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AUTHOR Godfrey, Leon D.
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ABSTRACT

In this report, the theoretical background and procedures for a study of concept learning are discussed. Several definitions of the term "concept" are analyzed, and the relations among concepts, chains of concepts, and hierarchies of concepts are explored. Conceptual learning is discussed from several points of view, and axiom systems for the learning process are described. The overall design of the study is outlined and the four treatments (rote reception, rote discovery, conceptual reception, and conceptual discovery) are described in detail. The results of this study are provided in Part 2. (SD)

Technical Report No. 307 (Part 1 of 2 Parts)

A STUDY OF THE ROTE-CONCEPTUAL AND RECEPTION-DISCOVERY DIMENSIONS OF LEARNING MATHEMATICAL CONCEPTS

Report from the Projection Conditions of School Learning and Instructional Strategies

By Leon D. Godfrey

Thomas A. Romberg and John G. Harvey
Principal Investigators

Wisconsin Research and Development
Center for Cognitive Learning
The University of Wisconsin
Madison, Wisconsin

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STATEMENT OF FOCUS

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ABSTRACT

The study reported in this thesis concerns the learning and use of mathematical concepts and the learning of relations described by a hierarchy of mathematical concepts.

The rote-conceptual and reception-discovery dimensions of learning were studied. The study was designed not only to allow study of the hypothesized main effects and interactions between these dimensions but also to determine the feasibility of studying the learning of relations between concepts.

Three types of relations that exist between mathematical concepts are designated which are used to define a hierarchy of mathematical concepts. The procedure used was to designate a hierarchy of mathematical concepts. Then, instructional units were prepared and taught to students in fifteen sections of college algebra by rote reception, rote discovery, conceptual reception, or conceptual discovery methods. The rote treatments allowed, but did not require, rote learning of single facts. In the conceptual treatments students were never given the same items more than once so that only conceptual learning was demonstrated. The procedure in the reception treatments was to give the S the correct definitions and examples of concepts; whereas, in the discovery treatements, the S had to discover the correct rule.

The results of the study indicated differences between rote and conceptual learning as well as between reception and discovery learning. No evidence of an interaction between the rote-conceptual and reception-discovery dimensions were found. An excellent fit for data from the conceptual reception treatments to theoretical values from Bower's model of paired-associate learning was found.

Conclusions drawn from the study are: (1) there are differences between rote and conceptual learning and between reception and discovery learning that can be studied using hierarchies of mathematical concepts as the content to be learned and by fitting observed data to different models of learning noting variation in parameters and fit, (2) rote learning does not hinder conceptual learning, and (3) if there is an interaction between rote-conceptual and rote-discovery learning, either in learning concepts or in learning relations between concepts, more refined methods are needed to analyze them.

Chapter I

THE PROBLEM

How can the psychological methods used in the study of concept learning and paired-associate learning be adapted to the study of mathematical concept acquisition and use? Will studies about the learning of mathematical concepts using these adapted methods produce results which can be applied to improve the teaching of mathematics? The research reported in this thesis sought answers to these questions.

This chapter includes a further discussion of this general problem, an overview of the study being reported, and description of the specific problem, of the experiment, and of the other chapters.

The General Problem

In general, the problem studied here is to gain information about the acquisition and use of mathematical concepts. This problem has two aspects: (1) to obtain facts in the form of experimental data gathered to answer theoretical questions and (2) to formulate theories about the acquisition and use of mathematical concepts. It is not necessary to carry out these activities in a specific order, but, unless each is done and related to the other, useful research will not have occurred; neither part is of value without the other. What is the value of a theory with invalid assumptions? What good is a set of data without a question to be answered?

An experiment to gather data that is designed without a theory in mind is of little value since the data cannot be processed in any reasonable way. A theory without factual underpinnings is a fantasy. Suppes (1972) paraphrased Hume by saying, "...general ideas about educational policy and practice contain little but sophistry and illusion, unless they can be defended by abstract reasoning from some other accepted general principles or be inferred in a definite manner from particular matters of fact (p. 1)." He continued by stating what he considers to be the task of research,

> Without proper evidence, alleged facts on which educational policy or practice is based can only be classed as fantasies. It is the task of research to convert the 'right' fantasies into facts and to show the others to be the unsubstantial fantasies they are (p. 2).

Marx (1970) said that facts were

> ...the basic raw materials from which theories are constructed. The primary scientific effort is at the empirical level and is expressed in terms of what is often called data language. Because the more or less direct sensory observations that are made at this level are likely to be agreed upon by all observers, they tend to become <u>facts</u>, or symbolic representations, usually verbal in nature, of such sensory observations (pp. 6-7).

It seems unnecessary to pursue this point further since it is unlikely that an advocate for the position of producing educational research without factual data could be found anyway. The second aspect of research, that of theoretical formulations, is somewhat different. In fact, in summarizing the discussion of a conference dealing with such questions, Pingry (1967) said that some conferees "...indicated that research efforts toward model building had such poor promise of success that priority efforts should be given to solving immediate practical problems at hand (p. 45)."

A strong case for theory and model construction has been made by various researchers. Suppes (1967), Scandura (1967), Forbes (1967), Heimer (1967), Romberg (1970), Long, Meltzer, and Hilton (1970) have all supported this position. Their arguments point out that theories are needed to provide research questions, to aid in prediction of optimum conditions for learning mathematics, to provide insight into the relation of the logical structure of mathematics to the psychological nature of learning mathematics, and in general to provide focus for the research process.

It is felt that these arguments have sufficient force to warrant serious attempts at theory and model building in the area of mathematical concept acquisition and use. To begin the task it is necessary to relate several psychological terms, each of which has been researched, but little has been done to form a cohesive theory combining them. The terms are <u>rote learning</u>, <u>meaningful learning</u>, <u>reception learning</u>, and <u>discovery learning</u>. The four terms can be separated into two dimensions of learning, rote-meaningful and reception-discovery. The rote-meaningful dimension describes how learned material articulates with a person's previous learning whereas the reception-discovery dimension describes how material is learned; whether the definition of a concept is given to the person or is discovered by him. Conceptual learning will be given a more precise definition later but it may be considered, for the time being, as meaningful learning of concepts.

Any theory that attempts to describe mathematical concept learning must be concerned with how the student relates material being learned to previously learned material since, if no connection is made by the student, learning is isolated and cannot be considered conceptual learning. A theory describing mathematical concept learning should take into account the way the concepts were learned (by being told or by discovering). Perhaps what is really the concern of such a theory is to relate the two dimensions; to explain the interaction between the two dimensions. This interaction presents some specific problems.

Since reception and discovery learning describe how material is presented to the learner it is necessary to study student behavior during instruction. It is not sufficient to simply measure if learning has or has not occurred at the end of instruction. Mathematical models of concept identification and paired-associate learning provide a method of comparing different types of learning as Ss progress from unlearned states to different levels of learning such as rote or conceptual.

These levels of learning are especially important in learning mathematics because of the logical structure of the subject. It was deemed important to consider the relations between concepts, not just single mathematical concepts, in conjunction with the psychological dimensions of learning. Thus it was necessary to determine these relations before proceeding to other considerations, but the relations were specified it was possible to use the learning theory models in studying interaction between the rote-conceptual and reception-discovery dimensions of learning mathematical concepts and hierarchies of mathematical concepts and in studying levels of learning in the rote-conceptual dimension. Stated in this form the general problem led to some specific considerations. These are discussed in detail in the second chapter and briefly outlined in the next section of this chapter.

Specific Problems

In proceeding to more specific questions terminology was a problem. It was necessary to relate various definitions of "concept" that appeared in the literature. This was done by giving a definition of "mathematical concept" and comparing other definitions to it. The next step was to describe what was meant by a "hierarchy of mathematical concepts" and show how these occurred in mathematics. After doing this it was possible to state two research questions:

1. Is there a significant interaction between the rote-conceptual and the reception-discovery dimensions of learning with respect to learning concepts that occur in a hierarchy of concepts?
2. Will mathematical models of paired-associate learning and concept identification provide information about learning mathematical concepts?

A full discussion of these questions is the subject of Chapter II. These questions led to the design and execution of an experiment which is described in detail in Chapters IV and V. The next section gives a brief description of the experiment.

The Experiment

The experiment was designed to gather data concerning the acquisition and use of mathematical concepts. The mathematical content of the experiment consisted of that portion of the college algebra course taught at the University of Wisconsin-Madison that deals with sequences and series. The subjects were students from fifteen class sections; all the students of five sections and students that volunteered from the remaining ten sections were included.

There were two parts to the experiment: (1) Part One was designed to provide data about Ss learning to recognize various sequences and series and (2) Part Two was designed to provide data about Ss' use of formulas and computations related to arithmetic and geometric sequences. In Part One students were randomly assigned to eight instructional groups; these groups were determined by the three dimensions: (1) pretest or no pretest, (2) definitions given or withheld, and (3) mastery of one segment of the instructional material required or not required before proceeding to the next segment. The instructional groups for Part Two were obtained by combining the instructional groups from Part One that were the same on the last two dimensions, that is by collapsing, on the pretest - no pretest dimension of Part One. The rationale for this design and further discussion of the experiment are given in Chapter IV. The other chapters are described next to give a perspective for what follows.

The Remaining Chapters

Chapter II gives a full discussion of the specific research problems. Chapter III cites research literature relevant to the research problems of Chapter II.

The questions of Chapter II led to the experiment described in Chapter IV. Chapter IV contains a complete description of the experimental design, instructional organization and materials, and other related matters. Chapter V describes the execution of the plans of Chapter IV.

Chapter VI presents the analysis of the data and relates it to the theoretical portion of the thesis.

Chapter VII contains a discussion of the implications of Chapter VI.

Chapter II

THE SPECIFIC PROBLEM

Consideration of the general problem discussed in Chapter I leads to questions that require research. The purpose of this chapter is to discuss the specific problem in detail. To do this it is necessary to give precise definitions to terms that will be used to formulate research hypotheses. These hypotheses are associated with a theory that is presented to account for the possible interaction of rote-conceptual and reception-discovery learning. Three mathematical models will be described that will be used to analyze different learning that may occur.

Concepts in Mathematics

It is most important to specify what kind of mathematical content is to be used in the research as there are a variety of things that students are called on to learn that are classified as mathematics. This section provides an explicit definition of "mathematical concept" and relates this definition to some of the other definitions that are presently being used.

Definition. A mathematical concept is the class of elements identified by a defined term in a mathematical discourse. Some terms representing mathematical concepts are "group," "vector space," and "topological space." Elements of a class that is a concept are called examples or positive instances of the concept and elements in the complement of the concept are said to be non-examples or negative instances of the concept.

The word "concept," in mathematics and elsewhere, has been defined in various ways by various researchers, some individuals have even chosen not to define the term at all (Fehr 1966, p. 225, Skemp 1971, p. 27).

Bourne (1966) gave the following: "As a working definition we may say that a concept exists whenever two or more distinguishable objects or events have been grouped or classified together and set apart from other objects on the basis of some common feature or property characteristic of each (p. 1)."

