students' progress and selections of mathematics courses. Mathematics departments in grades nine through twelve should develop systematic couseling programs and procedures for all students enrolled in mathematics.

Finally, it is recognized that the development of new mathematics technical equipment and multimedia materials will continue to have an impact on the mathematics program in grades nine through twelve. Secondary schools should prepare for (1) a substantial increase in the use of computers and minicalculators in many mathematics classes; (2) the establishment of mathematics resource centers, laboratories, and media centers; and (3) the increased use of aides or paraprofessionals in their mathematics programs.

Goals of Mathematics Instruction

The goals of the mathematics program in grades nine through twelve are the following:

- Develop, commensurate with each student's ability, the mathematical competence that is necessary to function in society. This includes the ability to (1) recall or recognize mathematical facts, definitions, and symbols; (2) count, measure, and handle money; and (3) conceptualize spatial properties.
- Develop, commensurate with each student's ability, the skill of performing mathematical manipulations. This goal includes

(1) the ability to do straightforward computation; and (2) the ability to manipulate relations or to perform the computations required in a variety of mathematical models.

- Develop, commensurate with each student's ability, the understanding of mathematical concepts and processes. This goal includes the ability to transform or translate from one form of symbolism to another, such as from words to symbols, symbols to words, equation to graph, physical situation to formula.
- Develop, commensurate with each student's ability, the skill to select knowledge, information, and techniques that are needed to solve a particular problem—social, technical, or academic—and to apply these selections in the actual solution of a problem.
- Develop, commensurate with each student's ability, the capability of using mathematics and mathematical reasoning to analyze given situations, to define or formulate hypotheses, to make optimum decisions, and to verify the validity of results.
- Develop an appreciation of the importance and relevance of mathematics as a substantial part of the cultural heritage of the human race that permits people to invent and discover relationships that influence and order their environment.

General Objectives of the Mathematics Program

The mathematics program in grades nine through twelve should provide for the following:

1. Aquisition of the skills and concepts presented in the framework for kindergarten through grade eight
2. Development of courses and curriculum organizations to provide the opportunity and encouragement for all students to continue their study of mathematics to meet their specific career and educational goals
3. Development of a series of topical minicourses as an alternative for the traditional yearlong general mathematics course for the noncollege-bound student
4. Development of alternatives for the traditional one-year blocks of algebra and geometry (to serve one of the needs of the large middle majority of nontechnically oriented secondary school students)
5. Development of a remedial clinic program for mathematics students who are achieving below their expected level of achievement
6. Development of mathematics resource centers or mathematics laboratories to be used as an integral part of the instructional program of each mathematics class

7. Acquisition by all students of knowledge about the nature of a computer and the roles computers play in our society; and for some students the opportunity to acquire skills and concepts in computer science, including career training
8. Development of programs for talented students leading to the completion of one year of calculus or another advanced elective course by the end of the twelfth year

Concepts of the Framework at the Elementary Level

Through the mathematics program in grades nine through twelve, students should have adequate opportunities to acquire, as necessary, the skills and concepts presented in the framework for kindergarten through grade eight. The increased need for mathematical learning on the part of citizens in a modern society is recognized in the framework for kindergarten through grade eight, which is devoted to the development of mathematical skills and concepts that all citizens should know to function satisfactorily in our rapidly expanding technological society. Some students in grades nine through twelve, in spite of their best efforts, will need additional study in the content of the kindergarten through grade eight mathematics program.

The program in grades nine through twelve must give ample opportunity for learning basic computational skills and applications of mathematics at the level of the students' needs. However, new materials and strategies are required for the students in grades nine through twelve who need study in the content of the kindergarten through grade eight program; high school teachers should not continue teaching these students, using the same methods which have proved unsuccessful in the earlier grades. New mathematical concepts should be included in each instructional unit, incorporating new approaches and techniques and thus recapturing the interests of students and indirectly improving their performance. A reorganization of staff (such as differentiated staffing; use of specialist teachers, teacher assistants, or individualization of instruction; use of nongraded classroom organization; or different grouping patterns) may be necessary to achieve this objective.

Encouragement to Study Mathematics

The grade nine through twelve mathematics program should provide for the development of courses and curriculum organizations that would provide the opportunity and encouragement for all students to continue their study of mathematics to meet their specific career and educational goals. The program in grades nine through twelve must meet the needs of students aiming for various careers in technical fields, as well as the needs of college-bound students interested in social sciences, humanities, economics, or biological and physical sciences. For example, a school could offer courses specially designed to assist students to prepare for examinations for apprentice programs, for industrial positions, or for civil service by utilizing modern technology, equipment, and media appropriate to the particular fields of employment.

The result of implementing this objective would be in sharp contrast to present practice in which students who have arithmetic difficulties are often permanently shut out from all other mathematics courses. Courses could be designed and offered to cover the usual content at different rates, or new approaches could be offered, depending on the background, motivation, and ability of the student. Statewide in the 1960s and 1970s, a minority of students successfully completed first-year algebra. In fact, state reports indicate that a majority of California students are permitted to take only a general mathematics course. In some areas, students are required to take mathematics in grades nine through twelve and may spend two or more years in general mathematics courses that are essentially grade-six or grade-seven arithmetic, with no possibility of studying concepts of algebra, geometry, statistics, computers, and so forth. These limitations cannot remain if the needs of the students and society are to be met.

Minicourses in Lieu of General Mathematics

The grade nine through twelve mathematics program should provide for the development of a series of topical minicourses as an alternative for the traditional yearlong general mathematics course for the noncollege-bound student in grades nine through twelve. The topical minicourse approach to general mathematics provides a way to accommodate the large numbers of above-average, average, or below-average students with diverse goals and abilities who elect to take general mathematics in grades nine through twelve.

Topical minicourses could be packaged into nine-week quarter blocks, allowing students to select up to four different minicourses in place of the usual yearlong course in general mathematics. Schools, for example, that now offer two identical yearlong general mathematics classes could offer up to eight different minicourses. Additional minicourses could increase the length, adaptability, and flexibility of this recommended program.

Some of the minicourses would have prerequisites, but prerequisites should be kept to a minimum so that these elective minicourses can be taken in a variety of sequences. Diagnostic tests could be used to measure student need for the minicourses. For example, students who demonstrate a need to improve their basic computational skills could be required to enroll in minicourses such as the following:

Mathematics clinic
Whole numbers, integers, and rational numbers
Ratios, properties, and percent
Geometry and measurement

Some other topical minicourses that could be offered are the following:

Calculating devices and minicalculators
Mathematics and living things
Measurement, measuring devices, and the metric system
Practical geometric constructions
Reading and using tables and graphs
Quality-control statistics
Flowcharts, computers, and programming
Credit and installment buying
Consumer economics
The mathematics laboratory
Mathematics and games

Alternatives to Algebra and Geometry

The mathematics program in grades nine through twelve should provide for the development of alternatives for the traditional one-year blocks of algebra and geometry to serve one of the needs of the large middle majority of nontechnically oriented secondary school students. Alternative courses should be true alternatives with equivalent college preparatory standing, not the conventional sequence of algebraic or geometric topics presented at a slower pace in "watered-down" courses. The usual grade placement of topics should be replaced by offering topics chosen from arithmetic, algebra, and geometry; the topics should be arranged in a logical sequence so that they provide mutual support. Topics such as functions, coordinate geometry, transformations, computer programming and flowcharts, and probability and statistics should be interwoven throughout. A strong effort should be made to make clear to the students the applications and relevance of mathematics to the real world.

For those students in the alternative mathematics courses who want a third or fourth year of mathematics, a third-year transition course should be offered that could prepare them to take such fourth-year courses as probability and statistics, computer programming, linear algebra, elementary functions, or a course to prepare for the AB Advanced Placement Examination in calculus. To provide another degree of flexibility, alternative courses could be organized into nine-week, semi-independent minicourses, thus allowing students a greater choice in the depth and direction that they could choose to follow.

Remedial Clinic Program

The mathematics program in grades nine through twelve should provide for the development of a remedial clinic program for mathematics students who are achieving below their expected level of achievement. A clinic program should be designed to provide individualized instruction aimed at meeting the needs of selected students whose mathematics achievement is significantly below their expected level of achievement. The program should be organized and planned to meet the identified needs of students at each school. A procedure for identification of students should include teacher/counselor recommendations and testing data. The program for each student should be planned individually and include diagnostic testing, pretesting, individualized instruction, and post-testing.

The clinic may operate as a "pullout" program or as a quarter or semester course. Aides should be provided to assist the teacher of the mathematics clinic. This aide(s) could be a paraprofessional, a parent, or a student assistant. A clinic should operate at a low pupil-teacher ratio (maximum 16:1) and should be adequately funded by state and local funds so as to provide for special materials and equipment and for optimum conditions for remediation and learning.

Mathematics Resource Centers

The mathematics program in grades nine through twelve should provide for the development of mathematics resource centers or mathematics laboratories that are used as an integral part of the instructional program of each mathematics class. Mathematics laboratories create the opportunity for and the encouragement of student research activities in applied mathematics. They also provide the environment for learning mathematical skills and concepts through the use of manipulative materials or the use of equipment and techniques that are a part of the daily procedures of business, industry, or science.

Sufficient funds should be provided to ensure that appropriate manipulative materials and up-to-date equipment are available and that adequate staffing is provided. In particular, teacher aides should be provided to maintain, organize, process, and control the use of materials and equipment. A professional staff member should also be designated as a director to train teachers in the use of the laboratory and to spearhead the development of new and innovative materials.

Computational aids such as desk calculators, slide rules, tables, electronic programmable calculators, and computer terminals should be an integral part of the laboratory learning system.

Knowledge of Computers

The mathematics program in grades nine through twelve should provide for acquisition by students of knowledge about the nature of a computer and the roles computers play in our society; and for some students, the opportunity to acquire skills and concepts in computer science, including career training. The average U.S. citizen has little idea of how computers work and how pervasive their influence actually is. The average citizen is, in short, culturally disadvantaged. It is essential that our educational system be expanded in such a way that every student becomes acquainted with the nature of computers because of the current and potential roles that computers play in our society. At a minimum, courses that include instruction in "computer literacy" should accomplish the following:

1. Give the student understanding about the way the computer works so that the student can understand what computers can and cannot do.
2. Include a broad sampling of the ways in which computers are used in our society, including nonnumeric as well as numeric applications. The impact of these various uses on the individual citizen should be made clear.
3. Introduce algorithms (and their representation by flowcharts). If time and equipment are available, computer programs representing the algorithms should be written and run on a computer, with printouts made available to the students.

Additional computer instruction should be designed to develop proficiency in the use of computers, particularly in the mathematical, physical, biological, and social sciences. Also, opportunities in vocational computer training should be more generally available. Currently, more than a million workers find employment in the computer industry, and this number will likely continue to increase.

Programs for Talented Students

The mathematics program in grades nine through twelve should provide for the development of programs for talented students leading to the completion of one year of calculus or another advanced elective course by the end of the twelfth year. The mathematics program in grades nine through twelve should include programs for talented students whose education or career plans require a strong mathematical background. It is estimated that at least 10 percent of the high school population, if interested, could complete a calculus course and receive college credit. However, it should be recognized that some capable students are not technically oriented and may not elect to take calculus. Alternative topics for these students could include linear programming, linear algebra, probability and statistical inference, introduction to logic, introduction to computers and computer programming, analytic geometry, or game theory.

Strands for Grades Nine Through Twelve

As in the program for kindergarten through grade eight, the mathematical content of the program for grades nine through twelve is classified by strands, seven strands for kindergarten through grade eight and nine strands for grades nine through twelve. The two additional strands in grades nine through twelve are *algebra* and *computers*. The name of one strand has been changed for grades nine through twelve to reflect a more sophisticated level of content; *arithmetic, numbers, and operations* becomes *the arithmetic of real numbers.*

Teachers in the program in grades nine through twelve are responsible to the students for instructional activities which reinforce and maintain all the concepts and skills identified in the kindergarten through grade eight program. Except for the two additional strands, each of the strands in the program in grades nine through twelve builds on the corresponding strands in the kindergarten through grade eight program. To comprehend fully the scope of each strand, the reader of this framework should first reread the corresponding strand and program-level objectives in the kindergarten through grade eight program.

