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ABSTRACT

This study compares and contrasts the status of the three main areas of mathematics in the secondary school and the education of mathematics teachers in the United States with that in Great Britain. Seven questions are posed at the start of the study. Comparisons are drawn between the two countries in view of these questions, and a chapter near the end summarizes the findings. Nine principles are stated as a basis for a method of training, and the final chapter evaluates the strengths and weaknesses of the training programs in both countries with regard to the stated principles. (LS)

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE FOURTEENTH YEARBOOK

THE TRAINING OF MATHEMATICS TEACHERS

FOR SECONDARY SCHOOLS IN ENGLAND AND WALES AND IN THE UNITED STATES

By Ivan Stewart Turner, Ph.D.

BUREAU OF PUBLICATIONS
TEACHERS COLLEGE, COLUMBIA UNIVERSITY
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1. Create and maintain interest in the teaching of mathematics.
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3. Help the inexperienced teacher to become a good teacher.
4. Help teachers in service to become better teachers.
5. Raise the general level of instruction in mathematics.

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EDITOR'S PREFACE

THIS is the fourteenth of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. The titles of the first thirteen Yearbooks are as follows:

1. A Survey of Progress in the Past Twenty-Five Years.
2. Curriculum Problems in Teaching Mathematics.
3. Selected Topics in the Teaching of Mathematics.
4. Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910.
5. The Teaching of Geometry.
6. Mathematics in Modern Life.
7. The Teaching of Algebra.
8. The Teaching of Mathematics in the Secondary School.
9. Relational and Functional Thinking in Mathematics.
10. The Teaching of Arithmetic.
11. The Place of Mathematics in Modern Education.
12. Approximate Computation.
13. The Nature of Proof.

The present study deals with the training of mathematics teachers in the United States and in England and Wales. It is to a large extent a comparative study and one in which all teachers of mathematics in these countries are sure to be interested. After all, our problems are very much the same. Moreover, it is becoming increasingly clear that one of the most necessary improvements in connection with the keeping of mathematics on a high plane in this country involves the better training of teachers to work in the schools.

In this study, Dr. Turner prepares a set of guiding principles to be followed and then goes on to attempt to analyze what kind of success is being met in the three countries concerned in these matters. All teachers of mathematics will be interested in this new yearbook.

I wish to express my personal appreciation as well as that of the National Council of Teachers of Mathematics to Dr. Ivan Stewart Turner for permitting us to publish this contribution to mathematical education as the Fourteenth Yearbook of the series.

W. D. REEVE.

ACKNOWLEDGMENTS

MANY PERSONS gave valuable assistance and advice to the author during the preparation of this study. Consequently it is with real pleasure that he welcomes this opportunity to express to them his sincere appreciation and to offer to them his thanks: to his wife Enid Florence Turner for continued encouragement and practical help without which the study could not have been carried to completion; to his sponsors, Professors W. D. Reeve and I. L. Kandel, Teachers College, Columbia University, for critical guidance and friendly counsel during all stages of the preparation of the study: to Professor H. R. Hamley, Institute of Education, University of London, for initial encouragement to undertake the study and for assisting in the collection of data; to Professor Homer Howard, University of Florida, for his personal interest, frank criticism, and continued encouragement; to Professor E. S. Evenden, Teachers College, Columbia University, for valuable criticisms of certain early sections of the study; to the large number of persons, many of them not known personally to the author, who gave of their time to complete a questionnaire prepared by the author; and finally to the large number of other persons with whom the author had the privilege to discuss many of the problems raised in this study.

Many institutions, public examining bodies, and publishing companies in England and Wales and in the United States also rendered valuable help by granting to the author permission to use data in their possession and to quote from their publications. For assistance of this kind the author gratefully acknowledges the contributions to the study made by the following: In England and Wales—The University of London; University of Liverpool; University of Wales, Institute of Education, London; University of Sheffield; Oxford and Cambridge Schools Examination Board; the Oxford Delegacy for Local Examinations; the Cambridge Local Examination Syndicate; the University of Bristol Schools Examina

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I. S. T.

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FOR SECONDARY SCHOOLS IN ENGLAND AND WALES AND IN THE UNITED STATES

Chapter I

INTRODUCTION

TEACHERS OF MATHEMATICS no less than teachers of any other subject in any type of school should possess certain qualifications which are not specific to the subject which they teach. They should be persons of character, of personality, and of experience. These qualifications or attributes, however, are already with teachers before they undertake any kind of training which is consciously directed toward preparing them for teaching. Persons with these desirable qualities can only be enlisted for teaching when, on the one hand, those who possess these qualities offer themselves for teaching, and when, on the other hand, the methods employed for selecting candidates for teaching are sufficiently refined that these qualities will be readily discerned in those candidates who possess them.

It must be assumed for the purposes of this study that these two conditions are satisfied, and that the persons chosen as prospective teachers already possess these qualities or qualifications. If they do not possess any or all of them at the age when they offer themselves as candidates for teaching there is little chance that the course in professional training in the narrower sense will make up the deficiency.

Another qualification of a general kind which prospective mathematics teachers should possess is a broad cultural education. Contributions to this broad cultural education should be made by each of the several educational institutions in which prospective teachers study as they climb step by step up the educational ladder. The desirability of such a broad general education as a prerequisite for mathematics teachers in particular has been stressed recently in reports prepared by mathematical associations in England and in the United States. The following quotation will illustrate the type of comment made:

A teacher, to be of maximum service to the community in which he lives, should be recognized as an educated man to whom adult members of the community may turn for consultation on intellectual matters. He should be able to participate in community activities, and assume his share of leadership. Certainly he cannot function satisfactorily if he is notably ignorant in what are commonly regarded as fundamentals of general culture. With these facts in mind we advocate a breadth of training for teachers of mathematics which will insure a degree of familiarity with language, literature, fine arts, natural science and social science, as well as mathematics.[1]

Richness of experience in the world beyond the walls of the several institutions in which their formal education has been carried on is also regarded as of the utmost importance for prospective teachers. From such rich and varied experiences mathematics teachers, no less than teachers of other subjects, should carry into their teaching a wealth of illustrations, a balanced judgment, and a knowledge of people.

Many educators have claimed, and still claim in some places, that the only additional qualification to be demanded of prospective teachers beyond the possession of the above mentioned qualities is a thorough knowledge of the subject they are going to teach, whether it be English, mathematics, languages, science, or what you will. In England and Wales, for example, it is not required by law that teachers before receiving appointments in secondary schools should have received any form of professional training. Indeed, many persons who possess scholarship and a broad general culture, but who lack any formal professional training, receive appointments to secondary schools at the present time. The emphasis in selecting teachers for the French lycées is also placed on qualities of scholarship and general culture to the almost complete neglect of formal professional training. These facts raise a question of considerable importance, namely: Is there any place for formal professional education in the narrower sense in the preparation for teaching of persons who already possess scholarship and a broad cultural education and who, in addition, possess in

[1] "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935), p. 272.

desirable measure the qualities of character, experience, and personality?

The thesis of this study is that there is a place of considerable importance for such formal professional education of teachers, in addition to the possession of the qualities already mentioned. The adequate training of mathematics teachers for secondary schools has been thought of as compounded of four parts: a broad general or cultural education, intensive training in the subject matter of mathematics, professional training in the narrower sense which restricts it to such studies as education and psychology, and to a period of practice teaching or interneship.[2] It is important to add to these aspects of training a fifth aspect, namely, that some training should also be given in a subject closely related to mathematics.

PURPOSE OF THE STUDY

It is the purpose of this study to examine in detail the methods employed for training mathematics teachers in secondary schools in England and Wales and in the United States. In so doing the study will examine, for the two countries, the kind of mathematics which is taught in the schools, the academic preparation in mathematics of mathematics teachers, and finally the professional education of mathematics teachers. In carrying out this plan an attempt will be made to answer a number of questions which previous studies in this field either have left unanswered or seem to have answered inadequately. These questions will be raised in the next section when the extant literature on the subject is examined. Further, a number of fundamental principles underlying the preparation of mathematics teachers will be made explicit, and they will be used to estimate the adequacy of the professional training of mathematics teachers in the two countries mentioned.

PREVIOUS STUDIES

Several previous studies have dealt with certain aspects of the training of mathematics teachers in England and Wales and in the

[2] These broad divisions in the training of teachers have been adopted by the North Central Association of Colleges and Secondary Schools in the United States, and by the Mathematical Associations of England and the United States respectively, in recent reports and studies prepared by them.

United States. Attention was drawn to the problem early in the present century by the International Commission on the Teaching of Mathematics which met for the first time in 1908. Arising out of the first meeting of this Commission in Rome, commissions were set up in various European countries and in the United States to report on the current practices of training mathematics teachers in these countries. Among the reports so prepared were reports on the subject for England and for the United States. These reports were published in 1911.[3] Again in 1917 a survey of the training of mathematics teachers was undertaken and the results were published by the United States Bureau of Education.[4]

Similar reports on the training of mathematics teachers in various countries were prepared again in 1932, under the auspices of the International Commission and published in *L'Enseignement Mathematique* in 1933.[5] The report for England was brief and was reprinted in the *Mathematical Gazette*.[6] The report on training of mathematics teachers in the United States did not appear at that time. However, attention was given to the same problem a little later by the Mathematical Association of America, and a commission of this association finally reported in May, 1935.[7] This was the first report which did more than describe existing conditions. While its main purpose was to make definite recommendations for the training of mathematics teachers for colleges, this report also included a section on the training of mathematics teachers for secondary schools.

In 1931 32 the National Survey of the Education of Teachers was conducted in the United States, and the mathematical data from this survey were dealt with by Sueltz.[8] He reported prevailing conditions as to certification, amount of academic and pro-

[3] United States Bureau of Education: Bulletins 1911, No. 7, 12 especially.

[4] R. C. Archibald, *The Training of Teachers of Mathematics*, United States Bureau of Education Bulletin, 1917, No. 27.

[5] *L'Enseignement Mathematique*, XXXII, (1933), pp. 1-20, 178-183, 205-207.

[6] "Report of the Committee of the Mathematical Association on the Teaching of Mathematics in Public and Secondary Schools," *Mathematical Gazette*, IX (December, 1919).

[7] "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935).

[8] B. A. Sueltz, *The Status of Teachers of Secondary Mathematics in the United States*, 151 pp.

fessional training, amount of in-service training of two groups of mathematics teachers in the United States.

Furthermore the North Central Association of Colleges and Secondary Schools of America undertook to survey the training of teachers in 1934, and in the numbers of the *North Central Association Quarterly*[9] reports of the findings have been made. In this survey opinions of mathematics teachers were sought on the type of training mathematics teachers should receive.

Because of their special purpose, or because of the necessity for brevity, the studies cited have left unanswered, both for England and Wales and for the United States, a number of relevant questions, as follows:

1. How much mathematics is studied by secondary school pupils in the two countries, and what standard is achieved?

2. Is the academic training of the mathematics teachers adequate to guarantee reasonably good teaching of school mathematics at the standard required in schools?

3. What is the nature of and how effective are the present methods of professional training of mathematics teachers in the two countries?

4. What facilities are provided for in-service training of mathematics teachers? How effective are these facilities?

5. What agencies other than the regular teacher training institutions contribute to the training of mathematics teachers, and what is the nature and importance of their contribution?

6. What assumptions are implicit in the methods of training adopted in the two countries?

7. On what principles should the methods of training be based? In the light of these principles what are the strengths and weaknesses of the training of mathematics teachers in England and Wales and in the United States?

These are the questions to which reference was made in stating the purpose of this study, and to which answers will be offered as the study proceeds.

[9] "Subject Matter Preparation of High School Teachers," *North Central Association Quarterly*, X (April, 1935), pp. 396-402; X (October, 1935), pp. 219-255; XI (January, 1937), pp. 265-297; XII (October, 1937), pp. 265-268; XII (April, 1938), pp. 439-539.

SOURCES OF DATA

The data for this study were obtained from a number of sources. In the United States the active interest during recent years of a number of important educational associations has resulted in the publication of a number of comprehensive and important studies on teacher training from which a research worker must draw heavily. These reports on some occasions have provided the data required for the study, and on other occasions they have suggested what additional data needed to be collected for the purposes of a specialized problem. The catalogues of the teacher training institutions in the United States were used as an abundant source of information, though here again sufficient details on important points for this study were lacking. An attempt was made also to supply the information which these two sources failed to provide by sending a questionnaire to a number of training institutions. The institutions to which the questionnaire was sent were chosen to be representative both of their geographical distribution throughout the country, and of the proportion of universities, colleges, and teachers colleges within each geographical region.

Data on mathematics in secondary schools in the United States were obtained from accrediting agencies, from school districts and state departments of education, and from the public examinations boards. Finally, visits were paid to a small number of institutions in eastern states which were considered representative of each of the three types of teacher training institutions mentioned above.

The data for the study of the methods of training mathematics teachers in England and Wales were obtained by visiting twelve of the twenty-two institutions in those countries which train secondary school teachers; by examining the catalogues and calendars of some of those institutions not visited; by writing to these institutions for additional information on specific questions; by examining the syllabi, regulations, and copies of the mathematical examination papers of the eight public examinations boards in England and Wales, and the syllabi and mathematical examination papers of representative universities in which prospective teachers receive their academic training in mathematics; and by examining the publications of the Mathematical Association of England.

Chapter II

PRINCIPLES

IN THIS CHAPTER certain principles which are fundamental to the training of mathematics teachers will be defined. Nine such principles follow.

I. Perhaps the most important principle for the training of mathematics teachers for secondary schools is that *prospective teachers should have a thorough course of training in mathematics.* This training, wherever given, should be such as to provide teachers with knowledge of and experience in the subject considerably beyond the material they are required to teach in school. This principle raises questions both as to the type of mathematics which prospective teachers of mathematics should learn, and as to where, when, and by whom this academic training in mathematics should be given.

II. A second principle is that *this training should be given in a university, college, or institution of equivalent rank by teachers who are themselves mathematicians of outstanding competence and who, in addition, appreciate and understand the difficulties inherent in mathematics, whether it be regarded as a subject of learning or of teaching.* While the former condition is commonly fulfilled, the latter is by no means usually found in those who teach mathematics to future teachers of that subject. This second condition, however, is one of great importance, and the history of mathematical teaching in the United States and in several European countries provides suitable examples of what can be accomplished by mathematicians who have not scorned to show interest both in the content of mathematics taught in the schools and in the teaching of it to pupils. In support of this contention, mention need only be made of the pioneer work of Felix Klein in Germany, John Perry in England, E. H. Moore, and J. W. A. Young in the United States. While their work has been responsible for many important changes both in content and in methods of teaching elementary mathematics in the schools of their respective countries, their more important contribution has perhaps been in directing attention to the kinds of mathematical study which prospective mathematics teachers should be required to undertake. While the length of the course of training in mathematics for prospective teachers varies from country to country, it should not be regarded as satisfactory unless the amount of mathematics taught by the end of the course complies with the conditions of the first principle.

There remains the question of the type of mathematics to be taught to prospective mathematics teachers. It should be understood clearly at the outset what the two principles already stated do not necessarily imply regarding the selection of the content of the courses in mathematics. They do not imply that no heed should be taken of the fact that the student is training to be a teacher of mathematics and not an engineer or other person who may perhaps make some use of mathematics in carrying out his professional duties. Nor do they imply that the connection between the advanced mathematics and that of a more elementary level should be ignored during the stage when the student is acquiring his fundamental knowledge of mathematics. While there are courses which can be taught to all students of mathematics, no matter to what use they may put them subsequently, there are other courses which teachers of mathematics should take but which may not be necessary for, though of course not harmful to, persons whose later contact with mathematics will be to use it mainly as a tool subject.

III. The third principle, then, concerns the type of mathematics which prospective mathematics teachers should learn during the period of their own fundamental training. To use the words of T. P. Nunn, they should take such courses as "represent adequately the essential genius of mathematics." This implies that *mathematics teachers should study the important branches of pure mathematics, mechanics, the history of mathematics; the applications of mathematics to a number of other fields of learning; statistics; the* *fundamental principles of mathematics as, for example, study of its logical foundations, and particularly the essential connection between the various branches of advanced mathematics and their counterparts at the more elementary stage.* Support for this widening of the mathematical courses for prospective teachers is given in the following extract from the previously mentioned report of the Mathematical Association of America. For

. . . . in view of modern trends in the content of high school courses in the United States of America a bare acquaintance with algebraic manipulation is no longer sufficient even for a deadly uninspiring presentation of eighth or ninth grade mathematics. In various courses in mathematics at these levels it is indispensable for the teacher to have at his command a thorough knowledge of trigonometry, college algebra and the typical methods of analytic geometry. We also find that eighth and ninth grade mathematics involve considerable content whose background is found in the college courses in elementary physics, statistics, economics and the mathematics of investment. For a truly inspiring presentation of ninth grade mathematics, particularly to students who may plan to prepare for college entrance, the teacher should appreciate the significance of elementary mathematics in the light of important applications of more advanced mathematics. Such equipment requires far more than the mere minimum of training sufficient for a mechanical presentation of subject matter.[1]

Pure Mathematics. There is a fairly general agreement among teachers of mathematics and among persons responsible for training them as to the content of the courses in pure mathematics which prospective teachers of mathematics should study. A "common core," so to speak, of knowledge in pure mathematics will include knowledge of higher algebra, modern geometry (Euclidean), analytic geometry of two and three dimensions, differential and integral calculus (elementary and advanced), and trigonometry.