Bourne's definition of concept is closely related to the definition of mathematical concept. Bourne required that the class have at least two elements; for mathematical concept it is difficult to conceive of any defined term that represents a singleton set. For example, consider "the" group of order three. What is the cardinality of the class of groups with three elements? There is just one isomorphism class, but there are many ways that groups with three elements occur in mathematics (the group of units of a field with four elements and the subgroup of even permutations of the symmetric group on three letters are two examples). What is usually meant by "the group of order three" is either a particular group of order three or the isomorphism class of groups of order three. In any case both the definition given by Bourne and that given for a mathematical concept agree in that there must be a set or class of examples.

Bourne's definition also requires a common property for the examples of the concept; this is also true of a definition of a mathematical concept. A definition of a mathematical concept simply gives a set of properties or characteristics for the defined term. So it is seen that Bourne's definition of concept agrees rather closely with that given for a mathematical concept.

Klausmeier, et al (1970) said, "...the point is made that one attribute of concept is definability (p. 3)." Klausmeier (1971) said the following about formal concepts:

> A high-level formal concept is inferred when the individual with normal language development can accurately designate certain objects or events as belonging to the same set and others as not belonging to the set, can give the name of the concept, and can name its intrinsic or societally accepted defining attributes (p. 3).

This statement does not constitute a definition of a concept, however, Klausmeier saw a set of examples, a definition, and a name as each being involved with behavior indicating a concept was learned. The definition of mathematical concept agrees on all three points.

In order to select concepts for inclusion in tests of concept attainment Romberg, Steitz, and Frayer (1971) used the following criteria: "... (a) the concept had to refer to a category of objects or events that could be defined on the basis of intrinsic characteristics common to members of the category, and (b) the concept had to have a one- or two-word name (p. 3)." Criterion (a) agrees closely with the definition of mathematical concept and criterion (b) simply limits the class of concepts to be included.

Gagné (1970) made an important distinction between concrete concepts and defined concepts:

> Many concepts cannot be learned in the manner described. . ., that is, as concrete concepts. Instead, they must be learned <u>by definition</u> and, accordingly, may be called <u>defined concepts</u>. Sometimes they are called abstract, in order to distinguish them from the concrete variety. More aptly, they may be called <u>rational concepts</u>, because they do in fact relate two or more simpler concepts. The concept <u>diagonal</u> is a defined concept, not a concrete concept. The statement, 'A diagonal is a line connecting opposite corners of a quadrilateral' represents a relation (connect) between the two concepts 'line' and 'opposite corners of a quadrilateral' [pp. 189-190].

Thus, in this paper, mathematical concepts are what Gagné calls <u>defined concepts</u>.

Consider what Skemp (1971) has said about defining the word <u>concept</u>:

> One other consequence of this principle, that concepts of a higher order in a hierarchy than those which a person already has cannot be communicated by definition, can now be deduced. This is, that <u>concept</u> itself cannot be defined: for any particular concept must be an example of <u>this</u> concept, which is therefore a higher order than any other concept. We can however describe some of the characteristics of concepts, discuss how they function, and build up a general understanding of the idea by relating it to other ideas. This is adequate for our purpose, as indeed it has to be (p. 27).

This argument may appeal to some but it should be pointed out that the hypothesis "that concepts of a higher order in a hierarchy than those which a person already has cannot be communicated by definition" should be questioned. It seems that in fact mathematical concepts, at least at some levels, are communicated by definition. For example, in an introductory group theory course, one of the things one does is define "group" and have the students identify various examples and non-examples of this concept.

Much of what is known about mathematics consists of relational statements about mathematical concepts. Most mathematical concepts are built from other mathematical concepts. "Rational numbers" is a mathematical concept; it is also an example of the mathematical concept "field." "Field" is used in the definition of "vector space," and is a subclass of the collection of "rings." Thus subclasses of mathematical concepts may be mathematical concepts. The following paragraphs will formalize these ideas.

Three types of relations that may exist between two mathematical concepts are: (1) mathematical concept A <u>is an example</u> of mathematical concept B, (2) mathematical concept C <u>is a subclass</u> of mathematical concept D, and (3) mathematical concept E <u>is used in the definition</u> of concept F. In each case the first named mathematical concept will be called <u>subordinate</u> to the second and the second will be called <u>supra-ordinate</u> to the first. These ideas are illustrated next.

Consider the conceptual organization of some introductory topology courses. "Topological space" is defined using the mathematical concept of "open set" and several examples are presented. "Open set" and each example are subordinate mathematical concepts to "topological space." Later other mathematical concepts are defined, say, "compact" and "countably compact." Then a theorem is presented: If a topological space is compact then it is countably compact. This theorem states that the class of compact topological spaces is a subclass of countably compact topological spaces. So that the mathematical concept "compact" is a subconcept of "countably compact." Thus, it is seen that very often in mathematics implications represent relations between classes of mathematical concepts. These three relations sometimes provide structure for sets of mathematical concepts; the following two definitions formally determine one kind of structure.

A chain of mathematical concepts is a finite set of mathematical concepts C_0, C_1, ..., C_n such that for each i=0, ..., n-1, there is a subordinate-supra-ordinate relation between C_i and C_{i+1}. It is not necessary that C_i be subordinate to C_{i+1}, often C_{i+1} will be subordinate to C_i.

A hierarchy of mathematical concepts is defined to be a class Q of mathematical concepts that satisfies the following condition:

For any pair of concepts, C_1, C_2 in Q there exists at least one chain between C_1 and C_2.

A hierarchy of mathematical concepts is a rule as defined by Gagné in that a hierarchy of mathematical concepts includes both the concepts and the relations that exist between them. Scandura (1969) has argued that the rule should be used as the fundamental unit for research in complex human behavior. Gagné (1970) concurred in this when he said:

> The ability of human beings to respond to the enormous variety of situations in which they operate effectively, despite almost infinite variety in the stimulation they receive, makes it at once apparent that rules are probably the major organizing factor, and quite possibly the primary one, in intellectual

> functioning. The Ss → R connection, once proposed as the unit of mental organization, has now been virtually replaced by the rule in the theoretical formulations of most psychologists. Even those who still favor the connection as a fundamental entity of neural functioning are forced to concede that the preponderance of observed human behavior occuring in natural situations is rule-governed (pp. 190-191).

In mathematics rule-governed behavior is usually desired; methods of relating the logical and psychological are needed. A hierarchy of mathematical concepts is defined strictly in terms of the logical structure of mathematics. This definition serves as a way of determining logically related mathematical concepts and seems to provide a useful first step in the process of studying concept acquisition and use. However there is more to the problem than this. Suppes (1966) said, "My present view, based partly on our experience and partly on conjecture, is that the psychological stratification of mathematical concepts will seldom, if ever, do violence to the logical structure of these concepts...." Thus Suppes saw two distinct structures with respect to mathematical concepts; the logical and the psychological. It is not always an easy matter to determine when a particular sequence of concepts is logically or psychologically sound. Weaver and Suydam (1972) agreed when they said:

> Past and present emphases upon aspects of mathematical structure contribute to confronting children with material which has logical meaning. This is a necessary but insufficient condition for that material to have psychological meaning for a pupil. Meaningful mathematics instruction facilitates meaningful mathematical learning, and each of these demands that we look beyond the logical meaning inherent in mathematical content *per se* (p. 60).

The Cambridge Conference on School Mathematics (1963) made no attempt to give answers to questions concerning the teaching of their proposed curricula. They felt that questions of what can be taught, as well as how; can only be answered by experimentation (p.4). However, organization of subject matter was considered to be very important. They said "...the question of what is teachable and what is not depends largely upon the organization of that subject matter (pp. 2-3)." And, "We therefore believe that the composition of problem sequences is one of the largest and one of the most urgent tasks in curricular development (p. 28)." What is referred to as "organization of the subject matter" should be expected to be consistent with the logical structure of the subject.

What is evident is that many feel there is structure in mathematics which naturally induces a structure on (or relations among) mathematical concepts, and this in turn is important in instruction. Thus it becomes necessary to relate the formal definitions of mathematical concepts to psychological dimensions.

Conceptual Learning

An item (this may be a stimulus in an S-R experiment or an example of a concept) may be either unlearned or learned by a student. If it is learned, the correct response is known, and it is assumed that it is given. If an item is in an unlearned state the correct response may or may not be given; the fact that an error is made is assumed to be evidence that the item is unlearned, but a correct response does not prove that the item is learned since the correct response may have occurred as a result of guessing.

Each item at some point is unlearned, call this level U. In level U, correct responses occur only by chance. When learning occurs it may be for a single item or for a set of items. In either case, it is assumed there will be no more mistakes on the item or set of items. Learning of a single item will be called learning at level P. This would generally be considered rote learning in that the learning has not included any other items and the item is learned as an isolated fact.

Suppose a student learns a concept or rule. Then the responses to a set of items are learned (the set is assumed to have more than one element), this will be called level R. Evidence of learning at level R would be correct responses to items not previously seen by the subject, where these correct responses did not occur by chance. It is important to keep in mind that learning at level R involves learning relations between previously learned concepts. It also involves more than rote learning since correct responses to items not seen before are required. In the following section the levels R and P are discussed with respect to paired-associate learning and concept identification.

Paired-Associate Learning and Concept Identification

Norman (1972) described some "more or less standard experimental paradigms" for learning models as:

> ...(consisting) of a sequence of trials. On each trial the subject is presented with a stimulus configuration, makes a response, and an outcome ensues. There are at least two response alternatives, and there may be infinitely many....If an outcome raises, or at least does not lower, the probability of a response in the presence of a stimulus, it is said to reinforce the association between stimulus and response (p. 1).

Paired-associate (P-A) learning experiments fit this general outline. The subject is presented with a list of S-R pairs, usually one at a time. In the anticipation method the stimulus member is presented before the response member. The subject is to indicate the response when the stimulus appears. After the subject gives his response he is shown the correct response. A trial consists of presenting each stimulus member of the P-A list together with the subject's response and the indication of the correct response.

Concept identification tasks also fit the general framework described by Norman. The subject is to classify stimulus items as examples or non-examples of concepts, the concepts are usually not defined to the subject, the object being for the subject to find the classification rule. After the subject responds, he is told the correct response.

Concept identification tasks differ from the usual P-A tasks in that there is a rule that determines the correct response for concept identification tasks but, at least originally, P-A lists were constructed to require separate learning of each pair in the list. In concept identification tasks a stimulus is not presented more than once (thus if learning occurs it cannot be on level P) whereas in P-A learning, trials are repeated until each item in the list is learned (and due to the nature of the P-A list, learning is usually assumed to occur at level P). Thus P-A and concept identification learning are seen to vary in the P-R levels discussed in the preceding section. Learning at level R will be related to a "meaningful learning" in the next section. Then P-A and concept identification learning will be seen to correspond to rote and meaningful learning respectively.