The preparation of the grade nine through twelve program in this framework did not include the identification of the major topics in each strand nor the development of program-level objectives as was done in the kindergarten through grade eight program. Those tasks, as well as the development of specific instructional activities, await the reaction of the educational community to this, the third framework.

The Arithmetic of Real Numbers Strand

The number concept in the program in kindergarten through grade eight is extended to the entire set of real numbers, rational and irrational. Through the discussion of roots, particulary of the irrationality of the square root of two, pupils intuitively understand that there are in any unit interval infinitely many points that correspond to irrational numbers. Such treatment of the number concept should not exclude practical applications of numbers at any ability level. It is important in this regard to continue the use of concrete applications, manipulative materials, multimedia materials, minicalculators, and exploratory laboratory activities.

Algebra Strand

Algebra is simultaneously tool and product—a prerequisite skill to studying mathematics and an end in itself. In both cases, algebraic concepts are an essential part of vocationally oriented mathematics courses. Learning the basic algebraic manipulations, as well as the logical system of algebra, is essential to all the strands in the program at the high school level.

The standard algebra course has undergone many changes that should be included in the program in grades nine through twelve. The concept of function has become the central theme throughout the algebra program. Inequalities and graphing now receive increased emphasis. New topics include linear algebra, linear transformations, matrices, and linear programming. Trigonometric functions are part of the current algebra program. The use of logical proof based on definitions, axioms, and postulates is now as essential to algebra as it once was to geometry (although care must be exercised by teachers to avoid overemphasis in the algebra program).

The algebra program should be allowed to continue to evolve, taking into account new ideas as they emerge.

Geometry Strand

The main functions of teaching geometry at the high school level are: organizing geometric facts into a more formal mathematical structure, extending and broadening the student's knowledge of mathematics, and applying geometric concepts in problem-solving situations.

Geometry has its roots deep in the historical development of mathematics. The evolvement of π, of irrational numbers, and of a postulational system as a model for logical reasoning are examples of mathematical ideas derived from the study of geometry. On the practical side, geometric ideas find wide application in such diverse career fields as clothing design, industrial design, architecture, construction skills, engineering, art, scientific research, advertising, and packaging. Geometry holds great potential for helping students gain understanding of and insight into arithmetic and algebra through a visual approach to learning number and algebra relationships. Not to be underestimated is the fact that for a large number of high school students, geometry provides a gateway to mathematics, creating an awareness of mathematics' breadth and depth and initiating many into logical reasoning.

The teaching of geometry in the high school can range from brief topical units to a full course along the lines of a traditional curriculum. Contemporary thinking about the nature of approaches to teaching formal geometry recognizes that there is no one best scheme. Pedagogically speaking, the degree of intuitive understanding desired relative to the amount of formal learning through logical reasoning depends on the ability and interest levels of students and teachers. Mathematically speaking, different approaches center on the choice of postulates that characterizes the system.

A majority of present-day geometry courses follow the postulational design of two- and three-dimensional Euclidean geometry modified to accommodate the natural relationship with the real numbers. Still other course designs emphasize coordinates, leading to applications in analytic geometry, calculus, and transformations and appealing both to manipulative experiences with concrete objects and to techniques of modern algebra. It is recommended that formal geometry courses be designed to maximize the mutual benefits in the understanding of both algebra and geometry; courses departing from this norm should be considered only where highly qualified teachers are available and where the total mathematics curriculum is able to accommodate different approaches.

Although no general agreement exists about what a geometry course should be, there is general recognition of some features that should characterize every geometry course. First of all, the course should be based on a set of postulates that is adequate, at a high school level of sophistication, to support proof of the theorems to be studied. The course should blend geometry of two and three dimensions and should contain substantial coverage of the topics of perpendicularity, parallelism, congruence, and similarity. The ability to use deductive methods to establish proofs of theorems is a desirable outcome. Mensuration theory, as applied to developing the usual formulas for measuring lengths, areas, and volumes, should be taught, and considerable practice in the use of these formulas should be given. Even in a course not emphasizing coordinate methods, an introduction to coordinates should provide the basis for an extended treatment in more advanced study.

Measurement Strand

Measurement skills and concepts are well-covered in the program in kindergarten through grade eight, but as pupils near the age of vocational preparation and consumer responsibility, measurement activities take on a more sophisticated nature. Some aspects of measurement (such as the approximate nature of measurement, precision, accuracy, and relative accuracy) then become worthy of consideration because these aspects exhibit the strengths and limitations of measurement. Every learner should emerge from the study of measurement with an understanding of its approximate nature, as well as with an ability to select and use basic measuring instruments correctly and efficiently.

The SI system of measurement (the International System of Units) is gradually and steadily gaining acceptance as the standard units of measurement in the United States. A shift to such a system became inevitable for the United States when it became the only industrial power that was not utilizing a metric system as its measurement standard. To prepare learners for this transition, the mathematics program should use SI units as the basis for instruction and practice in measurement. Initially, references to the U.S. Customary system may be useful in making the transition to metrics. In most cases, conversions between systems should be avoided and discouraged. Conversions within the SI system may be profitable in establishing familiarity with the units and nomenclature of the SI system.

Instruction should be activity oriented. However, the activities should relate as much as possible to the experience of students and should serve to improve their consumer skills and their ability to obtain gainful employment.

Some of the activities that promote measurement skill can and should improve buying skills. The school experience can be enlarged to include experiences in unit pricing and in comparisons between items to determine best buys. Weather-forecasting activities can be used to enhance skill in measuring temperature, barometric pressure, quantity of precipitation, and wind velocity.

Problem Solving/Applications Strand

The strand *problem solving/applications* is developed thoroughly in the program in kindergarten through grade eight and utilizes a five-step procedure which is appropriate also for the program in grades nine through twelve. At each of the five steps, important skills are identified, and activities are suggested that provide a systematic approach for solving all types of problems which arise in applied mathematics.

Problem-solving situations should be an outgrowth of student experiences involving phenomena that arise in the context of some natural event. A search for an "understanding of phenomena" has dominated human intellectual activity from the beginning of time.

This pursuit of knowledge, of structure, and of causation is usually motivated by the desire for comfort, by fear of the unknown, or by curiosity. The search for understanding is usually accompanied by a strong desire for predictability; that is, within reasonable bounds the theoretical predictions will agree with known results of the actual situation. The consistent development in all students of creative problem-formulation skills is essential if the search for understanding is to be motivated and improved.

A grade nine through twelve mathematics program should systematically assist students in devising strategies or tactics for analyzing problems that lead to some success in solving them. Problems arise in many situations that seem complicated and difficult to understand. By increasing the number of ways that a student can organize information presented in a given problem, the chances of understanding the essential features of the problem are increased. Some problem-solving strategies for the program in grades nine through twelve are the following:

1. Using diagrams or drawings to organize and analyze information
2. Using tables or graphs to organize and find new information
3. Using established forms of logical reasoning to discover characteristics of problems
4. Assigning numerical estimates to unknown quantities in a problem and using simple arithmetic to derive new information about the problem
5. Using simpler or similar problems to discover relationships in a given problem
6. Employing translation techniques

Attempts to teach specific applications of mathematics lead quickly to the identification of the following two major difficulties:

1. So many applications of mathematics exist that it is impossible to select any definite subset to represent all possible situations.
2. Nontrivial applications of mathematics usually require considerable teaching of nonmathematical topics.

To counteract these difficulties, the program in grades nine through twelve should, in general, be concentrated on teaching the skills and techniques associated with the process of applying mathematics rather than on teaching specific applications of mathematics.

To provide students systematic, in-depth experiences in which to apply mathematics realistically, it is recommended that problem-solving blocks be designed that last from a few days to two weeks during which the student has the opportunity to immerse himself or herself in a problem-solving situation, exercise investigative and experimental skills, and apply mathematics knowledge. Such experiences would enable students to develop enough of an understanding of the given situation that they could then answer more substantive questions about it and could see their mathematical skills used in a realistic manner.

Probability and Statistics Strand

Societal needs demand that students at the secondary level become knowledgeable about fundamental concepts of probability and statistics as methods of analyzing data. Statistical findings and graphic presentations are used in all facets of daily life. Significant decisions and predictions are systematically made at various levels of business, industry, and government on the basis of statistical interpretations and inferences. Similarly, students throughout their lives will be confronted with statistical presentations of data and hence will be forced to make decisions based on the analysis and interpretation of those data.

Interest and enthusiasm for the study of probability and statistics should be easy to foster among students, because the applications of statistics have widespread significance in almost every discipline. A resourceful teacher can introduce problems and experiments that will be relevant to the spectrum of experience possessed by any given group of students. When possible, experiments should be planned and conducted by students to enhance the opportunities for student understanding and motivation for learning statistical concepts.

Through actual experimentation, the program can enable students to develop the nature of probability from an empirical viewpoint, leading to the development of theoretical probabilities based on methods of counting outcomes.

Students should acquire concepts and methodology for calculating measures of central tendency, dispersion, and skewness. The binomial distribution, based on repeated independent trials of events, is a probability function that should provide insight about discrete variables. The central role played by the normal distribution in the study of observed data needs to be carefully developed. Students should develop facility in using tables of the normal distribution for the solution of inferential problems.

Learning experiences in the methodology of analyzing data should provide students with opportunities for collecting, organizing, presenting, graphically representing, interpreting, and making inferences about data. If statistical experiences are to be meaningful to students, then data collection should be the outgrowth of measuring and observing experiences pertinent to the students' environment.

Students who are interested in continuing their studies of probability and statistics should have opportunities for further exploration of more complex probability distributions, simple correlations between paired variables, and curve fitting for distributions of data.

Relations and Functions Strand

The notions of relations and functions are among the most significant and useful ideas in mathematics. Functions and relations should be identified in many fields such as economics, science, and education and in nature. The early introduction of the concepts of relations and functions makes it possible for students to unify parts of mathematics in a natural way and to apply mathematical techniques to many fields of study.

If students can learn to identify, represent, and use functions and relations, they gain the understanding and power to predict results from causes known or supposed. Furthermore, using the mathematical representations of functions and relations, they can literally reproduce a situation millions of times without having to perform experiments or use expensive equipment each time they want to acquire additional information.

The mathematics program in grades nine through twelve should provide for each student who studies mathematics the opportunity for a gentle but sustained development and use of these simple yet powerful unifying concepts.

As a minimum, the program in grades nine through twelve should provide the opportunity for each student to develop the following:

- A beginning understanding of relations and functions
- Some skill in representing relations and functions in word descriptions, in tabular form, in the use of formulas, in a diagram, in a graph, as a map, or as a set of ordered pairs
- Some skill in drawing graphs of simple functions and relations
- Some understanding of how functional relationships can be used to discover new information about a situation that may not have originally been apparent

As students progress in their study of mathematics, processes for constructing and manipulating functions should be identified, and the opportunity to study and use elementary functions (polynomial, rational, algebraic, exponential, logarithmic, and trigonometric) should be available. The notion of the inverse of a function should be emphasized, using the relationship between the graph of a function and the graph of its inverse. Functions and relations in the form of equalities and inequalities should be solved graphically as well as algebraically, and some introduction to the application of these relations should be given in the study of simple linear programming problems.

Logical Thinking Strand

The process of reasoning is basic to all mathematics, particularly to deductive reasoning. In fact, if anything typifies mathematics, it is the free spirit of making hypotheses and definitions, rather than a mere recognition of facts.

