DOCUMENT RESUME

ED 106 146 SE 019 168

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| TITLE | A Facilitation Model for the Transfer of Basic Mathematical Concepts. |
| PUB DATE | Mar 75 |
| NOTE | 14p.; Paper presented at the annual meeting of the American Educational Research Association (Washington, D.C., March 30 - April 3, 1975) |
| EDRS PRICE | MF-$0.76 HC-$1.58 PLUS POSTAGE |
| DESCRIPTORS | Curriculum; Elementary Education; *Elementary School Mathematics; *Learning; Mathematical Concepts; Measurement; *Number Concepts; *Research; *Transfer of Training |
| IDENTIFIERS | Research Reports |


ABSTRACT

In this paper the author proposes a learning model for basic mathematics based on measurement concepts. By analyzing mathematical concepts and citing related psychological and educational research, he develops and justifies this model. The basic premise on which the model is based is the notion that measurement provides an approach to number concepts which is more appropriate to young learners than the traditional counting approach. This premise is supported by the author's arguments that children do not readily grasp the concept of one-ness, and that ordinality is the central concept of arithmetic. The concepts of zero and continuity, assignment of numbers to related dimensions, accuracy in determining a quantity, and the arithmetic operations are developed. In discussing the problem of transfer and its facilitation through this model, the author refers to research on memory and learning set. (SD)

# A FACILITATION MODEL FOR THE TRANSFER OF BASIC MATHEMATICAL CONCEPTS*

302

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A learning model appropriate for basic mathematics must meet two requirements: (a) It must be a faithful and complete representation of the mathematical theory which it is meant to expose, and (b) the model must be consonant with the human disposition to learn, that is, be relatively more facilitating than other learning models. These two aspects will be examined separately here. If the model which this paper outlines can be elaborated in all of the necessary teaching-learning particulars, it should provide a substantially different means by which to introduce mathematics to children, a means which is meant to equip the learner at an early time with a set of quantitative concepts he would not otherwise have. The model is based upon teaching and learning activities that require the child to assign numbers to continuous variables, variables that are measured, and avoids entirely counting operations for the assignment of numbers to objects. The model is intended to assist the integration of basic mathematics concepts at an early point in the child's learning. The foundational repertoire will reflect a more complete understanding of quantitative concepts than has been true.

In the Tenth Yearbook of the National Council of Teachers of Mathematics, Brownell (1935) outlined the theories which, in various degrees, underlay the teaching of arithmetic at that time. There were three such theories, a drill theory, an incidental learning theory, and a meaning theory. According to Brownell, the complexity of arithmetic learning makes a meaning theory necessary. The shift in the schools to modern mathematics has been a response to a belief in meaning theory. Brownell argued the case this way:

> Arithmetic, when viewed as a system of quantitative thinking, is probably the most complicated subject children face in the elementary school. Number is hard to understand because it is abstract. No special 'arithmetic instinct' fits the child directly to learn arithmetic. Neither does nature provide the child with tangible evidence of number which he can apprehend immediately and thus come

*A working paper.

> easily to know through sense perception. There is no concrete quality of 'fiveness' in five dogs which may be seen, heard, and handled. Color, barking, weight, and shape may be grasped through the senses, but the 'fiveness' is not thus open to immediate observation. Neither is there any 'fiveness' in . . . or 'five' or in '5.' In each case the 'fiveness' is the creation of the observer; it is a concept or an idea which the observer imposes upon the objective data. . . . One way of putting 'fiveness,' 'sevenness,' 'tenness,' etc., into objective representations of number is to enumerate. The ability to count objects, the school does develop, but it does little more than this by way of providing children with other and more advanced ways of thinking of concrete numbers. Too commonly, instruction in counting is immediately followed by drill on the addition and subtraction combinations.
>
> This approach to primary number almost totally neglects the element of meaning and the complexity of the first stages in arithmetical learning. It even disregards the evidence provided by children themselves that they do not understand what they are learning and that they are in trouble. When children know a combination one day and do not know it the next, there is something wrong in the learning. (pp. 20-21)

Certainly, advances in modern mathematics have changed the picture since 1935, but a new difficulty is apparent. Experiences in manipulating quantities that are now given to primary school children are not as completely quantitative as one might at first think. "Fiveness" may not be any more apparent to the child of 1975 than it was in 1935, in spite of the important movement toward meaning theory.