Mechanics. The place of mechanics in a course of study varies in different countries. In the United States it is normally included in the physics course; in England and Wales and on the Continent in the mathematical courses. But whether it is included with

[1] "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935), p. 273.

physics or with mathematics prospective teachers of mathematics should have considerable acquaintance with the subject matter of mechanics because of the application of the mathematical principles and techniques which its study requires, and because of its richness in bringing together, illustrating, and using so many branches of elementary mathematics. The common core course in mechanics would involve topics in the fields of statics, dynamics, and hydrostatics.

Specialization. Beyond this common core of mathematical subjects some specialization is definitely desirable, this specialization to be in one of the many advanced branches of pure mathematics and mechanics that are now taught in the graduating classes in institutions of higher learning and in courses leading to a higher degree in mathematics.

Various opinions may be held as to the extent of these specialized courses in mathematics for teachers of mathematics in secondary schools. The point of view taken in this study is that they should carry the student sufficiently far into the subject to enable him to read relevant important treatises and monographs on some special aspect of the subject. Support for this point of view is to be found in the following quotations:

But if the teaching of mathematics is to be put on a really sound footing the teachers themselves must be trained to an appreciation of the research ideal in mathematics. Why should a teacher of mathematics be ignorant of the work that is going on in the world of science? . . . What is wanted is that every university student who intends to qualify for teaching mathematics should attend one course of lectures on one particular branch of higher mathematics, given by a professor who is an authority on that special subject and who has carried out researches in it. . . .

In addition the teacher requires some general kind of knowledge about certain important branches of current research. No man would be considered properly qualified to teach physics in a school who was not at least aware of the existence of Röntgen Rays, radium and wireless telegraphy. The mathematical master should at any rate be equipped with a similar small modicum of information regarding such modern fields of discovery as the Theory of Groups and Non-Euclidean Geometry. As he will largely be occupied in teaching geometry, his knowledge of the latter subject should extend at least to the contents of one of the little pamphlets on 'our conceptions of space' which are intended for non-mathematical readers. Otherwise he cannot form a proper appreciation of the geometry which he is teaching and he may lay undue stress on points which could with advantage be omitted.[2]

In the fields of his specialization, the student should make the acquaintance not only of the treatise-literature of his field; he should also know the monographic literature, both in some important corner of present day research and through the classic monographs in his field; he should know the principal technical journals reporting discoveries first hand and furnishing authoritative reviews of additions to the literature of his field. . . . In his junior and senior years (i.e. third and fourth years) of college work the acquaintance with the periodical literature of his field should extend to the journals of at least one non-English-speaking country.[3]

The course in algebra for teachers should have as one of its aims the acquaintance of the student with present day investigations in algebra. They should be introduced to topics in which they can do reading and research, and thus keep alive their interest in the subject which is going to feed and clothe them. The course should point out to them the beautiful woods and valleys through which the paths of mathematics may lead them, and certainly a few of the mountain peaks which have been as insurmountable as Everest.[4]

Depth of preparation is further promoted in most institutions by pursuing at least one field, or section of it, *to its frontier*, thus acquiring a mastery of its logic of organization and of its process of growth and expansion. Indeed, opportunity is provided for students to participate in the intensive study of frontier problems and to experience the thrill of intellectual discovery. Such participation is essential if they are to follow intelligently during subsequent years the important developments in their respective fields of specialization. The statement should be added that the basic purpose of frontier study by prospective teachers

[2] G. H. Bryan, "The Uses of Mathematics in the Training of the Mathematical Teacher," *Mathematical Gazette*, IV (March, 1908), pp. 221-225.

[3] W. C. Bagley, "Professionalized Subject Matter in Teachers' Colleges," *Mathematics Teacher*, XXVI (May, 1933), p. 273.

[4] W. H. Erskine, "The Content of a Course in Algebra for Prospective Secondary School Teachers of Mathematics," *American Mathematical Monthly*, XLVI (January, 1939), pp. 33-34.

is to insure greater depth of understanding rather than to develop the controls essential in laboratory research.[6]

It is desirable to have in a secondary school a mathematical staff trained in diverse fields of specialization in mathematics so that care can be taken of pupils of all stages of ability, including slow moving pupils, pupils of average ability, and pupils of genius.

History of Mathematics. A knowledge of the history of mathematics is regarded here as an essential part of the equipment of mathematics teachers. Without this knowledge many of the concepts which they teach must remain barren and unilluminated even to themselves. Because of the vast extent which this subject has assumed during recent years a selection of topics must be made for (or by) prospective teachers. The topics must not be discussed solely in chronological order but emphasis must be placed on those interconnections between topics which a study of the history alone can reveal. This aspect of the history of mathematics will be expanded in a later section. The selection of a minimum list of topics to be examined from the historical point of view will naturally be guided largely by the content of the school courses in mathematics. But even the topics in these school courses provide ample opportunity for historical study of the most valuable kind. No teacher for example can teach the binomial theorem as it should be taught if he does not believe from a study of its history

that the binomial theorem is romantic . . . Yet the binomial theorem is full of romance; it throbs with human interest. It is an adventure into the unknown. Lead the learner on by judicious encouragement to generalize: $(1 + x)^2 = 1 + 2x + x^2$, $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$, etc. into $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}, x^2 \cdots + x^n$, where x is any number and n is any positive whole number. He cannot but see in this a conquest of territory visible to the eye, not mysterious or clothed in the shadows of night but the necessary consolidation of empire, the appropriation of lands without which present possessions are only loosely held. But should not the young learner palpitate with excitement

[6] W. S. Gray, "Report on General and Specialized Subject Matter Preparation of Secondary School Teachers," *North Central Association Quarterly*, X (October, 1935), pp. 253–254.

when urged to venture into uncharted depths and take n to be absolutely anything? The series of terms

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots$$

has no end if n is not a positive whole number: it goes on and on indefinitely to infinity. Does this mean anything? Can, for instance, the square root of $1 + x$ be as much as a whole and unlimited series of terms, all following one another in logical and inexorable law; and taking us—who knows whither?[6]

The study of mechanics is illuminated by a knowledge of the historical development of the concepts of statics and dynamics as they have been collected and cogently presented by Ernst Mach in *The Science of Mechanics.*[7] The study of plane geometry and of the geometry of conic sections does not lose value and interest for teachers who have a close acquaintance with the critical translations of the works of Greek mathematicians by such scholars as T. L. Heath. The history of the calculus is a fascinating and illuminating story for teachers of mathematics to know, and a clear understanding of the historical development of the principles of the integral and differential calculus should be a necessary part of the training of all mathematics teachers, such an account, for example, as has been given recently by Boyer.[8] Nor should the teachers' acquaintance with the history of mathematics be confined always to competently written chronological histories of mathematics; it should include first hand study of such translations as are available of original papers and books on important topics in the elementary branches of mathematics. In this connection D. E. Smith's *Source Book in Mathematics* and *Rara Arithmetica* may be cited. Where the portions of original articles translated in this book are found to be too slight recourse may be had to the English translations of other works which are of great importance in the history of mathematics and which bear directly on the mathematics taught in schools. Books of this kind which have been translated into Eng-

[6] S. Brodetsky, *Sir Isaac Newton*, p. 16.
[7] E. Mach, *The Science of Mechanics.*
[8] Carl B. Boyer, *The Concepts of the Calculus.*

lish are Galileo's *Dialogues*, Descartes' *Geometrie*, Newton's *Principia*, Napier's *Logarithms*, and Dedekind's *Theory of Numbers*.

Support for the idea here presented is given by T. P. Nunn in a section of a report prepared by the Mathematical Association. In this article he states:

> The teacher of mathematics must learn that to teach mathematics is not merely to seize the pupil of a certain knowledge in Arithmetic, Algebra, Geometry and the rest, but to make him . . . an active intellectual adventurer in the realms of number and space, following up the traces of the great masters of mathematical thought and catching something of their spirit and outlook. To make the best use of this discovery the student must work over again the familiar fields of elementary mathematics, studying it this time as a critic interested in finding out whence mathematical thought springs, how it develops and whence it leads.[9]

Coordinating Principles and Foundations of Mathematics. Closely linked with the introduction of the history of mathematics in the mathematical courses for prospective teachers of mathematics is that of a course on what might be called the foundations of mathematics. While a knowledge of the point of view of the three principal schools of thought in the foundations of mathematics is desirable for teachers, it is not exclusively this kind of knowledge which is understood to be included under the above heading. The course of study contemplated here is to be one which reveals to the student the essential connection between topics in elementary mathematics and their counterpart in advanced mathematics. The essential starting points are the topics of elementary mathematics which are taught in schools. These topics will be pursued further and further into the field of advanced mathematics and will be illuminated by the discoveries in these fields. An illustration of what is meant here is nowhere more clearly provided than in the treatment of the various topics in Klein's *Elementary Mathematics from an Advanced Standpoint*, or in a more specialized field in Dantzig's *Number, the Language of Science*. It is important to realize that the coordination of topics discussed here implies that

[9] "Report of the Committee of the Mathematical Association on the Teaching of Mathematics in Public and Secondary Schools," *Mathematical Gazette*, IX (December, 1919), p. 417.

students who are to undertake this course shall have been thoroughly trained in the appropriate branches of higher mathematics. The argument for this kind of course and for the preparation necessary for profiting from it is given by Carslaw:

The commission also recommend emphatically that at the end of the general studies in pure mathematics a course be given *organizing the entire mathematical material according to its essential interrelations*, and as far as possible presenting the import of the higher branches for the different stages of school mathematics. For, in fact, *experience teaches that without such a course of study, the majority of students do not discover the inner bond that connects the various parts of mathematical sciences*, and thus the prospective teacher loses what should be for him the real gist of his mathematical studies. To avoid misunderstanding, we add expressly that *this course presupposes matured hearers, and should not be brought down to the level of those preparing to teach mathematics as a minor subject only.*[10]

An illustration of the way in which this treatment of an elementary topic in its various higher ramifications can be done was recently given in a course of lectures for teachers by Professor Richart Courant of New York University, and formerly of Göttingen University, Germany. In lectures on maxima and minima the topics ranged from classical problems soluble by elementary methods to problems on minimal curves and surfaces involving the use of calculus of variations. A final lecture was devoted to a demonstration of minimal surfaces made with the aid of wire frames and a soap solution. Here in a few short lectures was opened up and illustrated a whole set of ideas such as no intensive working at problems on maxima and minima in the orthodox course in the calculus could possibly yield.

Other topics for treatment by this method are readily found in all of the mathematical subjects of the school course—arithmetic, algebra, geometry, trigonometry, and mechanics.

Statistics. Because of the considerable use that is made of statis-

[10] H. S. Carslaw, "On the Constructions Which Are Possible by Euclid's Methods," *Mathematical Gazette*, V (January, 1910), p. 171 (footnote). Professor Carslaw is quoting from the report of a Commission of the Societies of German Natural Scientists and Physicists on the reform in the teaching of mathematics and the natural sciences in Germany. Passages were underscored by the writer of this study.

tics in connection with educational problems, and the misuse of them by persons not adequately trained in the mathematical theory and interpretation of statistics, it is becoming more and more important that teachers of mathematics should include in their preparation a course in statistics, where the emphasis is divided between a knowledge of the mathematical development of the subject and a sound training in statistical inference.

Applications of Mathematics. In keeping with the broad cultural training which is recognized as an essential qualification of a good teacher, a mathematics teacher should be aware of the important applications of his subject to as wide a variety of other subjects as possible. This implies that at the very least the teacher should know the language of the sciences or other subjects to which the processes of mathematics are applied, and that preferably he should have considerable knowledge of those particular phases of the subject to which the application of mathematical methods is appropriate. The following list of subjects to which mathematics is applied suggests two things. It suggests both the growing importance of mathematics and also the responsibility that is thereby placed on mathematics teachers of being able to make use of their knowledge of these applications of mathematics to enrich their own teaching:

Physical sciences	Navigation
Finance	Astronomy
Biological sciences	Social studies, particularly economics
Surveying	

It is obvious that to include in their mathematical preparation even the minimum requirement outlined in the foregoing pages prospective teachers of mathematics will need to spend a considerable period in study at an appropriate institution. At this point no attempt will be made to propose plans for the organization of courses to achieve the stated aims. However, suggestions for modifying the present organization of courses for mathematics teachers in training institutions in England and Wales and in the United States will be given in the appropriate places in this study.

IV. A further principle relating essentially to the academic training of mathematics teachers is that *during the period of their fundamental training they should make a less intensive study of some subject, preferably one closely related to mathematics.* Until recently it would have been assumed without question that the physical sciences were the only subjects which obviously suited this principle. However, the subjects mentioned in the previous section of this study show that the physical sciences, and in particular physics, no longer remain unchallenged as subjects in which applications of mathematics may be sought. Nevertheless, the importance and extent of the concepts of the physical sciences, their dependence on mathematics for their development on the theoretical side, and the ready applications they afford of even elementary mathematics make of the physical sciences, and particularly of physics, an admirable second study for persons whose chief interest is in mathematics. It must be repeated, however, and it is increasingly true in practice, that because of their pertinence in affording practical applications of mathematics, any one of the other subjects mentioned forms a suitable subsidiary study for students who are preparing to teach mathematics.

V. A fifth principle relates to academic training in service. No active teacher of mathematics can hope to keep pace either with the growth of the special branch of mathematics in which he is interested or with the increasing number of applications of mathematics to wider fields of learning. Nevertheless, it is a principle of this study that *the teacher, during his period of active service, should strive to progress in his acquaintance with and mastery of many aspects of mathematical knowledge.*

Thus teachers may continue the study of that branch of advanced mathematics in which they specialized in college, or they may begin the study of an entirely new branch of elementary mathematics, or devote attention to those portions of the history of mathematics which especially interest them, or read the literature of a semi-technical character which appears from time to time in the less advanced mathematical journals. By doing some or all of these things teachers will be promoting their own professional growth.

The training of mathematics teachers in the various branches of mathematics has long been accepted as a necessary part of their preparation for teaching. On the other hand, the need for training in the teaching of arts and crafts appropriate to mathematics has gained recognition much more slowly, and, likewise, the need for inculcating a professional point of view among mathematics teachers. This seems to be true of the representative European countries as well as of the United States. The need for training of this kind (here called professional training) was early recognized by teachers' associations, and the mathematical associations of the various countries have taken an increasingly active part in the campaign for better professional preparation of mathematics teachers.

It should be understood that, while these associations of mathematics teachers agree on the principle that professional training of some kind should be given, there is no general agreement among them either about the manner in which this training should be conducted or about the most suitable stage for commencing it. This fact, however, does not affect the principles herein discussed, and a critical examination will be made later of the position of each association.

The general position favoring professional training for mathematics teachers has been set forth by Nunn in a section of the report cited earlier in this chapter. The aim of professional training is

> not merely to turn out a competent craftsman, but to form a young man or a young woman into an enlightened member of a body that has an enormous responsibility for the well being of a nation. While he [the young graduate] is still warm with the generous and universal spirit that university life should have awakened in him, he should be led to inquire into the meaning of education and to understand something of its significance in relation to the many-sided business of life. He should learn how much wider education is than mere teaching, and should gain inspiration and right direction from those who have reflected most deeply upon it. . . . And he should have his bias as a specialist corrected by observing how all the major subjects of the curriculum answer to deep seated needs of the human spirit and represent essential currents of the great stream of movement called civilization.[11]

[11] "Report of the Committee of the Mathematical Association on the Teaching of Mathematics in Public and Secondary Schools," *Mathematical Gazette*, IX (December, 1919), p. 418.

. It is a major premise of this study that there is a definite need for professional training of mathematics teachers in the narrower sense in which the word professional is used here. In elaboration of this premise a number of principles will be formulated.