Rote-Meaningful and Reception-Discovery Dimensions of Mathematical Learning

It is not assumed that the P-R dimension is the same as the rote-meaningful dimension but it appears that they are closely related and at least intersect at "rote" since both rote learning and learning on level P implies learning isolated facts. Also, learning on level R and meaningful learning share the property that each requires that relations between concepts be learned. Brownell (1935) said, "The 'meaning' theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes (p. 19)" and that "...arithmetic instruction according to the 'meaning' theory helps to make number sensible is by emphasizing relationships within the subject (p. 25, emphasis added)." Thus learning a hierarchy of mathematical concepts would be meaningful in that specific relations are part of the hierarchy.

Reception learning is learning that occurs when a student learns a concept by being told the definition. Reception learning is sometimes called expository learning. Discovery learning occurs when a student learns a concept by inferring the definition from known examples and non-examples of the concept and not by being told the definition. It is important to note that reception and discovery learning refer to how learning takes place and that learning at level R, meaningful learning, and rote learning refer to what is learned.

Weaver and Suydam (1972) in discussing research dealing with meaning in elementary school mathematics cited reception-discovery as one of the dimensions that interact with the rote-meaningful dimension to confound results. The interaction has been considered, at least in theoretical terms. Shulman's idea (below) will be developed more fully in the next section.

Ausubel (1961, reiterated in 1968, p. 24) in a defense of reception learning said:

> The distinction between rote and meaningful learning is frequently confused with the reception-discovery distinction discussed above. This confusion is partly responsible for the widespread but unwarranted belief that reception learning is invariably rote and that discovery learning is invariably meaningful. Actually, each distinction constitutes an entirely independent dimension of learning. Hence, both reception and discovery learning can each be rote or meaningful depending on the conditions under which learning occurs (p. 17).

Perhaps Shulman (1970) stated the situation most clearly.

> Thus, the reception-discovery dimension reflects what the learner is doing in the course of instruction - the cognitive processes in which he is engaged as he learns. The rote-meaningful dimension represents the degree to which what is learned articulates with the learner's prior knowledge and cognitive structure, with no reference to how he learns it (p. 38).

When Scott and Frayer (1970) reviewed research on learning by discovery they pointed out that

> No experiment was performed to measure differences among Ss who discovered a generalization, Ss who were given a generalization, and Ss who learned facts alone. Such a comparison might suggest how much of the effect was due to Ss discovering a generalization as opposed to Ss using a generalization (p. 11).

Considering the basic nature of the two dimensions to learning it is surprising that to date not one single piece of research has been found dealing with the interaction of these dimensions in mathematics education.

The Interaction Problem

A simple theory relating the P-R and the reception-discovery dimension of learning is suggested by Shulman's statement in the preceding paragraph. Learning a concept involves acquiring the ability to classify correctly examples and non-examples of the concept. In learning to do this by discovery the theory is that the student must have well in mind a number of examples and non-examples to test hypotheses on. If there is not enough prior knowledge in the form of known examples and non-examples the discovery of the concepts will not occur. Thus where there is insufficient prior knowledge, discovery learning will be aided by rote learning of specific facts for the student to use in the discovery process. In reception learning after the student has been told the correct definition, practice is needed to provide understanding. Here learning a rote application of the concept would have limited value but having a larger number of examples and non-examples to apply the definition to would aid the learning greatly.

The theory predicts that what a student knows is important to the selection of appropriate learning activities. If the student has a basis of facts to draw on then discovery learning can occur; and if the student has been told a definition of a concept, to apply that definition he will need examples and non-examples of it. The theory will be applied to the following situation.

Learning in the paired-associate paradigm will be considered to be rote discovery in that if a concept is learned it must be discovered. There is reception learning at the P level in that the correct responses are told to the student until they are learned, but the concepts are never defined for the student. In the concept identification paradigm learning is assumed to be conceptual discovery since if correct responses occur as a result of learning, concepts must be discovered. If the definitions are provided to the students in each of these cases the learning would change from rote discovery and conceptual discovery to rote reception and conceptual reception respectively. The theory predicts that adding the definitions will aid conceptual learning more than rote learning. Thus the theory led to the hypothesis that the interaction is in the direction stated. The experiment as described in Chapter Four was designed to test this hypothesis. The experiment was also designed to provide data that would allow a detailed study of learning in each of the four situations. Three different models were selected to aid the study. These will be described next.

<u>Bower's Model</u>. Bower (1961) gave the following axioms for paired-associate learning:

1. Each item may be represented by a single stimulus element which is sampled on every trial.

2. This element is in either of two conditioning states: C_1 (conditioned to the correct response) or C_0 (not conditioned).

3. On each reinforced trial, the probability of a transition from C_0 to C_1 is a constant, c, the probability of a transition from C_1 to C_1 is one.

4. If the element is in state C_1 then the probability of a correct response is one; if the element is in state C_0, then the probability of a correct response is 1/N, where N is the number of response alternatives.

5. The probability c is independent of the trial number and the outcomes of preceding trials (p. 258).

This model was selected because it is known to fit data from paired-associate learning experiments very well. The model was modified for data from the concept identification experiments by allowing some items (concepts) to be represented by more than one stimulus element. This modification was necessary because more than one example of some concepts were included in order to study transfer from learning one example to learning a set of examples.

<u>Model II</u>. (Observable states paired-associate model). Model II is quite similar to Bower's model with the exception that the states are observable. Levine and Burke (1972) discuss construction of observable states models. It is important to have observable states

since these states are the levels of learning being investigated. Model II and Model III provide a means of analyzing probability of learning a concept rather than single items. The same modifications were made on these models as were made on Bower's model for the concept identification experiments. The axioms are as follows:

Axiom 1. Each item may be represented by a single stimulus element which is sampled on every trial.

Axiom 2. A stimulus element is in one of three states: U (unlearned with an incorrect guess for the response), C (unlearned with a correct guess for the response), or L (learned, hence a correct response).

Axiom 3. On each trial, the probability of a transition from U to L is a constant c greater than 0. The probabilities of a transition from C to L, from L to U and from L to C are all 0. The probability of transition from U to C is a constant d greater than 0 and the probability of a transition from C to U is a constant e greater than 0.

Axiom 4. The subject is in state U with probability f on trial one and in state C with probability 1 - f.

The item referred to in Axiom 1 will be either a specific example or a concept depending on the experiment, in paired-associate experiments an item is a single stimulus, in concept identification experiments an item is a concept which will be represented by one (or more for the modified model) example of the concept.

Axiom 3 formalizes the assumptions that there is no forgetting once learning has occurred and that learning occurs as an outcome of the last error. Axiom 4 implies that whatever the response on the first trial, it occurred by guessing.

<u>Model III</u>. (Observable states concept learning model). Model III is an attempt to develop an observable states model from a model developed by Batchelder (1966, pp. 29-32). The axioms for the model are:

Axiom 1. Each item may be represented by a single stimulus element which is sampled on every trial.

Axiom 2. A stimulus element is in one of four states: U (unlearned with an incorrect guess), C (unlearned with a correct guess), P (learned but the concept to which stimulus belongs is not learned), and R (learned and the concept to which the concept belongs is also learned).

Axiom 3. On each trial the probability of transition:

from P to R is a (a constant greater than 0).
from U to R is b (a constant greater than 0).
from U to P is c (a constant greater than 0).
from U to C is d (a constant greater than 0).
from R to any other state is 0.
from P to C or U is 0.
from C to P or R is 0.

Axiom 4. The subject is in state U with probability f on trial one and in state C with probability 1 - f.

It should be observed that Axiom 3 simply assumes there is no forgetting once learning occurs and learning may occur only after an error with the exception that an item may move to R from P without an error on that item, however, all examples of a concept will move to R at the same time, which will be after the last error of any example of concept.

These models were expected to aid the study by showing differences while learning was occurring. It was not expected that these models would be useful in analyzing the usual posttest data. Research hypotheses related to the posttests and to the learning of relations between concepts were to be tested using standard analysis of variance procedures. These research hypotheses are:

> Hypothesis 1a. There is a significant difference between reception and discovery learning on recognition of relations between concepts.
>
> Hypothesis 1b. There is a significant difference between learning that includes rote learning and learning that is conceptual on recognition of relations between concepts.
>
> Hypothesis 1c. There is a significant interaction between reception-discovery learning and rote-conceptual learning on recognition of relations between concepts.
>
> Hypothesis 2a. There is a significant difference between reception learning and discovery learning on recognition of examples of concepts learned.
>
> Hypothesis 2b. There is a significant difference between learning that includes rote learning and learning that is conceptual on recognition of examples of concepts learned.
>
> Hypothesis 2c. There is a significant interaction between reception-discovery learning and rote-conceptual learning on recognition of examples of concepts learned.
>
> Hypothesis 3a. There is a significant difference between reception and discovery learning on the ability to do computations related to sequences.
>
> Hypothesis 3b. There is a significant difference between learning that includes rote learning and learning that is conceptual on the ability to do computations related to sequences.
>
> Hypothesis 3c. There is a significant interaction between reception-discovery and rote-conceptual learning on the ability to do computations related to sequences.

Summary

In this chapter *mathematical concept* was defined and related to other definitions of *concept*. The definition made it possible to state three relations that occur between mathematical concepts and define a *hierarchy of mathematical concepts* using the relations. Following this was a discussion of rote, conceptual, reception, and discovery learning in the framework of the experimental paradigm often associated with learning models and paired-associate learning. A theory relating the rote-conceptual and the reception-discovery dimensions of learning was described. Three learning models were given and the research hypotheses to be tested by standard analysis of variance techniques were stated.

This concludes Chapter II. Chapter III contains a discussion of relevant and related research on rote, concept, reception, discovery learning and some brief remarks about the three learning models.

Chapter III

RELEVANT AND RELATED RESEARCH

A summary of relevant and related research could be voluminous or contained in one sentence, depending on the criteria used. By the most stringent criterion, that the research be about the interaction of the rote-conceptual and reception-discovery dimensions of learning and teaching mathematics, the discussion is simply, "No relevant research can be found." On the other hand, if the criterion that all research about any aspect of the two dimensions of learning be discussed, hundreds of pages would be required.

Since neither of these criteria are practical nor desirable, some limits between these extremes must be set. A reasonably complete review of research studying the differences between learning mathematics by discovery and either reception or rote is included. This discovery research is related to the dimensions of learning being studied. The difficulty of forming conclusions based on the research reported emphasizes the hazards of over simplifying research in complex learning situations. Thus, it becomes clear that there is a need to find different approaches for studying learning that is as complex as that required for mathematics. Some guiding principles must be chosen, but great care must be exercised in how it is done. One reasonable approach seems to be to hypothesize some, perhaps minimal, cognitive structure, and use this to guide development of further research, Fletcher (1969) provided one minimal structure when he postulated a model for describing cognitive processes. The model detailed by Klausmeier (1971) of learning a formal concept is related to Fletcher's model. Together these two models provide a focus for concept learning and only research relevant to these models will be discussed in the section on concept learning. Bower's model for paired-associate learning requires some discussion which is the last consideration of the chapter.

