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ABSTRACT

The objective of the study was to determine the effect which a unit dealing with recursive definitions would have on students' achievement in applying the principle of mathematical induction (PMI). Twenty-four secondary school students were randomly assigned to control and experimental groups. Students in the control group studied programmed units of the PMI program first and then programmed units on the recursive definition program. This procedure was reversed by the experimental group. Data were obtained from a posttest and a subsequent retention test. A 2 x 2 analysis of variance was used to analyze data. Changes in score were analyzed using "t" tests. The study indicates that it makes little difference whether a unit on recursive definitions precedes or follows a unit on the PMI. (Author/JBW)

REPORT ON A STUDY TO DETERMINE THE EFFECT OF KNOWLEDGE OF RECURSIVE DEFINITIONS UPON SUBSEQUENT APPLICATION OF THE PRINCIPLE OF MATHEMATICAL INDUCTION

by

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TABLE OF CONTENTS

INTRODUCTION

The Principle of Mathematical Induction (PMI) provides one of the most powerful methods of proof, not only for the research mathematician, but also for the student of more elementary mathematics.

Since a major objective of the secondary mathematics program for the college-capable is to provide these students with an adequate foundation for undergraduate mathematics, it follows that the method of proof by Mathematical Induction should be included in the program. The National Council of Teachers of Mathematics (NCTM), in its Program Provisions for the Mathematically Gifted Students (NCTM, 1957) lists the PMI as one of the essential topics of algebra for the college-capable. Both the School Mathematics Study Group (SMSG, 1960) and the University of Illinois Committee on School Mathematics (UICSM, 1960) included the topic in their materials.

It is clear that students have not mastered this topic in the secondary schools, or even as undergraduates. Calvin Long, in the introduction to his number theory text notes:

> The discussion of mathematical induction . . . is treated here in considerable detail since students at the sophomore-junior (college) level frequently have only the most rudimentary knowledge of these important and useful ideas. (Underlining added) (Long, 1965)

Beach (1963) indicated that many graduate students in the sciences do not understand proof by indirection. The opinion of Johnson (1960) is typical. He states that although mathematicians highly recommend the PMI, it is one of the most annoying and consequently avoided topics in algebra.

Various textbook authors warn the reader or the teacher of the difficulty encountered in teaching the PMI. SMSG (1960) cautions teachers not to attempt the topic with a below average class. A. A. Blank (1963) in the 28th NCTM yearbook states that the full impact of the PMI will not be realized until the student has reached greater mathematical maturity. In the teacher's edition of their text, Beberman and Vaughn (1966) caution the instructor not to expect thorough understanding of the techniques on the first (and possibly only pre-college) experience.

These statements are at odds with that of Poincaré (see Wilder, 1965) who felt that mathematical induction is forced on us by intuition. The Report of the Cambridge Conference (1963) indicates that at least an intuitive appreciation of the PMI is possible at the elementary school level.

Most of the literature dealing with the teaching of the PMI is opinion, not research. One finds that some individuals advocate a variety of approaches by the teacher. Shreve (1963) suggests using the well-ordering principle and proving theorems by either contradiction or contraposition. More examples, especially of "fallacious proofs" are proposed by Beach (1963). Koenen (1955) suggests that presentation and motivation are enhanced by using the puzzle called the "Tower of Hanoi". Smith and Henderson (1959) claim that students can produce proof by the PMI by following examples, but do not understand the logic of this type of proof.

Hoer (1922) states that the difficulty (for the student) is in making the connection between the statement of the proposition for n and n + 1. Polya (1957) emphasizes the importance of this point when he suggests that the PMI could be "called 'proof from n to n + 1' or still simpler 'passage to the next integer'."

Research dealing with the teaching of the PMI is limited. Several investigators (e.g. Wells, 1967; Nelson, 1962; Hildebrand, 1968) have utilized the PMI as the learning task in research related to modes of instruction. Their conclusions rightfully were drawn with respect to the modes of instruction and not to the teaching of the PMI.

Three recent studies have dealt with the teaching of the PMI.

Ward (1971) conducted an experiment to answer the following questions:

(1) Can an indirect inductive proof technique based on the Well Ordering Principle be effectively substituted for the method of mathematical induction based on the Principle of Mathematical Induction?

(2) If the two proof techniques are taught successively, what is their proper pedagogical placement?

(3) What constitute the primary sources of difficulty with inductive proofs? In particular, is lack of confidence in the Principle of Mathematical Induction one such source?

Two intact classes were treated as a single population and divided into two groups of comparable ability. One group assumed the Well Ordering Principle as an axiom; learned the indirect inductive technique, used this method to prove the Principle of Mathematical Induction, and, finally, studied the method of mathematical induction. The remaining group reversed the learning order, beginning with the Principle of Mathematical Induction as an axiom and concluding with the indirect inductive method of proof. Three criterion tests, each requiring proofs of four conjectures, were administered to each group at intervals of varying lengths throughout a ten-day study.

In addition to proving conjectures on the criterion tests, students were asked to fill out confidence scales following each of their proofs and to complete an information retrieval test designed to identify weakness in the learning sets requisite to understanding the method of proof.

The analysis of data revealed: (1) students performed significantly better using the method of mathematical induction rather than the indirect inductive technique, (2) those students who studied the Well Ordering Principle followed by the Principle of Mathematical Induction performed significantly better than those who reversed the order, and (3) the essential difficulty with inductive proofs was inability to perform the necessary algebraic manipulation in the inductive step, together with ignorance of the proper procedure for establishing a conditional. Lack of confidence in an axiom or method of proof was found not to be a source of difficulty with proving conjectures.

Walter (1972) hypothesized that students having the prerequisite knowledge of logic, and taught the PMI in terms of logic, would perform better on his criterion test over the PMI than students who were not taught the PMI in terms of logic.

The experiment was conducted twice, using pre-calculus college students first and college calculus students the second time. The results both times favored the research hypothesis. However, only the proof portion of the posttest was found to be significant at the .05 level and that only for the pre-calculus subjects.

A third investigator (Reeves, 1972) looked at two pedagogical aspects of the teaching of the PMI, viz; the introduction to the topic, and the method of presenting problems designed to induce facility with the proof technique.

The introductions compared were the "inductive sets" approach and a "semi-concrete" introduction which required students to consider hereditary situations of everyday life. Although both introductions were demonstrated to be effective, there was no difference in effectiveness between the two.

"Traditional" and "guided discovery" approach to the problem later verified by the PMI were compared. Neither was found to be more effective than the other.

Reeves (1972) found no significant correlation between understanding of the proof technique and ability to use the technique in proving theorems.

THE STUDY

Polya (1957), Hoer (1922), and Ward (1972) identified a difficulty as the understanding and manipulation of the inductive step. Since the recursive definition of a function is defined for any integer in terms of its successor, the relationship between recursive definitions and the PMI is obvious. (Youse (1964) ~~even~~ denotes recursive definitions as "inductive definitions"). Utilization of the relationship between these ideas was the focus of the study.

Specifically, the research hypothesis was that students who study recursive definitions prior to exposure to the PMI will have a better understanding of the PMI than those who reverse the procedure. A second hypothesis was that they (the experimental group) would retain this understanding longer.