If logical thinking is to be divided into categories, the most natural are deductive reasoning and inductive reasoning. In instruction at the secondary level, there should be conscious experience with both types of reasoning in order that students can understand and make applications. In solving problems, the students should be helped to realize whether they are starting from general premises and seeking consequences or whether they are aiming at universal conclusions by examining particular instances. Programs in mathematics must help students (1) to understand the relation between assumptions and conclusions and thus to test the implications of ideas; (2) to develop the ability to judge the validity of reasoning that claims to establish proof; (3) to generalize both of the preceding skills; and, finally (4) to apply all of the preceding skills to situations arising in many fields of thought.

Inductive reasoning. Inductive reasoning characterizes an early stage in the process of growth and maturation that culminates in a mature understanding of both induction and deduction. Inductive reasoning is usually informal and intuitive, but it includes several different modes of thought.

1. *Simple enumeration.* If enough cases are collected, there will be some assurance that the conclusion drawn from the evidence is reasonably certain. Simple enumeration is extremely worthwhile in that it gives the student reasonable assurance of correctness. The danger present is that students will tend to think that mere numbers constitute proof.
2. *Method of analogy.* Some of the most influential hypotheses have their origins in a person's ability to use analogous reasoning. One must be certain, however, that the analogy fits before conclusions are drawn.
3. *Extension of a pattern of thought.* Many ideas are the result not merely of enumeration and analogy but of extrapolation—extending ideas beyond the observed instances. The process of extension approaches the formality of deductive inference, but

extension does not carry the authority nor the necessity of the implication.

4. *Hypothesis.* Guessing has a place in any area of mathematics, from the earliest elementary school experiences to courses in the high school curriculum. When students have many experiences in making reasonable conjectures, they soon come to see the value of hypothesizing—extending their perception beyond what is immediately evident.

Deductive reasoning. The process of deduction involves moving from assumptions or reasons to conclusions, such a process being called an inference. Students are exposed to simple inferences in elementary school, and their experiences in daily living provide them with a wealth of this type of reasoning.

Any mathematics program should incorporate general procedures for presenting students with the basic rules for forming valid inferences. At times this may be done within the context of courses already in the curriculum, such as in a course in algebra or one in geometry. Other programs may include a course in logical thinking that will encompass the basic principles of deduction.

Computers Strand

The advent of the use of computers in high schools has raised questions which demand attention: How much should be taught about computers? To whom? By whom? What are the vocational responsibilities of the school? How can schools keep abreast of the rapid technological advances? Should the use of computers be applied to all areas of the curriculum? What are the effects of computer-assisted instruction on attitudes toward learning? Does the use of computers "dehumanize" society?

Regardless of the problems, the reality of computers in the educational environment has prompted the establishment of the following guidelines for the mathematics program in grades nine through twelve. When feasible, computers should be used in educational programs in the following three ways: (1) instruction about computers; (2) learning with the aid of computers; and (3) management of instruction.

Instruction about computers. Different student capabilities and interests will prescribe the scope of instruction about computers. A minimal level of computer literacy for all students includes the following:

1. Flowcharting
2. The functions (storage, computation, and control) of the

processing component of computers
3. The electronic method of coding (the binary system)
4. Input-output devices for communicating with computers (hands-on experiences)
5. The history and evolution of computers

For students with vocational interests in computers, instruction should be expanded to include the following:

1. Learning a compiler language (e.g., BASIC and FORTRAN)
2. Computer programming (e.g., alphabetizing and calculating)
3. Data processing (e.g., keypunching and sorting)
4. Business applications (e.g., inventory control and payroll)
5. On-site visits to computer installations

Finally, students who elect to pursue a career in computer science should be offered the following topics:

1. Advanced programming techniques
2. Additional skills in machine languages
3. Writing computer-assisted instruction programs
4. Writing simulation programs
5. Formulating and solving real problems in science, mathematics, social studies, economics, ecology, and so forth

Learning with the aid of computers. The diverse applications of computers to learning concepts and skills in mathematics preclude the establishment of a separate "computer department." Computers should be used in every mathematics class in the following ways:

1. Drill and practice with immediate correction
2. Remedial instruction with branching
3. Self-contained presentations of new material
4. Exploring interesting problems with repetitious calculations
5. Solving problems which arise in science, social studies, economics, ecology, and so forth
6. Simulation models (e.g.. predictions and games)

Management of instruction. Computers should be used extensively to aid the teacher in the classroom. Some possible services that can be provided are the following:

1. Scheduling and evaluating resource materials
2. Cataloging topics by cross-reference
3. Recording individual student progress
4. Prescribing individual instruction
5. Selecting test items from a bank of questions
6. Scoring and analyzing test results

Suggestions for Mathematics Programs

In the objectives of the framework for grades nine through twelve, several different mathematics programs were suggested. This section will attempt to clarify some of those suggestions. The suggestions are not intended to be definitive but should be considered only as a point of departure for local program development.

College-Preparatory Program for Nontechnically Oriented Students

An alternative two-year college-preparatory program can be offered for nontechnically oriented college capable students. The objectives of the curriculum are as follows:

1. The curriculum should be devoted almost entirely to those mathematical concepts that *all* citizens should know in order to function satisfactorily in our society.
2. The traditional grade placement of topics should be ignored. Instead, topics from arithmetic, algebra, and geometry should be interwoven in such a way that they illuminate and support each other.
3. The basic ideas about certain new topics, such as computer mathematics, functions, coordinate geometry, transformations, probability, and statistics, should be made available to all students.
4. It is important to make clear to all students that mathematics is indeed useful; that it can help us in understanding the world we live in and in solving some of the problems that face us.

The content of an alternative college-preparatory program could be as follows:

Grade nine

1. Structuring space
2. Functions
3. Informal algorithms and flowcharts
4. Problem formulation
5. Number theory
6. The integers
7. The rational numbers
8. Congruence
9. Equations and inequalities
10. Decimal representation for rational numbers
11. Probability
12. Measurement
13. Perpendiculars and parallels (I)
14. Similarity

Grade ten

15. The real number system
16. Area, volume, and computation
17. Perpendiculars and parallels (II)
18. Coordinate geometry
19. Problem solving
20. Solution sets of mathematical sentences
21. Rigid motions and vectors
22. Computers and programs
23. Quadratic functions
24. Statistics
25. Systems of sentences in two variables
26. Exponents and logarithms
27. Logic
28. Applications of probability and statistics

A third-year program could include the following:

1. Organizing geometric knowledge
2. Concepts and skills in algebra
3. Formal geometry
4. Equations, inequalities, and radicals
5. Circles and spheres
6. The complex number system
7. Equations of the first and second degree in two variables
8. Systems of equations
9. Logarithms and exponents
10. Introduction to trigonometry
11. The system of vectors
12. Polar form of complex numbers
13. Sequences and series
14. Permutations, combinations, and the binomial theorem[1]

Flexible Minicourse Program

A flexible minicourse program can be offered for noncollege-preparatory students. Following is a brief outline of topics that could be made available in a nine-week minicourse format. Some topics in the outline have prerequisites, while others do not. (No attempt has been made to arrange the topics in any fixed sequence.) The instructional materials for the minicourses should be written for average or below-average achievers in grades nine through twelve. A school with four general mathematics classes could offer a one- to

[1] *Newsletter No. 36: Final Report on a New Curriculum Project.* Prepared by the School Mathematics Study Group. Pasadena, Calif.: A. C. Vroman, Inc., 1972, pp. 7–8.

four-year program with minicourses such as the following. Students could select four courses per year, depending on their needs and interests.

1. Numerical Trigonometry and Introduction to Surveying
2. Geometric Constructions and Designs
3. Ratio, Proportion, and Variation
4. Problem Solving and Our Society
5. Practical Measurement and Measuring Devices
6. Mathematics and Nature
7. Mathematics and Games
8. Consumer Economics and Home Management Mathematics
9. Business Mathematics and Business Statistics
10. Flowcharts and Introduction to Programming
11. Credit and Installment Buying
12. Solutions of Equations and Inequalities and Applications
13. Desk Top and Minicalculators and Probability
14. Reading, Constructing, and Using Tables and Graphs
15. Practical Statistics
16. Coordinate Systems, Decision Making, and Solutions of Systems of Mathematics Sentences

Mathematics Clinic

A mathematics clinic program could be designed to provide individualized instruction aimed at meeting the needs of selected students whose math achievement is significantly below their expected level of achievement. The program could be organized and planned to meet the identified needs of students. A procedure for identification of students should include teacher-counselor recommendations and testing data. The program for each student is planned individually and includes diagnostic testing, pretesting, individualized instruction, and post-testing.

A clinic could operate in a pullout program (students are temporarily released from their regular classes) or as a quarter or semester course. It is desirable to provide aides to assist the teacher of the mathematics clinic. An aide might be a paraprofessional, a parent, or a student assistant.

Some examples of different students whose needs are met in clinic programs are as follows:

1. Students with an ability level within the average range who are achieving below their expected level in mathematics
2. Able students (including gifted) who are below their expectancy level of achievement in computational skills

3. Target students in mathematics as identified for the Elementary and Secondary Education Act, Title I, program

One example of a mathematics clinic program is a program designed for students with an average range of ability who are achieving below their expected level (see number 1 above). The program is designed to provide individualized instruction in basic concepts and computational skills suited to the needs of selected students. Offered as a nine-week quarter course, the clinic serves students whose test scores indicate deficiencies of approximately two years below expected grade level. The clinic is organized for a maximum number of 16 students per class period.

Some factors to be considered in recommending students to the clinic are the following:

1. *California Tests of Basic Skills* computational scores or some other appropriate test scores to determine a list of students from whom the clinic teacher will make recommendations to the counselors
2. Ability level of stanine three or above
3. Previous grades in math
4. Teacher recommendations

Recommended equipment for the clinic is as follows:

1. Four calculators that have floating decimal-point capability; the operations of addition, subtraction, multiplication, and division; and a percent key
2. Computational skills kit
3. Cassette players with tapes
4. Film-loop projector (Super-8) with tapes
5. Programmable calculators with a card reader and accompanying drill and practice instructional materials
6. Arithmetic facts kit with fact pacer

Recommended materials for the clinic are the following:

1. Programmed practice materials
2. Basic textbooks (or appropriate textbook for possible ungraded quarter courses) with accompanying workbooks

Another desirable equipment item for the math clinic is a videocassette player.

Math clinic activities are as follows:

1. Upon entering the mathematics clinic, a student is:
   a. Interviewed by the clinic teacher for interest in improving computational skills
   b. Given an orientation to the clinic procedures

c. Given standardized diagnostic tests to verify specific areas of difficulty
d. Involved in determining individualized work based on the deficiencies indicated by the diagnostic test
e. Involved in developing the student's individual program

2. When a student completes the individual program, he or she:
a. Is given a post-test
b. Discusses the results of the test with the teacher
c. Is programmed into the workbook accompanying the basic textbook that is used in the regular classroom (This interface assignment is given to bridge the gap between the clinic effort and current levels of classroom assignments.)
d. Can cover in detail some of the main topics studied in the student's regular classroom

Since each student is involved in developing an individual plan of progress, the teacher must observe the student's motivation and efficiency pursuant to correcting the student's problems.

College-Preparatory Program for Technically Oriented Students

Alternative college-preparatory programs can be offered for technically oriented college capable students. Two alternatives for including calculus in college-preparatory mathematics programs are suggested below. The first alternative program is designed to provide an articulated program beginning in grade seven and leading to advanced college placement for mathematically talented students. The pattern of courses is as follows:

Grade 7		*Grade 8*		*Grade 9*		*Grade 10*		*Grade 11*		*Grade 12*
Prealgebra	→	Algebra 1	→	Algebra 2	→	Geometry	→	Precalculus	→	Calculus

The course in grade twelve could be offered by a community college on the high school campus. While commercial textbooks are used at each level, other materials on topics not found in the books, such as mathematical systems, non-Euclidean geometry, source books of challenging problems, projects, and library references, need to be prepared.

