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ABSTRACT

Written in 1933, this book grew out of the author's concern that college matehmatics sequences of the day, although appropriate in algebra preparation, did not adequately prepare teachers of geometry. This book describes a course intended to remedy this by providing for both a comprehensive study of geometry as an axiomatically defined structure describing spatial relationships and a thorough consideration of the purposes and techniques of teaching geometry in the high school. The author intended that this course follow the study of educational psychology and professional ideals and responsibilities, and that students enrolled be familiar with the basic theorems of Euclidean geometry. After listing the deductive relationships among these theorems, the book deals with the history and function of geometry. Principles of teaching high school geometry including induction and deduction, laboratory methods, and heuristics are then discussed. The use of high school geometry materials is discussed at length. Nearly half of the book is devoted to the structure of geometry and methods of proof; the underlying principles of the analytic method are described and many problems suggested. Coordinate geometry and topics related to projective geometry are introduced. A bibliography is provided. (SD)

GEOMETRY PROFESSIONALIZED *FOR* TEACHERS

BY

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PREFACE AND ACKNOWLEDGMENTS

The integration of subject-matter and method has long been a puzzling problem in teacher training. This study is designed to give a teacher or prospective teacher a mastery of the subject-matter of geometry and simultaneously to train him in the method of teaching demonstrative geometry in the high school. By first setting up the objectives of such an integrated course, then establishing the pattern of teaching which is so peculiarly fitted to demonstrative geometry, the integration is effected by using the technique advocated in presenting the "essential theorems" of geometry.

A careful study of the materials included or outlined in Chapters II, III, IV, and sections II and V of Chapter V should, in the opinion of the author, constitute a minimum of material and method to be mastered by every teacher of geometry. The remainder of Chapter V and Chapter VI should not be omitted if the very best training is desired. Furthermore, liberal use should be made of the bibliography and references for more complete treatment of topics in which the reader may have special interest.

Adequate acknowledgment for the help of others in the preparation of this study is impossible. To President H. A. Brown I owe much for the stimulus in beginning this work. To Professor W. D. Reeve I am unpayably indebted for his kindly, constructive, enthusiastic, and tireless help. To Professor C. B. Upton for his keen insight into the problem and especially for his help with analysis and indirect proof, and to Professor W. C. Bagley for his stimulating conception of professionalized subject-matter, even to a casual reader, my debt is evident. To Professors John R. Clark and H. B. Bruner for their forceful, precise suggestions I am grateful. To Professor John P. Everett, Professor W. E. Anderson, and Dean E. J. Ashbaugh, for reading portions of the manuscript and for several challenging discussions which gave needed moral support, and to many friends and fellow students for evaluating problem material I acknowledge my indebtedness.

H. C. C.

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CHAPTER I

INTRODUCTION

I. The Definition of the Problem to be Solved

Present Training of Geometry Teachers. The academic mathematical training of the high-school teacher of mathematics, in addition to high-school algebra and geometry, consists at best of college algebra, trigonometry, analytic geometry, calculus, and perhaps differential equations, advanced algebra, surveying, mechanics, or astronomy. To this there is often added a special methods course in mathematics of two or three semester hours. The teacher is then supposedly trained to go into a high school and teach algebra and demonstrative geometry.

The weakness of this program of training for geometry teachers is apparent. The student is well trained in algebra because of his wide experience with its symbolism throughout his college course, but his training in geometry consists of very little more, if any, than his high-school course. Trigonometry makes use of similar triangles, analytic geometry deals with lines and circles, and calculus treats in part of areas and volume; but nowhere in the college course is the emphasis put on the proving or demonstrating of relationships in the way that is so widely emphasized in high-school geometry, nor are the relationships which are studied in high-school geometry widely used. The reason for this inadequacy in the training of high-school teachers lies in the fact that colleges of education have merely accepted the academic mathematics courses without modification or adequate adjustment to the needs of prospective high-school teachers of mathematics and have usually supplemented them with a brief "methods" course.

Needed Training of Geometry Teachers. Professor Bagley contends that "The high-school teacher of mathematics should surely undertake mathematics studies well in advance of those that he proposes to teach, and it is quite possible that the content of these advanced courses should be modified by the fact that he is to teach high-school mathematics. But in any case he needs courses in elementary algebra and plane geometry, which will not only refresh his mind with regard to elementary principles and processes, but also give him a much deeper and broader conception of those principles and a much more facile mastery of those processes than his elementary course could possibly give him. Such courses should emphasize the historical development of these elementary processes, and they should lay stress particularly upon the possibilities and methods of illuminating instruction by the applications of elementary mathematics to a variety of scientific, technical and industrial problems."[1]

Engineering schools have taken the academic courses in mathematics and have professionalized them to make them better suited to their uses. A corresponding professionalization is necessary in the courses for prospective teachers, if they are to prepare adequately for high-school teaching. Such a modification is eminently necessary for teachers of high-school geometry.

Textbooks in geometry have already been written ostensibly for the purpose of solving this problem. Altschiller-Court's *College Geometry* is advanced Euclidean geometry and is a pioneer effort in the direction of professionalization.[2] Johnson's *Modern Geometry* is a similar text.[3] Both of these books, along with Durell's, Godfrey and Siddon's, and others, are still quite academic in their point of view. They are really advanced geometry based on a thorough elementary course.

The Problem to be Solved. There remains then the problem of securing still further professionalization of subject-matter with more emphasis on the fundamental pattern of teaching geometry as well as on the foundations of geometry, more actual contact with high-school geometry, and more attention to the system of formulated reasoning and its application to non-geometric as well as geometric situations. Such is the problem whose solution is herewith attempted. The method of solution has been first to state the problem in terms of the specific objectives to be realized, and then to select the subject-matter which will best enable the student to attain those objectives. The subject-matter selected is of two types: The first is somewhat general and includes a presentation of problems concerning the function of geometry in the high school, a discussion of some specific problems which the high-school teacher has to meet, and finally the establishment of a philosophy of teaching and a technique of presentation recommended for high-school use; second, this philosophy and technique are applied to a few high-school geometry

[1] Bagley, W. C. *The Professional Preparation of Teachers for American Public Schools.* The Carnegie Foundation for the Advancement of Teaching. Bulletin No. 14, p. 152.

[2] Altschiller-Court, N. *College Geometry.* Johnson Publishing Co., Richmond, Virginia, 1925.

[3] Johnson, R. A. *Modern Geometry.* Houghton Mifflin Co., 1929.

theorems and college geometry theorems, and provision is made for an abundance of original exercises for practical application. Chapters I, II, and III present the first type of material which is largely professional in nature, and Chapters IV, V, and VI present the second type of material which applies the professional ideas to material that is largely mathematical.

II. The Objectives of this Study

The purpose of this study in the professionalization of geometry for teachers' colleges is to suggest mathematical and professional training for prospective teachers of high-school geometry. This general objective is broken up into the following specific objectives.

A. The Mathematical Objectives of the Course

1. To insure a thorough mastery of the subject-matter of the high-school geometry which the student is preparing to teach.
2. To develop an appreciation of the system of reasoning presented through the simple relationships of geometry by emphasizing the following facts:
   a. The foundation for building the structure called geometry is composed of undefined terms, definitions, axioms, and postulates.
   b. The structure itself is composed of conclusions which are reached by a process of reasoning based upon certain hypotheses and the foundations previously accepted.
   c. Sequence is therefore of paramount importance.
3. To develop and to extend the student's ability to discover and analyze space relationships, by the mastery of theorems and exercises of modern geometry more difficult than those used in high-school texts.
4. To develop an appreciation of the functional relationships in geometry.
5. To show the historical development of geometry and emphasize its early and continually prominent place in civilization as a universal language of thought.

B. The Professional Objectives of the Course

1. To give a new view of and, if necessary, to develop the abilities and concepts included in high-school geometry, with the conviction that "no one can teach what he does not know."
2. To present, to illustrate, and to use methods of teaching which correspond to those recommended by this study for high-school

teaching with the idea that "most teachers teach very largely as they have been or are taught rather than as they are told to teach."

3. To provide the student with problems in geometry relatively as difficult for him to solve as those in high school are for a high-school student, in order that he may appreciate anew the learner's point of view, and in order that he may be taught in the way that he is supposed to teach.
4. To emphasize the two-fold nature and purpose of a theorem in geometry, namely, to serve as a general law or principle used to prove "originals" and other theorems, and to serve as a reasoning pattern for solving geometric "originals" or for proving any conclusions which seem to be dependent upon given premises.
5. To teach, test, and diagnose; then, if necessary, to teach, test, and diagnose again to the point of mastery.
6. To emphasize the principle that learning takes place only during pupil activity or, more traditionally expressed, that "we learn to do by doing."

C. The Professional Assumptions upon Which the Course is Built

1. The student will have had a course in the introduction to teaching, emphasizing professional ideals and responsibilities, and also such routines as care of light, heat, ventilation of the class room, attendance, daily and term reports, marking systems, lesson planning, and the like.
2. The student will have had a course in educational psychology, and will understand and appreciate the laws of learning and the psychology of drill.
3. The student will have had a course in the principles of education which will stimulate him to adopt a philosophy of education and to appreciate the prominent place of education in the progress of civilization and in the future of democracy.
4. The student will have practice teaching after taking this course.

D. The Specific Mathematical and Professional Objectives of the Course. A mastery of the subject-matter in the detailed units as presented on the following pages and in the manner indicated, including a large portion of the original exercises, will be the specific, detailed objectives of the course. These details should not be emphasized so as to overshadow or minimize the general objectives, but so as to contribute to the realization of those general objectives.

III. The Selection of the "Essential" Theorems of High-School Geometry

The Technique of Selecting the Essential Theorems. Table I, which follows, shows the technique used for the selection of the "Essential Theorems of High-School Geometry." It shows an analysis of all the theorems and constructions in the Report of the National Committee on the Reorganization of Mathematics.[4] The analysis indicates that even in this short list of theorems there are many that are never used in the proof of later theorems. In the case of some that are used, the theorems in which they are used are often not referred to later, or a slight change in the proof of a later theorem may render a previous theorem useless.

For instance, theorem number 1a*, in Table I, "Two triangles are congruent if two sides and the included angle etc.—," uses in its proof only postulates 8 (p8), "Any figure may be moved from one place to another without changing its shape or size," and (p1), "Only one straight line can be drawn between two points." However, theorem 1a is itself used in the proof of a large number of theorems: 1c, 2, 3a, 4, 9b, etc. Again, theorem 7, "The sum of the angles of a triangle equals 180 degrees," uses in its proof, in addition to a definition and an axiom, construction 5 (c5), "Through a given point draw a line parallel to a given line," and theorem 6a, "When a transversal cuts two parallel lines, the alternate interior angles are equal," while it is used in the proof of theorems numbered 2, 5, c17, s4b, s5, s6, s8a, s14a, s18, s19, ce1, ce4. Theorems such as 9a, 9b, 11c, 13c, 14, given below are never used again in the proof of later theorems:

9a. "Any quadrilateral is a parallelogram (a) if its opposite sides are equal, (b) if two sides are equal and parallel."

11c. "The area of a trapezoid etc. —."

13c. "Two triangles are similar if their sides are respectively proportional."

14. "If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other."

These theorems and others like them, since they are not necessary for the proof of other propositions, are really not essential theorems in the sense that "essential" has been defined, but rather applications of the essential theorems.

[4] National Committee on Mathematical Requirements. *The Reorganization of Mathematics in Secondary Education.* Houghton Mifflin Co., 1927.

* These numbers refer to the theorems listed in the National Committee's report.

Theorem number 8, "A parallelogram is divided into congruent triangles by either diagonal," is used in the proof of s7a, which in turn is used in the proof of several theorems. Theorem s7a, "In any parallelogram the opposite sides are equal," can be proved without theorem 8 by including the proof of theorem 8 in its proof. Therefore theorem s7a is an essential theorem and theorem 8 is accordingly omitted from the list of essential theorems. Theorems 1a, 1b, 1c, are the three conventional congruence theorems; and they are postulated. The defense for such postulation is given later.

It is evident that a defensible minimum list of theorems could be made from these essential theorems. Such a list would be a perfectly logical system and would be composed of only the most important theorems of geometry. All others could be given a minor place or perhaps listed as exercises to be solved by the student. Such a choice has been made and the selection indicated by double asterisks in Table I. The proving of these theorems and the providing for applications of them make up the proposed direct contact with the field of high-school geometry which is included in Chapter IV of this study. A thorough mastery of these theorems and some of their applications is designed to provide confidence with the content of high-school geometry and experience with the technique of teaching geometry. The following list constitutes the theorems selected. In some cases the usual wording has been slightly changed.

It should be made clear, however, that the "Essential Constructions and Theorems" which are selected on the later-usage criterion are essential in a professional sense rather than in a mathematical sense. Their purpose is "to refresh" the reader's "mind with regard to elementary principles and processes," and to establish a pattern of teaching, but not to serve as a new list of the "fundamental theorems" of geometry. They are important theorems for a prospective teacher of Euclidean geometry to know, and fundamental in a professional sense only. They are designed to give the prospective teacher "a broader and a more accurate knowledge of the materials to be taught; . . . an appreciation of the teaching difficulties involved and an ability to apply the guiding principles of psychology and methodology to those difficulties; and . . . a richer cultural background of illuminating information and appreciation which will enliven and color the work of the teacher."*

* Evenden, E. S. "The Critic Teacher and the Professional Treatment of Subject-Matter: A Challenge," *Educational Administration and Supervision*, 15:373-82.

TABLE I

PLANE GEOMETRY

THE USE MADE OF THEOREMS IN THE PROOF OF LATER THEOREMS

Previous theorems, constructions, and postulates needed in the proof	*Propositions proved*	*Later theorems and constructions which need this theorem in their proof*
*p8, p1	1a	1c, 2, 3a, 4, 9b, 13b, 23b, c1, c3, c6b, c16, s8b, s9, ce5
p8	1b	3b, 5, 6b, 8, 10, 13a, 29, c6c, s7c, ce1
3a, 1a	1c	4, 9a, 11b, 11d, 13c, 23a, 24b, c1, c2, c3, c6a, ce6
p10, 3a, 7, 1a	**2	5, 11a, 24a, 26a, 26b, s13, ce3
c2, 1a	**3a	1c, 2, 17, 27, c17, $s14a_1$, $s14b_1$, ce5
c2, 1b	3b	17
1a, 1c	**4	c8, c9, s11
2, 7, 1b	**5	c10, s12
c4, 6b, p14	**6a	7, 8, 9b, 10, 17, s2a, s2b, s7c, s17
c1, c3, p12, 1b, p13	**6b	6a, 7, 9a, 9b, c5, s1a, s4a, s1b, s8a, s8b
c5, 6a	**7	2, 5, c17, s4b, s5, s6, s8a, s14a, s18, s19, ce1, ce4
6a, 1b	8	s7a, s7b
1c, 6b	9a	
6a, 1a, 6b	9b	
c5, s7a, p15, 6a, s2a, 1a	**10	12a
c3, p13, s7a, 2, p16	11a	s21
p2, p13, s7a, 1c, p16	**11b	11c, 11d, 18, c15, s21
11b	11c	
11b, 1c	11d	31
c5, 10	**12a	12b, 12c, 13a, 17, c12
c5, 12a	**12b	13b, 13c, s9
c5, s7a, 12a	12c	c7
c4, 1b, s[illegible], 12a	**13a	13b, 13c, 14, 18, 19, c14, s9, s20, s22
1a, 12b, 1[illegible]	13b	16b
12b, s2a, 1[illegible], 1c	13c	
p12, 28, 27, [illegible]	14	
axiom	15	30
axiom	16a	
axiom and 13b	16b	
p2, c5, 12a, 6a, s2a, 3b	17	
c4, c3, 13a, 11b	18	

* All numbers refer to the theorems in the list given by the *Report of the National Committee on the Reorganization of Mathematics in Secondary Education,* 1927, Houghton Mifflin Co., New York. c12 means construction number 12, s12 means subsidiary theorem number 12, pages 78-91, ce8 means theorem 8 of the additional College Entrance theorems on page 169.

** Selected for the minimum list of essential theorems.

TABLE I (*Continued*)

Previous theorems, constructions, and postulates needed in the proof	*Propositions proved*	*Later theorems and constructions which need this theorem in their proof*
c3, 13a	**19	20, c13
19	**20	
p8	21a	23b
p8	21b	22, 23a, 24a
21b	22	28
1c, 21b	23a	
21a, 1a	23b	29
2, 21b	**24a	26a, 26b, s16
1c	24b	
p5	**25a	29, s13, s16, ce3
p5	25b	c11, ce6
24a, 2	26a	ce6
2, 24a	26b	
3a, s5	**27	14, 28, 29, c11, s15, c16, s17, s18, s19, s20, ce4
27, 22	28	
23b, 27, 25a, 1b	29	31
15	**30	31
29, 11d, 30	**31	ce10
p1, 1c, 1a	**c1	6b, c11, c13
p1, 1c	**c2	3a, 3b
p1, 1c, 1a	**c3	6b, 11a, 11b, 18, 19, c11, c13, c16, s22
p1, 1c	**c4	6a, 13a, 18, c14
6b	**c5	7, 10, 12a, 12b, 12c, 17, c7, c12, c15, s17
1c	**c6a	
1a	**c6b	
1b	**c6c	
c5, 12c	c7	
4	c8	
4, p6	c9	ce2, ce5
5, p6	c10	
c3, p1, c1, 27, 25b	c11	
c5, 12a	c12	
c1, c3, 19	c13	
c4, 13a	c14	
c5, 11b	c15	
p1, c3, 1a, 27	c16	
3a, 7	c17	
p12, 6b	s1a	13a
6b	s1b	
p12, 6a	s2a	10, 13c, 17
6a	s2b	
p15, p13, p14	s3	s16
p2, 6b	s4a	
p12, 7	s4b	

TABLE I (*Continued*)

Previous theorems, constructions, and postulates needed in the proof	*Propositions proved*	*Later theorems and constructions which need this theorem in their proof*
7	s5	27
7	s6	ce7
8	**s7a	10, 11a, 11b, 12c, s7c, s10
8	s7b	
s7a; 6a, 1b	s7c	
7, 6b	s8a	
p12; 1a, 6b	s8b	
12b, 13a, 1a	s9	
s7a, s11	s10	
4	s11	s10
5	s12	
25a, 2	s13	
3a, 7	s14a_1	s14a_2, s14b_1, s14c_1
s14a_1	s14a_2	s14c_1
3a, s14a_1	s14b_1	s14b_2, s14d_1
s14b_1	s14b_2	
s14a_1, s14a_2	s14c_1	s14c_2
s14c_1	s14c_2	
s14b_1	s14d_1	s14d_2
s14d_1	s14d_2	
27	s15	
25a, s3, 24a	s16	
c5, 6a, 27	s17	s20
7, 27	s18	
7, 27	s19	
27, s17, 13a	s20	
11a, 11b	s21	
c3; 13a	s22	
7, 1b	ce1	
c9	ce2	
25a, 2	ce3	
7, 27	ce4	
c9, 3a, 1a	ce5	ce6
ce5, 1c, 26a, 25b	ce6	
s6	ce7	
p5	ce8	
post.	ce9a	
post.	ce9b	
31	ce10	
	**1	6, 8
	2	10, 12
	3	5
	**4	14
3	5	

TABLE I (*Continued*)

SOLID GEOMETRY

†*Previous theorems needed in proof*	*Theorem proved*	*Later theorems depending on this theorem*
	6	
	7	
1	**8	13, 15, 17a, 17b
	**9	13, 15, 17b
	10	
	11	
	12	
8, 9	13	14
13, 4	**14	16a, 27a
8, 9	15	
14	**16	27a
8	**17a	
8, 9	**17b	
	**17c	18
27, 17c	18	29a
	19	25
	20	23
	21a	
	21b	
51b	22a	
51a	22b	
20	23	
	24	
19	25	31
	**26	
14, 16	**27a	27b, 29a
27a	**27b	
	**28a	28b
28a	**28b	
18, 27	**29a	29b, 32
29a	**29b	
	**30	32
25	31	
29a, 30	**32	

† Only solid geometry theorems are listed here although only the double starred theorems of the previous part of this table were used.

The Essential Constructions and Theorems of Geometry.

A. Constructions

1. Construct a circle with a given radius and a given center.
2. Construct a triangle congruent to a given triangle using only the lengths of the three sides.
3. Construct an angle equal to a given angle.
4. Bisect a given angle.
5. Construct a perpendicular to a line at a point on the line.

6. Construct a triangle congruent to a given triangle using only two sides and their included angle.
7. Construct a triangle congruent to a given triangle using only one side and the two adjacent angles.
8. Construct a perpendicular bisector of a given line segment.
9. Construct a perpendicular to a line from a point not on the line.
10. Construct a line parallel to a given line through a given point.

B. Theorems on Straight Line Figures

1. The angles opposite the equal sides of an isosceles triangle are equal.
2. If two lines cut a third so that the alternate interior angles are equal, the lines are parallel.
3. The converse of 2. If two parallel lines cut a third line, the alternate interior angles are equal.
4. The sum of the angles of any triangle is 180°.
5. Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.
6. The opposite sides and angles of a parallelogram are equal.
7. If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.
8. If a straight line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionally.
9. Converse of 8. If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.
10. Two triangles are similar if two angles of one are equal respectively to two angles of the other.
11. In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

   Corollary 1. In any right triangle if a perpendicular be dropped from the vertex of the right angle to the hypotenuse,

   (a) the two right triangles formed are similar to the given triangle and to each other,
   (b) either leg of the given right triangle is a mean proportional between the whole hypotenuse and the adjacent segment,
   (c) the perpendicular is the mean proportional between the segments of the hypotenuse.

12. The area of a triangle is equal to half the product of the base times the altitude.
13. The locus of a point equally distant from two points is the perpendicular bisector of the line segment joining them.
14. The locus of a point equally distant from two intersecting lines is the pair of lines which bisect the angles formed by the lines.

C. Theorems Concerning Circles

15. A diameter perpendicular to a chord bisects the chord and the arcs of the chord.
16. An angle inscribed in a circle is equal to half the central angle having the same arc.
17. A line perpendicular to a radius at its outer extremity is tangent to the circle at that point.
18. If the number of sides of a regular inscribed polygon is indefinitely increased, its perimeter and area will both increase, while the perimeter and area of the circumscribed polygon, formed by drawing tangents to the circle, at the vertices of the inscribed polygon, will both decrease. The perimeters and areas of both polygons will each approach a limit.
19. The ratio of any circumference to its radius is constant and is equal to 2 pi.
20. The area of a circle is equal to pi times the square of the radius.

D. Theorems of Solid Geometry

21. If two planes intersect, their intersection is a straight line.
22. If two parallel planes cut a third plane, the lines of intersection are parallel.
23. If two angles not in the same plane have their sides parallel in the same sense, the angles are equal.
24. The lateral area of a prism is the product of a lateral edge by the perimeter of a right section.
25. The volume of any prism equals the product of its base by its altitude.
    *a. An oblique prism is equal to a right prism whose base is a right section of the oblique prism and whole altitude is a lateral edge of the oblique prism.
    *b. The volume of any parallelopiped equals the base times the altitude.

*c. A diagonal plane divides a parallelopiped into two equal triangular prisms.

*d. The volume of any triangular prism equals the product of the base and altitude.

e. The volume of any prism is the product of its base and altitude.

26. The lateral area of a regular pyramid equals ½ the product of its slant height and the perimeter of the base.

27. If a pyramid is cut by a plane parallel to the base and a distance (d) from the vertex,

(a) The lateral edges and altitude (h) are divided proportionally,

(b) The section is similar to the base, the "ratio of similitude" being d/h,

(c) The ratio of the area of the section to the area of the base is d^2/h^2.

28. The volume of a pyramid equals ⅓ the product of the area of its base (b) by its height.

*a. Two pyramids with equal bases and altitudes are equal.

*b. The volume of a triangular pyramid equals ⅓ bh.

c. The volume of any pyramid equals ⅓ bh.

29. The lateral area of a cylinder equals the product of an element by the circumference of a right section, and its volume equals the product of its base and altitude.

30. The lateral area of a right circular cone equals ½ the product of the slant height by the circumference of the base, and the volume of any cone equals ⅓ the product of the area of the base by the altitude.

31. The area of a sphere equals 4 pi times the radius squared.

*a. The area of the frustrum of a cone of revolution equals ½ the slant height times the circumference of a circle half way between the bases.

*b. The area of a surface of revolution formed by revolving a regular polygon about a diameter is 2 pi times the apothem times the diameter.

c. The area of a sphere equals 4 pi times the radius squared.

32. The volume of a sphere equals ⁴⁄₃ pi times the radius cubed.

* These propositions are usually given as theorems, but their only use is to lead up to the theorem just preceding. They are therefore given here as part of the theorem in the proof of which they serve.

The Use made of the Essential Theorems. In this study the theorems just given are used for a dual purpose. First, to provide a rapid review of high-school geometry and yet one that covers items of fundamental importance. It is impossible to emphasize each of 200 theorems in such a brief review, yet it is quite possible to emphasize each of 20 theorems and to use each one as a "mountain peak" from which the surrounding territory can be surveyed. Second, the theorems and constructions provide additional illustrations of the application of the heuristic pattern of teaching, which it is the purpose of this study to present. Two or three illustrations in a short chapter presenting the plan, philosophy, and purpose of heuristic teaching are entirely inadequate for a full development of such an important idea. In order that prospective teachers may be thoroughly indoctrinated with the philosophy of the "discovery and analysis" technique, they should be provided with many and varied illustrations of its application.

While it is not intended that high-school teachers shall actually use the same list of 10 constructions and 20 theorems in their teaching of plane geometry, yet it is evident that there are decided possibilities in the use of the same list of "essential theorems" in a high-school course. "It is the greatest triumph of philosophy to refer many varied phenomena to one or a very few simple principles."[5] So there is also a distinct advantage in having in a high-school geometry course a few large, important, useful theorems upon which all geometry depends. The possibilities are much greater for better teaching of theorems and for more applications of the techniques mastered, to real problems. Such a geometry has possibilities also for placing the major emphasis where it belongs, that is, upon the solution of "originals." Every theorem in geometry should serve two purposes. First, each theorem is a general truth, or principle, or law which is useful in solving problems. Second, each theorem illustrates a method of solution, since, before it was proved, the theorem was itself a problem. In other words, each theorem is not only a statement of a useful fact, but also a reasoning pattern to be used and followed in proving that a given conclusion follows from given premises.

[5] Davies, Charles. *The Nature and Utility of Mathematics.* A. S. Barnes & Co., 1875, p. 73.

CHAPTER II

HISTORY, FUNCTION, AND PROBLEMS OF GEOMETRY

I. A Brief History of Geometry

Extent and Origin. When we realize that United States Government as a school subject covers only the short span of years since 1776, that American history begins with 1492, affording for study a period of less than 450 years, and that even the English language itself was somewhat embryonic at the time of Chaucer, less than 600 years ago, we may begin to have some respect for a subject that has been a challenge to human thought and ingenuity since its earliest records in Egypt in 2300 B.C., a period of over 4000 years. Egypt, Babylonia, Greece, Rome, and medieval and modern nations have all used and contributed to geometry. It seems to be a universal language of thought and culture, disregarding time and geographical or political boundaries. Men, nations, and even languages seem to come and go, but the ideas of geometry remain, grow, and become more and more indispensable.

The word "geometry" is derived from two Greek words meaning "earth measure." In Egypt, where this science of measurement was begun, it was used largely to measure land in the Nile valley. Such measurement was a frequent task because of the semi-annual over-flow of the banks of the River Nile. Our present knowledge of Egyptian geometry comes from a papyrus, held in the British Museum, and written by Ahmes, who lived about 1600 B.C. This papyrus is largely a copy of an older document dating back to 2300 B.C.

Egyptian Geometry. Egyptian geometry was intuitive in nature and largely a list of rules and formulas. Some of these were inaccurate, as, for instance, the rule that the area of an isosceles triangle is equal to the product of the base by one of the equal sides. Ahmes used $(d-1/9\ d)^2$ in computing for the area of a circle, making 3.1605 as the value of pi.[6, 7] While these formulas are slightly inaccurate, we marvel that they were even so accurate at such an early age.

[6] Smith, D. E. *History of Mathematics.* Ginn and Co., 1925, Vol. II, p. 270.
[7] Sanford, Vera. *A Short History of Mathematics.* Houghton Mifflin Co., 1930, p. 231.

The Contribution of Greece. Geometry was carried to Greece by Thales of Miletus (640-550 B.C.). After a career in commerce, which brought him in contact with the Egyptian ideas, he founded a school in Miletus for the teaching and study of mathematics. It seems natural that the Greeks, who were so superior mentally, would be entirely dissatisfied with the inaccurate and intuitive rules and computations of the Egyptians. The challenge to them to prove that these rules were true and to perfect the science as a system of reasoning is apparent.

It was Thales in 600 B.C. who first logically demonstrated a theorem in geometry. In all he proved only five theorems,[8] but he is credited with being the first to organize geometry as a science and to prove his conclusions. While knowledge of many of the facts of geometry dates back to at least 2300 B.C. in Egypt, the demonstration portion of geometry comes from Greece and began in about 600 B.C.

Pythagoras (580-500 B.C.) was a student at Miletus and later founded a school of his own at Crotona, Italy. This school became a communistic brotherhood whose members were bound by an oath not "to reveal the teachings or secrets of the school." The Greek government, fearing its political influence, finally ordered it to disband; and Pythagoras, with many of his followers, was killed. Because of the secret nature of the society little is known of the work it did in mathematics, although the Pythagoreans are often credited with the proof of the right triangle theorem which is now usually known as the Pythagorean theorem. The facts of this theorem were known intuitively by the Egyptians for certain triangles. They used a loop of string 12 units long with knots separating it into three segments, 3, 4, and 5 units long, to lay off a right angle. There is evidence that the right angles of the pyramids were made this way, before 3000 B.C.[9] It remained for Pythagoras, however, to demonstrate that for all right triangles the square on the hypotenuse is equal to the sum of the squares on the other two sides.[10]

Plato (429-347 B.C.) was the next great mathematician, and his influence in making geometry a science of reasoning is often credited to Euclid. While Plato was not a student under Pythagoras, he no doubt learned much about the unwritten and secret work of the Pythagoreans. He founded a famous school, "The Academy," at Athens, over the entrance to which he placed these words: "Let none ignorant of geometry enter my door." Plato did much to systematize the thinking in geometry.

[8] Stamper, Alva W. "History of Teaching of Geometry," Teachers College, Columbia University, New York. *Contribution to Education*, No. 23, 1909, p. 11.

[9] *Ibid.*, p. 5.

[10] *Ibid.*, p. 12.

It is due to his influence that later mathematicians began the subject with a carefully worded series of definitions, postulates, and axioms. It was he who limited geometricians to the use of the straight edge and compasses, and his influence has kept curves other than the circles out of plane geometry.

Before the time of Plato the Greeks used the method of analysis in the solution of problems. They also used the idea of locus, the indirect or *"reductio ad absurdum"* proof, and the method of exhaustion for some problems.[11]

Euclid (300 B.C.) is one of the best known of the early contributors to geometry. Little is known about his life, but there is reason to believe that he studied at Athens before he became a teacher of mathematics at Alexandria, Egypt. Euclid collected all the mathematical knowledge of his time and organized it into a logical sequence.[12] "No doubt there were many propositions that were original with Euclid; but the feature which made his treatise famous, and which accounts for the fact that it is the oldest scientific textbook still in use, is found in its simple but logical sequence of theorems and problems."[13, 14] Over 1000 editions of this book have been published since it was first printed in 1482. His "Elements" was arranged in books, originally in scrolls: I Congruence, II Identities, III Circles, IV Inscribed and Circumscribed Regular Polygons, V Proportion, VI Similarity, VII-IX Arithmetic, X Incommensurables, XI-XIII Solid Geometry.[15] Euclid's influence on our present geometry is evident from the above list, and it seems almost uncanny to realize how little his geometry has changed in 2200 years.

Some of the features of Euclid's *Elements* are: (1) the omission of all practical work, (2) no original exercises, every proposition fully proved out, (3) hypothetical constructions not permitted (hence Euclid began his geometry with constructions), (4) a general plan for the proof of all propositions, (5) all constructions by means of the compasses and straight edge only (thereby barring all conic sections except the circle from plane geometry).

Contributions to Geometry after 300 B.C. Since the time of Euclid much has been discovered in geometry. Very little of this, however, has been put into high-school courses. Apollonius, about 200 B.C., did his

[11] Allman, George J. *Greek Geometry from Thales to Euclid.* Longmans, Green and Co., London, 1889, p. 111.

[11] Stamper. *op. cit.*, p. 23.

[12] Sanford, Vera. *op. cit.*, p. 269.

[13] Smith, D. E. *History of Mathematics.* Ginn and Co., 1925. Vol. I, pp. 103-106.

[14] Allman. *op. cit.*, p. 211; Stamper. *op. cit.*, p. 27.

[15] Smith. *op. cit.*, p. 106.

great work on conic sections and named them the *parabola, hyperbola,* and *ellipse*. He showed how these can be produced from sections of a cone. His contribution was much like that of Euclid in being a compilation of previous work with some original contribution.[16] Many of these contributions were originated by Archimedes. Euclid and Apollonius dominated geometry for nearly 2000 years. No other outstanding accomplishment was made until the time of Descartes.

In 1637 Descartes published his *La Geometrie* introducing to the world the next great geometry achievement. This was the use of a co-ordinate system in geometry to express algebraic relations: "The real idea of functionality as shown by the use of coordinates was first clearly and publicly expressed by Descartes."[17]

This discovery tended to widen the scope of geometry by associating many parts of it with algebraic symbols. It was the great forward step in modern mathematics which made possible the discovery of the calculus by Newton and Leibniz and opened vast fields of quantitative functional relationships. The influence on modern high-school geometry is still largely unfelt.

Mention should be made of the brilliant mathematician, Pascal. He made no new phenomenal discovery comparable to that of Descartes, but he wrote on conics, made some discoveries in physics, and discovered a famous theorem bearing his name that the opposite sides of a hexagon inscribed in a conic intersect in points that are collinear. From this theorem he deduced over 400 corollaries.[18]

Since some trigonometry is being included now in many high-school geometry texts, a brief account will be given of its development. Ahmes used a relation equivalent to the co-tangent of an angle in his shadow reckoning. Astronomers found that for a given angle in a circle of given radius the chord was constant. They made a table of chords (140 B.C.) and of half chords (510 A.D.) which would correspond closely to our sine.[19] As late as 1560 one writer uses *perpendiculum* for sine. Chord and half-chord were used with sine for many years, giving both a geometric and algebraic meaning to the function. The tangent originated as the result of shadow reckoning.

Trigonometry more than any other branch of mathematics seems to be the product of many men, no one of whom made any such outstanding contribution as did Euclid or Descartes for geometry. Although its

[16] Smith: *op. cit.*, Vol. I, p. 116f.
[17] *Ibid.*, p. 376.
[18] *Ibid.*, p. 382.
[19] *Ibid.*, Vol. II, p. 614.

origin dates back to Ahmes in 2300 B.C., trigonometry did not assume its present form until algebraic symbolism was perfected in the seventeenth century. It has been incidental to geometry because it depended on tables and consequently needed more than ruler and compasses for its use.

Other outstanding discoveries in the field of mathematics have accompanied geometry but have had less effect on it even than did trigonometry, although geometry has been indispensable to them. The contributions of Newton, Leibniz, Euler, Lagrange, Gauss, Laplace, Legendre, dating from 1680 to 1800, with their development of calculus, theory of numbers, least squares, and elliptic functions, mark bright spots in the progress of mathematics, without which our present civilization would not be possible. The work of these men would likewise have been impossible without the foundation work of Thales, Pythagoras, Plato, Euclid, Descartes, and other pioneers.

The brevity of this historical treatment may have left the impression that Euclid's *Elements* was handed down unmodified or unimproved from his time to the present. Such is far from the truth, as any study of the "Elements" will soon disclose. Many modifications were made by able mathematicians, mostly from France, Germany, and England.[20] Some of these were slight and some radical. Mention will be made here of only one of the most outstanding, that by A. M. Legendre in 1794. Legendre, although he abandoned to some extent the sequence of Euclid, was logically sound and so maintained the respect of mathematicians. He differed from Euclid in several other respects. He referred to arithmetic and algebra for the treatment of proportion and assumed the correspondence between line segment and number. Euclid insisted that all constructions as well as theorems be proved before they could be used; Legendre permitted "hypothetical constructions." In general, his modifications made the work simpler without sacrifice of rigor.

Non-Euclidean Geometry. Even a brief history of geometry is incomplete without some mention of the "non-Euclidean" as well as the modern algebraic and Euclidean geometries. Non-Euclidean geometry originated out of attempts to prove Euclid's parallel postulate. As stated by Euclid, his postulate was essentially this: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles."[21] The

[20] Stamper. *op. cit.*, pp. 55-103.

[21] Heath, T. L. *The Thirteen Books of Euclid's Elements.* Cambridge University Press, London, 1926. Second Edition. Vol. I, p. 202.

modern form, which is known as Playfair's postulate of parallels, is essentially this: "Through a given point only one parallel can be drawn to a given straight line."[22, 23] In the effort to prove the parallel postulate by an indirect proof, the opposite was assumed to be true, that more than one line can be drawn through a point parallel to a given line. No contradiction could be found, however, but the technique succeeded in building up another geometry and in showing that Euclid's postulate of parallels was independent of all others and consequently could not be proved by using them. This work probably originated with Gauss but was brought to a definite conclusion by two men working independently, Bolyai and Lobatchevsky (1823). They showed, for instance, that with the above assumption the sum of the angles of a triangle would be less than two right angles.[24] Analysis of their theorems shows them to be concerned with figures on a surface of negative curvature; and, consequently, ~~their~~ geometry has been called hyperbolic geometry.

In 1854 Riemann suggested a similar geometry based on the assumption that not even one parallel to a given line can be drawn through a given point. This resulted in a surface of opposite curvature and in a geometry called elliptic geometry. This leaves Euclid's geometry between them on a flat surface, and it is sometimes called parabolic geometry.[24]

In a stimulating article in the *Mathematics Teacher*, December 1922, Professor W. H. Bussey of the University of Minnesota gives an excellent summary of non-Euclidean geometries. He quotes some of the theorems which can be proved:—Lobatschevsky: (1) The sum of the angles of a triangle is less than two right angles; (2) In a trirectangular quadrilateral the fourth angle is acute; Riemann: (1) The sum of the angles of a triangle is greater than two right angles; (2) In a trirectangular quadrilateral the fourth angle is obtuse; (3) "There are no similar figures"; (4) "A straight line is the limit approached by a circle where the length of the radius approaches one-half of a line-length."[25]

It is easy to see that on a curved surface such as a sphere on which a straight line is a great circle, that not even one line could be drawn through a point which would be parallel to a given line. Although we are living on the spherical surface of the world, and Einstein has presented evidence to prove that space also is curved, yet what we see is so apparently flat that the Euclidean geometry is the accepted form. The

[22] Smith. *op. cit.*, Vol. II, p. 283.
[23] Heath. *op. cit.*, p. 220.
[24] Smith. *op. cit.*, Vol. II. p. 388.
[25] Bussey, W. H. "Non-Euclidean Geometry." *Mathematics Teacher*, Dec. 1922, p. 445.

others are interesting largely for their emphasis upon the necessity for accepting certain postulates without proof in order to build a system of logic based upon them.[26] They emphasize further the arbitrary nature of those postulates so far as the logic of the system is concerned.

Geometry has had a long and vigorous history. Probably the outstanding feature of it is the manner in which the ideas of Plato and Euclid have dominated plane demonstrative geometry for over 2000 years, barring most variations. As geometry teachers and students we have been blind followers of Euclid, unaware of other geometry materials equally valuable and other leaders equally challenging.

Geometry as a School Subject. A brief history of geometry as a school subject should be a fitting close to this section. The study of Euclid was prescribed at Oxford in the thirteenth century. By the latter half of the fourteenth century candidates for the master's degree were studying, at most, the first six books of the 'Elements.' For the bachelor's degree little or no Euclid was required. When Harvard was founded in 1636, arithmetic and geometry were taught one day a week for three-fourths of a year in the last year of the course. In 1726 printed texts began to appear, and Euclid was taught at Harvard and Yale in the fourth year. It was placed in the second year at Yale in 1744 and at Harvard in 1787, but not until 1818 was it a first year subject. In 1844 Harvard required geometry for entrance, but only the elementary notions. In 1887 all of plane geometry was required for entrance to Harvard. As early as 1818 Phillips Exeter Academy offered geometry in the fourth class of the classical college preparatory course. Geometry has constantly crept downward until now much of it is taught before students enter the senior high school.[27]

The early textbooks used, following Euclid's own compilation in 300 B.C., began with many translations and copies before printing was invented, first with that of Theon of Alexandria in 400 A.D. The first edition of Euclid to be printed in Latin was from the Adelard-Campanus translation from Arabic by Eunest Ratdolt at Venice in 1482.[27] The first English translation was by Henry Billingsley in 1570.[27] One of the most widely used English translations was by Isaac Barrow in 1655.[28] In 1795 the first course in plane and solid geometry was compiled in Playfair's Euclid, but the first radical departure from Euclid was by Legendre, who abandoned somewhat Euclid's sequence, in 1794. This

[26] Hassler and Smith. *The Teaching of Secondary Mathematics.* The Macmillan Co., 1930, pp. 89-101.
[27] Stamper. *op. cit.*, pp. 52, 96ff.
[28] Sanford. *op. cit.*, p. 275.

was followed by texts by Davies in 1840 and Chauvenet in 1870 and Todhunter in 1889.[29] Since then many geometry texts have been published differing in sequence, form, and applications, but only slightly in actual content from Euclid's *Elements*.

The method of teaching geometry has also a history. "The early Greeks used the Socratic method, while in the early universities the pupils learned by copying from dictation or lecture. During the seventeenth century in Germany demonstrative work began to be emphasized and in the eighteenth century the custom of students' explaining propositions was common. Although the nineteenth century saw the dogmatic method generally discredited, yet traces of it remained especially in England and the United States."[30]

It is comparatively recent that the possibilities of geometric originals have been realized and that the applications of geometry have been pointed out as a pattern of reasoning which may apply to other than geometric facts. Schultze claims that "a course in geometry should be principally a course in the methods of attacking original exercises; the regular book demonstrations should follow as by-products of such a course."[31] As indications of the applications of the pattern of geometric reasoning to social situations and consequently of geometry as training in analytic thinking, Keyser, *Thinking About Thinking*,[32] Keyser, *Pastures of Wonder*,[33] and Upton, *The Indirect Proof in Geometry and in Life*[34] are excellent illustrations.

II. The Function of High-School Geometry

Comparison of Intuitive and Demonstrative Geometry. The purpose of teaching demonstrative geometry in the senior high school is not primarily to give the student information concerning the facts of space relationships. He does not study demonstrative geometry merely to learn that the sum of the angles of a triangle equals 180°, that two right triangles are similar under certain conditions, that the area of a circle equals πr^2, or that the volume of a cone equals $\frac{1}{3}\pi r^2 h$. Many of these facts are included in the ordinary seventh, eighth, and ninth grade

[29] Stamper. *op. cit.*, p. 33.

[30] Stamper. *op. cit.*, p. 102.

[31] Schultze, Arthur. *The Teaching of Mathematics in Secondary Schools.* The Macmillan Co., 1914, p. 99.

[32] Keyser, C. J. *Thinking about Thinking.* E. P. Dutton and Co., 1922.

[33] Keyser, C. J. *Pastures of Wonder.* Columbia University Press, 1930.

[34] Upton, C. B. *The Indirect Proof in Geometry and in Life.* Fifth Yearbook of the National Council of Teachers of Mathematics. Bureau of Publications, Teachers College, Columbia University, New York, 1930, pp. 102-133.

courses. Table II shows the results of a comparative study of six sets of junior high-school textbooks together with three typical senior high-school geometry textbooks.

Table II, Part A, shows the geometry vocabulary found to be used in junior high-school textbooks in mathematics. The terms are classified as follows: terms used in connection with (1) Lines, Points, and Segments; (2) Angles; (3) Triangles; (4) Polygons; (5) Circles; (6) Solids; (7) Instruments; (8) General Terms. The table indicates the

TABLE II

Vocabulary, Abilities, and Theorems of Geometry Used in at Least One-half of the Junior High-School Courses as Found from an Analysis of Six Sets of Modern Junior High-School Mathematics Textbooks, and Additional Terms, Abilities, and Theorems Used by Two out of Three Typical Senior High-School Geometry Textbooks.