VI. One principle relating to the professional training of teachers is that *professional training* (in the restricted sense in which that term is here used) *is a necessary part of the pre-service training of mathematics teachers.* During recent years this principle has won the recognition of associations of mathematics teachers in the United States and in various European countries.

VII. A seventh principle may be stated thus: *The content of this course of professional training should be organized principally for the purpose of training teachers for teaching mathematics.* "It is not sufficient for him [the student] to learn the common arts of exposition and class management; he must learn what form these arts assume when applied, for example, in the field of mathematical teaching."[12]

In their "Report on the Training of Teachers of Mathematics" the Commission of the Mathematical Association of America emphasized the importance of what they called "specialized preparation for teaching mathematics." For example, referring particularly to the training of college teachers of mathematics, though they agree that the same arguments are applicable to the training of mathematics teachers in secondary schools, they say:

> We believe that it would be highly desirable for graduate students, however brilliant mathematically, to complete successfully the training described in (3a) and (3b) before they are given independent control of classes in college mathematics.[13]

The statements referred to in (3a) and (3b) deal respectively with observation and practice teaching in mathematics.

The implications of this principle are important for this study. In the first place it implies that adequate provision should be made

[12] *Ibid.*, p. 418.

[13] "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935), p. 268.

in the program of professional courses for a thorough study of the methods of teaching mathematics to pupils of various ages and of various grades of ability in mathematics. Among the more fundamental problems to be discussed in these courses on the teaching of mathematics are:

1. The position of mathematics in a liberal education.
2. Recent history of mathematical teaching and the present-day reform movements in the teaching of mathematics.
3. The principles governing the teaching of the various mathematical subjects—arithmetic, algebra, geometry, trigonometry, mechanics. This should include both a statement of the objectives to be realized in teaching each subject, and also an account of the teaching problems in the more fundamental topics in each subject.
4. The place, purpose, and methods of framing and marking examinations in mathematics.
5. The selection of the suitable school textbooks in mathematics.
6. The psychological principles involved in the learning process, when these seem to have been established after careful investigation.

It is possible also that some of the branches of mathematics listed earlier as courses for the academic training of mathematics teachers, such as the history of mathematics, statistics, coordinating principles and foundations of mathematics, might be studied with more profit at the time when the students are reviewing their acquaintance with and improving their knowledge of the mathematics they are preparing to teach than when they are busy acquiring their fundamental training in mathematics. Support for this alternative is to be found among certain teacher training groups in the United States. For, as Peik expresses it, the point of view of these groups is that "Professionalization of the teaching fields in specialized intensive courses is needed rather than treatment throughout the subject matter courses."[14] And in the second place it implies that arrangements should be made whereby prospective teachers of mathematics may have the opportunity both to teach mathematics

[14] W. E. Peik, "The Relationship of General and of Professional Education in the Preparation of Teachers," *American Association of Teachers Colleges*, Fifteenth Yearbook, 1936, p. 136.

for a considerable period to pupils at different stages in their mathematical education, and to observe the teaching of some competent teachers of mathematics. The point of view of this study is that purposeful, well supervised teaching should be done by the prospective teacher during this period. It is therefore regarded as of the greatest importance that a student should teach and observe under the direct supervision of a competent mathematics teacher in the secondary school who is willing and for whom the necessary additional time has been provided to do this kind of work.

This experience under the constant supervision of a responsible and competent teacher is to be thought of as providing opportunity for the student to observe good mathematical teaching to pupils of different levels of mathematical education, and to participate in the work of constructing and marking the class exercises and examination papers and other routine work in the mathematics classes.

Support for these implications of the second main principle relating to professional training is forthcoming from the mathematics teachers' associations both in England and in the United States. For example, in their report already cited, the Commission of the Mathematical Association of America say that professional training should include:

The equivalent of one year of observation and assisting three times a week in various college courses in mathematics which are taught by experienced members of the department of mathematics.

Practice teaching in college mathematics under the observation of, and with later criticism by, members of the department of mathematics. This teaching might advisably be done in different courses and should amount to the equivalent of at least a two semester-hour course. The practice teaching should involve participation in the construction and grading of examinations.[15]

Again in the article on the training of teachers already referred to Nunn says:

Nothing could be, in the long view, more unwise than to exclude from the professional course those elements of breadth and liberality in which,

[15] "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935), p. 130.

as every one agrees, the chief virtue of academic studies resides. Nevertheless the course will miss its point unless it also includes an adequate training in the teaching craft that belongs to the students' special subject.

The first thing he should discover here is that the art of teaching . . . is a process whose aim is to guide the pupil's mental activities along a path of development in which he repeats and makes his own some of the great historic achievements of the science.[16]

It is not the purpose of this discussion to choose between the possible methods of organizing practice or student teaching, but to stress the need for such teaching experience as a necessary part of the professional preparation of mathematics teachers. The methods for doing this which have been developed in England and Wales and in the United States will be discussed in later sections.

VIII. The principle that *teachers of mathematics should be equipped to teach at least a second (and preferably an allied) subject implies that they should undertake a course of professional training in this second subject.* The professional preparation in the second subject should deal with the same types of problems as have been mentioned above, though of course fewer opportunities for actual teaching in the second subject will be available.

IX. A further principle is that *the period of professional preparation of mathematics teachers should include some courses in the theory and practice of education, and in psychology.* The purpose of the education courses has been well stated by Nunn:

He [the prospective teacher] must have such courses in education as will lead him to inquire into the meaning of education and to understand something of its significance in relation to the many sided business of life.[17]

The point of view of the Mathematical Association of America on these requirements in education is contained in the following extracts from the report previously cited:

[16] "Report of the Committee of the Mathematical Association on the Teaching of Mathematics in Public and Secondary Schools," *Mathematical Gazette*, IX (December, 1919), p. 418. [17] *Ibid.*

However, we believe that in the fields of pure psychology and educational theory there is much material which should be valuable as training for teachers of secondary mathematics. We are inclined to think that, outside of foundation work in psychology, all of the theory of education presented to the candidate for a secondary teaching certificate in mathematics could best be given in courses definitely oriented with respect to his major teaching field and containing only students whose major or minor interests are in this field. . . . In making a recommendation concerning training in educational theory and practice teaching, we shall not approve by name any of the variously labeled courses which appear in the requirements in Education for the high school teaching certificate in different universities. Among such course names we might mention Educational Psychology, History of Education, Adolescent Psychology, Educational Measurements, and so forth.[18]

Their definite recommendation regarding the place of these subjects in the training of a mathematics teacher is as follows:

Training in the theory of education and practice teaching:

A one-year course in methods of teaching and practice teaching in secondary mathematics, together with any distinctly pertinent material concerning educational measurements and other content from educational theory. It is our belief that this essential part of the student's training should, if possible, be under the direction of professors who have had graduate mathematical training, who have taught mathematics at the secondary level, and who have maintained contacts with the secondary field.[19]

The need for training of this kind is also clearly recognized by the advocates in the United States of the method of teacher training in which subject matter and professional subjects are developed together. Thus while subscribing to the need for the studies mentioned in connection with this principle they would organize the courses differently. In particular, they would arrange for some professional courses in each of the four undergraduate years rather than concentrate them into the last two years or last year of the course of training as do some of the other teacher training institutions.

18 "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935), p. 275.

19 *Ibid.*

In this study it is regarded as of fundamental importance that mathematics teachers, as well as teachers of other subjects, should have an extensive knowledge of those whom they are to teach. It is assumed that such factors as wide experience, broad cultural education, and actual practice in teaching must each make a contribution to the prospective teachers' understanding of children and adolescents. Moreover, the present principle intends that these experiences shall be supplemented by and interpreted in the light of relevant contributions from the fields of guidance, general psychology, educational psychology, and related studies which comprise the principal studies in the so-called professional subjects.

In stating the foregoing principles the purpose has been to provide criteria in terms of which the present methods of training mathematics teachers in England and in the United States can be evaluated.

Chapter III

SECONDARY EDUCATION IN ENGLAND AND WALES AND IN THE UNITED STATES[1]

ENGLAND AND WALES

PRIMARY SCHOOL[2] EDUCATION in England and Wales ends between the ages of eleven and twelve. Then the stream of primary school pupils is divided into three other streams. The pupils comprising the largest stream are those less proficient in academic subjects. They are guided into what are called senior schools and senior departments of elementary schools. The pupils comprising the second and third streams are those whose interests and capabilities are considerably alike. Because of this authorities have difficulty in dividing them into the remaining two streams provided for them, namely, those leading respectively to the central schools and to the secondary schools. The division of pupils who remain after the first segregation is made on the results of a competitive examination, called the Free Place or Special Place examination, conducted by the local education authorities in England and Wales. The subjects of the examination are English, arithmetic, an oral examination, and, less frequently, a general (or psychological) examination. On the results of this examination free places (involving free tuition) in secondary schools are filled, and admission to central schools is determined. The free places in secondary schools are the more coveted, since admission to these schools means that one has a foothold on the ladder which leads to the university. Consequently those receiving the highest scores in the examination accept places

[1] In the preparation of this chapter continual reference has been made to I. L. Kandel, *Comparative Education*, Chap. 8, pp. 637–674, pp. 790–826.

[2] Primary school education in England is provided by public authorities for children between the ages of six and eleven or twelve. The education of children who do not attend primary schools is carried on in preparatory schools up to about the age of thirteen. At this age pupils from these schools transfer to so-called Public Schools, which are in reality private schools.

in secondary schools. Admission to a central school, however, does not automatically prevent a pupil from reaching the university, since the curricula of many of the central schools in cities in England and Wales closely parallel those in secondary schools. The number of free places available in secondary schools varies with the type of school. In *maintained* or *aided* secondary schools, i.e., in schools which receive all or part of their financial support from the local education authorities, 25 per cent of the pupils must be admitted to free places. In 1937 some 47 per cent of all pupils in maintained and aided secondary schools in England and Wales held free places. Pupils who are admitted to secondary schools as fee payers are not required to meet the same standards as are those who win free places. Indeed fee payers may, and frequently do, gain admission to secondary schools in preference to pupils who have failed to win a free place but who rank higher on the Free Place examination list. Independent schools, of course, need make no such provision for free places.

Curricula of Secondary Schools. The number of subjects studied in the secondary schools in England and Wales varies from school to school. From data obtained from representative secondary schools in London, it would seem that the following subjects are usually included in the curriculum:

Subject	*Periods per Week (45 min. each)*
English	5
Mathematics	6
Languages (2)	6–7
Science	2
History	3
Geography	3
Crafts	2
Music	2
Scripture	1
Physical training	2
Drawing	2
Recreation	2
Total	36–37

From these data it will be seen that mathematics occupies an important place in the curricula of secondary schools in England and Wales, about one sixth of the class time of the pupils.

Examinations. The content of the subjects taught in secondary schools is to a considerable extent affected by extramural examinations since many of the pupils in them take one or both of the public examinations discussed herewith. One of these public examinations, the First Certificate or School Certificate examination, is taken at the end of the first four or five years of the secondary school course by pupils about sixteen years old. The other public examination, the Higher Certificate or Second School examination, is taken at the end of a further two years of study in the secondary school, by pupils about eighteen years old. These examinations are conducted annually by eight examinations boards[3] throughout England and Wales. The subjects offered in the School Certificate examinations are arranged in groups as follows:

First Certificate[4] Examination

I. Scripture knowledge
   English
   History
   Geography
II. Latin
   Greek
   German
   Spanish
   Italian
   Arabic

Higher Certificate[5] Examination

A. *Principal subjects*
I. Greek
   Latin
   Ancient history
II. English literature
   French
   German
   Spanish
   Italian
   Russian
   Modern history

[3] University of Bristol.
University of Cambridge, Local Examinations Syndicate.
Central Welsh Board.
University of Durham, School Examinations Board.
University of London, Matriculation and School Examinations Council.
Northern Universities Joint Matriculation Board.
Oxford and Cambridge Schools Examination Board.
Oxford Local Examinations Board.

[4] Oxford and Cambridge Schools Examination Board, *Regulations* 1938-39, pp. 57-58.

[5] University of Durham, Schools Examinations Board, *Regulations for the Award of School Certificates*, 1938-39, pp. 3-4.

First Certificate Examination	*Higher Certificate Examination*
III. Elementary mathematics	III. Pure mathematics
Additional mathematics	Pure and applied mathematics
Physics	Geography
Chemistry	Art
Physics and chemistry	Economics and accountancy
General science	Physics
Botany	Chemistry
Biology	Botany
IV. Music	Zoology
Drawing	Geology
Geometrical and mechanical drawing	Biology
Handicraft	
	B. *Subsidiary subjects*[6]
	The subjects listed above are also examined at a subsidiary standard, in addition to the following subjects:
	Scripture knowledge
	Church history
	Music

Candidates for the First Certificate examination may be examined in six to eight subjects. Those who pass with credit[7] in four or five subjects (including languages) may be admitted as matriculated students to the universities in England and Wales. But pupils who thus matriculate for a university do not proceed at once to a university. Instead, they remain at school for two years while they prepare for the Higher Certificate examination in fewer subjects. The examinations boards require candidates to offer three principal subjects or two principal and two subsidiary subjects at this examination. During the period of preparation for the Higher

[6] A subsidiary subject is one which is studied less intensively than a principal subject. As the list of subjects shows the same subject may be taken either as principal or as subsidiary subject.

[7] A "pass with credit" means a pass at a standard higher than is necessary for a bare pass.

Certificate examination there is considerable specialization in each subject.

In addition to these school certificate examinations, Entrance Scholarship examinations are conducted annually in a number of subjects, including mathematics, by groups of colleges in the Universities of Oxford and Cambridge. Some pupils in secondary schools take these examinations besides the two certificate examinations.

Summary. The features of secondary education in England and Wales which have been stressed in this section are:

1. The selective and restrictive character of admission to secondary schools of pupils between the ages of eleven and twelve.
2. The academic, and relatively fixed, curriculum of the secondary school.
3. The considerable proportion of time allotted to mathematics in the curriculum.
4. The importance of extramural examinations as determining factors in the content of the various subject matter courses.
5. The period of specialization following matriculation to the university and prior to entrance to a university.

THE UNITED STATES

One of the outstanding features to be observed by students of education in the United States is the intense public interest and activity shown in school matters. This public interest finds expression in the general acceptance of the principle that educational opportunities should be open to boys and girls as long and as far as they are capable of availing themselves of them. A consequence of the acceptance of this principle is seen in the high percentage of the available age group which actually attends secondary schools in the United States. By far the greater majority of secondary schools are publicly supported and locally controlled. In all states some measure of control is exercised by state departments of education.

The school years in the United States are spoken of by numbers and they range from the first school year at the age of about six years to the twelfth school year at the age of eighteen. These school years are sometimes called grades, the twelfth grade meaning the same as the twelfth school year. According to the local practice in a given part of the United States the secondary school period extends either from and including the seventh grade through the twelfth grade, or from and including the ninth grade through the twelfth grade. Where the former organization obtains, grades seven, eight, and nine constitute the junior high school stage and grades ten, eleven, and twelve the senior high school stage; where the latter organization obtains, grades seven and eight belong to the upper division of the elementary school and the high school comprises grades nine through twelve.

Pupils automatically enter public secondary schools at the age of about eleven years if they proceed to a junior high school, and at the age of about thirteen years if they enter the ninth grade of a high school.

Curricula. An important characteristic of secondary education in the United States is the wide range of subjects taught in the high schools. The number of such subjects taught varies considerably with the size of the high school, a greater number of subjects being offered in the larger city high schools. This wealth of subjects has resulted in the adoption of the elective system by means of which pupils are permitted to participate in the selection of the subjects which they will learn. This selection, however, applies to only part of their school program since certain subjects are required of all pupils and certain others are restricted electives, especially in grades seven through nine. This elective system has led to the organization of school subjects into units for which pupils earn credits. It had its origin in the latter part of the last century when Harvard College pioneered in organizing its curriculum on the elective basis. In time the schools followed suit, and to bring order into the resulting confusion both as to attainments to be expected of pupils in a given subject and as to the length of time for which that subject was to be taught, a definite unit was adopted. This unit prescribed the length of a time a subject was to be studied, but it did not prescribe either the topics to be taught or the degree of mastery that pupils were to achieve. The unit as well as the elective system has prevailed to the present day.