Research on Rote, Meaningful, Reception, and Discovery Learning

Much of the research on discovery learning was based on the assumption that discovery learning is meaningful learning. The treatments that were defined as being the "discovery" or "meaningful" treatments often encouraged generalization whereas in the "rote" treatments generalizations were sometimes actively discouraged. Thus the terms "discovery" and "generalization" were used to describe treatments thought of as meaningful.

McConnell (1934) studied second graders who were taught 100 addition and 100 subtraction facts during a seven month period. The treatments were: (1) Rote - students were told to memorize the facts and (2) Discovery - students learned through "self-initiated discovery (p. 8)." The rote method produced higher mean scores for speed tests of retention, but the discovery method produced higher scores on transfer tests. It appears that McConnell promoted meaningful learning through the discovery of generalizations and that the resulting learning promoted transfer.

Thiele (1938) used a design and methodology similar to that employed by McConnell. The task of second grade students was to learn 100 addition facts either by a drill method corresponding to McConnell's rote method or a generalization method in which children were expected to find generalizations from known facts as the teacher introduced number facts using concrete materials and then proceeded "...in such a way that the perception of a useful generalization was required (p. 46)." The generalization group had higher mean scores on both retention and transfer tests.

Swenson (1949) used three methods, "generalization," "drill," and "drill-plus" to teach 332 second grade students the 100 addition facts. The materials for the generalization method were presented "...in such a way that the teacher could, by skillful instruction, lead the pupils to their own formulation of the generalization (p. 12)." The method was based on the meaning theory of teaching and learning arithmetic developed by Brownell and others during the thirties and forties.[1] The "drill" method was essentially rote learning in which speed was emphasized. In the "drill-plus" method new facts were verified by children counting or manipulating concrete objects; generalization was discouraged and rote learning of the facts after the initial introduction was expected.

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[1] Weaver and Suydam (1972) give an excellent treatment of the theory and research related to this theory.

The generalization method was superior in the production of initial learning, of transfer, and of retention to either of the other two methods. The drill method was generally better than the drill-plus method although the differences were not as marked as for the generalization method on standardized arithmetic tests, which were essentially retention tests. Students of low achievement and high ability did better on a test of mathematical thinking if they had received the generalization method of instruction rather than the drill method. Some caution should be exercised in viewing the findings of Swenson, Thiele, and McConnell as the Ss were not randomly assigned to the treatments in any of these experiments.

Brownell and Moser (1949) conducted a study comparing two methods of teaching ("mechanical" and "rational") two different subtraction algorithms (borrowing by either decomposition or borrowing by equal addition). The students were either taught to do borrowing mechanically or else they were taught so that the borrowing process was understood rationally. The rational method of presentation included using concrete objects, writing examples in expanded notation, writing crutch digits, and delaying verbalizing until the process was understood. It appears the treatments here should not be considered as discovery but rather reception since the degree of teacher control and direction was very substantial. In general, the results showed mechanical instruction produced higher scores on tests immediately following instruction but rational instruction produced increased retention and transfer.

Tredway and Hollister (1963) studied teaching percentage problems by rote or by meaningful methods. They found significantly better results at all levels of intelligence on posttests with significantly better retention for average students taught by the meaningful method.

Krich (1964) compared teaching of division of fractions to sixth grade students by two different methods. One method was to tell students the rules for inversion and explain how to use them. The other method was to explain the meaning of the number symbols for multiplication and division of fractions and then allow the inversion rules to be discovered. This is another case of meaning through discovery as one of the treatments while the other treatment was rote or mechanical. No significant differences were found between the low ability groups but average and high ability students did significantly better with the discovery treatment on tests of retention administered two months after the end of instruction.

Worthen (1968) studied teaching by discovery or teaching by expository methods. Concepts included in the study were "...(1) notation, addition, and multiplication of integers (positive, negative, and zero), (2) the distributive principle of multiplication over addition, and (3) exponential notation and multiplication and division of numbers expressed in exponential notation (p. 225)." The treatments were administered to 432 pupils in 16 fifth and sixth grade classes equally divided among eight elementary schools. Each teacher was carefully trained in both teaching by discovery and teaching by exposition. Each teacher taught both a discovery class and an expository class with careful monitoring to assure adherence to the appropriate techniques for each class. The study reported initial learning to be significantly higher for expository classes with significantly better retention and transfer for the discovery classes, but the individual student was used as the unit of analysis which Worthen and Collins later corrected. When data were analyzed with class means as the unit of analysis there were no significant differences (Worthen and Collins, 1971).

Hendrix (1947, 1950, 1961) taught the rule "The sum of the first *n* odd numbers is *n*-square (1947, p. 199)." In Method I the teacher stated the rule then illustrated it with examples. Method II asked students to find the sum of the first two odd numbers, then of the first three, then of the first four, etc. As soon as the student responses indicated he had discovered the rule he was asked to leave the room. Method III was the same as Method II except after the rule was recognized the teacher helped the student verbalize it in an accurate form. After two weeks a test was given with items that could be solved using the rule. The results favored Methods II and III over Method I but were not significant at the .10 level.

Kersh (1958, 1962) conducted two experiments. In each experiment the task was to learn two rules. The first rule was the same as the one Hendrix used; this rule is an example of the second rule which was the formula for the sum of the first n terms of an arithmetic sequence. In the 1958 experiment the treatments were termed "no-help," "direct-reference," and "rule-given." The *S*s in the "no-help" group were required to discover the rules without help. The "direct-reference" group was given examples and diagrams as aids. The "rule-given" group was told the rule directly and given practice in applying the rule without reference to why the rules were valid. The subjects were 60 college students from the experimenter's educational psychology class. A posttest immediately followed instruction and indicated perfect scores for Ss learning the rules. Four weeks later Ss were tested to determine if they used the rules or simply added. In the "no-help" group fewer Ss used correct rules on the posttest immediately following instruction than used correct rules on the test four weeks later. The number using correct rules in "direct-reference" and "rule-given" groups decreased. The difference led Kersh to speculate that the "no-help" group was motivated to learn the correct rules after the treatment and hence that apparent superiority of discovery learning in tests of retention may be accounted for in terms of motivation to continue learning.

In the 1962 study Kersh had a rote treatment in which Ss learned the rule without explanation, a directed learning treatment in which Ss were given careful explanations of the rules, and a guided discovery treatment where the rules were taught to the Ss tutorially using Socratic questioning. The rote group did significantly better on tests three days, two weeks, and six weeks after the instruction. Kersh concluded "Indeed, these data suggest that under certain conditions of learning, highly formalized 'lecture-drill' techniques, ordinarily considered sterile and meaningless, produce better results than techniques which attempt to develop 'understanding' (p. 69)."

Gagné and Brown (1961) taught 33 Ss in grades 9 and 10 formulas for the sum of the first n odd whole numbers, for the sum of the first n terms of the geometric sequence 1, 2, 4, 8, ..., and for the sum of the first n terms of the sequence 1, 2, 3, ... The treatments were Rule and Example, Discovery, and Guided Discovery. The Rule and

Example treatment began by giving Ss the correct formula and then providing practice until the formulas were used correctly. The Discovery treatment asked the Ss to find the rule and proceed by providing hints that were more and more explicit, but the Ss were never given the formulas. The Guided Discovery method was similar but the hints provided more help than did the Discovery method. The dependent variables were the time required to solve new but related problems, the number of hints required to solve the problems, a weighted average of these first two scores, and the proportion of mistakes. The proportion of mistakes provided "...few usable scores, and was not employed in analysis of results (p. 319)." The other measures indicated a significant difference favoring the Guided Discovery treatment over the Discovery and the Rule and Example treatment.

Scott (1970) conducted two experiments "...to determine the differential effects on immediate acquisition, retention, and transfer, of two methods of presenting geometry concepts to sixth graders (p. 81)." The treatments were Expository and Discovery. The studies had 256 sixth grade students as subjects. The first experiment dealt with the effects of method of presentation on retention and the second experiment dealt with the effect of method of presentation on initial acquisition and transfer. The results showed no difference between acquisition or transfer but the Discovery method produced better retention than the Expository method.

Most of the studies described up to this point were discussed by Wittrock (1966), Scott and Frayer (1970), Henderson (1963), or Weaver and Suydam (1972). Concerning the discovery learning studies Wittrock stated "... the current state of research on discovery is very disappointing and precludes any important conclusions about teaching or learning (p. 45)." Henderson said "One conclusion is that the evidence is not conclusive (p. 1020)." Scott and Frayer were able to conclude

> The general, though by no means universal, finding of these studies appears to be that discovery methods are not superior to rote or drill methods when the criterion is immediate learning or short-term retention but become superior when the criterion is either long-term retention or transfer (p. 11).

Of interest also is another statement by Scott and Frayer

> No experiment was performed to measure differences among Ss who discovered a generalization, Ss who were given a generalization, and Ss who learned facts alone. Such a comparison might suggest how much of the effect was due to Ss discovering a generalization as opposed to Ss using a generalization (p. 11).

It may be noted that the three methods mentioned are closely related to the conceptual discovery, conceptual reception, and rote reception treatments described in Chapter V. Why not include rote discovery learning? It appears likely that requiring Ss to learn a large number of related facts in a rote manner will result in the discovery of relations (it may even be true that Ss will create relations to simplify learning). Do students examine items learned and classify them by some conceptual rule (generalization)? If so, will this kind of learning be different from conceptual discovery where Ss are not asked to learn specific facts, but rather to guess a generalization and test it with new examples? Or, do both rote discovery and conceptual discovery involve basically the same mental processes? Consideration of questions of this sort lead directly to questions of the effects of different kinds of learning on cognitive structure.

It was, in fact, the effects of learning a given rule or discovering the rule on the cognitive structure of the learner that led Egan and Greeno (1973) to perform two experiments investigating an aptitude treatment interaction (ATI) between three levels of ability and rule given or discovery treatments. They stated the following:

> A simple hypothesis suggests that skills involved in solving problems and generalizing are more important to success in learning by discovery than in learning by rule. This idea leads to the expectation of an ATI such that the skills of subjects learning by discovery should be strongly related to their performance while the skills of subjects learning by rule should be less strongly related to performance (p. 85).

To test the hypothesized ATI, 57 subjects were taught to solve problems involving binomial probability using programmed texts. The rule given treatment included stating the formula on the first page and then proceeding with multiple choice questions through a branching program until the formula was used correctly. The Ss in the discovery treatment were given basically the same questions without being given the formula. Three tests of abilities were used in the analyses of the data: (1) a test of probability concepts, (2) a test of computational ability, and (3) the scores from the Scholastic Aptitude Test - Mathematics. For each test Ss were ranked as low, medium, or high to provide the three levels of aptitude. Three dependent measures were obtained: (1) the number of errors in answering the multiple choice questions in the instructional programs, (2) the time required to complete the instructional program, and (3) the proportion of errors made on the posttest. Analyses of the data gave weak support to the hypothesized ATI. This weak result led to a second experiment to replicate the result and to extend understanding of what was acquired in each type of instruction.