	In Junior High School	*Additional in Sr. H.S.*
A. Vocabulary. The number of terms in connection with		
1. Lines, points, and segments		
2. Angles	31	12
3. Triangles	11	5
4. Polygons	14	16
5. Circles	6	8
6. Solids	10	0
7. Instruments	9	1
8. General terms	25	25
B. Abilities. Number of different abilities involving		
1. Direct measurement or computation from direct measurement	21	2
2. Manipulation of measures	7	0
3. Use of instruments	10	0
4. Indirect measurement	8	1
5. Making constructions	18	30
6. Drawing designs based on geometric figures	4	0
7. Constructing graphs	3	0
8. Miscellaneous abilities	3	2
C. Geometric principles used.		
1. Axioms and postulates used	9	20
2. Theorems mentioned	38	106
3. Theorems Proved	5	139 (226)

List of texts analyzed:

1. Wentworth-Smith-Brown: *Junior High School Mathematics*. Ginn and Co. 1926.
2. Gugle, Marie: *Modern Junior Mathematics*. Gregg Publishing Co. 1920.
3. Schorling and Clark: *Modern Mathematics*. World Book Co. 1927.
4. Hamilton, Bliss and Kupfer: *Essentials of Junior High School Mathematics*. American Book Co. 1927.
5. Stone, J. C.: *The New Mathematics*. Benj. H. Sanborn and Co. 1927.
6. Breslich, E. R.: *Junior Mathematics*. Macmillan Co. 1927.
7. Nyberg, Joseph A.: *Plane Geometry*. American Book Co. 1929.
8. Wells and Hart: *Modern Plane Geometry*. D. C. Heath & Co. 1926.
9. Smith, Foberg, Reeve: *General High School Mathematics*. Book II. Ginn and Co. 1926.

number of different terms used by at least half of the books analyzed. All terms used by less than 50% of the books were not included in the table. Table II, Part B, shows the abilities of geometry developed, or at least taught, in the junior high school. These abilities are classified as: (1) direct measurement or computation from direct measurement, (2) manipulation of measures, (3) use of instruments, (4) indirect measurement, (5) making constructions, (6) drawing designs, (7) constructing graphs, (8) miscellaneous. Again all abilities not presented by at least 50% of the texts were not included in the table.

Part C of Table II gives the number of axioms, postulates, and both the theorems mentioned and those proved in the junior high-school texts. This shows that in so far as textbooks indicate what is taught in the junior high school, a large amount of geometry as information is included. However, Part C, 3, indicates that very little emphasis is placed upon proof of this material in the junior high school.

The last column of Table II is a summary of a similar study for senior high-school geometry, and indicates the number of additional terms, abilities, and theorems of plane geometry which are given in the plane geometry course but are not included in 50% of junior high-school texts. For instance, the 26 terms concerned with lines, points, and segments which were not included in at least 50% of the junior high-school textbooks were such terms as auxiliary lines, concurrent lines, median of a trapezoid, common chord, common tangent, line of centers, centroid, excenter, circumcenter, orthocenter, apothem, projection, tangent. Many of these were mentioned in some of the junior high-school texts but not in half of them. It is evident from this table that some information in the form of vocabulary used and abilities developed is added in the senior high-school geometry course, but it is also evident that the outstanding addition is the proving of theorems. Only 5 theorems were proved in the junior high school while 139 additional ones were proved by two out of three of the senior high-school demonstrative geometry texts analyzed. Further study of the data indicates that 226 additional theorems were proved by one of the three texts analyzed.

While it is not contended that a student of demonstrative geometry, by demonstrating facts and principles of geometry with which he was previously intuitively familiar, fails to improve his mastery of those facts, it is contended that such an increased mastery of facts is only a by-product. The outstanding contribution of geometry, the element which has made it interesting to thinking men for 4000 years, the part which thrills children when it is correctly taught, is its logical structure, its organized reasoning with simple concepts, its inherent possibilities for producing in children the satisfaction of significant achievement.'

The Chief Function of Demonstrative Geometry. The National Committee on the Reorganization of Mathematics states that the principal purposes of instruction in plane geometry are: "To exercise further the spatial imagination of the student, to make him familiar with the great basal propositions and their applications, to develop an understanding and appreciation of a deductive proof and the ability to use this method of reasoning where it is applicable, and to form habits of precise and succinct statement, of logical organization of ideas, and of logical memory."[35] The second of these, "to know the great basal propositions," is an objective involving largely information. The others are all of a more or less indirect, concomitant nature based on the system of logic represented by the simple concrete facts and principles of geometry.

Other statements of the function of demonstrative geometry could be quoted almost indefinitely but only a few will be presented here. Smith and Reeve,[36] in a chapter on demonstrative geometry state that "the real purpose of the subject is suggested more by the word 'demonstrative' than by the word 'geometry.' The chief purpose of this part of mathematics is to lead the pupil to understand what it is to demonstrate something, to prove a statement logically, to 'stand upon the vantage ground of truth.'" Reeve's statement of the purpose of geometry in the *Fifth Yearbook of the National Council* is even more forceful.[37] "The purpose of geometry is to make clear to the pupil *the meaning of demonstration, the meaning of mathematical precision, and the pleasure of discovering absolute truth.* If demonstrative geometry is not taught in order to enable the pupil to have the satisfaction of proving something, to train him in deductive thinking, to give him the power to prove his own statements, then it is not worth teaching at all."

Professor Upton in the *Fifth Yearbook* claims that, "some teachers may at first think that our purpose in teaching geometry is to acquaint pupils with a certain body of geometric facts or theorems or with the applications of these theorems in everyday life, but on second reflection they will probably agree that our great purpose in teaching geometry is to show pupils how facts are proved ... the purpose in teaching geometry is not only to acquaint pupils with the methods of proving geometric facts, but also to familiarize them with that rigorous kind of thinking which

[35] National Committee Report. *op. cit.*, p. 43.

[36] Smith, D. E. and Reeve, W. D. *The Teaching of Junior High School Mathematics.* Ginn and Co., 1927, p. 229.

[37] Reeve, W. D. "The Teaching of Geometry." Fifth Yearbook, p. 13f.

Professor Keyser has so aptly called 'the If-Then kind, a type of thinking which is distinguished from all others by its characteristic form. If this is so, then that is so.' . . . Our great aim in the tenth year is to teach the nature of deductive proof and to furnish pupils with a model for all their life thinking."[38]

Schlauch, in his chapter on the analytic method in the *Fifth Yearbook*, asserts: "Geometry seems, of all school subjects the best adapted to initiate a student into the meaning of mathematics as a science of necessary conclusions."[39]

The statement of the purpose of demonstrative geometry made by two Harvard men, Professors Birkhoff and Beatley, is interesting. "In demonstrative geometry the emphasis is on reasoning. This is all the more important because it deepens geometric insight. To the extent that the subject fails to develop the power to reason and to yield an appreciation of scientific method in reasoning, its fundamental value for purposes of instruction is lessened. There are, to be sure, many geometric facts of importance quite apart from its logical structure. The bulk of these belong properly in the intuitive geometry of grades VII and VIII, and are not the chief end of our instruction in demonstrative geometry in the senior high school."[40] Professor W. R. Longley,[41] of Yale, expresses much the same sentiment, although he stresses also the practical value of knowledge of the formulas, facts, relations, and methods used in geometry.

E. R. Breslich, of the University of Chicago, gives many specific objectives, not only for geometry but for the various units of geometry. The contributions of geometry to these various objectives include a knowledge of the facts of geometry, and also the power "to analyze geometric situations," "to attack and solve problems of space," "to establish geometric facts by proof," "to reason correctly."[42]

C. H. Judd, in his *Psychology of Secondary Education*, in speaking of the purpose of high-school education, claims: "Higher education is organized for the purpose of giving pupils insights. Ability to use numbers as necessary instruments of civilized life is usually acquired by the time the pupil comes to the high school, at least in sufficient measure to meet ordinary demands. Whatever justification there may be in ele-

[38] Upton, C. B. "The Use of the Indirect Proof in Geometry and Life." pp. 131-132.

[39] Schlauch, W. S. "The Analytic Method in the Teaching of Geometry." p. 134.

[40] Birkhoff, G. D. and Beatley, Ralph. "A New Approach to Elementary Geometry." *Fifth Yearbook*, p. 86.

[41] Longley, W. R. "What Shall We Teach in Geometry." *Fifth Yearbook*, p. 29.

[42] Breslich, E. R. *The Teaching of Mathematics in Secondary Schools*. University of Chicago Press, 1930. Vol. I, p. 203.

mentary arithmetic for a simpler type of treatment of mathematical ideas, *there is no justification in algebra and geometry for mere mastery of formulas and repetition of textbook demonstrations.* . . . The duty of higher education is to conserve all that has been achieved in the lower school and constantly to direct the pupil's attention to higher forms of generalized or scientific thinking."[43]

C. J. Keyser, Emeritus Professor of Mathematics at Columbia University, contends that "mathematics may be viewed either as an enterprise or as an achievement. As an enterprise it is characterized by its aim, and its aim is to think rigorously whatever is rigorously thinkable or whatever may become rigorously thinkable in the course of the upward striving and refining evolution of ideas. As a body of achievements, mathematics consists of all the results . . . from the prosecution of the enterprise."[44]

W. W. Hart, from the University of Wisconsin, claims that "Demonstrative Geometry uniquely develops the habit of deductive thinking, . . . more important than the 'habit of functional thinking.' This habit is based upon the appreciation of it and the use of it in geometry. . . . The training in demonstration should come from the solution of originals, and this must be made the chief aim of the course."[45]

From England the pen of John Perry informs us that "we pay teachers to give us something that will be useful in our education and useful to us in life, useful to us in understanding our position in the universe. . . . One use of Mathematics is giving men mental tools as easy to use as their arms or legs; enabling them to go on with their education." Further, he contends that mathematics should teach "a man to think things out for himself and so deliver him from the present dreadful yoke of au thority."[46]

Summary. In review of these statements of the function of demonstrative geometry in the tenth grade, and in restatement of the point of view represented by this study, the following organization can be used.

(1) Practical, immediate, or direct aims. There is no question concerning the practical value of knowing that the sides of similar triangles are proportional, or that an angle may be constructed equal to another

[43] Judd, C. H. *The Psychology of Secondary Education.* Ginn and Co., 1927, p. 111.
[44] Keyser, C. J. *Human Worth of Rigorous Thinking.* Columbia University Press, 1916, p. 3.
[45] Hart, W. W. "Purpose, Method, and Mode of Demonstrative Geometry," *Mathematics Teacher,* XVII, 1924, pp. 172-176.
[46] Perry, John. "British Association Report," *Teaching of Mathematics.* Macmillan and Co., 1901, pp. 4, 5.

by a definite process. Furthermore, there is also practical value in learning to make precise statements, to appreciate the need and value of definitions, to feel the power and the technique of rigorous deductive proof, to analyze a complex situation into simpler parts, to discover and to prove a general truth.

Then too, to know the meaning of "Pythagorean theorem" or an "inscribed regular polygon," as well as to understand a reference to "a pound of flesh" or "cosmic radiation," is an indication of culture. To see the beauty in the geometrical forms of nature, art, and industry, and to appreciate the power and perfection of logical reasoning are achievements which educated people like to possess.

(2) Indirect, transcendent, or concomitant values. The simple concepts with which geometry deals give it a peculiar function. Geometry achieves its highest possibilities if, in addition to its direct and practical usefulness, it can establish a pattern of reasoning; if it can develop the power to think clearly in geometric situations, and to use the same discrimination in non-geometric situations; if it can develop the power to generalize with caution from specific cases, and to realize the force and all-inclusiveness of deductive statements; if it can develop an appreciation of the place and function of definitions and postulates in the proof of any conclusion, geometric or non-geometric; if it can develop an attitude of mind which tends always to analyze situations, to understand their inter-relationships, to question hasty conclusions, to express clearly, precisely, and accurately non-geometric as well as geometric ideas.

There seem to be certain ordinary, practical, direct values which are easy to get, and also some superior, transcendent values which are possible but not certain. These superior values depend greatly, perhaps largely, upon the way geometry is taught, and consequently are not attained by all teachers, nor by all classes, nor perhaps completely by any teacher or class. They constitute an ideal, and depend upon the realization that geometry is not a bag of tricks to be performed, not merely information to be learned, nor is it a list of rules to be memorized. It is rather a fundamental system of logic to be understood, it is an organization of universal truth to be appreciated, it is a pattern of reasoning to be emulated.

III. Some Settled and Some Unsettled Difficulties

1. THE FOUNDATIONS OF GEOMETRY: POSTULATES, AXIOMS, UNDEFINED TERMS, AND DEFINITIONS; THEIR NATURE AND NUMBER

Postulates. The fallacious and inadequate conception that a postulate or an axiom is a "self-evident truth," is still present in many textbooks of geometry. It is a new idea even to many geometry teachers that postulates and axioms are not necessarily true, in fact there seems to be some evidence that some of Euclid's are false and that the world in which we live is not a Euclidean world. However that may be, the important fact about postulates is, not that they may be true or false, but that they are merely statements which are accepted without proof, and that as such, they constitute the foundation of geometry.

As expressed by Descartes, Pascal's conception of "the true method" was "to define all terms and to prove all propositions."[47] In contrast to Pascal's statement is the statement by Veblen that "in geometry each technical term is defined in terms of others. Hence at the beginning at least one term must be undefined else there would be no beginning. . . . Similarly every proposition is based on others from which it is deduced. Therefore, first ones must be assumptions."[48] Also, Aristotle stated that "every demonstrative science must start from indemonstrable principles; otherwise the steps of demonstration would be endless."[49]

In other words, since there must be a beginning somewhere, there must be some terms undefined and some relationships unproved; some "primitive ideas" and some "primitive propositions."[50] Since these first relations cannot be proved, we do not know that they are true, and consequently modern geometry makes no such claim. Furthermore, all theorems proved by the use of the postulates are true only provided the postulates are true; that is, they are true only relatively.

Influenced by the idea which Pascal has called "the true method," mathematical geniuses have constantly endeavored to reduce the number of postulates to as few as possible. This effort has thrown much light on the question of the truth of postulates. The parallel postulate has seemed of all postulates the most susceptible to proof. Efforts to prove it have been previously discussed. The three postulates, "Only one line can be

[47] Jevons, W. S. *Elementary Lessons in Logic.* Macmillan Co., 1900, p. 112.

[48] Veblen, Oswald. "The Foundations of Geometry." Chapter I of: Young, J. W. A. *Monographs on Modern Mathematics.* Longmans, Green and Co., 1911, p. 4.

[49] Heath. *op. cit.*, p. 119.

[50] Russell, B. and Whitehead, A. N. *Principia Mathematica,* Vol. I. Cambridge University Press, 1910, p. 1.

drawn through a given point parallel to a given line," "More than one line can be drawn through a given point parallel to a given line," and "Not even one line can be drawn through a point parallel to a given line," cannot all be true. Euclidean geometry is based on the first one of these postulates, and each of the two non-Euclidean geometries is based on one of the other two; yet all three geometries have their other postulates essentially the same. Each of these geometries is a perfectly logical system without contradiction; and, consequently, neither one can be said to be true and the others false. All that we can therefore say about these postulates or any others is that, since they are not necessarily true, they are merely assumptions upon which the rest of geometry is built.

Geometry becomes much more meaningful if the postulates are thought of as arbitrary statements, not necessarily true, but accepted as true without proof. They are, in a sense, merely the "rules of the game." They are the foundation principles upon which the whole reasoning structure is built. Forder speaks of them as "unproved propositions about undefined entities."[51]

Axioms and Postulates. It has been customary in high-school geometry to distinguish between axioms and postulates, axioms being rather general statements such as, "Equals may be substituted for equals," and postulates being considered as geometric statements such as, "Only one line can be drawn between two points." While this distinction still prevails in high-school geometries, it is no longer in accord with current usage in modern mathematics on the higher level. "Postulate, assumption, axiom, primitive proposition, and fundamental hypothesis . . . are being used interchangeably according to the taste of the author."[52]

Undefined Terms. The foundation of geometry consists, not only of relationships called axioms or postulates which are accepted unproved and which Einstein speaks of as "implied definitions,"[53] but also of terms which are accepted without rigorous definition, such as point, line, plane, solid, equal, greater than, less than, between, outside, length, distance, area, straight, direction, erect and draw. These terms may be described or explained but really cannot be satisfactorily defined by the use of concepts more simple than themselves.

[51] Forder, H. G. *Foundations of Euclidean Geometry.* Cambridge University Press, London, 1927, p. 4.

[52] Keyser, C. J. *Mathematical Philosophy.* E. P. Dutton and Co., 1922, p. 40.

[53] Einstein, Albert. *Sidelights on Relativity.* Methuen and Co., Ltd., 36 Essex St. W. C., London, 1922, p. 45.

Area is defined in *Webster's International Dictionary* as "the superficial contents of any figure, the surface included within any given lines." It would then be necessary to define "superficial contents" and "surface," which are more complicated than the term area itself. In his *Foundations of Euclidean Geometry*, Forder[54] gives a rigorous definition of area. He proves first by similar right triangles that the product of the base and altitude of a triangle is constant, and then defines one-half this product as the "measure" or area of the triangle. In this case the definition still depends on many complicated terms which may themselves be left undefined and this definition would consequently be useless in beginning geometry work. There is in this study a frank admission of the use of many undefined technical terms whose meaning is fairly well known. The reader must carry in his mind any image or meaning of an undefined term "which he can reconcile with what is said about it."[55] The statements that a point has no dimensions, a line has length only, a surface length and width, or a solid has three dimensions, help to clarify the meanings, but they are not definitions. They are explanations and descriptions only.

Definitions. The statement that many terms are used without definition must not be erroneously carried too far. Very carefully worded definitions are needed for many terms in order that their meaning may be clear and statements made concerning them may be understood. For instance, if a trapezoid is defined as a quadrilateral with one pair of parallel sides, or if defined as a quadrilateral with one and only one pair of parallel sides, the meaning is quite different. Definitions are a very important part of geometry: they make it precise, they make it unambiguous, they help to make it a science. Yet it is evident that there will of necessity be some beginning terms left undefined, and also that definitions, like postulates, are arbitrary. That is, one author may define a trapezoid one way and another author may define it differently; yet each, if consistent, would have an equally rigorous geometry. Fortunately, the recommendations of the National Committee on common and universal usage prevent significant variations in the definitions of terms used.

Furthermore, the second function of a definition, aside from its use in clarifying notions of common concepts, is its use as a "symbolic convenience," or as Bertrand Russell says, "a typographical convenience."[56] "From a strictly logical point of view a definition is the assignment of

[54] Forder. *op. cit.*, p. 261. [55] Veblen. *op. cit.*, p. 5.

a short name to a lengthy complex of ideas."[56] Frequently the full meaning of a statement which uses a newly defined term is made clear only by substituting the definition for the term. One needs only to read "Principia Mathematica" or even Ramsey's[57] defense of Whitehead and Russell's technique to realize the extensive use and great convenience of arbitrary symbols which are merely the shorthand for a lengthy complex of ideas. Beginning with the simple use of letters in formulas, mathematics makes constant use of arbitrary symbols which it defines in a certain way in order that in using those symbols time and energy may be saved. Even such simple devices as the use of three letters, "A.S.A.," for a whole theorem concerning the congruence of triangles is a legitimate substitution of a symbol for a "lengthy complex of ideas." In the next chapter the term "heuristic teaching" will be arbitrarily defined and used to mean a fourfold technique of teaching. Definitions therefore have a double function: they secure brevity and clarity.

In this study many terms will be left undefined yet careful definitions will be required for other terms. Postulates will be treated as unproved statements of relationships which are accepted and used to prove other relationships. It is therefore evident that the proved relationships are true only if the postulates and definitions are true, and that the science of geometry consists in establishing this dependence. In other words we are given certain postulates, undefined terms, defined terms, and perhaps certain other premises; then, assuming these without proof, other relationships are proved.

Number of Postulates. Since geometry is a system of reasoning built upon certain arbitrary definitions and postulates, it has seemed to mathematicians and logicians since the time of Euclid that it would be desirable to have just as few postulates as possible. Even now that is a real objective for pure mathematics. Forder's new book *The Foundations of Euclidean Geometry*, is an illustration of continued effort along this line. "The object of this work is to show that all the propositions of Euclidean geometry follow logically from a small number of axioms explicitly laid down, and to discuss to some extent the relations between these axioms."[58] Forder then proceeds to discuss the "many flaws" that have been discovered in Euclid's treatment during the last 2000 years, particularly his omission of the relations of "between," "inside," and "outside," which are so important in any deductive reasoning.

[56] Stebbing, L. S. *A Modern Introduction to Logic.* Thomas Y. Crowell Co., New York, 1930, pp. 180, 440.
[57] Ramsey, F. P. *The Foundation of Mathematics and Other Logical Essays.* Harcourt, Brace and Co., London, 1931, pp. xvii, 212-236.
[58] Forder, Henry G. *op. cit.*, p. 1.

The attempt by Euclid and many of his followers to prove the parallel postulate is an indication of the effort to reduce the number of postulates. Euclid had proved several theorems by using certain postulates, then had to add a new one to prove the parallel line relations. As worded by Playfair, this postulate is: "One and only one line can be drawn through a given point parallel to a given line." The effort to prove this true by assuming it false led to the discovery of one non-Euclidean geometry by Bolyai and Lobatschefsky, in which all of Euclid's axioms and postulates hold except the parallel postulate and in which all of Euclid's theorems except those based on this postulate are true. Further effort to prove this postulate led Riemann to discover a second non-Euclidean geometry sometimes called elliptical geometry.[59]

These great achievements in the history of human thought are largely the by-products of an effort to reduce the number of axioms and postulates. While that may be a desirable objective for mathematicians and logicians, it is not a defensible objective for high-school geometry. We have, in contradiction to this historic motive, the statement by the National Committee on the Reorganization of Mathematics that, not only should we postulate "all right angles are equal," but also such theorems as "the area of a rectangle equals the product of the base and height," and "vertical angles are equal." Furthermore, the report says that the list given is not exhaustive but "should be taken as representative of the type of propositions which may be assumed."[60] The trend in high-school geometry seems clearly to be that of postulating many relationships which were previously proved, although the very obviousness of the relationship made the proof in many cases rarely understood by high-school pupils. This trend is desirable and defensible in that it makes geometry more easily grasped at the beginning of the course. The training in reasoning is not less valuable because it increases the number of axioms and postulates, but rather more valuable because it emphasizes more the nature of the postulate basis upon which geometry is built.[61, 62]

The Use of Postulation as a Teaching Technique to Secure a Better Learning Situation. In this study there will be no attempt to limit the number of postulates to a minimum merely for the purpose of getting absolute "independence." There will be two criteria used in determining the system of postulates: (1) "Consistency" and (2) "Understandability" or simplicity. Wherever there seems to be some great gain through

[59] Heath, T. L. *op. cit.*, pp. 202-220, 280.
[60] National Committee Report, *op. cit.*, p. 79.
[61] Forder, H. G. *op. cit.*, pp. 4-5.
[62] Heath, T. L. *op. cit.*, pp. 117-124, 195-240.

temporary or permanent postulation of a proposition, such postulation shall be used in this study. For instance, in the beginning of the geometry course the principle technique to be gained is the use of the congruence theorems in solving problems. Their proof by superposition is not only long and difficult, but also a conflicting pattern that is not useful. Furthermore, their postulation makes it possible to begin geometry with simple constructions in which there is a motive for proving the construction correct. Consequently, the congruence theorems will be postulated in this study. The proof of the converse of the first principal theorem on parallels involves the use of indirect proof at a time when direct proof has not been completely mastered. It may be advisable to postulate this converse theorem until the next theorem involving indirect proof, and then prove both theorems by the indirect proof. By this plan the student will get a better presentation of and experience with indirect proof.

While this plan for using postulation as a means of making geometry more teachable is a slight extension of the notion of postulates as the foundation of geometry, yet it is fully in accord with the conception of a postulate, not as a self-evident truth, but as an unproved proposition which is assumed in order to get a beginning somewhere. For mature minds that beginning may well be with a minimum list of postulates, but for the mind of a high-school or college student it may make geometry more learnable to shift the foundation a little by following the recommendations of the National Committees for a greater number of postulates and even going a step further by using temporary or permanent postulation of theorems whenever there is a possible gain in interest or an improved learning organization.

Systems of Postulates. It is not the function of this study to present in great detail the various systems of postulates that have been devised since Euclid submitted his incomplete list. It must suffice here merely to list some of the more noted ones and suggest that the student who wishes to delve more deeply into these systems may do so.

1. Euclid—300 B.C.
2. Pasch—1882
3. Veronnese—1891
4. Hilbert—1903
5. Veblen—1911
6. Forder—1927

Of these the most famous is the system of 21 postulates given by Hilbert. His book will be interesting reading for any philosophically minded mathematician. Practically every textbook has a slightly different system of postulates and between Euclid and Pasch there were many changes made in Euclid's system by men like Wallis (1616), Saccheri (1733), Lambert (1728), Legendre (1752), Bolyai (1802), Riemann (1826). However, the first great extension of Euclid's system of postulates was made by Hilbert. Since then modern texts by men like Smith, Reeve, Clark, Stone, Hart, Hawkes, Strader, Nyberg, Mirick, Seymour, McCormick, Durell, Taylor, R. Smith, Hassler, and others, each have slight modifications of the Euclidean system, inclusions of some of Hilbert's and Veblen's ideas, and a phraseology and organization somewhat individual. This is also true of definitions and of explanations of undefined terms. Consequently, since it is not unprecedented to present a slightly different system of postulates and a new organization of theorems, no hesitation is felt in submitting the following organization, which is slightly different from any and all previous systems. Changes have been made for economy and for simplicity; to shorten the list of essential theorems and to make the beginning more teachable by placing the emphasis not so much upon tradition as upon simple proof patterns.

2. SUPERPOSITION

The Rigor of Proof by Superposition Questioned. The proof of two congruence theorems and several other theorems in Euclid's *Elements* and in most modern geometry textbooks is based on the postulate of superposition. As early as 1557 (Peletier's edition of Euclid) the rigor of this proof was attacked. In the *Mathematical Gazette* (Volume II, p. 165) Bertrand Russell speaks of superposition in no uncertain terms as "a tissue of nonsense."[63]

Essentially, the criticisms against superposition take one of three forms. First, since a triangle is formed by line segments joining by pairs three points in space, and since a point has position but no dimension, and the line has no dimension except length, the triangle cannot be moved. If you move a point, it is no longer the same point, since all a point has is position, and every point has a different position; consequently every different position is a different point. The argument is that, rigorously speaking, since neither a point nor a line can be moved, it is unfortunate to base a system of reasoning about relationships in empty space on the fallacy that they can be moved.[63]

The assumption of movability clearly presupposes, according to Schopenhauer,[64] other than empty space; that a triangle or a sphere is composed of material substance that can be moved. Then, as Dodgson[65]

[63] *Teaching Geometry in Schools.* Report of the Mathematical Association. G. Bell and Sons, 1925, p. 27.
[64] Heath, T. L. *op. cit.*, p. 227.
[65] Dodgson, C. L. *Euclid and His Modern Rivals.* Macmillan and Co., 1885, p. 101.

points out, the reasoning is absurd. Take two oranges with equal diameters and place them together so that their centers coincide, or even two cardboard triangles, and attempt to put them in the position of actual coincidence, and the best you can do is to put one triangle beneath or beside the other with their angles in parallel planes and points directly opposite each other. Clearly, motion of plane spatial figures in empty space is impossible; and, strictly speaking, even superposition of equal concrete movable figures *for coincidence* is equally impossible.[66] Why then base our geometric reasoning upon such a questionable foundation?

The second argument is to the effect that the assumption that the postulation of superposition makes the proof rigorous is itself fallacious and the proof is actually not a real proof. In proving the congruence of two triangles having side-angle-side respectively equal, we superpose one triangle on the other so that these given parts coincide, and, consequently, the rest of the triangles. In assuming that one triangle can be moved about without change of size or form when we know only that side-angle-side remain constant, we are really assuming that these facts determine the rigidity of the triangle; and we have, therefore, really postulated congruence itself in postulating the possibility of movement with rigidity. The proof is, therefore, but a camouflaged proof involving a vicious circle of reasoning.[67, 68, 69] In further support of this statement is the interesting note by Forder: "It is a scandal that in some examinations questions still demand proofs involving this vicious method."[68]

Superposition Pedagogically Unsound. A third argument, that congruence should be postulated for high-school pupils because the proof by superposition is too long and difficult for a first proof, has some claim for attention also. Certain it is that most classes seem to struggle rather painfully with these proofs, and, consequently, it would be a distinct advantage pedagogically to have them postulated. Furthermore, superposition gives the pupil a wrong notion of demonstration at the outset, by bringing in a new type of proof at a time when the pupil's entire attention should be directed to the type used in proving exercises by the congruent triangle method. As the proof is traditionally presented, pupils are compelled to unlearn the method of proof by superposition as soon as it is learned.

[66] Heath, T. L. *op. cit.*, pp. 225-231.
[67] *The Teaching of Geometry in Schools*, pp. 32-35.
[68] Forder, H. G. *op. cit.*, p. 91.
[69] Heath, T. L. *op. cit.*, p. 227.

The Postulation of Congruence for Simplicity at Beginning. In this course the argument will be avoided by using all three congruence theorems as postulates. This corresponds rather well with Hilbert's noted system of postulates in which congruence by side-angle-side is practically assumed without proof. The other congruence theorems are then proved by Hilbert using an indirect proof.[70, 71] Logically, Hilbert's system is superior to the postulation of all three congruence theorems because his system is independent and does succeed in avoiding superposition. Forder and Veblen prove congruence by the use of axioms on congruence of ordered couples.[72, 73] The chief defense for complete postulation of all three theorems, in addition to the abandonment of superposition, is simplicity and understandability at the beginning of the course. For those who insist on using the proof by superposition, such proof can be brought in at a time of review or at some appropriate later date.

This apparent reluctance to accept superposition as a postulate or as a method of proof, with the consequent postulation of the three congruence theorems, is not to be construed as an attack on high-school textbooks which use superposition, nor upon high-school teachers who use superposition. The sensible interpretation of the situation can be summarized in a few concise statements. The question is not so much a matter of right or wrong choice of postulates, as a matter of better or worse choice of postulates. Some choice of postulates must be made. Either superposition must be postulated or at least one of the con uence relationships. In order not only to emphasize the better choice of postulates, but also to emphasize, for prospective teachers, the necessity for a choice, as well as to emphasize the nature and function of postulates, the latter course has been chosen in this study.

Motion Different from Superposition. Furthermore, the argument against superposition is not to be interpreted by those versed in metaphysics, as a denial of motion. Perhaps a geometry could be built which did not use any such concept, but the idea of motion is so fundamental that it will be used in this study as an undefined term or principle. The following brief analysis of a geometry without motion is suggestive of its implications and difficulties.

A compass has two points, that is, two indefinable somethings with zero dimensions just beyond the smallest particle of steel, and through those two points there is a straight line. As the compass is moved about,

[70] Heath, T. L. *op. cit.*, p. 229.
[71] Hilbert, David. *Foundations of Geometry*. Open Court Publishing Co., 1902, p. 12.
[72] Forder, H. G. *op. cit.*, pp. 97-103.
[73] Veblen, O. *op. cit.*, pp. 27-32.

does the line through those points move, or does the compass merely succeed in getting its points on other lines, infinite in number and zero distances apart? Perhaps the compass does not have points, and those zero dimensional positions just beyond the *sharp ends* of steel, as the compass is moved, in some way jump to various points, infinite in number and separated by infinitesimal distances. The question seems to become one of deciding when a point is a point, or what a point really is. Are there points on or near the "points" of a compass? If points cannot be moved, if points are merely space positions, then the answer to this question must be negative. If points cannot move, then we must invent a new word for the sharp end of a needle or a compass. Perhaps there are two kinds of points, points which move and points which do not move, real points and mathematical points.

It seems to be quite possible for an automobile to move, even though it results in dire calamity to attempt to superpose one on or into another of exactly the same size and shape so that they both occupy the same space at the same time. Furthermore, as a car moves, solids, planes, lines, and points on that car move; or at least they do what we ordinarily describe by the word move; they rotate, they revolve, they are translated on lines or planes. Although geometry seems to be, at least partly, a study of space relationships, it is not a function of this study to settle the philosophical argument concerning the motion of spatial figures. Consequently, although other postulates are chosen to be substituted for the postulate of superposition, the motion of points, lines, and planes, by translation or rotation, is accepted as an undefined, commonly understood, and fundamental principle.

3. THE POSTULATION OF CONGRUENCE

Tradition. Since the superposition proof of congruence by side-angle-side and angle-side-angle has been seriously questioned by prominent mathematicians and logicians, and since the number of postulates need not be kept small, at least for high-school geometry, there seems no good reason, other than blindly following tradition, why these two theorems cannot themselves be postulated and the postulate of superposition be abandoned. At least such postulation would greatly simplify the beginning of geometry and make possible many easy exercises and constructions.

A Near Fallacy. Some high-school geometries have already postulated the first two congruence theorems, but at this writing few have yet postulated the third. Before advocating the postulation of congruence by three sides, it would be well to examine the present proof of this theorem. An examination of any high-school text will show that the proof depends upon the isosceles triangle theorem and congruence by side-angle-side. The isosceles triangle theorem in turn depends upon angle bisection and congruence by side-angle-side; and finally, angle bisection, unless postulated, depends for its proof upon congruence by three sides. This is an excellent example of reasoning in a circle, or what is logically known as *petitio principii* (begging the question), unless some link in the chain is postulated. Traditionally, this postulation has been in the hypothetical construction of the angle bisector. The reasoning involved takes some such form as the following: Since every angle has a bisector, draw BD and assume that it is the bisector of angle ABC. The concealment of the postulation is cleverly done in many texts; in others very crudely done by the bare statement, "Draw BD bisecting angle ABC," the author little realizing that such angle bisection depends for its proof upon the very theorem in the proof of which it is now being used.

The postulation of the possibility of angle bisection has some obstacles to overcome also. No mathematician would think of postulating angle trisection even though every angle may have a line cutting off one-third of it, as well as one-half of it. The selection of angle bisection for postulation is a purely arbitrary decision made by Benjamin Pierce,[74] in 1872 and followed ever since. A. M. Legendre proved the same theorem in 1860 by using the hypothetical bisector of the base. There is no reason why either of the other two major links in the chain of reasoning referred to above could not have been postulated instead of angle or base bisection. If the isosceles triangle theorem, that the angles opposite the equal sides of an isosceles triangle are equal, were postulated, then congruence by three sides could be proved, and finally angle bisection. Likewise, if congruence by three sides were postulated, then angle bisection could be proved and, following it, the isosceles triangle theorem.[75]

Choice of Postulate Arbitrary. Since the decision here is entirely arbitrary or traditional and any one of the three postulates makes the reasoning equally rigorous, there seems justification for postulating congruence by three sides along with the other two congruence theorems. "It is to some extent a matter of taste which are selected as a basis of the rest."[76] Such postulation makes possible a rigorous proof of several important constructions which are necessary for proving later theorems, and con-

[74] Dodgson, C. L. *op. cit.*, p. 222.
[75] Christofferson, H. C. "A New Beginning for Geometry," *Mathematics Teacher*, Vol. XXI, pp. 479ff. Also Vol. XXII, p. 19.
[76] Forder, H. G. *op. cit.*, p. 90.

sequently necessitates few, if any, hypothetical constructions. Probably the chief defense for the postulation of all three congruence theorems, and especially of congruence by "three sides," is not so much any logical advantage that it may have, but rather is pedagogical superiority.[77] The use of superposition as a pattern of proof is immediately abandoned in all texts as soon as it is used. Consequently, its mastery has little practical value. Furthermore, the postulation of congruence makes it possible to begin geometry with constructions and to prove the constructions. Such a beginning gives a simple and forceful purpose for the proof of a statement and serves to motivate demonstration and to make it an activity that satisfies a felt need.

4. HYPOTHETICAL CONSTRUCTIONS

The Use of Hypothetical Constructions in High-School Geometry. The hypothetical construction used in proving the isosceles triangle theorem has already been discussed. The rigor of such a procedure is questioned because it often involves the postulation of something which will later be proved and which may follow in sequence. In some cases the violation of sequence is only seeming.

In the proof for the sum of the angles of a triangle, the usual proof is to draw a line through one vertex parallel to the opposite side. Then almost invariably on a following page, often 20 or 50 pages farther on, the text shows how to draw a line parallel to another line. For instance, in one very carefully written book, the proof for the sum of the angles of a triangle occurs on page 64 and the method of construction of a line parallel to another line on page 113. We have here a hypothetical construction where the violation of sequence is only seeming. The construction indicated depends solely on a theorem previously proved on page 58 and not on any of the material from page 59 to 113. As far as sequence is concerned, the construction could readily have preceded the proof of angle sum for a triangle thus avoiding the hypothetical construction of the parallel.

There is a much more vicious and cleverly concealed hypothetical construction in the proof of congruence by superposition. The well-known one in the proof of congruence by three sides has been mentioned, but the one concealed in congruence by side-angle-side usually escapes attention. If this proof depends upon the assumption that it is possible to construct a triangle with side-angle-side equal to side-angle-side of a

[77] Beatley, Ralph, "First Year of Demonstrative Geometry in Secondary Schools," *Mathematics Teacher*, Vol. XXIV, p. 214.

given triangle in order to prove the two congruent, then it has two hypothetical constructions both of which are based on theorems which follow in sequence. First, the construction of two angles which are equal is a hypothetical construction based in reality upon congruence by three sides, which theorem, in turn, is based upon congruence by side-angle-side. Second, the construction of a line segment equal to another segment by means of the compasses is in reality based directly upon congruence by side-angle-side, where the two arms of the compasses and the angle between them are the equal parts of two triangles in which the two equal segments form the third sides. These concealed hypothetical constructions are clearly a double violation of sequence which ought to be condemned in any system of logic.

If the theorem is cleverly stated the construction of the triangle is concealed. For instance, "If two triangles in which side-angle-side of one are equal respectively to side-angle-side of the other, could be imagined or should happen to exist somewhere; then they would be congruent," is a statement that avoids the construction of the triangles to meet the given conditions. However, it is evident that the actual construction on the blackboard or on paper of two separate triangles in which the given conditions are true must involve hypothetical constructions which violate sequence.

Hypothetical Constructions not Used in This Study. It shall be the aim of this course to eliminate all hypothetical constructions by the use of three postulates of congruence and the consequent proving of all necessary constructions before any theorems are demonstrated. This abandonment of hypothetical constructions is not to be interpreted as a defense of Euclid against Legendre, nor as a desire to abandon such constructions in the secondary school. It is done here in order to lay more emphasis upon sequence and the logical structure of geometry. In fact, good teaching which adjusts the difficulties to the individual abilities may find it necessary sometimes to use not only hypothetical constructions but the temporary postulation of theorems too difficult to prove at an early stage.

This abandonment of hypothetical constructions, of the form described above in which there is violation of sequence, is not to be construed as the abandonment of hypothetical constructions of the more refined type in which sequence is not involved. For instance, the theorem, that the angles opposite the equal sides of an isosceles triangle are equal, need not depend upon the accurate construction of the triangle nor even upon the existence of the triangle. Consequently, the free-hand drawing of a triangle, and even a bisector of the angle, merely to represent the possible triangle could in a sense be called a hypothetical construction. It is a construction that the hypothesis states or previous theorems or constructions have shown to be possible. Such constructions, included in the hypothesis or shown to be possible by previous proofs, are quite different from the more crude constructions made in the proof of a theorem which must itself be used to prove the construction. If the definition of hypothetical constructions would limit them to this more refined type, then no objection to them could possibly be raised.

5. SEQUENCE

Sequence Fundamental. Since geometry is essentially a system of reasoning, sequence is of the utmost importance. In proving any theorem only previously stated definitions and postulates or previously proved theorems can be used. Failure to observe this without exception is fatal to the logic of the system. There is, however, no best and necessary sequence of theorems all the way through the course.[78] Areas may be taught before or after similar triangles as far as straight line figures are concerned except for the relationship between the areas of similar figures. The locus theorems can be proved as soon as the congruent triangle theorems are completed, but need not be, since it may seem advisable to leave them until students have had experience with more geometry relationships.

No One Best Sequence. The following statements are all trite, but they seem necessary as a preface to the statement that there is no one demonstrated best sequence so long as there is no violation of sequence. Nearly every textbook has a sequence of its own; and, consequently, it seems justifiable to use another new sequence in this study since there is a definite reason for such a procedure. The postulation of congruence and the abandonment of hypothetical constructions makes possible many simple constructions easily made and rigorously proved. Upon these constructions and upon certain axioms, postulates, and undefined terms, as well as defined terms this study presents the essentials of the entire geometry course.

6. NUMBER OF THEOREMS

Many Theorems in High-School Ge try. Table III gives the number of theorems, corollaries, and postulates in six widely used or recent text-

[78] Dodgson, C. L. *op. cit.*, p. 101.

books in geometry. These must be considered more or less in the nature of general laws or principles with which the high-school student is to solve original exercises. The surprising fact about Table III is the great number of relationships which the authors deem of sufficient importance to be classified as theorems or corollaries. Many of these, even though they have been dignified by the name of theorems, are never

TABLE III

THE NUMBER OF THEOREMS AND POSTULATES IN HIGH-SCHOOL GEOMETRY TEXTBOOKS*

Names of Authors	*Axioms and Postulates of Plane Geometry*	*Constructions, Theorems, and Corollaries of Plane Geometry*	*Axioms and Postulates of Solid Geometry*	*Theorems and Corollaries of Solid Geometry*
Durell Arnold	38	196	11	153
Nyberg	52	198	15	127
Otis Clark	38	165	9	114
Seymour Smith	48	218		
Foberg Reeve	33	162	(16)	(59)
Wells Hart	29	231		
Average	40	195	12	131
This Study	32	30	4	12

* This count may be slightly in error because of the cases in which it was not clear whether the author intended a statement for a definition, a postulate, or a theorem. Some authors used also "principles," "properties," and problems; and again it was not always clear whether these were intended for definitions, postulates, or theorems.

again used in the proof of later theorems. Reference to Table I will show this to be a fact. Furthermore, it is often impossible to tell whether the author meant a statement to be a definition, a postulate, or a theorem.

A Minimum Number of Theorems Used in this Study. Rather than to increase the number of theorems to discover how many it is possible to demonstrate for a pattern of thought, *it is one of the purposes of this study to discover how few really fundamental theorems are needed, upon which to build the entire structure of geometry.* Reference to Section III, Chapter I and to Chapter IV will show that in this study the entire plane geometry structure has been placed upon ten fundamental constructions and twenty theorems. To require the mastery of ten simple constructions and twenty theorems upon which to build a course in reasoning seems at least an attainable objective and more in accord with the number of fundamental laws in other sciences such as physics, chemistry, and biology.

A further defense for this minimum list, as already stated, is that it acquaints the prospective teacher with the entire field of high-school geometry in a short period of time. A random choice of theorems for this purpose would be difficult to defend, and some choice is necessary since the time available for a teacher's course does not permit the use of all the high-school work.

7. COMMENSURABLE AND INCOMMENSURABLE CASES

While it is recognized that the number of cases of incommensurable magnitude is as infinity to one in comparison with those which are commensurable, and while it is recognized that no treatment can be rigorous which does not consider the incommensurable cases; yet, in order to be sure not to shed a reactionary influence, the incommensurable cases will be omitted at first in Chapter IV but presented near the end of Chapter V of this study. In all cases of measurement, such as ratios, areas, and angles, the assumption of commensurability will be made. This is done to agree with the trend which seems to consider it impossible for most high-school students to grasp the significance of incommensurability. At the same time the comprehensive treatment of incommensurables as a unit gives the prospective teacher an opportunity to master its technique and significance and to be prepared to teach it if any occasion requires that it be taught.

8. INEQUALITIES

Since the theorems on inequalities are included in only the subsidiary list given by the National Committee and our analysis disclosed no great fundamental need for them, they are presented later as a unit in connection with indirect proof.

IV. Problems for Review and Discussion

1. What is meant by the "Essential Theorems" of geometry?
2. Compare United States history and geometry on the basis of (a) age, (b) universality, simplicity of concepts, (c) truth of conclusions, (d) definiteness of conclusions, (e) the use of hypothesis and postulates; (f) applications.
3. What was Plato's contribution to geometry? What in general did the Greeks contribute to the Egyptian beginnings?
4. How has Descartes' invention of coordinates affected the modern Euclidean geometry?
5. Discuss: Geometry has evolved from a study for adults to a study for children; yet some principles governing its contents have not been modified to meet its new function.
6. Discuss: The statement that a straight line is the shortest distance between two points is not necessarily true.
7. Discuss the following phrases from definitions of a point.
   (a) An indefinitely small space
   (b) That which has neither parts nor magnitude
   (c) That which has position but neither length, breadth, nor thickness
   (d) The limit of a line as it decreases indefinitely
   (e) That by the motion of which a line is generated
8. Does the refusal to use superposition in this study repudiate the practice of constructing two triangles on paper under the conditions given, cutting out the triangles, and placing one on the other to show equality of all parts? Is this actual coincidence, if by coincidence is meant occupying exactly the same space?
9. Could an angle be trisected if the hypothetical construction of a Conchoid of Nicomedes were granted? See Sanford, V., *History of Mathematics*, p. 262.
10. What are "primitive ideas" and "primitive propositions"?
11. Is analytic geometry, based on coordinates, Euclidean or non-Euclidean geometry?
12. How can a person ever get the meaning of an undefined term?
13. What is meant by the statement that the reasonableness and truth of the theorems proved by means of postulates prove the postulates to be true rather than the truth of the postulates proves the theorems true?
14. Just what is a trapezoid? Do all texts agree that two sides of a trapezoid must be non-parallel?