Another alternative program is a highly sequential, integrated six-year program beginning in grade seven. Students may leave the program at any grade level. This program removes the traditional barriers between separate courses, such as between algebra and geometry, and it provides students with the equivalent of two full years of college-preparatory mathematics.

Appendix A

Mathematics Program Strands and Objectives for Kindergarten Through Grade Eight

Arithmetic, Numbers, and Operations Strand

Counting

Counting: Readiness

Exploration and identification of the number of objects in a set
Exploration of the concept of "how many"
Exploration and identification of equal and unequal sets of physical objects
Acquisition of the skill of counting concrete objects

Counting: Kindergarten–Grade Three

Acquisition of the skill of counting pictorial representations of objects
Exploration and identification of predecessor and successor elements for whole numbers
Exploration and recognition of the uses of the empty set
Demonstration of counting by multiples of numbers
Illustration of the conservation of numbers
Exploration of the process of locating points on the number line
Development of the construction and use of the number line
Demonstration of the order relation for the whole numbers
Extension of the number line concept to include integers
Intuitive development of concepts of equality and inequality, using sets of physical objects
Exploration and identification of equivalent and nonequivalent concrete sets, using one-to-one correspondence
Acquisition of the skill of reading and writing numerals and number names written in words
Demonstration of ordering of sets of objects by size comparisons
Exploration of reading and counting pictorial representations of objects
Maintenance of counting skills through review and practice

Counting: Grades Four–Six

Exploration of simple infinite sets (whole numbers, even numbers, and so forth)
Acquisition of the skill of reading and writing nonnegative rational numbers
Utilization of counting by multiples to aid in the understanding of multiplication and division

Extension of the number line concept to include rational numbers
Demonstration of the order relation for the nonnegative rational numbers
Demonstration of the order relation for the integers

Counting: Grades Seven–Eight

Exploration, acquisition, and utilization of concepts involving comparisons and correspondences between sets–both finite and infinite
Demonstration of the order relations for the rational numbers
Acquisition of the skill of reading and writing rational numbers

Operations

Operations: Readiness

Intuitive development of joining and separating sets, using concrete objects
Intuitive understanding of addition and subtraction, using sets of concrete objects

Operations: Kindergarten–Grade Three

Development of addition as joining sets and subtraction as separating sets
Exploration of addition and subtraction, using Venn diagrams
Presentation of appropriate use of symbols (+, –, X, =, >, <, and so forth) between numbers
Introduction of addition and subtraction facts
Mastery of addition and subtraction facts
Acquisition of the skill of adding or subtracting without regrouping
Acquisition of the skill of adding or subtracting with regrouping
Acquisition of the skill of adding and subtracting amounts of money
Identification of addends and sums in addition algorithms
Maintenance of addition, subtraction skills for whole numbers by review and practice
Intuitive understanding of the concept of multiplication
Acquisition of, and review of, basic multiplication facts
Acquisition of an understanding of, and use of, multiplication algorithm
Exploration of addition and subtraction of decimal fractions
Exploration of, and acquisition of, skill in working with number sentences
Exploration of the addition and subtraction of integers
Acquisition of the skill of adding and subtracting like fractions with sums less than one
Exploration of solving simple linear equations and inequalities involving + and –
Maintenance of multiplication skills for whole numbers through review and practice

Operations: Grades Four–Six

Illustration of addition as "joining" and subtraction as "separating"
Application of Venn diagrams as an aid in understanding and solving problems, using addition and subtraction
Development of mastery of the addition and subtraction facts

Acquisition of an understanding of, and application of, the operational and relational symbols (+, −, ÷, =, >, <, and so forth)
Acquisition of problem-solving skills for problems that involve the addition and subtraction of whole numbers
Development of mastery and accuracy in using addition and subtraction algorithms
Maintenance of addition and subtraction skills for whole numbers by review and practice
Intuitive understanding of the concept of division
Development of basic division facts
Development and mastery of the basic multiplication and division facts
Acquisition and usage of the division algorithm
Explanation of the various steps used in solving division problems
Acquisition of problem-solving skills in multiplication or division of whole numbers
Exploration of the use of integers in everyday situations
Identification of factors and products in multiplication and division algorithms
Development of the skill of adding and subtracting integers
Acquisition and maintenance of skill in working with number sentences
Maintenance of multiplication, division skills for whole numbers by review and practice
Acquisition of computational skill for addition and subtraction of decimal fractions
Acquisition of computational skill for multiplication and division of decimal fractions
Representation of a number in its equivalent fraction or decimal fraction form
Acquisition of computational skill for the addition and subtraction of common fractions
Acquisition of computational skill for multiplying and dividing common fractions
Exploration of percents through related work with fractions
Development of the use of whole numbers as exponents
Exploration of the skill of multiplying and dividing integers

Operations: Grades Seven–Eight

Application of operational and relational symbols (+, −, ×, ÷, >, <, and so forth)
Application of addition and subtraction skills
Maintenance of addition and subtraction skills through review and practice
Acquisition of problem-solving skills related to consumer mathematics
Explorations of shortcuts in the basic algorithm operations
Development of the skill of multiplying and dividing integers
Acquisition of the skill of adding and subtracting rational numbers
Maintenance of multiplication and division skills for whole numbers and fractions through review and practice
Maintenance of computational skills for addition and subtraction of decimal fractions
Maintenance of computational skills for the multiplication and division of decimal fractions

Skills development in estimating sum, difference, product, and quotient of rational numbers
Development of skills in rounding off a rational number written in decimal form
Acquisition of the skill of using integers as exponents
Representation of, and use of, numbers written in percent form
Recognition of, and skill in simplifying, complex fractions
Acquisition of the skill of using number operations to solve number sequences
Acquisition of the skill of squaring a rational number
Acquisition of the skill of estimating the square root of a nonnegative rational number

Place Values

Place Values: Readiness

Exploration of grouping and counting concrete objects
Demonstration of grouping and counting sets of concrete objects
Exploration of grouping and counting concrete objects by tens
Exploration of grouping and counting concrete objects by tens and ones

Place Values: Kindergarten–Grade Three

Exploration and use of the digits zero through nine
Exploration of the role of numbers containing decimal points (e.g., money, metric notation)
Development of skills in counting concrete objects and recording tens and ones
Development of the concept of place value by grouping objects
Exploration of place value of digits in numerals
Identification of place value for any digit in a three-digit numeral
Representation of a number in expanded notation form
Development of skills of counting by tens and hundreds

Place Values: Grades Four–Six

Utilization of counting by tens, hundreds, thousands . . . to develop the concept of place value
Identification and role of a decimal point
Identification of place value for any digit in a numeral
Interpretation of place value of digits in any numeral
Representation of a number in expanded notation form and in exponential form
Exploration of the representation of numbers by scientific notation
Utilization of the place value concept to help develop an understanding of the algorithms for the basic operations with whole numbers

Place Values: Grades Seven–Eight

Interpretation of the order of the powers of ten in expanded notation
Recognition and interpretation of integers as exponents (powers of ten)
Development of, and application of, scientific notation
Application of the properties of the decimal numeration system to the metric system of measurement

Representation of rational numbers, using scientific notation
Interpretation and use of zero and negative exponents

Patterns

Patterns: Readiness

Exploration of simple patterns made with objects
Discovery of patterns made with objects and the extension of those patterns
Exploration of even and odd whole numbers

Patterns: Kindergarten–Grade Three

Discovery of simple patterns using pictures and drawings
Exploration of pattern recognition in sequences of numbers
Exploration of sequences of even and odd numbers and their properties
Exploration of methods of counting, using multiples of numbers
Discovery of patterns of odd and even numbers

Patterns: Grades Four–Six

Discovery and completion of number sequences
Identification and applications of multiples and divisors of numbers
Exploration of number patterns that represent the basic properties of the real number system

Patterns: Grades Seven–Eight

Discovery of a variety of number patterns based on concrete and pictorial models
Recognition of the patterns in sets of ordered pairs
Exploration of the use of variables to represent number patterns

Nature of Numbers

Nature of Numbers: Readiness

Exploration of the reading of numerals
Exploration of the use of numbers and numerals in daily life
Identification of the numeral that names "how many" in a set
Recognition of halves of objects
Recognition of half of a set of objects

Nature of Numbers: Kindergarten–Grade Three

Comparison of equivalent and nonequivalent sets using symbols
Development of the order relation for numbers
Selection of relational symbols to make true statements (=, <, >)
Illustration of the property of betweenness for numbers
Exploration of life situations involving negative numbers
Exploration of fractional parts of whole concrete objects
Recognition of the fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$ represented by the shaded region of a figure

Identification of fractional parts $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$
Description of fractional parts of a whole
Decimal equivalents of fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$
Comparison of equivalent fractions

Nature of Numbers: Grades Four–Six

Definition of addends and sums, factors and products
Illustration of multiplication of whole numbers as repeated addition and powers as repeated multiplication
Exploration of the property of betweenness for numbers
Introduction of integers
Development of fraction notation and the properties of fractions
Selection of a number in fraction form that corresponds to part of a whole concrete object
Acquisition of the skill of changing fractions to higher or lower terms (equivalent fractions)
Definition of least common multiple and greatest common factor
Exploration of the technique of finding common multiples in terms of intersection of sets
Acquisition of the skill of computing the least common multiple of two or more whole numbers
Acquisition of the skill of computing the greatest common factor of two or more whole numbers
Identification of prime and composite numbers
Definition of prime and composite numbers
Exploration of techniques to determine if a given number is prime
Identification of prime factors of a composite number
Definition of relatively prime numbers
Acquisition of the skill of ordering fractional numbers
Demonstration of the conversion of fractions or decimals to percents and the converse
Exploration of the basic concepts of ratio, proportion, and percent
Demonstration of the conversion of percents or decimals to fractions in lowest terms

Nature of Numbers: Grades Seven–Eight

Definition of factors in terms of divisibility
Utilization of techniques to find missing factors
Utilization of a technique to determine if a given number is prime
Exploration of some of the simple properties of primes
Acquisition of computational skills using negative numbers
Description of rules for divisibility
Prediction of the number of possible whole number factors of a given number
Utilization of a technique to determine the prime factors of a number
Applications of least common multiple and greatest common factor
Exploration of techniques to find sequential prime numbers
Utilization of greatest common factors in solving proportions

Acquisition of the skill of converting percents, decimals to fractions in lowest terms
Acquisition of the skill of converting fractions or percents to decimals
Exploration of the periodicity of the decimal form of rational numbers
Application of the use of percents, fractions, and decimals in problem situations

Properties

Properties: Readiness

Exploration of the order relation of numbers

Properties: Kindergarten–Grade Three

Intuitive presentation of the commutative property of addition
Intuitive exploration of commutative and associative properties of addition and multiplication for whole numbers
Intuitive exploration of the distributive property of whole numbers
Exploration of the use of parentheses in grouping
Informal development of the definition of the set of whole numbers and the set of integers
Exploration of the special properties of zero and one

Properties: Grades Four–Six

Demonstration of the distributive property
Utilization of parentheses to demonstrate the associative and distributive properties
Demonstration of the properties of operations for nonnegative rational numbers
Exploration of the closure property for addition and multiplication on the set of whole numbers
Exploration of the role of identity elements in addition and multiplication
Exploration of the role of inverse elements in addition and multiplication
Demonstration of the use of inverse elements in operations with integers and fractions
Recognition of the binary nature of our four basic operations
Illustration of subtraction and division as inverse operations of addition and multiplication, respectively
Exploration of presence or absence of the property of denseness in different subsets of the real numbers
Exploration of the use of properties of number systems to develop algorithms for basic operations when using scientific notation
Intuitive development of definitions of the set of integers and the set of nonnegative rational numbers
Development of the special properties of zero and one
Exploration of the identity element for the operations of addition and multiplication on various sets of numbers