A unit of study in any subject is a year's study in that subject in a secondary school, constituting approximately a quarter of a full year's work. It requires the study of a subject for four or five hours a week for a thirty-six-week year. Thus four units of work represent the usual number taken by high school pupils in the United States in any one year. These four units, however, may be made up of half units as well as whole units. A half unit represents four or five hours work a week for a half year (semester). At the end of a four-year high school course a pupil should have earned credit for fifteen or sixteen units of work. In many institutions in different states pupils are required to have earned credits for English, social studies, some mathematics, some science, and sometimes a foreign language among the fifteen or sixteen credits necessary for graduation.[8]

Owing to the profusion of subjects from which pupils may make selections and the loss of prestige which mathematics as a school subject has suffered in recent years, a comparatively small percentage of pupils in high schools in the United States elect mathematics as a subject of study beyond the ninth year. For example, of all pupils enrolled in high schools in 1934 about 17.06 per cent took geometry and about 30.41 per cent took algebra.[9]

The mathematical subjects available for selection are usually elementary algebra, plane geometry, intermediate algebra, trigonometry, and solid geometry. A few schools, following the recommendations of the National Committee on Mathematical Requirements, have experimented with school courses in analytic geometry and calculus, and in some schools these subjects are now offered regularly. Most pupils take elementary algebra, intermediate algebra, and plane geometry.

Examinations. From the point of view of numbers of pupils directly taking them, it is true to say that public examinations do not loom very large in the scheme of secondary education in the United States, but as is shown later their indirect influence is considerable. There are two principal external examining bodies in the country, namely, the College Entrance Examination Board and

[8] P. Roy Brammell, *Articulation of High-School and College*, p. 47, Table 15.

[9] Carl A. Jessen, *Offerings and Registrations in High School Subjects, 1933-34*, p. 30, Table 2.

the Board of Regents of the University of the State of New York.[10] The former conduct examinations for entrance to universities and colleges, and they examine about 14,000 candidates twice annually. Success in the examinations conducted by this Board insures admission to colleges and universities; indeed, it represents the only avenue of admission to certain of these institutions. The latter conducts examinations called the Regents Examinations in the State of New York for pupils from the high schools of that state. Success in the Regents Examinations insures admission to colleges, universities, and teachers colleges.

The following groups of subjects are examined in the New York Regents Examinations:

English	Social studies
Classical languages	Commercial subjects
Modern languages	Physical science
Mathematics	Biological science

High school graduates who do not take these examinations gain admission to all state universities, colleges, and teachers colleges and to some private institutions on the basis of earned credits in high school, supported by the endorsement of the high school principal, and the evidence of a personal interview. Most of these institutions in the different states in the United States accept high school graduates who have earned fifteen or sixteen credits in high school. Where there is competition for places, however, private institutions and a few public institutions admit pupils who rank in the upper 50 per cent of their high school graduating class and who have included in their course certain specified subjects. In some states, state-wide tests have been devised for many subjects and satisfactory performance on these tests is accepted as the qualification for admission to universities, colleges, and teachers colleges. It should be said, however, that the standards set by the external examining bodies exert a considerable influence on the qualifications for admission to institutions of higher learning. An important influence on standards in school subjects is also exerted

[10] The University of the State of New York is the controlling authority in education in the State of New York, and is not a university in the sense in which that term is used in England and Wales or elsewhere in the United States.

by the five great accrediting associations[11] which prescribe conditions which schools must satisfy before they are accepted by these associations as duly accredited schools.

Important Educational Movements. The local character of the interest in and control of education in the United States has developed an attitude of experimentation in a number of local educational units. The experiments have usually sought to test some definite educational theory or to follow some new educational development. At the moment in the United States there are two important educational movements each of which has a substantial following. One movement, the essentialist movement, follows the more traditional path in education, and the other, sometimes called the progressive education movement, follows a different path. The precise function of each movement seems to be a matter of controversy at the moment. Some educators see only a difference in emphasis in the two movements.

So far as subject matter is concerned the position taken by exponents of the two movements is that for the essentialists subject matter is an end in itself and for the progressives it is a means. The latter tend to abandon the organization of the curriculum by subjects and to substitute for it an organization based on activities and centers of interest. Of special interest for this study is the fact that thirty schools are engaged in an experiment with curricula and procedures in keeping with the progressive movement. An important part of the arrangement is that pupils from these schools will be accepted into colleges and universities without examination after completing the curricula on which the schools have been working.

Summary. For the purposes of this study the following features of secondary education in the United States have been considered:

1. The large percentage of the appropriate age group in high schools with the corresponding heterogeneity of the high school population.

[11] The Association of Colleges and Preparatory Schools of the Middle States and Maryland.
The Association of Colleges and Preparatory Schools of the Southern States.
The New England Association of Colleges and Preparatory Schools.
The North Central Association of Colleges and Secondary Schools.
The Northwest Association of Secondary and Higher Schools.

2. The elastic nature of the high school curriculum and the influence of the system of electives in the subjects chosen for study by high school pupils.

3. The relatively unimportant place occupied by mathematics in the high school curriculum, and the fact that a majority of high school students study only elementary algebra and plane geometry.

4. The relatively small proportion of high school pupils which prepares for external examinations. It was pointed out, nevertheless, that the external examining bodies and the five large accrediting agencies exert a considerable influence on the standards in many of the high school subjects.

5. The arrangements whereby pupils gain admission to institutions of higher learning, namely, by means of (*a*) external examinations, (*b*) state-wide examinations in specific subjects, (*c*) the acquisition of fifteen or sixteen high school credits in approved subjects earned in high schools accredited by one of the associations of colleges and secondary schools.

6. The influence of two dominant philosophies of education on teaching in high schools in the United States.

7. The allotment of time units to each subject in the secondary school. A unit consists of four or five hours' work a week in a subject for a school year. Thus time devoted to a subject and not attainment in that subject becomes the important factor.

COMPARISONS

In the preceding brief sections on secondary education in England and Wales and in the United States it has been shown that in the former countries admission to secondary schools is highly selective while in the latter country admission is not selective. Consequently, the character of the school population in the secondary schools in the two countries differs widely both in native ability and in attainments. While the secondary schools in the United States enroll a group of pupils comparable in ability to the English and Welsh secondary school pupils, they also enroll a much larger group of pupils whose interests are not academic and whose native abilities vary widely. These latter pupils, therefore, create problems of organization and of curricula which are unknown in secondary schools in England and Wales, but which are cared for in the senior and central schools of those countries.

Since in England and Wales it has been necessary only to cater to a selected group of pupils in the secondary schools, a relatively fixed and academic curriculum has developed in these schools. Faced with the problem of educating pupils with a wider range of abilities, the schools in the United States have developed a very wide range of curricula and subjects of study to suit the interests and abilities of all pupils. At the same time they have organized a type of secondary school in which all subjects are offered for study and pupils are given the opportunity of electing some at least of the subjects which they will study.

The position of mathematics in the secondary schools of these countries reflects the above-mentioned difference in organization very well. In England and Wales it is prescribed and it occupies a relatively large proportion of the pupils' time; in the United States it is an elective, and consequently a small percentage of pupils studies mathematics in the upper grades of the secondary school.

The remaining feature to be compared in the secondary schools of the two countries is that of examinations. On the whole public examinations do not loom so large in the secondary schools of the United States as in the secondary schools in England, but in both countries they have considerable influence on the curricula and on the subject matter taught to those pupils who ultimately go on to college.

Finally, in secondary schools in England and Wales at the present time there is no movement which corresponds at all closely to that known as the progressive education movement in the United States.

Chapter IV

MATHEMATICS IN SECONDARY SCHOOLS IN ENGLAND AND WALES AND IN THE UNITED STATES

ENGLAND AND WALES

MOST OF THE CHILDREN who attend secondary schools in England and Wales take the First School Certificate examination, which includes mathematics, at the end of four years in school. (See Table I.) Therefore the subject matter which they are taught in mathematics lessons is made to conform in large measure to that covered by questions set in the public examination papers. Because of this, the examinations boards have, over the years, exercised a dominating influence over the mathematics curricula in secondary schools. While protests have been made from time to time against this dominance, especially by the Mathematical Association, it is true that these boards have modified their requirements in mathematics to some extent to conform to the recommendations of various reform movements, and they have encouraged schools to submit syllabi of their own for approval. In spite of the encouragement thus given to schools, and in spite of the boards' awareness of the need for reform, it is still true to say that the mathematics which the pupils learn in secondary schools is largely that required to pass examinations. It is inevitable, too, that the organization of the courses in mathematics during the years preceding the School Certificate examination should be such as to cover, in a systematic fashion, the range of work required for this examination. This does not imply that systematic cramming for an examination four years ahead is indulged in; indeed, many schools claim that no conscious effort is directed toward teaching for examinations till the last school year.

The point here made is that the requirements in mathematics of the various public examinations directly influence the kind of mathematics taught in the schools. Consequently, an analysis of the mathematics syllabi and papers of these public examinations will serve adequately to show how much mathematics is taught in secondary schools in England, and what level of attainment is expected in the various mathematical subjects.

It should be noted that in their recent report on secondary education (the Spens Report), the Consultative Committee of the Board of Education has made suggestions which may lead to a change in the position now occupied by mathematics in the secondary schools, and also in the type of mathematics to be taught therein. This is shown by the following quotations from the report:

We have said above that we believe that Mathematics should be taught as Art and Music and Physical Science are taught because it is one of the main lines which the creative spirit of man has followed in its development, and that if Mathematics is taught in this way it will no longer be necessary to give the number of hours to the subject that are generally assumed to be necessary.[1]

Again:

It is unfortunate that the mathematical teaching in Grammar [secondary] Schools has always tended . . . to pay far more attention to the logical arrangement and development of mathematical ideas in the abstract than to the utility of these ideas in actual life.[1]

Speaking of the separation of mathematics into separate subjects for teaching purposes the report says:

We have already deplored this separation of branches as distorting the characteristic architecture of mathematics. . . . . It means on the one hand that important mathematical ideas are presented piecemeal in so narrow a field that the pupils' conception of them lacks clarity. . . . In the same way we hold that the ideas of the calculus, both differential and integral, should be reached through the graph and through the course in algebraical methods before the majority of pupils leave school, and that the mathematical ideas or topics which are included in the school course of study should be deliberately selected to make this possible.[2]

[1] *Report on Secondary Education*, Consultative Committee of the Board of Education, 1939, pp. 235, 236.
[2] *Ibid.*, p. 237.

Finally:

We think we have said sufficient to show how the amount of time now given to Mathematics may be reduced without loss to the value of the subject. We realize that there are serious difficulties in carrying our proposals into effect both in finding teachers who are willing to abandon the traditional methods in which they have been brought up, and in the requirements of existing examinations.[3]

Mathematics for the School Certificate Examinations. The examinations boards prepare syllabi in elementary mathematics, additional mathematics, and mechanics for the School Certificate examination; and they also prepare syllabi in pure mathematics, applied mathematics, and pure and applied mathematics for the Second or Higher Certificate examination. The number of pupils taking the School Certificate and Higher Certificate examinations from grant-aided secondary schools in England and Wales are given each year in the report of the Board of Education.

TABLE I*

NUMBER OF PUPILS TAKING SCHOOL CERTIFICATE EXAMINATION, 1937

Examination	Subject	Number of Entries	Per Cent of Passes	Per Cent of Entrants Offering Subject	Total Number of Secondary School Pupils
First School Examination	Elementary Mathematics.....	71,124	55.5†	92.0	
	Additional Mathematics......	3,906	45.8	5.1	466,625
	Mechanics..................	1,660	50.3	2.1	
Second School Examination	Mathematics................	5,262	72.7	44.0	

* Compiled from: Board of Education, *Education in 1937*, pp. 143-144, Table 49.
† Pass with credit, i.e., at a standard rather higher than is necessary for a bare pass.

Table I shows that the overwhelming majority of pupils in secondary schools do not study mathematics beyond the School Certificate stage. Consequently the greater number of mathematics teachers in secondary schools are required to teach mathematics only to this standard. Nevertheless, a sufficient number of pupils elect additional mathematics at the School Certificate stage and also Higher Certificate mathematics to make it neces-

[3] *Report on Secondary Education*, Consultative Committee of the Board of Education, 1939, p. 242.

sary to train a considerable number of teachers who are qualified to teach mathematics at this higher standard.

Mathematics in the School Certificate Examination. Table II has been prepared to show the mathematical subjects comprising elementary and additional mathematics in the School Certificate examination. In addition to the names of subjects examined, the table also shows the time allowed for each paper and the number of questions in each paper candidates are required to attempt.

The Board of Education in 1933 published a report on the School Certificate examination. In this report an analysis was made of the mathematical examination papers. On the whole the report approved the standards attained and the type of examination papers set in elementary methematics. In connection with the examinations on additional mathematics the report stated,

> There is much diversity of practice among the Examining Bodies in regard to the conduct of the examination, the content of the syllabus, and the standard of difficulty of the papers. . . . A little more Algebra and Trigonometry may be needed, but time should be devoted mainly to the Calculus, Mechanics, and Coordinate Geometry.[4]

A. *Elementary Mathematics.* As Table II shows, arithmetic, algebra, and geometry are the chief mathematical offerings in elementary mathematics. Where trigonometry is not mentioned specifically as an examination subject, it may be included with any of these other three subjects.

Syllabi. The syllabi for the subjects of elementary mathematics, with the exception of the syllabus in geometry, are given by the examinations boards merely in outline. The geometry syllabus, however, contains a complete statement of propositions which must be known, and distinguishes clearly between those which may be assumed without proof and those for which a proof is necessary. Permission to quote the following extracts from syllabi and examination papers has been granted by the examinations boards concerned. Only one syllabus for each subject is quoted because the requirements of the various boards in each of the subjects comprising Elementary Mathematics are very much alike.

[4] *The School Certificate Examination*, p. 118.

TABLE

ANALYSIS OF EXAMINATION SUBJECTS IN MATHEMATICS AT THE SCHOOL CERTIFICATE EX-

Examination Board	Elementary Mathematics		
	Subject of Examination	Time Allowed in Hours	Number of Questions to be Answered
University of Bristol†	Arithmetic and trigonometry	2	8
	Algebra	2	8
	Geometry	2½	8
Oxford Local Examinations	Arithmetic	1½	8
	Geometry	2¼	7
	Algebra	1¾	9
	Trigonometry and mensuration	2	7
Oxford and Cambridge	1. Arithmetic, algebra, and geometry	2	7 (10)‡
	2. Arithmetic, algebra, and geometry	2	7 (10)
	3. Arithmetic, algebra, and geometry	2	7 (10)
Cambridge Local Examination Syndicate	Arit etic	2	9 (11)
	Geometry	2½	7
	Algebra	2	10
University of Durham	1. Arithmetic and trigonometry	2	10 (13)
	2. Algebra	2	8 (9)
	3. Geometry	2¼	7 (8)
Central Welsh†	Arithmetic	2½	7
	Algebra	2½	7
	Geometry	3	8
	Trigonometry (optional)	3	8
Northern University	Arithmetic	2	ઠ (11)
	Algebra	2	12 (14)
	Geometry	2½	8 (10)

* Syllabi and papers from seven of the eight Boards are here examined with the permission of the respective boards.

† In the syllabi of the University of Bristol Examination Board and the Central Welsh Board this subject is referred to as mathematics and not as elementary mathematics. Elementary mathematics there has a special significance which does not concern us here.

II

AMINATION OFFERED BY EXAMINATION BOARDS IN ENGLAND AND WALES: SUMMER 1938*

	Additional Mathematics		Mechanics	
Subject of Examination	Time Allowed in Hours	Number of Questions to be Answered	Time Allowed in Hours	Number of Questions to be Answered
Geometry and trigonometry	3	9 (10)‡	3	9 (10)‡
Algebra and calculus	3	9		
Higher geometry and trigonometry	2½	8		
Higher algebra and calculus	2	9		
Applied mathematics (statics and dynamics)	2½	7		
Geometry, algebra, trigonometry	2	8 (11)		
Statics and dynamics	2	8 (10)		
Coordinate geometry and elementary calculus	2	8 (10)		
1. Algebra, trigonometry, calculus	2	9		
2. Geometry, trigonometry, analytical geometry, calculus	2	9		
3. Statics	2	6		
4. Dynamics	2	6		
Two papers on algebra and elementary calculus	2		3	
Geometry, trigonometry	2			
1. Algebra and trigonometry				
2. Coordinate geometry and calculus	3	7 (10)	3	
3. Mechanics (whole paper or part taken)				

‡ Numbers in parentheses following the number of questions to be answered indicate the total number of questions on the paper.

ARITHMETIC[5]

Syllabus

Elementary principles and processes of arithmetic.

The principles of vulgar and decimal fractions (excluding recurring decimals).

Knowledge and use of the tables required, both in the English and Metric Systems, for the measurement of length, area, volume, capacity, weight, and time.

Mensuration of the rectangle, parallelogram, triangle, trapezium, and circle.

Averages, ratio, proportion, percentages.