## The Beginning Learner

The child entering school will bring with him certain informal information that is prequantitative as, for instance, some knowledge of size comparisons, i.e., large, larger, superlatives, i.e., largest, and certain number uses like calling the natural numbers, 1, 2, 3, 4, 5, etc., and usually some ability to enumerate objects, whereby the numbers are vocalized in order to give a cardinal value to the collection of objects counted. Schools generally emphasize those number tasks that will enlarge the child's existing repertoire, namely, more practice in size comparisons, larger, smaller, that are based upon visual judgments and practice in counting.

## The Transfer Model for Learning Introductory Mathematics

The neophyte needs most of all a repertoire of concrete referents stored in his memory. Organization of these referents is dependent upon how experience is organized by those who teach the child and whether the mathematical content of the experience is both memorable and a true and complete representation of the basic mathematical concepts. Numerous observers of mathematics learning have noted a deficit in concrete experience (Lamon, 1972). We will call these early learnings through tangible experience "the transfer base" for it is out of these organized manipulative experiences that transfer can be anticipated. Transfer in this instance represents the integration and consolidation of concrete activities into an abstract organization of mathematical concepts.

A fact of human learning is that highly organized teaching activity does not necessarily translate into highly organized learning. The information must be in an assimilable form--a focus of this paper, and the ideas to be learned need repeating with variations in their form so as to provide a context for the storage of new information.

The form of learning activities required by a transfer model must now be described and in the description, I will include an analysis of a transfer model in terms of general numbers theory. After that, I provide a learning theory analysis.

## Measuring Activities Vice the Counting of Objects

In the transfer model, practice with numbers will be limited to the taking of measurements. Enumeration of objects will not be practiced. Measurement of linear distance is a propitious form of measurement, for it can be performed by small children to the end that the measured dimension will represent the number concepts we wish the child to learn. Such measurement should be practiced by requiring that the learner take in hand some convenient object to serve as a unit of measure. A blackboard eraser is such an object. The learner should then be instructed on how to use the eraser to measure length or width of several objects. If the first measured object is a table length, the learner will align the eraser at the edge of the table and marking the forward edge of the eraser, transport the eraser forward one unit distance or eraser length. He will be shown how to alternately mark and transport the eraser forward until the table has been fully traversed. The number series will be called by the child as he marks off the unit distances. Though the procedure is quite simple, the symbolic content which can be drawn from such a measuring activity is quite profound. Some of the initial content follows, which should be the focus of attention.

## Unit of Measure--The Concept "1"

The concept "1" is not readily grasped and it is quite possible for a child to generate a misleading concept that will remain with him for sometime. The mathematical meaning of oneness is abstract and cannot be represented appropriately by any tangible object. This meaning of one is not taken from the natural numbers, which do make reference to particular objects and which construe a number as the object itself. The measurement of a continuous quantity can lead to a different view of number. In measuring distance, one is represented, again, each time a quantity is incremented by the unit:

$$(((1) + 1) + 1) + 1)$$

According to Gal'perin and Georgiev (1969), measuring is the best activity by which to convey to the child a correct meaning of one. These observers have acquired evidence to show that failure of a child to conserve quantities is due, in part, to learning "oneness" by counting objects having idiosyncratic qualities, and failure to properly associate a unit to the quantity to be measured.

When measuring distance, as a way to introduce the number series, the object representing a unit should be changed frequently and the linear unit spaces, along the distance marked off, should be made the focus of explanation and practice until the child comes to realize that the unit is not the measuring instrument. Such a practice will represent a radical departure from the use of a number series formed by counting objects as the means by which to introduce quantitative ideas to children. In time, "one" will come to mean an arbitrary distance marked off in a continuous series.

## Ordinality--A Basic Concept Intrinsic to Measuring Distance

Brainerd (1973) and Hempel (1945) have shown that ordinality, as represented in the succession of the natural number series is the central concept of the arithmetic of the natural number series. Brainerd has further shown that children acquire competence in this concept before the easiest addition and subtraction skills and before they understand cardinality as a concept.

In measuring a linear distance where the measuring unit is transported forward, each ordinal value retains its relative position, so that when the learner has traversed the progressive series, it is easy to require him to relocate the fourth ordinal or the seventh or the successor of some other value. He can, therefore, learn, quite explicitly, the meaning of ordinal position in a continuous series. Note that the concrete activity of measuring differs from counting in this respect. In enumerating, there is no natural order. One can count the objects in any order he wishes. Then, too, the distinction between ordinal and cardinal concepts can be made explicitly clear in the operations of measuring, for they each have their concrete reference; namely, the cardinal end of the series and the respective ordinal positions enroute to the cardinal terminus.