Properties: Grades Seven–Eight

Demonstration and utilization of the properties of rational numbers
Demonstration and application of the closure property for the operations of addition and multiplication on various sets of numbers

Demonstration and application of the identity elements for the operations of addition and multiplication on various sets of numbers
Demonstration and application of the inverse elements for the operations of addition and multiplication on various sets of numbers
Development of an understanding of the definition of absolute value
Demonstration of presence or absence of the property of denseness in different subsets of the real numbers
Application of field properties to solve mathematical sentences
Exploration of group properties of some subsets of real numbers
Exploration of the use of the field properties in the development of algorithms for basic operations with integers and rational numbers
Demonstration of the use of properties of number systems to develop algorithms for basic operations when using scientific notation
Intuitive development of the definition and properties of real numbers
Recognition and use of the special properties of zero and one
Illustration and utilization of one-to-one correspondence between real numbers and points on the number line
Exploration of field and group properties, using finite number system
Acquisition of ability to identify irrational numbers

Geometry Strand

Geometric Figures

Geometric Figures: Readiness

Exploration with familiar physical objects representing three- and two-dimensional geometric shapes
Exploration through design building with three-dimensional materials, including pattern formation
Informal identification of geometric shapes according to their properties
Informal identification of special qualities: inside, outside, on, flat, curved, and straight
Informal identification of properties, using premeasuring activities involving simple comparisons; e.g., round, square, tall, or deep

Geometric Figures: Kindergarten–Grade Three

Exploration with three- and two-dimensional geometric shapes in the environment, including picture design and patterns
Identification of properties of geometric shapes, using more precise language, written and oral
Identification of special qualities of two- and three-dimensional geometric figures and perpendicular and parallel relationships of lines through models and pictures
Recognition of curves: open, closed, regular, and irregular
Development of classification skills with plane geometric configurations according to their properties, using informally the ideas of similarity and congruence

Development of pattern design with reproduction of formalized patterns; e.g., tessellations
Acquisition of ability to demonstrate properties of geometric shapes

Geometric Figures: Grades Four–Six

Exploration with geometric shapes, including movement of forms as an intuitive introduction to transformations
Identification of closed and open curves, closed and open surfaces, cube, sphere, triangular prism, and so forth, using precise language and appropriate symbols
Identification of properties, leading to development of geometric models
Development of an understanding of lines, planes, and space as sets of points and perpendicular and parallel relationships of lines
Development of skills of construction of models and patterns to illustrate geometric ideas using appropriate instruments
Development of ability to classify and name geometric figures, using congruence and similarity
Acquisition of ability to illustrate properties of shapes and their interrelationships

Geometric Figures: Grades Seven–Eight

Exploration with geometric figures, using transformations to investigate properties of reflections, rotations, translations, and dilations
Exploration of geometric models in relation to problem solving
Identification of geometric models applicable to specific problem situations
Acquisition of ability to demonstrate concepts of congruence and similarity, and perpendicularity and parallelism
Acquisition of skill of model building
Application of laws and principles related to geometry in practical situations
Applications using constructions and models to pertinent situations in two- and three-dimensional space

Reasoning

Reasoning: Readiness

Exploration through the sorting of three-dimensional and two-dimensional shapes, leading to short chains of reasoning

Reasoning: Kindergarten–Grade Three

Exploration with physical objects for informal verification of similarity and congruence relationships

Reasoning: Grades Four–Six

Explorations with materials verifying similarity and congruence
Exploration of geometric patterns illustrating inductive reasoning, leading to generalizations
Identification of ways to verify congruence and similarity
Identification of short chains of deductive and inductive reasoning
Acquisition of ways to verify similarity and congruence

Reasoning: Grades Seven–Eight

Exploration with ideas of congruence and similarity
Exploration of angle measurement relations for triangles and quadrilaterals
Exploration of deeper meanings of the reasoning process
Development of simple deductive and inductive reasoning processes, especially those related to similarity, congruence, and sum of angle measures for angles of a triangle
Acquisition of ability to perform short, sequential reasoning exercises

Coordinate Geometry

Coordinate Geometry: Readiness

Exploration with two-dimensional materials as a foundation for coordinates; e.g., tile floor or checkerboard
Informal exploration with simple patterns involving symmetry in the environment; e.g., blot patterns or leaves

Coordinate Geometry: Kindergarten–Grade Three

Informal exploration with coordinates, using appropriate vocabulary and symbols in the first quadrant
Explorations with simple patterns of picture symmetry; e.g., paper folding
Identification of symmetry in the natural world
Identification of points on a line corresponding to positive and negative numbers

Coordinate Geometry: Grades Four–Six

Explorations to develop concepts related to the coordinate plane, such as map reading, graphing, and Cartesian products
Identification of patterns of symmetry on the plane
Identification of points on the coordinate plane with appropriate language and symbols in all four quadrants
Demonstration of ability to locate points in all four quadrants
Utilization of graphing and ordered pairs in the *problem solving/applications* and *relations and functions* strands

Coordinate Geometry: Grades Seven–Eight

Explorations that continue to develop concepts of two-dimension and three-dimensional space
Identification of symmetry on the plane
Identification of points on the coordinate plane in all four quadrants
Demonstration of the ability to use coordinate geometry in all four quadrants
Acquisition of the skill of graphing, using coordinates and symmetry
Application of coordinate geometry to real-life situations

Measurement of Geometric Figures

Measurement of Geometric Figures: Readiness

Exploration with physical objects, using comparison and premeasuring activities

Measurement of Geometric Figures: Kindergarten–Grade Three

Exploration through informal physical activities (e.g., pacing) of concepts of length and perimeter, using arbitrary units
Exploration with materials, using standard units
Identification of the concepts of length and perimeter
Development of skill for measuring
Application of skills of geometric measuring in problem solving

Measurement of Geometric Figures: Grades Four–Six

Explorations with regular and irregular geometric shapes, using comparisons with arbitrary units
Identification of such concepts as area, volume, and measurement of angles
Development of geometric measurement concepts in relationship to computation skills
Application of skills of geometric measurement or problem solving

Measurement of Geometric Figures: Grades Seven–Eight

Explorations with regular and irregular geometric shapes, using comparisons with arbitrary units
Explorations with standard units in measuring geometric figures
Identification of need for developing skill in measuring
Development of the skill for measuring and using measurements
Application of skills and understandings of geometry in numerical problem solving, including use of the Pythagorean formula

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Measurement Strand

Arbitrary Units of Measurement

Arbitrary Units of Measurement: Readiness

Exploration of the attributes of measurement together with the development of appropriate vocabulary
Exploration activities that informally use arbitrary units of measure
Exploration activities that develop a beginning of an understanding of the need for measuring objects and materials in the environment

Arbitrary Units of Measurement: Kindergarten–Grade Three

Exploration of measurement, using a variety of arbitrary units of measure
Development of useful vocabulary for measurement
Development and organization of techniques related to measuring, using selected arbitrary units of measure
Application of techniques of measuring, using selected arbitrary units

Arbitrary Units of Measurement: Grades Four–Six

Exploration of a variety of measuring experiences within the child's environment
Introduction of appropriate vocabulary of measurement
Acquisition of skills in measuring, using arbitrary units of measure

Arbitrary Units of Measurement: Grades Seven–Eight

Exploration of the measuring situations arising in daily living
Demonstration of proficiency in using correct language for measurement
Utilization of techniques in the solution of real-life problems

Standard Units of Measurement

Standard Units of Measurement: Readiness

Exploration of comparative measures within the child's environment, using unmarked objects which are of standard metric unit size

Standard Units of Measurement: Kindergarten–Grade Three

Demonstration of a familiarity with informal methods of measuring
Exploration of comparative measures within the child's environment, using objects marked with standard metric units
Exploration activities with concrete objects that lead to the selection of the appropriate unit for measuring the objects
Exploration of measuring to the nearest whole unit
Development of appropriate vocabulary
Development of measuring techniques, using standard units of measure, including temperature and time
Development of skills, using simple measuring instruments
Exploration activities that involve the expression of measurements in decimal notation
Application of techniques of measuring, using selected standard units

Standard Units of Measurement: Grades Four–Six

Exploration activities leading to the development of concepts related to scale drawings and interpretation of maps
Demonstration of an understanding of measuring to the nearest unit
Development of the ability to choose the appropriate unit for measuring objects
Introduction of conversion between units within the SI (Système International) metric system
Development of the ability to convert units within the SI metric system
Application of measuring skills, using standard metric units
Development of correct vocabulary of measurement
Development of skills in representing measurements in decimal notation
Application of measuring skills that require use of simple measuring instruments

Standard Units of Measurement: Grades Seven–Eight

Development of correct vocabulary
Demonstration of proficiency in converting between units within the metric system
Demonstration of proficiency in measuring to nearest unit
Application of measuring to objects and situations that occur in the pupil's environment

Development of common formulas for measuring objects that are represented by geometric figures
Utilization of common measuring instruments to find the measures of objects in the pupil's environment
Presentation of U.S. Customary units in their historical perspective and development of informal comparisons between them and SI metric units

Approximate Nature of Measurement

Approximate Nature of Measurement: Kindergarten–Grade Three

Exploration of measurement activities, leading to an understanding of the approximate nature of measurement
Identification of the approximate nature of measurement

Approximate Nature of Measurement: Grades Four–Six

Exploration of the approximate nature of measurement
Exploration of the relation between the size of the unit and the "error" in the measurement
Acquisition of an understanding of the approximate nature of measurement
Recognition of the relationship between the approximate nature of measurement and rounding off
Recognition of generalizations relative to the approximate nature of measurement
Acquisition of skill in choosing the appropriate unit of measure

Approximate Nature of Measurement: Grades Seven–Eight

Demonstration of an understanding of the approximate nature of measurement
Demonstration of an understanding of "error" in measurement
Demonstration of skill in choosing appropriate units of measure
Application of the approximate nature of measurement in the solution of problems

Estimation

Estimation: Readiness

Exploration of estimation of the size of objects compared to familiar objects within the range of a child's environment

Estimation: Kindergarten–Grade Three

Exploration of estimation of the size of objects compared to familiar objects within the range of a child's environment
Recognition of the use of guessing in making valid estimations of measurement
Recognition of the importance of choosing the correct unit of measure for valid estimates
Demonstration of the skill of estimation of the size of objects in the child's environment
Development of techniques of estimation, using standard units of measure
Utilization of estimation techniques in the solution of problems involving measurements

Estimation: Grades Four–Six

Demonstration of the skill of estimation of the size of objects in the child's environment

Development of techniques of estimation, using standard units of measure

Recognition of the importance of choosing the correct unit of measure for useful estimates

Utilization of estimation techniques in the solution of problems involving measurements

Estimation: Grades Seven–Eight

Demonstration of techniques for the refinement of an estimate

Application of estimation skills in situations found inside and outside the classroom

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Problem Solving/Applications Strand

Problem Formulation

Problem Formulation: Readiness

Exploration of meaningful situations in which questions would arise such as How many? How far? or How long?

Exploration of meaningful situations in which questions would arise such as What does it look like? or What shape?

Exploration of the formulation of questions that lead to new information about a given situation such as Which is the greater? or What comes first?