Within the contexts of their classes, and without any experimental design, two local "advanced math" teachers attempted to evaluate the hypothesis. Both were of the opinion that the first hypothesis was correct, i.e. "These classes learned the PMI better than previous classes which studied recursive definitions after the PMI". Encouraged by these results, arrangements were made to conduct a formal experiment.

The objective of the study was to determine the effect a unit dealing with recursive definitions would have upon students' achievement in application of the PMI.

The subjects in the study were 24 advanced mathematics students at The Ravenscroft School, Raleigh, N. C. The students were randomly assigned to either experimental or control groups.

Programmed units on the PMI and recursive definitions were prepared. Students in the control group studied the PMI program first and then the recursive definition program. The experimental group reversed this procedure. Three classes (of 50 minutes each) were allotted for each programmed unit. All of the participants easily finished the units in the allotted time. On the seventh class day, the posttest was administered. After three weeks, an equivalent form, the retention test, was administered. Appropriate statistical procedures were utilized to evaluate the following null hypothesis:

HO-1 There is no significant difference between the scores of the control and experimental groups on the posttest.

HO-2 There is no significant difference between the scores of the control and experimental group on the retention test.

HO-3 There is no significant difference between control and experimental groups in the change of scores between the posttest and the retention test.

Much of the material in the two tests was adapted from that of Walter (1972). A score of 80 was possible.

RESULTS AND ANALYSIS

A 2 X 2 analysis of variance was utilized to analyze the data. Analysis of the raw data revealed:

1) Posttest mean of the control group was 65.6 with a range of 56 to 78.
2) Posttest mean of the experimental group was 63.8 with a range of 51 to 78.
3) Retention test mean of the control group was 64.50 with a range of 54 to 77 and a DECREASE in mean score of 1.08.
4) Retention test mean of the experimental group was 64.4 with a range of 53 to 75 and a INCREASE in mean score of 0.60.
5) Posttest - Retention test coefficient of correlation was 0.67.

Table I gives the results of the analysis of the Posttest.

ANALYSIS OF VARIANCE SCORES ON THE POSTTEST

SOURCE	D.F.	S.S.	M.S.	F.
SECTION	1	63.375	63.375	1.119
METHOD	1	18.375	18.375	0.324
INTERACTION	1	30.375	30.375	0.536
RESIDUAL	20	1132.833	56.642	---
CORRECTED TOTAL	23	1244.958	54.128	---

From the data above, it is concluded that:

1) There is no significant difference (at the 0.05 level) in the mean scores due to section.

2) There is no significant difference (at the 0.05 level) in the mean scores due to method.

3) There is no significant interaction between method and section.

Table II summarizes the analysis of the results on the retention test.

ANALYSIS OF VARIANCE OF SCORES ON THE RETENTION TEST

SOURCE	D.F.	S.S.	M.S.	F.
SECTION	1	155.04	155.04	3.049
METHOD	1	0.04	0.04	.001
INTERACTION	1	92.04	92.04	1.810
RESIDUAL	20	1016.83	50.84	---
CORRECTED TOTAL	23	1263.96	54.95	---

The results of the analysis indicate that there is no significant difference on the retention test attributable to either section or method. The analysis also indicates a lack of interaction.

The question of retention was investigated by looking at the changes in score between the two administrations of the test.

"t" tests were run to see if the change scores differed significantly from zero. Results of these analysis are below.

Summary of Change Scores

Group	Mean	Standard Deviation
Control	$\overline{x}_c = -1.08$	$s_c = 2.51$
Experimental	$\overline{x}_e = +0.58$	$s_e = 1.57$

Summary of t-Tests

Hypothesis	t-value	d.f.
$\bar{x}_c = 0$	-1.49	11
$\bar{x}_e = 0$	1.57	11
$\bar{x}_e = \bar{x}_c$	0.893	11

The t values indicate that the following null hypothesis could not be rejected.

Ho1 The change in scores of the control group was not significantly different from zero.

Ho2 The change in scores of the experimental group was not significantly different from zero.

Ho3 The change in scores of the experimental group was not significantly different from the change in scores of the control group.

These results were verified by a 2 X 2 repeated measures analysis of variance performed by the North Carolina State University Computing Center.

IMPLICATIONS AND RECOMMENDATIONS

The study indicates that it makes little difference whether a unit on recursive definitions precedes or follows a unit on the PMI.

The scores of the criterion tests indicate that the two units together form an appropriate introduction to the PMI.

It is recommended that the study be modified and replicated. The suggested modification is that a test on recursive definitions be developed and given after completion of that unit. The suggested design is given below.

R	P	T_1	D	T_2	T_3
R	D	T_2	P	T_1	T_3

R = random assignment

P = PMI unit   D = Recursive definition unit

T_1 = PMI test T_2 = R. Def. test T_3 = Retention test covering both PMI and RD

Figure I

Since the students in the study were high ability subjects at a private school, and relatively few in number, it is suggested that the study be replicated in public schools with a larger and more heterogeneous population.

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APPENDIX A

RECURSIVE DEFINITION UNIT

INSTRUCTIONS

This lesson has the following pattern. Some commentary (explanation) followed by a frame of questions. The frames of questions are very important, and you should try to answer each question before you look at the answer. The answers appear immediately below the frame of questions. The index card supplied with this instructional material is to be used to cover the answers until you are ready to check them. If you answer a question incorrectly, be sure that you understand why you have made the error before you continue. In some instances this might require rereading some of the material you have already covered. CORRECT your answers before continuing!

With each frame of questions there is associated a number, and if a frame contains more than one question, the questions will be designated by the frame number followed by a letter of the alphabet. For example:

61:

61a. The product of -3 and -2 is ____________.

61b. If 3a = o, then a is equal to __________.

ANSWERS: 61a. 6 61b. o

When you have completed this unit, a test will be administered to determine whether or not you have acquired the knowledge this unit intends to impart. Proceed at your own rate. There are no rewards for finishing first, however, you should keep moving along without unnecessary waste of time.

RECURSIVE DEFINITIONS

Mathematicians sometimes use three dots to indicate missing parts of an expression. For example, look at the definition of x^n.

$$x^n = x \cdot x \cdot x \cdot \ldots \cdot x \text{ where there are n x's.}$$

Usually, this is fairly clear, but a more exact definition is found by using a recursive formula like this:

$$x^1 = x \text{ and for all } n>1, x^n = x \cdot x^{n-1}.$$

Using this definition, let's find x^5.

$$\begin{aligned} x^5 &= x \cdot x^4 \\ &= x \cdot (x \cdot x^{4-1}) = x \cdot x \cdot x^3 \\ &= (x \cdot x)\ (x \cdot x^{3-1}) = x \cdot x \cdot x \cdot x^2 \\ &= (x \cdot x \cdot x)(x \cdot x^{2-1}) = x \cdot x \cdot x \cdot x \cdot x \end{aligned}$$

which is what you expected.

Now let's look at n!.

Recursively, we define n! as follows

$$1! = 1 \text{ and for } n>1,\ n! = n \cdot (n-1)!$$

Now apply the definition to find 5!