Practical applications of arithmetic.

Questions on simple Numerical Trigonometry will be set and will be alternative to other questions.

(Note.—Great importance will be attached to correct working. The use of logarithms is allowed except in questions where they are expressly forbidden.)

The examination papers in arithmetic deal for the most part with problems on the work set out in the syllabus. Pure computation is either excluded or restricted to one or two simple questions. Following are typical examination questions in arithmetic taken from School Certificate papers for 1938:

Examination Questions

1. (i) Find in francs, correct to the nearest franc, the difference between £73:10*s*. and 10,250 francs, assuming that £1 = 138.75 francs.

   (ii) One-sixth of a man's weekly wage is spent on rent and five-eighths of the remainder on food. If 15 shillings then remain, find his weekly wage.

   (iii) A number is divided by 21 by dividing by the factors 3 and 7. The successive remainders are 1 and 4 and the quotient is 68. Find the number.

   (Central Welsh Board, 1938)

2. A long-distance flier kept up an average of 122.7 miles per hour, and was in the air for 2 days 2 hours 12 minutes. Find the length of the flight in miles.

   (Oxford Local Examinations, 1938)

[5] *Regulations for Inspection and Examination of Schools*, University of London, 1939, p. 116.

3. A company announced that in 1936 it made a net profit of £14,675; that it proposed to reserve for taxes £3470, to put £2000 to reserve, to pay a dividend of 60 per cent on the ordinary, 10 per cent on the preference shares, and that the credit balance would then be £1039.

The corresponding figures for 1935 were: profit, £11,172, taxes £2443, reserve £1500, dividends 50 per cent and 10 per cent, balance £272. Find

(i) the amounts distributed as dividend each year.

(ii) the amounts of ordinary and preference shares respectively.

(Bristol, 1938)

4. A rectangular piece of thin sheet metal $ABCD$, in which $AB = 13.2$ cm. and $AD = 30.5$ cm., is bent into a tube of circular section by bringing AD to coincide with BC. Find

(i) the radius of the cross-section of the tube,

(ii) the volume of water, correct to the nearest c.cm., that the tube could hold if it were closed with plane ends.

(Cambridge Local Examinations, 1938)

5. A manufacturer sells a bicycle to a wholesale distributing firm at a profit of 15 per cent. The distributor sells it at a profit of 25 per cent to a retailer who in turn sells it to a customer at a profit of 20 per cent. In each case the profit is calculated as a percentage of the cost price to the seller. If the customer pays £6. 18*s*., what was the cost to the manufacturer?

(Oxford and Cambridge, 1938)

ALGEBRA[6]

Syllabus

The construction and use of formulae; their inversion (change of subject).

The index laws for positive indices.

Equations of the first degree with one or two unknowns; easy problems.

Factors of simple algebraic expressions; remainder theorem; fractions.

Graphs of statistics and of linear and quadratic expressions; the equation of a straight line in the form $y = mx + c$; gradient of a line; maximum and minimum values.

[6] *General Regulations and Examination Schedule* Central Welsh Board, Annual Examination, 1939, pp. 21-22.

Quadratic equations with one unknown; also with two unknowns (one linear and one quadratic); easy problems.
Elementary examples of ratio and proportion.
Direct and inverse variation when one magnitude depends upon not more than two others; elementary graphical representations.
Arithmetical and geometrical series (not infinite).
Use of logarithms.

Numerical Trigonometry

Trigonometrical ratios for angles up to 90°; special cases of 0°, 30°, 45°, 60°, and 90°.
Graphical representation. Easy applications involving right-angled triangles; simple problems in heights and distances; bearings, such as N.W.; N. 60°E.
Use of trigonometrical tables.
The Algebra and Geometry papers may each contain a question on Numerical Trigonometry.

In the algebra papers about 40 per cent of the questions on five papers examined required a knowledge of the formal processes in algebra (factorization and fractions). The remaining 60 per cent of questions dealt with graphs, solutions of problems, and application of algebra generally. The following examination questions in algebra, selected from School Certificate papers for 1938, seem to be typical:

Examination Questions

1. Resolve each of the following expressions into factors:
   (i) $6x^2 + 11xy - 35y^2$;
   (ii) $3ac + 3bd - 9ad - bc$;
   (iii) $x^2 - 6cx + 9c^2 - 9y^2$.

(Oxford and Cambridge, 1938)

2. (i) If $u = \sqrt{a + bt}$, express t in terms of a, b, u.
   (ii) Solve the equation

$$\frac{5x + 2}{7} - \frac{x - 3}{5} - \frac{2x + 3}{2} = 0,$$

and verify that your solution satisfies the equation.

(Cambridge Local, 1938)

3. Find the first term and common difference of an arithmetical progression in which the 30th term is twice the 8th term, and the sum of the first 8 terms is 111.

(Oxford Local, 1938)

4. Draw the graph of $y = 2 + 2\frac{1}{2}x - x^2$ for values of x from -2 to $+4$.

Use your graph (i) to find the values of x when y is equal to $\frac{1}{4} - \frac{1}{2}x$, and (ii) to solve the equation $2 + 5x - 2x^2 = 0$

(Central Welsh, 1938)

5. A train covers the last third of a journey of 420 miles at an average speed 10 m.p.h. higher than the first two-thirds and does the whole journey in 7 hours. Find the average speeds over the two parts of the journey in miles per hour to 2 decimal places.

(Bristol, 1938)

6. ABC is a triangle, right-angled at B, and $AC = 14$ inches, angle $BCA = 67°$. On AC is drawn a rectangle $ACXY$ (XY is on the opposite side of AC from B), whose diagonals meet at M. Given that $AY = 9$ inches, calculate

(i) the distance from X to AB,

(ii) the size of angle YCB to the nearest degree, and

(iii) the distance from M to BC.

(Central Welsh Board, 1938)

GEOMETRY

Syllabus

As mentioned above, the syllabus in geometry is set out in much more detail than are the the syllabi in other subjects.

The theorems are arranged in groups each centering around the topics in the following list:

* Angles at a point.
* Parallel straight lines.
* Congruence theorems.
Loci.
Parallelograms.
Areas.
* Circles.
Similar triangles.

* Proofs of certain theorems in groups marked with an asterisk are not required.

The constructions which need to be done are included in the following list:[7]

Bisection of angles and of straight lines.
Construction of perpendiculars to straight lines.
Construction of an angle equal to a given angle.
Construction of parallels to a given straight line.
Simple cases of the construction from sufficient data of triangles and quadrilaterals.
Division of straight lines into a given number of equal parts or into parts in any given proportions.
Construction of a triangle equal in area to a given polygon.
Construction of tangents to a circle and of common tangents to two circles.
Construction of circumscribed, inscribed, and escribed circles of a triangle.
Simple cases of the construction of circles from sufficient data.
Construction of a fourth proportional to three given straight lines and a mean proportional to two given straight lines.
Construction of regular figures of 3, 4, 6, or 8 sides in or about a given circle.
Construction of a square equal in area to a given polygon.

The questions in the geometry papers usually require the writing out of a theorem and the solution of a rider (original) on this theorem. Data for questions are sometimes given on figures, and occasionally constructions are given. Following are typical examination questions in geometry taken from School Certificate papers for 1938:

Examination Questions

1. Construct a pentagon $ABCDE$ in which $AB = BC = CD = 1$ in., $\angle B = \angle C = 120°$. $\angle A = \angle D = 90°$. Produce DC, DE to meet AB in X, Y respectively. Calculate the magnitudes of the angles X, Y and show that A is the middle point of XY.

(Durham, 1938)

2. Assuming no other property of a parallelogram than the fact that its opposite sides are parallel, prove that its opposite sides and angles are equal.

[7] *Regulations for the Award of School Certificates*, University of Durham, School Examinations Board, 1938 and 1939, p. 15A.

Show that a parallelogram whose diagonals are equal is a rectangle, and construct a rectangle in which the diagonals are each 4 inches long and the area is 6 square inches, giving a brief explanation of your construction.

Measure the sides of the rectangle.

(Oxford Local, 1938)

3. Prove that the ratio of the areas of two similar triangles is equal to the ratio of the areas of squares on corresponding sides.

ABC is a triangle in which $AB = AC = 13$ inches and $BC = 10$ inches. The perpendicular from A to BC meets BC in P. S is the mid-point of AP and a line through S parallel to AC meets BA in E and BC in F.

Calculate the area of the triangle BEF.

(Central Welsh, 1938)

4. Prove that if one angle of a triangle is bisected internally, the bisector divides the opposite side in the ratio of the two sides which include the angle.

D is the middle point of the side BC of a triangle ABC; DX and DY bisect the angles ADB and ADC respectively, meeting AB at X and AC at Y. Prove that XY is parallel to BC.

(Cambridge Local, 1938)

5. In Figure 1 the angle BAC is a right angle; $ABDE$ is a square; BF is perpendicular to BC and meets ED produced at F; $AGHK$ is parallel to BF, and FK is parallel to BC. Prove that (i) $BF = BC$, (ii) the square $ABDE =$ the rectangle $BFKG$.

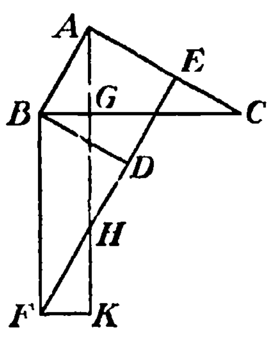


Figure 1.

(Oxford and Cambridge, 1938)

B. *Additional Mathematics.* Mathematics to the standard indicated above is taken by nearly all pupils in secondary schools in England. Those pupils who wish subsequently to study mathematics at a university are also examined in Additional Mathematics, the subjects of which are shown in Table II above. The extra work involved in this subject is illustrated by the following extracts from syllabi, and by the typical questions from examination papers in Additional Mathematics.

Algebra

Syllabus

(Additional to the algebra presented for the Elementary Mathematics)

Theory of quadratic equations.

Theory of indices and logarithms.

Permutations and combinations.

Binomial theorem for a positive index.

Topics less frequently required are:

Partial fractions, infinite geometric series, harder graphical work.

Examination Questions

1. In the expansion of $\left(2x^2 - \frac{1}{x}\right)^{15}$ find the term which is independent of x. Express your answer as the product of prime numbers.

   If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, prove

   (i) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

   (ii) ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$.

   (Oxford Local, 1938)

2. (i) Find the sum of n terms of the series whose nth term is $\frac{23}{10^{2n}}$.

   (ii) Express $0.\dot{2}\dot{3}$ as a vulgar fraction.

   (iii) Find the sum of n terms of the series $.23 + .2323 + .232323 + .23232323\ldots$, by first finding $(1 - 10^{-2})$ times this sum.

   (Bristol, 1938)

3. A prize for Mathematics and a prize for Science are awarded to the same class of thirty boys, and any boy may win both prizes.

   Stating your reasons clearly and without quoting any formula, find the number of different ways in which the prizes can be awarded.

   (Cambridge Local, 1938)

Geometry

Syllabus

The geometry syllabus requires harder work on the geometry of the elementary mathematics syllabus.
The geometry of the triangle.
Elementary solid geometry.

Examination Questions

1. L, M, N are the feet of the perpendiculars from a point D to the sides BC, CA, AB respectively of a triangle ABC; if D is on the circumcircle of the triangle ABC, prove, by quoting any theorem required, that $BL.CM.AN = BN.CL.AM$

(Cambridge Local, 1938)

2. Prove that, if R is the radius of a circumcircle of a triangle ABC,

$$R = \frac{a}{2\sin A} = \frac{abc}{4S},$$

where S is the area of the triangle.

(Consider the cases in which (i) A is acute, (ii) A is obtuse; you may assume the expression for the area of a triangle.)

If r, r_1, r_2, r_3 are the radii of the incircle and excircles of a triangle, prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

(Oxford Local, 1938)

Trigonometry

Syllabus

Graphs of trigonometrical functions.
Circular measure.
Additional formulas.
Solution of trigonometrical equations.
Solution of triangles.

Examination Questions

1. The angle of elevation of the top of a pole viewed from a point on a level with its foot is 40°. From a point on the same level 20 yards further away from the pole the angle of elevation is 25°. Ca' ulate the height of the pole and the distance of the first point of observation from its base.

(Oxford Local, 1938)

2. (i) Prove that

$$\frac{\sin 2A}{1 + \cos 2A} = \tan A = \frac{1 - \cos 2A}{\sin 2A}.$$

(ii) Find two angles between 0° and 180° that satisfy the equation

$$3 \cos 2x^\circ + \cos x^\circ + 1 = 0$$

(Oxford and Cambridge, 1938)

3. Show that $\cos x - \sin x = \sqrt{2} \cos (x + \pi/4)$.

Draw the graph of $y = \cos x - \sin x$ from $x = -2\pi$ to $x = 2\pi$ and use it to obtain the numerically least positive and negative angles satisfying $\cos x - \sin x = \frac{1}{8}$.

Deduce two solutions of this equation between 8π and 10π.

(Bristol, 1938)

Calculus

Syllabus

Differentiation of x^n.

Integration of x^n $(n - 1)$.

Application to problems on maxima and minima, areas, volumes, centers of gravity, and rate problems.

Examination Questions

1. Calculate the coordinates of the center of gravity of the area enclosed by the straight lines $x = 0$, $y = 0$ and the portion of the curve $y = 9 - x^2$ which lies in the first quadrant.

(Oxford and Cambridge, 1938)

2. The radius of a sphere is 5 in. Two parallel planes are drawn at distances 2 and 3 in. respectively from the center and 1 in. apart. Use the calculus to determine the volume of the slice of the sphere between the two planes. (Regard the sphere as formed by the rotation of the circle $x^2 + y^2 = 25$ about the x – axis.)

(Oxford and Cambridge, 1938)

Analytic (Coordinate) Geometry

(Required only by one Examining Authority)

Syllabus

Analytic Geometry of the straight line and circle.

Equations of tangents and normals to a circle.

Use in loci.

Elementary conic sections (parabola, ellipse, rectangular hyperbola).

Examination Questions

1. Prove that the line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

if $c^2 = a^2m^2 + b^2$

Find the equations of the tangents to the ellipse

$$4x^2 + 9y^2 = 1$$

which are perpendicular to $y = 2x + 3$.

(Oxford and Cambridge, 1938)

2. Find the equation of the tangent at the point $(2t, 1/t)$ to the rectangular hyperbola $xy = 2$.

Show that the product of the perpendiculars from the points (2, 2) and $(-2, -2)$ to this tangent is equal to -4.

(Oxford and Cambridge, 1938)

MECHANICS (Optional)

Syllabus: Statics. Lever, inclined plane, machines, friction, equilibrium.

Dynamics. Composition of velocities and accelerations, laws of motion, work and energy, momentum, projectiles.

Examination Questions

1. Prove that forces represented in magnitude, direction, and line of action by the sides XY, YZ, ZX of a triangle reduce to a couple whose moment is represented by twice the area of the triangle.

$ABCDEF$ is a regular hexagon. Eight forces proportional to AB, BC, CD, DE, EF, FA, $3AE$, $3DB$ act along these lines in the directions indicated. Prove that the forces are in equilibrium.

(Oxford and Cambridge, 1938)

2. A mast of length 60 ft. weighing 400 lb. is free to turn about its lower end and is kept in equilibrium at an angle of 60° to the horizontal by a rope, the lower end of which is attached to the upper end of the mast; the rope is inclined at an angle of 30° to the horizontal. If the center of gravity of the mast is 20 ft. from its lower end, calculate the tension in the rope.

(Cambridge Local, 1938)

3. A particle is projected from a point with a velocity of magnitude u at an inclination α to the horizontal. Show that after a time u/g

sin a the direction of its velocity has turned through a right angle and that its distance from the point of projection is then equal to that which it would have described in the same time, falling freely under gravity.

(Oxford Local, 1938)

4. An athlete holds a 16 lb. weight at rest at a height of 4 ft. above the ground. He then hurls it so that it leaves his hand at a height 6 ft. 3 in. above the ground with a velocity of 34 ft. per sec. Calculate in foot-pounds the total energy, kinetic and potential, which he has imparted to the weight.

If this action takes place in 1 sec., at what horsepower is the athlete working during that time?

(Oxford and Cambridge, 1938)

5. Two small equal spheres hang side by side from the same point on two strings each 4 ft. long; one of the spheres is then held with its string taut and making an angle of 45° with the vertical. This sphere is then released; it strikes the other sphere and adheres to it so that both move on together. Use the principle of conservation of energy to find the velocity immediately before impact.

Find also the common velocity of the spheres immediately after impact.