15. Can a point be moved? A line? Can two points be made to coincide? Two lines? Could there be two kinds of mathematical points, fixed and moving ones, just as the "points" made in a speech, the good "points" about a man's character, or the "points" which a given stock rose or fell yesterday involve different notions of "points"? The moving points seem possessed with the peculiar limitation that they always coincide with some fixed point, yet do in some way get from one fixed point to another. Explain.
16. Why is sequence important?
17. How does reducing the number of fundamental theorems make geometry more simple and more like other sciences?
18. Explain how there can be more incommensurable than commensurable line segments. What does commensurable mean? Incommensurable?
19. Discuss: To begin geometry with constructions not only provides a beginning that is simple and concrete, but also provides a "felt need" to prove that the construction is correct; and consequently, this new beginning motivates demonstrative geometry because the activity of demonstrating becomes purposeful.
20. In this study the number of theorems is reduced to a minimum, yet the number of postulates is not reduced, but rather, extended. Defend this.

CHAPTER III

PRINCIPLES OF HIGH-SCHOOL GEOMETRY TEACHING

I. Induction and Deduction

Definition of Induction. Induction and deduction are two terms frequently used in discussions of thought processes and in professional literature on methods of teaching. Carefully worded definitions should therefore be given. In *An Introduction to Reflective Thinking* the Columbia Associates in Philosophy define induction, after showing how a scientific investigator is enabled to draw a general conclusion from one or more restricted individual cases, by asserting that the "transition from particular facts to a general knowledge about these facts is known as the 'process of induction.'"[79] It is a process widely used in science. Every investigator must study specific cases. From these he makes an inference or draws a conclusion, he sets up a hypothesis or a theory, or, in other words, he makes a generalization. He then often studies more cases in the light of the general truth he has discovered. If the generalization is correct, these cases will be much simplified and illuminated. If it is not correct, it must be modified in accordance with these added cases.

L. S. Stebbing in a recent book on logic feels that induction has two slightly different meanings. "In one sense 'induction' is used for that process by means of which we apprehend a particular instance as exemplifying an abstract generalization. In the second sense 'induction' means a form of reasoning in which we establish a generalization by showing that it holds of every instance that is said to fall under it. In both senses induction is concerned with particular instances."[80] The first of these has sometimes been called "intuitive induction," "perfect," "complete" or "summary" induction. It is the type used in mathematics. The second sense of the word fits the conception of induction as used in all the other sciences, since it is essentially the scientific method for

[79] Columbia Associates in Philosophy. *An Introduction to Reflective Thinking.* Houghton Mifflin Co., 1923, p. 74.

[80] Stebbing, L. S. *A Modern Introduction to Logic.* Thomas Y. Crowell Co., London, 1930, pp. 243ff.

handling empirical data. The second use of the word is concerned with "enumeration of particular instances."[80] In this sense of the word a generalization is not necessarily invalidated by contradictory instances, in fact the theory of probability enters as part of the technique of scientific method in all empirical sciences. Stebbing claims that "it seems now to be generally agreed that induction essentially consists in generalization from particular instances, and that scientific method involves not only induction but deduction."[80]

John Stuart Mill gave the terms a broader definition: "the operation of discovering and proving general propositions." Consequently when Mill and Nicod disagree on induction they are not talking about the same idea although they use the same word. Bacon and Mill would contend "induction which proceeds by simple enumeration is childish; its conclusions are precarious, and exposed to peril from a contradictory instance. In science it carries us but a little way. We are forced to begin with it; we must often rely on it provisionally. But, for the accurate study of nature, we require a surer and more potent instrument,"[80] namely, induction based on analysis of causes and conditions. Nicod in attacking Bacon and Mill would hold that "induction by simple enumeration is a fundamental mode of proof and all those who have thought that they can do without it have done so only by the aid of sophisms."[81]

It is evident that even logicians cannot agree when they use the same word to mean somewhat different ideas. In mathematics the argument is of no concern because the truth of conclusions in mathematics does not depend upon enumeration. We may use enumeration to discover a conclusion which seems to be true, but we establish the truth or falsity of it by deductive reasoning.

In all mathematics, and especially in geometry, one is constantly drawing general conclusions about triangles, parallelograms, or circles from studying one or more specific figures. Therefore induction is inherently and inescapably a fundamental part of geometry and of geometry teaching. The following examples illustrate the process of drawing general conclusions from specific facts, which process is called *induction*.

1. Given that 3 over 6 equals 5 over 10. It seems to be true also that 3 over 5 equals 6 over 10. The following query at once suggests itself: Would this always be true of any four numbers related in the same way? Or, stated symbolically, is it true, if a over b equals c over d, that a over c equals b over d?

[81] Nicod, Jean. *Foundations of Geometry and Induction*. Harcourt, Brace and Co., London, 1930, p. 201.

2. Given that 3 over 6 equals 5 over 10, it seems to be true that (3 + 6) over 6 equals (5 + 10) over 10, or that 9 over 6 equals 15 over 10. Again this suggests a generalization that if a over b equals c over d, then $(a + b)$ over b equals $(c + d)$ over d.

3. Suppose a triangle is cut out of paper, its angles torn off and placed together so as to show that for this triangle the sum of its angles seems to be a straight angle. Or suppose the angles were very carefully measured with a protractor and the sum found to be very close to one hundred eighty degrees. Or suppose that each of the three angles of an equilateral triangle were known to be sixty degrees and, therefore, their sum would be one hundred eighty degrees. These special individual cases suggest a general conclusion that for any triangle the sum of its angles is one hundred eighty degrees, no matter if the triangle be equilateral, isosceles, scalene, acute, obtuse, or right, black or white, standing up or lying down, in Florida, Alaska, or New Zealand.

4. By using a cone and a cylinder with the same base and altitude, it is easy to illustrate that the volume of the cylinder seems to be exactly three times that of the cone. Is it then generally true that every cone is ⅓ of a cylinder with the same base and altitude? Since the volume of a cylinder equals its base times its altitude, does the volume of a cone equal ⅓ the product of its base and altitude? Again this generalization is possible of proof and illustrates the function of a possible inductive approach in a deductive science.

Definition of Deduction. If geometry were exactly like other sciences, these generalizations could never be completely proved to be true. They could only be assumed to be true so long as no contradictory evidence was forthcoming, or the probability that they were true might be .8. "One of the chief glories of mathematics is that it can take its theorems out of the realm of inductive probability into the realm of deductive certainty,"[82] as no empirical science can do. Geometry can prove its generalizations to be true by showing their dependence upon other relationships which have been previously proved, upon definitions which have been stated, or upon certain axioms and postulates which have been previously accepted. This whole process of proving a general truth by showing its relation to other general statements which are accepted as true is called *deduction*. As worded by the Columbia Associates in Philosophy, deduction is "the whole process of following the network of relations which bind truths together."[83]

[82] Young, J. W. A. *The Teaching of Mathematics.* Longmans, Green and Co., 1920, p. 57.
[83] Columbia Associates in Philosophy. *op. cit.*, p. 98.

In developing further the nature of deduction the Columbia Associates contend that "mathematics is concerned with that structure of things which by its existence makes it possible to proceed from one truth to another deductively. . . . The relations with which the mathematician deals seem to be a part of the very foundation of the world we live in, so that we have discovered that, if any proposition that holds of experience is elaborated in accordance with the rules of mathematics, the conclusion thereupon reached will also hold of experience. . . . It does seem to be true that the more highly developed a science becomes, . . . the more its beliefs tend to fall into mathematical form, and to admit of treatment by purely mathematical methods. So true is it that a science is successful just in so far as it is able to formulate its beliefs mathematically that many men have naturally come to think that in mathematics is to be found the exemplar of all true knowledge."[83]

Geometry is essentially a deductive science. It deals with general truths and relationships which may have been suggested by induction, but which it proves by means of other general relationships. An "if-then-science" is a phrase commonly used in describing geometry. In every case, however, both the "if" and the "then" clauses are general statements. The following examples illustrate the function and method of deductive reasoning. They are not to be interpreted as desirable patterns for teaching, but merely as illustrations of deductive reasoning.

1. It was suggested inductively that if a over b equals c over d, then a over c equals b over d. Let us proceed to prove this deductively.

   Proof: (1) $a/b = c/d$ by hypothesis.

   (2) $a/b \cdot b/c = c/d \cdot b/c$. Both terms of equation (1) multiplied by b/c.

   (3) $a/c = b/d$. Equals multiplied by equals make equals.

2. If $a/b = c/d$ then $\dfrac{a+b}{b} = \dfrac{c+d}{d}$ was also suggested inductively.

   Proof: (1) $a/b = c/d$ by hypothesis.

   (2) $a/b + 1 = c/d + 1$. Equals added to equals make equals.

   (3) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$. Both terms of equation (1) changed to improper fractions.

3. By induction it was suggested that the sum of the angles of any triangle equals 180°. (Suggestion: The reader may need to draw a figure in order to follow the proof easily.)

Proof: (1) Through any vertex, such as C, draw a line parallel to opposite side, AB. Three angles will be formed. Call them x, y, and z, using x for the angle nearest A.

(2) $x + y + z = 180°$ by definition of a straight angle.

(3) $x = A$ and $z = B$, because if two parallel lines are cut by a third, the alternate interior angles are equal.

(4) $y = c$, because they are the same angle.

(5) Therefore $A + B + C = 180°$, because equals may be substituted for equals.

(6) Therefore the sum of the angles of any triangle equals 180°.

It should be evident that the truth of these three statements has been established by means of reference to other relationships or definitions previously accepted. The absolute truth of the propositions depends entirely upon the absolute truth of the secondary propositions. However, in the entire process of reasoning no use is made of the specific nature of any fact or thing. All statements are completely general. Incidentally, all three illustrations are also synthetic in organization. They could as well have been analytic except that they would then have been longer.

T. Percy Nunn, in discussing the teaching of algebra, contends that the "business of algebra is to disengage the essential features of an arithmetical process of given type from the numerical setting which a particular case presents."[84] If this is true of algebra, it is even more true of geometry. Every proposition and exercise in geometry seeks to disengage a general truth from a specific setting.

Use of Induction and Deduction in Geometry. The meaning and natural function of induction and deduction have now been concisely stated. The drawing of general conclusions from specific cases is induction. The process of dealing with a general conclusion in proving it by means of other general relationships or of applying it to specific cases is deduction. Induction is the natural way of presenting or discovering a general conclusion, and deduction is the rigorous, useful, economical, and forceful way of proving or applying it.

Demonstrative geometry is essentially a deductive science, involving the proving of general conclusions which have been discovered inductively from specific figures. It should therefore utilize constantly the natural relationship between the specific and the general, between induc-

[84] Nunn, T. Percy. *The Teaching of Algebra.* Longmans, Green and Co., London, 1927, p. 2.

tive and deductive thinking. The simultaneous use of both of these forms therefore becomes a general controlling principle in the teaching of geometry.[85, 86]

This joint use of induction and deduction is illustrated and emphasized with several theorems in Chapter IV and Chapter VI of this study. As typical cases see theorems 8, 9, 33, 34, 35, 36, 37, 38. Note the inductive "approach." In fact the purpose of this entire study is to provide patterns for the use of induction as well as deduction in the teaching of geometry, and also to provide patterns for the use of analysis and synthesis, which are to be discussed presently. The use and function of induction and of deduction in geometry will be more completely appreciated when, in Section IV of this chapter, its place in the general plan of "heuristic" teaching is indicated.

II. Laboratory Work in Geometry

Laboratory Work Essential for Induction. The preceding defense for an inductive approach to a deductive science should give laboratory work in geometry a prominent place. It should also put it in its right place, that is, as a means of discovering and suggesting possible conclusions to be proved.

Laboratory work in geometry, which is a matter of making drawings by following directions given in a book, is not laboratory work at all in the sense here intended. It is rather mechanical drawing, a by-product of geometry, and contributes little toward the realization of the chief function of geometry—that of discovering and proving space relationships. So-called laboratory work of this kind has values that are significant, such as the development of the ability to handle the drawing instruments, to follow directions, and to make neat and accurate drawings. These are not generally conceded, however, to be the best possible aim of geometry. Laboratory work, if not controlled and made to serve its proper purpose, may become the end rather than the means to an end.

At the other extreme is the conception of laboratory work in geometry as a way for the child to discover, entirely independent of suggestion, that the area of a circle equals pi times the radius squared and other relationships of geometry. To expect a child to discover entirely by himself, in 180 hours, what it has taken brilliant mathematicians of the race thousands of years to discover, is of course an impossibility.

[85] Schultze, Arthur. *The Teaching of Mathematics in Secondary Schools.* Macmillan Co., 1914, pp. 37-41.

[86] Hassler, J. O. and Smith, R. R. *Teaching Secondary Mathematics.* Macmillan Co., 1930, pp. 136-139.

If by laboratory work is meant the actual handling of concrete figures in such a way as to discover the relationships existing, with some guidance by the text or teacher, then it performs its most useful service to geometry. The natural way of making general conclusions is through dealing with specific cases. The laboratory work must then be limited to this function and must merely supplement, rather than supplant, the deductive analytic reasoning. "After a consideration of a sufficient number of cases it is a relief, a simplification, to abstract, to generalize. Abstractions and generalizations are rather the crowning products than the foundation stones."[87] "Laboratory methods form an exceedingly valuable supplement to the teaching of mathematics. Students doing some work of this nature will have more interest in and understanding of mathematics."[88]

An Illustration of a Laboratory Lesson. Laboratory work in geometry need not be individual, but may be more or less group work. The following is a description of a laboratory approach to a theorem, which the author witnessed recently.

Teacher: "Draw a straight line on your paper, and on it lay off three or four equal segments."

The teacher also did this on the board and saw to it that all had done likewise.

Teacher: "Now draw a line through one of these points and then a line through each of the other points parallel to the line through the first point. You need not construct the parallels accurately but draw them with the ruler to get the lines straight."

The teacher followed her own directions at the board and each member of the class did likewise.

Teacher: "Look at your figure; you should have at least three parallel lines cutting equal segments on a transversal."

She held up several drawings in various positions for the class to see.

Teacher: "What do you think would be true of this figure?"

Responses were almost unbelievably rapid, (1) "The parallels are the same distance apart." (2) "If you draw another line across them, the parts of it would be equal." Other statements of equivalent nature were made.

Teacher: "Suppose you try drawing another line across the parallel lines, but don't make it parallel to the first one nor even perpendicular to the parallels, just any other line crossing the parallel lines."

[87] Young, J. W. A. *op. cit.*, p. 105f.
[88] Schultze, Arthur. *op. cit.*, p. 49.

She did this at the board, and all did likewise at their seats.

Teacher: "Now what would be true of this line?"

Many of the volunteers suggested, "Its segments would be equal."

Teacher: "Why do you think the segments would be equal?"

Various pupils answered, mostly to the effect that they looked equal, or that they were equal on the first transversal, or that they could be proved equal.

Teacher: "Would it always be true that if parallel lines were drawn in this way they would cut equal segments on any transversal?"

There was general agreement. Then the teacher said, "Suppose we prove it. Before we start to do that, will someone make a good sentence stating the conditions and the conclusion?"

The first attempt was, "If parallel lines cut equal segments on one line, they cut equal segments on any line." These criticisms were offered: "The 'any line' would have to cross the parallels." "There would have to be at least three parallels." Finally the statement was modified into, "If three or more parallel lines cut off equal segments on one transversal they will cut off equal segments on any other transversal."

The teacher then proceeded with the proof, the pupils doing the analyzing and discovering of the steps in it. They gave a new proof but one entirely correct. It will be given in full in a later section.

Summary and Conclusion. The above illustration indicates that as the term laboratory work is used in this study it means experimentation with specific figures for the purpose of discovering relationships. As such it forms an integral part of the pattern of teaching which is herewith advocated. Its place in the general plan of teaching will be more apparent after the next section on analysis and synthesis has been mastered, and the section, "The Heuristic Method in Teaching Geometry" has indicated the place and function of each type of work in the general plan.

III. Analysis and Synthesis

Analysis and Synthesis in Chemistry. Analysis and synthesis are terms commonly used in chemistry. A chemist analyzes a substance and perhaps discovers that it contains iron, sulphur, and oxygen. That is, he puts a sample in a test tube, a beaker, or a retort, and subjects it to various processes in order to break it down into simpler compounds or into elements.

Synthesis in chemistry refers to the putting together of elements or compounds by subjecting them to various processes in order to make some new or desired product. Frequently, in fact usually, a synthetic process is suggested by and follows analysis. Soil is analyzed to determine what elements it needs, then by supplying these elements the soil is built up so that it can produce more efficiently. Iron ore is analyzed to determine how much carbon or other ingredients must be added to make the best steel. In other words analysis is a breaking down process used to discover something, and synthesis is a building up process based on analysis and used to produce a desired product.

Analysis and Synthesis in Geometry. In geometry, analysis is a mental process of tearing down a geometric statement to discover the relationships upon which its truth or existence depends. In geometry, analysis is based on the Principle or Law of Sufficient Reason, which Leibniz expressed by saying that "nothing happens without a reason why it should be so rather than otherwise."[89, 90] Analysis is a systematic process of discovering this sufficient reason why a statement or a relationship is so rather than otherwise. This is done by a technique which says: "This will be so if that is so; that will be so if something else is so, etc." On the other hand synthesis is the building up of sufficient reasons to establish or prove the conclusion.

Analysis Used by Plato. The analytic method is not a new method of teaching geometry. Plato is credited with being its originator, and Apollonius and Archimedes were very successful with its use. The method invented by Plato differs slightly from the modern method. It is based upon the following definitions of analysis and synthesis. "Analysis is an assumption of that which is sought as if it were admitted and the passage through its consequences to something admitted to be true. Synthesis is an assumption of that which is admitted and the passage through its consequences to the finishing or attainment of what is sought."[91] To illustrate the meaning of this definition, examples can be used from the preceding section in which certain general conclusions were suggested as the result of inference from particular cases.

An Illustration of Plato's Method of Analysis. Since $3/5 = 6/10$ and $\frac{3+5}{5} = \frac{6+10}{10}$, it was thought possible, in the section on induction, that it might always be true for any proportion that, if $a/b = c/d$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

[89] Jevons, W. Stanley. *Logic.* D. Appleton and Co., 1890, p. 125.
[90] Enriques, Federigo. *The Historic Development of Logic.* Henry Holt and Co., New York, 1929, p. 256.
[91] Heath. *op. cit.*, I, p. 138.

HYPOTHESIS: (1) $a/b = c/d$

CONCLUSION: (2) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$

ANALYSIS: "Assume the conclusion true; then pass through its consequences to something admitted to be true."

If (2) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$, then (3) $(a+b)d = (c+d)b$. Why?

If (3) is true, then (4) $ad + bd = bc + bd$. Why?

If (4) is true, then (5) $ad = bc$. Why?

But 5 is true from (1).

Therefore (2) is true if all steps used are reversible.

SYNTHESIS (PROOF): "Assume the hypothesis and pass through its consequences to the desired conclusion." That is, retrace the steps discovered in the analysis. [Use the previous hypothesis (1) and conclusion (2)].

(3) $ad = bc$. Multiplying (1) by bd.

(4) $ad + bd = bc + bd$. Adding bd to both terms of (3).

(5) $d(a + b) = b(c + d)$. By factoring (4).

(6) Therefore $\dfrac{a+b}{b} = \dfrac{c+d}{d}$. By dividing both terms of (5) by bd and cancelling.

There are other possible analyses and syntheses for this same generalization. For example:

HYPOTHESIS: (1) $a/b = c/d$.

CONCLUSION: (2) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$.

ANALYSIS:

If (2) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$, then (3) $\dfrac{a}{b} + \dfrac{b}{b} = \dfrac{c}{d} + \dfrac{d}{d}$.

If (3) is true, then (4) $\dfrac{a}{b} + 1 = \dfrac{c}{d} + 1$.

But (4) is true from (1) by adding equals to equals, and therefore (2) can be proved.

SYNTHESIS (PROOF):

(3) $a/b + 1 = c/d + 1$. By adding 1 to both terms of (1).

(4) $a/b + b/b = c/d + d/d$. By changing to same denominator.

(5) Therefore $\dfrac{a+b}{b} = \dfrac{c+d}{d}$. By adding the fractions in (4).

Analysis and synthesis supplement each other; the first, discovering the steps for the proof, and the second, putting together the ideas discovered by analysis so as to form a concise, rigorous proof. Analysis, as Plato is credited with using it, is effected by assuming the conclusion true, discovering the results of this assumption, then trying to establish these results without using the conclusion. This method begins by saying, "if the conclusion is true," and concludes each step with, "then certain results follow."

The Modern Method of Analysis. The modern analytic method differs rather markedly in point of view from the older method although it is based upon the same fundamental notion of working from the conclusion back to known facts. According to D. E. Smith the modern analytic method asserts that "a proposition is true if another is true, and so on, step by step, until a known truth is reached."[92] In the modern synthetic method, according to the same author, "known truths are put together in order to obtain a new truth."[93] Using the analytic method, a student will say, "I can prove this if I can prove that; I can prove that if I can prove . . ."; and so on until he reaches a proved or accepted proposition.[94, 95, 96] Illustrations will help to clarify this theory.

(a) HYPOTHESIS: (1) $a/b = c/d$.

CONCLUSION: (2) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$.

ANALYSIS: Equation (2) will be true if (3) $(a+b)d = (c+d)b$.
Equation (3) will be true if (4) $ad + bd = bc + bd$.
Equation (4) will be true if (5) $ad = bc$.
But $ad = bc$ from (1). Therefore (2) can be proved.

Again there are other alternatives than equation (3). This will usually be true in every analysis regardless of its form. The synthetic proof is the same as in the previous case. In this type of analysis the "if" comes with the second clause in each step of the analysis, stating that the desired conclusion will be true if something else is true.

Comparison. The analysis used by Plato and the type designed as the Modern Method of Analysis are essentially identical; the only difference physically is the placement of the "if"; and, theoretically, the difference consists in a point of view, with less confusion in the modern method

[92] Smith, D. E. *Teaching Geometry.* Ginn and Co., 1911.
[93] Smith, D. E. *Essentials of Plane and Solid Geometry.* Ginn and Co., 1923, pp. 93 94.
[94] Beman and Smith. *New Plane Geometry.* Ginn and Co., 1899, p. 152.
[95] Hassler and Smith. *op. cit.*, pp. 131-136.
[96] Schlauch, W. S. *Fifth Yearbook*, p. 134.

because of less reversal of thinking. Furthermore, in the method credited to Plato, converses are involved, and hence more care needs to be taken to be sure that all operations are reversible. If one assumes a conclusion true, certain results may follow which may not themselves be adequate to be the "sufficient reasons" for the conclusion. For instance, suppose the desired conclusion is that $ABCD$ is a square. Plato's analysis might assume that $ABCD$ is a square, and as a consequence angle A is a right angle. But, reversing the statement, that angle A is a right angle is not sufficient to make $ABCD$ a square. In the modern method the direct form is retained by reasoning that $ABCD$ is a square if certain conditions are true. That is, the Plato analysis begins by saying, "If the conclusion is true, then certain results follow." The thought is that in reversing the thinking the results will become the causes which make the conclusion true. In the modern form the analysis begins by saying, "The conclusion will be true if certain other conditions are true." Then these conditions will be true if still others are true, and so on until facts have been reached which are true or accepted. It should be evident that the modern method avoids many dangers which were inherent in the older form.

Symbolically expressed the modern form of analysis and synthesis is as follows:

ANALYSIS, modern form

First, Given A, to prove B.
Second, B will be true if C is true.
C will be true if D is true.
D will be true if E is true.
Third, But E is true because of A.
Therefore B is true or at least B can be proved.

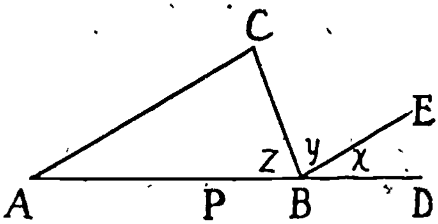
SYNTHESIS

A is true because it is the hypothesis, a definition, a postulate, or a previous theorem.
E is true because of A or other accepted facts.
D is true because of A, E, or other accepted facts.
C is true because of A, E, D, or other accepted facts.
B is true because of A, E, D, C, or other accepted facts.
Therefore, if A is true, B is true.

Geometric Illustration of Modern Analysis. It is desired to prove that the sum of the angles of triangle ABC is a straight angle.

ANALYSIS: (1) It is evident that this could be done if one angle could be shown supplementary to the sum of the other two, or the three angles equal respectively to three other angles whose sum is known to be 180°.

(2) If AB, in triangle ABC, is extended to D, then angle ABC is supplementary to angle CBD, or ABD is a straight angle. Now it is desired to prove that angle A plus angle ABC plus angle C = angle ABD.



(3) Clearly angle ABC (angle z) is already part of angle ABD. Therefore angle A + angle z + angle C = angle ABD if angle A + angle C = angle CBD.

(4) In angle CBD it would be possible to draw an angle at B equal to angle A. Call it x and the rest of angle CBD call y. Call the line drawn, BE. Since angle x + angle y = angle CBD and angle A = angle x, then angle A + angle C would equal angle CBD if angle C = angle y.

(5) Finally angle C would equal angle y if BE were parallel to AC since the angles are alternate interior angles.

(6) But BE is parallel to AC because of the fact that angle x was drawn equal to angle A.

(7) Therefore angle A + angle B + angle C can be proved equal to a straight angle.

Note: What other alternatives might there be in place of step (2)? Try taking some point on AB such as P, drawing lines parallel to AC and BC, and then proving the three angles at P equal to the three angles of the triangle. There are many ways of getting three angles equal to angles A, B, and C. The usual form of this proof is synthetic and because of its familiarity will not be reproduced here, but it is clearly the reverse of this analytic form.

We have then established by these specific illustrations an inductive basis for a sensible conclusion, namely, that analysis and synthesis are supplementary methods in geometry, the one used for purposes of discovering the proof and the other used for concisely stating it. The general principle which is here illustrated is that since geometry is essentially the science of proving relationships, it must of necessity use both analysis and synthesis, the analysis for discovering the proof and the synthesis for presenting it. The analytic method of going from the unknown to the known, furnishes a powerful instrument for reasoning, while the synthetic presentation, going from the known to the unknown by short familiar steps, is the sensible, easy, rigorous, accepted, and efficient way of presenting the proof. A geometry devoid of either of these methods would be unfortunately handicapped if it could exist at all.* [97, 98, 99]

Synthesis a Check on the Reversibility of Analysis. It may be thought by a casual reader that the synthetic form of the proof is unnecessary. This is probably often true; in fact the synthetic proof seems always to repeat the analytic statements in reverse order and to be merely a refined way of concealing the real thinking process. However, the synthetic proof has a second function which is very important although not as applicable to the modern form of analysis as to the form inaugurated by Plato. That function is to insure reversibility, since there are some processes that cannot be reversed. This is shown most easily by illustrations from algebra.

(a) CONCLUSION: $+2 = -2$.

ANALYSIS: $+2 = -2$ if $(+2)^2 = (-2)^2$ or if $4 = 4$.

But $4 = 4$.

Therefore the erroneous conclusion can seemingly be proved.

PROOF: (Snythesis)

(1) $4 = 4$.

(2) $\pm\sqrt{4} = \pm\sqrt{4}$.

(3) $+2 = +2$ or $-2 = -2$.

Notice that the analysis does not reverse because the process of squaring two signed numbers, when reversed, will not produce the erroneous result.

(b) HYPOTHESIS: $a = b$.

CONCLUSION: $2 = 1$.

* The use of analysis and synthesis is further illustrated in Chapters IV, V, and VI of this study.

97 Schlauch, W. S. *op. cit.*, pp. 134-144.

98 Schultze, Arthur. *op. cit.*, pp. 30-36, 228-244.

99 Young, J. W. A. *op. cit.*, pp. 262-263.

ANALYSIS: (1) $2 = 1$ if $a + a = a$ or $a + b = a$.
(2) $a + b = a$ if $(a + b)(a - b) = a(a - b)$ or if $a^2 - b^2 = a^2 - ab$.
(3) $a^2 - b^2 = a^2 - ab$ if $b^2 = ab$ or if $b = a$.
(4) But $b = a$, therefore, if the processes used are reversible, $2 = 1$.

In step (2) multiplying by $a - b$, since $a = b$, is multiplying by zero, and the reverse of this, division by zero, is impossible. The synthetic form of these statements quickly reveals this error.

(c) CONCLUSION (Identity to be proved): (1) $\tan A = \sqrt{\dfrac{1 - \cos 2A}{1 + \cos 2A}}$

ANALYSIS: Equation (1) will be true if (2) $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$.
Equation (2) will be true if (3) $\tan^2 A = \dfrac{1 - (1 - 2\sin^2 A)}{1 + (2\cos^2 A - 1)}$.
Equation (3) will be true if (4) $\tan^2 A = \dfrac{\sin^2 A}{\cos^2 A}$.

But (4) is true because $\tan A = \dfrac{\sin A}{\cos A}$.

Therefore (1) can be proved.

Synthetic proof of this will show that equation (1) is true only if a plus and a minus precede the radical sign; otherwise it is true only for angle A acute, since a radical without a sign is assumed to be positive.

These illustrations reveal that the chief function of the synthetic proof is not merely to secure conciseness and beauty of form, but rather to insure rigor. The chief function of analysis lies in discovering the steps for synthetic proof, rather than in being a form of proof.

Integration of Induction, Deduction, Analysis, and Synthesis. The function of induction and deduction, of analysis and synthesis has been illustrated. Induction, with laboratory work, is the natural and usual manner for suggesting a general, deductive conclusion. Deduction is the more inclusive and usable form for conclusions and is highly to be desired. In proving a general or specific conclusion, the natural, most powerful technique is to analyze the desired conclusion in an effort to discover a basis for proof in the realm of known facts or conclusions. After the analysis has been completed the proof is most beautifully, rigorously, and efficiently stated in the form of a synthetic development, building up from known facts and by permissible processes a series of steps resulting in the desired conclusion being inescapable. It shall now be the purpose of the next section to show how these four processes with concomitant techniques form integral parts of a technique of teaching geometry.

Before going to the next section we might well state that no attempt has been made in this introductory presentation of analysis to develop in the student the ability to analyze problems. The objective in this section has been merely that of explaining what analysis and synthesis mean. The objective of the entire study is to develop the reader's ability to use the analytic method in teaching geometry. The illustrations in this section are purposely simple in order to put more emphasis on the method than on the content. They do not, therefore, deserve the condemnation which some treatments of analysis so aptly merit, that "analysis works beautifully on simple problems whose solution is known. Show us how to use it when the solution is unknown." Furthermore, the illustrations given are admittedly inadequate for a complete presentation, the reason being that a complete presentation depends upon the facts and principles of geometry given in the next chapter. All of Chapters IV and VI is devoted to providing further "patterns" of analysis, while Chapter V presents a more extended and complete form of the analytic method and provides ample experience with its application.

IV. The Heuristic Method in Teaching Geometry

Definition and Illustration. The term heuristic is derived from a Greek word meaning "to find." As used here, it will be defined as, first the "finding," by means of an inductive approach (perhaps through laboratory work), of a deductive or general conclusion, second, the "finding" of its deductive proof by an analysis of the relationships upon which the conclusion depends, and finally, the synthetic statement of these relationships in a deductive proof. The "finding" takes place in two parts of this development, first, in drawing the general conclusion from the specific cases studied, and, second, in determining the steps necessary in the proof through an analysis of the conclusion.

The heuristic method does not mean that the student must discover, unaided, the proofs for theorems which the mathematical geniuses of the race have discovered in a period of over 4000 years. Such an interpretation of heuristic teaching is impossible. It is intended that heuristic teaching should mean that each pupil be given an opportunity to discover as much as possible. Such discovery, even though small, gives a feeling of satisfaction and of creative power that is stimulating. A few examples will illustrate this meaning of heuristic teaching.

The examples given in the previous sections on induction and laboratory work illustrate the heuristic approach, or the discovery of a conclusion by the students. By setting up a concrete situation and making the possibility of drawing the conclusion not too difficult, as was done in the illustration of a laboratory lesson, we are often surprised at the discoveries students will make. The remainder of that lesson completes an excellent illustration of good heuristic teaching. The previous section reported progress only up to the discovery of a conclusion which seemed to be worth proving. The analysis and synthesis which followed proceeded somewhat as follows:

Teacher: "We seem to need to prove certain line segments equal. Do we have a way of doing that which we have used before?"

Pupil: "If they were corresponding parts of congruent triangles, they would be equal; but there are no triangles here."

Two boys seemed possessed with a stimulating idea at about the same time.

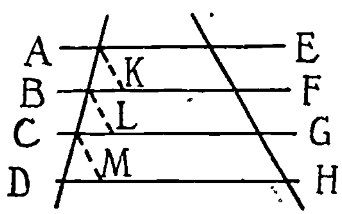
Teacher: "Well, Swaford, suppose you give us your idea."

Swaford: "If you draw lines through the upper ends of those segments parallel to the other transversal, you will have triangles which may be proved congruent. There will be some parallelograms, and the triangles will have one side equal and enough parallel lines to show the other angles equal."

This seemed to be a remarkable response, in fact so remarkable that one can forgive the teacher for making what seemed a mistake in handling it.

She said, "Suppose you come to the board and show us what you mean."

Swaford went to the board and completed the figure so that it looked like the one below:



Teacher: "Won't you go back a little bit now and state the hypothesis and conclusion so as to be sure we are all following you. Then tell us what you have done and complete your proof."

Swaford: "*Hypothesis:* $AB = BC = CD$ and lines AE, BF, CG, and DH are parallel through A, B, C, and D, respectively.

Conclusion: "The segments on any other transversal such as EH are equal, or EF, FG, and GH are equal.

Proof: "Through A, B, and C draw lines parallel to EH. Then these three triangles (pointing to them) will be congruent. These upper angles (pointing to them) are equal because they are corresponding angles of parallel lines AK, BL, and CM cut by the transversal AD, and these lower left angles (pointing to them) will be equal because they are corresponding angles of parallel lines BF, CG, and DH cut by the transversal AD. Therefore the triangles are congruent by angle-side-angle. That makes $AK = BL = CM$. Then $AK = EF$, $BL = FG$, and $CM = GH$, because they are opposite sides of these three parallelograms (pointing to them). Therefore $EF = FG = GH$ by substituting equals for equals."

Teacher: 'Very well done. What is it, Sanford?"

Sanford: "He didn't need to draw those lines parallel to EH. He could have drawn them parallel to AD through E, F, and G and made triangles on the other side."

Teacher: "That's a fine idea. How many see Sanford's point here?"

A dozen or more hands went up.

After a few questions here and there on the proof and suggestions by various students of other ways to do parts of it, the teacher made her assignment: "For tomorrow I want you to write up the proof for this theorem, and let us have it in good form and neatly done. To be sure we all know what to do, will you tell us again just what the hypothesis and the conclusion are, Tilly?" Tilly did so. "You may write up Swaford's proof or invent a new one of your own. In addition to that see if you can prove exercises 191, 192, and 193, but you need not write them out; I think these problems will be clear to you without any help. Now we will use the rest of this period to begin our work so that I can help any of you who may need it."

The rest of the period (15 minutes) was a real study period. It should be noted that these students did not use any textbook; so they had not studied this theorem beforehand. The proof suggested by the lad, Swaford, practically proves no previous study, since most texts draw the parallels on the other line. The teacher handed out mimeographed sheets of originals.

The statement was made on a previous page that this teacher made an error in technique even though her presentation produced such unusual results. Perhaps this cannot be proved. As the lesson was conducted, at least two boys, but probably not over six students, had the thrill of discovering the proof or a large portion of it. Swaford *told* the rest of them. The lesson would have been conducted to better advantage had the assignment been made without the complete proof being given. Then more pupils would have had a chance to discover it. Perhaps the presence of a visitor made it excusable to have someone show how well he was reasoning. Faulty or not faulty, this was a rather unusual and successful illustration of heuristic teaching, such as you seldom see except in the classrooms of master teachers.

Heuristic Teaching of Difficult Theorems and Problems. Sometimes it may be impossible or inadvisable to have the first portion of the presentation of a new theorem or problem heuristic; in fact it is conceivable that for a proof such as that for the area of a sphere or volume of a pyramid that neither part be wholly heuristic. It may be necessary for the teacher to break up the major problem by more or less analytic methods into smaller problems, each of which may be attacked heuristically. For heuristic teaching to be successful, the pupils must of necessity make discoveries. This can be achieved in harder problems by breaking them up into smaller problems whose solution can be more readily discovered, or often by giving several specific examples. Some textbooks achieve this in the exercises preceding a theorem, when these exercises are designed to build up a background in special cases for the theorem.

Such a modified heuristic lesson is often necessary in studying the area of a circle. Usually someone will know the formula from intuitive geometry, and therefore the inductive heuristic approach is valueless. It remains to make the analysis as heuristic as possible. This can be done by inscribing a regular polygon and drawing radii from the vertices to the center. Then the lesson should proceed somewhat as follows:

Question: What is the area of one triangle?

Answer: $\frac{1}{2}ab$.

Question: What is the area of n triangles?

Answer: n times $\frac{1}{2}ab$ or $\frac{1}{2}nab$.

Question: Suppose we change this to $A = (\frac{1}{2}a)(nb)$.
What is nb on the polygon?

Answer: Perimeter.

Questions to follow: Then we can write $A = \frac{1}{2}ap$. Now suppose we had a polygon of twice as many sides, would its area still be $\frac{1}{2}ap$? Would its a and p, however, be the same as before? What would then be true if the sides were increased greatly so that the polygon would be practically the same as the circle? What is the area of the polygon approaching as n increases? What is a approaching? p? What then is the area of the circle?

The problem of finding the area of a circle is broken up into many smaller problems, all of which are, however, attacked with the idea of permitting the student to discover as much as possible. After getting one proof of this type as a pattern of analysis, the student will be much more successful in the discovery of the solution of other problems involving the notion of limits.

If time permits, a laboratory exercise of actually cutting up a circle of cardboard into triangular sections and then fitting them together into a parallelogram-shaped figure whose base is half the circumference and whose altitude is the radius, will help to discover the formula. Then after the formula to be proved has been discovered more or less inductively and analytically, the proof could be developed as previously shown.

Slogans. In summary of this heuristic method of teaching geometry, a few general slogans can be presented which emphasize the general principles involved.

(1) "Teach, don't tell."

(2) "Construct the figure so as to indicate the hypothesis."

(3) "Let the student discover the conclusion by experimenting with specific figures."

(4) "Commands to think, or questions with yes or no for an answer usually indicate failure on the part of the teacher."

(5) "Lead the student to analyze the conclusion and to discover the proof."

(6) "Each question should always be a problem whose solution is to be discovered."

(7) "The larger the discovery the better, but a small discovery is better than none."

(8) "Teachers should know their geometry, its philosophy, its logic, its general educational value, and be fearless in applying the heuristic principle."

(9) "Use induction for the discovery of the deduction and use analysis for the discovery of synthesis."

(10) The heuristic pattern has many elements which are "identical" with non-geometric, life situations; and consequently it should be possible for students to "generalize" from the geometric pattern of reasoning and apply that pattern to many and varied situations.[100, 101]

Summary. The "heuristic pattern," as defined and described in the preceding pages, is the technique recommended in this study for teaching high-school geometry. Note its salient features: First, it has an inductive approach to help the student discover a generalization worth proving. This inductive approach may sometimes be omitted and when

[100] Young. *op. cit.*, pp. 61, 69-80.

[101] Schultze. *op. cit.*, pp. 44-45.

present usually involves laboratory work. Second, an analysis of the conclusion is used to discover the relationships upon which the proof depends. The student's discoveries may be large or small, but must be real discoveries, and the larger the better. Third, the proof is presented in characteristic synthetic, deductive form. This final step, although important, really has little of the heuristic spirit in it, since it is merely a summary, in logical form, of the previous discoveries. Thus the heuristic pattern is composed of induction, deduction, analysis, and synthesis.

Chapter IV, immediately following the next short section, contains the essential theorems of high-school geometry, some of which are completely presented so as to illustrate the heuristic pattern. For additional exercise material reference should be made to some good modern text in geometry, preferably one whose author is dominated by the heuristic ideal of teaching. A few exercises combined with careful mastery of the materials of Chapters IV, V, VI should provide learning experience in the spirit of heuristic teaching which will make the principles laid down in this chapter a functioning part of a student's educational philosophy and even of his everyday thinking.

Chapter V contains a complete presentation of the method of analysis and provides for the prospective teacher ample experience with problems to be analyzed, using both direct and indirect proof.

Chapter VI contains theorems in geometry more difficult than those used in high school. Its purpose is to impress still more firmly upon the student's mind the power and possibilities of the analytic method which forms the heart of the heuristic teaching technique. Here the college student will be confronted with new theorems and new definitions which will be relatively as difficult for him as those of high-school geometry are for the high-school student. He will, therefore, learn new geometry material by a technique of teaching which is recommended for him to use in high-school teaching. If learners are likely to teach as they are taught, then Chapters IV, V, and VI should insure the use, by readers of this study, of the heuristic pattern of teaching, which it has been the purpose of this chapter to explain and defend.

V. General Techniques, Philosophy, and Principles

It is not the purpose of this section to emphasize the importance of classroom ventilation and temperature; of dispatch in taking attendance, in collecting or distributing papers, and in beginning the lesson; of seating arrangement; of general co-operative, friendly, and yet businesslike attitude; or even of marking systems, averages, and the normal curve; however important these items may be. These items constitute subject-matter rather for the general course in principles or methods of teaching, or perhaps supplementary material to be given by the teacher of this course.

It is the purpose of this section, however, to indicate a general plan of procedure which can of course be modified to conform with varying situations or with a teacher's individual preferences. Every mathematics teacher should be guided by the following general principles:

A. A teacher should have a general plan for every unit of work. (1) The new material must be presented and work assigned for pupils to master. (2) Pupils should study during part of the class time under the direction of the teacher who can then give skillful help to meet individual needs. (3) The teacher will need to check, in some way, the pupils' mastery of the unit before beginning the next unit. If a unit of work is completed in one day, then step (3) would probably come on the next day before new work is begun. If the unit lasts for several days, step (2) may last for more than one day. In any case, the three major divisions should be present, and the second one, study under the direction of the teacher, should not be minimized.

B. Every teacher should have a testing program that is designed more for helping the teacher to discover and reteach what the child does not know than for the purpose of giving marks on a report card. Perhaps every teacher should adopt some such teaching and testing slogan as: "Teach, test, diagnose; then, if necessary, teach, test, and diagnose again to the point of mastery."[102]

C. Every teacher should have a philosophy of education which realizes first, the function of mathematics as a whole, and of each portion of mathematics in contributing to that whole; second, the importance of using the laws of learning, and of having the child interested, happy, and successful; third, the truth of the Comenian axiom, "We learn to do by doing," and the force of the statement that successful teaching is measured, not so much by "teacher activity," as by the resulting "pupil activity."

In acquiring a philosophy of education let the teacher ponder over the following suggestive quotations. Bacon: "No one truly and fundamentally possesses any knowledge that he has not, so to speak, created for himself"; Comenius: "Let the main object of our art of teaching be to seek and to find a method of teaching by which teachers may teach less, but learners learn more"; M. A. Jullien: A child is not "an empty vessel to be filled to over-flowing so as to make him appear rich in

[102] Morrison, H. C. *The Practice of Teaching in the High School.* University of Chicago Press, 1926, p. 79.

borrowed plumage";[103] Rousseau: "In order to make his (the child's) curiosity grow, do not be in a hurry to satisfy it. Put problems before him, and let him work them out. Let him know nothing because you have told it to him, but because he has learned it for himself. Let him not be taught science, but discover it";[104] "It is not so much a matter of teaching him the truth as of showing him how he must go about so as always to find truth";[105] and Horace Mann: "Unfortunately education amongst us at present consists too much in telling, not in training. . . . Never tell a child what he can be led to discover for himself."

VI. Summary of Chapters I-III

The purpose of the first three chapters of this study has been largely three-fold. First, there has been presented a philosophy of teaching geometry which makes geometry primarily a course in analytic reasoning with many opportunities for students to analyze relationships and to discover proofs, and secondarily a course providing useful information.

Second, Chapter II attempted to give a brief history of geometry and a summary of the problems confronting the teacher of geometry. This included a statement of the primary function of the subject, based upon its peculiar nature and organization, and a discussion of some unsettled difficulties which were at least disposed of, if not always settled.

Third, Chapter I began with the statement of the problem which this study seeks to solve and a statement of the method of solution. Based upon the postulate that a teacher of geometry should know the subject-matter of geometry and be thoroughly conversant with its possibilities as a school subject before attempting to teach it, this study has selected a minimum list of theorems covering the whole field of geometry, and in Chapter IV, following, will present some of them as patterns of teaching and patterns for teaching. This use of the actual subject-matter of geometry as a pattern for teaching geometry is based upon the theory that teachers are more apt to teach geometry as they were or are taught geometry than as they are told to teach it. The subject-matter following, therefore, has a double function: to insure mastery of the content of high-school geometry and to establish a philosophy and technique of teaching geometry through the use of that philosophy and technique in the actual presentation of subject-matter. This double use of subject-matter, namely, to insure mastery and to establish a pattern for teaching, is what has been defined for this study as professionalized subject-matter.

[103] Ferriere, A. *The Activity School.* The John Day Co., New York, 1928, p. 34.
[104] *Ibid.*, p. 18.
[105] *Ibid.*, p. 19.