1a $5! = 5 \cdot (\quad)!$

1b $= 5 \cdot (\quad) \cdot (\quad)!$

1c $= 5 \cdot (\quad) \cdot (\quad) \cdot (\quad)!$

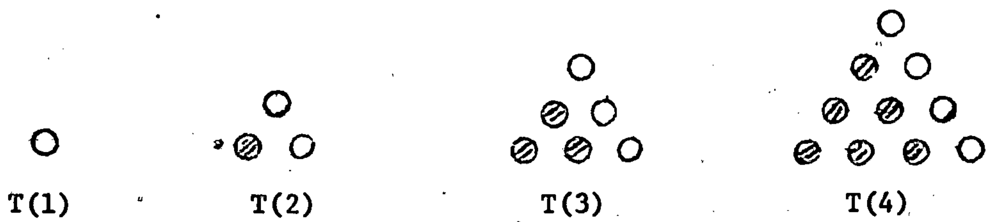
1d =

1e =

Answers: 1a. 4!. 1b. $4 \cdot (3)!$. 1c. $4 \cdot 3 \cdot (z)!$ 1d. $5 \cdot 4 \cdot 3 \cdot 2 \cdot (1)!$
1e. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =]20$.

The interesting thing about recursive definitions is that after an initial definition is made, all other values depend upon the value of the preceeding value.

The Ancient Greeks discovered some interesting properties about the positive integers by looking at "figurate numbers"--called that because they can be geometrically arranged. Consider the following arrangements, and particularly note the shaded dots.



These are called the first four trianglelar numbers.

T(1)=1 total number of dots in the triangle.

T(2)=3=T(1) + 2.

2a. T(3)=6=T(2) + ____

2b. T(4)=10=___ + ____

Answers: 2a. T(2) + 3 2b. T(3) + 4

Using the pattern above as a guide, what is T(5)?

3. T(5)=

Answer: 3. T(5)=T(4)+5=10+5=15.

4a. If we know T(n) for some n, how can we define T(n+1)?

T(n+1)=

4b. If T(20)=210, what is T(21)?

Answers: 4a. T(n+1)=T(n)+n+1 4b. T(21)=T(20) + 21=231.

---

5. What is a recursive definition of T(n)?

Answer: 5. T(1)=1 and for n>1, T(n)=T(n-1)+n.

At this point, we recognize that T(n) is the sum of the integers from 1 to n. We can express this as indicated:

$$T(n)=\sum_{i=1}^{n} i$$

Let's try to get an idea of how to calculate T(n) for any n. Complete the following chart.

n	n+1	T(n)	2·T(n)
1	2	1	2
2	3	3	(6a)__
3	4	6	(6b)__
4	5	(6c)__	(6d)__
5	6	(6e)__	(6f)__

Answers: 6a. 6 6b. 12 6c. 10 6d. 20 6e. 5 6f. 30

We see that each entry in the last column is the product of the two entries in the first two columns in the corresponding row. For example:

$$2\cdot T(3)=3\cdot 4=12 \text{ so } T(3)=\frac{3\cdot 4}{2}$$

If the pattern holds, the row for integer k looks like:

7a. k k+1 T(k) 2·T(k)=______

7b. and T(k)=______

Answers: 7a. k·(k+1). 7b. $\frac{k(k+1)}{2}$

Later, we shall prove that

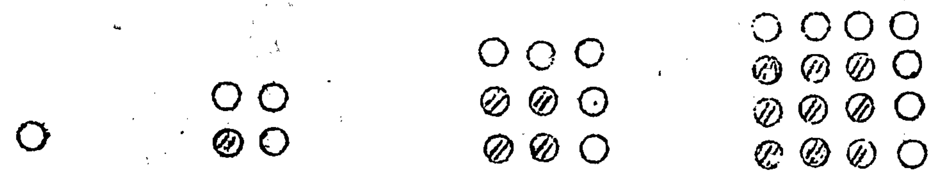
$$\sum_{i=1}^{n} i = \frac{(n)\ (n+1)}{2}$$

If the sum of the integers from 1 to k is $\frac{(k)\ (k+1)}{2}$,

8a. what is the sum of the integers from 1 to (k+1)?________

Answer: 8a. put (k+1) in the place of k. We get $\frac{(k+1)\cdot[(k+1)+1]}{2}$ or $\frac{(k+1)\ (k+2)}{2}$

In a similar fashion, the Greeks answered the question "What is the sum of the first n odd numbers?" by looking at the geometric figures on the next page.



S(1) S(2) S(3) S(4)

Complete the statement by looking at the figures.

S(1) indicates the number of dots in the first figure, S(2), the second, etc.

9. $S(1)=1$ $S(2)=4=1+3=S(1)+2(2)-1$

9a. $S(3)=9=4+5+S(2)+2(\ \ )-1$

9b. $S(4)=16=9+\_\_=\_\_+\_\_$.

ANSWERS: 9a. $S(2)+2(3)-1$ 9b. $9+7=S(3)+2(4-1)$

If $S(20) = 400$, how do we find $S(21)$?

10. $S(21)= S(20) + \_\_\_\_$

$= \_\_\_\_ + \_\_\_\_$

$= \_\_\_\_\_$

ANSWER: 10. $S(21)=S(20)+2(21)-1 = 400+42-1 = 44-1$

---

11. How do we define $S(n)$ in terms of $S(n-1)$?

ANSWER: 11. $S(n) = S(n-1) + 2n-1$

---

12. By looking at the chart, can you guess a short way of calculating $S(n)$?

ANSWER: 12. $S(n) = n^2$

Suppose that $R(n)$ is defined recursively as follows: $R(1) = 3$ and for all $n > 1$, $R(n) = R(n-1) + 3$.

Write the first four values of $R(n)$.

13a. $R(1) =$

13b. $R(2) =$

13c. $R(3) =$

13d. $R(4) =$

Answers: 13a. 3 13b. 6 13c. 9 13d. 12

14 if $R(13)=39$ what is $R(14)=$ ______?

Answer: 14. $R(14)= R(13) + 3 = 39 + 3 = 42$.

---

15 if $R(n) = 3n$ what is $R(n+1) =$ ______?

Answer: 15. $R(n+1)= R(n) + 3 = 3n + 3 = 3(n+1)$.

In frame 15, we see that $R(n) = 3n$ is an <u>explicit</u> formulation. By this we mean a formula which tells us how to find the value of $R(n)$ for any value of <u>n</u>. Look at the explicit formulas below and write the expression for n+1 asked for.