Development of beginning techniques of questioning that lead to the placement of a given situation in a mathematical context involving counting

Problem Formulation: Kindergarten–Grade Three

Exploration of the formulation of questions that identify useful or extraneous information given in the description of a situation

Recognition of simple questioning techniques for placing a given situation in a mathematical context that involves whole numbers and the operations of addition, multiplication, and subtraction

Recognition of simple questioning techniques for placing a given situation in a mathematical context that involves linear or weight measurements, time, or money

Identification of typical questions about a given situation that lead to the relations between numbers of $<, =$, or $>$

Identification of simple questioning techniques that lead to the placement of a given situation in a mathematical context involving line or circle graphs

Development of skills in posing questions about the geometric properties of a given situation that lead to the informal use of the relationships, similarity and congruence, or the properties of familiar geometric shapes

Problem Formulation: Grades Four–Six

Exploration of situations in which questions arise that place a given situation in a mathematical context involving the coordinate plane

Exploration of questioning techniques that lead to the recognition of simple functional relationships in a given situation

Exploration of meaningful situations in which questions arise that place the given situation in a mathematical context involving the use of set diagrams to show logical relationships

Exploration of meaningful situations in which questions arise that place the given situation in a mathematical context involving percent

Exploration of situations in which questions arise that place the given situation in a mathematical context involving powers of ten or scientific notation

Identification of simple questioning techniques that lead to the placement of a given situation in a mathematical context involving linear and area measurements, weight measurements, time, money, or indirect measurements such as speed

Development of skill in posing questions that place a given situation in a mathematical context involving whole numbers, integers, the fractional and decimal forms of rational numbers, and the operations of addition, subtraction, multiplication, and division

Development of skill in posing questions that lead to the placement of a given situation in a mathematical context involving mathematical sentences or formulas

Development of skill in posing questions that place a given situation in a mathematical context that involves the arrangement of information in pictographs, line graphs, or circle graphs

Development of beginning skill in posing questions that place a given situation in a mathematical context involving the probability of an event, odds, tree diagrams, or permutations of combinations

Acquisition of skill in posing questions that place a given situation in a mathematical context that involves concepts such as divisibility, factors, multiples, primes, or powers or numbers

Acquisition of skill in posing questions that lead to the informal use of parallel and perpendicular relationships between lines, informal use of similarity and congruence, and properties of figures

Problem Formulation: Grades Seven–Eight

Exploration of situations in which questions arise that place the given situation in a mathematical context involving the concepts of ratio and proportion

Exploration of meaningful situations in which questions arise that place the situation in a mathematical context involving sampling techniques, frequency tables, histograms, range, median, mean, standard deviation, or probability of an event

Identification of questioning techniques that lead to the placement of a given situation in a mathematical context involving the coordinate plane

Development of skill in posing questions that lead to the identification of the functional relationships in a given situation

Development of skill in posing questions that lead to the recognition of logical relationships in a giv[illegible]ation

Development of skil[illegible]osing [illegible]uestions that lead to the identification of the concept of percent [illegible] a given situation

Development of skill in posing questions that place a given situation in a mathematical context involving expanded notation, powers of ten, or scientific notation

Development of skill in posing questions that identify the need for the concepts of arbitrary standard units of measure, subdivisibility of units of measure, the additive nature of units of measure, or the accuracy and precision of measurements

Acquisition of skill in posing questions that identify the need for the techniques of arranging data in statistical graphs such as line graphs, pictographs, circle graphs, histograms, or frequency polygons

Acquisition of skill in posing questions that identify the need for the concepts of permutations and combinations, probability of an event, conditional probability, or mathematical expectation in the interpretation of a given situation

Utilization of skill in posing questions about a given situation that lead to mathematical situations involving mathematical formulas or mathematical sentences and their solution sets

Utilization of questioning techniques that lead to the placement of a given situation in a mathematical context involving linear, area, and volume measurements, weight and capacity, time, temperature, or indirect measurements such as velocity

Utilization of skill in posing questions indicating need to involve the concepts of range, standard deviation, median, mode, or mean in analysis of a given situation

Utilization of skill in posing questions identifying need for geometric concepts (parallelism, perpendicularity, congruence, similarity, or properties of figures) in analysis of a given situation

Problem Analysis

Problem Analysis: Readiness

Exploration activities involving students in drawing pictures to develop understanding of a formulated problem

Exploration activities for the manipulation of physical objects to develop understanding of a given problem

Exploration activities for the use of geometric shapes and their properties to develop an understanding of a given problem

Exploration activities for the use of "guesses" or estimations to develop understanding of a given problem

Development of skill related to counting or one-to-one matching as a technique for the understanding of a given problem

Problem Analysis: Kindergarten–Grade Three

Exploration of the use of tables, drawings, or diagrams to develop understanding of a formulated problem

Exploration of oral discussion activities to develop techniques for restating a problem and for identifying clearly the object of the problem, the given information, and the unknowns

Exploration of techniques for subdividing a problem into simpler subordinate problems and then integrating the analysis of each part into a general analysis of the given problem

Development of skill in using addition, subtraction, or multiplication of whole numbers, fractions, or decimals to represent parts of a given problem

Development of skill in using "guesses" to help identify the relationships between parts of a given problem

Development of skill in using the manipulation of physical materials to simulate the features of a given problem to help identify the characteristics of the problem

Problem Analysis: Grades Four–Six

Exploration activities that involve the written restatement of a problem and the written identification of the unknowns and of the given information

Exploration activities of the identification of similar or simpler problems that have been analyzed before and the use of these results to identify the significant relationships in a given problem

Exploration activities of experiments with physical models of problems to identify significant relationships in a given problem

Exploration activities in which students mathematically generate new data to identify significant relationships in a given problem

Development of skill in using graphs to identify the relationships between parts of a given problem

Development of skill in using patterns, diagrams, drawings, or geometric figures to develop an understanding of the relationships in a given problem

Acquisition of skill in using numerical experimentation (guessing) to identify the important relationships in a given problem

Problem Analysis: Grades Seven–Eight

Exploration of the process of identifying unknowns as variables in a given problem

Exploration of the process of identifying functional relationships, if they exist, in a given problem

Exploration of the process of graphing relationships in the coordinate plane to identify significant features of a given problem

Development of skill in the written analysis of a problem, including the clear identification of the hypotheses or given information and the identification of the unknowns

Development of skills that use the analysis of simpler or similar problems in the analysis of the given problem

Development of skill in using physical simulations of the problem to identify significant relationships in a given problem

Acquisition of skills using arithmetic or geometric patterns, tables, diagrams, drawings, or geometric constructions to develop an understanding of the relationships in a given problem

Utilization of skills in which students estimate the solutions of a given problem in order to identify the significant relationships in the problem

Utilization of skills in which students use and design physical experiments to simulate the features of a given problem in order to identify the significant relationships in the problem

Problem Models

Problem Models: Readiness

Exploration of the construction of models of problems that only involve writing counting numbers (to ten)
Exploration of the construction of models of problems that use the order relations between counting numbers (to ten)
Exploration of the construction of physical objects shaped as simple geometric figures as models of given problems
Exploration of the construction of physical objects, drawings, or pictographs as models of given problems
Exploration of the construction of models that require measurements made by using nonstandard units, such as a hand, a step, or a stick

Problem Models: Kindergarten–Grade Three

Exploration of the construction of models of problems involving the sum, difference, or product of two whole numbers
Exploration of the construction of models of problems involving the sum or difference of two rational numbers in fractional or decimal form
Exploration of the construction of simple number sentences as models of given problems
Recognition of tables, line graphs, circle graphs, or pictographs as models of given problems
Development of skill in using simple geometric shapes such as lines, circles, triangles, squares, or rectangles as models for given problems
Development of skill in using tables of data as models of given problems

Problem Models: Grades Four–Six

Exploration of the number line or the coordinate plane as a model
Exploration of the construction of set diagrams as models
Exploration of the construction of models that use variables, functions, or relations and translation of phrases or sentences
Exploration of the construction of models of problems that involve percent
Exploration of the construction of number patterns or geometric patterns as models of given problems
Exploration of modifying or simplifying mathematical models of problems in order to solve them
Development of skill in drawing geometric figures and using their properties (parallelism, perpendicularity, congruence, or similarity) as models of problems or parts of problems
Development of skill in drawing pictures, array diagrams, or tree diagrams as models of problems
Development of skill in using simple formulas or of constructing simple mathematical sentences as models of problems

Acquisition of skill in using tables of data or simple statistical measures such as mean, median, mode, or range as models of problems

Problem Models: Grades Seven–Eight

Exploration of the use of scale drawings and ratio and proportions as models of given problems

Exploration of the use of concepts of probability and statistics as models of problems

Development of skill in using the graphs of simple functions or relations as models of given problems

Development of skill in using Venn diagrams as models of a given problem

Development of skill in using percent in a model of a given problem

Development of skill in using measurements such as linear, area, volume, weight, time, capacity, or velocity as models of a given problem

Acquisition of skill in translating a problem into mathematical sentences or formulas

Acquisition of skill in using numerical patterns such as arrays or sequences or using geometric patterns as models of given problems

Acquisition of skill in using properties of geometric figures such as similarity, congruence, parallelism, or perpendicularity as models of problems

Problem Solution

Problem Solution: Readiness

Exploration of counting or one-to-one matching as a method of solving a problem

Exploration activities with physical objects, the relations $<$, $=$, or $>$, and the counting numbers (to ten) as a method of solving a problem

Informal development of skills, using nonstandard units of measure such as a hand, a stick, or a step to solve a given problem

Informal development of techniques of comparing the solutions of given problems with a guess

Problem Solution: Kindergarten–Grade Three

Exploration of the use of the computation of sums, products, or differences of positive rational numbers as a means of solving problems

Exploration of the use of simple mathematical sentences to solve problems

Exploration of methods of using tables, drawings, pictographs, and line or circle graphs to find solutions of problems

Development of skill in using properties (such as similarity or congruence) of simple geometric shapes to find solutions

Development of skill in using standard units of measure such as millimeter, centimeter, meter, gram, kilogram, minute, and hour to find solutions

Development of skill in performing simple physical experiments and collecting data empirically to find solutions

Problem Solution: Grades Four–Six

Exploration of graphing on the coordinate plane to solve given problems
Exploration of Venn diagrams to find solutions
Exploration of computation with percents to find solutions
Exploration of the use of simple mathematical sentences to find solutions
Exploration of simple number sequences or series to find solutions
Development of skill in using basic operations on rational numbers to find solutions
Identification and use of simple formulas to find solutions
Development of skill in using properties of geometric figures such as perpendicularity, parallelism, congruence, or similarity to find solutions
Development of skill in using sketches, arrays, or tree diagrams to find solutions
Utilization of data arranged in tabular form and statistical measures such as mean, median, mode, or range to find solutions

Problem Solution: Grades Seven–Eight

Exploration of scale drawings and the use of ratio and proportion to find solutions
Development of skill in using simple probability and statistics to find solutions
Development of skill in using graphs of functions or relations to find solutions
Development of skill in using Venn diagrams to find solutions
Development of skill in computing with percent to find solutions
Development of skills of making or computing measurements such as area, volume, weight, time, capacity, or velocity to find solutions
Acquisition of skill in finding solution sets of mathematical sentences or formulas
Acquisition of skill in using series, sequences, or geometric patterns to find solutions
Acquisition of skill in using properties of geometric figures such as similarity, congruence, parallelism, or perpendicularity to find solutions

Solution Interpretation

Solution Interpretation: Readiness

Exploration of the method of testing counting number solutions or problems back in the original situations to see if they reflect the actual situation
Exploration of the method of testing guesses or estimates of solutions of problems in the original situation

Solution Interpretation: Kindergarten–Grade Three

Exploration of checking the solutions of simple mathematical sentences
Exploration of testing geometric solutions or relationships identified in drawings, pictographs, or line or circle graphs in the original situation
Development of skill in testing positive rational number solutions of problems

Solution Interpretation: Grades Four–Six

Development of skill in checking solutions of simple mathematical sentences or formulas

Development of skill in testing geometric solutions of problems
Development of skill in interpreting variations in a model in terms of the original problem situation
Acquisition of skill in testing rational number solutions of problems
Acquisition of skill in performing experiments to test solutions of problems involving measurements

Solution Interpretation: Grades Seven–Eight

Exploration of using solutions of problems to see if the variables in a given problem are related directly or inversely
Development of skills to verify the solutions found using data arranged in tables, geometric figures, or graphs
Acquisition of skill in checking solutions of mathematical sentences
Utilization of solutions of problems to predict solutions to related problems