In similar fashion, transitive relations can be easily demonstrated when C > B and B > A. It is possible to show C > A in an obvious way in measuring distance where one has arranged duplicate copies of the unit of measure in an end-to-end series. One will simply identify points A, B, and C on the row of measured units and show their quantity relations visibly arrayed before the learner with the overlapping distances represented in the linear segments. Like the concept "unit of measure," the concepts of "asymetrical order" or a "series of successors" and the transitive relations made possible by succession of the unit are very important in theories of number. The measurement model is more completely consistent with the concept of succession than is true of object enumeration.

## Zero and Continuity as Related Concepts in the Measurement Model

In the natural numbers series, as represented in the counting of objects, zero is a number, but not the successor of any number. In continuous quantities, the relation of zero to other numbers is similar, but there is perhaps a more ready grasp of its meaning. Regardless of the unit size, minuscule or great, zero appears in contrast to some quantity or position on the scale; in contrast to some of any unit size selected. Zero, viewed in this way, probably has no special merit in mathematical theory, but because zero is, in a sense, more readily present as the point of origin in the measured continuum, it likely has special pedagogical merit. The learner can see zero in relation to the infinitely divisible continuum. Rational numbers, fractionality, ratio, and proportion are each appropriately explainable within the measurement of overlapping or parallel linear distances, where, in each case, zero holds a comprehensible position.

## Accuracy in Determining a Quantity

Discrete quantities, as counted, can be reported with complete accuracy; that is, one can obtain a ready consensus for the cardinal value obtained by counting objects. Not so with the measurement of continuous quantities, for we almost always measure with inaccuracy and can never know if a completely accurate measure has been taken, regardless of the quality of the measuring instrument. Neophyte learners are prone to cavalier use of a measuring device and the process of teaching them that in measuring length, the unit distances must coincide at coterminal points is a kind of fundamental concrete exercise in the meaning of accuracy. Two advantages of measurement exercises are immediately apparent in a discussion of accuracy. The child will learn something of the need to preserve the integrity of the unit of measure--that the unit is always constant. He may also acquire a habit of compulsive carefulness in the practice of accurate measurement. A trait peculiarly valuable for one learning mathematics.

## Unit-Quantity Relations and the Assignment of Numbers to Measured Dimensions

There is possibly a reason to view conservation ability as a degree of sophistication in quantity relations. Measurement, in its various forms, requires first that one select an appropriate unit by which to take the quantity. When a distance is to be measured, the learner must choose the unit of length that will yield a happy compromise between accuracy of the measure and time saving efficiency--not too short and not too long. The decision is one which requires a rudimentary choice of ratio or unit-quantity relation. Knowledge about the determination of quantities within various dimensions is apparently gained rather slowly as demonstrated in the vast number of conservation studies and focused in the work of Piaget.

Naturally, small children tend to regard quantities in a gross way, in terms of perceptual comparisons of larger than, smaller than, largest, etc. The disposition of the child to look and declare A to be larger than B or to fail to conserve a quantity under transformation was demonstrated by Piaget to be typical of small children. He has said that they fail to decenter their perceptions. The child's attention is drawn only to visual transformations he sees in, for instance, a pile of rice that is spread out and made to cover more area or a dimensional change created in some other quantity. According to Piaget, measurement is a rote exercise until the child has developed conservation ability and, therefore, measurement should be delayed until that time (Piaget, 1960; Sawada & Nelson, 1967). Probably, what the developing child eventually does is what Piaget's term, decenter, implies. He comes to acknowledge compensating changes at the same time in two or more dimensions. Or simply speaking, the child sees more than one change at a time, one dimension adjusting for the other, leaving the quantity unchanged.

But what does it mean to say that a child does or does not decenter his perception? Success in conserving quantity might be given a meaning quite different from the one Piaget has given it. Namely, that there are particular learnings in assessing quantity that can come about by assigning numbers to quantities, so that the learner will acquire a grasp of unit-quantity relations. The child will then no longer simply look to form a quantity judgment. He will measure until he has formed the generalization that a transformation does not change the quantity. Mind you, this is probably the more sophisticated way of doing it. Small children do not typically experience formal measurement training, and yet they eventually come, in some unknown way, to conserve.

Mathematics books in the elementary school generally treat measurement as an ancillary topic or a skill learning to be acquired periodically and largely independently from other topics. In the measurement model for teaching and learning mathematics, the apprehension of unit-quantity relations is a superordinate concept whether for taking dry and liquid measure, for taking two dimensional space and linear distance, or for measurement within other dimensions. That measurement activities, in their various forms, should be the concrete representation of the most fundamental mathematics ideas is in agreement with the view that complex subject matter needs a common and continuous organizer, one to which the learner will be referred time and again.