(Cambridge Local, 1938)

Review of the Examination Questions. The examination questions in the foregoing papers have been chosen to represent as fairly as possible the standard and extent of mathematical knowledge of pupils at the end of the first four years in a secondary school. For this purpose the questions were chosen from different places in each paper on the assumption that in general the easier questions occur in the earlier part of the paper and the harder questions toward the end. In this way also a wider sampling of the topics tested was obtained. The examination questions in each subject cover adequately the syllabus of work in that subject. They test manipulative skill in processes whose usefulness does not go unchallenged, at least in the United States; memorization of proofs (as in geometry) and ability to apply these skills and known results in the solving of problems. These problems do not reflect that the examiners are greatly concerned with "life situations." Indeed they do not always quite represent the type of problems to be found in the more modern textbooks which pupils use in preparation for the examination; this fact is responsible for some of the criticisms of the examinations boards that are made from time to time.

Mathematics in the Higher Certificate Examination. The foregoing account of the mathematics taught up to the First School Certificate stage by no means tells the full story of mathematical instruction in secondary schools in England and Wales. Pupils in secondary schools who proceed to the universities remain at school for two years beyond the First Certificate examination stage. At the Higher School Certificate examination, pupils take fewer subjects than they do in the First Certificate examination. The usual alternatives are to take three principal subjects, or two principal and two subsidiary subjects. Table III gives an analysis of the mathematical papers set in Higher School Certificate examinations by the four largest examinations boards in England and Wales.

Table III shows that candidates for the Higher School Certificate in mathematics may take mathematics either as two principal subjects or as one principal subject. Candidates who aim at gaining distinction in mathematics or at winning a University Entrance Scholarship or Exhibition in mathematics usually take mathematics as two principal subjects. When mathematics is taken as one principal subject one of the sciences is usually taken as the other principal subject. In the former case six or seven three-hour examination papers are set, covering a wide range of subjects in Pure and Applied Mathematics; in the latter case three or four three-hour examination papers either in Pure Mathematics or in Applied Mathematics are set, covering portions of the syllabi of the more advanced work. The table shows also that the main subjects of examination are algebra, geometry, trigonometry, calculus (integral and differential), analytic geometry, statics, dynamics (kinematics and kinetics), and hydrostatics.

The examinations boards prepare syllabi for each of the mathematical subjects in the Higher Certificate examination. To gain an adequate picture of the most advanced mathematical work done in the secondary schools at the Higher Certificate standard it is

TABLE III

EXAMINATION FOR HIGHER CERTIFICATE IN MATHEMATICS AS GIVEN BY THE FOUR LARGEST EXAMINATION BOARDS IN ENGLAND AND WALES

Examination Board	Higher Certificate Examination in Mathematics: Subject	Time Allowed (Hours)	Number of Questions to Answer	Remarks
University of London, Matriculation and School Examinations Council	*Group C.*			Candidates must take both Pure and Applied Mathematics in Group C. A subsidiary subject must be taken also.
	Pure Mathematics (4 papers)			
	Algebra	3		
	Geometry	3		
	Plane Geometry	3		
	Analytic Geometry and calculus	3		
	Applied Mathematics (2 papers)			
	Statics and dynamics	3		
	Hydrostatics	3		
	Group D.			Candidates may take Pure and/or Applied Mathematics in this group. In all, three subjects are to be taken in Group D.
	Pure Mathematics (3 papers)	3 ea.		
	Applied Mathematics (3 papers)			
	Algebra, geometry, trigonometry	3		
	Calculus, statics, dynamics	3		
	Hydrostatics	3		
Oxford and Cambridge Schools Examination Board	1. Arithmetic, algebra, and trigonometry	2½	10	For a pass in Higher Certificate Mathematics a candidate must pass in papers 1-4. A candidate for distinction must satisfy the examiners in papers 1-7.
	2. Pure and analytic geometry	2½	8 (10)*	
	3. Differential and integral calculus	2½	8 (10)	
	4. Statics and dynamics	3	9 (10)	
	5. Mathematics Distinction Paper I	3	8 (9)	
	6. Mathematics Distinction Paper II	3	8 (10)	
	7. Mathematics Distinction Paper III	3	8 (10)	
	8. Essay on mathematical subject	1½		
Central Welsh	1. Advanced Mathematics (6 papers 3 Pure, 3 Applied)	3 ea. 3 ea.	9, 9, 8 8, 8, 8	Advanced Mathematics is equivalent to two principal subjects; each of the others is equivalent to one principal subject only.
	or			
	2. Pure Mathematics (3 papers)	3 3 3	9 9 9	

TABLE III—*Concluded*

Examination Board	Higher Certificate Examination in Mathematics: Subject	Time Allowed (Hours)	Number of Questions to Be Answered	Remarks
Central Welsh—*Continued*				
	or			
	3. Applied Mathematics	3	9	
	(2 papers)	3	9	
	or			
	4. Pure and Applied	3	9	Pure and Applied Mathematics consists of geometry, trigonometry, algebra, calculus, kinematics, statics, kinetics.
	Mathematics	3	9	
	(3 papers)	3	9	
Northern Universities	1. *Pure Mathematics* Algebra, trigonometry, plane geometry,	3	10	See Column 2, No. 4: *Pure Mathematics* Differential Calculus, Algebra and Trigonometry *Applied Mathematics* Statics, Dynamics. Candidates must take three principal subjects, two principal and one subsidiary subject or Higher Mathematics and one subsidiary subject. All eight papers must be taken in Higher Mathematics.
	solid geometry, differential and integral calculus (2 papers)	3	10	
	2. *Applied Mathematics* Kinematics, dynam-	3	10	
	ics, statics (2 papers)	3	10	
	3. *Pure and Applied Mathematics* Algebra, trigonometry, geometry, cal-	3	9	
	culus, statics, dynamics (2 papers)	3	9	
	4. *Higher Mathematics*			
	Pure Mathematics I	3	10	
	Pure Mathematics II	3	10	
	Pure Mathematics III	3	10	
	Scholarship Paper	3	10	
	Applied Mathematics I	3	10	
	Applied Mathematics II	3	10	
	Applied Mathematics III	3	10	
	Special Paper A	3	10	
	Special Paper B	3	10	

* Figures in parentheses represent the number of questions on the paper.

necessary to give some account of the most advanced mathematical work done at the Higher Certificate standard, for, as Table IV shows, 95 per cent of the exhibitions and scholarships in mathematics at Oxford and Cambridge for the year 1937 were awarded to pupils from grant-aided secondary schools.

TABLE IV*

PREVIOUS EDUCATION OF STUDENTS GAINING AWARDS AT OXFORD AND CAMBRIDGE UNIVERSITIES

Number of Entrance Scholarships and Exhibitions in Mathematics Awarded by the Colleges of Oxford and Cambridge during the Year ended August, 31, 1937

Subject	Secondary School on Grant List	Other Secondary Schools	Other Institutions	Total	
				Scholarships	Exhibitions
Mathematics	40	36	4	49	31

* Board of Education, *Education in 1937*, p. 145, from Table 50.

Syllabus and Examination Papers. To illustrate the type of work required for the Higher Certificate Examinations in Mathematics, the 1939 syllabus of the University of Cambridge Local Examinations Syndicate is reproduced below by permission of the Local Examinations Syndicate.

MATHEMATICS (AND HIGHER MATHEMATICS)

In each subject the syllabus must be taken to include the more elementary parts which are not specified below.

Subject 11. Higher Mathematics. The syllabus for the 'ordinary' papers consists of the whole of that set out below with the exception of the portions in **sections (c).**[8] In the 'advanced' papers of subject 11 questions may be set on any part of the syllabus including that in sections (c).

Subject 12. Mathematics. The syllabus for the 'ordinary' papers is printed in sections (a).[8] The syllabus for the 'advanced' papers of subject 12 consists of sections (a) together with section (b)[8] (except for the Pure Geometry and a portion of the Analytic Geometry, which apply only to subject 11 and are set out in a separate section.)

[8] The terms section (a), section (b), section (c) are used here in place of the different types of print which distinguish the corresponding items in the document from which these quotations were made.

ALGEBRA AND TRIGONOMETRY

(a) Permutations and combinations. Mathematical induction. The summation of simple finite series. The binomial series for positive integral index. Solution of linear equations involving not more than three unknowns, and of quadratic equations involving not more than two unknowns. Partial fractions.
Trigonometrical ratios of angles of any magnitude, and their graphs. Solution of triangles by standard methods; area of a triangle. Addition formulae. Radian measure; small angles. Applications to simple three-dimensional problems.

(b) Use of the binomial series for any rational index; and of the expansions of e^x, $\log(1 + x)$.
Solution of trigonometrical equations. Use of the series for sin x, cos x. (Circumcircle, incircle, excircles, and nine-point circle of a triangle.)[9]

(c) Solutions of one quadratic and two linear equations involving three unknowns. Geometrical representation of a complex variable. Determinants of the third and fourth order. Relation between the roots and coefficients of an equation; transformation of equations. (Questions will *not* be set on recurring series, convergence, probability, theory of numbers, continued fractions, or the solution of cubic and biquadratic equations.)
Summation of finite trigonometrical series. De Moivre's theorem and easy applications.

CALCULUS

(a) Differentiation of simple algebraic, trigonometric, exponential, and logarithmic functions. Integration as the reversal of differentiation; informal treatment of definite integrals. Integration of rational functions and of simple functions involving trigonometrical functions; integration by simple substitution.
Application to tangents and normals of curves given by rectangular cartesian equations, parametric or otherwise; easy curve tracing; application in one variable to maxima and minima, and to small errors.
Determination of simple plane areas and volumes.

(b) Derivatives of inverse trigonometrical and of hyperbolic and inverse hyperbolic functions. Integration by parts. Integration of functions involving $\sqrt{ax^2 + 2bx + c}$.

[9] Not for subject 12.

(Use of polar coordinates.)[10]

The differential equation $f(x)\frac{dy}{dx} + \phi(y) = 0$; the linear equation $P\frac{dy}{dx} + Qy = R$, where P, Q, and R are functions of x; and the differential equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, where a, b, c are constants.

(c) Development of the properties of the logarithmic function $\int_1^x t^{-1}\,dt$. Curvature. Simple envelopes.

Further curve tracing including easy singular points and asymptotes. Graphical treatment of mean-value theorems. Expansions (including use of Taylor's and Maclaurin's theorems). Easy partial differentiation (first order) and easy change of variable. Reduction formulae. Lengths of arcs of curves and areas of surfaces of revolution.

Solution of the differential equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$, where a, b, c are constants and $f(t)$ is such that one solution can be found by trial.

GEOMETRY (Subject 11 only)

(b) Simson's line, the circle of Apollonius, simple properties of the orthocenter and median point of a triangle, and of the circumcircle, incircle, excircle, and nine-point circle. The theorems of Ceva and Menelaus. Elementary ideas in solid Geometry with applications to lines, planes, spheres, and circular cone. The orthogonal tetrahedron. Harmonic pairs of points and lines. Properties of the ellipse deduced from its definition as the orthogonal projection of a circle.

Elementary treatment of the straight line and of the loci given by the equations

$$ax^2 + 2hxy + by^2 = 0, \quad y^2 - kx = 0, \quad (x-a)^2 + (y-b)^2 - r^2 = 0, \quad xy - c^2 = 0, \quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} - 1 = 0,$$

including conjugate diameters and pole and polar.

[10] Not for subject 12.

(c) Co-axal circles and inversion. Properties deduced from the definition of a conic as the conical projection of a circle or as the locus of a point determined by the focus and directrix property. Pole and polar. Reciprocation. The general equation of the second degree. Parametric equations. The use of polar coordinates. Tangential (envelope) equations. (Questions will *not* be set on homography, involution, oblique axes, or homogeneous coordinates.)

ANALYTIC GEOMETRY (Subject 12 only)

(a) Elementary treatment of the straight line and of the loci given by

$$y^2 - kx = 0, \qquad (x - a)^2 + (y - b)^2 - r^2 = 0,$$

including the use of parameters. (Rectangular cartesian coordinates only are required.)

(b) Elementary treatment of

$$xy - c^2 = 0 \qquad \text{and} \qquad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} - 1 = 0,$$

including the use of parameters but excluding conjugate diameters and pole and polar.

APPLIED MATHEMATICS

(a) Addition and subtraction of vectors.
Kinematics and dynamics of a single particle moving in a straight line (where variable accelerations or forces occur the questions will be soluble graphically although calculus methods will be accepted). Kinematics and dynamics of a single particle moving in a plane with uniform acceleration (including projectiles), or with uniform circular motion (including the conical pendulum). This syllabus is intended to include Newton's laws of motion, mass, momentum, force, work, energy, power, but it excludes questions on impulse.
Conditions of equilibrium of a rigid body under coplanar forces (questions will not be set involving more than two connected bodies); center of gravity (calculus methods will be accepted); friction; simple machines.

(b) Relative motion of two particles moving with constant velocity in a plane. Simple harmonic motion. Simple problems involving two particles, e.g., Atwood's machine. Impulses. The direct impact of smooth elastic spheres. The dynamics of a single particle moving in a circle in a vertical plane (including the simple pendulum).

Rotation of a rigid body about a fixed axis (excluding finding the reaction at the axis).

Application of Bow's notation to simple frameworks of light rods.

Hydrostatic pressure, center of pressure, principle of Archimedes. Boyle's law.

(c) Rectilinear motion when the resistance is a simple function of the velocity. Formulae for velocity and acceleration of a particle moving in a plane; principles of conservation of energy and of linear and angular momentum for a system of two particles moving in a plane. (A knowledge of central orbits will not be expected and questions will not be set on oblique impact.

Bending moments and shearing force. Catenaries. Virtual work; conditions of stability for a system of one degree of freedom.

N.B. The changes in the Applied Mathematics syllabus for the 'advanced' paper of subject 11 will allow simpler questions to be set on a wider range than under the old regulations.

It will be noted that papers at what are called "ordinary" and "advanced" standards are to be set in both Mathematics 11 and 12. To illustrate the standards which candidates for these examinations are expected to reach, selections will be made only from the papers in algebra, calculus, and analytic geometry. With this selection it will be possible to make direct comparisons with the mathematical work done in high schools and colleges in the United States.

Illustrations of the examination items conforming to the so-called "ordinary" and "advanced" standards will be found in the following selections.

ORDINARY STANDARD

CALCULUS

1. A solid is formed by rotating about the x-axis the area between the graphs of the two functions

$$\left[1 - \frac{1}{2\sqrt{x}}\right] \quad \text{and} \quad \frac{1}{2\sqrt{x}}$$

for values of x from 1 to $a^2 (a > 1)$. Prove that the centroid of the solid divides its axis in the ratio $(5a + 5);(3a + 1)$.

(Oxford Higher School Certificate Examination, 1938)

2. Evaluate the following integrals:

$$\int_1^8 (x^{1/3} - x^{-1/3})\,dx, \qquad \int_0^{\frac{1}{2}\pi} (\sin x - \cos x)^2\,dx,$$

$$\int_4^7 \frac{x-1}{x^2 - 2x - 3}\,dx$$

Water flows from a tap at the bottom of a cylindrical tank 9 ft. deep, the rate of flow being proportional to the square root of the depth of water in the tank at any instant. If the tank was initially full, and the water level sinks 5 ft. in an hour, find the time taken to empty the tank.

(Durham Higher School Certificate Examination, 1938)

3. Solve the equations

(i) $$y\frac{dy}{dx} + x\sqrt{y^2+1} = 0;$$

(ii) $$(1 + x^2)\frac{dy}{dx} + xy = 3x + 3x^3.$$

(Cambridge Local Higher School Certificate Examination, 1938)

ALGEBRA

1. Express $(2n^2 + 1)3^n$ in the form

$$An(n-1)3^{n-2} + Bn\cdot 3^{n-1} + C\cdot 3^n,$$

where A, B, and C are independent of n.
Sum to infinity the series whose nth term is $\dfrac{(2n^2+1)3^n}{n!}$.

(Oxford Higher School Certificate Examination, 1938)

2. Find the sums of the series

(i) $$1 + \frac{nx}{2} + \frac{n(n-1)x^2}{2\cdot 4} \ldots + \left(\frac{x}{2}\right)^n;$$

(ii) $$1 + 2x + 3x^2 + 4x^3 \ldots + (n+1)x^n.$$

(Cambridge Local Higher School Certificate Examination, 1938)

3. Define $\log_e x$, and deduce from the definition that

$$\log_e \frac{ay}{z^2} = \log_e a + \log_e y - 2 \log_e z$$

Find the first four terms of the expansion of

$$\log_e \frac{2 + x}{(1 - x)^2}$$

in a series of ascending powers of x; and state the range of values of x for which the expansion is valid.