16 $G(n) = n^2 + n$ $\quad G(n+1)=(n+1)^2 + (n+1)=n^2+ 2n + 1 +n+ 1 = n^2 + 3n +2$

16a $H(n) = \dfrac{n^2-n}{2}$ $\quad H(n+1)=$

16b $K(n) = \dfrac{n^2 + n + 3}{n}$ $\quad K(n+1)=$

16c $L(n) = \dfrac{(n)\ (n+1)\ (n+2)}{6}$ $\quad L(n+1)=$

Answers: 16a. $H(n+1)=\dfrac{(n+1)^2 - (n+1)}{n+1} = \dfrac{n^2 + 2n +1 - (n + 1)}{n+1} = \dfrac{n^2 + n}{n+1}$

16b. $K(n+1)=\dfrac{(n+1)^2 - (n+1)+3}{2}= \dfrac{n^2 + 2n + 1 + n + 1 + 3}{2} = \dfrac{n^2 + 3n + 5}{2}$

16c. $L(n+1)= \dfrac{(n+1)\ (n+2)\ (n+3)}{6}$

Let's look at some recursive definitions and guess at (mathematicians say "make a conjecture about") the explicit formula determined by this definition. First

$Q(1) =1$ and for n I, $Q(n) = Q(n-1) + [8\cdot(n-1)]$

$Q(1) =1$

17a $Q(2) = 1 + [8(2-1)] =$ __________

17b $Q(3) =$

17c $Q(4) =$

17d $Q(5) =$

17e It appears that $Q(n) =$ __________.

ANSWERS: 17a. $Q(2) = 9$ 17b. $Q(3) = 25$ 17c. 49 17d. 81

17e. $Q(n)$ = Square of n^{th} odd integer or $Q(n) = (2n - 1)^2$.

---

18. If $Q(n) = (2n - 1)^2$ find $Q(n + 1)$

ANSWER: 18. $Q(n+1) = [2(n+1)-1]^2 = [2n+2-1]^2 = [2n+1]^2$

$V(1) = 7$ for $n > 1$, $V(n) = V(n-1) + 4$

19a. $V(2) = V(1) + 4 = 7 + 4 =$ __________

19b. $V(3) = V(2) +$ ________ $=$ __________

19c. $V(4) =$ ________ $=$ __________

19d. $V(5) =$ ________ $=$ __________

ANSWERS: 19a. 11 19b. $V(2) + 4 = 15$ 19c. $V(3) + 4 = 19$

19d. $V(4) + 4 - 23$

---

20. Explicitly, $V(n) =$ __________ $+ 3$.

ANSWER: 20. $4n + 3$

---

21. If $V(n) = 4n + 3$, find $V(n + 1)$

ANSWER: 21. $V(n + 1) = 4(n + 1) + 3 = 4n + 7$

---

22. $f(1) = \frac{1}{2}$; for $n > 1$, $f(n) = [f(n-1)] \frac{n}{n+1}$.

22a. $f(2) = \frac{1}{2}$ ________ $=$

22b. $f(3) =$ ______ 22c. $f(4) =$ ______ 22d. $f(n) =$ ______

ANSWERS: 22a. $f(2) = \frac{1}{3}$ 22b. $f(3) = \frac{1}{4}$ 22c. $f(4) = \frac{1}{5}$

22d. $f(n) = \frac{1}{n+1}$

$P(1) = 2$, $P(n) = 2P(n-1)$

23a. $P(2) = 2P(1) =$ ______ 22b. $P(3) =$ ______ 22c. $P(4) =$ ______

23d. $P(n) =$ ______

ANSWERS: 23a. $P(2) = 4$ 23b. $P(3) = 8$ 23c. $P(4) = 16$

23d. $P(n) = 2^n$

Our last example of a recursive definition is called the Fibonacci numbers.

$F(1) = 1$ $F(2) = 1$ and for all $n > 2$, $F(n) = F(n-1) + F(n-2)$

$F(3) = F(2) + F(1) = 2$

24a. $F(4) = F(3) + F(2) =$ ______

24b. $F(5) =$ ______

24c. $F(6) =$ ______

24d. $F(7) =$ ______

24e. $F(8) =$ ______

24f. $F(9) =$ ______

ANSWERS: 24a. $F(4) = 3$ 24b. 5 24c. 8 24d. 13 24e. 21

24f. 34

---

25. If $F(n) = F(n-1) + F(n-2)$ what is $F(n+1)$?

ANSWER: 25. $F(n+1) = F(n) + F(n-1)$

---

26. In summary, how do we evaluate an expression explicitly defined in terms of <u>n</u> for the next integer namely ($n+1$).

ANSWER: 26. Substitute ($n+1$) for n in each place n occurs.

APPENDIX B

MATHEMATICAL INDUCTION UNIT

We have already learned that special sets of numbers can easily be described by their characteristics. For example what are the numbers described by the following sets?

1. $N = \{x|x \text{ is an integer and } x > 0\}$.

ANSWER: 1. The positive integers or natural numbers.

---

2. $\{x|x = 2k \text{ where } k \in N\}$.

ANSWER: 2. Positive even Integers

---

3. $\{x|x = 2k + 1 \text{ where } k \in N\}$.

ANSWER: 3. Positive odd integers

---

4. $\{x|x \in N \text{ and if } p \text{ is a factor of } x \text{ then } p = 1 \text{ or } p = x\}$.

ANSWER: 4. Primes

---

5. Now, using set notation describe the set of positive powers of two

$\{x|x \qquad\qquad\qquad \}$

ANSWER: 5. $\{x|x = 2^n \text{ where } n \in N\}$.

---

6. Now describe the positive multiples of nine

$\{x|x \qquad\qquad\qquad \}$

ANSWER: 6. $\{x \mid x = 9k \text{ where } k \in N\}$

Suppose that S is a subset of N and has the following properties.

$1 \in S$, and

whenever a number $k \in S$ then I know

that $(k + 1)$ also belongs to S

Symbolically $S \subseteq I^+$, $1 \in S$, and $k \in S \Rightarrow (k + 1) \in S$.

7. Can you show that 2 is an element of S?

ANSWER: 7. $1 \in S$ so $(1 + 1) \in S$ that is $2 \in S$

---

8. Does 3 belong to S? Why

ANSWER: 8. $2 \in S$ therefore $(2 + 1) \in S$ that is $3 \in S$.

---

9. Does 817 belong to S?

ANSWER: 9. Yes, but using the above method, it would take a long time to show it.

---

10. Is there any positive integer which does not belong to S?

ANSWER: 10. No, although showing that a large integer belongs to S would take an unusually large amount of time.

Since any positive integer belongs to S, $N \subseteq S$ and we know $S \subseteq N$, therefore we can say that S=N or that a set with the given properties is the set of positive integers.

This is the idea behind a very important mathematical tool, THE PRINCIPLE OF MATHEMATICAL INDUCTION (or The PMI). We summarize the PMI as follows:

Any set S which

1. is a subset of N,
2. contains 1, and
3. contains x + 1 whenever it contains x

is the set of all natural numbers (i.e. S = N).

Let us look at an example of proof using the PMI. We want to prove the statement that for all positive integers n, $(n^2 + n)$ is divisible by 2. Now we could start checking all integers, but we would never finish. So we use the PMI. If we let $G = \{x \in N | x^2 + X$ is divisible by two$\}$, we want to know if G = N

First, $G \subseteq N$ by definition,

Second, we check to see if $1 \in G$.

since $1^2 + 1 = 2$ and 2 is divisible by 2, then $1 \in G$.

Now we have established the $G \neq \phi$. Hence suppose $k \in G$. This means that $(k^2 + k)$ is divisible by 2. We need to check that $(k + 1)^2 + (k + 1)$ is also in G, that is, that $(k + 1)^2 + (k + 1)$ is divisible by 2.