CHAPTER IV

GEOMETRY MATERIALS FOR THE APPLICATION OF THE PATTERN OF TEACHING

The Need for Mastery of High-School Materials. It is the contention of this study that the subject-matter for a professional course should be determined by the needs of the student preparing to teach rather than by traditional subject-matter standards in academic mathematics courses. In support of this contention, which has really been postulated in this study, there has already been quoted the statement by Professor Bagley that, in addition to other needs, the high-school teacher "needs courses in elementary algebra and plane geometry, which will not only refresh his mind with regard to elementary principles and processes, but also give him a much deeper and broader conception of those principles and a much more facile mastery of processes than his elementary course could possibly give him." R. B. Buckingham states that "it is a tragic mistake on the part of the academic professor ... to suppose that the moment the bare knowledge of a fact is attained it qualifies him to teach it to others."[106] There would be little difficulty in presenting overwhelming objective evidence to the effect that the college junior or senior has as a rule much too "bare a knowledge" of high-school geometry to teach it to others with confidence. Such evidence has been considered unnecessary and is therefore omitted. However, since it is conceded to be a bit difficult to teach what one does not know, the prospective teacher should not feel embarrassed if he has to spend much time with the essentials of high-school geometry which are outlined on the following pages of this chapter.

In case the teacher in a teachers' college is tempted to slight the high-school material, perhaps the testimony of Dean Emeritus James E. Russell, of Teachers College, Columbia University, will help to bolster up this emphasis on subject-matter. In speaking of professionalized subject-matter, Dean Emeritus James E. Russell says that such materials "cannot be judged by academic standards. The needs of the practitioner

[106] *Journal of Educational Research.* Vol. 16, p. 214.

in his practice are the sole standards for determining what shall be taught."[106] Dean William F. Russell in the *Teachers College Report of the Dean*, 1929, discusses "The Professional School Ideal," which forms "the basis for university work on which the professionally minded professor arranges his program of studies. It matters little to him whether the work be easy or hard, graduate or undergraduate, two-point or five-point, resident or extra-mural. His test is whether or not the work given prepares the student to practice his profession. . . . If taxidermy or blacksmithing be necessary, it is just as good as anthropology or Sanskrit to him. . . . His teaching depends upon research, he respects it, and he may even be a skilled investigator himself; but he has no faith in an academic tradition that forces his students to spend a major portion of their time painfully acquiring techniques that they will never use."[107] It is the plan of this chapter to present in outline form the essential theorems of high-school geometry because students preparing to teach need this material.

The Plan Used for Presenting High-School Materials. While it is recommended in this study that the essentials of high-school geometry be thoroughly mastered by the prospective teacher of high-school geometry, it is, nevertheless, recognized that in a dissertation which is not a textbook it would probably be difficult to defend the inclusion of any great amount of high-school geometry material. Liberal reference will therefore be made to good high-school texts. The postulation of the three congruence theorems, which has been advocated and defended in Chapter II, makes possible a much more simple and direct organization of materials than is found in most high-school texts, and consequently it has been thought advisable to outline in rather detailed form this very brief section on the fundamentals of high-school geometry.

The outline following will give a list of the most important undefined terms, definitions, and postulates used for the first two constructions. For the others the student will make his own lists. For the actual wording of the definitions and the actual making and proving of the constructions the student is referred to any high-school text. It is recommended, however, that the prospective teacher master very thoroughly this material in the order in which it is outlined. The material is purposely given in an order slightly different from the usual order so that it may be a challenge worthy of the ability of a college student, and so that it may provide

[107] Russell, Dean William F. *Teachers College Report of the Dean*, 1929.

experience with sequence. Care must be taken that no theorem or construction be used in any proof until that theorem or construction has itself been proved.

Furthermore, in order to emphasize the pattern of teaching geometry which this study advocates, some of the constructions and theorems are fully developed in the heuristic pattern. In getting others from high-school texts, usually only a synthetic proof will be found. The contrast, when considering the possibilities of each method for developing in the student the ability to reason things out for himself and really to understand what he is doing rather than to memorize the results of the thinking of other people, should make more emphatic than mere words could ever do the power and educational value of geometry taught by the heuristic method.

The list of constructions and theorems depends for its order upon the postulation of each of the three congruence theorems. If this postulation fails to meet the approval of any one using this outline, then it is entirely possible to prove each of the theorems by means of its traditional proof at or before the point in the sequence at which it is postulated. Furthermore, some readers may be interested in a different proof that is not traditional. If in place of postulate 9* the following postulate is accepted, the proof of "congruence by three sides" can be rather satisfactorily established.

The Construction Postulate, a Substitute for Superposition. If, at a given position on a given base line, only one triangle can be constructed by using a given set of conditions, then all triangles which have been constructed from, or which conform to, the given set of conditions are congruent.

It should be evident that "congruence by three sides" can be readily proved by means of this postulate and postulates 2 and 8. Then, following the construction of an angle equal to a given angle, the other two congruence theorems—"side-angle-side" and "angle-side-angle"—can be established by the use of this new postulate and postulates 2, 4, and 10. There seem to be three ways of handling the congruence theorems, (a) postulating them, (b) proving them by the postulation of both superposition and angle-bisection, and (c) establishing their reasonableness by this new "Construction Postulate," or by means of a somewhat different set of postulates such as those of Veblen or Forder.

* Postulates given on the following pages.

I. Construction 1

A. *Undefined terms used in the definitions, postulates, construction, and in the simple applications of construction 1:* point, line, straight line, curved line, measure, length, distance, equal, compass, describe a circle, rigid, ruler, lay off or cut off, definition, center, given, problem, procedure, proof, intersecting, greater than and less than and the symbols for these ideas, sum, difference, plane, straight edge, draw.

B. *Technical terms needing definition:* construct, circle, line segment, radius, diameter, postulate, axiom.

C. *Postulates:*

1. With a given center and radius one and only one circle can be drawn in a given plane.
2. By means of the compasses a length may be measured off equal to any given line segment. Or, a circle cuts its radius at one and only one point.
3. Line segments may be added or subtracted by the use of the compasses.
4. Only one straight line can be drawn between two points.

D. *Construction 1:* Construct a circle with a given center and a given radius.

Note: The construction is too simple for comment, yet a fundamental one. Notice how the definitions and postulates form a basis for the construction and for its applications.

II. Construction 2

A. *Undefined terms used:* the notations "line a" and "angle A," intersecting in a point, "equal respectively," corresponding parts, sides of a triangle, angles of a triangle, side opposite, direction, plane figure, end points, coincide, point or line in common, shape, size.

B. *Terms needing careful definition:* triangle, congruent triangles, angle, isosceles triangle, equilateral triangle, arc, vertex of an angle and of a triangle.

C. *Postulates:*

5. A straight line is the shortest distance between two points, and the shortest distance between two points is a straight line.
6. A line segment may be extended indefinitely in either direction.
7. Two circles can intersect in not more than two points.

8. If the distance between the centers of two circles is less than the sum of their radii and yet greater than the difference between their radii, the circles or arcs of the circles will intersect in two points, one on each side of the line joining their centers.
9. If two triangles have three sides of one equal respectively to three sides of another the triangles are congruent. (Abbreviation "3S")

D. *Construction 2:* Construct a triangle congruent to a given triangle, measuring only the lengths of its three sides.

Note: See any high-school text, and use postulate 9 above in the proof. Notice that the postulates form a basis for the construction and proof. Postulates 7 and 8 are frequently used yet rarely mentioned in high-school texts. This omission is not to be condemned but rather approved since too great a refinement in a system of postulates would make geometry much too difficult for high-school students.

III. Constructions 3-7

The first three of the following constructions can now not only be made, but very readily proved to be correct on the basis of the given postulates and the two previous constructions. For constructions 6 and 7, three new postulates will be needed in the proof. They are given below. The reader should make his own list of the undefined and defined terms needed. These constructions should all be rigorously proved to be what is claimed for them. This can easily be done by using postulates 1-12.

Postulate 10. Two straight lines can intersect at only one point.

Postulate 11. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent. (Abbreviation "S.A.S.")

Postulate 12. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent. (Abbreviation "A.S.A.")

Construction 3. Construct an angle equal to a given angle.

Construction 4. Bisect a given angle.

Construction 5. Construct a perpendicular to a line at a point on the line.

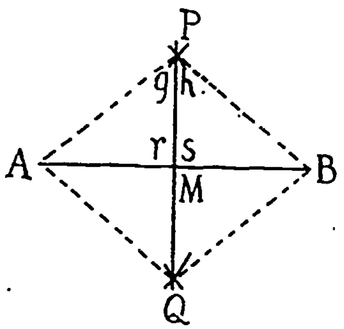
Construction 6. Construct a triangle congruent to a given triangle, measuring only two sides and their included angle.

Construction 7. Construct a triangle congruent to a given triangle, measuring only two angles and their included side.

IV. Construction 8

The following construction is the first one in which the proof is at all difficult and therefore it will be given in full. A second reason for giving this proof in full is to have it serve as a pattern for the proof of all construction problems. Since the technique of making this construction has probably been learned in the junior high school, no analysis of it will be given. However, the proof is not easily discovered; therefore, an analysis of the proof will be given to show how a proof may be discovered.

Problem: To construct a perpendicular bisector of a line segment.



*Procedure (Construction): (1) Take any line segment *AB*.

(2) With *A* and *B* as centers and with a radius more than half of *AB* cut arcs above and below line *AB* intersecting at points *P* and *Q*.

(3) Draw line *PQ*. It is the required perpendicular bisector.

Analysis: (Usually not written out, but entered here as a pattern for thinking out the solution of the exercises.)

(1) *PQ* would be perpendicular to *AB* and $AM = BM$ if triangle *APM* were congruent to triangle *BPM*.

(2) Triangles *APM* and *BPM* have $AP = BP$ by construction and $PM = PM$ by identity, and therefore they would be congruent if angle *g* equals angle *h*, or if $AM = BM$.

(3) Angle *g* would equal angle *h* if they were corresponding parts of a pair of congruent triangles other than *APM* and *BPM*.

(4) Angles *g* and *h* are corresponding parts of triangles *APQ* and *BPQ* and would therefore be equal if these triangles could be shown to be congruent.

(5) But triangles *APQ* and *BPQ* have $AP = BP$ and $AQ = BQ$ by

construction, and PQ common to each. They are therefore congruent by "three sides," Postulate 9.

(6) Therefore PQ can be proved to be the perpendicular bisector of AB.

*PROOF:

(1) Connect points A and B to P and Q thus forming two triangles APQ and BPQ.

(2) Triangle $APQ \cong$ triangle BPQ. "3 S" (Postulate 9).

(3) Consequently angle g = angle h. "C.p.c.t.e." (Corresponding parts of congruent triangles are equal.)

(4) Then in triangles APM and BPM,
$AP = BP$ by construction (Postulate 2).
$PM = PM$ by identity.
Angle g = angle h. See (3) above.

(5) Therefore triangle $APM \cong$ triangle BPM. "S.A.S." (Postulate 11).

(6) Therefore $AM = BM$ and angle r = angle s. C.p.c.t.e.

(7) Since angle r + angle $s = 180°$, each must be 90°. Each is half of 180°.

(8) PQ is therefore the perpendicular bisector of line AB.

* The headings, "Problem," "Procedure," and "Proof"—the "Analysis" will usually be mental, not written—are used because they seem more meaningful than the traditional headings for the corresponding parts of a construction problem.

V. CONSTRUCTIONS 9 AND 10

Construction 9 below can be readily made, and is proved much as construction 8 was proved. Construction 10 must be made by using only perpendiculars, or any other of the first nine constructions, until after the theorems on parallel lines in the next chapter have been proved. Needed definitions and postulates are to be stated by the reader. If it is postulated that "only one perpendicular to a given line can be drawn in a given plane and through a given point," then it will be evident that two lines in the same plane each perpendicular to a third line cannot meet and must therefore be parallel by the usual definition of parallel.

Construction 9: Construct a perpendicular to a line from a given point not on the line.

Construction 10: Through a given point construct a line parallel to a given line.

VI. THEOREMS 1-7

The first seven of the twenty "essential" theorems of plane geometry are stated below. Reference is made to any high-school geometry for their proofs. Theorems 2 and 3 are proved in various ways. A stimulating paper could be written on the different ways of proving these two theorems on parallel lines. Whichever one is proved first, the second is usually proved indirectly. Consequently it is recommended for this study that the second one be temporarily postulated, and that its proof be taken up at the same time that an indirect proof is given for theorem 9.

Suggestions for the proof of theorem 4 have been given in Chapter III. Next to theorem 10, theorem 4 is probably the most important theorem in plane geometry. It provides information which makes possible the solution of many problems. The applications of this important theorem merit careful study. Numerous corollaries, that is, numerous generalizations which are very readily deduced from this main theorem, could easily be added. A few suggestive ones are given. It is probably stretching the meaning of corollary a bit to include numbers 4 and 5, yet the suggestion given makes the conclusion very simply deduced from the main theorem.

Theorem 5 is not easy to prove, but theorem 6 is very simple. Any good textbook will give these proofs for a reader or student who has difficulty in recalling them or in discovering them anew. Theorem 7 is really the beginning of similarity. Its proof has been given in full in Chapter III, where it was used as an illustration of heuristic teaching.

Theorem 1. The angles opposite the equal sides of an isosceles triangle are equal.

Theorem 2. If two lines cut a third line so that the alternate interior angles are equal, the lines are parallel.

Theorem 3. The converse of theorem 2. If two parallel lines cut a third line, the alternate interior angles are equal.

Theorem 4. The sum of the angles of any triangle is 180°.

Corollaries:

(1) The acute angles of a right triangle are complementary.
(2) Each angle of an equilateral triangle is 60°.
(3) An exterior angle of a triangle equals the sum of the two non-adjacent interior angles.
(4) The sum of the interior angles of a polygon of n sides is $(n - 2)$

straight angles. (Suggestion for the proof: draw lines to the vertices of the polygon from any point within it.)

(5) The sum of the exterior angles of any polygon is four right angles.

(6) Two right triangles are congruent by any side and an acute angle of one being equal respectively to the corresponding parts of the other.

(7) Two isosceles triangles are congruent by "any side and one angle."

(8) Any two triangles are congruent by "one side and any two angles."

(9) No triangle can have more than one right angle or obtuse angle.

(10) Two isosceles right triangles are congruent if any side of one equals the corresponding side of the other.

Theorem 5. Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.

Theorem 6. The opposite sides and angles of a parallelogram are equal.

Theorem 7. If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

VII. Theorems 8 and 9

These two theorems are presented here in detail. They involve some new concepts in proportion and therefore furnish excellent examples of an inductive approach to a relatively new idea. They are consequently included as patterns for teaching a new portion of geometry. Theorem 9 involves an indirect proof and it is given as a sample of indirect proof. The indirect proof for theorem 3 should be taken up along with that of theorem 9 for additional experience with indirect proof. The work on proportion is probably the most difficult, yet the most interesting, and without doubt the most important part of geometry. It is the basis for trigonometry, for surveying, for map drawing, for all types of similar figures, and for all proportions in geometry. Its great and varied applications cannot be over emphasized, and consequently if any pattern theorems are to be presented it is well to present these because of their difficulty and also because of their importance.

In the proofs which follow it is assumed that the previous theorems and constructions have been proved and that all the definitions and postulates which were necessary have been given. A few new terms will be necessary, however.

A. *Definitions of new terms needed for theorems 8 and 9.*

1. *A ratio* between quantities is the result of dividing one by the other, such as a/b or 2/3.
2. *A proportion* is a statement of equality between two or more ratios, such as $a/b = c/d$ or $a:b = c:d$ or $2/3 = 10/15$, sometimes read "a is to b as c is to d," but usually "a over b equals c over d." Four quantities are required for a proportion.
3. *Two segments* of one line are *proportional* to those of another if the ratios between the segments are equal. Thus if a line six inches long has segments two inches and four inches, and another line nine inches long has two segments three inches and six inches, the segments are proportional because $2/4 = 3/6$, or $2/6 = 3/9$. Two lines with their segments proportional are often said to be *divided proportionally.*
4. Of the four terms of a proportion, the first and the last are often called the *extremes*, and the second and the third, the *means*. In the proportion, $a/b = c/d$, a and d are *extremes*, b and c are *means*.
5. If the second and third terms of a proportion are the same, such as $a/b = b/c$, $3/6 = 6/12$, then that term is called the *mean proportional* between the other two. b is the *mean proportional* between a and c, and 6 is between 3 and 12.

*APPROACH (for theorem 8):

(1) If a line bisects one side of a triangle and is parallel to a second side, what does it do with the third side? Use theorem 7.

(2) Draw a triangle and cut off one-third of one side. Construct a line through this point of trisection parallel to the second side. It will cut off what part of the third side?

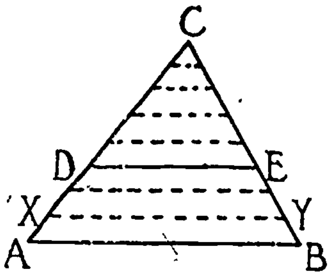
(3) Try (2) again, using one-fourth instead of one-third.

(4) Try constructing a triangle with two sides 8 and 16 inches or units long. Three inches from one end of the 8-inch side draw a line cutting the 16-inch side and parallel to the third side. What will be the segments on the 16-inch side? Will their ratio equal 3/5 as with those on the 8-inch side?

(5) Suppose in any triangle a line is drawn parallel to one side cutting the other two, what is true of the four segments formed?

(6) State this as a theorem.

*THEOREM: If a straight line is drawn through two sides of a triangle parallel to the third side it divides the two sides proportionally.



*HYPOTHESIS: Triangle ABC with DE cutting AC and BC, and parallel to AB.

*CONCLUSION: $\dfrac{AD}{DC} = \dfrac{BE}{EC}$ or $\dfrac{AC}{DC} = \dfrac{BC}{EC}$

*ANALYSIS:

(1) First let us analyse the specific case given in the approach under (4) to discover a method of analyzing the general case.

(2) In (4) the line corresponding to AD is 3 inches and to DC, 5 inches and the ratio of $\dfrac{AD}{DC} = \dfrac{3}{5}$. We need now to discover the ratio of $\dfrac{BE}{EC}$ with DE parallel to AB.

(3) BE is clearly not 3 inches long nor is EC 5 inches long, but if they could be shown to contain some other unit of measure 3 and 5 times respectively, then their ratio would be 3 to 5.

(4) If we lay off the 3 and 5 units on AC and through the points of division draw lines parallel to the base, these lines will cut equal segments on CB by theorem 7. There will be the same number of segments on BE as on AD and the same number on CE as on CD because the same parallels cut both lines. Therefore the ratio of $BE/EC = 3/5$, and therefore $AD/DC = BE/EC$ since both equal $3/5$.

(5) Now for the general case in which any line DE is parallel to the base and cuts the side AC in any point D. AD and DC would probably not have exactly 1 inch as a common unit of measure. However, they might have 1/4 inch, or 1/10 inch, or even

1/1,000,000 inch as a common unit which would be contained in each a whole number of times without a remainder.

(6) If AD and DC have a common unit of measure, no matter how small, the plan above of drawing parallel lines through the points of division of AC would always cut segments on BC which would be equal to each other. Also the number of segments on BE and EC would be the same number as on AD and DC since the same system of parallels cuts both sides of the triangles.

(7) If the ratio $\frac{AD}{DC} = \frac{m}{n}$ then $\frac{BE}{EC} = \frac{m}{n}$ and $\frac{AD}{DC} = \frac{BE}{EC}$.

(8) Or $\frac{AC}{DC} = \frac{BC}{EC}$ by adding 1 to each side of equation (7).

*PROOF:

(1) Assume that AD and DC have a common unit of measure such as AX which will be contained in AD and DC an integral number of times without remainders. (See *note* below.)

(2) Then that unit will be contained in AD some whole number of times, such as m, and in DC a whole number of times, such as n, and $\frac{AD}{DC} = \frac{m(AX)}{n(AX)} = \frac{m}{n}$.

(3) If parallels to AB be drawn through the points of division of AC, these parallels will cut m segments on BE and n segments on EC, all of which will be equal. See theorem 7.

(4) One of these segments, say BY, can now be used as a unit of measure, and $\frac{BE}{EC} = \frac{m(BY)}{n(BY)} = \frac{m}{n}$.

(5) Therefore $\frac{AD}{DC} = \frac{BE}{EC}$ since both ratios equal $\frac{m}{n}$.

(6) Also $\frac{AD}{DC} + 1 = \frac{BE}{EC} + 1$ or $\frac{AC}{DC} = \frac{BC}{EC}$, adding 1 to each side of (5) and simplifying.

*Various names are used in various texts for these portions of the demonstration. Since the headings used here are at least all nouns they have some advantage over such headings as "Given" and "To prove." However, this is merely a matter of form, and a teacher should use any he prefers, but probably those used in whatever text he is using.

Note: This assumption excludes incommensurable segments. The problem of incommensurables is discussed fully in Chapter V. It is omitted here because it is no longer considered suitable subject-matter for high-school courses. (See National Committee's Report, p. 49.) It should be pointed out, however, that the number of cases of incommensurability in comparison to the number of commensurable cases is as infinity to one. For instance, if you draw two line segments and say that one is $3\frac{5}{16}$ inches long and the other $8\frac{3}{4}$ inches long, these segments would have $\frac{1}{16}$ inch as a common unit of measure. However, the $3\frac{5}{16}$ inches is merely a crude approximation to the actual length, and merely indicates that its real length is perhaps closer to $3\frac{5}{16}$ inches than to $3\frac{11}{32}$, but we have no assurance that it is not 3.31246 inches long. The diagonal of a square and its side make a good illustration of incommensurability because the diagonal can be computed to a thousand or more decimal places if necessary, but will never be exactly determined in terms of the same unit which measures the side.

The assumption of commensurability is, however, a very fundamental one, in fact it is the assumption upon which all our measurements are based. We say a certain field is 20 rods long, a certain trip took $2\frac{1}{2}$ hours, a certain person weights $153\frac{1}{2}$ pounds, or a certain angle is 76° 38.6″. In each case we are giving merely a crude approximation based on the assumption of commensurability. If we did not make this assumption, all our measurements, except a very few, would be endless decimals. In Euclid's time, without our marvelous number system, the problem of incommensurable magnitudes was a major one. Now approximations to three or four decimal places satisfy all practical purposes and make incommensurability much less significant.

VIII. Exercises Following Theorem 8

Directions: Theorem 8 should be used, not so much as a pattern, but as information for proving the following exercises. Whenever line segments are to be proved proportional, try to get them parts of two sides of a triangle cut by a line parallel to the third side.

1. The corresponding segments cut off on two transversals by a series of parallels are proportional.
2. A line parallel to the one side of a triangle cuts the other two sides so that either side has the same ratio to either segment as the other side has to its corresponding segment.
3. If two parallel lines cut two intersecting lines, the two segments on one line, formed by the parallels and the point of intersection, will

be proportional to the two corresponding segments of the other line.

4. If a line is drawn from vertex A of the parallelogram $ABCD$, cutting BC in F and CD produced in G, then $AF/DC = FG/CG$.
5. In theorem 8 prove that since $CD/DA = CE/EB$ then $CA/DA = CB/EB$ and also that $CA/CD = CB/CE$.
6. Divide any line segment into parts proportional to two or more given segments.
7. Construct a "fourth proportional" to three given line segments.
8. What is the "mean proportional" between 4 and 25? 3 and 75? 7 and 28?
9. The base of a triangle is 20 feet. The other sides are 16 feet and 10 feet. A line parallel to the base cuts off 2 feet from the lower end of the shorter side. Find the segments of the other side.
10. The base of an isosceles triangle is 12 inches and the equal sides are each 16 inches. A line parallel to the base cuts off 3 inches on one of the equal sides. How much does it cut off on the other? A line parallel to one of the equal sides cuts off 3 inches on the other. How much does it cut off on the base?

IX. Theorem 9

Approach: (1) Draw a triangle, trisect two of its sides, and connect a pair of corresponding points. What seems to be true of that line and the base?

(2) Divide two sides of another triangle into segments having some other ratio and connect the points of division. Is the line still parallel to the base?

(3) State the theorem. How is it related to theorem 8?

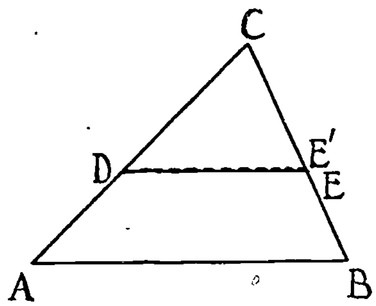
Theorem: If a line divides two sides of a triangle proportionally it is parallel to the third side. (Converse of theorem 8.)

Hypothesis: $\dfrac{CD}{DA} = \dfrac{CE}{EB}$

Conclusion: DE is parallel to AB.

Analysis: (1) Converses are usually proved by the indirect method, therefore, we can start by stating that DE is either parallel to AB or it is not parallel.

(2) Let us assume DE not parallel to AB and see whether that assumption leads to an absurd or impossible situation which would make the assumption itself impossible.



(3) If DE is not parallel to AB, it would be possible to draw DE' parallel to AB cutting BC in E'.

(4) Then $\frac{BE'}{E'C} = \frac{AD}{DC} = \frac{BE}{EC}$, or $\frac{BE'}{E'C} = \frac{BE}{EC}$

(5) From (4) we note that both E and E' divide BC into segments which measured in the same direction have the same ratio. But this is no more possible than for a line to have two middle points or two different points which cut off one-third of it from the same end.

(6) Therefore, the assumption that DE is not parallel to AB is untenable and consequently DE is parallel to AB.

(7) Or, instead of (3) and (4), if DE is not parallel to AB, it would be possible to construct AB' parallel to DE cutting CB or CB extended in B'. Then $CE/EB' = CE/EB$ which is impossible.

PROOF: Left to the student to obtain from the analysis, with the suggestion that he look up this proof in some good geometry text for a different form making more use of the ratios.

X. Exercises Following Theorem 9

DIRECTIONS: The following exercises can be proved indirectly as was done with theorem 9. However, if the information gained in theorem 9 is used, the exercises can all be proved directly, using theorem 9.

11. The base BC of the triangle ABC is divided into five equal parts and each division point connected with A. Line DE is drawn through the middle points of AB and AC. What is true of DE and the four lines drawn?

12. Prove that:
If a line is drawn through the midpoints of two sides of a triangle, it is parallel to the base and equal to half of it.
13. Prove that:
If the middle points of the sides of an isosceles triangle are joined to the middle point of the base, a rhombus is formed.
14. By the use of Exercise 12 find a way of measuring the distance across a lake. Note: Let the unknown distance be the base of the triangle.
15. In the rectangle, $ABCD$, let M and N trisect AB; O and P trisect BC; R and S trisect CD; and X and Y trisect DA. What is true of the lines NO, MP, AC, YR, and XS? Prove it.
16. If E is the midpoint of side AB of the isosceles triangle ABC, and EF is perpendicular to the base BC, show that BF is $\frac{1}{4}$ BC. Note: Draw EG to G, the midpoint of BC.
17. If G and H are the midpoints of DC and AB respectively of parallelogram $ABCD$, prove that DB is trisected by AG and CH at points M and N. Suggestions: Show that AG is parallel to CH, that AG bisects DN, and that CH bisects MB.
18. Prove that:
The median of a trapezoid is parallel to the bases and equal to one-half their sum. Note: Extend the lower base DC to G so that $CG = AB$. Connect AC and BG, forming a parallelogram $ABGC$. Why will the diagonal AG go through the midpoint of BC? Then can you show that the median $EF = \frac{1}{2}$ the sum of AB and DC?
19. Prove theorem 3 using an indirect proof.
20. Make up a problem, geometric or non-geometric, which can be solved or proved indirectly.

XI. Theorems 10-14

The treatment of the following theorems in any good high-school text is usually very adequate, except of course that the inductive approach and the analysis are omitted. Two other theorems are usually given following soon after theorem 10. These involve the other conditions for similarity and are important. They are, however, really more or less of the nature of corollaries of theorem 10 and are not themselves theorems of outstanding importance. Theorem 10 is without doubt the most important theorem in geometry. Theorem 11 has a very interesting history, many simple applications, and has over 200 different proofs.[108] All theorems on areas depend on theorem 12, the area of a triangle, which in turn depends upon the postulate that the area of a rectangle equals the product of its base and altitude. The postulation of this relation for the area of a rectangle is recommended by the National Committee. Theorems 13 and 14 are fundamental theorems on locus. In connection with these two theorems and their application the double-headed meaning of the word locus needs to be carefully appreciated. Notice how one of these meanings is the converse of the other. See any good high-school text, and Chapter V, section IV Converses, of this study.

Theorem 10: Two triangles are similar if two angles of one are equal respectively to two angles of the other.

Corollaries:

1. Two triangles are similar if the sides of one are (a) parallel to the sides of the other, or (b) perpendicular to the sides of the other.
2. Two right triangles are similar if an acute angle of one equals an acute angle of the other.
3. Two isosceles triangles are similar if an angle of one equals the corresponding angle of the other.
4. All isosceles right triangles are similar.
5. All equilateral triangles are similar.
6. Two triangles are similar if an angle of one equals an angle of the other and the including sides are proportional.
7. Two triangles are similar if the sides of one are proportional to the sides of the other.
8. In any right triangle ABC with given acute angles A and B the ratios a/b, a/c, and b/c, are constant. (They are respectively the tangent, sine, and cosine of angle A.)

Theorem 11: In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Corollary:

1. In any right triangle if a perpendicular be dropped from the vertex of the right angle to the hypotenuse,

   (a) the two right triangles formed are similar to the given triangle and to each other,

[108] Loomis, E. S. *The Pythagorean Theorem.* Masters and Wardens Association of 22nd Masonic District of the M. W. Grand Lodge F. & A.M., Ohio, 1110 Webster Ave., S.E., Cleveland, Ohio, 1927. Price $2.00.

(b) either leg of the given right triangle is a mean proportional between the whole hypotenuse and the adjacent segment,

(c) the perpendicular is the mean proportional between the segments of the hypotenuse.

Theorem 12: The area of a triangle is equal to half the product of the base times the altitude.

Note: Corollaries concerning the area of a parallelogram and a trapezoid could be readily proved by means of theorem 12.

Theorem 13: The locus of a point equally distant from two points is the perpendicular bisector of the line segment joining them.

Theorem 14: The locus of a point equally distant from two intersecting lines is the pair of lines which bisect the angles formed by the given lines.

XII. Theorems on Circles, 15–20

The following six theorems are usually considered very easy and are well treated in most texts, except again for the heuristic trend. Several new definitions and postulates will be necessary. Theorems 15 and 17 each have two important converses, all four of which are very easily proved indirectly. The student should attempt to prove these theorems and make his own list of definitions and postulates, then finally compare with the proofs and lists of definitions and postulates given in some high-school text. An abundance of easy original exercises can also be found in almost any text. Most textbooks present the first three of these theorems before similar triangles. The proof of these three depends in no way upon similar triangles, nor do any of the proofs for the theorems 7 to 14 depend upon these three theorems. Consequently, their placement is purely arbitrary. They are probably easier than the theorems involving proportions and therefore are usually placed early. In this study, they are placed last so as to have all theorems on straight-line figures together and the six theorems on circles together. The last three theorems are concerned with the measurement of the circle. For these theorems the circumference and area of a circle are defined respectively as the limits described in theorem 18.

Theorem 15: A diameter perpendicular to a chord bisects the chord and the arcs of the chord.

Corollaries:

(1) A diameter which bisects a chord is perpendicular to the chord.

(2) A line which is the perpendicular bisector of a chord passes through the center of the circle.

Theorem 16: An angle inscribed in a circle is equal to half the central angle having the same arc.

Corollaries:

(1) An angle inscribed in a semi-circle is a right angle.

(2) An angle between a tangent and a chord is equal to half the central angle having the same arc.

Note: Corollaries, readily proved by a slight extension of the main theorem, such as the measurement of the angle between two chords, secants, or tangents, could probably be legitimately added. There might be some question concerning the use of the term corollary for classifying them. They are probably rather in the nature of subtheorems or exercises which depend almost entirely upon theorem 16 for their proof.

Theorem 17: A line perpendicular to a radius at its outer extremity is tangent to the circle at that point.

Corollaries:

(1) A radius of a circle drawn to the point of contact of a tangent is perpendicular to the tangent.

(2) A perpendicular to a tangent at its point of contact with the circle passes through the center of the circle.

(3) Problem: To construct a tangent to a circle (*a*) at a point on the circle, (*b*) from a point outside the circle.

Theorem 18: If the number of sides of a regular inscribed polygon is indefinitely increased, its perimeter and area will both increase, while the perimeter and area of the circumscribed polygon, formed by drawing tangents to the circle, at the vertices of the inscribed polygon, will both decrease. The perimeters and areas of both polygons will each approach a limit.

Theorem 19: The ratio of any circumference to its radius is constant and is equal to 2 pi.

Theorem 20: The area of a circle is equal to pi times the square of the radius.

Definitions and postulate needed for theorems 18, 19, and 20.

In order to provide a pattern proof for theorems using the idea of approaching a limit, and to present rigorously the new terms used in theorems 18, 19, and 20, the proof for theorem 19 is given in full. Theorem

18 is easily proved, its proof depending upon the postulate that a straight line is the shortest distance between two points and the following reasoning. The perimeter of the circumscribed polygon, *Pc*, is always greater than that of the inscribed polygon, *Pi*, therefore if *Pc* decreases and *Pi* increases because of increasing the number of sides and yet *Pc* always remains larger than *Pi*, then both *Pc* and *Pi* must approach a limit beyond which neither can go.

Variable and Constant. The term variable is used in mathematics with two meanings. Broadly speaking, any quantity which varies, changes in value, is a variable. In theorem 18, *Pc* and *Pi* are variables, yet neither one depends upon the other, although both of them depend upon the radius of the circle and the number of sides of the inscribed polygon.

The second meaning of variable takes into consideration two related quantities. When two quantities are so related that changes in the value of one of them causes changes in the value of the other, the first is called an *independent variable* and the second a *dependent variable; e.g.* the radius and the circumference of a circle are two such variables, changes in the radius being accompanied inevitably by changes in the circumference and area. The two proofs for theorem 19, which follow will illustrate these two uses of the term variable, one meaning commonly used in geometry, the other more commonly used in algebra. In contrast to a variable is a *constant,* a quantity which does not change, such as the number 2. (Reference: Any good geometry and college algebra textbook.)

Approach a limit: When a variable approaches a constant in such a way that the difference between it and the constant becomes and remains less than any given positive quantity, however small, the variable is said to approach the constant as a limit.

The *circumference* of a circle (crudely defined) is the length of the curved line forming the circle. However, since length as measured by the compasses or by any unit of measure is always on a straight line and there are no curved units of length, it is necessary to have a more precise definition.

The circumference of a circle is the limit approached by the perimeter of its regular inscribed polygon as the number of sides is indefinitely increased.

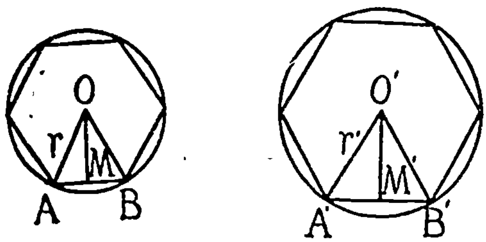
The *area of a circle,* since it is impossible to fit square or triangular units into a curved surface, is defined as the limit approached by the area of the inscribed regular polygon as the number of sides is indefinitely increased.

The *apothem* of a polygon is the radius of its inscribed circle. (Needed in theorem 20.)

Pi (π) is the ratio between the circumference of a circle and its diameter. Therefore $c/d = \pi$, $c = \pi d$, or $c = 2\pi r$. Pi is the first letter of the Greek word meaning circumference. The purpose of theorem 19 is to show that π is constant, no matter how large or small the circle.

Postulate of Limits. If two variables are always equal and each approaches a limit, the limits are equal.

Theorem 19: The ratio between the circumference and the radius of a circle is constant and is equal to 2π.



HYPOTHESIS: A circle with radius r.

CONCLUSION: $\frac{c}{r} = 2\pi$ or $\frac{c}{d} = \pi$, which is a constant.

PROOF (A):

(1) Draw a second circle with any radius r' and inscribe regular polygons of n sides in both circles.

(2) Draw radii from a pair of adjacent vertices, thus forming two triangles, ABO and $A'B'O'$.

(3) These triangles are similar. Why?

(4) Then $\frac{r}{r'} = \frac{AB}{A'B'}$ or $\frac{p}{p'} = \frac{r}{r'}$. Why? ($p$ means perimeter).

(5) Therefore $\frac{p}{r} = \frac{p'}{r'}$ Multiplying (4) by $\frac{p'}{r}$.

(6) Equation (5) is true regardless of the number of sides providing the inscribed polygons are regular, and each has the same number of sides (n).

(7) Let the number of sides (n) increase indefinitely. Then $\frac{p}{r}$ will ap-

proach $\frac{c}{r}$ and $\frac{p'}{r'}$ will approach $\frac{c'}{r'}$ (r and r' being constant).

(8) Therefore $\frac{c}{r} = \frac{c'}{r'}$, because $\frac{p}{r}$ and $\frac{p'}{r'}$ are two variables which are always equal as they approach certain limits. Therefore the limits are equal by the postulate of limits.

(9) To say that $\frac{c}{r} = \frac{c'}{r'}$ merely means that regardless of the size of the circle the ratio between its circumference and radius is constant. This constant is usually written as 2π, therefore

(10) $\frac{c}{r} = 2\pi$ or $c = 2\pi r$, also $c = \pi d$.

Proof (B):

(1) Inscribe a regular polygon of n sides in the given circle, draw radii to two adjacent vertices forming triangle AOB, and draw OM perpendicular to AB.

(2) Then OM bisects AB (s) and angle AOB.

(3) Angle $AOM = \frac{180°}{n}$. Why?

(4) $\frac{s}{2} = r \sin \frac{180°}{n}$. By definition of sine.

(5) Then $p = 2\,nr \sin \frac{180°}{n}$. Why?

(6) Or $p = 2r(n \sin \frac{180°}{n})$. In this equation p and n are the variables; 2, r, and 180° are constants. As n varies, p will vary, but will always be evaluated by the formula expressed in (6). It remains to show that $n \sin \frac{180°}{n}$ approaches a constant. It is not possible to prove this by the methods of elementary mathematics, yet the following table should establish it beyond any reasonable doubt, because it is evident that as n increases beyond even 100, there is very little increase in $n \sin \frac{180°}{n}$ which seems to approach a definite constant as a limit. Shanks in 1873 computed to 707 deci-

mal places the value of this constant. To 50 places it is 3.141,592, 653,589,793,238,462,643,383,279,502,884,197,169,399,375,1.

n	$n\,(\sin \frac{180°}{n})$*
6	3.000000
12	3.105827
24	3.132628
50	3.139527
100	3.141075
200	3.141463
500	3.141571
1000	3.141587
2000	3.141591
10,000	3.141592

*Computed by means of seven-place logarithm tables.

XIII. Theorems 21 and 22 (Solid Geometry)

These theorems require for their proof additional definitions and postulates. The student should write these as needed.

Theorem 21: If two planes intersect, their intersection is a straight line.

Theorem 22: If two parallel planes cut a third plane, the lines of intersection are parallel.

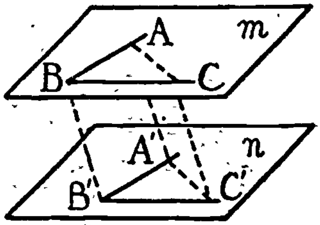
XIV. Theorem 23

Theorem 23 illustrates beautifully the dependence of solid geometry upon plane geometry and also the difficulty of representing solid figures on a flat surface. It serves admirably as a "solid pattern" for proving theorems.

Theorem: If two angles not in the same plane have their sides parallel in the same sense, the angles are equal.

Hypothesis: Angles ABC and $A'B'C'$ with AB parallel to $A'B'$ and BC parallel to $B'C'$.

Conclusion: Angle ABC = Angle $A'B'C'$.



PROOF: (1) Lay off $AB = A'B'$ and $BC = B'C'$.
(2) Draw AC, $A'C'$, AA', BB', CC'.
(3) $BB'A'A$ is a parallelogram. Two sides equal and parallel.
(4) $BB'C'C$ is a parallelogram. Why?
(5) $AA' = BB' = CC'$ and AA' is parallel to BB' and CC'. Why?
(6) Therefore $ACC'A'$ is a parallelogram. Why?
(7) Therefore $AC = A'C'$. Why?
(8) Therefore triangle $ABC \cong$ triangle $A'B'C'$. 3S.
(9) Therefore angle ABC = angle $A'B'C'$. Why?

XV. THEOREMS 24–28

Theorem 24: The lateral area of a prism is the product of a lateral edge by the perimeter of a right section.

Theorem 25: The volume of any prism equals the product of its base by its altitude.

a. An oblique prism is equal to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism.

b. The volume of any parallelopiped equals the base times the altitude.

c. A diagonal plane divides a parallelopiped into two equal triangular prisms.

d. The volume of any triangular prism equals the product of the base and altitude.

e. The volume of any prism is the product of its base and altitude.

Note: Theorem 25 is outlined as traditionally proved and the proof is long and difficult. Some modern texts shorten the proof very greatly by the postulation of Cavalieri's Theorem or its equivalent: "If two solids have equal altitudes and if sections of one made by planes parallel to the base are equal respectively to the corresponding sections of the other, then the solids are equal in volume." The use of this theorem as a postulate to simplify the proof of theorem 25 is highly commended. Theorem 25 depends upon yet a second theorem which, can be postulated if the recommendations of the National Committee are to be accepted: "The volume of a rectangular solid is the product of its three dimensions ($V = lwh$)." Since for any prism a right prism with a rectangular base equal to the base of the first can be constructed, it is evident that the use of Cavalieri's theorem makes theorem 25 easily proved, omitting *a, b, c, d,* and *e.*

Theorem 26: The lateral area of a regular pyramid equals 1/2 the product of its slant height and the perimeter of the base.

Theorem 27: If a pyramid is cut by a plane parallel to the base and a distance d from the vertex,

a. The lateral edges and altitude (h) are divided proportionally.
b. The section is similar to the base, the "ratio of similitude" being d/h.
c. The ratio of the area of the section to the area of the base is d^2/h^2.

Note: Theorem 27 is simple yet provides the solution for many very tricky and puzzling exercises which most textbooks have in abundance.

Theorem 28: The volume of a pyramid equals 1/3 the product of the area of its base by its height.

a. Two pyramids with equal bases and altitudes are equal.
b. The volume of a triangular pyramid equals 1/3 bh.
c. The volume of any pyramid equals 1/3 bh.

Note: Theorem 28 brings into geometry a very important fraction, 1/3. In plane geometry the only fraction used anywhere in formulas was 1/2. Now, as also in theorems 30 and 32 following, 1/3 comes into use, and it is important that students appreciate fully where it comes from. The proof of theorem 28 is made more simple if Cavalieri's theorem is used as with theorem 25.

XVI. Theorem 29

In order to have another pattern proof using the postulate of limits, which was also used for theorem 19, theorem 29 is completely proved. Furthermore the definitions of the areas and volume of solids with curved surfaces are not always clearly stated. Therefore, this proof with its definitions and postulate will serve a double purpose.

Definitions:

1. A *cylindrical surface* is a surface generated by a straight line moving along a curved line in such a way as to remain always parallel to some line not in the plane of the curve. A closed cylindrical surface has a closed curve, such as a circle or an ellipse, to direct the generating line. Any single position of the generating line is called an *element.*
2. A *cylinder* is a portion of a closed cylindrical surface included between two parallel planes cutting all the elements. What is a circular cylinder? A right circular cylinder? A section? A right section of a cylinder?
3. The terms area and volume as heretofore used have applied only to flat surfaces or straight line solids. The *area and volume of a cylinder* will now be defined as the limit approached by the corresponding area or volume of a circumscribed prism, as the number of sides is indefinitely increased.

THEOREM: (The "Approach" is omitted because the formula is an old familiar one.) The lateral area of a cylinder equals an element times the circumference of a right section ($L.A. = ec$), and the volume of a cylinder equals the area of the base times the altitude ($V = bh$).

HYPOTHESIS: Any cylinder.

CONCLUSION: (a) $L.A. = ec$

(b) $V = bh$

ANALYSIS: Since the lateral area and volume of a cylinder correspond with those of a prism, if the formulas for the lateral area and volume of a prism are taken and the number of the sides of the prism indefinitely increased, the effect on the formulas of this increasing number of sides could be found.

PROOF IS:

(a) (1) Circumscribe a prism about the cylinder, its lateral area $L.A.P. = ep$.

(2) Let the number of sides of the prism increase indefinitely, then its lateral area will vary and approach the lateral area of the cylinder as a limit, by the definition of lateral area of a cylinder. Furthermore, the perimeter of the right section of the prism approaches the circumference of the right section of the cylinder as a limit, by theorem 18.

(3) But $L.A.P. = ep$ always regardless of the number of sides.

Therefore *L.A.* (cylinder) $= ec$ by the postulate of limits.

(b) The corresponding proof for the volume of a cylinder is left to the student.