## Arithmetic Operations

The arithmetic operations of addition, subtraction, multiplication and division can each be carried out by the manipulation of line segments and other forms of measurement. There may well be advantages for mastery of these basic operations in tangible space, as in adding line segments and subtracting them, or for doing similar operations with dry and liquid measure before the learner is required to do paper-pencil operations of arithmetic. If the abstract paper computations are delayed until the child has numerous concrete referents for the symbolic operations, it is conceivable that a more comprehensible form of the knowledge may emerge. Paper-pencil operations can be taught, conceivably, so as to call up, in the child's memory, images of a tangible dimension that serves as a check on the reasonableness of paper-pencil outcomes. If the transfer model is to afford unique advantages, these measurement images must, of course, be more adequate representatives of number theory than obtain with object counting.

## The Transfer of Basic Mathematical Concepts

The concept of transfer of training is a relatively old one in pedagogy and in the psychology of learning but it is an awkward concept, one which may eventually be represented by a large number of laws of learning. The progress of knowledge about transfer has been slow and its processes seem immensely complex. Generalizations that represent transfer in today's literature are crude statements. Central among these generalizations are those which emphasize similarity, similarity of stimulus conditions from an original to a transfer task and similarity of response conditions between those two tasks. Ellis (1965) has provided a summary of transfer literature and has listed what he called principles of transfer. Among these principles is one which anticipates that learning of one task will benefit or transfer to another task when the training conditions are highly similar. Among the remaining principles are these: (a) learning to learn--cumulative practice from task to related task eventuates in an increased capacity of the learner to learn new tasks; (b) emphasis on early tasks in a series--more transfer will occur when practice and emphasis are given to the early tasks in a series of connected tasks; (c) learner insight--insight or quick solutions to problems comes about only through extensive practice in a limited range of related tasks; (d) practice with a variety of stimuli--variation in the training experience facilitates transfer.

### Aspects of Memory and the Transfer Base

The evidences, which Ellis has brought together, show the improtance to transfer of establishing a large working repertoire of organized experience in the learner. Early learning should center on closely related concrete tasks that are memorable in the sense that the mathematical content is salient and can be easily organized to represent mathematical concepts. The first part of this paper outlined attributes of a kind of measuring that embodies intrinsic mathematical meanings, like the important unit-quantity relation or the continuous nature of measured variables. Memory will benefit from the assignment of numbers accompanied by tangible cues but, paradoxically, the basis for one of the first and most important abstractions will have been laid. The learner must come to know that "one" is not really the unit by which he is measuring at that instant, but the idea of oneness as taken from all of the measures and that the quantity can be given by any of the unit sizes. Obviously, the learner must repeat the measuring operations many times before he will gain a correct understanding of number. The activities must be organized to demand interpretations of oneness, continuity, unit-quantity relations, and perhaps through these interpretations the concept of a stable quantity. Note that while the conditions of practice vary with the measuring object, the dimension measured, and the problem to be solved (i.e., the sum of two line segments to be taken), the manipulanda and the circumstances remain much the same. Overlearning is, therefore, likely. One who has formed a discrimination and is required, subsequently, to redemonstrate many times his knowledge of the discrimination is said to have overlearned (Travers, 1972).

Learning Set

There is a wide consensus that overlearning contributes to the formation of a transfer base. Consider the problem of explaining learning set or "learning to learn" in the now famous Harlow (1949) experiment wherein monkeys made two choice discrimination responses. Harlow's monkeys were presented a problem: In which of two food wells can raisins be found? The principle finding was that when many trials (50) were given on each of the first 32 problems, the animals began to learn how to solve the problems, so that as problems were combined into blocks, the monkeys came to solve each successive block more rapidly than the previous block and only six trials were then given on each problem. The monkey's second encounter with a particular problem revealed what had been learned from his immediately prior choice. There was an orderly and systematic increase in the proportion of these second encounters for which the monkey made a correct response. The progressive improvement was described by Harlow as the formation of a learning set. The monkey had learned how to attend to the relevant stimulus dimension. But what were the conditions that permitted the learning phenomenon to occur? They were similarity of circumstances from problem to problem and massive experience with the discriminanda. Probably, the concept "overlearning" describes or explains the result best. Unfortunately, a microanalysis was not possible to show which small stimulus changes were accompanied by learning increments.