(Oxford and Cambridge Higher School Certificate Examination, 1938)

ANALYTIC GEOMETRY

1. Prove that

$$(a_3h + b_3k + c_3)(a_2x + b_2y + c_2)$$
$$+ (a_2h + b_2k + c_2)(a_3x + b_3y - a_3h - b_3k) = 0$$

if the equation of the diagonal, which does not pass through (h, k), of the parallelogram whose sides are $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$, and the parallels to these lines through the point (h, k).
Hence, or otherwise, obtain in the form

$$\lambda (a_2x + b_2y + c_2) + \mu (a_3x + b_3y + c_3) = 0$$

the equation of the median through the middle point of the side $a_1x + b_1y + c_1 = 0$ of the triangle whose sides are $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$.

(Oxford Higher School Certificate Examination, 1938)

2. Show that the equation of the common chord of the two circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

What is the significance of this result if the circles do not intersect in real points?
If the circles are tangential at the origin, show that

$$c_1 = c_2 = 0, \text{ and that } g_1/g_2 = f_1/f_2.$$

Hence, or otherwise, find the equation of the common tangent at the origin to the two circles

$$x^2 + y^2 - 12x + 16y = 0, \qquad x^2 + y^2 + 6x - 8y = 0,$$

and the coordinates of the points from which the tangents to the two circles are 10 units in length.

(Durham Higher Schools Certificate Examination, 1938)

3. Prove that the length of the perpendicular from the point (x_1, y_1) to the straight line $ax + by + c = 0$ is

$$\pm \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}.$$

The coordinates of two points are (x_1, y_1) and (x_2, y_2).
Interpret geometrically the statements (i) $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign, and (ii) they have opposite signs.
One of the points (1, 3), (3, 3), and (3, 4) lies inside the triangle with vertices (0, 0), (5, 1), (2, 5) and sides

$$4x + 3y - 23 = 0, \qquad 5x - 2y = 0, \qquad x - 5y = 0.$$

Find which it is, and check your result by a figure.

(Oxford and Cambridge Higher School Certificate Examination, 1938)

ADVANCED STANDARD

Algebra

1. (i) Given that $(x - a)(x - b)(x - c) \equiv x^3 + px^2 + qx + r$, express p, q, r in terms of a, b, c.
Solve the equation

$$x^3 - 21x^2 + 126x - 216 = 0$$

given that the roots are in geometric progression.
(ii) Solve the simultaneous equations

$$3x^2 + 15xy - 56y^2 + 36 = 0$$

$$2x^2 + 9xy - 33y^2 + 28 = 0$$

(Oxford and Cambridge Higher School Certificate Examination, 1938)

2. If $(1 + x)^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, prove that:
(i) $a_0^2 - a_1^2 + a_2^2 - \cdots + (-1)^n a_n^2$ is equal to 0, when n is odd, and is equal to

$$(-1)^{n/2} \frac{(n+2)(n+4)\cdots 2n}{2\cdot 4 \cdots n},$$

when n is even;

(ii) $$a_r - 3a_{r-1} + 6a_{r-2} + \cdots + (-1)^{\frac{r(r+1)(r+2)}{1\cdot 2}} a_0$$

is equal to ${}_{n-3}C_r$, when $r \leq n - 3$,
and is equal to 0, when $r > n - 3$.

(Cambridge Local Higher School Certificate Examination, 1938)

3. Prove that, if x, y, z are determined by the simultaneous equations

$$a_1x + b_1y + c_1z = u_1, \qquad a_2x + b_2y + c_2z = u_2,$$

$$a_3x + b_3y + c_3z = u_3, \text{ then the ratio}$$

$$(l_1x + m_1y + n_1z - \phi_1) / (l_2x + m_2y + n_2y - \phi_2)$$

is D_1/D_2, where D_1, D_2 are the values of the two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 & u_1 \\ a_2 & b_2 & c_2 & u_2 \\ a_3 & b_3 & c_3 & u_3 \\ l_1 & m_1 & n_1 & \phi_1 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & b_1 & c_1 & u_1 \\ a_2 & b_2 & c_2 & u_2 \\ a_3 & b_3 & c_3 & u_3 \\ l_2 & m_2 & n_2 & \phi_2 \end{vmatrix}$$

(Cambridge Local (General Paper) Higher School Certificate, 1938)

CALCULUS

1. Differentiate

(i) $e^{ax^2 \log bx}$, (ii) $\sin^{-1} \dfrac{ax}{\sqrt{(1 + a^2x^2)}}$.

If $y = e^{a \sin bx}$, and y' and y'' are the first and second differential coefficients of y with respect to x, show that

$$(yy'' - y'^2)^2 = b^2y^2(a^2b^2y^2 - y'^2).$$

Calculate the values of y' and y'' when $x = 0$.

(Oxford and Cambridge Higher School Certificate, 1938)

2. If $y = \sin(m \sin^{-1} x)$, show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$$

Hence, or otherwise, show that the expansion of $\sin m\theta$ as a power series in $\sin\theta$ is

$$m \sin\theta\left[1 - \frac{m^2 - 1}{3!}\sin^2\theta + \frac{(m^2 - 1)(m^2 - 3^2)}{5!}\sin^3\theta - \cdots + (-1)^n \frac{(m^2 - 1)\cdots\{m^2 - (2n - 1)^2\}}{(2n + 1)!}\sin^{2n}\theta + \cdots\right]$$

(Oxford and Cambridge (Distinction Paper) Higher Certificate, 1938)

3. The coordinates of a point of a curve are given in the form $x = f(t)$, $y = \phi(t)$, where t is a parameter. Prove that the radius of curvature is equal to
$[\dot{x}^2 + \dot{y}^2]^{3/2}/[\dot{x}\ddot{y} - \ddot{x}\dot{y}]$, where dots denote differentiation with respect to t.
Find the radius of curvature of the curve given by

$$x = t^2 + 2t,\ y = 2t^3 + 3t^2$$

at the point whose parameter is t.

(Cambridge Local Higher School Certificate, 1938)

4. Obtain a reduction formula for $\int \cos^n\theta\, d\theta$,

and evaluate $\int_0^{\frac{1}{2}r\pi} \cos^n\theta\, d\theta$,

where r and n are positive integers.
Prove that, if n is a positive integer,

$$\int_0^{\frac{1}{2}\pi} \frac{1 - \cos^n\theta}{1 - \cos^2\theta}\, d\theta = \frac{(n-1)(n-3)\cdots 3}{(n-2)(n-4)\cdots 2}\cdot\frac{\pi}{2}$$

$$\text{or } \frac{(n-1)(n-3)\cdots 4}{(n-2)(n-4)\cdots 3}\cdot\frac{2}{1}$$

according as n is even or odd.

(Oxford Higher School Certificate Examination, 1938)

Analytic Geometry

1. Prove that the equation of the normal to the hyperbola $xy = c^2$ at the point (ct, ct^{-1}) is

$$t^3x - ty + c - ct^4 = 0.$$

Prove that the parameters of those points of $xy = c^2$ which lie on the curve $y^2 - x^2 - gx - fy = 0$ satisfy the equation

$$ct^4 + gt^3 + ft - c = 0,$$

and that the normals to $xy = c^2$ at these points pass through the point $(-g, f)$.

Prove that $t_1t_2t_3t_4 = -1$ and $\sum t_1t_2 = 0$ together form sufficient conditions for the normals to $xy = c^2$ at the points whose parameters are t_1, t_2, t_3, and t_4 to be concurrent.

(Oxford and Cambridge Higher School Certificate, 1938)

2. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, explain why $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin parallel to these lines.

Prove that the area of the parallelogram formed by the lines

$$10x^2 - 7xy + y^2 - 32x + 13y + 22 = 0 \text{ and}$$

$$10x^2 - 7xy + y^2 = 0, \text{ is } \tfrac{2}{3} \text{ units.}$$

(Cambridge Local Higher School Certificate Examination, 1938)

3. Prove that the focal distances of the point $(a \cos \alpha, b \sin \alpha)$ on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, are $a(1 - e \cos \alpha)$ and $a(1 + e \cos \alpha)$. S is the focus $(ae, 0)$ of this ellipse and PP' is any diameter of the ellipse. Prove that the locus of the intersection of PP' with the internal bisector of the angle PSP' is the ellipse

$$b^2x(x - ae) + a^2y^2 = 0.$$

Textbooks. The mathematical textbooks used in secondary schools in England and Wales conform to a fairly uniform pattern, inasmuch as they are influenced by the requirements of external examinations boards. The subject matter in each subject which they cover has already been indicated in the syllabi quoted earlier. In order to indicate somewhat more definitely the way in which subject matter is organized in the more widely used textbooks, brief mention will be made of texts in geometry and elementary algebra.

The geometry textbooks follow the plan of having an introductory section on informal geometry which requires the use of mathematical instruments, followed by the logical development of the subject in accordance with some definite sequence. The chief emphasis is placed on riders (originals) of which a very large number of graded and miscellaneous examples is provided. Little or no attempt is made to apply the methods of geometrical reasoning to data which are non-geometrical in character.

In algebra the textbooks published during recent years have followed the lead given by Nunn in his epoch-making books *The Teaching of Algebra* and *Exercises in Algebra, Parts I and II*. The development in the earlier chapters in these books is psychological rather than logical. It proceeds through a study of notation, formulas, graphs, equations, directed numbers, graphs of functions to quadratic equations. A characteristic of these early chapters is that illustrations and problems are drawn from a wide variety of sources, and they tend in a real manner to correlate and to coordinate the various branches of mathematics with those subjects to which mathematical methods are applicable. The exposition in the more advanced textbooks follows a more strictly logical development. An important characteristic of the more advanced mathematical textbooks, including algebra texts, is the number and difficulty of the problems provided for solution.

Summary. The purpose of this section has been to state and to illustrate the main features of secondary school mathematics in England and Wales with a view to giving a clear picture of the type of mathematics which teachers are required to teach.

1. It was shown that the mathematics syllabi on the whole conform to the requirements of the external examinations boards; consequently a fairly detailed analysis was made of the mathematics syllabi and examinations of these boards.

2. Most of the mathematics teaching in secondary schools is concerned with arithmetic, algebra, geometry, and trigonometry since a majority of the pupils take only the elementary math-

ematics examination in these subjects at the School Certificate Examination.

3. Since a considerable number of pupils take Additional Mathematics at the School Certificate Examination and Mathematics at the Higher Certificate Examination, a discussion of the work done for these examinations was also included. The subjects examined in the examinations are algebra, pure geometry, analytic geometry (of two dimensions), calculus (differential and integral), mechanics (statics and dynamics), and elementary differential equations.

4. Mathematics is examined in public examinations by means of a number of three-hour examinations containing essay-type questions which fairly adequately cover the range of mathematics suggested by the syllabi, but which tend not to examine some of the more interesting applications of mathematics.

5. The mathematical textbooks used in England and Wales are somewhat formal in character; and yet the better ones among them provide considerable opportunities for correlating the several branches of elementary mathematics with one another and with allied subjects. Little or no attempt is made in them to provide for the application of mathematical types of thinking to non-mathematical material.

THE UNITED STATES

In approaching the study of mathematics in the secondary schools in the United States it is necessary to take into account the diversity which marks the schools. Agencies, organizations, aims, curricula, and the like exhibit considerable variation both among the states and among communities within the states. However, certain coordinating forces are operating, not indeed to produce anything like nation-wide uniformity either of requirements or of standards, yet each with its own definite influence.

These coordinating forces are supplied by:

1. Mathematical textbooks, expecially those which have a market on a state-wide or nation-wide scale.
2. The five accrediting associations of colleges and secondary

schools which exert an important influence on the curricula of schools which they accredit.

3. Leaders and writers whose work affects the mathematics curriculum in the schools through commissions on which they serve, through the direct influence of their teaching in various types of teacher training institutions, and through textbooks which they produce.
4. Extramural examinations boards, such as the College Entrance Examination Board, the Board of Regents of the University of the State of New York, and departments of education in various states which sponsor Every-Pupil tests within their own state.
5. State syllabi and courses of study, and state-wide examinations.
6. College entrance requirements as set up for colleges.

In spite of the variety of influences thus operating to affect the kind of mathematics taught in the secondary schools of the United States, a good picture of the mathematical work in these schools can be obtained by considering in detail the influence of the external examinations boards on the teaching of mathematics. The reasons for this statement are:

1. That these boards have modified their requirements to incorporate the principal recommendations of the commissions on mathematical teaching.
2. That textbook writers in most states claim to organize the material in their books to suit the recommendations both of the commissions on the teaching of mathematics and of the examinations boards.
3. That accrediting agencies state whether or not mathematics shall be taught in schools which they accredit, but they do not prescribe any particular organization of material; consequently the material taught tends to follow the textbooks which in their turn conform largely to the examination requirements.
4. That state syllabi of courses of study are developed around the subjects algebra, geometry, trigonometry, and general mathematics, the content of which also follows more or less closely

the recommendations of the Commissions on the Teaching of Mathematics.

5. That when mathematics is required as a subject for entrance to college the type of mathematics taught is determined either by the textbooks or by the examination requirements.

In this section, therefore, attention will be directed chiefly to a consideration of the syllabi and examination papers of the examinations boards. In addition it will be necessary to consider the mathematical work done in certain progressive schools in the country and also to point out to what extent the general picture outlined above does not fit particular regions in the country.

The National Committee on the Teaching of Mathematics. A statement of the mathematical requirements in high schools in the United States would be incomplete if reference were not made to the important leadership offered from time to time by national committees on the teaching of mathematics. Of first importance for this study is the report of the National Committee on the Teaching of Mathematics, issued in 1923 under the title *Reorganization of Mathematics in Secondary Education.* This report was the first really national report because the members of the National Commission came from all parts of the country, both from high schools and from colleges, and its tentative reports were studied and criticized by groups of teachers all over the United States. An account of the influence of this report is included here because frequent reference will be made to it in the sequel. The report furnished important leadership for the future direction of mathematical education in the United States: it not only provided a statement of aims and objectives, but it suggested several different methods of organizing subject matter in mathematics for the junior and senior high schools, it encouraged experimentation with new subject matter in the senior high school, and it provided teachers with an account of the latest developments in certain aspects of mathematical education. In short, it summarized the important trends in mathematical thinking at the school stage and showed teachers of mathematics in the United States the way toward better teaching of the subject. Two other commissions are at present engaged in carrying on the work started by the National Committee in 1923.

These are the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, and the Commission on the Secondary School Curriculum of the Progressive Education Association. The influence of the tentative reports of these two more recent commissions will be discussed later in this chapter. The importance of the work of these three commissions in the professional training of mathematics teachers is considered in a later section of this study. The 1923 Committee recommended that plane demonstrative geometry, algebra, solid geometry, trigonometry, elementary statistics, elementary calculus, history and biography, and additional elective subjects such as shop mathematics, surveying, and navigation should be taught in the schools. In spite of these forward-looking recommendations and in spite of the prestige which the Commission framing them undoubtedly had, it has become apparent since 1923 that the older tradition was hard to alter. For example, the survey of secondary education, made ten years after the National Committee had reported, showed that

> The traditional college-preparatory subjects plane and solid geometry, algebra and trigonometry are offered in practically all schools, but statistics, calculus, history, and biography are not offered at all under these names. . . . However, materials from these fields are sometimes presented in connection with other courses.[11]

Mathematics as a High School Subject. The fact that mathematics is an elective subject in the upper grades of the high school, and also the fact that many states do not require credit in mathematics for entrance to college have led to a decrease in the percentage of pupils taking mathematical subjects in high schools in the United States during the present century. For example, the percentage of pupils taking algebra in the ninth grade has been decreasing since 1905, and during the period 1922 28 it decreased from 40.15 per cent to 35.22 per cent of all pupils enrolled in high schools. For geometry the corresponding figures during the same period were 22.08 per cent and 19.80 per cent.[12] The latest figures,

[11] Edwin S. Lisle, *Instruction in Mathematics*, p. 44.