XVII. Theorems 30-32

The proofs for these theorems are quite standard and can easily be found even though theorem 31 is rather involved.

Theorem 30: The lateral area of a right circular cone equals 1/2 the product of the slant height by the circumference of the base, and the volume of any cone equals 1/3 the product of the area of the base by the altitude.

Theorem 31: The area of a sphere equals 4 pi times the radius squared.

a. The area of the frustrum of a cone of revolution equals 1/2 slant height times the circumference of a circle half way between the bases.

b. The area of a surface of revolution formed by revolving a regular polygon about a diameter is 2 pi times the apothem times the diameter.

c. The area of a sphere equals 4 pi times the radius squared.

Theorem 32: The volume of a sphere equals 4/3 pi times the radius cubed.

XVIII. Conclusion and Summary

The purpose of this chapter has been two-fold: (1) to outline work for the prospective teacher in order to insure his mastery of the essential theorems of plane and solid high-school geometry, (2) to familiarize the reader more fully with the heuristic method of teaching and learning geometry. This has been attempted through the use of the heuristic method in presenting a few of the theorems of high-school geometry. Other important theorems were given in the outline, but their proof was left to the student to get by himself or to find in some high-school text. It is assumed that the careful student will not only master these few theorems but also that he will prove the corollaries and will work a liberal number of "originals" found in the high-school texts.

The following chapter will present a few features of geometry and an abundance of original exercises for practice. These exercises should serve to strengthen and test a student's mastery of the technique and the content of high-school geometry.

CHAPTER V

SOME FEATURES OF GEOMETRY AND EXERCISES FOR ANALYSIS AND INDIRECT PROOF

Introductory Statement. The purpose of this chapter is to present in a fairly complete form certain outstanding features of geometry, especially the analytic method and the indirect proof. Students preparing to teach, or teachers in service, will find in this chapter not only these vital topics but also ample problem material carefully selected and difficult enough to test and to insure mastery of the subject-matter of geometry. A further function of this material is that of providing an application for the methods advocated in Chapter III and illustrated in Chapter IV. The material would fail to achieve its highest function if it failed to make possible the use of the heuristic pattern of teaching and learning which it is the purpose of this study to present. The section on continuity brings out a fascinating concept, and the treatment of incommensurables presents briefly a problem of rapidly decreasing importance. The discussion of the structure of geometry emphasizes the nature of the science and its dependence upon postulates and definitions.

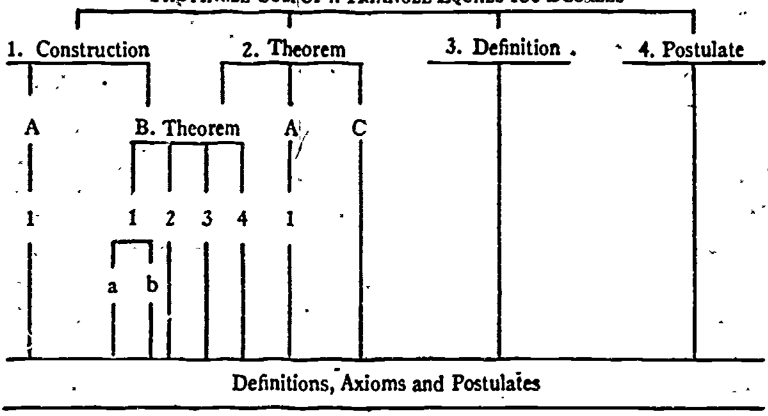
I. The Structure of Geometry

Analysis of the Proof of an Important Theorem. The structure of geometry is well shown by a simple project consisting of the analysis of the proof for theorem IV: The sum of the interior angles of a triangle is 180°. Its proof depends upon four other relationships, namely, (1) the construction of a line through one vertex parallel to the opposite side, (2) the theorem that alternate interior angles are equal when two parallel lines cut a third line, (3) the definition of a straight angle, and (4) the axiom that equals may be substituted for equals. The construction (1) and theorem (2) are both dependent upon other relationships. The definition and the axiom are arbitrarily accepted without proof. The following outline and chart indicate the various interrelationships and the fundamental dependence of one theorem upon other theorems, and eventually the dependence of all upon arbitrary definitions and postulates.

Theorem to be proved: The angle-sum of a triangle is 180°.

1. Construct a line parallel to AB.
    - A. Construct an angle equal to angle B.
        - (1) Congruence by three sides. (Postulate)
    - B. Theorem: If two lines cut a third making the alternate interior angles equal the two lines are parallel.
        - (1) Construction of midpoint of line segment.
            - a. congruence by 3S (Postulate)
            - b. congruence by S.A.S. (Postulate)
        - (2) Vertical angles are equal. (Postulate)
        - (3) Congruence by angle-side-angle. (Postulate)
        - (4) Two lines perpendicular to the same line are parallel. (Postulate).
2. Theorem: If two parallel lines cut a third line the alternate interior angles are equal.
    - A. The construction of an angle equal to a given angle.
        - (1) Congruence by 3S. (Postulate)
    - B. Theorem: If the alternate interior angles are equal the lines are parallel. (See B under 1 above.)
    - C. Postulate of parallels.
3. Definition of a straight angle.
4. Substitution of equals for equals. (Axiom)

THE ANGLE-SUM OF A TRIANGLE EQUALS 180 DEGREES



This chart shows the dependence of theorems and constructions upon other theorems and constructions, and eventually that definitions and axioms form the foundation for the entire structure in reasoning. The letters and figures refer to the previous outline.

Conclusions and Suggested Problem. It is evident from this illustration that "demonstrative geometry" is a structure composed of (1) definitions, (2) axioms and postulates, (3) constructions, and (4) theorems. The theorems and constructions are based on definitions, postulates, and previously proved theorems. These theorems in turn are based upon other definitions, postulates, and perhaps theorems, until all rest finally on an ultimate basis of arbitrary definitions and accepted postulates or axioms.

The elements that are of outstanding importance in this organization are the seeing of relationships, the building of the structure by the analysis of relationships, and the appreciation of the dependence of one relationship upon others. The very nature of geometry makes it necessarily and fundamentally a course in reasoning, a course in discovering and proving relationships on the basis of known facts and known relationships.

The student should take some important theorem such as number 8 or 10 and make a similar analysis to show this fundamental nature of the science he is studying.

II. The Analytic Method Applied to Geometric Originals

The Plan for Completing the Presentation of Analysis. In Chapter III, analysis and synthesis were presented as essential parts of the heuristic method which has been advocated in this study as efficacious in teaching high-school geometry. In Chapter 4, several illustrations of analysis have been given and if these illustrations have served as patterns for the proof of originals, then the analytic method has been amply illustrated with familiar material. However, the real test of the power of the analytic method comes in the attack of difficult originals. The purpose of this part of Chapter V is to provide that experience, but to preface it by further illustration and discussion of analysis. First, however, in order not to break the continuity of thought later and also for review purposes, a list is given of the methods or devices used in proving easy originals and theorems. The coordination of these methods into a composite plan of attack will constitute the analytic method as applied to the solution of difficult problems.

Summary of Devices or Methods Available for Proving Geometric Statements.

1. Two line segments are equal if they are
   (1) equal by hypothesis, by construction, or by identity.
   (2) corresponding parts of congruent triangles.
   (3) both equal to the same segment or to equal segments.

(4) sides of a triangle opposite the two equal angles of the triangle.
(5) opposite sides of a parallelogram.
(6) parallels cut off by parallels.
(7) chords having equal arcs or equal central angles.
(8) chords equally distant from the center of the same circle or of equal circles.

2. Two angles are equal if they are
(1) equal by hypothesis, by construction, or by identity.
(2) corresponding parts of congruent triangles.
(3) angles opposite the equal sides of an isosceles triangle.
(4) alternate interior angles formed by two parallel lines cutting a third.
(5) corresponding angles formed by two parallel lines cutting a third.
(6) alternate exterior angles formed by two parallel lines cutting a third.
(7) Angles whose sides are respectively parallel "left to left and right to right."
(8) angles whose sides are respectively perpendicular "left to left and right to right."
(9) opposite angles of a parallelogram.
(10) both equal to the same or to equal angles.
(11) corresponding angles of similar figures.
(12) inscribed in the same arc or in equal arcs.
(13) measured by the same arc or by equal arcs.

3. Two angles are supplementary if they
(1) have their sum equal to 180°, a straight angle.
(2) are so given in the hypothesis.
(3) are interior angles on the same side formed by two parallel lines cutting a third.
(4) are exterior angles on the same side formed by two parallel lines cutting a third.
(5) have their sides parallel "left to right and right to left."
(6) have their sides perpendicular "left to right and right to left."
(7) are consecutive angles in a parallelogram.

4. Two angles are complementary if they
(1) have their sum equal to 90°.
(2) are so given in the hypothesis.
(3) are the acute angles of a right triangle.

5. Two lines are perpendicular if
   (1) they are so given in the hypothesis.
   (2) they are constructed perpendicular.
   (3) they meet so as to form a right angle.
   (4) one bisects the straight angle formed by the other.
   (5) they form equal supplementary angles.
   (6) each is parallel to one of two other lines which are perpendicular.
   (7) each is perpendicular to one of two other lines which are perpendicular.
   (8) they are sides of an angle inscribed in a semicircle.
6. Two lines are parallel if
   (1) they are so given in the hypothesis.
   (2) they are constructed parallel.
   (3) they are everywhere equally distant.
   (4) the alternate interior angles, formed by the two lines cutting a third, are equal.
   (5) the corresponding angles are equal.
   (6) the alternate exterior angles are equal.
   (7) the interior angles on the same side are supplementary.
   (8) the exterior angles on the same side are supplementary.
   (9) they are opposite sides of a parallelogram.
   (10) they are both perpendicular to a third, or both parallel to a third line.
   (11) one is the base of a triangle and the other line divides the other two sides of the triangle proportionally.
7. Any two triangles are congruent if
   (1) three sides of one equal respectively three sides of the other (3S.)
   (2) side-angle-side of one equal respectively side-angle-side of the other. (S.A.S.)
   (3) angle-side-angle of one equal respectively angle-side-angle of the other. (A.S.A.)
   (4) two angles and any side of one equal respectively the corresponding parts of the other.
8. Any two right triangles are congruent if
   (1) the hypotenuse and an acute angle of one equal respectively the hypotenuse and an acute angle of the other. (H.A.)
   (2) the hypotenuse and a side of one equal respectively the hypotenuse and a side of the other (H.S.)

(3) any side and an acute angle of one equal respectively the corresponding parts of the other.
(4) any two sides of one equal respectively the corresponding sides of the other.

9. Any two isosceles triangles are congruent if
    (1) one side and any angle of one equal respectively the corresponding side and angle of the other.
    (2) the base and altitude of one equal respectively the base and altitude of the other.
    (3) they are right triangles and one side of one equals respectively the corresponding side of the other.
    (4) the base and one of the equal sides of one equal respectively the base and one of the equal sides of the other.

10. A quadrilateral is a parallelogram if
    (1) its opposite sides are parrallel.
    (2) its opposite sides are equal.
    (3) one pair of opposite sides is both equal and parallel.
    (4) its opposite angles are equal.
    (5) any two consecutive angles are supplementary.

11. Two segments on one line will be proportional to two segments on another
    (1) if their ratios are equal to the same quantity or to equal quantities.
    (2) if they are cut off by parallels.
    (3) if they are parts of two sides of a triangle cut off by a line parallel to the third side.

12. Any two segments will be proportional to two other segments
    (1) if their ratios are equal.
    (2) if they are corresponding parts of similar triangles.
    (3) if two segments are two sides of a triangle and the other two are the segments of the third side formed by the bisector of the angle between the first two.

13. Two triangles are similar
    (1) if they satisfy the definition of similar triangles.
    (2) if they have two angles of one equal to two angles of the other.
    (3) if they have one angle of one equal to one angle of the other, and the including sides proportional.
    (4) if they have the three sides of one proportional to the three sides of the other.

(5) if they are right triangles with an acute angle of one equal to an acute angle of the other.

(6) if they are isosceles triangles with one angle of one equal to the corresponding angle of the other.

(7) if they have their sides respectively parallel, or respectively perpendicular.

14. The product of two segments equals the product of two others

(1) if one pair is the means and the other pair is the extremes of a proportion.

(2) if they are parts of similar triangles and are so related that a proportion may be set up with one pair the means and the other pair the extremes.

(3) if they are parts of intersecting chords or secants.

15. Two areas will be equal

(1) if they are inclosed by triangles or parallelograms with the same base and altitude.

(2) if they are measured by two pairs of products which can be shown to be equal.

16. Since an angle is a right angle if inscribed in a semi-circle, a perpendicular can be constructed by constructing a semi-circle with an angle inscribed.

17. The locus of a point a given distance from a point or a line, or the locus of a point equally distant from two points or two lines or any combination of two points and two lines can be found by the simple application of the definition of locus and theorems 13 and 14.

18. The locus of a point satisfying two or more sets of conditions may be found by finding first a locus for each set of conditions separately, and then the intersection of these loci will satisfy all the sets of conditions simultaneously.

19. In situations involving limits, set up an equation from a known analagous situation, then by some process make the two sides of the equation become variables. If these variables can then be shown to remain equal as they approach their limits, these limits will be equal.

Note: These nineteen statements of conclusions to be proved with the several ways in which each may be proved are not given with the thought that they constitute a complete list. The conditions which make a triangle isosceles, a line straight, and others, are omitted. The given list is suggestive rather than exhaustive.

20. General devices.

(1) Often it may not be possible to prove two triangles congruent at once because the equality of needed parts is not known. In such cases there may be other triangles whose congruence establishes the desired equality. See constructions 8 and 9.

(2) To prove lines or angles equal when no triangles are available the construction of an auxiliary line often solves the difficulty. See constructions 3, 4, 5, and theorems 1, 2, 4, 5, 6, 8, 15, and 16. This auxiliary line may join crucial points, bisect an angle, bisect a line, be parallel or perpendicular to a line and through a crucial point, or be drawn so as to involve any known construction.

(3) To prove a converse theorem or statement an indirect proof is often very useful.

(4) To make a construction to satisfy certain conditions, assume the construction made and then analyze the figure to discover the relations which determine the figure from the given facts, and upon which the procedure depends. Note: This technique is shown in the illustrative construction problems solved in the next section.

(5) To get similar triangles often auxiliary lines may be drawn at crucial points, perhaps perpendiculars or parallels. See theorem 10.

(6) Make the most general figure possible, so as to avoid extending the hypothesis.

(7) Construct figures accurately.

(8) State the hypothesis and conclusion clearly.

(9) Begin by assuming the theorem true; see what follows from that assumption; then see if this can be proved without the assumption; if so, try to reverse the process.

(10) Or begin by assuming theorem false, and endeavor to show the absurdity of that assumption.

(11) To secure clearer understanding follow Pascal's advice and substitute the definition for the name of the thing defined; *e.g.* to say that CM is a median to the base of isosceles triangle ABC is the same as saying that in triangle ABC, $AC = BC$ and M is taken so that $AM = BM$.[109]

(12) Sometimes the proof for an exercise is apparently discovered

[109] Beman and Smith. *op. cit.*, p. 35.

synthetically by trying out various possible ideas, more or less blindly, until suddenly a combination is discovered that works. While this trial and error method seems to be synthetic in nature, the trials are probably governed by a subtle analytic touch. The trial and error process with known facts is really but the working form of analysis trying to link the unknown fact with the known. Consequently most trial and error attacks on exercises, unless succeeding by chance, are probably essentially analytic in character.

(13) Analyze the conclusion to be proved in order to discover the relationships upon which it depends. In the analysis be guided by the methods outlined under 1 to 19 above, and by the "X will be true if M is true" technique, which was discussed in Chapter III and will be presented more completely on the following pages.

The Analytic Method, Coordinating the Devices Previously Used Into a Scientific Plan for the Discovery of the Proof in Original Exercises. In the theorems of the preceding chapter there has been occasionally a need for analyzing the conclusion to discover the manner in which its proof depended upon the hypothesis or other known facts. In many cases the proof was evident at once. Before trying some rather difficult exercises to get experience with this much discussed and all powerful analytic method, it will be well to summarize the theoretical basis of analysis as well as to list the devices or methods already available for proving geometric conclusions.

Analysis, the presentation of which was introduced in Chapter III, Section III, begins with the conclusion or some part of the conclusion, X, and then reasons that X will be true if some other condition, C, is true. Then it continues, C will be true if B is true, B will be true if A is true, but A is true, therefore X can be proved. (See Chapter III, Section III.)

The method of analysis is not new. As previously stated, it was probably invented by Plato, but most extensively used by Archimedes and Apollonius. In fact, Apollonius became so expert with it that his analysis of geometry problems, without the use of algebra, has not been excelled to the present time.[110] Analysis, as a major part of the heuristic method, has been strongly advocated in this study as *the* method of discovery in teaching geometry. This advocacy is based upon the feeling that one

[110] Heath. *op. cit.*, Vol. III, p. 246.

of the major contributions which geometry is able to make to the education of an individual is the development of an analytic attitude toward the solution of all problems, non-geometric as well as geometric.[110, 111, 112]

Synthesis begins with the hypothesis, a part of the hypothesis, or some known fact or construction. From this it proceeds by means of other known or accepted facts or processes to the unknown conclusion. These steps may or may not have been discovered by formal analysis, but if not, they probably were discovered by an unconscious analysis. In many of the easy exercises of high-school geometry following the theorems already given, the proof was no doubt discovered by a more or less unconscious, intuitive analysis. Proofs are not often discovered by synthesis. The analytic method is the method of discovery. It begins with the unknown or desired conclusion, and analyzes it, breaks it up, to discover other facts or relationships upon which it depends. These it analyzes further until a dependence upon known or accepted relationships is discovered. Finally, for a concise statement of these discovered relationships the synthetic arrangement, practically reversing the order of the analytic results, is very desirable. Analysis is the method of discovery, synthesis the rigorous and concise method of presenting the discovery.

Elementary Pattern of Analysis. In the preceding discussion of analysis in Chapter III the general pattern presented was as follows.

HYPOTHESIS: A is true.
CONCLUSION: B is true.
ANALYSIS: B will be true if C is true.
C will be true if D is true.
D will be true if E is true.
But E is true because of A.
Therefore B is true, or at least can be proved.

Complete Pattern of Analysis with Difficult Originals.
HYPOTHESIS: A is true.
CONCLUSION: B is true.
ANALYSIS:

(1) B will be true if C is true, or if M is true, or if X is true.
  a. C will be true if D is true, or if G is true, or if K is true. But perhaps neither of these can be proved.
  b. Then M will be true if N is true, or if H is true, or if P is true, or if Y is true.

[111] Hassler and Smith. *op. cit.*, pp. 131-170.
[112] Breslich, E. R. *Problems in Teaching Secondary School Mathematics.* University of Chicago Press. 1931, pp. 268-323.

But even now perhaps none of these can be proved.
c. Then finally X is true if J is true, or if R is true.
(2) J will be true if L is true, or if Q is true, or if Z is true, etc.
(3) But Z is true, therefore B is true, or at least can be proved.

In other words, in a real problem where the solution is unknown there may be several ways of proving any one statement and each of these may have to be investigated. Many presentations of the analytic method in textbooks on the teaching of mathematics, as well as high-school textbooks, fail to recognize this multiplicity of possibilities. They show how beautifully it works on some problem whose solution they know beforehand. When the solution is unknown there will be at nearly every step several possible steps that might be taken. For instance, the list of "Devices" gave fifteen different possible ways of proving one angle equal to another and twelve ways of proving two triangles congruent. Whenever there are various possible steps it may be necessary to exhaust several possibilities before the right one is found.

However, if we accept the principle that "nothing happens without a sufficient reason why it should be so rather than otherwise," then the reason exists and can be found if the analysis is complete. If the analytic method of geometry is to be applicable to and useful for the solution of problems whose solution is unknown, then the rather complex pattern just given for analysis will necessarily be needed. Illustrative problems are given below showing the use of the analytic method in the solution of difficult originals.

Problems Analyzed as Patterns. The exercises immediately following involve more than one step. They give some opportunity for analysis and should be a real challenge to a student who is anxious to become expert in geometric analysis. Five exercises (two constructions and three exercises involving proof only) will be analyzed to serve as patterns for the analysis of the others. Notice that analysis proceeds only to the point of discovery of known relationships, and then the proof can be written down synthetically. The headings used for the constructions are not the traditional ones, but they are suggestive and the alliteration is novel.

1. *Illustrative Construction Problem Solved by Analysis.*

PROBLEM: Construct a trapezoid, given the 4 sides, a, b, c, d, with b and d the parallel sides.

ANALYSIS: Assume a trapezoid, $ABCD$, drawn and analyze the relationship between its sides.

(1) The construction could be made if there were some way of

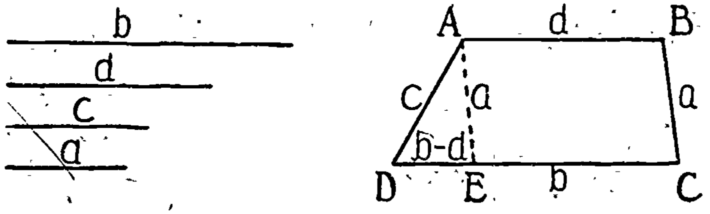
getting the angle between any two adjacent sides such as c and b, or if there were some way of getting the distance between the parallels.

(2) Now the angle between c and b could be found if there were some way of getting a triangle with three sides given. With A as a center and line a as a radius, cut CD at E. Draw AE.

(3) Then triangle ADE has two sides known and could be drawn if DE were known.

(4) But $ABCE$ is a parallelogram, therefore EC equals d and $DE = b - d$.

Therefore the triangle ADE can be constructed.



PROCEDURE (Construction):

(1) Construct a triangle EAD with sides EA, AD, and DE equal respectively to a, c, and $b - d$.

(2) Extend DE to C making $DC = b$.

(3) Through A draw a line AB parallel to DC and make $AB = d$.

(4) Connect points B and C.

(5) Then $ABCD$ is the required trapezoid.

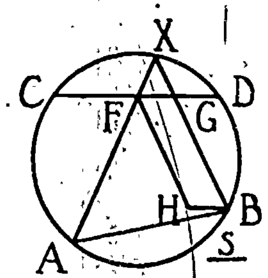
PROOF: Left to the student.

POSSIBILITIES: It should be evident that, unless a, b, c, and d are of such a nature that a triangle can be constructed with sides a, c, and $b - d$, the solution is impossible. If, for example, $a = 6$, $c = 10$, and $b - d = 2$, then no solution is possible. It is also evident that AB and DC must be laid off in the same direction, but that, if both had been laid off in the opposite direction from that indicated in the figure, the trapezoid would be reversed.

Note: The second alternative in statement (1) of the analysis, if followed through, would lead to about the same result. The solution above uses the technique of assuming the construction drawn for the purpose of analysis. Then a parallel is drawn from a crucial point. Often the connecting of crucial points or the drawing of parallels or perpendiculars at crucial points helps in the analysis. Perpendiculars from A and B to line CD would have obtained about the same result.

2. *Illustrative Construction Problem Solved by Analysis.*

PROBLEM: Given two chords AB and CD in a circle, to find a point X so that XA and XB will cut off on CD a given segment s.



ANALYSIS: Assume the construction done. Then parallels to FA through G or to GB through F suggest a parallelogram $FHBG$. Construct such a parallelogram. It is now evident that, even though angle X is unknown, BH can be drawn equal to s and parallel to CD. If F or G could be determined, the construction would be solved. But angle X = angle AFH = ½ arc AB, and therefore F can be located by the construction for the locus of the vertex of a given angle whose sides pass through two given points. This then determines the parallelogram and the steps in the analysis can be reversed.

PROCEDURE:

(1) Draw BH parallel to CD and equal to the required line segment s.

(2) Construct a circle through A and H with an inscribed angle equal to an angle measured by ½ arc AB.

(3) This circle will cut CD either in two points, F and F', one point F, or no points at all.

(4) Draw AF and extend it to cut the circle in X. Also AF' if desired, cutting the circle in X'.

(5) Draw XB cutting CD in G, also draw $X'B$ cutting CD in G' if desired.

(6) FG is the required segment. $F'G'$ also if desired.

PROOF:

(1) Angle AFH = angle X by construction.

(2) Therefore FH is parallel to BG, since the corresponding angles are equal.
(3) BH is parallel to FG by construction.
(4) Therefore $FGBH$ is a parallelogram because its opposite sides are parallel.
(5) Therefore $FG = BH$ since the opposite sides of a parallelogram are equal.
(6) Therefore $FG = s$ because both equal BH.

POSSIBILITIES:[113]

(1) The circle through A and H with inscribed angle equal to an angle measured by ½ arc AB might cut CD in (a) no points; (b) one point, if tangent; (c) two points, both within the circle; (d) two points, both on CD extended; or (e) two points, one within and one outside the circle. In (a) there would be no solution possible, or in other words there is no point X such that AX and BX would cut on CD a segment as large as the one desired. In (b) only one solution would be possible while in (c) there are clearly two solutions. In (d) AF would cut the circle in X, which would be on the major arc of CD, and AX and BX would intersect between the chords. There would still be two solutions, although the segments would be reversed in their lettering. For (e) one segment would be reversed or a negative segment, and the other, positive, but still two solutions.

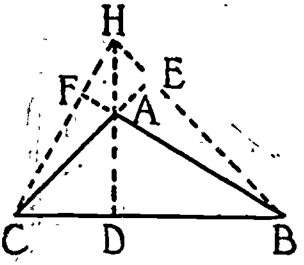
(2) If AB and CD are parallel, perpendicular, intersect within the circle or intersect outside the circle as in the figure, the solutions would be modified slightly but would be essentially unchanged.

(3) If the segment s is too large the solution becomes impossible as in 1 (a) above. If the segment s is zero, the other extreme, then the point X becomes either C or D.

3. *Illustration of an Original Exercise Solved by Analysis.* The altitudes of a triangle intersect so that the product of the segments of one equals the product of the segments of the other.

HYPOTHESIS: Any triangle, even an obtuse one such as ABC with altitudes AD, BE, CF intersecting at H.

[113] Petersen, Julius. *Methods and Theories for the Solution of Geometrical Constructions.* G. E. Stechert and Co., 1923, p. 102.



CONCLUSION: $AH \times DH = BH \times EH = CH \times FH$

ANALYSIS: (1) $AH \times DH = BH \times EH$ if $\dfrac{AH}{BH} = \dfrac{EH}{DH}$ or $\dfrac{AH}{EH} = \dfrac{BH}{DH}$

(2) Now $\dfrac{AH}{EH} = \dfrac{BH}{DH}$ if triangles AEH and BDH are similar, but $\dfrac{AH}{BH} = \dfrac{EH}{DH}$ if triangles AHB and EHD are similar.

(3) The first condition in (2) looks easier to prove. The triangles AEH and BDH are similiar if any one of several conditions are true. First, however, one is likely to observe that the triangles are both right triangles and they are therefore similiar if an acute angle of one equals an acute angle of the other.

(4) It is evident, however, that the angle at H is common to both triangles, therefore the triangles can be proved similar and steps (3), (2), and (1) can be reversed for a synthetic proof.

PROOF: Left to the student to complete.

EXERCISE: Draw a second figure having the altitudes intersecting within the triangle, and see if the same analysis holds.

4. *Illustration of an Original Exercise Solved by Analysis.* If equilateral triangles are constructed upon the sides of any triangle, the lines drawn from their outer vertices to the opposite vertices of the given triangle are equal.

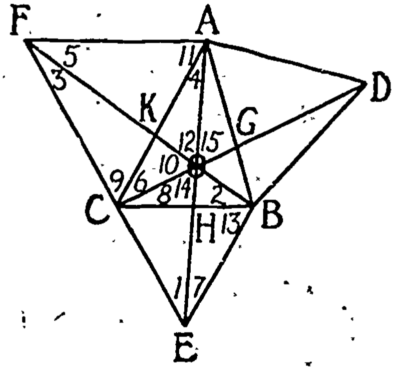
HYPOTHESIS: Any triangle ABC with ABD, BCE and ACF equilateral.

CONCLUSION: $BF = CD = AE$

ANALYSIS:

(1) $BF = CD$ if triangles BFC and CDB are congruent, or if triangles BFA and CDA are congruent, or perhaps if other pairs of triangles are congruent.

(2) Triangles BFC and CDB have no parts equal by hypothesis, except $BC = BC$, but triangles BFA and CDA have $AF = AC$ and $AB = AD$. Therefore triangles BFA and CDA will be congruent if either $BF = CD$ or angles FAB and CAD are equal.



(3) The first of these conditions is the conclusion to be proved, therefore we try the second one. However, a glance at these angles reveals that the angles are each equal to a 60° angle plus angle CAB, and angles FAB and CAD are therefore equal.

SYNTHETIC PROOF:

(1) Angle FAC = angle BAD since both equal 60°.
(2) Angle CAB = angle CAB by identity.
(3) Therefore angle FAB = angle CAD by adding (1) and (2).
(4) $AF = AC$ by hypothesis.
(5) $AB = AD$ by hypothesis.
(6) Triangle $ABF \cong$ triangle ADC by S.A.S.
(7) Therefore $BF = CD$ by c. p. c. t. e.
(8) Similarly $BF = AE$, and therefore $AE = BF = CD$.

5. *Illustration of an Original Exercise Solved by Analysis.*

HYPOTHESIS: Any triangle ABC with ABD, BCE, and ACF equilateral. (Same as illustration 4. Use figure for 4.)

CONCLUSION: AE, BF, and CD are concurrent and meet so as to form six 60° angles.

ANALYSIS:

(1) The three lines will be concurrent if CD passes through the point of intersection of the other two.

(2) If the lines are concurrent, the angles will all be 60°, providing two adjacent angles can be proved to be 60° each.
(3) Let O be the point of intersection of BF and AE and draw OC and OD, then COD must be proved to be the straight line CD.
(4) Line COD will be a straight line if the sum of three angles on one side at O is 180°, or if angle ACO = angle ACD.
(5) Suppose we take the second alternative in (4). Angle ACO = angle ACD if any one of a dozen or more conditions is true: both equal the same angle, or parts of congruent or similar triangles, etc.
(6) From the previous proof (4 above) angle 1 = angle 2, angle 3 = angle 4, angle 5 = angle ACD and angle 7 = angle BCD. Now angle ACD will equal angle ACO or KCO if angle KCO = angle 5.
(7) Angle KCO = angle 5 if triangle KCO is similiar to triangle FKA.
(8) Since these triangles have the angles at K equal, but nothing yet known about the other angles, they will be similar if the sides at K are proportional..
(9) But triangle FKC is similar to triangle AOK because two angles are equal respectively and therefore $\frac{FK}{AK} = \frac{CK}{OK}$, or $\frac{FK}{CK} = \frac{AK}{OK}$ by interchanging the means.
(10) Therefore triangle OKC is similar to triangle AKF by one angle equal and including sides proportional, and therefore angle KCO = angle 5 = angle ACD, and therefore COD can be proved a straight line.
(11) The angles at O will be 60° if they can be proved equal to the angles of the equilateral triangles.
(12) But angle 9 = angle 12 and angle 11 = angle 10 from the similar triangles already used, therefore angles 10 and 12 are 60° and their vertical angles are also.
(13) But angle 14 equals angle 15, therefore each is 60°.
(14) Therefore all the angles at O can be proved equal to 60° and the lines AE, BF, and CD can be proved concurrent at O.

Proof: Left to the student.

Summary Discussion: The analyses in the foregoing illustrative solutions are given exactly as they were worked out by the writer, except that many of the leads that seemed to fail were omitted to save space. The discovery in step (9) of the last exercise really came as the result of an abandoned attempt to follow the first suggestion in (4). A multiplicity of suggestions is usually present in any problem. As various ones are carried out and abandoned, interrelationships are discovered which may help later. Every difficult original will always have several possibilities at every step. The writer worked several hours, extending over odd study periods for more than a week, in solving the last exercise. The student should not be too readily discouraged if in trying to solve some of the following exercises, he finds that the solution is elusive. *A few hard originals really worked out will teach more about analysis than can possibly be done in many pages of discussion.*

III. Exercises for Analysis

Directions: The following problems are designed to provide experience with the analytic method for constructions and for problems to be proved. They have been arranged in order of difficulty on the basis of the combined weighting of twelve judges who are teachers of experience. Enough of the exercises should be worked, both constructions and proofs, to make sure that the analytic technique had been mastered. For the constructions the subheadings used in the patterns are helpful and suggestive: problem, analysis, procedure, proof, and possibilities. In both construction problems and other problems the analysis may be oral and therefore may be omitted from the written form.

A. Problems for Construction

1. Given the base, the smaller adjacent acute angle, and the difference between the other two sides of the triangle, construct the triangle.
2. Construct a line through a given point D within a given acute angle so as to form with the sides of the given angle an isosceles triangle.
3. Construct a trapezoid, given its bases and its diagonals.
4. On the side AC of triangle ABC to find the point P such that the parallel to AB from P, meeting BC at D, shall have $PD = AP$.
5. To construct a trapezoid, given the four sides.
6. Given a point A on one side of angle ABC, find a second point on this side whose distances from the other side and from A shall be equal.
7. Construct a square, given the sum of a side and a diagonal.
8. From two given points to draw lines meeting a given line in a point

and making equal angles with that line, the points being on (1) the same side of the given line, (2) opposite sides of the given line.

9. Construct a parallelogram having its perimeter and area equal respectively to those of a given triangle.
10. Inscribe a square within a given right triangle having one of its angles coincident with the right angle of the triangle and the opposite vertex lying on the hypotenuse.
11. Euclid's construction for the tangent to a circle with center M and from a point A outside is as follows:
    (1) Draw the circle with center M and radius MA.
    (2) Draw MA intersecting the given circle at B.
    (3) Draw BC perpendicular to MA at B, meeting the larger circle at C.
    (4) Draw MC, intersecting the given circle at D.
    Conclusion: AD is tangent to the given circle. Make the construction and give the proof.
12. With a given radius, to describe a circle having the center on one side of a given angle and determining a chord of given length on the other side of the angle.
13. Through a given point to draw a line so that the two chords intercepted on it by two circles of equal radii shall be equal.
14. Through a given point of a circle to draw a chord which shall be twice as long as its distance from the center.
15. Construct a circle which will be tangent to each of two parallels and will pass through a given point lying between the parallels.
16. Construct a circle with a given radius which will be tangent to a given circle and will pass through a given point inside the circle.
17. Describe two circles of given radii r_1 and r_2, tangent to one another and both tangent to a given line on the same side.
18. Construct a triangle given
    (a) $b + c, a, A$.
    (b) $b + c, B$, altitude CF
    (c) $b + c, C$, altitude BE
    (d) $b + c, a$, altitude BE or CF
19. Construct a triangle given
    (a) $b - c, a, C$.
    (b) $b - c, A, B$.
    (c) $b - c, a$, altitude BE.
    (d) $b - c, A$, altitude BE.

20. Construct a triangle, given an angle, the bisector of the angle, and the ratio of the two segments into which this bisector divides the opposite side.
21. Construct a trapezoid, given the ratio between the parallel sides, $m:n$, the length of both non-parallel sides, and the angle between the non-parallel sides extended.
22. Construct a parallelogram equal to a given triangle having one of its angles equal to a given angle.
23. Construct a parallelogram so that two given points shall constitute one pair of its opposite vertices and the other pair of vertices shall be on a given circle.
24. Construct a triangle given t_a, h_a, and A; t_a is the bisector of angle A, and h_a is the altitude to side a.
25. Construct a square which shall have two of its vertices on a diameter of a given circle, and the remaining two vertices on the semicircle constructed on this diameter.
26. Through one of two points of intersection of two circles, to draw a line on which the two circles determine two chords of equal length.
27. Through one of the two points common to two circles draw a line so that the two chords which the two circles determine shall subtend equal angles at the respective centers of the circles.
28. Through one of the points of intersection of two circumferences to draw a chord of one circle which shall be bisected by the circumference of the other.
29. Draw a line dividing a quadrilateral into two equal quadrilaterals.
30. Construct a triangle, given the base, the opposite angle, and the sum of the two altitudes to the other two sides. $(a, A, h_b + h_c)$. (See Altshiller Court, *op. cit.*, p. 27, problem 23.)
31. Construct a triangle given, (a) $A, a, b/c$; (b) $A, a/c$, altitude CF.
32. With two given points as centers and equal radii, to describe two circles so that one of their common tangents shall be tangent to a given circle.
33. Hypothesis: Two circles, O and O', intersecting at P and line AB equal to the sum of two chords through P.
Construction: Draw a line through P making chords whose sum is equal to the required sum, AB.
34. Through a given point P of the diameter AB to draw a chord CPD so that arc $BD = 3$ arc AC.
35. To construct a polygon of n sides in a circle with diameter AB.
(1) Construct an equilateral triangle, ABC, on AB.

(2) Divide AB into n equal segments, and the end of the second one from A along AB call D.

(3) Let line CD cut the circle, on the portion opposite from the triangle, in the point P.

(4) AP is the side of a polygon of n sides.

Note: The arc seems to be the fractional part $2/n$ of the semicircle, and therefore $1/n$ of the whole circle. Try out this construction with different values for n. Try to prove that it is true for any value of n.

36. To construct a pentagon.

(1) Draw a circle with two perpendicular diameters, AB and CD, intersecting at O.

(2) Bisect AO at M and on AB from M lay off MN equal to MC.

(3) Then CN is the side of a pentagon in a circle with radius OA. Make the construction and prove it to be general. Compare it with the construction of a decagon.

37. Construct a triangle given the base, the angle opposite, and the point where the bisector meets the base.

38. Construct a square so that each side shall pass through a given point.

39. To describe a circle with a given radius and

(a) passing through two points.

(b) passing through one given point and tangent to a given line.

(c) passing through a given point and tangent to a given circle.

(d) tangent to two given lines.

(e) tangent to two given circles.

(f) tangent to a given line and to a given circle.

40. Construct a circle*

(a) through three points. (*P. P. P.*)

(b) tangent to three given lines. (*L. L. L.*)

(c) through two points and tangent to a given line. (*P. P. L.*)

(d) through a given point and tangent to two given lines. (*P. L. L.*)

(e) through two points and tangent to a given circle. (*P. P. C.*)

(f) through one given point and tangent to a given line and a given circle. (*P. L. C.*)

(g) through a given point and tangent to two given circles. (*P. C. C.*)

* Note: These ten constructions constitute what is historically known as the problem of Apollonius. See Altshiller-Court, N., *op. cit.*, pp. 173-180. There are impossible situations for each set of conditions. All possibilities should be presented.

(h) tangent to two given lines and one given circle. (*L. L. C.*)
(i) tangent to one given line and to two given circles. (*L. C. C.*)
(j) tangent to three given circles. (*C. C. C.*)

B. PROBLEMS FOR PROOF

41. If one side of a parallelogram is produced in one direction and the opposite side is produced by the same length in the opposite direction, then the line joining their terminal points passes through the point of intersection of the diagonals of the given parallelogram.
42. If from any point in the base of an isosceles triangle perpendiculars are drawn to its sides, their sum equals the perpendicular from either base angle to the opposite side.
43. Prove that the sum of the perpendiculars drawn from any point within an equilateral triangle to the sides of the triangle is equal to the altitude of the triangle.
44. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD and CD respectively, meeting at D; prove that angle $CDB + 1/2$ angle $A =$ a right angle.
45. If the base AB of triangle ABC is produced to X, and if the bisectors of angle XBC and angle BAC meet at P, what fractional part is angle P of angle C?
46. If equiangular triangles be constructed upon the sides of any triangle, the lines drawn from their outer vertices to the opposite vertices of the given triangle are equal and concurrent. (Prove this without referring to the proof given on a preceding page.)
47. If D is the midpoint of leg BC of right triangle ABC, prove that the square of the hypotenuse, AB, exceeds 3 times the square of CD by the square of AD.
48. If BE and AF are the medians drawn from the extremities of the hypotenuse AB of right triangle ABC, prove $4\overline{BE}^2 + 4\overline{AF}^2 = 5\overline{AB}^2$
49. If ABC and ADC are angles inscribed in a semicircle, and AE and CF are drawn perpendicular to BD extended, prove $\overline{(BE)}^2 + \overline{(BF)}^2 = \overline{(DE)}^2 + \overline{(DF)}^2$
50. If lines be drawn from any point P to the vertices of rectangle $ABCD$, prove that $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$.
51. ABC, DBA are two triangles with a common side AB. If P is any point on AB, and PX parallel to AC, and PY parallel to AD meet BC and BD in X and Y respectively, prove that triangle YBX is similar to triangle DBC.

52. In the triangle ABC the side BC is bisected at E, and AB at G; AE is produced to F so that EF equals AE, and CG is produced to H so that GH equals CG. Prove that F, B, H, are in one straight line and that FB equals BH.
53. In triangle ABC, altitudes AD and BE intersect at O. The perpendicular bisectors FK and HK of AC and BC respectively, meet at K.
    (a) Prove that triangle ABO is similar to triangle FHK
    (b) Prove $AO = 2\,HK$
    $BO = 2\,FK$
54. If one of the equal sides, CB, of an isosceles triangle ABC is produced through the base, and if a segment BD is laid off on the produced side, and an equal segment AE is laid off on the other equal side, then the line joining D and E is bisected by the base. (Consider the case in which $BD > CB$, $BD = CB$, $BD < CB$.)
55. The sum of the three medians of a triangle is greater than three-fourths of its perimeter.
56. Circle D is tangent internally at B to a larger circle whose center is E. If a line through B cuts circle D at C, and circle E at A, prove that AE is parallel to CD.
57. The diameters of two circles are 12 and 28, respectively, and the distance between their centers is 29. Find the length of the common internal tangent.
58. If from the extremities of any chord perpendiculars to that chord are drawn, they will cut off equal segments, measured from the extremities, on any diameter.
59. AB is a fixed chord of a circle, and XY is any other chord having its midpoint P on AB. What is the greatest and what is the least length that XY can have?
60. Given two pairs of parallel chords, AB parallel to $A'B'$, and BC parallel to $B'C'$. Prove that AC' is parallel to $A'C$.
61. If $ABCD$ is a quadrilateral circumscribed about a circle whose center is O, prove that angle AOB + angle $COD = 180°$.
62. $ABCD$ is a parallelogram: from A a line is drawn cutting BD in E, BC in F, and DC produced in G, prove that AE is a mean proportional between EF and EG.
63. In triangle ABC, CM is a median; angles BMC and CMA are bisected by lines meeting a and b in R and Q, respectively. Prove that QR is parallel to AB.
64. $DEFG$ is a square having its vertices D and E on sides AB and BC

respectively, of triangle ABC and its vertices F and G on side AC. Let BH be parallel to AC, meeting AE extended at H; let HK be perpendicular to AC and BT perpendicular to AC. Prove $BHKT$ is a square.

65. The square of one of the equal sides of an isosceles triangle is equal to the square of any line drawn from the vertex of the triangle to the base increased by the product of the segments cut off by the line.
66. ABC is a triangle; AC is bisected at M; BM is bisected at N; AN meets BC at P; MQ is drawn parallel to AP to meet BC at Q. Prove that BC is trisected by P and Q.
67. Prove that in any triangle three-fourths the sum of the squares of the sides equals the sum of the squares of the medians.
68. Two parallel chords are 10 inches and 12 inches long and are 1 inch apart. Find the radius of the circle.
69. If any two chords cut each other perpendicularly the sum of the squares of the four segments equals the square of the diameter.
70. If the sides BC, CA, AB, of triangle ABC are produced to X, Y, Z, respectively, so that $CX = BC$, $AY = CA$, $BZ = AB$, prove that triangle XYZ = seven times triangle ABC.
71. E is any point on diagonal AC of parallelogram $ABCD$. Through E, parallels to AD and AB are drawn, meeting AB and CD at F and H respectively, and BC and AD at G and K respectively. Prove parallelogram $FBGE$ = parallelogram $EHDK$.
72. If E is any point inside BC of parallelogram $ABCD$ and DE is drawn meeting AB extended at F, prove triangle ABE equals triangle CEF.
73. $ABCD$ is a quadrilateral inscribed in a circle. If the sides AB and DC extended intersect at E, and AD and BC extended intersect at F, prove that the bisectors of angle E and angle F are perpendicular.
74. $ABCD$ is a quadrilateral inscribed in a circle. Another circle is drawn upon AD, a chord, meeting AB and CD extended at E and F respectively. Prove chords BC and EF parallel.
75. The sides AB and CA of a triangle are bisected in C' and B' respectively; CC' cuts BB' at P. Prove that triangle PBC equals quadrilateral $AC'PB'$.
76. A right triangle has for its hypotenuse the side of a square and lies outside the square. Prove that the straight line drawn from the center of the square to the vertex of the right angle of the right triangle bisects the right angle.
77. If a circle is circumscribed about a right triangle, and on each of the

legs of the triangle as diameters semi-circles are drawn, exterior to the triangle, the sum of the areas of the crescents thus formed equals the area of the triangle.