## A Basic Learning Set: Whatever the Dimension and the Unit, Give the Quantity

Quite naturally, children do not come with ready-made conceptions of quantity. Curiously, failure of small children to conserve quantity is sometimes met by unusual effort on the part of investigators to show that the child already possesses the ability but for some reason cannot exhibit what he knows; i.e., that the child does not know the language or understand the question. A different view is that early experience is qualitative and that the child obtains from time to time bits of informal information that is only prequantitative. Preschool and early-school experience is marked by references to comparatives and superlatives that imply quantities, but only gradually is the child given quantitative information and then usually limited to enumerated quantities from objects counted.

A direct way to inform a child of the meaning of quantity is to give him a large amount of experience in determining quantities. Practice in taking measurements, like determining linear distance, weight and volume, will, of course, cumulate to the end that the learner has a general set of rules and the appertinent mathematical concepts required for assigning numbers to concrete quantities.

## The Form of Memory and Its Restructuring as Organized Knowledge

Well educated adults show a remarkable capacity to utilize directly abstract information without first forming concrete referents. Children are particularly dependent upon concrete formulations. These tangible experiences are built up slowly in a child's memory and called up by a teacher to provide reference points for an explanation she will give of a novel concept. From high dependence on concrete experiences organized for them, to practically no dependence, how do humans restructure information so as to become increasingly more competent? Paivio (1974) has discussed two theories of the organization of memory. In the theory he favors, knowledge is believed to be acquired as perceptual images of things or events together with verbal commentaries. The other view holds that stored information is more abstract and organized, like a computer program. Paivio has cited evidence to show that the brain accepts imaginal and verbal information differently, and further, that the dominant hemisphere stores verbal information and the other stores spatial and nonverbal information. Communication between the halves varies with the concreteness of the information. Concrete representations, as verbal labels, form direct associations to the imaginal side as a picture might elicit a term for its central figure. Abstract nouns make only indirect connections to the imaginal system. Paivio has cited evidence that the two connected systems can act independently, as when engaging visual imagery in one channel while the other is listening to speech. Interference occurs, on the other hand, when both a memory task and a perceptual task involve the same perceptual motor system. Memory comes to be restructured or reorganized as the particular visual experiences merge. Imaginal memory may evolve, gradually, to the eventual retention of only imaginal prototypes that best represent categories of experience. These are the most salient analogues by which to summarize related activities and the subconcepts which form the prototype image. According to this view, the prototype image is based upon many experiences with variations of the class it represents. Again, as in the evidences cited by Ellis, transfer and the integrative processes follow much practice and it is reasonable to assume that efficiency of the practice depends on the nature of the task, its manipulanda, and how representative these are of the universe of concepts the learner is meant to acquire. Then, too, faulty and incomplete notions are certain to arise from sporadic and disjointed practice. Practice must be relatively intensive and the tasks must be fairly homogeneous for mastery of the initial experiences and for the purpose of forming a transfer base. Overpractice, of an initial set of measurement tasks, will be required before a child can separate an appropriate abstract concept of unit and number from the real dimensions the child measures. Nonetheless, these learnings should develop more quickly by measuring than where objects are counted. In counting, the prototype image of number is likely to remain for a long time as apples, or bottles, or whatever is counted. The basis for that persisting image is that numbers are associated, directly, with the objects counted and it is not easy to think of a way to wean the learner from the image of a counted object.

In measuring, however, the child marks off the distances, assigning numbers as he measures. As the child learns to select an appropriate unit size, he has the basis for grasping the abstract meaning of number, a meaning independent of the unit size. Shaeffer (in press) has recently argued for a theory of memory organization, unlike the theory of images. Though the theory is unlike that of Paivio, it, too, provides a means of explaining the formation of higher order concepts from the particulars of experience called, "skill representatives" or "memory locations." When skills are thoroughly practiced, it is said that they are automated and can be used by the learner without requiring much of his attention. Before a skill is automated, it can be practiced only with the support of cues. When a child counts objects, he may use the common mnemonic device of sequential pointing of the finger as a means of keeping track of which objects were counted.

Integration of skills requires multiple skill representatives, that is, a great many repetitions at each level of the hierarchy of skills. Two or more skills can be integrated if they are activated together, that is, "kept in mind and used together." If they integrate, the new skill will have a new memory location and be capable of integration with still other skills. Shaeffer illustrated integration as a case where a child counts three objects and recognizes also by visual pattern the three objects and so forms the cardinality rule. Both counting and pattern recognition being well automated, the child is able to focus on the terminal value--three as counted and the visual pattern three.

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