Now lets look at $(k + 1)^2 + (k + 1) = k^2 + 2k^2 + 1 + k + 1$

$$= (k^2 + k) + (2k + 2)$$

$$= (k^2 + k) + 2(k + 1)$$

11. Is $k^2 + k$ divisible by two?

ANSWER: 11. Yes, since $k \in G$.

---

12. Is $2(k + 1)$ divisible by two? Why?

ANSWER: 12. Yes, because 2 is a factor.

We can now see that $(k^2 + k) + 2(k + 1)$ is divisible by two since this expression is the same as $(k + 1)^2 + (k + 1)$ we see that $(k + 1) \in G$.

To summarize, $G \subseteq N$.

$1 \in G$ and

$k \in G \Rightarrow k + 1 \in G$ so $G = N$.

Therefore, we conclude that for all positive integers n, $n^2 + n$ is divisible by two. Since $n^2 + n = n(n + 1)$ we can verbalize this fact as follows: The product of two consecutive positive integers is an even positive integer.

Before we proceed further, we will adopt some useful notation.

Notation: P(n) (read, "P of n") will represent some statement about positive integers, such as, "$n^2 + n$ is divisible by two." Then to say P(1) is true, is to say "$1^2 + 1$ is divisible by two" is a true statement. Similarly P(k) will mean, "$k^2 + k$ is divisible by two."

13. Let P(n) be: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2\ (n + 1)^2}{4}$

13a. P(1): ______________________

13b. P(2): ______________________

13c. P(5): ______________________

13d. P(k): ______________________

13e. P(k + 1): ______________________

ANSWERS: 13a. $1^3 = \frac{1^2\ (1 + 1)^2}{4}$ 13b. $1^3 + 2^3 = \frac{2^2\ (2 + 1)^2}{4}$

13c. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \frac{5^2\ (5 + 1)^2}{4}$

13d. $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2\ (k + 1)^2}{4}$

13e. $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k + 1)^3 = \frac{(k + 1)^2\ (k + 2)^2}{4}$

We now restate the PMI using our new notation.

The PMI:

Let P(n) be a statement about positive integers.

If
(i) P(1) is true, and
(ii) for every positive integer k, if P(k) is true, then P(k + 1) is true,

then P(n) is true for all positive integers.

In order to see that this formulation of the PMI is a consequence of our first one (p. 7) we let

$G = \{n \in N \mid P(n) \text{ is true}\}$

and we suppose that both (i) and (ii) in the above formulation hold.

14:

14a. (i) tells us that G contains ______________.

14b. (ii) tells us that if $k \in G$ then ______________________________.

14c. Therefore, by our first formulation of the PMI we can conclude that G ____.

ANSWERS: 14a. 1 14b. $(k + 1) \in G$ 14c. $G = N$, or that $P(n)$ is true for all positive integers.

We will look at another proof by use of the PMI, first however, let us review the procedure in light of our new notation.

Suppose we are given some statement $P(n)$ about positive integers. To prove that $P(n)$ is true for all positive integers we must show that $P(n)$ satisfies conditions (i) and (ii) of the PMI.

15:

15a. To show that condition (i) is satisfied we must show that _______ is true.

15b. To show that condition (ii) is satisfied we must assume that _______ is true, and then show that _____________ is true.

ANSWERS: 15a. $P(1)$ 15b. $P(k)$, $P(k + 1)$.

16: Consider the following particular statements

$P(1)$ $1^2 = 1$

$P(2)$ $2^2 = 1 + 3$

$P(3)$ $3^2 = 1 + 3 + 5$

16a. P(4) 4^2 = ________________

16b. P(5) 5^2 = ________________

16c. P(10) 10^2 = ________________

ANSWERS: 16a. 1 + 3 + 5 + 7 16b. 1 + 3 + 5 + 7 + 9

16c. 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19

We note that each of these particular statements is true. From these particular statements one might obtain (inductively) some general statement. Then we should ask, "How can we be certain that our general statement is true?"

We will inspect some of these particular statements more carefully.

17: P(2) says, "The sum of the first two odd positive integers is equal to 2^2."

P(4) says, "The sum of the first four odd positive integers is equal to 4^2."

17a. P(5) says, ________________________________

17b. P(10) says, ________________________________

17c. What general statement can be obtained from the particular statements given in this example? "The sum of ________________

____

ANSWERS: 17a. "The sum of the first five odd positive integers is equal to 5^2." 17b. "The sum of the first ten odd positive integers is equal to 10^2." 17c. The first k odd positive integers is equal to k^2."

Recall that any odd positive integer can be written as an even positive integer, decreased by one. That is, for each positive integer n, 2n is even, and 2n - 1 is odd.

18: Observe that the third odd positive integer is $5 = 2 \cdot 3 - 1$.

18a. The fourth odd positive integer is $5 + \_\_\_\_\_ = 7 = 2 \cdot 4 - 1$.

18b. The fifth odd positive integer is $7 + \_\_\_\_\_ = \_\_\_\_\_ = 2 \cdot \_\_\_\_\_ - 1$.

The k^{th} odd positive integer is $2k - 1$.

18c. The $(k + 1)^{th}$ odd positive integer is $2k - 1 + \_\_\_\_\_ = 2(k + 1) - 1$.

$$= 2k + 1$$

ANSWERS: 18a. 2 18b. 2, 9, 5 18c. 2.

Therefore in mathematical symbols, our general statement becomes:

$$P(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

We arrived at this general statement inductively, hence, we do not know that it is true for all positive integers n. We do, however know that P(n) is true for n = 1, 2, 3, 4, 5 and 10. In order to prove that P(n) is true for all positive integers, we will appeal to the PMI. Review the procedure for applying the PMI in the paragraph preceding and including Frame 11.

We have our statement P(n) about positive integers which is

$$P(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

We must show that P(n) satisfies conditions (i) and (ii) of the PMI. We know that P(1), P(2), P(3), P(4), P(5), and P(10) are true so condition (i) is certainly satisfied. To show that condition (ii) is also satisfied we assume that P(k) is true and we need to show that this implies that P(k + 1) is true.

19: P(2) true, means that $1 + 3 = 2^2$

19a. P(3) true, means that $1 + 3 + 5 =$ __________

19b. P(4) true, means that ____________________

19c. P(k) true, means that $1 + 3 + 5 + \cdots +$ __________ = ____________

19d. P(k + 1) true, means that ____________________________________

ANSWERS: 19a. 3^2 19b. $1 + 3 + 5 + 7 = 4^2$ 19c. $(2k - 1)$, k^2

19d. $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$

Now let us suppose that P(k) is true.

20: P(k): $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

Now add $(2k + 1)$, the $(k + 1)^{th}$ odd positive integer to both members of P(k). (To go from $k = 3$ to $k = 4$, we added the fourth odd number, 7, to both members.)

20a. $1 + 3 + 5 + \cdots + (2k - 1) +$ ________ $= k^2 +$ ________________

20b. The left member of 20a is the sum of the first ______________ (how many) odd positive integers.