78. Prove that the square inscribed in a semi-circle is equal to two-fifths the square inscribed in the entire circle. Suggestion: Let R equal the radius of the circle. Compute the areas of the two squares.
79. AB and AC are tangents to a circle from the point A, and D is any point in the smaller of the arcs subtended by the chord BC. If a tangent to a circle at D meets AB at E and AC at F, prove that the perimeter of triangle $AEF = AB + AC$.
80. If from any point P, on the diameter AB, PX and PY are drawn to the circumference on the same side of AB and making angle APX equal to angle BPY, then triangles APX and YPB are mutually equiangular.
81. Prove that the bisector of any angle of an inscribed quadrilateral and of the opposite exterior angle meet on the circumference.
82. Triangle ABC is inscribed in a circle of which AD is the diameter. A tangent to the circle at D cuts AB extended at X and AC extended at Y. Prove triangle ABC similar to triangle AXY.
83. In a parallelogram the sum of the squares on the four sides equals the sum of the squares of the diagonals.
84. A' and B' are the feet of the perpendiculars from A and B to a and b in triangle ABC; M is the midpoint of AB. Prove that angle $B'A'M =$ angle $A'B'M =$ angle C.
85. In triangle ABC, P is any point in AB, and Q is such a point in CA that $CQ = PB$; if PQ and BC, produced if necessary, meet at X, prove that $CA : AB = PX : QX$.
86. ABC is a triangle, and through D, any point on AB, DE is drawn parallel to BC to meet AC in E; through C, CF is drawn parallel to EB to meet AB produced in F. Prove that AB is a mean proportional between AD and AF.
87. Prove that the lines joining the midpoints of the opposite sides of a quadrilateral and the line joining the midpoints of the diagonals of the quadrilateral meet in a point.
88. If a line be extended from vertex C of isosceles triangle ABC meeting base AB extended at D, prove $(CD)^2 - (CB)^2 = AD \times BD$.
89. If AD and BE are the perpendiculars from vertices A and B, respectively, of acute angled triangle ABC to the opposite sides, prove $AC \times AE + BC \times BD = (AB)^2$.
90. The perpendiculars drawn from the vertices of a triangle to the

opposite sides are the bisectors of the angles of a triangle formed by joining the feet of the perpendiculars.

91. If D is the midpoint of side BC of triangle ABC, E the midpoint of AD, F of BE and G of CF, then triangle $ABC = 8$ times triangle EFG.
92. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector. Prove $AB \times AC = BD \times CD - (AD)^2$.
93. In any inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.
94. The feet of the three perpendiculars dropped upon the sides of a triangle from a point P in its circumcircle are collinear. (This is Simpson's Line.)
95. The Simpson Lines of two diametrically opposite points are perpendicular.
96. The Simpson Lines of three points form a triangle similar to the triangle determined by the three points.
97. The six segments determined by a transversal on the sides of a triangle are such that the product of three non-consecutive segments is equal to the product of the three others.
    Note: This is the Theorem of Menelaus.
98. The lines joining the vertices of a triangle to a given point determine on the sides of the triangle six segments such that the product of three non-consecutive segments is equal to the product of the three other segments.
    Note: This is Ceva's Theorem.
99. Does Ceva's Theorem hold if the point is outside the triangle? On one side? On one side produced? At one vertex?
100. If d is the distance from a point to the center of a circle and r the radius of the circle, write a formula for computing the length of the tangent (t). $t = f(d, r)$. If r is constant, then $t = f(d)$. Show what happens to t as d approaches r and finally becomes less than r and approaches zero.

IV. Converses

A. Converses and Partial Converses

Definition of Converses and Partial Converses. One of the major contributions of geometry to the science of reasoning is the method of the indirect proof. Since one of the chief uses of indirect proof is for converses, it seems necessary that a full understanding of converses should precede the use of indirect proof. The following definition of converse and the analysis of the fundamental relation between a direct statement and its converse are vital to an understanding of the direct or *reductio ad absurdum* (reduction to an absurdity) proof.

The usual definition, "to interchange the hypothesis and conclusion," often leads to an absurd or impossible statement. It is only when there is but one simple condition in the hypothesis and one simple consequence in the conclusion that such interchange is possible. Strictly speaking, such interchange is never possible. At first thought the converse, according to this definition, of "If two sides of a triangle are equal, the angles opposite these sides are equal," is "If two angles of a triangle are equal, the sides opposite these angles are equal." Close inspection reveals, however, that "triangle" remains in the hypothesis for both statements. This can be brought out by a different statement of the theorem. (1) If a polygon has three sides and two of these are equal, then the angles opposite are equal. The converse, according to the usual definition, would not be less than this:—"If a polygon has two equal angles, then it has three sides and two of these are equal." This converse is impossible to prove.

Since the usual definition of converse is sometimes impossible, clearly a more carefully worded one must be framed. The actual and usual conception of a converse is that a *converse* of any theorem may be stated by *interchanging any one consequence in the conclusion with any one condition given in the hypothesis.* Such a definition would permit several converses for some theorems. In cases where there is more than one converse of a theorem each one is sometimes called a *partial converse.*[114] The following proposition illustrates the definition:

A. The direct theorem

HYPOTHESIS: (1) The curve $ABCD$ is a circle.
(2) CD is a chord.
(3) AB is a diameter.
(4) AB is perpendicular to CD.

CONCLUSION: (X) AB bisects CD.

B. Converse by the usual definition (interchange of hypothesis and conclusion)

HYPOTHESIS: (X) AB bisects CD.

[114] Heath, T. L. Vol. I, *op. cit.*, p. 256.

CONCLUSION: (1) *ABCD* is a circle.
(2) *AB* is a diameter.
(3) *CD* is a chord.
(4) *AB* is perpendicular to *CD*.

C. Converses by the more precise definition

(a) Interchange of (4) and (X)

HYPOTHESIS: (1) *ABCD* is a circle.
(2) *CD* is a chord.
(3) *AB* is a diameter.
(X) *AB* bisects CD.

CONCLUSION: (4) *AB* is perpendicular to CD.

(b) Interchange of (3) and (X)

HYPOTHESIS: (1) *ABCD* is a circle.
(2) *CD* is a chord.
(X) *AB* bisects CD.
(4) *AB* is perpendicular to *CD*.

CONCLUSION: (3) *AB* is a diameter.

(c) Interchange of (2) and (X)

HYPOTHESIS: (1) *ABCD* is a circle.
(X) *AB* bisects *CD*.
(3) *AB* is a diameter.
(4) *AB* is perpendicular to *CD*.

CONCLUSION: (2) *CD* is a chord.

(d) Interchange of (1) and (X)

HYPOTHESIS: (X) *AB* bisects *CD*.
(2) *CD* is a chord.
(3) *AB* is a diameter.
(4) *AB* is perpendicular to *CD*.

CONCLUSION: (1) *ABCD* is a circle.

The usual definition of converses, interpreted literally, results in an absurd and utterly impossible situation as soon as there is more than one condition in the hypothesis and only one result in the conclusion, as in proposition *A* above. The interchange of hypothesis and conclusion, as in *B* above, is an incomplete statement of the relation between a proposition and its converse. On the other hand, propositions *a*, *b*, *c*, and *d*, under *C* above, are perfectly sensible and legitimate converses, and not only illustrate but establish the modified definition. They can even be proved, and this is not always true of converses, as the next section will reveal.

B. PROBLEMS ON STATING CONVERSES

101. In a spherical polygon the sum of the interior angles is greater than (n-2) straight angles, but less than n straight angles. With this in mind, state the converses of the following modified statements of familiar theorems.
    (a) If a polygon has 3 sides and these sides are straight lines, then the sum of its angles is one straight angle.
    (b) If a polygon has n sides and these sides are all straight lines which lie in the same plane, then the sum of its angles is (n-2) straight angles.
    (c) The sum of the exterior angles of any plane polygon is two straight angles.
102. State the converse or converses of each of the twenty "essential theorems," and also of each of the corollaries. Be careful of the converses of theorems 7, 10, 12, 18, 19, and 20.
103. State the converse of exercise 41 of the examples for analysis. Try doing the same for each exercise from 42 to 50. In some cases the converse is very difficult to state without practically including the conclusion in the hypothesis; for example, number 43. Converses often are not true.

C. THE LAW OF CONVERSE EXPRESSED IN DIFFERENT WAYS

Relation between Direct and Converse Statements for Proof. The proving of converse theorems involves an interesting problem in logic. Converses are not always true, although in most high-school geometry courses no converses are mentioned which are not true. It will be interesting to discover the conditions under which a converse will be true.

Two illustrations, one geometric and one non-geometric, will help to clarify the situation: (1) If a triangle has three equal sides, it has two equal angles. (2) If a man is rich he can buy a two-cent stamp. It is evident that the converse of neither of these statements is true. The reason for this is that the hypothesis is more generous than is necessary. The two words, necessary and sufficient, are advantageously used in explaining this situation.[115]

Three sides equal is a sufficient condition for two angles being equal but not a necessary condition, because two sides equal is all that is necessary. That is, three sides equal is a more generous limitation than is

[115] Garabedian, Carl A. "Necessary, Sufficient, and Necessary and Sufficient Conditions," *Mathematics Teacher,* XXIV, pp. 345-352.

necessary to make only two angles equal. Therefore, in the converse, which reverses the hypothesis and conclusion, its hypothesis, two angles equal, will not be a sufficient limitation to make the three sides equal.

Richness is a sufficient condition to enable a man to buy a two-cent stamp, but not a necessary condition, since it is more limiting than is necessary. Consequently, the converse is false: If a man can buy a two-cent stamp, he is rich. For any statement to be true the hypothesis must be sufficient to make the conclusion true. For the converse statement to be true the hypothesis of the first statement must be necessary in order that the hypothesis of the converse may be sufficient. For a statement and its converse both to be true, the hypothesis must be both sufficient and necessary.

Hypothesis Sufficient but Not all Necessary, Converse not True. Further illustrations, both geometric and non-geometric, will help to make clear the conditions under which a converse statement will be true. For instance, in the following statement it is evident that the hypothesis is more limiting than is necessary: If all points on a line are equally distant from a point within, the line is a curved line. That is, the hypothesis is sufficient to make the conclusion true; yet not necessary. Consequently, the converse is not true, because when the conclusion becomes the hypothesis, that new hypothesis will not be adequate.

"If a quadrilateral is a square its adjacent sides are equal," is a statement whose converse is clearly not true. The reason is evident. While the present hypothesis is sufficient to make the conclusion true. it is not necessary; that is, it is more generous and more limited than is necessary. Therefore, in the converse, the hypothesis would be inadequate to make the conclusion true.

"If a quadrilateral is a square, its diagonals are equal, or its diagonals are perpendicular bisectors of each other," is also a statement in which the hypothesis contains more limiting conditions than are necessary, and consequently, although either conclusion is true from the given hypothesis, yet in the converse, either hypothesis would be insufficient to justify the conclusion. It is evident that if the diagonals of a quadrilateral are equal, the figure could be a rectangle, and if the diagonals are perpendicular bisectors, the figure might be a rhombus.

Hypothesis Sufficient and Necessary, Statement and Converse Both True. However, the statement that, if a quadrilateral is a square, then the diagonals are perpendicular bisectors of each other and are also equal, has an hypothesis that is not merely sufficient, but just barely sufficient, with no extras. That is, in this statement all the limitations in the hypothesis are necessary, consequently, in the converse, the hypothesis will be sufficient for its conclusion.

If all points on a line are equally distant from a point within, the line is a circle. Clearly in this statement the hypothesis is sufficient and just barely sufficient, that is, it is both sufficient and necessary to make the conclusion true. Consequently, the converse is true.

Hypothesis Necessary, but Insufficient; Statement False, but Converse True. A statement in which the hypothesis is not sufficient, but is necessary, has a conclusion that is false; yet the converse will be true. This is so because of the fact that in the converse the original conclusion and hypothesis are interchanged, and the new hypothesis will therefore be sufficient to make the new conclusion true. For example, the statement that all equal angles are right angles, is false. Yet its converse, all right angles are equal, is true. Similarly, it is not true that any quadrilateral that has its opposite sides parallel is a rectangle. Yet the converse of this statement is true. In these statements, the hypothesis is necessary, but not sufficient for the conclusion, and in the converse, since the hypothesis and conclusion are interchanged, the hypothesis is amply sufficient.

Again, it is not true that if two sides of a triangle are equal, the triangle is equiangular. The reason for this falsity is that the equality of two sides is necessary but inadequate to make the conclusion true. That is, the hypothesis is not sufficient. It is necessary, however, even though inadequate, and consequently the converse is true. "If a man has two cents, he can buy an ice cream soda," is not a true statement because the possession of two cents is not a sufficient, although a necessary condition, for buying the ten-cent article. Here again the converse is true.

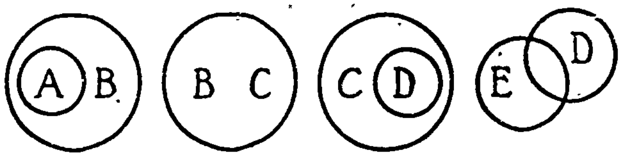
Hypothesis Neither Necessary Nor Sufficient. Furthermore, if the hypothesis is neither necessary nor sufficient, neither the direct statement nor the converse is true: If a quadrilateral is constructed with white chalk on a blackboard, it is a rectangle. It is evident that the white chalk and blackboard are trivial, and, while insufficient to make the figure a rectangle, are also unnecessary; and consequently, neither form of the statement is true.

Summary of Relation Between any Statement and its Converse. It should therefore be evident that for any statement to be true the conditions in the hypothesis must be sufficient, whether just barely sufficient or more than sufficient; for a converse to be true the conditions in the hypothesis must be necessary, yet not more than necessary, whether sufficient or not sufficient. Consequently, for both a statement and its converse to be true, the hypothesis must be both sufficient and necessary.

(1) Any statement is true if the hypothesis is sufficient.
(2) Any statement has its converse true if the original hypothesis is necessary.
(3) Any statement and its converse are both true if the hypothesis is both sufficient and necessary.
(4) Any statement and its converse are both false if the hypothesis is neither sufficient nor necessary.

A graphic representation of the relations between a generalization and its converse will help to emphasize and clarify. The following four statements will be pictured.

(1) Let A represent all triangles with three equal sides, and B those with two equal angles. Then all *A* is *B*, but the converse of this is false.
(2) Let C represent all triangles with two equal sides, and B again those with two equal angles. Then all *C* is *B*, and the converse is true: All *B* is *C*.
(3) Let *D* represent all triangles with three equal angles, and *C* again those with two equal sides. Then the statement, all *C* is *D*, is false; yet the converse is true: All *D* is *C*.
(4) Let *E* represent all triangles made with white chalk on a blackboard, and *D* again, all triangles with 3 equal angles. Both the statement that all *E* is *D*, and its converse are false.



(1) All A is B, but not all B is A.
(2) All C is B and all B is C
(3) Not all C is D, but all D is C,
(4) Not all E is D, and not all D is E.

It is evident in figure (1) that all *A* is *B*, but that there are many *B's* that are not *A's*. If we substitute the specific meanings for the symbols, figure (1) represents that all triangles with three equal sides have two equal angles, but that there are many triangles with two equal angles that do not have three equal sides. Being an *A* is sufficient to make a triangle a *B*, but more limiting than necessary. Therefore not all *B's* are *A's*.

These letters can now be disassociated from their specific meanings, and may apply to any "if-then" statement. Even though all A is B, all B is not A unless being A is not only sufficient, but also entirely necessary for being B.

Figure (2) shows the condition of necessity as well as sufficiency between a conclusion and its hypothesis. It is evident that having two sides equal (B) is not only sufficient, but entirely necessary in order that a triangle may have two equal angles, (C). Therefore, the converse is true. Symbolically, if B is both necessary and sufficient for C, then all B is C, and all C is B. Again this statement may be generalized by giving B and C any meanings which satisfy the conditions above; that is, the hypothesis must be both necessary and sufficient for the conclusion.

Figure (3) represents the relation where C is a necessary but not a sufficiently limiting condition to make all C's be D's, even though all D's are C's. If we substitute the specific meanings for C and D given above, the application of this relationship to a specific situation is evident. However, the relationship pictured is completely general and applies wherever C is necessary but not sufficient to make D true.

Figure (4) represents the situation in which the hypothesis is neither necessary nor sufficient for the conclusion and consequently in which neither the direct statement nor the converse is true. E and D may overlap, but not all E is D, and also, not all D is E.

Formulas for Determining the Truth of a Converse. This necessary and sufficient condition upon which the truth of the converse depends can be expressed in a formula.[116]

If all X is Y and
all *non-X* is *non-Y*,
then all Y is X.
(Formula 1.)

This condition for the truth of a converse is stated a bit more precisely by Augustus De Morgan in his text on logic.[117] It is reported by C. B. Upton[118] as follows:

If it has been proved that,

X less than Y makes A less than B,
X equal to Y makes A equal to B, and
X greater than Y makes A greater than B,

then it follows logically that the converses of all three of these state-

[116] Heath, T. L. *op. cit.*, Vol. I, p. 256.
[117] De Morgan, Augustus. *Formal Logic.* Taylor and Walton, London, 1847, p. 25.
[118] Upton, C. B. *op. cit.*, p. 117.

ments are true without further proof. (Formula 2, the Law of Converse.)

Let us illustrate the meaning of these two formulas for determining the truth of a converse statement, from the examples previously given. We shall apply formula 1 by first selecting cases where the converse is not true and then cases where the converse is true. The statement; that if three sides of a triangle are equal, then two angles will be equal, is true, but its converse is not true. Test: That all triangles with three sides equal have two angles equal is true, but the statement that all triangles with not three sides equal have not two angles equal, is not true. It is true that all rich men can buy a postage stamp, but it is not true that all non-rich men, all men who are not rich, cannot buy a postage stamp. However, not only is the direct form but also the converse true in the following cases. All triangles which have two sides equal have two angles equal, and, since all triangles which do not have two sides equal do not have two angles equal, the converse is true and the equality of two angles determines the equality of two sides. Furthermore, all men with two cents can buy a two-cent stamp; also all men who do not have two cents cannot buy a two-cent stamp. Naturally this statement bars the facetious response that perhaps the postmaster would extend credit or even donate a stamp. It is merely an illustrative statement to show the conditions under which a converse is true. Here again the converse is true; that is, the buying of a two-cent stamp indicates the possession of two cents.

Formula 2 above can also be applied. For the triangle the wording had best be changed somewhat. If the first three statements below can be proved, then the converse of all three will be true without further proof.

> If a greater than b makes angle A greater than angle B,
> a equal to b makes angle A equal to angle B,
> and a less than b makes angle A less than angle B,

then the converses of these three statements will all be true:

> If A is greater than B, then a is greater than b.
> If A equals B, then a equals b.
> If A is less than B, then a is less than b.

An indirect proof very readily establishes the truth of each of these conclusions.

Furthermore, if having more than two cents makes it possible to buy more than one two-cent stamp, having two cents makes it possible to buy one such stamp; and having less than two cents makes it possible to buy less than one such stamp, then the converse of each of these three statements is true. That is, if one can buy more than one two-cent stamp, only one stamp, or less than one stamp, he must have respectively more than two cents, two cents, or less than two cents.

Applying these formulas to statements whose converses are false helps to make clear the general nature and applicability of the formulas.

Formula 2:

If a man has more than 10 cents, he can buy more than one two-cent stamp;

If he has 10 cents, he can buy one two-cent stamp;

If he has less than 10 cents, he can buy less than one two-cent stamp.

If the three statements above were all true, then the converses would be true by the law of converse. Since not all of them are true, not all of the converses are true.

Formula 1:

If a man has 10 cents, he can buy a two-cent stamp;

If he has not 10 cents, he cannot buy a two-cent stamp.

If the second of the two statements were true, the converse of the first one would be true. Since the second statement is false, the converse of the first is false.

Opposites and Converses. Formula (1), "If all X is Y and all *non-X* is *non-Y*, then all Y is X," has introduced a new concept, opposites. The statement that all *non-X* is *non-Y* is the opposite of the statement that all X is Y. The theorem that all triangles with two equal sides have two equal angles, has for its opposite the statement that all triangles with "not two" (without two) equal sides have not two equal angles. In general, if X represents any hypothesis and Y any conclusion, then the following symbols can be used.

Theorem: (a) All X is Y.

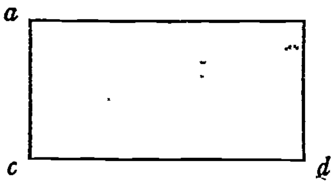
Converse: (b) All Y is X.

Opposite: (c) All *non-X* is *non-Y*.

Converse of the Opposite or Opposite of the Converse:

(d) All *non-Y* is *non-X*.

It is interesting to note that the truth of the converse makes the opposite true. This can readily be proved indirectly. If all Y is X, then all *non-X* is *non-Y*, because if some *non-X* were Y, then not all Y could be X. Similarly, the truth of the statement all X is Y makes the converse of the opposite true, all *non-Y* is *non-X*, because if some *non-Y* were X, then not all X could be Y. In other words (*a*) and (*d*) are equivalent, as are also statements (*b*) and (*c*). Therefore it should be evident that if a theorem (*a*) and its converse (*b*) are proved, that the opposite (c), and the converse of the opposite (d), will be true. Also, if (a) and (c) are proved, (b) and (d) will be true; if (c) and (d) are proved, then (a) and (b) will be true; and if (b) and (d) are proved, then (a) and (c) will be true.



If the letters, representing the above statements: (a), (b), (c), and (d), are placed around a rectangle as in the figure above, then diagonally opposite statements are equivalent, and consequently, any two adjacent statements are sufficient to establish the truth of the remaining statements.

Applying this theory to the illustration of triangles, we have the following statements, in which S stands for triangles with two sides equal, and *non-S* stands for triangles with not two sides equal, A stands for triangles with two angles equal and *non-A* for triangles with not two angles equal.

(a) Theorem: All S is A.
(b) Converse: All A is S.
(c) Opposite: All *non-S* is *non-A*
(d) Converse of Opposite: All *non-A* is *non-S*.

Again it is evident in the above arrangement that if any one statement is true the diagonally opposite one can easily be established by indirect proof; and, therefore, if any two adjacent statements are true, the remaining ones will also be true. This can be illustrated figuratively as before by two circles, S and A. If all of circle S is within circle A, then (a) all S is A, and (d) all *non-A* is *non-S* because if some *non-A* were S, then not all S would be A. If A is within S then (b) all A is S and (c) all *non-S* is *non-A*. In order that (a) all S be A and (b) all A be S or (c) all *non-S* be *non-A*, it is evident that the circles would have to coincide. In other words, if S be the necessary and sufficient condition for A, regardless of the assigned meanings of these two letters, then (a) all S is A and (b) all A is S or (c) all *non-S* is *non-A*.

Converse and Opposite in Locus Problems. The equivalence of the converse and opposite is used extensively in locus problems. In proving a locus problem one must prove not only that all points on the locus satisfy the conditions, but also either the converse or the opposite of this statement; namely, that all points which satisfy the conditions are on the locus, or all points not on the locus do not satisfy the conditions. In fact, if the four statements are arranged in the form above, the proving of any two adjacent statements, *a* and *b*, *a* and *c*, *b* and *d*, or *c* and *d*, proves the locus.[119, 120]

To say that the locus of a point equally distant from two points, *A* and *B*, is the perpendicular bisector *PQ* of the line segment *AB*, means that, of the following statements, either *a* and *b*, *a* and *c*, *c* and *d*, or *b* and *d* must be proved true to prove the locus, but that *a* and *d*, or *b* and *c* would not prove the locus problem.

a. Direct Statement: All points on *PQ* are equally distant from *A* and *B*.
b. Converse: All points equally distant from *A* and *B* are on *PQ*.
c. Opposite: All points not on *PQ* are not equally distant from *A* and *B*.
d. Converse of Opposite: All points not equally distant from *A* and *B* are not on *PQ*.

In high-school texts statements corresponding to *a* and *b* are most commonly used in proving locus theorems, although the equivalent of *a* and *c* is sometimes used. It is evident that if *a* is true, *d* can readily be proved by indirect proof. If (*a*) all points on *PQ* are equally distant from *A* and *B*, then (*d*) all points not equally distant from *A* and *B* are not on *PQ*, because, if they were on *PQ*, they would be equally distant from *A* and *B*. Similarly, if *d* is true, *a* is true; if *b* is true, *c* is true; and if *c* is true, *b* is true.

Furthermore, these letters can be generalized and interpreted as referring to general statements as well as to the parts of the particular locus problems given. The following problems illustrate the relation between a geometric statement and its converse and also the methods of proof which have been given.

D. PROBLEMS ON PROVING CONVERSES

104. Write the converse, the opposite, and the converse of the opposite for the following propositions:
   a. If two lines are parallel and are cut by a third line, the corresponding angles are equal.
   b. If a quadrilateral is a rectangle, its diagonals are equal.
   c. If two triangles have three sides of one equal to three sides of another respectively, the triangles are congruent.

[119] Schultze, Arthur. *op. cit.*, pp. 144-146.
[120] Beman and Smith. *op. cit.*, pp. 34, 39.

d. If a quadrilateral is a rectangle, one of its angles will be a right angle.
e. If a triangle is a right triangle, the square on its longest side equals the sum of the squares, on the other two sides.
f. If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

105. Apply the various conditions for proving converses to each of the six parts of exercise 104 and indicate in which cases the converses are true. Use both formula 1 and formula 2.
106. Theorems 13 and 14, two fundamental locus theorems, can be proved in any one of four different ways. Outline the proof for both theorems and carry it out in detail for one of them.
107. The locus of the vertex of the right angle of a right triangle is a circle with the hypotenuse for a diameter. Prove in four different ways.
108. The locus of the midpoint of the hypotenuse of a right triangle is a circle whose center is the vertex of the right angle and whose radius is one-half the hypotenuse. Prove by using statements (a) and (b), and (a) and (c).
109. The locus of the vertex of a given angle opposite a given side of a triangle is an arc of a circle cut by the given side of the triangle as a chord.
110. The locus of a point whose coordinates satisfy an equation of first degree is a straight line.

Note: Further practice with proving converses and detecting their falsity will be provided in the exercises following the next section on indirect proof.

E. THE PURPOSE FOR CONSIDERING NECESSARY AND SUFFICIENT CONDITIONS

Indirect Proof Preferable to the Test of Necessary and Sufficient Conditions. It should be evident to the careful reader that the two formulas given are but a concise way of expressing exactly the same idea that was discussed under the head of necessary and sufficient conditions. All *non-X* being *non-Y* is a test for the necessity of X for Y. (Formula 1). Then too, when the relation between A and B is determined by the relation between X and Y as in formula 2, then X and Y are not only sufficient but also necessary for the relation between A and B. For some situations one formula seems to apply more simply than the other, and in some cases the general test for the necessity of the hypothesis is preferable. In either case it is quite evident that it may be as difficult to discover whether or not these conditions are satisfied as to prove the converse statement itself. A converse statement can often be proved rather easily by an indirect proof.

Law of Converse Postulated. Furthermore, these statements of the conditions for the truth of a converse have not been proved in this study. They have merely been illustrated and explained, and consequently are really postulates taken from logic, which in turn really "grew out of the critical work of the mathematicians who reflected about the nature and structure of mathematical truths."[121] If the conditions of being sufficient and necessary, or their equivalent expressed in formulas (1) or (2), are used to establish a converse or to make any indirect proof, these conditions must be considered as logical postulates.

Applications of Ideas Involved in the Law of Converse. However, while it may seem incongruous to use the generalizations of logic to help establish the generalizations of geometry, when those principles of logic are, without doubt, deductions derived from a study of the specific generalizations of geometry; yet the generalizations derived do throw considerable light on the relation between a statement and its converse. Then too, if geometry is a reasoning pattern for non-geometric situations as well as geometric situations, teachers of mathematics need to be familiar with the general terminology and the general relationships. Professor Hedrick[122] claims that a knowledge and appreciation of "the ideas of necessary and sufficient conditions, and the difference between conditions that are necessary and those that are sufficient," are valuable contributions of mathematics to the education of an individual. "The resulting confusion among those not properly trained is notorious, and this confusion is certainly transferred to every field of thought, from cookery to politics."[122]

A study of the following "if-then" statements will quickly reveal the generalized nature of the above conclusions concerning necessary and sufficient conditions, formulas 1 and 2, and the relation between statements and their converses.

(1) If you would be a great man, you must be willing and able to work hard.

(2) If you use butter in a frying pan over a hot wood fire, it will fry food well.

[121] Enriques, Frederigo. *The Historic Development of Logic.* Translation by Jerome Rosenthal. Henry Holt and Co., 1929, p. 4.

[122] Hedrick, E. R. "The Reality of Mathematical Processes," *Third Yearbook of the National Council of Teachers of Mathematics.* Bureau of Publications, Teachers College, Columbia University, New York City, 1928, p. 37.

(3) If a man is a good golfer, he will buy the best clubs made.
(4) If a man is an expert caster, he must have a light, flexible, strong rod.
(5) If you are a great man, then you will be so busy and do things so rapidly that you will become a poor penman.
(6) If you are a great statesman, then you will be courteous and considerate of the rights of others.
(7) If you use a gallon of water, then you can cook a half dozen potatoes.
(8) If you write a dissertation, then you will work hard and be willing and able to take suggestions.
(9) If you have $1200, you can buy a new Studebaker.
(10) If you go up in the air, then you will come down.
(11) If your skin is white, then you are an honorable man.
(12) If you multiply 3 by 4, then you will get an answer of 12.
(13) If b and c are constant and A increases, then a will increase.
(14) If a fly has six legs, then a bear has four.
(15) If the present market prices are below the average of the last ten years, then the future market prices will be above the average of the last ten years.
(16) Most of the people in *Who's Who* have a college education.
(17) Three meals a day are sufficient to keep a man alive.
(18) In China, people who drink tea and no water do not get typhoid fever.
(19) Four years of academic training in Mathematics through Calculus, Mechanics, and Elliptic Integrals may be sufficient for making a good teacher of mathematics.
(20) If each laborer works 8 hours a day and 7 days a week, that is sufficient to keep the factory in continuous operation.

The above statements include typical conclusions which are often thought to be true in converse form, and in some cases the converse is true. However, in each case the converse is clearly not true unless the conditions in the hypothesis are necessary, nor is the statement itself true unless the hypothesis is sufficient. The statements above also include some sets of conditions which are sufficient and, consequently, thought by many people to be necessary, when in many cases those conditions are largely extraneous and incidental, but include within them the necessary element. For instance, it is the boiling of the water, not the tea, that kills the typhoid germs; it is probably the native ability which makes it possible for a man to master Elliptic Integrals that is also largely the determinant of a good teacher, rather than the knowledge of the subject-matter of the advanced course.

V. The Technique of the Indirect Proof

Present Attitude Toward Indirect Proof. The method of indirect proof in geometry has in the past been a great source of grief to most students and teachers. The reason for this is no doubt largely the result of its early use in the older geometries. If a child has just begun geometry and does not fully realize the purpose of direct proof, and if with the second or even the fourth theorem he is confronted with an indirect proof, there is little reason to question the cause of his bewilderment. One widely used text of several years ago, after ten pages of definitions, one page of axioms, and one page of postulates, has for proposition I, which is proved directly: "All right angles are equal," and for proposition II: "At a given point in a given line not more than one perpendicular can be drawn to that line in the same plane." The second proposition is proved by a full page indirect proof; that is, an indirect proof was presented before the technique of direct proof had been established. Such flagrant violation of the fundamental principle of presenting only one major difficulty at a time in any subject could result in little else than the present disparaging attitude toward indirect proof.

In support of the contention that teachers of today fear, neglect, do not understand, and underestimate the value of indirect proof the following evidence is interesting. In the preparation of this study fifty exercises involving indirect proof were submitted to thirty-one graduate students at Teachers College, Columbia University. A few of these exercises were ridiculous, two or three were impossible, most of them were fairly difficult converses to which the indirect method very directly applied. The students were asked to rank them in one of the following five classes.

E—Easy to work by indirect proof.
M—Moderately difficult by indirect proof.
V—Very difficult by indirect proof.
X—Unable to prove by the indirect proof.
D—Easier to solve by direct proof than by indirect proof.

The first impossible exercise was marked *E, M, V,* and *D* by 6, 19, 10, and 6 percent respectively; the rest marked it *X* or omitted it. A second exercise, which ought to have been marked *E,* was given the five marks listed above by 16, 32, 7, 13, 10 percent respectively, and 22 percent failed to weigh it at all. Furthermore, less than one-fifth of the thirty-one students completed the entire task. Such results suggest that even some teachers of experience dislike the indirect method, have no confidence in its use, and do not appreciate its importance and function in geometry or in life.

The Indirect Method a Major Contribution of Geometry. If geometry is to be taught largely because of its inherent possibilities to provide experiences in the science of reasoning through applying that reasoning to the simple concepts of geometry, then surely it is a mistake to omit or neglect to emphasize the method of the indirect proof. Much of the reasoning which we do in life is indirect; therefore much of the value of geometry must be in its treatment of indirect proof. It shall, therefore, be the function of this section to attempt to overcome some of the effects of the reader's previous unfavorable experiences with indirect proof, by establishing, through presentation and illustration, the various forms and the relatively simple technique and organization of indirect reasoning.

The Underlying Principles of Indirect Proof. Indirect proof is based upon the fact that one of two opposite statements must be true and only one can be true, or upon the exhaustion of all possible cases except one by proving all false except that one conclusion. For instance, a certain point P is either on line AB or not on it; a segment PQ either equals QR or does not equal QR; today is either September 30 or it is not September 30; the prisoner either committed the crime or he did not commit it. Notice that each of these statements contains two contradictory propositions, both of which can neither be true at the same time nor false at the same time. One of two contradictory statements must be true, and only one can be true. There is no middle ground. Therefore, in logic this principle is called the "Law of Excluded Middle."

In interesting seeming contradiction to this statement is the contention by Bogoslovsky that "the old reasoning is a generalization of experience in a static universe, where motion is incidental, where everything is absolute, where crossbreed forms are deformities. . . . The new reasoning is based on a dynamic universe with motion as its essence, with ceaseless change its characteristic aspect, a universe conceived as a continuous succession of different phases of one process which are all related to each other. Logic of this reasoning must have as its foundation principle and root the law, '*A* is *B* and *non-B* at the same time.'" However, "Dynamic Logic of the 'Included Middle' is not a flat contradiction of the Static Logic of the 'Excluded Middle,' but includes it as one pole or extreme of dynamic thinking where A is 99.9999% B and .0001% *non-B* corresponds closely to static logic."[123]

In other words, Bogoslovsky's thinking is influenced by Einstein's ideas on relativity. While it is true that a point P is either on line AB or not on it, still "on-ness" may be in a sense a relative matter if it is considered that some points are more nearly on AB than are others. So also, if equality is interpreted as a relative matter, it is true that two segments 6 inches and 5.9 inches long are more nearly equal than are two segments 6 inches and 3 inches long. However, in non-geometric situations the idea of more than two alternatives, in fact a whole series of gradations, is more defensible than in geometric situations. For instance, is it true that a man is either honest or not honest; that a piece of cloth is either linen or not-linen; or that a man rises either early or not early in the morning? Are not honesty and "linen-ness" and earliness, as well as many other qualities, a matter of degree, of relativity, rather than of absolute fact?

Furthermore, if statements are not made in contradictory form, then there are many situations in both geometry and life where there is a middle ground. For instance, to say that a line is either curved or straight involves a situation in which there is no contradiction. It is evident that if a line had part of it straight and part of it curved, then it would be neither all curved nor all straight. A point is either inside, on, or outside a circle; one angle is either less than, equal to, or greater than another; an angle is either acute, right, or obtuse. We may say that a piece of linen is either black or white, but that may not be true since white linen gets soiled and black linen may fade. Similarly to say that a car is either worth \$400 or \$500 is equally fallacious; it may be worth \$450. To say that either Jones or Smith stole the money is dangerous because both of them together or neither of them might have stolen it. However, linen is either white or not white even though there be degrees of whiteness; the car is worth either \$400 or not \$400, whether \$5 or \$100 more or less than \$400; Jones either stole the money or he did not steal it, whether with or without the help of Smith. Similarly, in the geometric illustrations it is possible in each case to have but two alternatives. That is, a point is either on a circle or not on it; two angles are either equal or not equal; an angle is either a right angle or not a a right angle; a line is either straight or not straight. In either case one of these possibilities will be proved true if it can be established that the

[123] Bogoslovsky, Boris B. *The Technique of Controversy, Principles of Dynamic Logic.* Harcourt, Brace and Co., 1928, pp. 12, 18.

other possibility is false; or, if there are three or more possibilities, if it can be established that all but one are false.

In other words, the generalizations of Bogoslovsky apply only to situations in which there is no contradiction, or in which contradiction is interpreted in a relative sense. For indirect proof, both in geometric and non-geometric situations, the Law of Excluded Middle is assumed to be true; that is, this law is postulated. It is of little concern in indirect reasoning that the class of *"non-B"* is large and the class of *"B"* is small, or *vice versa*. It is postulated, however, that two contradictory statements in geometry cannot be both true or both false at the same time.

It is therefore evident that it may be possible to prove one of two contradictory statements true by proving the other false or to prove a statement false by proving its contradictory true. This type of proof is called "Indirect Proof" because the reasoning is indirect. The validity of indirect proof depends upon three postulates, usually called principles in logic:

Postulate I. A thing must either be or not be.

Postulate II. If one of two contradictory statements is proved to be false, it immediately follows that the other statement must be true; similarly, if one of two contradictory statements is proved to be true, then the other must be false (the Law of Excluded Middle); or, if there are only three possibilities, one of which must be true and only one of which can be true, then if two of these are proved false, the third must be true, and if one is proved true, the other two must be false.

Postulate III. If certain premises and a correct process of reasoning necessarily reach a conclusion which is false; then at least one of the premises must be false.

The following illustrations will help to clarify the first two of these postulates; the first ones have but two alternatives, the last ones each have three possibilities, only one of which can be true:

(1) AB either equals PQ or does not equal PQ.
(2) Line XY is either straight or not straight.
(3) Point P is either on the circle or not on the circle.
(4) AB is either greater than, equal to, or less than PQ.
(5) Line XYZ is either straight, broken, or curved.
(6) Point P is either within, on, or outside the circle.

These illustrations indicate the meaning of Postulates I and II above. These two postulates do not violate the contention of Bogoslovsky for a law of "Included Middle," because all reasoning here is on the idealistic basis in which close approximations do not count, and relativity is neither denied nor necessary. Wherever it is necessary to prove XYZ either straight or not straight, there are no degrees of straightness which are significant. Furthermore, just as it is unknown whether or not actually more than one parallel to a line can be drawn through a given point, yet we accept a certain traditional postulate for Euclidean geometry, so also the law of Excluded Middle is postulated here as a basis for indirect proof in geometry. Therefore it is evident that in the first three statements, if one of the contradictory propositions could be proved false, the other would be true, and *vice versa*. In other words, if the assumption that AB does not equal PQ could be proved false, then its contradictory would be true and AB would equal PQ without further proof. Similarly, to prove that XY is a straight line or to prove point P on a circle, it is necessary to prove only that the contradictory statement is false. Also, in the second group of statements, if two of the three only possibilities given in each statement could be proved false, then it must follow that the remaining one would be true without further proof.

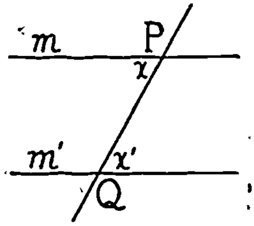
It is therefore evident that it may sometimes be very desirable both in geometry and in life to prove a statement false. Geometry, without indirect proof, has been taken up largely with proving statements true, not false. The technique for proving statements false is a very simple one based upon Postulate III. For instance, to prove that $AB = PQ$, if the assumption that AB does not equal PQ would necessarily and inevitably lead to a conclusion which is false, then the assumption upon which the correct reasoning is based must itself be false. Therefore, without further argument, $AB = PQ$. Similarly, to prove XY a straight line, or point P on a given circle, if the contradictory statements lead inevitably by correct reasoning to conclusions which are false or impossible, then the original statements must be true. This method of reasoning is called the method of the Indirect Proof.[124]

Types of Indirect Proof. While all indirect proofs depend on the three postulates given above, and are, in a major sense, the same in their outstanding features; yet, in minor characteristics, there are five types of indirect proof which may be distinguished. They are illustrated below. The first and simplest form is one that can be used to prove any converse theorem that is true. In it there are but two possible conclusions, one the contradictory of the other. This type of proof has already been

[124] Upton, C. B. "The Use of the Indirect Proof in Geometry and in Life," *Fifth Yearbook of the National Council Teachers of Mathematics*, 1930, pp. 102-133.

illustrated in the proof of theorem 9 that if a line divides two sides of a triangle proportionally, it is parallel to a third side. Indirect proof is also used in proving theorem 3 on parallels, which should be postulated in the early treatment of it in order to avoid the intricacies of indirect proof at a time when all attention is needed for mastery of direct proof. In the proof below it is assumed that the first theorem on parallels, theorem 2, has been proved: "If two lines cut a third so as to make the alternate interior angles equal, the lines are parallel."

Type I. Proving the Opposite False. Theorem: If two parallel lines cut a third line, the alternate interior angles are equal. (Converse of theorem 2.)



HYPOTHESIS: m is parallel to m' and cut by PQ.

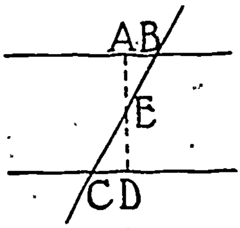
CONCLUSION: Angle x equals angle x'.

PROOF:

(1) Angle x either equals, or does not equal angle x'.
(2) Assume that angle x is not equal to angle x'.
(3) Then at P imagine drawing a line AP so that angle APQ equals angle x'.
(4) Such a line AP would then be parallel to m' by theorem 2.
(5) However, this would be impossible because there would then be two lines through P parallel to m'. See parallel postulate.
(6) The assumption that angle x is not equal to angle x' leads inevitably to an impossible conclusion, and must be false.
(7) Therefore angle x equals angle x'.

Type II. Using Formula 1 for Converse. This same proof can be effected more easily, although not more simply, by using the facts concerning the proof of converses from the previous section. If, in the direct statement in which it was proved that the equality of two angles made certain lines parallel, it had also been proved that the "non-equality" of those angles made the lines "non-parallel," then the converse would be true at once. This is true from the preceding section in which it was shown that converses are always true if opposites are true.

THEOREM: (Direct proof first, indirect proof in the last two steps.) If two lines cut a third line so as to make a pair of alternate interior angles equal, the lines are parallel.



HYPOTHESIS: Angle *ABE* equals angle *DCE*.
CONCLUSION: *AB* is parallel to *CD*.
PROOF:

(1) Bisect *BC* at *E*. Draw *ED* perpendicular to *CD* and extend it to *AB* at *A*.
(2) Triangle *ABE* is congruent to triangle *DCE* by A.S.A.
(3) Therefore *AD* is perpendicular to *AB*. Why?
(4) Therefore *AB* is parallel to *CD*. Why?
(5) If angle *B* is not equal to angle *C*, then the angles at *A* and *D* would be unequal since the angles at *E* are equal. *AD* would therefore not be perpendicular to *AB*, and therefore *AB* would not be parallel to *CD*.
(6) Therefore the hypothesis is necessary and the converse is true: if the lines are parallel, the alternate interior angles will be equal.

Type III. A Proof That Begins Like an Indirect Proof but Ends in a Direct Form. Using the figure, hypothesis, conclusion, and the first four steps in the proof for the illustration for Type I, the rest of the proof would then be as follows.

(5) But *m* is parallel to *m′* by hypothesis.
(6) Therefore *m* and *AP* coincide by the postulate of parallels.
(7) Therefore *m* is parallel to *m′* since it coincides with *AP* which was drawn parallel to *m′*.

Summary of Type I, II, III. Note the characteristics of these three types of indirect proofs.

1. In each case there are but two alternatives one of which must be true.
2. Type I assumes the false statement true in order to prove that it leads inevitably by correct reasoning to a false conclusion and that it must therefore be false. If the first conclusion chosen cannot be proved false, then either there is some error in reasoning or the other conclusion must be the false one.

3. The first and third types use the direct form of the theorem and other theorems to prove the supposedly false statement false.
4. In the more general form, II, using Formula 1, which was given in the discussion of converses, the truth of the converse is established by the proof of the opposite of the original theorem.
5. In the third type the proof culminated in a direct proof resulting from an attempt to reconcile the facts with the erroneous result which was obtained by assuming the truth of the false conclusion.

Exercises Using Indirect Proofs of Types, I, II, and III.

111. State and prove the converse of theorem 8, which is theorem 9, using Type I, Type II, and Type III.
112. State the two converses for each of theorems 15 and 17 and prove at least one of them as above.
113. State and prove, if possible, the converse of theorem 7. Remember that not all converses are true.
114. The proof of a locus problem really involves proving a direct and a converse statement. Show how theorems 13 and 14 can be proved using the technique of the types given.

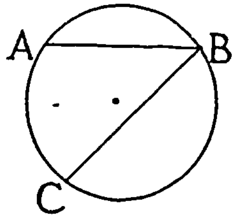
Type IV. Three Possibilities. This type of indirect proof involves a situation where there are three, and only three possible conclusions, one of which must be true, and only one of which can be true. To illustrate the indirect proof in such a case the converse of theorem 16, and one of the inequality theorems will be used. Theorem 16 has an interesting and little known converse. The theorem that an inscribed angle has the same measure as half its arc, has a converse that would be stated and proved in about the following way.