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Probability and Statistics Strand

Collection, Organization, and Representation of Data

Collection, Organization, and Representation of Data: Readiness

Exploration of counting physical objects and grouping in various quantities
Exploratory discussions of simple inferences drawn from collected data

Collection, Organization, and Representation of Data: Kindergarten–Grade Three

Exploration activities for manipulation of concrete objects to generate data and subsequent discussions of inferences
Exploration experiences in the construction and interpretation of simple bar graphs
Exploration activities for construction and interpretation of simple line graphs
Informal development of ideas for construction and interpretation of circle graphs
Development of experiences drawing inferences from simple graphs

Collection, Organization, and Representation of Data: Grades Four–Six

Exploration activities for manipulation of objects to generate data; subsequent discussions of inferences drawn from data collected
Development activities for construction and interpretation of pie or circle graphs
Development activities for construction and interpretation of ideographs or pictograms
Recognition of techniques for drawing inferences from collected data
Recognition of techniques for developing tables for the collection and organization of data

Collection, Organization, and Representation of Data: Grades Seven–Eight

Exploration activities for manipulation of objects to generate data and discussions of inferences drawn from data collected

Exploration experiences in using sampling techniques
Exploration and use of the technique of random sampling
Acquisition of techniques for drawing inferences from data
Acquisition of techniques for construction and interpretation of frequency tables
Acquisition of techniques for construction and interpretation of histograms

Interpretation of Data

Interpretation of Data: Readiness

Exploration of drawing inferences from collected d

Interpretation of Data: Kindergarten–Grade Thre

Exploration experiences in determining the range fo[illegible] of data
Recognition of techniques for drawing inferences from a set of data

Interpretation of Data: Grades Four–Six

Exploration of techniques for calculating the arithmetic average or mean for a set of data
Development of the ability to calculate the median, mode, range, and mean for a given set of data
Development of techniques for drawing inferences from the treatment of the data set

Interpretation of Data: Grades Seven–Eight

Exploration of the interpretation of a normal curve
Exploration of the meaning of standard deviation
Exploration of the meaning and interpretation of quartiles and percentiles

Counting Techniques

Counting Techniques: Readiness

Exploratory experiences in collecting data for one-to-one relationships as well as sorting and grouping of data

Counting Techniques: Kindergarten–Grade Three

Exploratory experiences in collecting data for one-to-one relationships as well as sorting and grouping of data
Utilization of tally marks or objects to record data

Counting Techniques: Grades Four–Six

Exploratory experiences using manipulative materials in data-counting procedures
Development of techniques for the calculation of the number of combinations (selections) that can be made from a given set of objects taken n at a time
Development of techniques for the calculation of the permutation (arrangements) of n things taken r at a time

Development of fundamental counting procedure
Development of tree diagrams to count selections

Counting Techniques: Grades Seven–Eight

Exploratory experiences, using manipulative materials in data counting procedures
Exploration of the determination of a sample space for a particular occurrence
Acquisition of techniques for the calculation of permutations of n things taken r at a time
Application of fundamental counting procedure
Utilization of tree diagrams to count selections

Probability

Probability: Readiness

Exploratory experiences in guessing, hypothesizing, and making predictions, followed by experimentation and discussion

Probability: Kindergarten–Grade Three

Exploratory experiences in guessing, hypothesizing, and making predictions, followed by more complex experimentation and discussion

Probability: Grades Four–Six

Exploration activities that lead to an understanding of the definition of probability
Development of an understanding of odds
Development of the concept of an event

Probability: Grades Seven–Eight

Development of an understanding of the probability of an event that is certain to occur
Development of an understanding of the probability of an event that is certain not to occur
Development of experiences with $P(A \cap B)$ and $P(A \cup B)$
Development of an understanding of independent events
Development of an understanding of complementary events
Development of an understanding of mutually exclusive events

Relations and Functions Strand

Patterns

Patterns: Readiness

Exploration activities for the completion or construction of pictorial representations of patterns
Exploration activities for the identification of patterns in the relationships between objects (such as shape, location, size, time, weight, and temperature)

Patterns: Kindergarten–Grade Three

Exploration activities involving the recognition of patterns of symmetry and repetition in geometric objects or drawings
Exploration activities involving the recognition of patterns in simple numerical sequences
Development of skills in recognizing patterns in the relationships between, or properties of, objects such as shape, location, size, time, weight, temperature, and so forth
Development of skills in identifying missing terms in numerical sequences
Acquisition of meaningful vocabulary of comparison

Patterns: Grades Four–Six

Exploration activities for the recognition of specific mathematical patterns
Exploration activities for the use of mathematical patterns
Acquisition of skills in using mathematical patterns that can serve as models of given problems

Patterns: Grades Seven–Eight

Development of skills in identifying patterns in numerical sequences
Development of skills in using variables in the representations of mathematical patterns
Utilization of skills in using mathematical patterns as models of given problems
Maintaining the meaningful vocabulary of comparison

Relations

Relations: Readiness

Exploration activities involving the concept of a set of ordered pairs
Exploration of the concept of pairing by associating names with objects
Exploration activities that illustrate various examples of relations

Relations: Kindergarten–Grade Three

Exploration activities that will involve a description of a set of ordered pairs through pictorial representation
Exploration activities that will involve comparison of sets through matching
Development of a simple mathematical language for sets of ordered pairs
Development of skills in recognizing equivalent sets
Development of skills in determining the rules for finding the second number of an ordered pair
Development of skills in using the comparison relationships between sets by determining if the number of a set is greater than, less than, or equal to the number of another set
Acquisition of skills in identifying the properties of order on the number line by (a) determining numbers that come before and after a given number; (b) determining one more than and one less than a given number; and (c) comparing fractions on the number line
Identification of the symbols $<, >, =, \neq$

Relations: Grades Four–Six

Exploration of activities for the identification and interpretation of ordered pairs of numbers
Exploration of activities for the identification and demonstration of various types of relations
Development of skills in expressing a given relation, using a mathematical sentence; and in graphing the solution set on the Cartesian coordinate plane
Development of skills in recognizing the one-to-one correspondence of sets
Development of skills in recognizing the equality and inequality of rational numbers
Acquisition of skills involving the location of points on the number line
Acquisition of skills that involve the comparison of numbers on the number line
Acquisition of skills involving the graphing of solution sets of simple and compound mathematical sentences

Relations: Grades Seven–Eight

Exploration activities involving the collecting of data for different types of relations
Development of skills in defining one-to-one correspondence between sets, both finite and infinite
Development of skills involved in recognizing the domain, range, and rule of a relation
Development of skills used in recognizing the three properties of equivalence relations
Acquisition of skills used in graphing solution sets of number sentences on the number line
Acquisition of skills in recognizing the relations between sets (inclusion and so forth)

Functions

Functions: Readiness

Exploration of activities that involve sets of ordered pairs
Exploration of activities that involve the use of pictorial representations related to functions

Functions: Kindergarten–Grade Three

Exploration of activities for the identification of a rule for different graphs of functions
Exploration of activities involving addition and subtraction, using function machines
Exploration of activities involving the many functional relations existing in nature
Development of skills in identifying graphs of functions

Functions: Grades Four–Six

Exploration of activities for determining rules for function machines
Development of activities for recognizing functions

Development of activities for defining functions
Development of skills in generating addition and multiplication tables from function machines
Acquisition of skills in illustrating formulas as functions (length, area, volume)

Functions: Grades Seven–Eight

Exploration of activities involving recognition of the fundamental operations as functions
Acquisition of skills to apply the concept of pairing to relations
Exploration of activities involving recognition of formulas, statements, graphs, or tabulated data as functions

Graphs

Graphs: Readiness

Exploration activities involving graphs related to physical objects.
Exploration activities that will use simple charts for reference, comparison, and recordkeeping
Development of skills leading to the recognition of patterns through special activities involving pictorial representation of relations

Graphs: Kindergarten–Grade Three

Exploration activities for development of a mathematical language through pictorial representation
Exploration activities that include recognition and construction of various kinds of graphs
Development of activities for interpretation of graphs and tables
Exploration activities for the application of the use of tally marks for counting
Exploration activities that include the illustration and construction of frequency tables for use in recordkeeping
Exploration activities that include representation of number pairs in tabular and graphical form
Exploration activities that include tabulation of data
Development of activities for locating points by using ordered pairs of numbers

Graphs: Grades Four–Six

Exploration activities for interpretation of data by bar and double bar graphs
Exploration activities that will include interpreting and graphing of data given as sets of ordered pairs
Exploration activities that include construction of the Cartesian product for any two sets of whole numbers
Development of skills that generate sets of ordered pairs from tables, relations, formulas, and so forth
Development of activities for determining a rule for the graph of a relation
Development of skills that include graphing of ordered pairs
Acquisition of skills for identifying, interpreting, and constructing the coordinate plane

Acquisition of skills for recognizing graphs of functions
Acquisition of skills involved in graphing linear functions, using whole numbers, rationals, and integers
Acquisition of skills involved in interpreting and constructing line and circle graphs

Graphs: Grades Seven–Eight

Exploration activities that involve recognition, construction, interpretation, and demonstration of various kinds of graphs
Exploration activities that include collecting data for sets of ordered pairs
Exploration activities that include interpreting charts and graphs from data
Development of skills needed for construction and use of the coordinate plane
Development of skills that will include construction of graphs of relations, including inequalities
Acquisition of skills that will include plotting of linear and quadratic functions, step functions, and constant functions
Acquisition of skills that will include defining the Cartesian plane as the set of all ordered pairs of real numbers

Logical Thinking Strand

Formal and Informal Reasoning

Formal and Informal Reasoning: Readiness

Exploration of methods of sorting and matching objects, using appropriate vocabulary
Development of ability to make comparisons
Development of concepts of more, fewer, and same number as, using sets of objects

Formal and Informal Reasoning: Kindergarten–Grade Three

Exploration using manipulatives to express short chains of logical reasoning
Exploration of the vocabulary of logic in simple mathematical sentences
Development of visual discrimination and informal reasoning
Exploration of patterns for logical reasoning
Development of the concepts of between, before, after, and so forth
Experiences in logical reasoning in situations with one or two conditions
Development of the vocabulary of logic (and, or, not, if . . . , then)

Formal and Informal Reasoning: Grades Four–Six

Exploration using more formal reasoning applied to manipulative games and materials
Exploration of the logical meaning of all, some, and none
Development of conjunction, disjunction, negation, and conditional
Development of the mathematical meaning of or, and, not, if . . . , then, all, or some
Application of patterns of reasoning to more complex situations involving several true statements

Application of flowcharts to show steps in operations and solution of word problems

Formal and Informal Reasoning: Grades Seven–Eight

Exploration of deductive and inductive arguments with and without manipulatives
Exploration of number puzzles and games to extend concepts of logical thinking
Identification and construction of simple deductive arguments
Identification of the use of precise statements in logical reasoning procedures

Patterns in Mathematics

Patterns in Mathematics: Readiness

Exploration of pattern recognition, using concrete objects
Recognition of likenesses and differences
Recognition of objects on the right or left and first or last in a series

Patterns in Mathematics: Kindergarten–Grade Three

Exploration of patterns, using geometric figures and simple applications in nature
Recognition of patterns and practice in making logical choices
Identification and extension of patterns of numbers
Development of patterns for multiplication and division readiness