In the second experiment, 72 Ss were taught to solve problems involving joint probability by a computer-assisted instruction (CAI) system. The treatments and tests of ability were similar to those in Experiment I, however, the posttest was more carefully constructed. It consisted of 18 problems chosen to test different aspects of transfer. Egan and Greeno were able to conclude

> The present results suggest that subjects lacking in skills necessary to solve problems may learn more efficiently when instructed by techniques requiring interpretation and application of a rule. By every measure, subjects low in relevant abilities performed better when instructed by the rule method. That the rule method used in this study was not inherently better can be inferred from two results found in Experiment I and replicated in Experiment II (pp. 92-93).

Evidence of real differences (as indicated by statistical significance as well as observation) in cognitive structure seemed apparent.

More emphasis will be given to cognitive structure in the next section than has been possible here.

Models of Concept Learning

A model that was useful in suggesting the effect of learning on cognitive structure was described by Fletcher (1969). There are four stages:

Stage 1. Attentional Processes ...includes all processes which serve to detect those cues which are relevant to the particular problem (p. 7).

Stage 2. Transformation Processes ...includes all those processes which serve to encode appropriate information. In the trivial sense, S responds only to encoded information, never to actual stimuli, so transformations are fundamental. However, we use the term 'transformation' in the nontrivial sense to refer to those initial active processes which convert cues into meaningful information. These rule-governed processes are, of course, subject to change or enrichment as, for example, when the child transforms stimuli into first letters, then words, then sentences. Of the potentially many ways in which stimuli can be transformed into meaningful information, there does seem to be one easily identifiable 'dimension', which can be labelled as the analytic-synthetic dimension Appearing as two separate cognitive factors in Bloom's taxonomy, these labels refer to those cognitive operations which serve either to break down stimuli into individually meaningful elements or to impose initial overall meaning upon discrete elements (p. 8).

Stage 3. Generation Processes ...[includes] all the processes or operations which serve to generate solutions by systematically going beyond the already transformed information. Appropriate processes would include all logical manipulations, the detection of relationships, and the identification of rules or patterns of sequential stimuli (p. 8).

Stage 4. Evaluation Processes ...contains all processes which serve to determine whether or not solution has been achieved. Interestingly, an argument may be made that only the single evaluative process of comparing exists, and that one is always comparing two units of information against each other or one unit of information against some internal or external criterion. In any event, the stage functions as the decision point. ...if a negative evaluation occurs, there is feedback to each prior stage and the entire information-processing sequence may recycle with different rules involved at each stage.

> Some general comments about the model are in order. Clearly, memory is a cognitive process and, as such, must be involved in processing information. The model explicitly admits this --in fact emphasizes the point--by showing that memory is involved at all stages. Stored in memory are not only currently generated outputs but also the more permanent types of information and solution rules which may be utilized at each particular stage (p. 8).

Fletcher stresses the importances of memory at each stage. Figure 3.1 shows a diagram of the model. It seems that rote learning should be included in the Transformation Processes stage and discovery learning and generalization should be included in the Generation Processes stage.

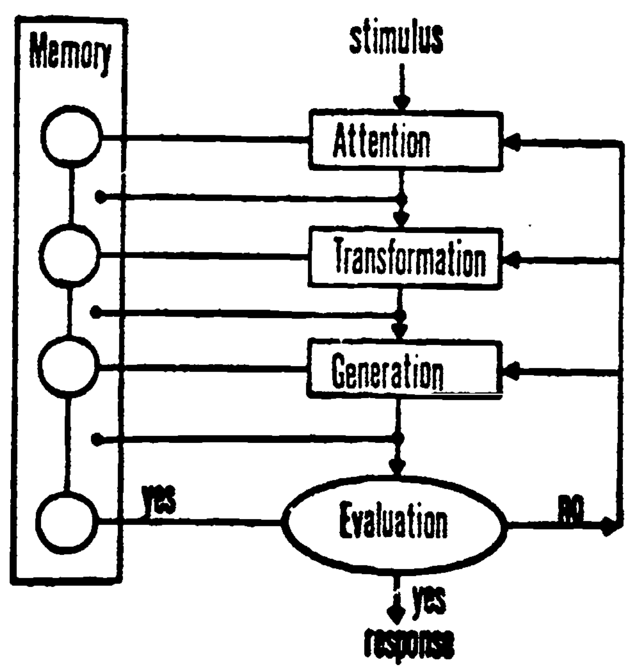

Figure 3.1. Schematic representation of a general operational model of information processing with four functional stages of cognitive processes (Fletcher, 1969, p. 8).

Klausmeier (1971) gave a model for cognitive operations in the attainment of a formal concept. The first successive operations were:

attending to objects, discriminating one object from another, remembering discriminated object, generalizing that two or more perceptible forms of the same thing are equivalent, generalizing that two or more instances are equivalent in some way, and discriminating the attributes of a formal concept. Concurrent with discriminating the attributes of a formal concept, the process of acquiring and remembering the attribute labels occurs. Next to infer the concept either the operation of cognizing common attributes of positive instances may occur or else a processs of forming a hypothesis and testing the hypothesis with positive and negative instances may occur. In either case, the formal concept is inferred and a concept label is acquired and remembered to complete the operations in the attainment of a formal concept. Once the concept is inferred, generalization may occur. Superordinate, coordinate, subordinate relations, and relationships involving cause and effect, correlation, and other contingencies may be cognized. The concept may also be generalized to problem solving situations. Figure 3.2 is a diagram of this model that combines the parts of two diagrams (Klausmeier, 1971, p. 2 and p. 3) that represent learning related to formal concepts (the parts not included represent lower level concept attainment).

The operations of discriminating and labeling the attributes of the concept were felt to be implied by Klausmeier (1971) as the result of studies by Kalish (1966), Lynch (1966), Fredrick and Klausmeier (1968), and Klausmeier and Meinke (1968) which showed that "...instructions making explicit the attributes that define the concept population do facilitate subsequent concept attainment (p. 3)."

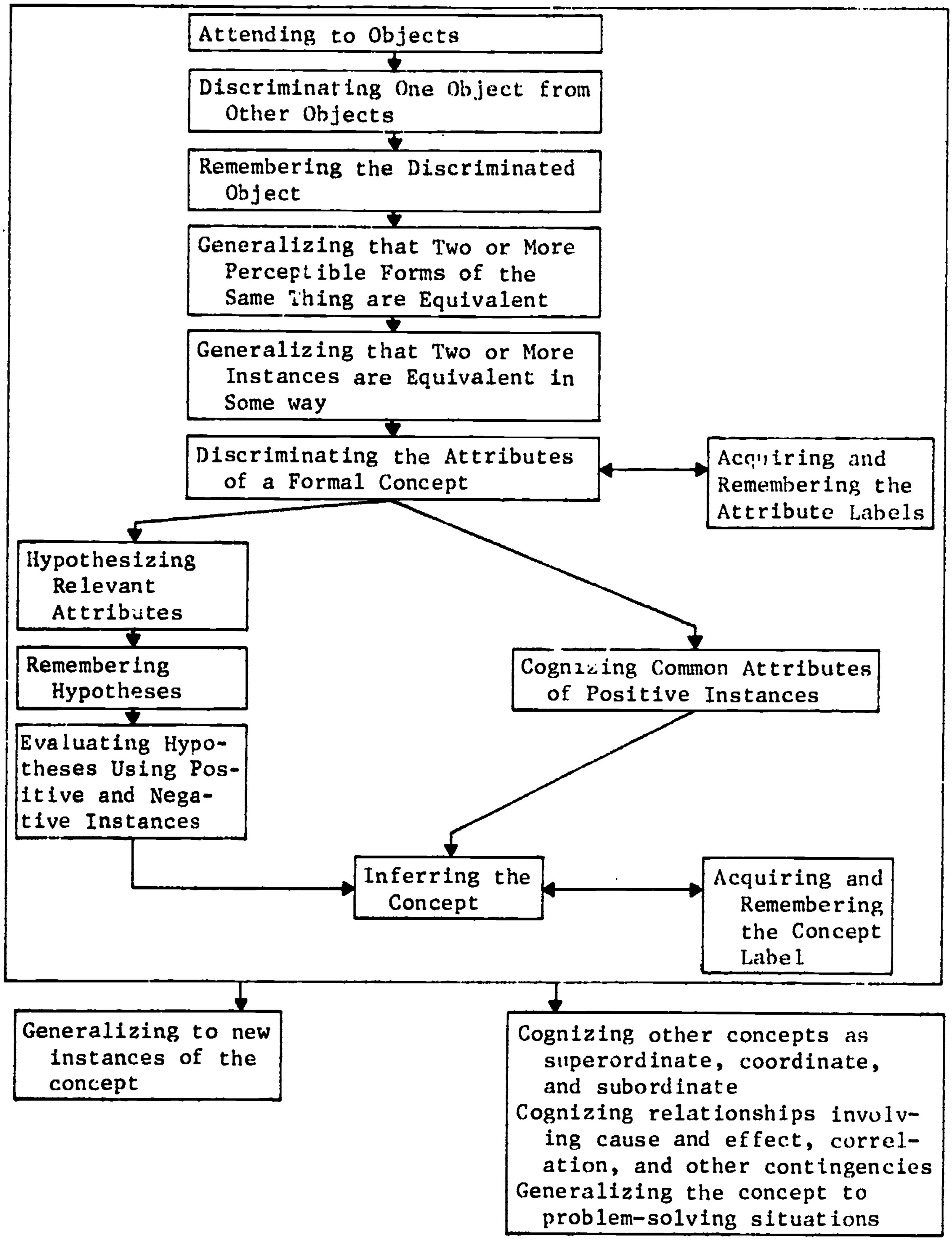

Figure 3.2. A diagram of Klausmeier's model showing cognitive operations in the attainment of a formal concept, extension and use.

That a concept may be inferred by hypothesizing relevant attributes, remembering the hypothesis, and then evaluating positive and negative instances was argued using research by Levin (1967) and Klausmeier, Harris, Davis, Schwenn, and Frayer (1968). Research by Tagatz (1967) provided support for the operation of using only positive instances to infer a concept at least in children.

It appears that Klausmeier has provided reasonable support for his model through efforts of researchers at the Wisconsin Research and Development Center for Cognitive Learning. (He indicates other supporting research as well.) Some other research will now be discussed that provides further support.

Anderson and Kulhavy (1972) found that "...people can easily learn concepts from definitions, provided the meanings in the definitions are accessible to the reader (p. 389)." The study used undergraduates as subjects in an experiment to "...explore to what extent people can acquire concepts from exposure to definitions (p. 385)." Subjects were given definitions of words they did not know. A random half of the Ss received instructions to create and say aloud a sensible sentence using the defined word. The other half was instructed to read each definition aloud three times. Ss composing the sentences did significantly better.

Requiring S to read aloud the definition three times increases focus on only a portion of the operations in Klausmeier's model, specifically, knowing relevant attributes. Telling the definition is essentially showing Ss the correct hypothesis, which is also only a portion of the operations involved in learning a concept. Whereas using the word in a sentence does require the understanding of relevant attributes it also involves the other operations. Note the implications for Fletcher's model. Reading the definition aloud, without using it in a sentence, involves only the first two stages of the model but using the word in a sentence requires all four stages.