(a) *First Illustration of Type IV.*

PROBLEM: If an angle has the same measure as one-half the concave arc cut off by its sides, then its vertex is on the circle.

HYPOTHESIS: Angle $ABC = \frac{1}{2}$ arc AC. Arc AC is concave to B and cut off by the sides of angle ABC.

CONCLUSION: B is on the circle.



PROOF:

(1) B is either on the circle, inside the circle, or outside the circle.
(2) If B is inside the circle then AB and CB extended would cut the circle in two points, P and Q. Then angle B would equal ½ (arc AC + arc PQ). This is contrary to the hypothesis and therefore B cannot be inside the circle.
(3) If B is outside the circle then angle ABC will be an angle between two secants cutting the circle in points A, R, C, and S, and angle B would then be equal to ½ (arc AC — arc RS). This conclusion would also be contrary to the hypothesis and consequently the premise upon which it is based must be false. Therefore B cannot be outside the circle.
(4) Since the assumptions that B is inside or outside the circle both lead to impossible conclusions, they are false by Postulate III, and B is on the circle, by Postulate II.

COROLLARY: If the opposite angles of a quadrilateral are supplementary, the quadrilateral is inscriptible.

(b) *Second Illustration of Type IV.*

DIRECT THEOREM ASSUMED TO HAVE BEEN PROVED: If a triangle has two unequal sides, the angle opposite the greater side is the greater.
CONVERSE THEOREM: If a triangle has two unequal angles, then the side opposite the greater angle is the greater.
HYPOTHESIS: A triangle ABC with angle A greater than angle B.
CONCLUSION: Side a is greater than b.

PROOF:

(1) a is either less than, equal to, or greater than b.
(2) If a is less than b, then angle A is less than angle B from the direct theorem.
(3) If $a = b$, then angle A = angle B from theorem I (the isosceles triangle theorem).
(4) Both of these conclusions, (2) and (3), are impossible since by hypothesis angle A is greater than angle B, yet both conclusions are the inevitable results respectively of the first two assumptions. Therefore these assumptions are both false by Postulate III.
(5) Consequently a is greater than b by Postulate II.

Type V. Based upon the Law of Converse. The proof here is carried on with the original theorem as in type III. The original theorem for the first illustration under type IV was that an inscribed angle B has the same measure as half its arc AC. If it can be shown that

(1) B on the circle makes angle $B = \frac{1}{2}$ arc AC,
(2) B within the circle makes angle B greater than $\frac{1}{2}$ arc AC, and
(3) B outside the circle makes angle B less than $\frac{1}{2}$ arc AC,

then the converse is true without further proof by the Law of Converse, which was discussed in the preceding section.

In the second illustration for type IV, if it can be shown in the direct theorem (If a triangle has one side greater than another, the angle opposite the greater side is the greater.) that

(1) a greater than b makes angle A greater than angle B,
(2) $a = b$ makes angle A = angle B, and
(3) a less than b makes angle A less than angle B,

then the converse will be true by the Law of Converse. These three statements can easily be shown to be true from the proof of the original theorem and consequently the converse is true.

Summary of Types IV and V. Note the characteristics of these two types of indirect proofs.

1. They involve situations in which there are three or more possibilities, only one of which can be true and one must be true.
2. Two of these possibilities are proved false in type IV by using the direct form of the theorem and other theorems.
3. In type V the hypothesis was shown to be the necessary and sufficient condition for the conclusion to be true, and consequently the converse is true without further proof by the Law of Converse.
4. Attention should be called to the fact that by a slightly different wording and organization type IV may be made type I, and type V may be made type II. That is, in the first illustration, point B is either on or not on the circle. If not on the circle, it would be either inside or outside the circle, and consequently if neither inside nor outside the circle then it must be on the circle. Similarly, in the second illustration a is either greater than b or not greater than b. If not greater, then a is either equal to or less than b, etc.

Note: An analysis of thirty modern textbooks, the details of which are not reported in this study, shows that types I, III, and IV are used exclusively. Types II and V are the cleverest forms and in some cases much the shortest of them all, yet the process of evolution in geometry in the United States has gradually left them out. Types II and V are given here chiefly as background material and to make the presentation reasonably complete.

Furthermore, the similarities of these five types should be noted. By a different set up of conditions, types IV and V could be treated as in type I and II. Type III is exactly like type I, except in the ending, so that in a sense there may be but two types, one using the materials at hand, and the other using the general solution covered by the Law of Converse.

Exercises on Converses for Types IV and V.

115. Prove that if 2 angles of a triangle are equal, the triangle is isosceles. Note: If angle A = angle B, AC is either greater than, equal to, or less than BC. Assume that the direct form of the theorem given as the second illustration for Type IV above, precedes this exercise. Prove it independently, both directly and indirectly, as though it were an exercise following theorem 4.

116. State and prove indirectly the converse of theorem 6. Note: The reader should not be discouraged if he finds it difficult to prove this indirectly, yet easy to prove directly. Some of the exercises following will also be of that nature and yet for most of them the indirect proof is the more concise and simple if not the only possible proof.

117. If 2 triangles have 2 angles of one equal to 2 angles of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second. Note: See any high-school text for proof of this. State and prove the converse of this theorem.

118. Equal chords are equally distant from the center, and of two unequal chords the greater is nearer the center. Assume this statement proved, then state its converse and prove it by indirect proof, not by using exercise 117 above.

119. If in triangle ABC, $a^2 + b^2 = c^2$, then angle C is a right angle.

120. Given a triangle ABC with A', B', and C' the midpoints of its sides and D, E, and F the feet of the altitudes. A circle through A', B', and C' will also pass through D, E and F. Assume this conclusion proved, then state and prove its converse.

General Discussion and Further Illustration. The preceding illustrations have been purposely selected so as to cover the customary uses of indirect proofs in high-school geometry. In order to give the reader more experience with indirect proof certain exercises have been designed. Most geometry texts have no exercise material whatever requiring the use of indirect proofs. Where such proofs might otherwise be possible, a theorem, proved by the indirect method, precedes the conventional list, all the exercises of which are easily proved directly by quoting the theorem just proved.

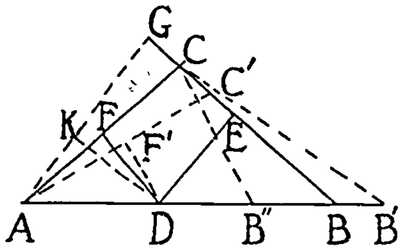
In the examples selected the student may use any of the five patterns just illustrated. Where there are two contradictory possibilities he may use one of the first three forms, preferably one of the first two. Where there are three possibilities he may use either of the last two types. Most of the following exercises are converses of exercises given in the previous section. Since that is the case one must remember that not all converses are true. Whenever the hypothesis is unnecessarily generous the converse is not true. In solving the following problems the reader may assume the direct statement to have been proved, whether he proved it himself or not.

If the formulas for converses are used, the required statements must be carefully proved. If the other types of proof are used, they involve, in most cases, a construction which satisfies the conditions. *If so, the construction should be made so as to effect the least possible change in the figure.* Since the examples previously given are more or less traditional and the proofs therefore somewhat familiar, two additional illustrations of indirect proof follow. The precise delimitation of the conditions to be proved false should be especially noticed.

a. *Illustrative Indirect Proof.* If the sum of the perpendiculars from any point in the base of a triangle to its two sides equals the perpendicular from the vertex of one of the base angles to the opposite side, the triangle is isosceles. (Converse of exercise number 42.)

HYPOTHESIS: Triangle ABC with D any point in base AB, AG and DE are perpendicular to BC, DF to AC, and $AG = DF + DE$.

CONCLUSION: Triangle ABC is isosceles.



PROOF:

(1) ABC is either isosceles or not isosceles.

(2) If ABC is not isosceles, there are several ways of making a triangle that is isosceles. One way is to construct CB' or CB'' equal to AC. However, less change in the given figure is effected by assuming angle A not equal to angle B, and constructing angle BAC' = angle B.

(3) Draw DF' perpendicular to AC', and DK perpendicular to AG.
(4) Then by Exercise 42 (which is assumed to have been proved), $AG = DE + DF'$ or $AK = DF'$.
(5) But this is impossible, since DF' does not equal DF, because otherwise triangle ADF' would be congruent to triangle ADF and to triangle DAK, and then angle $F'AD$ = angle KDA = angle B = angle A. This is contrary to (2).
(6) Therefore the triangle is isosceles. Postulate II.

This exercise could be easily proved by direct proof. Triangles ADK and DAF are congruent by H.S., and angle FAD = angle ADK = angle ABC. The direct proof would involve at least six steps just as the indirect proof does. Consequently, the indirect proof is equally concise and equally desirable, since it is a useful pattern of proof. Furthermore, any converse, even though it can be proved directly, can also be proved indirectly.

The proof given above, Type I, is really much longer than is necessary. The concise and clever proof for an exercise of this kind is effected by the use of formula 1, involving the opposite; that is, by the use of Type II. The direct proposition must be proved first; and then, by the use of the opposite, the converse is proved in two short statements.

A Second Proof for (a) Using Type II. Exercise 42 is the proposition of which illustration (a) above is the converse.

HYPOTHESIS: Triangle ABC with $AC = BC$ and D any point on AB. Also DE and AG perpendicular to BC and DF perpendicular to AC.

CONCLUSION: $DE + DF = AG$

PROOF:

(1) Draw DK perpendicular to AG. (See the figure above.)
(2) $DE = KG$. Why?
(3) DK is parallel to BC. Why?
(4) Angle KDA = angle B = angle FAD. Why?
(5) Therefore triangle $ADF \cong$ triangle DAK by H.A.
(6) Therefore $DF = AK$.
(7) Therefore $DE + DF = AG$ by adding (2) and (6).
(8) But if ABC were not isosceles, then angle B would not equal angle A, triangle ADF and DAK would not be congruent, and therefore $DE + DF$ is not equal to AG.
(9) Therefore the converse of the proposition in Exercise 42 is true since $AC = BC$ makes $DE + DF = AG$ and AC not equal to BC makes $DE + DF$ not equal to AG.

b. *Illustrative Indirect Proof.* However, the beauty, brevity, and power of indirect proof comes in exercises like the following one in which direct proof, if not impossible, is at least involved and much longer than the indirect proof. Such is the case with theorems 3 and 9 as well as some of the converses of the inequality theorems.

Theorem: In triangle ABC, (a) if $a^2 + b^2 = c^2$, then angle C is a right angle; (b) if $a^2 + b^2 > c^2$, angle C is acute; and (c) if $a^2 + b^2 < c^2$, angle C is obtuse.

(a) Hypothesis: $a^2 + b^2 = c^2$

Conclusion: Angle C is a right angle.

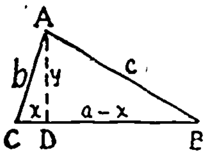


Fig. 1

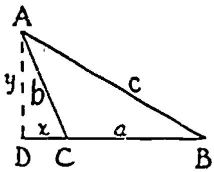


Fig. 2

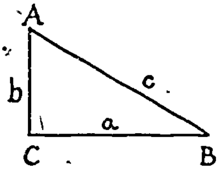


Fig. 3

Proof:

(1) Angle C is either acute, obtuse or right.

(2) Assume angle C acute (Fig. 1) and draw y perpendicular to BC.

(3) Then
$$c^2 = y^2 + (a - x)^2$$
$$= y^2 + a^2 - 2ax + x^2$$
$$= b^2 + a^2 - 2ax, \text{ since } x^2 + y^2 = b^2.$$

(4) Therefore $c^2 < a^2 + b^2$. But this is contrary to the hypothesis and therefore assumption (2) is false.

(5) Assume angle C obtuse (Fig. 2) and draw y perpendicular to BC extended.

(6) Then
$$c^2 = y^2 + (x + a)^2$$
$$= y^2 + x^2 + 2ax + a^2$$
$$= b^2 + a^2 + 2ax, \text{ since } x^2 + y^2 = b^2.$$

(7) Therefore $c^2 > a^2 + b^2$, which is also contrary to the hypothesis and therefore assumption (5) is false.

(8) If angle C is neither acute nor obtuse, it must be a right angle.

(b) Hypothesis: $a^2 + b^2 > c^2$

Conclusion: Angle C is acute.

Proof: Left to the reader to show, as above, that angle C can be neither a right nor an obtuse angle and must therefore be acute.

(c) HYPOTHESIS: $a^2 + b^2 < c^2$

CONCLUSION: Angle C is obtuse.

PROOF: Left to the reader to follow the pattern above.

Suggestion: Try formula 2, Law of Converse, for proving all three converses, a, b, and c, at once.

VI. EXERCISES FOR INDIRECT PROOF

DIRECTIONS: The following problems are designed to provide experience with indirect proof. They have been arranged in order of difficulty according to the combined weightings of a class of twelve college juniors who were preparing to teach mathematics. Enough of the exercises should be worked to insure mastery of the technique of indirect proof. Proofs of types I and IV should predominate, although the use of types II and V is often very effective.

121. If there were some way of proving theorem III without using theorem II, or if theorem III were postulated, show how theorem II could be proved indirectly, using theorem III or the theorem that the exterior angle of a triangle is greater than either non-adjacent interior angle.
122. Theorem 7 states that if a series of parallel lines cut equal segments on one transversal, they will cut equal segments on all transversals. State and, if possible, prove its converse.
123. If the diagonals of a trapezoid intersect in a point of trisection, then one of the parallel sides must be twice the other.
124. If the distance (t) from a point A on the circle to a second point P outside the circle is given by the formula $t = \sqrt{d^2 - r^2}$, in which d is the distance from the point P to the center of the circle, then the line AP is a tangent. (Converse of 100.)
125. In triangle ABC, side BC is trisected by points P and Q. If QM is drawn parallel to PA then it will bisect AC.
126. The theorem of geometry concerned with a line bisecting two sides of a triangle has a second converse not often given: If a line is parallel to the base of a triangle, equal to half the base, and has its end points on the other two sides, it bisects these two sides.
127. If two lines AX and BY are drawn from the two vertices of triangle ABC to points X and Y on the opposite sides and intersecting in point G, so that $\frac{AG}{GX} = \frac{BG}{GY} = \frac{2}{1}$ then the two lines are medians of the triangle.
128. If in quadrilateral $ABCD$ a line is drawn parallel to diagonal BD

cutting AB and AD in X and Y respectively and if XP parallel to BC cuts AC in P, then PY is parallel to CD. (Converse of 51.)

129. If a circle is drawn using side AB of quadrilateral $ABCD$ as a chord and cutting AD and BC extended, if necessary, in points E and F respectively and if EF is parallel to CD, then quadrilateral $ABCD$ is inscriptible. (Converse of 74.)
130. If the bisector of an exterior angle of an inscribed quadrilateral is tangent to the circle the two sides adjacent to this vertex are equal. State and prove the two converses.
CONCLUSION: (First converse) The bisector is tangent.
(Second converse) The tangent bisects the exterior angle.
131. If two opposite angles of a quadrilateral are right angles the bisectors of the other two angles are parallel. State and try to prove the converse of this.
132. If the bisector of angle C of the inscribed quadrilateral $ABCD$ cuts the circle at E then EA bisects the exterior angle at A. (Converse of 81.)
133. If the bisector of the exterior angle at C of the inscribed quadrilateral $ABCD$ cuts the circle at E then EA bisects the interior angle at A. (Converse of 81 and of 132.)
134. If the diagonals of a quadrilateral divide it into two pairs of similar triangles, then the quadrilateral is inscriptible.
135. If the lines PA, PB, PC and PD drawn from any point to the vertices of a quadrilateral are of such a nature that $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$ then the quadrilateral is a rectangle. (Converse of 50.) Prove if possible.
136. In triangle ABC, CM is a median, MR bisects angle AMC, and QR is parallel to AB. Prove MQ bisects angle BMC. (Converse of 63.)
137. If in quadrilateral $ABCD$, with diagonals AC and BD, a line is drawn parallel to BD and intersecting AB and AD in X and Y respectively and XP and YP are drawn parallel to BC and CD respectively, then P is on diagonal AC. (Converse of 51 and of 128.)
138. In triangle ABC a distance CE is laid off on AC extended and the same distance is laid off on BA toward A and called BF. If EF is bisected by the base BC, then the original triangle is isosceles. (Converse of 54.)
139. If through any point in the common chord of two circles two other chords are drawn, one in each circle, the four extremities will lie on a third circle.
140. The bisector of the angle between two chords intersecting within the circle bisects the arcs if and only if the bisector is a diameter.

141. State and prove the converse of exercise 82 using as a conclusion the statement that AD is a diameter.
142. State and prove the converse of: The orthocenter of ABC is the incenter of the pedal triangle DEF.
143. In circumscribed triangle ABC, with AD perpendicular to BC and cutting BC in D and the circle in P, if DH is laid off on DA equal to DP, then H is the orthocenter.
144. If the projections of a point upon the sides of a triangle are collinear, the point lies on the circumcircle of the triangle. (Converse of 94.)
145. Of all triangles with equal perimeters and the same base the isosceles triangle has the maximum area.
146. If a square is drawn on side AB of scalene triangle ABC and a line drawn from the vertex C to the center of the square bisects the angle C, then angle C is a right angle.
147. If a line is drawn from the point of intersection of the medians of a quadrilateral bisecting one of the diagonals, it will, if extended, bisect the other diagonal also. (Converse of 87.)
148. State and prove the converse of Ptolemy's Theorem: In any inscriptible quadrilateral the product of the diagonals equals the sum of the products of the pairs of opposite sides.
149. State and prove the converse of the following theorem: The six segments determined by a transversal on the sides of a triangle are such that the product of three non-consecutive segments is equal to the product of the three others. Note: This is the Theorem of Menelaus.
150. State and prove the converse of the following theorem: The lines joining the vertices of a triangle to a given point determine on the sides of the triangle six segments such that the product of three non-consecutive segments is equal to the product of the three other segments. Note: This is Ceva's Theorem.

VII. The Principle of Continuity

A. Geometric Interpretations

Continuity and Discontinuity in Coordinate Geometry. The principle of continuity is an interesting, beautiful, and useful part of geometry. The conception of continuity used in coordinate geometry is quite different from, although in a sense reconcilable with, the Euclidean geometry conception. For instance, we speak of a function as being continuous if there are no gaps in its graph, otherwise it is discontinuous. The relationship $Y = 1/X$ is discontinuous at the point $X = 0$. The function $Y = 1/(X - 2)$ is discontinuous at the point $X = 2$, because then Y = infinity. The functions $Y = \tan X$, $\cot X$, $\sec X$, or $\csc X$ are all discontinuous at periodic intervals, when for certain values of X, Y becomes infinitely large. Although in our mathematics courses most functions seem to be continuous there are many discontinuous functions. The cost (c) of any number of articles (n) at some price (p) is discontinuous unless an unlimited fractional division of the article is permitted. The cost of eggs at 5 cents apiece is discontinuous between eggs, because fractional parts of an egg are not purchased. The cost of mailing a letter by first class postage is discontinuous at regular intervals because the cost jumps by 2 cents at each advance. So every cost function is in a sense discontinuous, since no cost can jump by less than one cent at a time, and usually it jumps by considerably larger amounts. However, the equation $x + y = 5$ shows y to be a continuous function of x; and the formula, $d = rt$, shows distance traveled to be a continuous function of rate and time.[125, 126, 127]

Continuity in Euclidean Geometry. In Euclidean geometry discontinuity seems to have little place and meaning, while continuity refers to the constancy or permanency of certain properties which have been demonstrated to be true regardless of the change in the form of the figure. As expressed by Thomas Holgate, the principle of continuity, "first assumed by Kepler and later by Desargues, asserts that a property which can be demonstrated for a particular figure will hold true if the figure should change its form in any manner subject to the conditions under which it was first constructed."[128]

In other words, subject to the limitations stated, the principle of continuity applies to every proposition in geometry. In some cases it amounts merely to a statement that the proof of the proposition is perfectly general, or in other words that Aristotle's Dictum applies: "Whatever is predicated universally of any class of things, may be predicated, in like manner of anything comprehended in that class."[129] The principle of continuity is but an illustration in geometry of the more general principle often called "The Permanence of Mathematical Laws."

[125] Carver, Walter B. "Functions in General, and the Function [X] in Particular," *Mathematics Teacher,* Vol. XX, pp. 429-434.

[126] Lovitt, W. V. "Continuity in Mathematics and Everyday Life," *Mathematics Teacher,* Vol. XVII, pp. 31-34.

[127] Davis, E. W. and Brenke, W. C. *The Calculus.* The Macmillan Co., 1923, p. 11.

[128] Young, J. W. A. *Monographs on Topics of Modern Mathematics.* Longmans, Green and Co., 1911, p. 60.

[129] Davies, Charles. *The Nature and Utility of Mathematics.* A. S. Barnes and Co., 1875, p. 73.

This principle is beautifully expounded by David Eugene Smith: "What I learned in chemistry, as a boy, seemed true at the time, but much of it today is known to be false. What I learned of molecular physics seems at the present time like children's stories, interesting but puerile. What we learn in history may be true in some degree, but is certain to be false in many particulars. So we may run the gamut of learning, and nowhere, save in mathematics alone, do we find that which stands as a tangible symbol of the immortality of law, true 'yesterday, today, and forever.'

"We may change the symbols, . . . they are temporary expedients to convey the idea; we may speak in different tongues, . . . they are local expedients to convey thought; but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere, . . . a tangible symbol of the immortality of law.

" . . . all geometry is a science of invariance. We prove a law for a general plane triangle and it never varies, whatever we do to the figure. If we prove that $a^2 = b^2 + c^2 - 2bc \cos A$, then, however A may change, the law itself will never vary. In it the pupil comes into touch with the unchangeable, with the absolute.

"It is the same with all other laws of geometry. In any convex polyhedron, whatever its shape, the law remains that the number of faces plus the number of vertices is equal to the number of edges increased by two."[130]

Illustrations. The theorem that "if two straight lines intersect, the vertical angles are equal," asserts that the equality of the vertical angles is a property which will hold true of two intersecting lines regardless of changes in the lines so long as they intersect and are straight. Two lines will remain parallel or perpendicular if the conditions which made them parallel or perpendicular remain unaltered. Two angles will remain equal or an angle will be bisected so long as the conditions for equality or for bisection remain constant, regardless of other changes in the size or form of the angles or sides. Clearly this is yet merely a statement to the effect that geometry proofs are general, that they approach very near to "absolute truth," being dependent only upon accepted hypotheses, definitions, and postulates.

Extensions of the Principle of Continuity. The full novelty and beauty of the principle of continuity are evident only when the principle is applied in a more extended form. Let us take some of the twenty essential

[130] Smith, D. E. "Religio Mathematica," *Mathematics Teacher*, Vol. XIV, Dec. 1921, pp. 416-417.

theorems given in chapter 4, more or less in sequence, and show the extensions possible through the application of the principle of continuity. That the sum of the angles of a triangle is two right angles, whether the triangle is acute, obtuse, or right angled, whether the triangle is large or small, whether represented by wood, iron, paper, or by chalk on the blackboard, whether in the arctic regions or in the tropics, is an interesting fact. But, extend the idea a little. Take any triangle and lengthen its base until the base angles become acute and finally very small, in fact, approach zero, and the property still holds. Let the base shrink, the other two sides become equal, the angle between them approach zero, and the property still holds. Let the base be fixed and let the vertex move freely, above, below, to the left, or to the right, if the definition of a triangle is extended to include a figure with one or two angles equal to zero and consequently sides and parts of sides coinciding, the property demonstrated still holds true.

All the other theorems concerning straight lines can be made more interesting and the all-inclusiveness of their sweeping generalizations more impressive, if the figures are made to change, to grow, to shrink or to move about and still continue to exhibit the truth just demonstrated. In order to be unhampered in the application of the principle it would be well to define parallel lines as lines which intersect at infinity, then two lines will always intersect and the continuity of the relationship is not broken by the one exception of parallels. This definition is needed in both projective geometry and in coordinate geometry. If this definition is not accepted the property of intersecting could be thought of as discontinuous at the point where the lines are parallel. Perhaps other definitions will need to be made more general, as for instance, the definitions of the trigonometric functions whose meaning may thereby be extended to angles in other quadrants than the first.

Continuity Depends upon Motion for Clarity only. It is recognized that there is a school of thought, as indicated in a previous chapter, that is opposed to using motion in geometry. Yet it should be pointed out that motion has not been used in the proof of the theorems to which continuity has been applied. In no case does the proof of any proposition depend upon the principle of continuity. Its use has been to emphasize the general nature of the propositions proved, to reconcile apparent conflicts between closely related propositions, to integrate various supplementary propositions thereby decreasing the number of different ideas, and to make the study of geometric relationships more interesting to young people, who naturally are more concerned with a dynamic than with a static geometry. Hence, the rigor of the geometry presented is unaffected by the outcome of the argument on motion. Whether each position of the vertex of the triangle is a different point or the same point moved; whether a secant can move and become a tangent or the apparent motion can be nothing but different lines like the pictures in a motion picture film; whether or not there is motion is not the issue here, and therefore, the rigor is unaffected by the decision. It is enough that the use of continuity helps to clarify, to simplify, to generalize, and to make geometry more interesting. In fact, the analogy to motion pictures[131] can well be carried a step farther. Just as figures appear to move on the screen, although they do not really move since they are but successive pictures on a film; so the various apparent motions in geometry can be thought of as successive pictures with parts which therefore seem to change and yet exhibit a continuance of certain properties.

Further Illustration and Extension of the Principle of Continuity. Similar triangles can be made to exhibit continuous properties by apparently pivoting at imagined joints, but retaining the conditions for similarity. The sides of a right triangle can be made to vary and yet the relations expressed by the Pythagorean Theorem continue to be true. The theorems on circles furnish the most interesting examples of the continuity of demonstrated properties. In some cases the property can be shown to be continuous through several apparently conflicting theorems.

An angle whose sides extended cut a circle has a certain continuous relation to its arcs, regardless of the position of the vertex. Similarly, the product of the segments formed by two intersecting lines is independent of the position of the point of intersection.[132]

A. An angle between two lines which intersect a circle has the same measure as half the sum of the intercepted arcs, regardless of the position of the vertex of the angle. The vertex may be in any of the following positions.

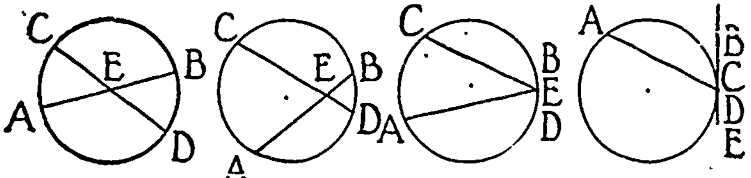


FIG. 1 FIG. 2 FIG. 3 FIG. 4

[131] Smith, D. E. and Bakst, Aaron. *The Play of Imagination in Geometry* (Motion Picture). Department of Education Talking Pictures Division of Research, Electrical Research Products, Inc., 250 West 57th Street, New York City, 21 pp.

[132] Reeve, W. D. *General Mathematics*, Book II. Ginn and Co., 1922, pp. 225, 228, 229, 240, 368.

1. At the center, as in Figure 1.
2. Within the circle, not at the center, as in Figure 2.
3. On the circle and both sides chords, as in Figure 3.
4. On the circle and one side a tangent, as in Figure 4.

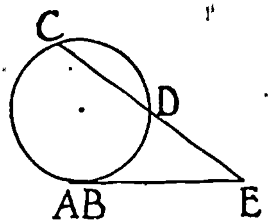


FIG. 5

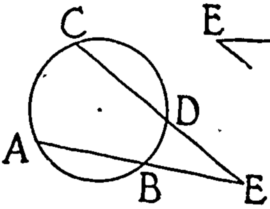


FIG. 6

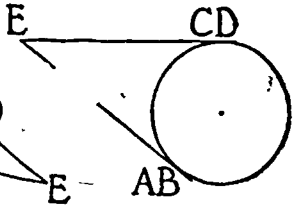


FIG. 7

5. Outside the circle with both sides secants, providing the one arc convex to the vertex is interpreted as a negative arc, as in Figure 5. This interpretation is of course not according to Euclid because negative numbers had not been invented at the time of Euclid.
6. Outside the circle with one side a tangent, providing again the smaller or concave arc is considered negative. See Figure 6.
7. Outside the circle with both sides tangents, providing the smaller or concave arc is considered negative. See Figure 7.

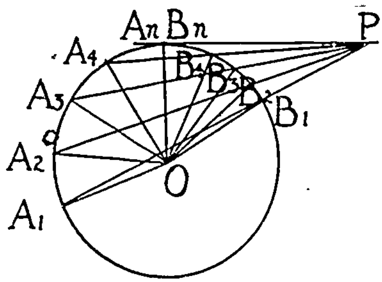
B. If two lines AB and CD intersect a circle at points A and B, and C and D respectively, and the lines themselves intersect at E, then the product of the segments of one line equals the product of the segments of the other; that is, $(AE)\ (BE) = (CE)\ (DE)$. This will be true regardless of the position of the point of intersection, even though the lines may one or both be chords, tangents, or secants. The various positions of E and the lines may be as follows: (The preceding figures can be used.)

1. E at the center.
2. E within the circle, but not at the center. Chords AC and BD would make two similar triangles.
3. E on the circle. One segment of each chord, AB and CD, is now zero.
4. E on the circle and one line a tangent. Both segments of the tangent cut off by the circle would be zero and the point of tangency would be a triple point and, in addition, a point of intersection.
5. E outside and both lines secants. Then $(AE)\ (BE)$ would be the whole secant times its external segment. Chords AC and BD would still make two similar triangles.

6. E outside the circle, AB a tangent and CD a secant. The tangent can be thought of as intersecting the circle in two points, A and B, which coincide, and therefore (AE) (BE) is the tangent squared. Chords AC and BD would still complete two similar triangles.
7. Same as 6 but CD also a tangent. Then CE and DE are the same segment and (CE) (DE) is the tangent squared. AC and BD would now make two congruent triangles.

Note: The reader should prove these two sets of seven propositions and show that the continuity holds in each set. Try proving each one by drawing chords AD and BC instead of AC and BD.

C. A secant cuts a circle in two points.



If a secant from P cuts a circle in two points which are connected with the center, an isosceles triangle is formed, such as A_1OB_1, A_2OB_2, . . . A_nOB_n. In these triangles, since they are always isosceles, what angles are equal? What happens as the angle at O decreases and approaches zero, and the two points, A and B, approach coincidence? When the secant becomes a tangent at what angle does it meet the radius? Could it be perpendicular to a radius at the outer extremity of the radius if not a tangent? Could a line perpendicular to the tangent fail to pass through the center of the circle?

The statement that "a straight line intersects a circle in two points" is one in which the principle of continuity extends the meaning of intersection to include the algebraic or coordinate geometry conception. Otherwise the property would be discontinuous. If the line is a tangent the two points of intersection with the circle can readily be thought of as coincident, but when the line fails to touch the circle there is no geometric representation of the intersections. In coordinate geometry the solution and representation are simple. Given a circle, $x^2 + y^2 = 9$, and a line, $x + y = 10$, the simultaneous solution gives:

$$x^2 + (10 - x)^2 = 9$$
$$x^2 - 10x + 45.5 = 0$$
$$x = 5 \pm 4.5i$$
$$y = 5 \mp 4.5i$$

Clearly the two points are $(5 + 4.5i, 5 - 4.5i$ and $(5 - 4.5i, 5 + 4.5i)$, and they are imaginary. Therefore it is evident that the principle of continuity forces the extension of the Euclidean geometry if exceptions to the principle are to be avoided.

The concept of the slope of a moving secant as the basis for a graphic presentation of the derivative in calculus is a tempting illustration which lies beyond the scope of this study. The slope of the secant as it approaches a tangent is always the ratio of the increment in the function to the increment in the variable as the increment in the variable approaches the limit zero. This is an interesting application which shows the dependence of calculus upon the principle of continuity.

The entire theory of limits with all its applications in higher mathematics and with its applications to incommensurables is dependent upon this postulate, first assumed and expressed by Kepler. The use of continuity in coordinate geometry, where functions are often spoken of as continuous or discontinuous, can now be somewhat extended by the statement that $y = 1/x$ becomes continuous through points at infinity; and that even costs are continuous for integral points, that is, the relation between cost and price is continuous, although there will be gaps in the correct graph.

Conclusion. No teacher of geometry, or of any branch of mathematics for that matter, should fail to utilize the principle of continuity. Clearly, the wide application of the generalizations of mathematics is one of its fundamental characteristics, as well as one of its most fascinating charms. The conclusions of mathematics are all-inclusive, yet relative. Every conclusion is based upon certain fundamental assumptions and definitions. Yet the properties demonstrated are general, or continuous, and often penetrate realms unthought of when first presented in specific form. The use of the principle of continuity should make geometry more interesting, more alive, more general in its applications, and more powerful in its pattern of reasoning.

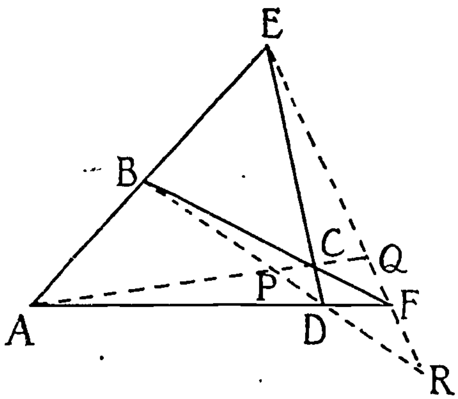
B. PROBLEM MATERIAL

151. Solve the two problems, A and B, in this section.

152. Apply the principle of continuity to the theorem that the sum of the angles of any polygon is $(n - 2)$ straight angles. Let n vary,

let the polygon become concave, let it flatten out completely or in parts, so that adjacent sides coincide, and yet show the property continuous.

153. Define a quadrilateral as the figure formed by four intersecting lines; a vertex as the point of intersection of any two sides whether adjacent or opposite; and a diagonal as a line connecting any two vertices. Then any quadrilateral *ABCD* has three diagonals, *AC*, *BD*, and *EF*, which intersect in three points, *P*, *Q*, and *R*. Show how this can be true for a parallelogram; a trapezoid; a convex quadrilateral with no sides parallel; a concave quadrilateral (one angle greater than 180°); a cross quadrilateral (like *ABDC* in figure given); a solid quadrilateral (one vertex not in the same plane as the other three).



154. Make up further illustrations. For instance: (1) What happens to the two tangents to a circle from a point *P* outside the circle as *P* moves toward the circle, is on the circle, or passes inside the circle? (2) What happens to the four tangents to two non-intersecting circles as the centers of the circles approach each other; as the circles become tangent externally; intersect in two points; become tangent internally; become concentric?

155. Take the circles in the problem above and let them rotate about their line of centers. What happens to the tangents for different positions of the centers and for different radii?

156. Show how the principle of continuity applies to the trigonometric functions; for example, $\sin A = y/r$, as the angle varies from 0° to 90° to 180° to 270° to 360°.

157. Apply the principle of continuity to the Sine Law of trigonometry:

$$\left(\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\right)$$

158. The Cosine Law ($a^2 = b^2 + c^2 - 2\,bc \cos A$) is a most excellent illustration of the force and beauty of the principle of continuity. Since $\cos 0° = 1$, $\cos 45° = +.7$, $\cos 90° = 0$, $\cos 135° = -.7$, $\cos 180° = -1$, $\cos 225° = -.7$, $\cos 270° = 0$, $\cos 315° = +.7$, $\cos 360° = +1$, show how side a varies with angle A. When angle A is 90°, triangle ABC is a right triangle with side a the hypotenuse. Show how the principle of continuity makes the Pythagorean theorem just a special case of the Cosine Law.[138]
159. The length of a tangent (t) from point P to a circle with center C and radius r is: $t = \sqrt{d^2 - r^2}$ in which $PC = d$. What happens to t as P approaches C or as d approaches zero? Show that the imaginary tangent is a half-chord.
160. If equal arcs AC and AD are laid off on each side of one extremity of a diameter and lines drawn from C and D to any point on the diameter AB or AB extended, the lines are equal, they make equal angles with the diameter, and they cut off equal arcs on the circle measured from B; and conversely. (More than one converse.)
161. The area of an ellipse is πab. Show that this formula is continuous as a approaches b.
162. The volume of an ellipsoid is $4/3\ \pi abc$. Show that this formula is continuous as the ellipsoid approaches a sphere.
163. In the next chapter exercises 210, 218, 219, 229, 230, 235 involve relations to which continuity applies very well.
164. Apply the principle of continuity to the circles of Apollonius given in problem number 40 of the exercises for analysis.
165. Find one good illustration of a relationship which is continuous through various changes in the form of the figure. Either select it from exercises 41-150, from some other geometry, or make it up.

Note: In the next section are many examples in geometry. As these are being worked the principle of continuity should be applied. In Chapter 6 each major theorem is an excellent example of continuous properties. In Ceva's Theorem point P, and in Menelaus's Theorem the crucial line DEF can be taken in various positions and yet the property demonstrated is continuous.

[138] Reeve, W. D. *General Mathematics*, Book II. Ginn and Co. 1922, pp. 368-372.

VIII. Incommensurables

Meaning. A clear understanding of the meaning of the term incommensurable is quite necessary. The derivation of the word would suggest that if two segments are commensurable, they can be measured together, that is, by means of the same unit. The favorite illustrations of incommensurability are the diagonal of a square which cannot be measured with the same unit used to measure the side, and the circumference and diameter of a circle which cannot be accurately measured by means of the same unit of measure. Another form of expressing the idea is to say that two segments are commensurable if they have a common divisor, and incommensurable, if not. The following definition and theorem help to clarify the problem as will also the illustrations which follow.

Definition. "Those magnitudes are said to be commensurable which are measured by the same measure and those incommensurable which cannot have any common measure."[134]

Theorem. "If, when the less of two unequal magnitudes is continually subtracted from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable."[135]

Discussion and Illustration. Because of our versatile common fractions and decimal fractions, which provide approximate measures for all line segments correct to any degree of accuracy desired, the problem of commensurability and incommensurability, which loomed so large for Euclid and Pythagoras, has faded greatly in significance. If one tries to express the square root of two by means of Roman numerals or by means of the awkward Greek system of numbers, the truth of this statement becomes apparent. It is in the attempt to express the diagonal of a square in terms of its side that incommensurable magnitudes were discovered. If s and d represent the side and diagonal of a square respectively, it is evident that $d^2 = 2s^2$. That is, the square on the diagonal is exactly twice the original square, yet the ratio between d and s is not expressible in terms of rational numbers. This would indeed be puzzling to any thoughtful man without a number system so flexible as to be able to account for it satisfactorily by a theory of approximations. The diagonal of a 10 inch square is the square root of 200 or 14.14213, correct to five decimal places; yet since 14.14213 squared is only 199.99984, to five decimal places, it is evident that 14.14213 is not the exact length of the diagonal. Furthermore, that length can never be expressed exactly by using the same unit used in measuring the side.

[134] Heath, T. L. *op. cit.*, Vol. III, p. 10.
[135] Euclid X, 2. (Heath, T. L. *op. cit.*, Vol. III, p. 17.)

Proof that $\sqrt{2}$ is Incommensurable with Unity. The proof of the incommensurability of $\sqrt{2}$ with unity is a relatively simple indirect proof. It was referred to by Aristotle, so is not new.

(1) Assume s and d commensurable.
(2) Then $s/d =$ some fraction p/q reduced to lowest terms.
(3) But $s^2/d^2 = p^2/q^2 = \frac{1}{2}$. Since $d^2 = 2s^2$.
(4) Therefore q^2 is divisible by 5 and q must be also.
(5) Let $q = 2r$, then $4\ r^2 = 2p^2$.
(6) Consequently $2r^2 = p^2$ and p^2 is divisible by 2, and therefore p must be.
(7) But if p/q is reduced to lowest terms and q is divisible by 2, then p must be odd and yet be divisible by 2.
(8) This is impossible, therefore s and d are not commensurable.[136]

Similarly, any other irrational number can be proved incommensurable with unity. Let a and $2a$ be the sides of a rectangle with diagonal b, and prove a and b incommensurable, or the $\sqrt{5}$ incommensurable with unity.

(1) Assume a and b commensurable or that $a/b =$ the fraction p/q reduced to lowest terms.
(2) $b^2 = (2a)^2 + a^2 = 5a^2$ or $b = a\sqrt{5}$
(3) Then $a^2/b^2 = p^2/q^2 = 1/5$ or $q^2 = 5p^2$.
(4) Therefore q^2 is divisible by 5 and q must be also.
(5) Let $q = 5r$, then $q^2 = 25r^2$.
(6) Then $25r^2 = 5p^2$ or $5r^2 = p^2$. Therefore p^2 is divisible by 5 and consequently p must be also.
(7) Therefore p/q can be reduced, which is contrary to the assumption in (1).
(8) Therefore a and b are not commensurable. That is, $b = a\ \sqrt{5}$, and $\sqrt{5}$ is incommensurable with unity.

It is furthermore interesting to note that the $\sqrt{2}$ and $\sqrt{5}$ are just as incommensurable with each other as either of them is with unity. This suggests the unlimited complexity of incommensurability.

Incommensurables to be Omitted from High School. The National Committee recommends that "the formal theory of limits and of incommensurable cases be omitted, but that the ideas of limit and of incommensurable magnitudes receive informal treatment."[137] There are at least two good reasons for this recommendation. First, our number sys-

[136] Heath. *op. cit.*, Vol. III, p. 2.
[137] National Committee. *op. cit.*, p. 49.

tem makes possible approximate measurements which satisfy all practical needs. Second, the proofs are too hard, and consequently require more time and energy than they merit in comparison with other more interesting and more useful material.

Other Cases of Incommensurability. In order to establish a little more defense for including the following treatment of incommensurables, and also in further defense of the statement that incommensurables are infinitely more common than commensurable magnitudes, as well as to avoid the common error of presenting incommensurables as though the $\sqrt{2}$ and π were the only ones, the following examples are given as "lead on" suggestions.

Next after $\sqrt{2}$, no doubt π is the most famous incommensurable number. In 1853 W. Shanks made himself famous by carrying out the value of π to 707 decimal places. To 25 places $\pi = 3.141,592,653,589,793,238,462,643$. It is evident that this is not the exact value, although for most practical work $3\frac{1}{7}$ is a close enough approximation. While the circumference of a circle with a definite radius is easily drawn, yet that circumference is incommensurable with the radius or diameter. Furthermore, if the circumference were commensurable with some unit, the radius would then be incommensurable with the circumference. The ratio, expressed as π, is incommensurable, even though one or the other of the quantities may be commensurable with some arbitrary unit. All true, in spite of the rumor that a bill was once introduced before a legislative body in the United States proposing to make π commensurable and equal to three so as to correspond with the Biblical value found in I *Kings*, VII, 23.

If you draw two line segments you may accidentally get one exactly 2 inches and the other exactly 3 inches long or even $1\frac{53}{64}$ and $2\frac{61}{64}$ inches long respectively. However, the chances are infinity to one that these would be but close approximations to the actual lengths and that it would be impossible accurately to measure even these segments. The dimensions of a standard door, a table, the dimensions of this page, the length of a foot rule, are all only relatively exact; that is, strictly speaking they are incommensurable in most cases.

The trigonometric ratios illustrate the problem. Sine of 0° is 0, of 30° is ½, and of 90° is 1. In these three cases the numbers which give the size of the angle in degrees and the ratio between the side opposite and the hypotenuse are commensurable. For no other angle between 0° and 90° is this true. Also, for 0°, 45°, and 90° the ratio between the side opposite and the side adjacent is commensurable with the angle. This is true for no other angle between 0° and 90° measured in integral degrees, although there are an infinite number of angles in which these sides are commensurable.

The logarithm of one is zero, of 10 is 1, of 100 is 2, of 1000 is 3, but for no other number between 1 and 1000 is a number commensurable with its logarithm. In the natural system the logarithm of one is zero and in no other case whatever is the logarithm commensurable with the number. Furthermore, logarithms are usually incommensurable with each other as well as with their number and base.

These few illustrations suggest the comparative frequency of incommensurability and in addition to being a defense for its treatment should help to make the meaning clear and to suggest the nature of the problem to be solved. However, before presenting proofs for any theorems involving incommensurable magnitudes, it is interesting to note that out of eight modern geometry textbooks for college courses in geometry, all but one written since 1920, not one considers the problem of incommensurability of enough importance even to mention it. It seems to be more or less a closed issue because any property which is true of commensurable magnitudes will also be true for incommensurable magnitudes by means of the pattern proof which follows. Furthermore, our very efficient decimal system makes informal treatment of the problem amply satisfying for all practical and most cultural purposes. In addition to this, the recent development of complex numbers has presented a problem of such magnitude as to throw commensurable and incommensurable quantities into one class by comparison. However, the problem has enough historic and intrinsic importance to merit the limited treatment which follows and the limited mastery of the problem which the following presentation makes possible.

General Plan of Attack. It is one matter to prove that the area of a rectangle whose dimensions are commensurable, such as 3 and 5, is the product of these dimensions. It is quite a different matter if the dimensions are $\sqrt{3}$ and $\sqrt{5}$, which might occur as frequently. The technique of proof for all incommensurable cases will consist of three steps: First, take the proof for commensurable magnitudes. Second, set it up so that it will have variables which remain equal as they approach the incommensurable magnitudes as limits. Then, finally, by means of the postulate of limits, draw the evident conclusion.