20c. The right member of 20a is $k^2 + (2k + 1) = k^2 + 2k + 1$ (Factor)

= ____________________

20d. Hence, 20a becomes

$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$ which is P(_____)

20e. The expression in 20d says that the sum of the first _____(how many) odd positive integers is equal to ____________________

ANSWERS: 20a. $(2k + 1)$, $(2k + 1)$ 20b. $k + 1$ 20c. $(k + 1)^2$

20d. $(k + 1)$ 20e. $k + 1$, $(k + 1)^2$.

Hence, we have deduced the truth of $P(k + 1)$ after assuming that $P(k)$ was true. So condition (ii) of the PMI has been satisfied. Therefore, by the PMI, $P(n)$ is true for all positive integers n.

We have proved the following theorem: For every positive integer n, the sum of the first n odd positive integers is equal to n^2. We should note that the source of this theorem was induction, the assertion was found experimentally, and the proof by the PMI was an example of deduction.

Let us consider a different type of problem using the PMI. Recall that

$1! = 1$ and $2! = 2 \times 1$ and $n! = n \times (n - 1) \times \cdots \times 2 \times 1$

a. What is $(3!)$?

b. What is 2^3?

c. Then $3!$ is $>$, $<$, or $= 2^3$ (circle correct answer)

(a. 6 b. 8 c. $<$)

What is the relationship of $4!$ and 2^4?

$24 = 4! > 2^4 = 16$

Check $5!$ and 2^5 and we see $5! \boxed{}\ 2^5$

$120 = 5! > 2^5 = 32$

It appears that if $n > 3$ then $n! > 2^n$

To prove it, let $P(n)$ be the statement

$(n + 3)! > 2^{(n + 3)}$

by our work above, $P(1)$ becomes $4! > 2^4$ which is true.

21. What is $P(k)$?

ANSWER: 21. $(k + 3)! > 2^{k + 3}$

---

22. We need to show that if P(k) is true then P(k + 1) is true.

What is P(k + 1)?

ANSWER: 22. $[(k + 1) + 3]! > 2^{[(k + 1)+3]}$ i.e., $(k + 4)! > 2^{k + 4}$

---

* $(k + 3)! > 2^{k + 3}$ since P(k) is true

** $k + 4 > 2$ since k is an integer

recall that if $a > b$ and $c > d > o$ then $ac > bd$.

Applying this fact to equation * and ** we get

$(k + 4)(k + 3)! > 2(2^{k + 3})$

that is

$(k + 4)! > 2^{(k + 4)}$ but this is P(k + 1)

therefore P(1) is true. P(k) true implies P(k + 1) true so P(n) is true for all n.

Lets look at the following geometric problem. If we have a segment of length 1 given to us, can we, using a compass and a ruler, construct a segment of length $\sqrt{n}$ for each integer n? We shall attempt to answer this by using the PMI.

23. What does P(n) become?

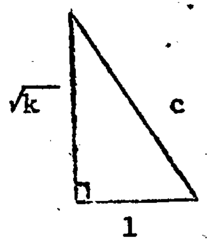
ANSWER: 23. P(n) is the statement, a segment of length $\sqrt{n}$ can be constructed using a compass and a ruler.

Is P(1) possible? Yes, since we are given a segment of length 1 and $\sqrt{1} = 1$.

24. Suppose P(k) is true, then what do we know?

ANSWER: 24. A segment of length $\sqrt{k}$ can be constructed.

Now consider the following. We can construct a right triangle with a ruler and a compass. Mark off the legs with lengths 1 and $\sqrt{k}$ as shown. (Remember that $\sqrt{k}$ is possible since that is what P(k) tells us.)



Then, by the Pythagorean Theorem,

$$c^2 = (\sqrt{k})^2 + 1^2$$

$$c^2 = k + 1$$

$$\text{so, } c = \sqrt{k+1}$$

Thus a segment of length $\sqrt{k+1}$ can be constructed so P(k + 1) is true when P(k) is true.

So, P(1) and P(k + 1) is true whenever P(k) is true tells us by applying the PMI, that P(n) is true for all $n \in N$.

We will conclude this lesson with one additional application of the PMI.

Let P(n) be the statement

$$P(n): \quad 2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

We wish to show that P(n) is true for every positive integer n. Recalling that the PMI states that, if

25:

25a. ____________ is true, and

25b. for every positive integer k, if ____________________

then ____________________

25c. then ______________________________

We should agree that the PMI is suited to answer our question.

ANSWERS: 25a. P(1) 25b. P(k) is true, P(k + 1) is true.

25c. P(n) is true for all positive integers.

26: Write out each indicated statement where we have

$P(n)$: $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

26a. P(1): ______________________________

26b. P(2): ______________________________

26c. P(k): ______________________________

26d. P(k + 1): ______________________________

26e. Is P(1) true? ______________________________

ANSWERS: 26a. $2 = 1(1 + 1) = 2$ 26b. $2 + 4 = 2(2 + 1) = 2 \cdot 3 = 6$

26c. $2 + 4 + 6 + \cdots + 2k = k(k + 1)$ 26d. $2 + 4 + 6 + \cdots + 2k + 2(k + 1) = (k + 1)(k + 2)$ 26e. Yes.

27: In order to show that P(k) true implies P(k + 1) true, we assume that P(k) is true.

27a. The left member of P(k) is the sum of __________ (how many) terms.

27b. The left member of P(k + 1) is the sum of ________ (how many) terms.

27c. To show that P(k) true implies P(k + 1) true, we add __________ to both members of P(k) and show that we have arrived at P(k + 1).

ANSWERS: 27a. k 27b. k + 1 27c. 2(k + 1)

28: Perform the operations of 27c, thus showing that P(k) true implies P(k + 1) true.

ANSWERS: 28.
$$\begin{aligned} 2 + 4 + 6 + \cdots + 2k + 2(k + 1) &= k(k + 1) + 2(k + 1) \\ &= k^2 + k + 2k + 2 \\ &= k^2 + 3k + 2 \\ &= (k + 1)(k + 2) \end{aligned}$$

29: What is your conclusion concerning P(n)? Why?

ANSWERS: 29. P(n) is true for all positive integers because P(n) satisfies the conditions of the PMI.

In order to help prepare for the test over this unit, the following questions are proposed.

1. What is the statement of the PMI?
   a. In terms of sets.
   b. In terms of statements.
2. What is the procedure for using the PMI to prove a theorem?

EXERCISE I. Prove the statement that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof by PMI

P(n) is statement $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

P(1) is true since $1 = \frac{(1)\ (1+1)}{2}$

Suppose P(k) is true, then

$$\sum_{i=1}^{k} i = \frac{(k)(k+1)}{2}$$

To show P(k+1) is true, complete the following

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i \quad + k+1 = \frac{(k)\ (k+1)}{2} + k+1$$

$$= \frac{k^2 + k}{2} + \frac{2(k+1)}{2}$$

$$=$$

ANSWERS: $\frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$

P(k+1) is true so by PMI, P(n) is true for all $\underline{\underline{n}}$.

EXERCISE II

Sum of Squares of the first n integers is $\frac{(n)(n+1)(2n+1)}{6}$

To prove this by PMI, we first need to show P(1) is true.