Patterns in Mathematics: Grades Four–Six

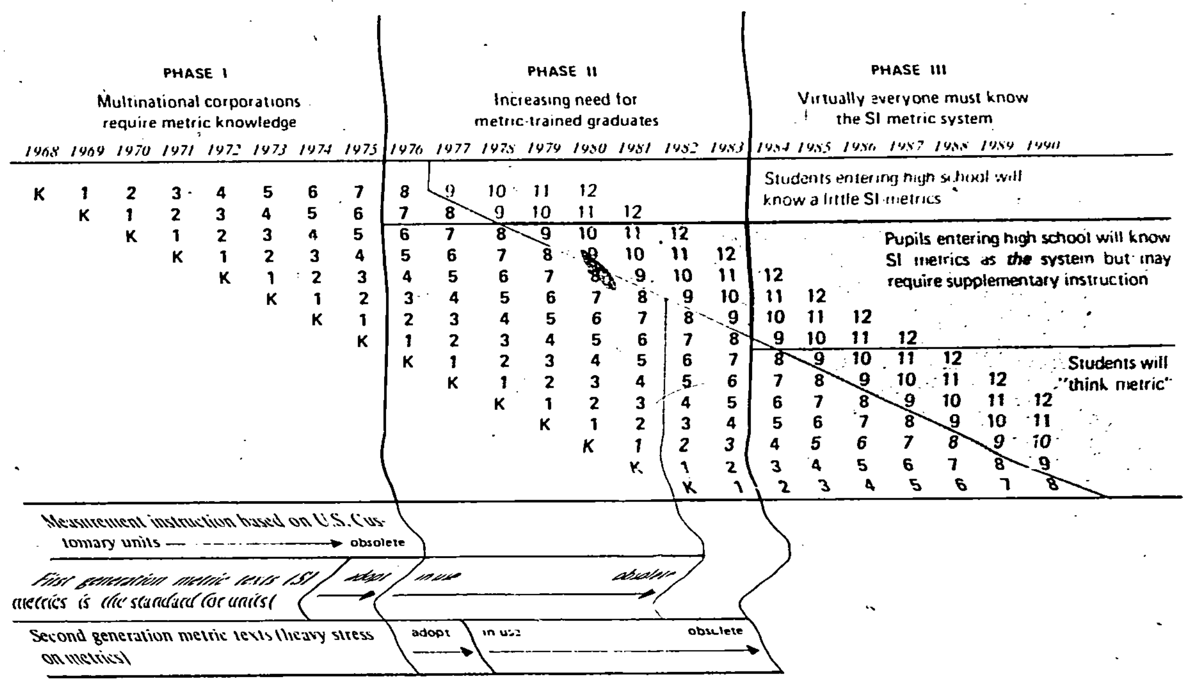
Recognition and use of patterns and sequences
Recognition and use of patterns in basic operations and algorithms and in number theoretic concepts such as factors, primes, powers, and so forth
Application of pattern recognition to puzzle situations
Utilization of patterns for the determination of rules for function machines

Patterns in Mathematics: Grades Seven–Eight

Exploration of complex patterns in nature, art, architecture, and so forth
Recognition of patterns, using different numeration systems antecedent to the decimal system
Application of number properties to extend the basic facts
Comparison of systems of numeration to find basic principles

Appendix B

Time Line for Implementation of Metrics



NOTE: The progress of a class in metrics education can be horizontally traced in the above illustration. For example, the class that will graduate in 1984 will receive metrics instruction in grades four through twelve. The first class to receive metrics instruction throughout its twelve years will be the class of 1988, which will enter kindergarten in 1976 when metrics texts are first introduced in California elementary classrooms.

Appendix C

Criteria for Evaluating Instructional Materials in Mathematics for Kindergarten Through Grade Eight

Approved by the State Board of Education, May 9, 1974

I. Content Coverage

A. *Arithmetic, Numbers, and Operations.* The materials shall provide for:

1. Use of the concept of one-to-one correspondence as a basic tool for developing the concepts of number, counting, and order
2. Presentation of the number line and number plane as an aid in development of the concept of number and operation with numbers, both positive and negative
3. Earlier and greater emphasis upon decimal notation and computation with decimals prior to formal computation with numbers in fraction form
4. Memorization and use, on a regular basis, of the basic arithmetic facts of addition and multiplication
5. Use of the basic addition facts in the development of the operation of subtraction
6. Use of the operations of subtraction and multiplication in developing the division algorithm
7. Learning and use of the equality and order relations
8. Use of properties of operations in the development of computational skills to the extent that those properties are required for understanding
9. Development of an understanding of elementary number theory concepts
10. Development of computational skills with positive and negative numbers
11. Selection of the operation(s) appropriate to solving given problems

12. Doing mental arithmetic
13. Development of the concept of place value in the decimal numeration system
14. Development of exponential notation
15. Development of scientific notation
16. Development of an understanding of real numbers which includes square roots and cube roots
17. Development of the representation of rational numbers as repeating decimals
18. Activities that develop the following:
    a. Understanding of decimal notation
    b. Computational skills with decimals
    c. Fraction concept and fraction notation
    d. Concepts of ratio, proportion, and percent
    e. Skill in rounding off numbers

B. *Geometry.* The materials shall provide for:
1. Intuitive, informal development of basic geometric concepts utilizing the environment as a source for models
2. Introduction of the concepts of similarity, congruence, and transformations
3. Opportunities to explore the concepts of parallelism, perpendicularity, and skewness
4. Classification of geometric shapes
5. Activities for using simple geometric instruments
6. Use and construction of two- and three-dimensional models
7. Intuitive development of reasoning procedures beginning with simple arguments
8. Activities for computing length, circumference, perimeter, area, volume, and angle measures of common geometric figures
9. Introduction of the use of the Pythagorean Formula
10. Informal development of elementary concepts of coordinate geometry

C. *Measurement.* The materials shall:
1. Provide "hands-on" experiences in measuring familiar objects.
2. Provide flexibility in the choice of a unit for measuring, with the introduction of arbitrary units preceding instruction in standard units.
3. Present standard units as a uniform way of reporting measurements.
4. Employ the metric system known as the International System of Units (SI) as the standard units of measurement.

5. Avoid computational conversions between the U.S. Customary Units System and the International System of Units (SI). The material may provide for informal comparisons of metric units with U.S. customary units.
6. Offer many opportunities for development of skills and practice in estimating common measurements.
7. Develop an understanding of the approximate nature of measurement.
8. Provide numerous activities for pupils to improve their skill in reading various measuring instruments.
9. Provide opportunities for pupils to make and interpret scale drawings and maps.
10. Provide opportunities for pupils to develop formulas for determining measurements such as perimeter, area, and volume, and provide exercises for using these formulas.

D. *Problem Solving/Applications.* Materials shall provide:

1. Activities that permit experimentation and investigation in open-ended situations
2. Problems drawn from everyday situations
3. Opportunities for students to compare and contrast, summarize, order events sequentially, develop an awareness of cause and effect relationships, and predict outcomes
4. Experiences in the organizing of information into tables, charts, and graphs
5. Situations which require problem formulation, mathematical model building, development of solution strategies, and solution interpretation
6. Development of a variety of solution strategies or tactics
7. Problems for which several alternative solution strategies exist
8. Explicit opportunities for students to use different solution strategies when solving problems
9. Open-ended and challenging problems to encourage conjecture, data recording, analysis and discerning of patterns, and making of generalizations

E. *Probability and Statistics.* Materials shall provide:

1. Activities for collecting, organizing, and representing data derived from real-life situations
2. Activities that develop the fundamental counting procedure through the use of tree diagrams
3. Activities that develop the concepts of permutations (arrangements) and combinations (selections)

4. Experiences in making guesses about patterns or trends that might appear among data
5. Activities which lead to making statistical inferences
6. Activities which lead to an understanding of the various measures of central tendency and dispersion
7. Systematic development of vocabulary pertinent to the topic of probability and statistics
8. Concrete activities that lead to the development of elementary concepts of probability
9. Opportunities for students to make predictions based on samples of data they have collected and for promoting the discussion of how reliable those predictions might be

F. *Relations and Functions.* Materials shall provide:

1. Activities that develop skills in constructing and interpreting tables, charts, graphs, and schedules
2. Introduction and use of mappings, correspondences, ordered pairs, and "rules" leading to an intuitive development of the concepts of function and relation
3. Gradual development of notations for the function concept
4. Use of the concept of a function to make experimental inferences, using situations drawn from areas that represent applications of mathematics
5. Experiences to encourage pupils to look for and discover patterns and relationships and to form generalizations

G. *Logical Thinking.* Materials shall provide:

1. Manipulative activities, games, and puzzles which stimulate and afford opportunities for developing elementary reasoning patterns
2. Activities involving trial and error that permit students to explore and discover logical patterns
3. Opportunities for children to discover and apply reasoning patterns to nonmathematical situations
4. Activities for exploring direct and indirect reasoning patterns

II. Manner of Presentation

A. *Pupil Needs.* The materials shall provide:

1. Experience in consumer decision making
2. Role models that expand the vocational and social horizons of both boys and girls
3. Enrichment experiences for children at all ability levels
4. Multisensory approach to learning

5. Historical development of some concepts and skills and historical references to important mathematical discoveries
6. Systematic approach toward developing reading skills in mathematics
7. Opportunities for the development of new and unfamiliar concepts, when appropriate, in a manner which proceeds from the concrete to the abstract
8. Activities at all levels which develop concepts and skills through the use of manipulative aids
9. Presentation of concepts and skills at levels which meet the needs of individual learners
10. Opportunity for a learner to progress at a rate and in a style commensurate with his learning abilities and interests
11. Opportunity for development and use of appropriate mathematical vocabulary
12. Some answers for students to facilitate self-appraisal
13. Experiences and settings representative of all socioeconomic levels, ethnic groups, urban and rural environments
14. Respect for the emotional, physical, and intellectual needs of children from all cultures, socioeconomic levels, family structures, and diverse backgrounds, including suburban, rural, migr[illegible], and inner-city experiences
15. Help for [illegible]dent in identifying values and value systems of our m[illegible]ed society
16. Conside[illegible] the relationship between people and their physi[illegible]al [illegible]onment, and promotion of a responsible attitude to[illegible]d that environment

B. *Content Organization.* The materials shall:

1. Pro[illegible] [illegible]tuations which develop students' investigative and exp[illegible]tory skills.
2. Dev[illegible]p the interrelated skills of communication.
3. Include problem solving based upon students' experiences in school, home, and community.
4. Provide recreational activities, including games which have appeal to all students, which are designed to satisfy a basic mathematical objective.
5. Include innovative approaches in the presentation of computational skills which are designed to stimulate interest and motivate learning.
6. Be designed so that the format clearly indicates the concepts and skills being developed.

7. Provide for the study of the concepts of sets only to the extent that sets are required for an adequate understanding of the content outlined in the framework.
8. Be designed so that there is a smooth transition from concrete learning experiences to abstract learning experiences.
9. Be designed to develop an appreciation for the beauty, history, and language of mathematics.

III. Teacher Materials

The teacher materials shall:

1. Provide an adequate interpretation of tables, charts, graphs, schedules, and maps.
2. Suggest strategies for teaching students to learn to read instructions.
3. Suggest strategies to use in organizing a class into groups for the purpose of instruction, tutoring, or evaluation.
4. Describe a variety of alternative activities for potential use in accomplishing the student objectives stated by the publisher.
5. Outline specific classroom activities that require pupils to use or develop skill in elementary addition, subtraction, multiplication, and division.
6. Provide a list of the student or program objectives for the instructional materials presented.
7. Suggest strategies and activities for developing mental arithmetic skills and skill in estimating solutions of problems.
8. Provide teachers with materials and guidance to facilitate evaluation of the classroom program as well as individual pupil progress. These evaluation recommendations shall describe the construction, use, and limitations of a variety of measurement tools, including standardized tests, publisher-made tests, teacher-made tests, item-sampling for group assessment, diagnostic tests, and observation/interview techniques.
9. Provide a description of the mathematical development of the content included in the pupil program, expand upon these concepts, and provide historical perspective whenever appropriate.
10. Include suggestions for the development of interest, motivation, and favorable attitudes with regard to the learning of mathematics.

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