Johnson and Stratton (1966) cite four ways used to teach a concept: by using it in a sentence, by requiring students to classify objects as positive or negative instances, by defining the concept for the student, and by giving synonyms. The four methods and a fifth method that combined the four were used to teach Ss the concepts. Learning of concepts by each of the four stated methods were not found to be significantly different but the combined method was better ($p < .01$). However, more learning did occur ($p < .01$) in each of the groups when they were compared to a control group. The first four treatments involve single operations in Klausmeier's model of use or hypothesizing and testing positive and negative instances, but the combined method involves more operations which aid in cognizing a concept.

Feldman and Klausmeier (1974) used the concept "equilateral triangle" to study the effects of two kinds of definitions on attainment of the concept. One definition was stated in "technical" terms with all defining attributes specified. The other definition was stated in common language and did not include all of the defining attributes. The subjects were fourth grade and eighth grade students. A 2 x 2 analysis of variance indicated a significant main effect for grade level and a weak interaction between grade level and kind of definition ($p < .07$).

The fourth grade students performed better using the common language definition, and the eighth grade students performed better using the "technical" definition. In terms of the model, this study indicates that formal definitions are of more value in understanding attributes of the concept to eighth grade students than to fourth grade students while the common language definition help fourth grade students more than eighth grade students.

In viewing research on concept learning and discovery learning it seems that there is a need to consider more than just differences in posttest scores produced by different treatments. It is important to recognize that one way treatments produce different effects is through exercise of different cognitive operations in the learner. Thus different patterns of learning promise to be useful dependent variables in the study of learning. Some of the patterns can be described mathematically.

Mathematical Considerations

The axioms for Bower's model of paired-associate learning were stated in Chapter II. These axioms imply distribution functions for two statistics that are used in Chapter VI to test the fit of Bower's model to data gathered in the study described in Chapter IV and Chapter V.

A discussion of the statistics used is given below that summarizes a thorough treatment in Atkinson, Bower, and Crothers (1965).

Let x_n be a random variable with value 0 if a correct response is given for an item on trial n and 1 if an error is made. The probability of an error on trial n is given by:

$$P(x_n = 1) = (1 - c)^{n-1} (1 - g)$$

where $(1-c)^{n-1}$ is the probability that the subject is in the unlearned state and $(1-g)$ is the probability of an incorrect response given that the subject is in the unlearned state. (Recall that g is the probability of a correct guess and c is the probability of changing from the unlearned state to the learned state.)

Let T be the total number of errors made before learning an item so that

$$P(T = 0) = \sum_{i=1}^{\infty} [g(1 - c)]^{i-1} gc.$$

This follows from observing first that the probability of learning on trial i with no preceding errors is $[g(1-c)]^{i-1}gc$ since $g(1-c)$ is the probability of guessing correctly and not learning on each of the preceding i-1 trials and gc is the probability of a correct guess followed by learning. But $P(T=0)$ is the sum of the probabilities of learning on each trial without preceding errors. Next observe that

$$\sum_{l=1}^{\infty} [g(1-c)]^{i-1} gc \text{ simplifies to } \frac{gc}{1-g(1-c)}.$$

To simplify further computations let

$$b = \frac{c}{1 - g(1 - c)} .$$

The probability of no further errors following an error is b, so that to have exactly k errors an error must occur (the probability is 1-gb), there must be k-1 errors followed by at least one more error (the probability is $(1-b)^{k-1}$) and there must be no further errors after the

k^{th} error (the probability is b) giving

$$P(T=k) = (1-gb)(1-b)^{k-1}\ b = \frac{(1-b^k)b}{1-c},\ \text{for } k \geq 1.$$

The parameters g and c are estimated in Chapter VI by minimizing

$$D^2 = \sum \frac{[\text{observed} - \text{theoretical}]^2}{\text{theoretical}}$$

which is asymptotically distributed as a Chi-square random variable with degrees of freedom equal to three less than the number of independent observations, since there are two parameters estimated (Brunk, 1960; Levin and Burke, 1972). Due to the small sample size it is not desirable to assume that D^2 has a Chi-square distribution, however, for an exploratory study, this method gives sufficient information in that what is sought is an indication that complex learning can be analyzed with these methods. This analysis will be presented in Chapter VI.

Summary

In this chapter research related to the study has been considered. Experiments that were designed to study various aspects of rote, meaningful, discovery, and reception learning were described. Two models of learning, with supporting research, were presented. The experiments cited in this chapter provide a background for the next chapter which contains a description of the development of the study.

Chapter IV

DEVELOPMENT OF THE STUDY

This chapter gives a detailed description of the study. The considerations of Chapter II indicated a particular experimental design and certain procedures; these are described first, followed by a description of the selection and preparation of the unit and materials.

Experimental Design and Procedures

The experiment was divided into two parts. In Part One the students were to learn to identify examples and non-examples of the mathematical concepts in the unit and in Part Two, they were to learn to use formulas related to these mathematical concepts.

The basic unit of instruction was a trial. A trial consisted of reading the definitions (in Part One) or reading the formulas (in Part Two) if the student was in a reception treatment, otherwise this part was omitted. Next the student was given a list. A list consisted of a set of eight mathematical concepts and twelve examples and non-examples (to be referred to as items) for Part One and twenty computations to perform (five computations for each of four sequences) for Part Two. The student then responded to items on the list and was then given the correct responses. The lists will be described in more detail later. A trial was completed when the student had finished studying the correct responses for the list.

Part One - Concept Identification

The concept identification part of the experiment (Part One) was a factorial design with two levels in each of three dimensions; pretest given or not given, rote-conceptual, and reception-discovery. The rote-conceptual dimension was determined by whether or not it was possible to respond correctly to some of the items by having previously learned that specific item by rote. In the rote treatments it was possible to learn a concept. In fact, trials were repeated until the concepts were learned. In conceptual treatments no item was repeated so it was impossible to give correct responses by having previously learned a particular item by rote. The reception-discovery dimension was determined by whether or not the definitions were given to the students before giving them the lists. In the rote reception, rote discovery, conceptual reception, and conceptual discovery treatments described next, half the students received pretests and half did not to constitute the pretest given or not given dimension.

The rote reception treatment. In the rote reception treatment students were first given instructions to follow. These instructions are shown in Figure 4.1. They were then given definitions of the concepts, examples of the concepts, and an explanation of the relations that existed between the concepts (see Figure 4.2). Each

A Unit on Sequences and Series

You will be given a set of definitions. You are to learn to identify the different kinds of sequences and series that are defined there.

DIRECTIONS:

1. After you finish reading the directions, spend a few minutes reading the definitions. It is not necessary to memorize each one since you will find as you use them that you will learn them easily.

2. After you have read the definitions, return them to the instructor and get a booklet containing examples from him/her. In the booklet place an X in front of those terms that describe the example. For instance:

   Phydeau (a local dog)

   X animal X dog
   plant

3. The correct responses are marked on the back of the page. Your answers are not used to determine your grade. PLEASE DO NOT CHANGE ANY OF YOUR ANSWERS!

4. When you have finished your booklet be sure your name is on it and give it to your instructor.

5. Get a list of definitions from the instructor and reread any you wish, then return the definitions and get a new booklet.

6. Repeat this fun process until you can use the definitions correctly. Don't spend more than 20 minutes per table since a certain amount of speed is necessary.

PLEASE WORK INDIVIDUALLY.

Figure 4.1. Instructions for the rote-reception treatment of Part One.

DEFINITIONS

A set of real numbers a_1, a_2, a_3, ... in a definite order and formed according to a definite rule is called a sequence. Each number in a sequence is called a term of the sequence. Any sequence with a finite number of terms is called a finite sequence, and any sequence with an infinite number of terms is called an infinite sequence. Often an ellipsis (...) is used to indicate some terms are not written but understood to be included. For instances 1, 2, ..., 10 means 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and the counting numbers can be written 1, 2, 3, 4, ...,

An arithmetic sequence is a sequence in which any pair of successive terms differ by a fixed number (the first term of the pair is always subtracted from the second to find the fixed number). This fixed number is called the common difference for the arithmetic sequence. The arithmetic sequence 1, 11, 21, 31 is formed by starting with 1 then adding 10 to get the second term, 11, then adding 10 to get the third term, 21, and the fourth term is 21 plus 10 or 31.

A geometric sequence is a sequence in which the quotient of any pair of successive terms is a fixed number (the first term of the pair is always divided into the second). This quotient is called the common ratio for the geometric sequence. The geometric sequence 1, 2, 4, 8 is formed by multiplying by 2; 1, 1 x 2 = 2, 2 x 2 = 4, 4 x 2 = 8.

A series is the indicated sum of the terms of a sequence. Finite series, infinite series, arithmetic series and geometric series are formed from finite sequences, infinite sequences, arithmetic sequences, and geometric sequences respectively.

NOTE:
- A. Every sequence and series is either finite or infinite but not both.
- B. Every example given in the unit will be either a sequence or a series but not both.
- C. Each kind of series is defined by the kind of sequence that it is formed from.
- D. The only sequence that is both an arithmetic sequence and a geometric sequence is a sequence that has every term the same (for instance 4, 4, 4,...). In such a case the common difference is 0 and the common ration is 1. No examples like this will be given in the unit.

Figure 4.2. Definitions given in reception treatments of Part One.

student was allowed to study the definitions as long as he wished and when finished he was given a list in booklet form. After choosing a response to an item the S was then given the correct response and compared his response to it. Then the S proceeded to the next item in the booklet and repeated this process until the booklet was completed. Then the experimenter checked the S's responses on the booklet to determine whether or not there were mistakes. If there were mistakes, the student was given the definitions to read again and the routine continued using the same form of the booklet until he completed a list with no mistakes. Upon completion of the first form of the booklet without mistakes the student was again given the definitions to read and then given a different form of the booklet which he marked and checked as before. The procedure continued until a student completed two different forms in succession with no mistakes. At this point the student took the posttest and went on to the second part of the experiment. An example of this procedure might be: a student reads the definitions and marks form A (the different forms were indicated by A through J) making some mistakes. He then repeats the trial and again makes mistakes but on the third time through form A he makes no mistakes. He then does a trial using form B and makes mistakes the first time but not the second time. On his first trial in form C he makes no mistakes and so is ready to take the posttest for Part One and then go to Part Two. The first few lists could be done correctly by rote learning but to complete Part One the student had to eventually learn the concepts.

The rote discovery treatment. The rote discovery treatment was the same as the rote reception treatment except that the definitions of the mathematical concepts were not given to the students of the group. To complete Part One it was necessary to discover the correct definitions.