1. What is P(1)? Is it true?

ANSWER: 1. $1 = \frac{(1)(2)(3)}{6} = 1$; yes

---

2. We now suppose that P(k) is true. What are we assuming?

ANSWER: 2. $P(k) = \sum_{i=1}^{k} i^2 = \frac{(k)(k+1)(2k+1)}{6}$

---

3. How do we state P(k + 1)?

ANSWER: 3. $\sum_{i=1}^{k+1} 1^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

To show P(k+1) is true, start with

$$\sum^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 = \frac{(k)(k+1)(2k+1)}{6} + (k+1)^2$$

By appropriate algebraic manipulation, it can be shown that this expression is equal to $\frac{(k+1)(k+2)(2k+3)}{6}$

$$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k^2+3k+2)(2k+3)}{6} = \frac{2k^3+9k^2+13k+6}{6}$$

and

$$\frac{(k)(k+1)(2k+1)}{6} + (k+1)^2 = \frac{2k^3+3k^2+k}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{2k^3+3k^2+k+6(k^2+2k+1)}{6}$$

$$= \frac{2k^3+9k^2+13k+6}{6}$$

Thus P(1) is true;

P(k+1) is true whenever P(k) is true; so

P(n) is true for all n.

APPENDIX C

POSTTEST

EXAM

Problems 1 and 2: Refer to the following set G.

Let $G = \{n \in N | P(n) \text{ is true}\}$

1. To say $1 \in G$ we must show that

   A. P(1) is true
   B. P(k) is true implies P(k+1)
   C. $G \subseteq N$
   D. None of the above

2. If we know $1 \in G$, then G = N if we also show

   A. $G \subseteq N$
   B. if P(k) is true then P(k+1) is true
   C. P(1), P(2), P(3) are all true
   D. G is infinite.

3. Let P(n) be a statement for which P(1) is true and P(k+1) is true whenever P(k) is true. Which of the following is (are) true?

   A. There is a positive integer s such that P(k) is not true.
   B. For any positive integer m, P(m) is true.
   C. Both A and B
   D. Neither A nor B

4. Let $G = \{n \in N | (2n^3 - n) \text{ is divisible by } 7\}$ then

   A. $1 \in G$
   B. $2 \in G$
   C. $3 \in G$
   D. all of the above

5. The set G (in No. 4) is not equal to N because

   A. $1 \notin G$
   B. $k \in G$ does not imply $(k+1) \in G$
   C. $G = \emptyset$
   D. none of the above

6. Let P(n) be the statement $n^3 - n$ is divisible by 6.

   A. 0 is divisible by 6.
   B. $G \neq \emptyset$
   C. $G \subseteq N$
   D. P(k) implies P(k+1)

In Problems 7 and 8, Let P(n) be the statement about positive integer: $4^{n+1} + 5^{2n-1}$ is divisible by 11

7. To show P(3) is true we need to show that

   A. $4^3 + 5^3$ is divisible by 11.

   B. $4^4 + 5^5$ is divisible by 11.

   C. P(3) is false and cannot be shown true.

   D. $3 \cdot 11 = 4^{n+1} + 5^{2n-1}$.

8. It is easy to show that P(1) is true. In order to show that P(n) is true for all positive integers n, (using the PMI), we must show for all positive integers k,

   A. P(k) is true
   B. P(k+1) is true
   C. P(k+1) is true whenever P(k) is true
   D. None of the above

9. Which of the following statements would be appropriate for proof by the PMI?

   A. For every $n \in N$, if n^2 is divisible by 5, then n is divisible by 5.
   B. For every $n \in N$, if n is odd then n^1 is odd.
   C. For every $n \in N$, $6^{n+2} + 7^{2n+1}$ is divisible by 43.
   D. None of the above.

10. Let P(n) be the following statement about positive integers.

    $P(n): 2 + 4 + 6 + \cdots + 2n = n^2 + n + 1$

    P(n) is NOT true for all positive integers because:

    A. P(1) is not true.
    B. Assuming P(k) true, it is impossible to deduce P(k+1) true.
    C. P(1) is true but P(2) is false.
    D. P(n) is true for all positive integers n.

11. Let P(n) be the following statement about positive integers.

    $P(n): 1 + 2 + 3 + \cdots + n = \frac{n^2 + 1}{2}$

    P(n) is NOT true for all positive integers because:

    A. P(1) is not true.
    B. Assuming P(k) true, it is impossible to deduce P(k+1) true.
    C. It is impossible to determine the truth value of P(n).
    D. None of the above.

12. Let $P(n)$ be a statement about positive integers. In order to show that if $P(k)$ is true, then $P(k+1)$ is also true, we take the truth of $P(k)$ as a premise and then deduce:

A. $P(1)$
B. $P(k+1)$
C. $k + 1$
D. We cannot logically accept $P(k)$ as a premise.

The following questions are "short answer" and will require that you fill in the blanks.

Let $P(n)$ be the following statement about positive integers.

$P(n)$: $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

We wish to show that $P(n)$ is true for every positive integer n.

13. First we must show that ____________ is true.

14. Secondly, we assume _________ is true and deduce that _________ is true.

Write out each indicated statement.

15. $P(1)$: ____________________________

16. $P(2)$: ____________________________

$P(3)$: $2 + 2^2 + 2^3 = 2^{3+1} - 2$.

17. $P(k)$: ____

18. $P(k + 1)$: $2 + 2^2 + 2^3 + \cdots + 2^k +$ ____________ = ____________

19. Is $P(1)$ true? __________

20. Is $P(2)$ true? __________

In order to show that $P(k)$ true implies that $P(k + 1)$ is true, we assume that $P(k)$ is true and then show that this implies that $P(k + 1)$ is true.

21. The left member of $P(k)$ is the sum of ______ (how many) terms.

22. The left member of $P(k + 1)$ is the sum of _______ (how many) terms.

23. To show that $P(k)$ true implies $P(k + 1)$ true, we add ____________ to both sides of $P(k)$ and show that we have arrived at $P(k + 1)$.

24. Perform the operations of 23, thus show that $P(k)$ true implies $P(k + 1)$ true.

25. What is your conclusion concerning $P(n)$? Why?

Now let $P(n)$ be the following statement about positive integers.

$P(n)$: For every positive integer n, $n^3 - n$ is divisible by 3.

26. In order to show that $P(n)$ is true for all positive integers n, we must show that

(i) ____________________

(ii) ____________________

Write out each indicated statement.

27. $P(1)$: ____________________

28. $P(2)$: ____________________

29. $P(k)$: ____________________

30. $P(k + 1)$: ____________________

31. Is $P(1)$ true? ________

32. Is $P(2)$ true? ________

We wish to show that $P(k)$ true implies that $P(k + 1)$ is true. Assume $P(k)$ is true and consider:

$$(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1)$$
$$= (k^3 - k) + 3(k^2 + k)$$

33. Is $k^3 - k$ divisible by 3? ________ Why? ____________________

____

34. Is $3(k^2 + k)$ divisible by 3? ________ Why? ____________________

____

35. Therefore, $P(k)$ true implies ____________________

36. What conclusion can you make regarding P(n)? Why? ______________

____

____

37. Let P(n) be some statement about positive integers. Explain how you would prove that P(n) is true for all positive integers using the PMI. (Be very concise.)