Very little difficulty seems ever to have been recognized in adding or subtracting incommensurables, although it seems difficult to see why $\sqrt{3} - \sqrt{5}$ and $\sqrt{3} + \sqrt{5}$ should be less significant than $\sqrt{3} \cdot \sqrt{5}$ or

$\sqrt{3} / \sqrt{5}$. Probably the explanation lies in the fact that areas and proportions involve multiplication and division, while addition and subtraction are little used in geometry. The general threefold plan of proof would nevertheless be used for addition or subtraction of incommensurables as for multiplication and division.

The Postulate of Limits. If two variables are always equal as they approach their limits the limits are equal.

Note: This postulate will be needed in each of the following theorems. The first of these theorems was postulated in the preceding chapter. While nothing was said about the nature of the dimensions of a rectangle, yet all explanations naturally assumed them commensurable. The question then arises concerning the continuity of this relationship if those dimensions become incommensurable.

THEOREM A. The area of a rectangle is the product of the base times the altitude. (Incommensurable dimensions.)

HYPOTHESIS: A rectangle of area A with its base and altitude, b and h, incommensurable.

CONCLUSION: $A = bh$.

PROOF:

(1) Take a convenient unit of measure and lay it off on b and h as many times as it will go integrally leaving a remainder less than the unit of measure in each case. The segments thus laid off, b' and h', will be commensurable and will form a rectangle with area A'.

(2) Then $A' = b'h'$. The area of a rectangle equals the base times the altitude. (Commensurable case.)

(3) Now take a unit of measure one-half, one-tenth, or one-hundredth as great and lay it off on b and h in the same manner. The new A', b' and h' will be larger than before. By continuing to use smaller and smaller units of measure A', b', and h' can be made to vary and to approach A, b, and h, respectively, as limits.

(4) That is, $A' = b'h'$ regardless of the unit of measure.

(5) But A' approaches A, and $b'h'$ approaches bh by definition of a limit and by the construction of b' *and* h'.

(6) Therefore $A = bh$ by the postulate of limits.

EXERCISES ON THEOREM A

171. Show that the volume (V) of a rectangular solid, whose dimensions l, w, and h are incommensurable, is found by this formula: $V = lwh$.

172. Show that the area of a triangle is ½ the base times the height, even though these quantities are incommensurable. Similarly, for the area of a parallelogram, a trapezoid, and a circle.
173. What is true of the volume of any prism and pyramid, or cylinder and cone, even though their dimensions are incommensurable, in the light of exercise 171 above? Apply the pattern proof above in the proof of this.

THEOREM B. A line parallel to one side of a triangle divides the other two sides proportionally.

HYPOTHESIS*: A triangle ABC with l parallel to b and dividing a and c into incommensurable segments m and n, and p and q, respectively, with n and q nearer the base.

CONCLUSION: $\frac{m}{n} = \frac{p}{q}$

PROOF:

(1) Take some unit of measure that will be contained in m integrally and lay it off on n as many times as it will go making a segment n', which is commensurable with m, and a remainder less than the unit of measure. Draw a new base, b', parallel to b through the end point of n' and it will cut off a segment q' on q such that

(2) $\frac{m}{n'} = \frac{p}{q'}$ (Commensurable case, theorem 8.)

(3) Now by the process of successively decreasing the unit of measure, n' and q' can be made to vary and to approach n and q as limits.

(4) That is, $\frac{m}{n'} = \frac{p}{q'}$ regardless of the unit of measure.

(5) But m/n' approaches m/n, and p/q' approaches p/q by definition of a limit.

(6) Therefore $\frac{m}{n} = \frac{p}{q}$ by the postulate of limits.

* The student should construct his own figure.

EXERCISES FOR THEOREM B

176. Show that in any right triangle with a given acute angle (A) the ratio of the side opposite to the side adjacent is constant even though these sides are incommensurable. In other words show that the tangent of A is constant.

177. Show the sin A constant. Cosine A.
178. Show that two triangles are similar if two angles of one equal respectively two angles of the other, even though the sides are incommensurable.
179. Name other relationships involving proportions and incommensurable segments which theorem B completes. What effect does theorem B have on the Pythagorean relation as proved from similar triangles?

THEOREM C. In equal circles or in the same circle, arcs have the same ratio as their central angles.

HYPOTHESIS:* Two arcs AB and CD in the same circle or in equal circles, and their central angles p and q respectively.

CONCLUSION: $\frac{AB}{CD} = \frac{p}{q}$

PROOF:

(1) Divide AB into an integral number of parts and using one of these as a unit of measure, lay it off on arc CD as many times as it will go leaving a remainder less than the unit of measure, and arc CD' commensurable with AB. Let q' be the central angle of arc CD'.

(2) Then $\frac{AB}{CD'} = \frac{p}{q'}$ (Commensurable case)

(3) Now by the process of reducing the unit of measure through bisection or some other process of dividing it integrally, arc CD' and angle q' can be made to vary and to approach arc CD and angle q as limits.

(4) That is, $\frac{AB}{CD'} = \frac{p}{q'}$ regardless of the unit of measure.

(5) But $\frac{AB}{CD'}$ approaches $\frac{AB}{CD}$ and $\frac{p}{q'}$ approaches $\frac{p}{q}$ by definition and construction.

(6) Therefore $\frac{AB}{CD} = \frac{p}{q}$ by the postulate of limits.

Note: Because of the above correspondence between the arcs and their central angles it is readily seen that if one of the two angles were a unit angle and its arc correspondingly a unit arc that the second angle would

* The student should construct his own figure.

have the same ratio to the unit angle that its arc has to the unit arc, hence the statement that a central angle has the same measure as its arc and conversely.

EXERCISES FOR THEOREM C

181. Show that in equal circles or in the same circle inscribed angles have the same ratio as their arcs, even though incommensurable.
182. State the converse of theorem *C* and also of exercise 181 above.
183. How does theorem *C* affect the other theorems which state relationships between angles and arcs?

IX. Summary and Conclusion

The outstanding portions of this chapter, from the point of view of a prospective teacher, are the three sections: Analysis, Indirect Proof, and Continuity. It has been the purpose of this chapter to present these three and other topics so as to provide a more complete experience for the student, not only with the subject matter of geometry, but more particularly with the heuristic method of teaching and learning geometry. Study of this chapter, including the solution of a liberal number of problems by means of the methods presented, should insure a comprehensive knowledge of geometry and of the reasoning patterns which geometry so concretely provides, as well as an unforgettable experience with the analytic method of discovering proof and with the indirect method of proof. The sections on the Structure of Geometry and Continuity are presentations of interesting points of view with respect to geometry. The final section on Incommensurables has been included for the sake of completeness and tradition.

CHAPTER VI

MODERN EUCLIDEAN GEOMETRY

I. Introduction

The Selection of Subject-Matter. Not only should a prospective teacher of high-school geometry be thoroughly familiar with the subject-matter to be taught and at the same time appreciate its possibilities when correctly taught, but he should have mastered as well other more advanced and more difficult material of a similar nature so that he may have a background that will enrich his teaching and give him additional professional experience with the heuristic method. There is so much splendid material in the field of modern Euclidean geometry, algebraic geometry, and projective geometry available for this purpose that several years of study could be spent on it. The material herewith presented is consequently selected with no attempt at a complete presentation of modern geometry. Two criteria for selection have been used. First, the material was selected so as to depend upon the preceding chapters and in some cases to complete and amplify them. Second, the material was selected so that it would involve some new definitions and concepts which are relatively as difficult for a college student to master as those of high-school geometry are for the high-school student.

The Professionalization of Subject-Matter. No claim is made that the theorems and exercises presented in this chapter are the best that could be selected. They are important ideas since they are included in all modern geometry textbooks such as Durell, Godfrey and Siddons, Johnston, and Altshiller-Court. Furthermore, each theorem selected serves a definite function in the plan for this chapter. The Theorems of Ceva and Menelaus are interesting extensions of plane geometry theorems and serve to unify the ideas of concurrency and collinearity. The nine-point circle and Euler's line theorems serve to extend the ideas about crucial points in a triangle and to show the relation between the various "centers" of a triangle. The theorem on coaxal circles serves to provide experience with entirely new concepts, new definitions, new relationships. It really opens up the whole interesting subject of "Inversion," but is used in this chapter merely for the purpose of having the prospective teacher experience the difficulty of learning something entirely new to him. The concept of "power of a point," or "radical center" does not immediately have meaning for him just as in high-school geometry "median," "parallelopiped," "ratio," "mean proportional" may be new to a high-school pupil and need to be developed by extended contact.

In other words the material in this chapter is selected, not so much for its mathematical content and completeness, as to fulfill a professional purpose. The two previous chapters have applied the heuristic teaching pattern to material with which the student has had contact and with which he is at least slightly familiar; this chapter applies the same pattern to new material in order more completely to establish the power and educational value of the inductive-deductive-analytic-synthetic processes which, combined, we have called the heuristic method of teaching.

II. Ceva's Theorem and the Theorem of Menelaus

1. Ceva's theorem (33)

History. The theorems of Ceva and Menelaus, with their converses, are fundamental for the science of projective geometry. Ceva's Theorem is given first here, although historically it came later. Giovanni Ceva was an Italian engineer and mathematician. He discovered the theorem named after him in 1678 and published at the same time the closely related Theorem of Menelaus, which had been little used since its discovery by Menelaus in 100 B.C.[138]

Since it has been recommended that high-school teachers use an inductive approach in discovering a deductive conclusion, it will be consistent to use that pattern with this theorem and the following theorems. It should likewise be possible to use an analytic method of discovering the proof, which can then be presented in synthetic form.

Definitions Needed. In the extended applications of these theorems it will be necessary to use directed line segments corresponding to the directed numbers of algebra, positive and negative. It will also be convenient to use some new terms, consequently a few definitions become necessary.

1. *Negative line segments* are segments taken in the opposite direction from those considered positive. Thus $AB = -BA$ for any segment AB, also if AP and BP are both positive, P is on AB extended, but if one is negative, then P is between A and B on segment AB.

[138] Johnson, Roger A. *Modern Geometry*. Houghton Mifflin Co., 1929, p. 148.

Illustrate by drawing a line segment AB and locating P so that AP and BP are both positive, then both negative, finally only one negative.

2. *Concurrent lines* are lines which have a common point.
3. *Collinear points* are points which lie on the same line.

APPROACH TO CEVA'S THEOREM:

(1) Draw a triangle with sides 6, 8, and 11 in.

(2) Divide the 11 inch side into 2 segments 5 and 6 inches, and the 8 in. side into segments 3 and 5 in., in consecutive order around the triangle.

(3) Draw lines from the opposite vertex to each of these points of division. Let these lines intersect in P. Draw a line from the remaining vertex through P to the 6 in. side.

(4) Measure the segments on this side, they seem to be 4 inches and 2 inches.

(5) Notice that 5 x 3 x 4 = 6 x 5 x 2

(6) Notice the order of selection of the segments to be multiplied together.

(7) Use sides 9, 8, and 11 in. long and go through the same construction.

(8) Take any other triangle and select any point P, draw lines to the vertices as above, extend them to the opposite sides, get the products corresponding to those above and see what happens.

(9) Can this be stated as a general theorem? Try it.

12, x 22

$\frac{3 \times 5 \times 6}{6 \times 3 \times 5}$

18 x 10 x 22

$\frac{6}{12} \times \frac{10}{6} \times \frac{12}{10}$

CEVA'S THEOREM: Three concurrent lines from the vertices of a triangle divide the opposite sides into segments so that the product of three non-adjacent segments equals the product of the other three.

HYPOTHESIS: Triangle ABC with any point P and lines AP, BP, and CP cutting the opposite sides in points D, E and F. (Figure, next page.)

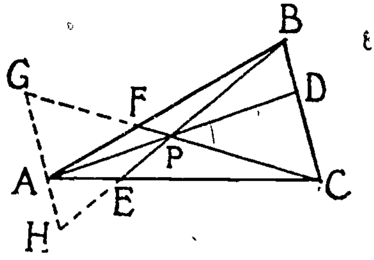
CONCLUSION: $(AF)\ (BD)\ (CE) = (FB)\ (DC)\ (EA)$

ANALYSIS: Three factors on each side of the equation suggest that dividing by the right-hand member would give three ratios, which in turn suggest the use of similar triangles.

(1) $(AF)\ (BD)\ (CE) = (FB)\ (DC)\ (EA)$ if

(2) $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ or

(3) $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = -1$



(4) Since the three factors in the denominator are now all negative, their product is thus made negative. Equation (2) or (3) suggests similar triangles obtained in some way so that these ratios can be obtained. Since AF and FB are corresponding sides it suggests that we draw GH through A parallel to BC and extend lines BE and CF as in the figure.

(5) Now if $\dfrac{AF}{FB} = K$, $\dfrac{BD}{DC} = L$, and $\dfrac{CE}{EA} = M$ so that

$(K) \cdot (L) \cdot (M) = 1$ then the solution would be obtained.

(6) But $\dfrac{AF}{FB} = \dfrac{AG}{BC}, \dfrac{BD}{DC} = \dfrac{HA}{AG}, \dfrac{CE}{EA} = \dfrac{BC}{HA}$. Why?

(7) And $\dfrac{AG}{BC} \cdot \dfrac{HA}{AG} \cdot \dfrac{BC}{HA} = 1$

(8) Therefore $\dfrac{AF}{FB} \cdot \dfrac{BD}{CD} \cdot \dfrac{CE}{EA} = 1$ or

$$\frac{AF}{DF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = -1$$

PROOF: The synthetic statement of this proof is now left to the student.

2. EXERCISES FOLLOWING CEVA'S THEOREM

201. State the converse of Ceva's Theorem. It would contend that if the sides of a triangle are divided so that the products stated would be equal, then lines joining the points of division to the opposite vertices would be concurrent.

Note: In proving this use the indirect method. Assume that two of these lines meet in a point, then use this point in applying Ceva's

Theorem, and it will divide the third side in a way that will indicate that the third line goes through the same point. Note that this converse theorem forms a basis for proving lines concurrent and consequently becomes rather a fundamental theorem. The point formed by the intersection of three lines through the vertices of a triangle is often referred to as the "Gergonne Point."

202. Use exercise 201, the converse of Ceva's Theorem, in proving that the medians of a triangle are concurrent.
203. Try the converse of Ceva's Theorem for proving concurrency of the angle bisectors.

   Note: Remember that the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.
204. Try the converse of Ceva's Theorem to prove concurrency of the altitudes.

   Note: Use similar right triangles.
205. Prove the theorem if point P is outside of the triangle.

   Note: Be careful of direction of line segments, since lines from this point through the vertices will intersect two of the opposite sides externally. Take various positions of the point.
206. Would the converse of exercise 205 also be true for external segments?
207. Suppose the point P is on one of the sides; what happens?
208. Prove that the bisectors of two exterior angles and the other interior angle are concurrent.
209. Lines joining opposite vertices with the points of contact of the excircles of a triangle are concurrent. (This point is called Nagel's Point.)
210. Show that the relationships described in Ceva's Theorem and its converse are continuous. Let the point P move to any position: on one side or at a vertex; on a median, a bisector, or an altitude; at the incenter, an excenter, the circumcenter, the orthocenter, or the centroid; let the triangle be equilateral, isosceles, scalene, right, or obtuse.

3. THE THEOREM OF MENELAUS (34)

APPROACH:

(1) Draw a triangle ABC with sides 11, 8, and 6 inches. Divide the two longer sides AB and BC into segments 7 and 4 inches and 1 and 7 inches at points D and E.

(2) Draw line DE and extend it until it meets AC extended at F.

Measure AF and CF, they are 8 and 2 inches.

(3) Note that $\frac{7 \times 1 \times 8}{4 \times 7 \times 2} = 1$ or $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$

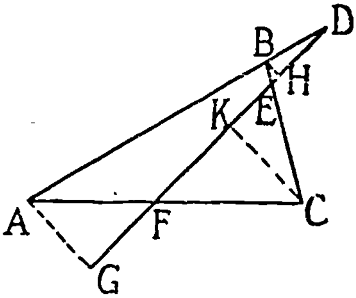
(4) Try it over with $AC = 12$ inches, instead of 6 inches.

(5) Try it drawing any line DF across any triangle intersecting two sides internally and the third externally.

(6) Try it intersecting all three sides externally.

(7) State the Theorem of Menelaus.

THE THEOREM OF MENELAUS: A transversal of a triangle divides the sides into six segments in such a way that the product of any three non-adjacent segments equals the product of the other three (disregarding direction).



HYPOTHESIS: DF is any transversal cutting the sides of triangle ABC in points D, E, and F respectively.

CONCLUSION:

(1) $(AD)(BE)(CF) = (BD)(CE)(AF)$ or

(2) $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$

ANALYSIS:

(1) The above ratios suggest getting similar triangles. Again this can be done in many ways. Suppose we draw perpendiculars to DF from A, B, and C. Call their feet G, H, and K.

(2) Now if the three ratios can be shown equal to other ratios whose product is 1 or -1 then we have the solution.

(3) $\frac{AD}{BD} = \frac{AG}{BH}, \frac{BE}{CE} = \frac{BH}{CK}, \frac{CF}{AF} = \frac{CK}{AG}$. Why?

(4) Now $\frac{AG}{BH} \cdot \frac{BH}{CK} \cdot \frac{CK}{AG} = 1$

(5) Therefore $\frac{AD}{BD} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = 1$ Why?

(6) or $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$ Why?

(7) or $(AD)\ (BE)\ (CF) = (BD)\ (CE)\ (AF)$
$= -(DB)\ (EC)\ (FA)$ Why?

PROOF:

(1) Draw perpendiculars from A, B, and C to DF. Call their feet G, H, and K.

(2) $\frac{AD}{BD} = \frac{AG}{BH}$ Why?

(3) $\frac{BE}{CE} = \frac{BH}{CK}$ Why?

(4) $\frac{CF}{AF} = \frac{CK}{AG}$ Why?

(5) Therefore $\frac{AD}{BD} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = 1$ Multiplying together equations (2), (3), (4) or

(6) $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$ Why? or

(7) $(AD)\ (BE)\ (CF) = (BD)\ (CE)\ (AF)$ from (5)
$= -(DB)\ (EC)\ (FA)$ from (6)

Therefore the Theorem of Menelaus is true for any triangle with any line cutting it.

4. EXERCISES FOLLOWING THE THEOREM OF MENELAUS (35)

211. Prove the Theorem of Menelaus if line DF cuts all three sides externally.

212. State and prove the converse of Menelaus' Theorem.
Suggestion: Apply the idea given for proving the converse of Ceva's Theorem. Note that this converse is fundamental for proving points collinear.

213. The bisectors of two interior angles and the other exterior angle intersect the opposite sides in points which are collinear.
Note: Use the converse of the Theorem of Menelaus.

214. The bisectors of the three exterior angles intersect the opposite

sides in collinear points. Try one interior and two exterior angles. Explain.

215. In triangle ABC let D, E, and F be the middle points of the sides AB, BC, and CA, and P be the middle point of DF. Let BP cut AC in Q. What is the ratio of AQ to QF?

    Note: Use triangle ADF and get the ratio of AQ to QF by Menelaus's theorem.

216. Pascal's Theorem: The opposite sides of an inscribed hexagon intersect in points that are collinear.

    Note: For $ABCDEF$, extend AB, CD, and EF to form triangle PQR. Now if the opposite sides intersect in points L, M, and N, prove LMN a line across PQR by considering BC, DE, and FA as transversals of PQR.

217. Desargnes' Theorem: If two triangles, ABC and DEF, are so situated that lines through corresponding vertices intersect in point P, i.e., ADP, BEP, and CFP are concurrent in P, and AB and DE intersect in Q, BC and EF in R, and CA and FD in S, then Q, R, and S are collinear.

218. Show how the Principle of Continuity applies in the Theorem of Menelaus. What happens as line DEF passes through one vertex; two vertices; bisects an angle; bisects a side; passes through AA', BB', or CC'; is parallel to one side; perpendicular to one side; or the triangle is a right, obtuse, scalene, isosceles or equilateral triangle.

219. Apply the Principle of Continuity to the converse of the Theorem of Menelaus by means of an analysis similar to the one above.

220. Make up an example using either of the last two theorems or their converses.

III. Nine-Point Circle Theorem and Euler's Line

1. The Nine-Point Circle (Theorem 36)

Definitions Needed. Before taking up the next theorem it will be well to study the standard notation for a triangle and its various points.

Notation for a triangle ABC with sides a, b, c, and with A', B', C' the midpoints of those sides respectively.

AA', BB', CC' are its medians.

$A'B'C'$ is the medial triangle.

AD, BE, CF are the altitudes.

DEF is the pedal triangle.

I is the intersection of the angle bisectors or, the center of the inscribed circle, the incenter.

G is the intersection of the medians, known as the centroid or center of gravity.

H is the intersection of the altitudes, the orthocenter.

O is the intersection of the perpendicular bisectors of the sides, the circumcenter.

N is the center of the nine-point circle.

APPROACH TO THE NINE-POINT CIRCLE THEOREM.

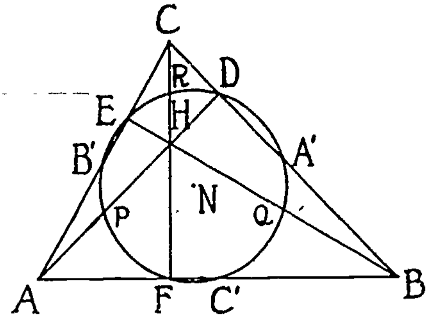
(1) Draw any triangle and find the middle point of each side.

(2) Draw a circle through these three points. This circle will cut the three sides in three other points unless the triangle is equilateral or isosceles. Connect these points with the opposite vertices. These lines seem to be perpendicular and therefore to be altitudes. Call their intersection H.

(3) AH, BH and CH seem to be bisected by the circle drawn. Call these points P, Q, and R respectively. Then it seems possible that A', B', C', D, E, F; P, Q, and R are nine points all on the same circle.

(4) State the theorem for the nine-point circle.

THEOREM: The midpoints of the three sides, the feet of the altitudes, and the midpoints of the segments joining the orthocenter to the vertices, all lie on the same circle.



HYPOTHESIS: Any triangle ABC with points A', B', C', D, E, and F as defined and P, Q and R the midpoints of AH, BH, and CH.

CONCLUSION: These nine points are on the same circle.

ANALYSIS: It will be possible to draw a circle through three of these points, then if the others can be shown to be on this circle, perhaps even one at a time, our problem is solved. Further analysis is left to the student.

PROOF:

(1) Construct a circle through A', B', and C'.
(2) In quadrilateral $A'DB'C'$, $A'D$ is parallel to $B'C'$. Why?
(3) Also $B'D = A'C'$, both being ½ of AC.
(4) Therefore $A'DB'C'$ is an isosceles trapezoid, its opposite angles are supplementary and consequently it is inscriptible.
(5) Therefore D is on circle $A'B'C'$.
(6) Similarly E and F can be shown to be on circle $A'B'C'$.
(7) In quadrilateral $A'B'PC'$, $B'P$ is parallel to CF, because in triangle ACH, $B'P$ joins the midpoints of AC and AH.
(8) Angle $A'B'P$ is a right angle because $A'B'$ is parallel to AB which is perpendicular to CF.
(9) Similarly PC' is parallel to BE which is perpendicular to AC and also to $A'C'$.
(10) $A'C'P$ is also a right angle.
(11) Therefore $A'B'PC'$ is an inscriptible quadrilateral and P is on the circle $A'B'C'$.
(12) Similarly Q and R can be shown to be on the circle $A'B'C'$.
(13) Therefore all nine points are on the same circle.

Note: If this is hard to follow draw three separate figures just alike and use the first one for steps 1 to 5, the second for step 6, and third for steps 7 to 12. Then actually draw the lines to form the quadrilateral discussed.

2. EULER'S LINE (THEOREM 37)

APPROACH:

(1) Draw several triangles of different shapes and in each locate I, O, H, G, and N.
(2) What seems to be true of these points?
(3) Is I ever on the line OH? Is C? Is N?

THEOREM: The circumcenter, orthocenter, centroid, and nine-point center all lie on the same line-segment OH which is bisected by the nine-point center, N, and divided in the ratio of 1 to 2 by the centroid, G.

HYPOTHESIS: Triangle ABC with points H, N, G, and O.

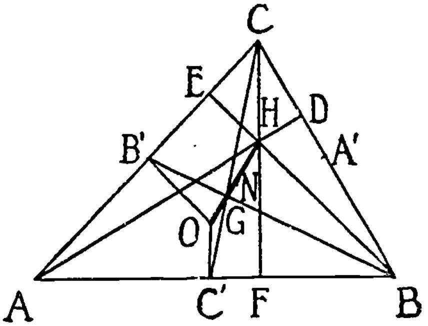
CONCLUSION:

(1) $HNGO$ is a straight line,
(2) $OG = ⅓\, OH$, and
(3) $ON = ½\, OH$.

ANALYSIS:

(1) If we take the straight line HG and prove that O and N are on it then $OGNH$ will be a straight line.

(2) Also if HG is extended ½ its length to O', and O' shown to be O then $OG = \frac{1}{3} OH$.



(3) Similarly if the middle point of OH can be shown to be the center of the nine-point circle then it must be N, and $ON = \frac{1}{2} OH$.

PROOF:

(1) Draw HG and extend it to O' so that $O'G = \frac{1}{2} HG$.

(2) Draw $O'B'$ and $O'C'$ (O' and O will be the same point; therefore think of O as O' for the present).

(3) Triangles $C'O'G$ and CHG are similar and also

(4) Triangles $B'O'G$ and BHG are similar, because 2 sides are proportional and the included angles equal.

(5) Therefore angles $GO'C'$ and GHC are equal, also

(6) Angles $GO'B'$ and GHB are equal. Why?

(7) Therefore $O'C'$ is parallel to CF which is perpendicular to AB, and

(8) $O'B'$ is parallel to BE which is perpendicular to AC.

(9) Therefore O' is the circumcenter and coincides with O. Why?

(10) Consequently $OG = \frac{1}{2} GH = \frac{1}{3} OH$.

(11) The perpendicular bisectors of lines $B'E$ and $C'F$ will both bisect OH and therefore must intersect at the midpoint of OH. Why?

(12) But $B'E$ and $C'F$ are chords on the nine-point circle and therefore the midpoint of OH is the center of the nine-point circle, or N.

(13) Therefore (a) $HNGO$ is a straight line,

(b) $OG = \frac{1}{3} OH$, and

(c) $ON = \frac{1}{2} OH$

Note: This line, $HNGO$, is called Euler's line.

3. EXERCISES FOLLOWING THE THEOREMS ON THE NINE-POINT CIRCLE AND EULER'S LINE

221. Construct a given line OH and lay off $OG = \frac{1}{3} OH$ and $ON = \frac{1}{2} OH$. Then construct the rest of the triangle having given also OA and GA.
222. State other conditions connected with the five notable points of a triangle which determine the triangle.
223. Prove triangles PQR and $A'B'C'$ congruent.
224. Prove that the nine-point circle bisects any line from the orthocenter to any point on the circumcircle.
225. The Theorem of Feuerbach. The nine-point circle of a triangle is tangent to the inscribed circle and to each of the escribed circles.

    Note: This is a difficult theorem to prove and is one of the famous theorems of geometry. It was discovered by Feuerbach in 1822 and since then by Steiner and many others. There are several proofs. See Johnston: *Modern Geometry*, pp. 200-205.
226. Draw a large scalene triangle, determine points G and O, from them determine N and H. Also determine I and draw the incircle, circumcircle, nine-point circle and the three excircles. The relationships stated in exercises 223, 224, 225 should be evident.
227. The circumcircle bisects each of the six lines joining the points I, I_1, I_2, I_3, (I_1 is the center of the excircle tangent to BC, etc.).
228. If from a point P, on the circumcircle, perpendiculars PX, PY, PZ be drawn to the sides of triangle ABC, then X, Y, and Z are collinear.

    Note: The line XYZ is called the Simson Line, named after Robert Simson, Glasgow, who first discovered it.
229. Show that the relationships described in the nine-point circle theorem and Euler's line are "continuous".
230. Show that the relationships described in the exercise above on "Simson's Line" are continuous.

IV. Coaxal Circles

Definitions. While the previous theorems have involved some new terms and relationships, they have been slightly familiar. The theorem and the definitions of this section open up a whole new field of geometry. The ideas are quite new and unique to students familiar with only high-school geometry. They furnish, therefore, excellent learning and teaching experience in geometry, and serve as a climax for the professionalization of geometry through the application of the heuristic pattern to the subject-matter to be learned. Several of the new terms will need to be defined.

1. *The power* (p) *of a point* P with respect to a circle is equal to the square of the tangent from P to the circle. If OP is drawn from the center O to an external point P, and OT to the point of tangency, it is evident that $p = t^2 = (OP)^2 - r^2$.

   The definition given above, while simple, is not quite complete. It holds true only for points outside of or on the circle. However, if p be defined more completely as $(OP)^2 - r^2$, then the power of p will be negative when OP is less than r, zero when OP equals r, and positive when OP is greater than r. Also every point P has a power with respect to a given circle and that power is constant.

   If the point P is within the circle $(OP)^2 - r^2$ is negative. A little study of a circle will reveal that again the power may be represented by a definite line segment. If a chord CD is drawn perpendicular to OP at P then $r^2 - (OP)^2 = (OC)^2$ or $OP^2 - r^2 = -(OC)^2 = -(\frac{1}{2}\text{ chord})^2$. Therefore if P is on the circle, its power is zero; if P is outside of the circle, $p = t^2$; if P is within the circle, $p = -(\frac{1}{2}\text{ chord})^2$; but always $p = (OP)^2 - r^2$.
2. *The minimum chord* (m) of P is a chord of a circle through an interior point P perpendicular to a diameter through P. As its name implies it is the shortest chord through P, and being perpendicular to a diameter is bisected by P. The power of P within the circle can now be stated as $P = -(\frac{1}{2}m)^2$, where m is the minimum chord through P, just as the power of P outside the circle can be stated as $p = t^2$, where t is the tangent to the circle from P.
3. *The radical axis* of two circles is the locus of a point P whose powers with respect to the two circles are equal.
4. *The radical center* of three circles is the locus of a point P whose powers with respect to the three circles are equal.
5. *The common secant* of two intersecting circles is a secant passing through their points of intersection. This corresponds to the term common chord, and of course really is a common chord extended.

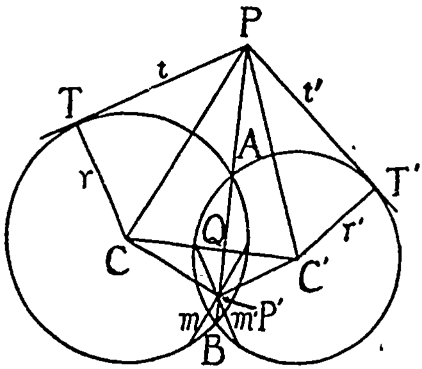
1. THEOREM ON RADICAL AXIS (38)

APPROACH to some generalization concerning the locus of a point P whose "power" with respect to two circles is the same:

1. Construct a circle with center C, radius r, and a tangent PA, and the radius to the point of tangency, AC.

2. Construct several tangents to the circle, with the same given length k.
3. What is the length of PC? Is it the same for every tangent with the given length k and on the same circle?
4. What then is the locus of point P whose power with respect to a given circle is constant. Show that $(CP)^2 = k^2 + r^2$.
5. Construct a second circle of different size not intersecting the first one. Construct the locus of point P' whose power with respect to the second circle C' is equal to the constant k. Show that $(C'P)^2 = k^2 + (r')^2$.
6. Suppose these loci intersect in 2 points, then each of the two points of intersection has the same power with respect to each circle. Call the points P and Q, and draw line PQ. PQ seems to be perpendicular to the line of centers. Is it? Prove it. Call the point where PQ cuts CC', A. Find AC and AC'.
7. Take some other point on PQ and try to prove that it has equal powers with respect to the two circles.
8. Take any other power k' and find its locus with respect to the two circles. Show that if the loci intersect at all that the line connecting their points of intersection coincides with PQ since it is perpendicular to CC' at A.
9. Take two intersecting circles and find the locus of a point P whose powers with respect to the two circles are equal.
10. It seems to be the common secant.
11. State this apparently true fact as a theorem to be proved.

THEOREM: The radical axis of two intersecting circles is their common secant.



HYPOTHESIS: Two circles intersecting at A and B with a variable point P having $p = t^2$ for each circle.

CONCLUSION: The radical axis, which is the locus P, is the secant through A and B.

ANALYSIS:

(1) Every point on the secant AB must have equal powers, and

(2) Every point having equal powers with respect to the two circles must be on the secant AB.

(3) $t = t'$ if they are parts of congruent triangles, or if any one of several sets of conditions is true. However, since both tangents have a secant from P, that fact suggests that $t^2 = (PA)\ (PB)$ and $t = t'$ if $(t')^2$ also $= (PA)\ (PB)$, which is clearly true.

(4) Similarly $m = m'$ if $r^2 - (CP')^2 = (r')^2 - (C'P')^2$ or if $m^2 = (AP')\ (P'B) = (m')^2$. Clearly this last statement is true because AB is a common chord for both circles.

(5) Then any point P such that $t = t'$ will lie on AB or AB extended if the contradictory assumption can be shown to be false, or if the conditions in (3) and (4) above can be shown to be necessary as well as sufficient.

PROOF A:

(1) Assume P to be a point on the secant outside the circles and draw tangents PT' and PT.

(2) $(PT')^2 = (PA)\ (PB) = (PT)^2$ Why?

(3) Therefore $PT = PT'$.

(4) Therefore every external point of AB has the same power with respect to both circles.

(5) For A and B the powers are both zero.

(6) For points between A and B, $(\frac{1}{2}m)^2 = (P'A)\ (P'B) = (\frac{1}{2}m')^2$ since if two chords intersect the product of the segments of one equals the product of the segments of the other.

(7) Therefore $(\frac{1}{2}m)^2 = (\frac{1}{2}m')^2$ and every point between A and B has the same power with respect to both circles.

(8) Therefore all points on secant AB are on the locus.

PROOF B:

(1) Using the indirect method assume that some point P' not on AB or AB extended has equal power with respect to both circles.

(2) Draw the tangents or minimum chords.

(3) Then $t^2 = (P'A')\ (P'B') = (t')^2$ for these tangents or half-chords.

(4) Then these two circles would have a second common secant $P'A'B'$, which is impossible.

(5) Therefore all points of the locus are on the secant AB.

2. EXERCISES FOLLOWING THEOREM 38

231. The radical center of three intersecting circles is the intersection of their common secants. First take the simple case where the centers are not collinear and where two circles do not intersect but the third intersects both the others. Then vary the intersections and relative positions and see what happens to the radical center. What if the centers are collinear? What if the circles are tangent? Concentric?
232. Find the radical center of three non-intersecting circles.
233. The radical axis of two non-intersecting, non-concentric circles is a straight line perpendicular to their line of centers.

    Note: Show that the perpendicularity of these lines is unaffected as the distance between the centers increases up to $r + r'$, and therefore is constant regardless of the distance between centers.
234. If two lines AB and CD meet at P so that $(AP)\ (BP) = (CP)\ (DP)$ then A, B, C, and D are concyclic (lie on the same circle).
235. What is the radical axis of two circles tangent externally; tangent internally; one within the other, yet not concentric; two concentric circles? Apply the principle of Continuity to this situation.

3. PROPOSITION 39 (A CONSTRUCTION)

There are several important theorems relative to coaxal circles, conjugate coaxal circles and their orthogonal relationships. Only two will be given here, one, a construction and one, a theorem.

DEFINITIONS: *Orthogonal circles* are circles which intersect so that the tangents to the two circles at the points of intersection are perpendicular.

Coaxal circles are circles so arranged that any pair of circles has the same radical axis. For example, all circles with a common secant form a coaxal system whose radical axis is that secant. In other words $(PC)^2 - r^2 = p = k$ for each circle. What would be true of a system of circles all tangent to a line at the same point?

PROBLEM: Construct a system of non-intersecting coaxal circles.

CONSTRUCTION:

(1). Since the line of centers is perpendicular to the radical axis their point of intersection, O, provides a value of p which must be the same for each pair of circles.

(2) Draw two perpendicular lines.

(3) Take a series of values for radii of circles such as $a, b, c, d, \ldots$ to O

(4) Take some constant value for $p = k^2$.

(5) Compute OA, OB, OC, etc. from

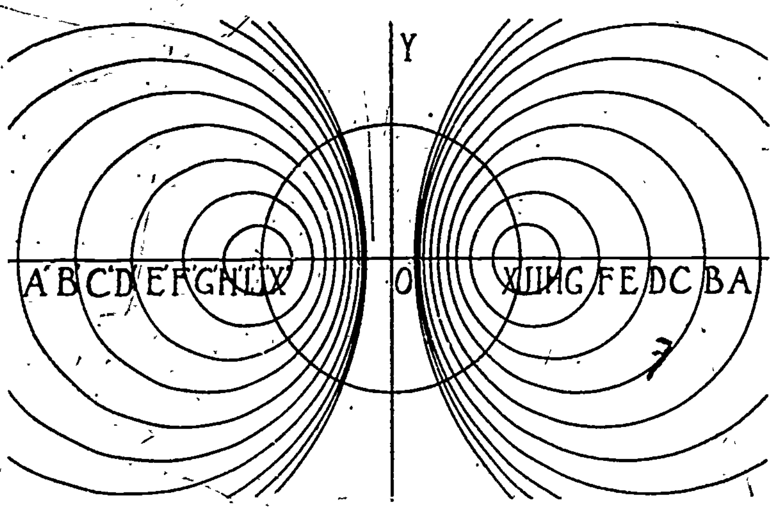
$$p = k^2 = (OA)^2 - a^2 = (OB)^2 - b^2 = (OC)^2 - c^2 = \ldots.$$

or

$$(OA)^2 = k^2 + a^2,\ (OB)^2 = k^2 + b^2, \text{and } (OC)^2 = k^2 + c^2$$

(6) OA, OB, OC are the distances from O to the centers of each of the coaxal circles whose radii are a, b, c, respectively.

(7) What would happen if $r = O$? If $r = k$? If $r = 2k$?



Proof:

(1) Each pair of circles has the same radical axis by construction and therefore by definition they form a coaxal system.

(2) When $r = o$, the circle is a point or sometimes called a limiting point, and $OX = k$ or in other words the tangent coincides with the line of centers.

4. THEOREM 40

A circle whose diameter is the line segment joining the limiting points of a coaxal system of non-intersecting circles is orthogonal to each circle of the system.*

HYPOTHESIS: To be stated by the student.
CONCLUSION: To be stated by the student.
ANALYSIS:

(1) What is necessary in order to prove circles orthogonal?
(2) Is a tangent perpendicular to a radius drawn to the point of contact?
(3) Referring to the figure of theorem 39, how long is each tangent from O to the circles of the coaxal system? What is the radius of the circle passing through the limiting points? Will the circle with radius k then go through the points of tangency?

PROOF: Left to the student.

* (Use the figure of proposition 39 from which the theorem really is suggested.)

5. EXERCISES FOLLOWING THEOREM 40

241. Construct a system of coaxal circles with a common secant for a radical axis.
242. The same with a common tangent for a radical axis.
243. A circle orthogonal to two fixed circles is orthogonal to each circle coaxal with them.

   Note: Why is its center on their radical axis? Why is its radius equal to their tangents from the center?
244. The circles orthogonal to two circles constitute a coaxal system.
245. Two given circles determine two coaxal systems, one is composed of all circles coaxal with them, the other, of all circles orthogonal to them.
246. Show how to construct a circle orthogonal to another circle. To each of two given circles. To each of three given circles.

V. SUMMARY STATEMENT OF PURPOSE AND CONTENT OF CHAPTER VI

The objective of this chapter has been to extend the work of Chapter IV and to provide a limited experience with a few of the theorems of modern geometry which use new concepts and relationships. This was done so that the student might encounter learning and teaching difficulties as a college student comparable to those of high-school geometry for a high-school student. No attempt has been made to make this exhaustive from the standpoint of geometry, although the theorems have been selected with care.

For further work of this nature special reference is given to the following books in order of their probable usefulness.

(1) Johnson, R. A. *Modern Geometry*. Houghton, Mifflin Co., 1929
(2) Durell, C. V. *Modern Geometry*. Macmillan and Co., 1926
(3) Altshiller-Court, N. *College Geometry*. Johnson Publishing Co. Richmond, Virginia, 1925
(4) Godfrey, C. and Siddons, A. W. *Modern Geometry*. Cambridge University Press, 1923
(5) Peterson, Julius. *Methods and Theories of Geometrical Constructions*. G. E. Stechert Co., New York, 1923
(6) Forder, H. G. *Foundations of Euclidean Geometry*. Cambridge University Press, 1927

CHAPTER VII

SUMMARY AND CONCLUSIONS

It has been the purpose of this study to present a detailed outline of the subject-matter of high-school demonstrative geometry in such a way as to provide professional training simultaneously with a mastery of mathematical content. This has been done through presenting the subject-matter of geometry by means of the technique of teaching which this study recommends as peculiarly fitted for geometry.

In the solution of this problem many other problems have presented themselves. If the function of geometry is merely to provide information about space relationships and a little practice in mechanical drawing, then its technique of teaching would be determined. If, on the other hand, geometry is conceded to be a course in the science of reasoning, then the methods of teaching it would be modified to achieve this aim. Because of the simple and concrete nature of geometry concepts, because of the clear-cut relation between its conclusions and the premises upon which these conclusions are based, and because of the unique history of the subject as a science of reasoning, it has been concluded for this study that the function of demonstrative geometry in the high school should be only secondarily informational and training in making constructions, and primarily a course for providing experience with the nature, method, and power of deductive reasoning. In Chapter II, Section II, this function of geometry has been presented and defended.

If geometry is to be taught as a science of reasoning in which the place of definitions, postulates, premises, and the method of proof are to have a conspicuous part, then this function will largely determine the philosophy and technique of teaching. On this basis the interrelationship between inductive and deductive thinking should be kept constantly prominent, and the analytic method of discovering the proof for a proposition as well as the precise synthetic proof should receive major consideration. A technique of teaching, called heuristic teaching, which gives proper recognition to these four complementary methods of reasoning has been advocated in this study as the ideal for geometry teaching. In Chapter III this heuristic technique of teaching has been presented.

Since in any course for training geometry teachers the time is greatly limited, it was deemed impossible to outline in an appropriate professional setting the entire two hundred or more theorems of modern plane and solid geometry. Some selection had to be made, and was finally made by means of a unique plan which gave due recognition to sequence of theorems and to the major plan of geometry as a science of reasoning. The theorems outlined by the National Committee and supplemented by the College Entrance Examination Board were all proved and the use made of theorems in the proof of later theorems noted. On the basis of this later usage criterion the "Essential Theorems of Plane and Solid Geometry" were selected. This list is comprised of ten constructions, twenty theorems of plane geometry, and twelve theorems of solid geometry. Such a minimum list of important theorems was considered to be sufficient to serve the dual purpose of providing a re-view from a professional point of view of the essential content of high-school geometry, and also of providing opportunity for applying and establishing the ideal technique of teaching geometry. In Chapter IV the "Essential Theorems" of plane and solid geometry have been outlined, and in Chapter I the method of selection has been described.

Since merely high-school geometry materials seemed inadequate to provide as complete a knowledge of geometry and its methods as a geometry teacher should have, it was deemed necessary to provide additional and more difficult problems. A more detailed presentation of the analytic method with many difficult problems for its application, and also a more detailed presentation of the powerful technique of indirect proof with an abundance of carefully selected problems for practice in its use have both been presented in Chapter V. Along with these major ideas have also been presented some interesting facts concerning continuity, converses, incommensurables, and the structure of geometry.

Finally, as a climax for this professionalized-subject-matter project, a few important theorems from modern college geometry, involving some new concepts and relationships, have been introduced and proved by means of the heuristic technique. This material in its professional setting has been placed in Chapter VI, the final chapter.

In Chapter II, besides the discussion of the function of geometry, there has been included a brief history of geometry as a science and as a school subject, and also a presentation of "Some Settled and Some Unsettled Difficulties". Here has been included such topics as the foundations of geometry, superposition, the postulation of the congruence theorems, hypothetical constructions, and sequence. The outline of theorems presented in Chapter IV is based on the postulation of the congruence theorems, yet such postulation is not necessary. If these three theorems are proved in the traditional way they must be added to the list of "Essential Theorems" given in Chapter IV.

The statement of the problem to be solved and the objectives of the study have been presented in Chapter I. These objectives determine the content of the other chapters. In general the problem has been to outline the training, both in subject-matter and method, for a high-school teacher of plane and solid demonstrative geometry. This outline consists of a pattern of teaching established by applying that pattern to the materials of a minimum list of theorems, to a liberal supply of difficult original exercises, and to a few difficult and new theorems of college geometry. Some of the material is necessarily subjective in nature, yet the sequential content has been objectively determined, and even the original exercises must of necessity be objectively placed in the correct sequence.

As training for high-school teachers the material outlined in Chapters II, III, and IV should be an irreducible minimum. Chapters V and VI add much valuable experience which should give a student a reasonably excellent professional preparation for teaching geometry in the high school.

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