The conceptual reception treatment. The conceptual reception treatment resembled the rote reception treatment in that the students were given the definitions but differed in the way the lists were used. The lists were in grid format (see Figure 4.3) instead of a booklet. The same list was never given to a student more than once. After reading the definitions the student marked the entire list and gave it to the experimenter. The experimenter gave the student the correct responses to study. If there were mistakes the student was given the definitions to read before he was given a different form of the grid list. If there were no mistakes he took the posttest for Part One and went on to Part Two. A student, after reading the definitions the first time, might make some mistakes on form A of the grid list and would be given the correct responses to study. When he finished studying the correct responses he would study the definitions and mark his responses on form B of the grid list. Again, if there were mistakes he would repeat the process except he would mark his responses on form C of the grid list, if this time there were no mistakes he would take the posttest for Part One and begin Part Two.

The conceptual discovery treatment. The conceptual discovery treatment was the same as the conceptual reception treatment except

FORM J	Arithmetic Series	Geometric Series	Finite Sequence	Infinite Series	Geometric Sequence	Finite Series	Infinite Sequence	Arithmetic Sequence
5, 14, 23, 32								
6, 14, 23, 33								
$3 + 2 + \frac{4}{3} + \frac{8}{9} + \ldots$								
$4, \frac{8}{7}, \frac{16}{49}, \frac{32}{343}, \ldots$								
7 + 13 + 19 + 25								
8 + 14 + 20 + 26								
$9, -\frac{63}{8}, \frac{441}{64}$								
$\frac{1}{7}, \frac{1}{3}, \frac{1}{10}, \frac{1}{13}, \ldots$								
6 + 11 + 17 + 24								
$\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \ldots$								
$2 + \frac{8}{5} + \frac{32}{25} + \frac{128}{125}$								
6, 8, 10, 12								

Figure 4.3. List in grid form for Part One

the definitions of the mathematical concepts were not given to the students of the group. To complete Part One it was necessary to discover the correct definitions.

<u>Part Two - Concept Use</u>

The concept use part of the experiment (Part Two) was a factorial design with two dimensions and two levels in each dimension; rote-conceptual and reception-discovery. As in Part One the rote-conceptual dimension was determined by whether or not it was possible to respond correctly to some of the items by having previously learned that specific item by rote. In the rote treatments trials were repeated until the computations were performed correctly on new examples so that conceptual learning eventually occurred even though the initial learning may have been rote. In the conceptual treatments no item was repeated so that it was impossible to give correct responses by having previously learned a particular item by rote. The reception-discovery dimension was determined by whether or not the formulas and examples of their use were given to the students before giving them the lists (see Figure 4.4). The treatments were the same as the corresponding treatments in Part One except a list, either booklet or grid format, was successfully completed if no more than two mistakes were made (as opposed to no mistakes in Part One).

The Unit and Materials

This section gives a description of the selection of the unit and the methods used to construct the materials used in the study.

FORMULAS

If a is the first term of an arithmetic sequence and d is the common difference then:

a. The nth term is $a + (n-1)d$. The 20th term of 1, 3, 5, ... is $1 + 19 \cdot 2 = 39$ since $a = 1$ and $d = 2$.

b. The sum of the first n terms is $\frac{1}{2}n[2a + (n-1)d]$. The sum of the first 20 terms of 1, 3, 5, ... is $\frac{1}{2} \cdot 20[2 \cdot 1 + 19 \cdot 2]$.

If a is the first term of a geometric sequence and r is the common ratio then:

a. The nth term is ar^{n-1}. The 20th term of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... is $1(\frac{1}{2})^{19}$.

b. The sum of the first n terms is $\frac{a(1-r^n)}{1-r}$. The sum of the first 20 terms of 1, $\frac{1}{2}$, $\frac{1}{4}$, ... is $\frac{1(1 - \frac{1}{2}^{20})}{1 - \frac{1}{2}}$

c. For $|r|<1$, the sum of the terms of an infinite sequence is $\frac{a}{1 - r}$. The sum of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... is $\frac{1}{1 - \frac{1}{2}} = 2$.

Figure 4.4. Formulas given in the reception treatments of Part Two.

Criteria for Selection of the Unit

Four criteria for selecting the unit were used. These are described and discussed below.

Criterion 1. The unit was to have content that is taught as a part of the regular school mathematics curriculum. This criterion was included because the study deals with learning and teaching mathematics of the kind that is taught in regular school programs. If the material in the unit is not usually taught, generalizations would be required to make the results of the study apply to the classroom. These generalizations may be quite sound but the point is the problem could be avoided altogether. Another reason for the criterion, which was perhaps more important, was that the study would be carried out using students in mathematics classes and would deal in part with ways of organizing course content and teaching course material.

Criterion 2. The unit was to have a hierarchy of concepts as the main content to be taught. One question to be investigated concerned students' learning the structure of a hierarchy of concepts and so this criterion was absolutely essential.

Criterion 3. The unit was not to require extensive previous mathematical knowledge. This criterion was a practical consideration. It was possible that the unit might be taught (for the purposes of the study) at a time that would not be in the usual textbook sequence for some of the classes. The criterion was also intended to minimize the effect of previous knowledge interacting with the learning of the unit.

Criterion 4. The unit was to be appropriate for various levels of the curriculum, preferably for junior high school, high school, and college. This was to facilitate acquisition of subjects. It would also be possible to compare learning of students in these different groups.

These criteria led to the selection of the following unit.

The Instructional Unit

The instructional unit was approximately the material covering sequences and series in the college algebra courses at the University of Wisconsin - Madison. Immediately following this unit, mathematical induction was usually taught. One week was allowed for the instruction.

The eight mathematical concepts that were chosen were: (1) infinite sequence, (2) infinite series, (3) finite sequence, (4) finite series, (5) arithmetic sequence, (6) arithmetic series, (7) geometric sequence, and (8) geometric series. Students were to be able to recognize examples and non-examples of each of these mathematical concepts as well as be able to find the twentieth term, the formula for the nth term, the sum of the first twenty terms, and the formula for the sum of the first n terms of arithmetic and geometric sequences. The students were also to be able to find the sum of the terms of an infinite geometric sequence if it exists.

It should be noted that while the definitions of sequence (see below) is somewhat imprecise it is essentially the one used by Groza and Shelly (1969), which was the text for the course and it does communicate the basic ideas. The more correct definition that a sequence is the range of a function from the natural numbers into the real numbers was considered but not used because of criteria three and four.

The following are the definitions and formulas that were included in the unit:

A set of real numbers a_1, a_2, a_3, ... in a definite order and formed according to a definite rule is called a <u>sequence</u>. Each number in a sequence is called a <u>term</u> of the sequence. Any sequence with a finite number of terms is called a <u>finite sequence</u>, and any sequence with an infinite number of terms is called an <u>infinite sequence</u>. Often an ellipsis (...) is used to indicate some terms are not written but understood to be included. For instance 1, 2, ..., 10 means 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and the counting numbers can be written 1, 2, 3, 4, ...

An <u>arithmetic sequence</u> is a sequence in which any pair of successive terms differ by a fixed number (the first term of the pair is always subtracted from the second to find the fixed number). This fixed number is called the <u>common difference</u> for the arithmetic sequence. The arithmetic sequence 1, 11, 21, 31 is formed by starting with 1 then adding 10 to get the second term, 11, then adding 10 to get the third term, 21, and the fourth term is 21 plus 10 or 31.

A <u>geometric sequence</u> is a sequence in which the quotient of any pair of successive terms is a fixed number (the first term of the pair is always divided into the second). This quotient is called the <u>common ratio</u> for the geometric sequence. The geometric sequence 1, 2, 4, 8 is formed by multiplying by 2; 1 x 2 = 2, 2 x 2 = 4, 4 x 2 = 8.

A <u>series</u> is the indicated sum of the terms of a sequence. <u>Finite series</u>, <u>infinite series</u>, <u>arithmetic series</u>, and <u>geometric series</u> are formed from finite sequences, infinite sequences, arithmetic sequences, and geometric sequences.

If a is the first term of an arithmetic sequence and d is the common difference then the nth term is a + (n-1)d and the sum of the first n terms is $\frac{1}{2}n[2a + (n-1)d]$.

If a is the first term of a geometric sequence and r is the common ratio then the nth term is ar^{n-1}, the sum of the first n terms is $\frac{a(1-r^n)}{1-r}$, and for $|r| < 1$, the sum of the terms of an infinite sequence is $\frac{a}{1-r}$.

This concludes the definitions; next, the hierarchy of concepts determined by the foregoing definitions will be discussed.

<u>The Hierarchy of Concepts</u>

In Chapter II three possible relations that could exist between mathematical concepts were given. They were: (1) mathematical concept A <u>is an example</u> of mathematical concept B, (2) mathematical concept C <u>is a subclass</u> of mathematical concept D, and (3) mathematical concept E <u>is used in the definition</u> of mathematical concept F. In each case the first named mathematical concept will be called <u>subordinate</u> to the second and the second will be called supra-ordinate to the first.

The mathematical concept "sequence" was used in the definition of "finite sequence," "infinite sequence," "arithmetic sequence," "geometric sequence," and "series." "Series" and "finite sequence" were used in the definition of "finite series"; "series" and "arithmetic sequence" were used in the definition of "arithmetic series"; and "series" and "geometric sequence" were used in the definition of "geometric series." This determined the subordinate - supra-ordinate relations described in (3) above.

"Finite sequence (series)," "infinite sequence (series)," "arithmetic sequence (series)," and "geometric sequence (series)" are all subclasses of "sequence (series)," these are the relations described in (2). The relations described in (1) will not be included here but there were many examples of each concept in the unit. The examples and method of selecting them will be given later. Figure 4.5 is a diagram of these relations.

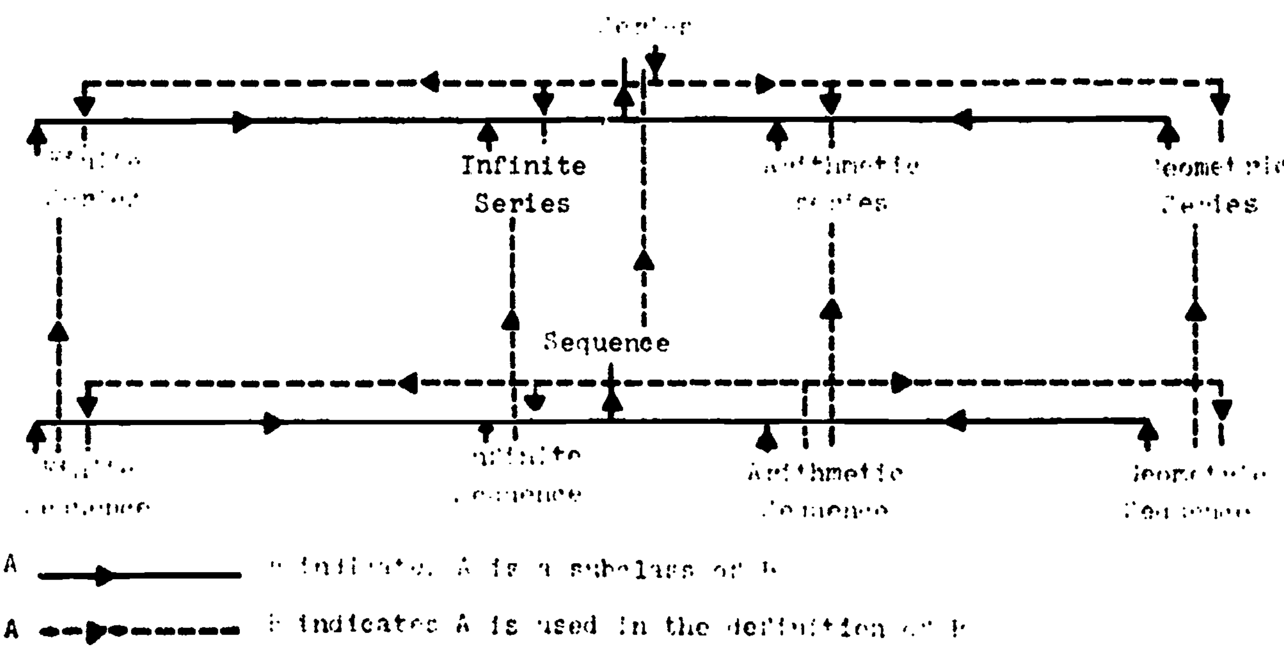

Figure 4.5. Hierarchy of Concepts in the Instructional Unit.