38. Prove the following by the PMI.

let P(n) be the statement,

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n-1).$$

that is

$$\sum_{i=1}^{n} 2^i = 2(2^n-1).$$

APPENDIX D

RETENTION EXAM

RETENTION EXAM

Problems 1 and 2: Refer to the following set G.

Let $G = \{n \in N | P(n) \text{ is true}\}$

1. To say $1 \in G$, we must show that

   A. P(1) is true
   B. P(k) is true implies P(k+1)
   C. $G \subseteq N$
   D. None of the above

2. If we know $1 \in G$, then $G = N$ if we also show

   A. $G \subseteq N$
   B. If P(k) is true then P(k+1) is true
   C. P(1), P(2), P(3) are all true
   D. G is infinite.

3. Let P(n) be a statement for which P(1) is true and P(k+1) is true whenever P(k) is true. Which of the following is (are) true?

   A. There is a positive integer s such that P(k) is not true.
   B. For any positive integer m, P(m) is true.
   C. Both A and B
   D. Neither A nor B

4. Let $G = \{n \in N | n^2 - n + 13 \text{ is prime}\}$ then

   A. $1 \in G$
   B. $2 \in G$
   C. Neither A nor B
   D. Both A and B

5. The set G (in No. 4) is not equal to N because

   A. $1 \in G$
   B. There is a $k \in G$ with $(k+1) \notin G$
   C. $G = \phi$
   D. None of the above

6. Let P(n) be the statement $n^2 - n$ is odd. $G \neq N$ because

   A. $1 \notin G$
   B. $G = \phi$
   C. P(k) implies P(k+1)
   D. False $G = N$

In Problems 7 and 8, Let P(n) be the statement about positive integer: $4^{n+1} + 5^{2n-1}$ is divisible by 11

7. To show P(3) is true we need to show that

   A. $4^3 + 5^3$ is divisible by 11.

   B. $4^4 + 5^5$ is divisible by 11.

   C. P(3) is false and cannot be shown true.

   D. $3 \cdot 11 = 4^{n+1} + 5^{2n-1}$

8. It is easy to show that P(1) is true. In order to show that P(n) is true for all positive integers n, (using the PMI), we must show for all positive integers k,

   A. P(k) is true
   B. P(k+1) is true
   C. P(k+1) is true whenever P(k) is true
   D. None of the above

9. Which of the following statements would be appropriate for proof by the PMI?

   A. For every $n \in N$, if n^2 is divisible by 5, then n is divisible by 5.
   B. For every $n \in N$, if n is odd then n^1 is odd.
   C. For every $n \in N$, $6^{n+2} + 7^{2n+1}$ is divisible by 43.
   D. None of the above.

10. Let P(n) be the following statement about positive integers.

    $$P(n):\ 2 + 4 + 6 + \cdots + 2n = n^2 + n + 1$$

    P(n) is <u>NOT</u> true for all positive integers because:

    A. P(1) is not true.
    B. Assuming P(k) true, it is impossible to deduce P(k+1) true.
    C. P(1) is true but P(2) is false.
    D. P(n) is true for all positive integers n.

11. Let P(n) be the following statement about positive integers.

    $$P(n):\ 1 + 2 + 3 + \cdots + n = \frac{n^2 + 1}{2}$$

    P(n) is <u>NOT</u> true for all positive integers because:

    A. P(1) is not true.
    B. Assuming P(k) true, it is impossible to deduce P(k+1) true.
    C. It is impossible to determine the truth value of P(n).
    D. None of the above.

12. Let P(n) be a statement about positive integers. In order to show that if P(k) is true, then P(k+1) is also true, we take the truth of P(k) as a premise and then deduce:

A. P(1)
B. P(k+1)
C. k + 1
D. We cannot logically accept P(k) as a premise.

The following questions are "short answer" and will require that you fill in the blanks.

Let P(n) be the following statement about positive integers.

$P(n)$: $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2.$

We wish to show that P(n) is true for every positive integer n.

13. First we must show that ____________ is true.

14. Secondly, we assume __________ is true and deduce that __________ is true.

Write out each indicated statement.

15. P(1): ______________________________

16. P(2): ______________________________

$P(3)$: $2 + 2^2 + 2^3 = 2^{3+1} - 2$

17. P(k): ______________________________

18. $P(k + 1)$: $2 + 2^2 + 2^3 + \cdots + 2^k +$ ____________ = ____________

19. Is P(1) true? __________

20. Is P(2) true? __________

In order to show that P(k) true implies that P(k + 1) is true, we assume that P(k) is true and then show that this implies that P(k + 1) is true.

21. The left member of P(k) is the sum of ______ (how many) terms.

22. The left member of P(k + 1) is the sum of _______ (how many) terms.

23. To show that P(k) true implies P(k + 1) true, we add __________ to both sides of P(k) and show that we have arrived at P(k + 1).

24. Perform the operations of 23, thus show that P(k) true implies P(k + 1) true.

25. What is your conclusion concerning P(n)? Why?

Now let P(n) be the following statement about positive integers.

P(n): For every positive integer n, $n^3 - n$ is divisible by 3.

26. In order to show that P(n) is true for all positive integers n, we must show that

(i) ______________________________

(ii) ______________________________

Write out each indicated statement.

27. P(1): ______________________________

28. P(2): ______________________________

29. P(k): ______________________________

30. P(k + 1): ______________________________

31. Is P(1) true? __________

32. Is P(2) true? __________

We wish to show that P(k) true implies that P(k + 1) is true. Assume P(k) is true and consider:

$$\begin{aligned}(k + 1)^3 - (k + 1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1)\\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

33. Is $k^3 - k$ divisible by 3? __________ Why? ______________________________

34. Is $3(k^2 + k)$ divisible by 3? __________ Why? ______________________________

35. Therefore, P(k) true implies ______________________________

36. What conclusion can you make regarding P(n)? Why? ____________

____

____

37. Let P(n) be some statement about positive integers. Explain how you would prove that P(n) is true for all positive integers using the PMI. (Be very concise.)

38. Prove the following by the PMI.

Let P(n) be the statement:

$$n^2 \geq 2n - 1$$

APPENDIX E

RAW DATA FOR STUDENTS IN THE STUDY

RAW DATA FOR SUBJECTS IN THE STUDY

Column—Heading—Code

1. Student Number
2. Section 1 or 2
3. Method 1 (Recursive) or 2 (PMI) first
4. Score on Posttest
5. Score on Retention test
6. Change Score

1	2	3	4	5	6
01	1	2	70	61	-9
02	2	2	56	61	+5
03	2	2	61	54	-7
04	1	2	60	67	+7
05	1	2	64	75	+11
06	2	2	67	65	-2
07	2	2	58	67	+9
08	2	2	63	60	-3
09	1	2	66	62	-4
10	2	2	72	53	-19
11	1	2	78	77	-1
12	1	2	72	72	0
13	1	1	51	53	+2
14	2	1	52	54	+2
15	2	1	56	60	+4
16	2	1	63	68	+5
17	1	1	65	65	0
18	1	1	58	61	+3
19	1	1	69	71	+2
20	2	1	66	70	+4
21	2	1	65	56	-9
22	1	1	73	74	+1
23	1	1	70	66	-4
24	2	1	78	75	-3