The preceding discussion has shown that the mathematical concepts of the unit do form a hierarchy of concepts so that criterion two for the selection of the unit was satisfied. Criterion one was satisfied since the content of the unit was taken from an existing mathematics course that is a standard part of the college mathematics curriculum. It seemed that criteria three and four were also met, three being more or less obvious after examining the mathematical concepts and four by the fact that the high school curriculum also often includes these concepts in either a second year algebra course or a fourth year mathematics course. It was deemed that the unit probably was not appropriate for junior high school, not because of the difficulty but rather the material leads to problems that require mathematical induction which may not be appropriate for junior high school.

The Lists for Part One

On each list of Part One the eight mathematical concepts were given. The items (the examples of the mathematical concepts) for the lists were chosen so that each list had one item that was an infinite sequence (series) but was neither an arithmetic nor geometric sequence (series), one item that was a finite sequence (series) but neither an arithmetic nor a geometric sequence (series), one infinite geometric sequence (series), one finite geometric sequence (series) and two finite arithmetic sequences (series). Thus the twelve items represented six sequences, six series, four finite sequences, four finite series, two infinite sequences, two infinite series, two arithmetic sequences, two arithmetic series, two geometric sequences, and two geometric series.

Selection of the Examples

It was estimated that students would master the mathematical concepts of Part One in at most ten trials. Thus it was necessary to have a pool of at least twelve examples of each concept (examples were needed for the pre- and posttests also). The following rules were established to generate these examples.

Rule for infinite sequences. The examples of infinite sequences that were non-examples of arithmetic and geometric sequences were of the form 1/a, 1/b, 1/(a+b), 1/(2a+3b), ...where a and b were randomly chosen integers between 0 and 10.

Rule for finite sequences. The examples of finite sequences that were non-examples of arithmetic and geometric sequences were of the form a, a+b, a+2b+1, a+3b+3. The method of selection of a and b was the same as for the infinite sequences.

Rule for infinite series. The examples of infinite series that were non-examples of arithmetic and geometric series were of the form a/b + a/2b + a/3b + a/4b + ... Selection of a and b was the same as for the infinite sequences.

Rule for finite series. The examples of finite series that were non-examples of arithmetic and geometric series were of the form a + (a+b) + (a+b+1) + (a+2b+3). Selection of a and b was the same as for the infinite sequence.

Rule for selecting geometric sequences and series. All examples of infinite and finite geometric sequences and series were determined by the same rule. The first term was a randomly selected integer between 0 and 10. The ratio c/d was reduced to lowest terms and randomly chosen to be positive or negative. The absolute value of c was less than the absolute value of d. A pair of integers between 0 and 10 was selected randomly, the larger was assigned as the value for d and the other c, the fraction was then reduced and the predetermined sign affixed.

Rule for selection of arithmetic sequences and series. Examples of arithmetic sequences and series were selected using the same rule. The first term and common difference were randomly selected integers between 0 and 10. The common difference was then randomly determined to be either positive or negative.

The booklets. A booklet consisted of a cover page with a place for the student's name and section number (from the college algebra classes) and form (A through J). The odd numbered pages had an example of one of the mathematical concepts from the pool of examples and under this example the eight mathematical concepts from the hierarchy of concepts were listed in random order. (The order was the same for each page of a booklet, but was different between forms.) On the back of each of the pages the same example and the mathematical concepts were reproduced in the same order with the correct mathematical concepts marked. Thus, a student could mark responses on one side of the page and then turn it over to check them. There were 24 pages, 12 for the student to mark and 12 with answers already given (see Figures 4.6 and 4.7).

The grids. The grid forms used the same examples as the corresponding booklet forms. Across the top of an $8\frac{1}{2}$" x 11" sheet of paper were wr'tten the eight mathematical concepts and down the left side the items were given. The students were to mark the intersection of the row and column if an item in the column of examples corresponded to the mathematical concept in the column above (see Figure 4.3). After marking the responses, the students were given the correct responses.

The Lists for Part Two

Ten forms of the lists for Part Two were prepared in booklet and grid format. Form A of Part Two had the examples of the arithmetic and geometric sequences for form A of Part One and each of the other forms for Part Two had examples from the corresponding forms of Part One; two arithmetic and two geometric sequences.

The booklets. The front of each page of the booklet had an example of either an arithmetic or a geometric sequence at the top of the page with a place to write the twentieth term, the formula for the nth term, the sum of the first twenty terms, and the sum of the infinite series if it existed. On the back of the page the example was repeated and the correct answers given (see Figures 4.8 and 4.9).

5 + 14 + 24 + 35

Infinite Sequence

Arithmetic Series

Geometric Sequence

Geometric Series

Finite Sequence

Finite Series

Arithmetic Sequence

Infinite Series

1

Figure 4.6. Page from booklet list for student responses, Part One.

5 + 14 + 24 + 35

Infinite Sequence

Arithmetic Series

Geometric Sequence

Geometric Series

Finite Sequence

✓ Finite Series

Arithmetic Sequence

Arithmetic Series

2

Figure 4.7. Page from booklet list showing correct responses, Part One.

6, 4, 2, 0, ...

20th term =

nth term =

sum of 1st 20 terms =

sum of 1st n terms =

sum of the terms of the
infinite sequence =

7

Figure 4.8. Page from booklet list for student responses, Part Two.

6, 4, 2, 0, ...

20th term = −32

nth term = 6 + (n − 1) (−2)

sum of 1st 20 terms = −240

sum of 1st n terms = $\frac{n}{2}$ [12 + (n − 1) (−2)]

sum of the terms of the
infinite sequence = x

8

Figure 4.9. Page from booklet list showing correct responses, Part Two.

The grids. The grids were constructed with the examples down the left side and the indicated computations across the top. Figure 4.10 shows a typical grid for Part Two.

The Tests

The pretest. The pretest was either in booklet or grid format and had the same eight concepts as the lists of Part One. The only difference was that the pretest had only six items instead of the twelve items in Part One and answers were not given. The six items were examples of an infinite sequence that was neither an arithmetic nor a geometric sequence, an infinite series that was neither an arithmetic nor a geometric series, a finite arithmetic sequence, a finite arithmetic series, a finite geometric sequence, and an infinite geometric series.

The posttest for Part One. The posttest for Part One was either in booklet or grid format, had the same eight concepts as the lists and had 12 items chosen from the same categories as the lists but no answers were provided. The first six items were a finite series that was neither an arithmetic nor a geometric series, a finite geometric sequence, a finite arithmetic sequence, an infinite series that was neither an arithmetic series nor a geometric series, a finite geometric series, and a finite arithmetic series. These six items were on form A of the lists and in some cases were originally learned in a rote manner. The other six items were new

Name: ______ H.	20th term	nth term	sum of 1st 20 terms	sum of 1st n terms	sum of the terms of the infinite sequence
4, -3, -10, -17, ...					
8, 4, 2, 1, ...					
9, 11, 13, 15, ...					
6, $\frac{-21}{4}$, $\frac{147}{32}$, ...					

Figure 4.10. List in grid form for Part Two.

examples of a finite arithmetic sequence, a finite arithmetic series, a finite sequence that was neither an arithmetic nor a geometric sequence, an infinite sequence that was neither an arithmetic nor geometric sequence, an infinite geometric series, and an infinite geometric sequence.

The posttest for Part Two. The posttest for Part Two was either in booklet or grid form and had the same computations as the lists of that part. The first two items were a geometric sequence and an arithmetic sequence from form A of the lists. The next two items were new examples of a geometric sequence and an arithmetic sequence.

The final examination questions. Five problems were included on the final examination for the college algebra course that was given ten days after the completion of the study. These questions were to compute the twentieth and nth terms for an arithmetic and a geometric sequence and find the sum of the terms of an infinite geometric sequence.

Summary

A description of the planning of the study has now been given. This chapter contains a discussion of the experimental design, the planned procedures including the twelve treatments (eight for Part One and four for Part Two, the instructional materials, both tests and lessons, are in Appendix A), the criteria for selecting the unit to be taught, the unit that was selected, the hierarchy of concepts in the unit, the procedures for selecting the examples, the construction of the lists (both booklet and grid), and closes with a description of the posttests. Information concerning the execution of the study will be found in Chapter V.

National Evaluation Committee

Helen Bain
Past President
National Education Association

Lyle E. Bourne, Jr.
Institute for the Study of Intellectual Behavior
University of Colorado

Sue Buel
Dissemination and Installation Services
Northwest Regional Educational Laboratory

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Professor Emeritus
University of Chicago

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College of Education
University of Toledo

Chester W. Harris
Graduate School of Education
University of California

Hugh J. Scott
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National Evaluation Committee

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R & D Center

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Director
R & D Center

Len VanEss
Associate Vice Chancellor
University of Wisconsin–Madison

Dan Woolpert
Director, Management Systems
R & D Center

Faculty of Principal Investigators

Vernon L. Allen
Professor
Psychology

B. Dean Bowles
Associate Professor
Educational Administration

Frank H. Farley
Associate Professor
Educational Psychology

Marvin J. Fruth
Associate Professor
Educational Administration

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Associate Professor
Mathematics

Frank H. Hooper
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Child Development

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Educational Psychology

Gisela Labouvie
Assistant Professor
Educational Psychology

Joel R. Levin
Associate Professor
Educational Psychology

L. Joseph Lins
Professor
Institutional Studies

James Lipham
Professor
Educational Administration

Wayne Otto
Professor
Curriculum and Instruction

Robert Petzold
Professor
Curriculum and Instruction

Thomas A. Romberg
Associate Professor
Curriculum and Instruction

Dennis W. Spuck
Assistant Professor
Educational Administration

Richard L. Venezky
Associate Professor
Computer Science

Larry M. Wilder
Assistant Professor
Communication Arts