[12] W. D. Reeve, "Modern Curriculum Problems in the Teaching of Mathematics in Secondary Schools," *Mathematics Teacher*, (March, 1939), p. 118.

which have already been quoted, show that of pupils enrolled in high schools in 1934, 17.06 per cent took geometry and 30.41 per cent algebra.[13]

Some information on the mathematical requirements for college entrance is available from the bulletin on articulation of high school and college which was issued as part of the National Survey of Secondary Education. For example, the survey showed that

1. The average number of units of mathematics required at present [1932] for entrance to college ranges from 1.7 among the teachers' colleges and normal schools to 2.4 among the institutions in New England and the South.
2. The requirement in mathematics is affected more by the region in which the institutions are located and by the type of institution than by the size of the institution.
3. Twenty-seven of a total of thirty-six zero requirements are reported by institutions in the Middle West.[14]

Data on the more usual mathematical courses taken prior to entering college were also obtained. The findings were as follows:

The influence of college entrance on pupil election in mathematics is reflected to some degree in the comparative enrollments in elementary algebra and plane geometry which are the usual requirements for college entrance, and in the courses following them. In New York City for example, Orleans reports from 38.6 to 43.2 per cent of all pupils enrolled in elementary algebra and plane geometry in 1929, an enrollment of 12% in intermediate algebra, and from 0.9 to 3.6 per cent enrolled in solid geometry, advanced algebra and trigonometry.[15]

Although this suggests that proportionately little attention is given to intermediate algebra, solid geometry, advanced algebra, and trigonometry in high schools in the United States, the syllabi in these subjects and the standard of work demanded in them need to be examined in this study to show the type of mathematical work which some mathematics teachers are required to teach in high schools. When these subjects are taken, the recognition they

[13] Carl A. Jessen, *Offerings and Registrations in High School Subjects, 1933-34*, p. 30, Table 2.

[14] P. Roy Brammell, *Articulation of High School and College*, p. 36.

[15] Edwin S. Lide, *Instruction in Mathematics*, p. 64.

receive in terms of points or credits toward graduation is substantially as follows:

Elementary algebra	1	Plane trigonometry	½
Intermediate algebra	1	Advanced algebra	½ or 1
Plane geometry	1	Calculus and analytics	1
Solid geometry	½		

Grade Placement of Mathematics in Schools. The survey of secondary education also revealed[16] that most pupils in grades seven and eight in the junior high schools in the United States are required to take arithmetic and or general mathematics. It should be noted that the term *general mathematics* as used in this survey is somewhat ambiguous. It appears from the context to include what has variously been called parallel mathematics, correlated mathematics, and cooperative mathematics as well as general mathematics in the sense in which the chief proponents of that term meant it to be used. For example, in his study McCormick stated the meaning to be attached to general mathematics as follows:

General mathematics first originated as a somewhat less highly correlated course in mathematics. It represented an effort to select the best from the different subjects, present it in such a way that the pupil of average intelligence can learn it, and place emphasis upon correlation only to the extent that the correlation represents the natural relationships of the subjects.[17]

In grades seven and eight general mathematics[18] consists of arithmetic, some algebra, and informal geometry. In these grades mathematics is usually taught for five periods[19] a week for two semesters, i.e., for a school year. The offering in the ninth grade is either elementary algebra or general mathematics, the latter differing from the former only by the addition of the elements of numerical trigonometry. The time devoted to ninth grade mathe-

[16] Edwin S. Lide, *Instruction in Mathematics*, p. 21.

[17] C. McCormick, *The Teaching of General Mathematics in the Secondary Schools of the United States*, p. 4.

[18] The term general mathematics in the remainder of this chapter has the connotation intended by the National Survey of Secondary Education.

[19] The length of a period varies from forty to sixty minutes.

matics is mostly five periods a we k for a school year. The content of this course is outlined a little later in this section.

Traditionally, the tenth grade subject is plane demonstrative geometry. This subject usually occupies five periods a week for the school year, and covers the range of work indicated in the geometry syllabi quoted later. In the eleventh and twelfth grades solid geometry, intermediate algebra, advanced algebra, and trigonometry are the mathematical subjects usually offered, and each also usually occupies five periods a week for one semester. From this discussion it is seen that while there is some continuity of mathematical studies in grades seven through nine in secondary schools in the United States, this continuity is broken during the last three years of high school when mathematical subjects are elective and are taught in relatively isolated units.

It will be convenient at this point to give the evidence on which the claim was made earlier in this section that a discussion of external examinations in mathematics would almost certainly cover the contribution which mathematical textbooks make to the mathematical teaching in secondary schools in the United States.

Whitcraft has shown that the most recent mathematical textbooks in this country on the whole adhere closely to the syllabi and requirements of the examinations boards and to the suggestions of the report of the National Committee. From his study Whitcraft concludes, for algebra, that

> The general tendency in the preparation of textbooks in elementary algebra has been to follow the requirements of two outstanding syllabuses, namely, the Report of the National Committee on Mathematical Requirements and the 1923 definition of the requirements of the College Entrance Examinations Board in the subject of algebra. Of these two syllabuses, the requirement of the College Entrance Examinations Board carried the greater weight in those textbooks in elementary algebra which were published between the years 1920 and 1930, and which were included in this study.[20]

and for geometry, that

[20] L. H. Whitcraft, *Some Influences of the Requirements and Examinations of the College Entrance Examination Boards on Mathematics in Secondary Schools of the United States*, p. 50.

An examination of twenty-one textbooks published since 1925 shows that the greater proportion of the authors of geometry textbooks included in this study were influenced to the degree that they tried to fulfill the requirements of the College Entrance Examination Board and of the National Committee on Mathematical Requirements. Of these two bodies the College Entrance Examination Board had the greater influence.[21]

In view of these findings it will be assumed here that an adequate discussion of the syllabi and examination papers of the external examinations boards will also cover the contributions of textbooks to the mathematical instruction in secondary schools in the United States.

External Examinations Boards. In the United States there is only one examining body with a nation-wide influence, namely, the College Entrance Examination Board. The Board came into being in 1900, and since that date it has examined an increasing number of candidates for entrance to college. At present it examines in English, history, Latin, Greek, French, German, Spanish, mathematics, biology, physics, chemistry, biological sciences (two years), physical sciences (two years). In addition it conducts a Scholastic Aptitude Test and, on request, scholarship examinations. The purpose of the Scholastic Aptitude Test is prognostic. It is intended to give an index of the candidate's prospects in college work. The items in this test are of the objective type and in addition to items intended to measure the general intelligence of the examinee, they sample a wide range of information. The mathematical examinations conducted by this Board are discussed later in this chapter.

In several states of the United States external examinations are held on a state-wide basis. These examinations are of two types. The first type is represented by the Regents Examinations conducted by the University of the State of New York, and the second type by the Every Pupil Tests conducted, for example, in the states of Ohio and Iowa. While the Regents Examinations examine pupils in the state of New York alone, they examine more pupils than does any other single examining body in the United States.

[21] *Idem.*

TABLE V

ANALYSIS OF MATHEMATICS EXAMINATION PAPERS FOR SECONDARY SCHOOL PUPILS SET BY EXTERNAL EXAMINING BODIES IN THE UNITED STATES

Examination Board	Subject Examined	Time Allowed in Hours	Organization of the Paper and Number of Questions Set	Number of Questions to be Answered	Number of Candidates
Regents, New York	Mathematics, Third Year	3 1½	Group I 20 short answers‡	20	(1936)
		1½	Group II 6 problems Group III 3 problems	5	
	Plane Trigonometry*	3 1½	Group I 20 short answers	20	13,727
		1½	Group II 5 problems Group III 4 problems	5	
	Solid Geometry*	3 1½	Group I 14 short answers	14	5,303
		1½	Group II 6 problems	3	
			Group III 3 problems	2	
	Plane Geometry	3 1½	Group I 12 short answers	12	
		1½	Group II 5 theorems and originals	2	84,253
			Group III 3 originals	2	
	Intermediate Algebra†	3 1½	Group I 20 short answers	20	
		1½	Group II 8 problems	5	51,648
	Advanced Algebra†	3 1½	Group I 20 short answers	20	
			Group II 9 problems	5	4,254

* Alternative subjects, as each is set down for examination at the same time: may be taken in successive examinations.

† Alternative subjects.

‡ Short-answer questions of which candidates do as many as they can. Short-answer questions in

TABLE V—*Concluded*

Examination Board	Subject Examined	Time Allowed in Hours	Organization of the Paper and Number of Questions Set	Number of Questions to be Answered	Number of Candidates
College Entrance Examination Board	Mathematics A (El. Algebra)	3	Graded essay type questions	18	(1938) 2,353
	Mathematics A* (algebra to quadratics)	2		12	162
	Mathematics C (plane geometry)	3		12	1,534
	Mathematics Alpha	3 1¼ 1¾	Part I 160‡ Part II 10 problems	variable 10	209**
	Mathematics Beta	3 1¼ 1¾	Part I 150‡ Part III 10 problems	variable 10	1,406**
	Mathematics Gamma	4 1¼ 2¾	Part I 100‡ Part IV 10 problems	variable 10	1,159**
Ohio State Department of Education	Mathematics (grades 7 and 8) (arithmetic and mensuration)	¾ min. 10 30	Part I 20 questions Part II 5 questions Part III 7 questions	20 5 7	(1936) 22,167
	Elementary algebra (grade 9)	¾ min. 15 10 10	Part I 20 short answers Part II 10 multiple choice Part III 10 problems	20 10 10	6,867
	Plane geometry (grade 10)	¾	Part I 10 general information Part II 7 (reasons) Part III 6 numerical problems Part IV 5 constructions	10 7 6 5	3,427

mathematics are items in which the examinee is expected to perform some highly specific operation, such as computation, construction of part of a figure, completion of a formula.

** At the 1938 examination Advanced Algebra, Quadratics and Beyond, Plane and Solid Geometry, were offered also. Henceforth these will be omitted and it is expected that the candidates who would usually have taken these subjects will take Mathematics Alpha, Beta, and Gamma. After 1941 only Mathematics Alpha, Beta, and Gamma will be offered in mathematics.

A pass in the Regents Examination is required for admission to state universities and colleges.

In order to test the effectiveness of the learning of the secondary school subjects the Departments of Education in some states in the United States have arranged for Every Pupil Tests to be conducted in some of these subjects. These tests are not designed to serve primarily as a means of selecting pupils for college. As an example of these types of tests an account is given later of those conducted by the state of Ohio.

As a first guide to the standards reached in the mathematical subjects in high schools in the United States an account will be given of the syllabi and examinations in mathematics of the College Entrance Examination Board, the New York State Regents, and the State Department of Education, Ohio. Thereafter an account will be given of practices which seem to differ fundamentally in spirit, content, and standards of work from those set up by these examinations boards.

Table V summarizes data concerning the mathematical subjects examined by the three examining bodies mentioned above. The data summarized are the time allowed for each examination and the organization of each examination paper in terms of number and type of questions asked. It will be seen at once that the type of examination paper most favored combines the objective short-answer type of question with the essay or problem type of question.

College Entrance Examination Board. The general purposes served by the College Entrance Examination Board as an examining body have already been described. While the Board examined pupils in forty-five out of the forty-eight states of the Union, their statistics show that 77.6 per cent of candidates examined in 1938 attended schools in the New England and Middle Atlantic States, and that 86.1 per cent were seeking entrance to colleges and universities in these same states. The remaining 32.4 per cent of the candidates attended schools in the North Central, Western, and Southern States. Sixty-six per cent of the candidates were from private schools and 34 per cent were from public schools.

For the future the important mathematics examinations set by the College Entrance Examination Board will be the Mathematics

Attainment Tests, known as Mathematics Alpha, Mathematics Beta, and Mathematics Gamma. For the present, additional mathematical subjects are examined, namely, algebra, plane geometry, and algebra to quadratics. The syllabi of these examinations and representative questions from them will now be discussed.

The general features of the examination papers for the Mathematics Attainment Test are shown in Table V. Each paper consists of two parts. Part 1 contains 160 short-answer questions. Candidates taking Mathematics Alpha begin at question 1, those taking Mathematics Beta begin at question 31, and those taking Mathematics Gamma begin at question 61: all candidates work as many of the remaining questions as they can in the time allowed ($1\frac{1}{2}$ hours). Parts II, III, and IV each contain ten questions; Parts II, III, and IV are for candidates taking Mathematics Alpha, Mathematics Beta, and Mathematics Gamma respectively.

In describing the attainment tests, the official handbook of the College Entrance Examination Board says:

Examination Alpha, Beta and Gamma will be of such a character as to provide the colleges with an appraisal of each candidate's competence in mathematics. . . . These three examinations together will comprise a sort of ladder or attainment test in mathematics. . . .

The character of each examination will be such as to combine the advantages of the longer essay type, multiple step question with those of the single-step question of the type now used in the mathematical section of the Scholarship Aptitude Test.[22]

To indicate both the range of mathematical knowledge examined and the standard which is expected of pupils, the following syllabi and examination questions are quoted.

SYLLABI

Mathematics Alpha

The scope of the examination will be arithmetic and the simpler concepts and techniques of algebra, numerical trigonometry, and plane geometry.

[22] College Entrance Examination Board, *Definition of the Requirements*, Edition of December, 1937, pp. 39–50.

This examination may include additional topics in algebra and geometry. . . .

Further indication of the scope of this examination is furnished by the following list of specific attainments:

1. Understanding of important ways of employing convenient symbolism for the communication of ideas.
2. Skill in arithmetical operations and elementary algebraic manipulation.
3. Knowledge and understanding of the facts of mensuration.
4. Understanding of functional dependence, its mathematical formulation and its applications.
5. Ability to apply general rules to particular cases
6. Ability to state a proposition in logical form, indicating hypothesis and conclusion.
7. Understanding of what constitutes a mathematical proof; ability to demonstrate simple propositions. There will be no distinction between "book theorems" and "originals."
8. Appreciation of the nature of a deductive science, that is, the place of undefined terms, definitions, and postulates, and the dependence upon them of a logical chain of theorems.
9. Ability to apply trigonometric functions to the solution of simple problems.

Mathematics Beta

The scope of the examination represents the preparation in mathematics expected of students intending to do at least the minimum work in mathematics and the natural sciences required in colleges. The main body of the examination will be concerned with algebra, plane geometry, and numerical trigonometry. This examination may include questions from solid geometry, from more advanced parts of algebra and trigonometry, and from elementary statistics, but it will be so constructed that a candidate cannot obtain a higher score by sacrificing thorough competence in the main body of the examination in order to show acquaintance with a wider range of topics.

Sharp separation between algebra, geometry, and numerical trigonometry will not be maintained. Understanding of the interrelations between these subjects should be part of the candidate's equipment. The following list serves to indicate the general scope of the examination and the point of view from which the material is to be considered rather than to provide a catalogue of topics.

Algebra

1. Language, ideas, and processes of algebra.
   (a) Symbolism and vocabulary (e.g., coefficients, exponents, use of parentheses, use of subscripts).
   (b) Number systems (e.g., natural numbers, signed numbers, rational, real, complex numbers).
   (c) Laws of operation (e.g., rational operations in various number systems).
   (d) Techniques (e.g., the summing of progressions, the use of the binomial theorem).
2. Functional dependence
   (a) The formula (e.g., formulas which appear in geometry, in physics, or in other fields within the experience of the candidate).
   (b) The equation (e.g., linear and quadratic equations in one variable, systems of equations in several variables, their connection with the identity and with the formula).
   (c) The graph (e.g., representation of statistical data and of relations between two variables given by equations).
3. Problem solving, i.e., the translation of a question into algebraic form, its solution and interpretation.

Geometry

1. Congruence of rectilinear and of circular figures.
2. Perpendicularity and parallelism of lines.
3. Comparison and measurement of angles in rectilinear figures and in circles.
4. Areas of rectilinear figures and circles.
5. Similarity of figures.
6. Constructions by means of straight-edge and compasses.
7. Locus theorems and problems.
8. The nature of a deductive science; that is, the place of undefined terms, definitions, and postulates, and the dependence upon them of a logical chain of theorems.

Trigonometry

1. The solution of right triangles by the use of four-place tables (natural functions and logarithms).
2. Some understanding of the technique of calculation, of the importance of checks, and of the limitations imposed upon the accuracy of results by the use of approximate data and of tables.

Mathematics Gamma

The main body of the examination will be concerned with trigonometry, solid geometry, and advanced algebra. In order to measure those candidates who have been able to assimilate a wider mathematical training, this examination may include questions from analytic geometry and the calculus.

The examination will not observe strict separation of the subjects trigonometry, solid geometry, advanced algebra. Candidates will be expected to be able to use any one or more of these subjects in the solution of a problem with which they are confronted.

Trigonometry

1. The six trigonometric functions of angles from any magnitude. The computation of five of these functions from any given one. Functions of 0°, 30°, 45°, 60°, 90°, and of angles differing from these by multiples of 90°.
2. Determination, by means of a diagram, of such functions as sin $(A + 90°)$ in terms of the trigonometric functions of A. Use of trigonometric tables, with interpolation.
3. Circular measure of angles; periodicity; length of an arc in terms of the central angle in radians.
4. Solution of right and oblique triangles (both with and without logarithms) with special reference to the applications. Value will be attached to the systematic arrangement of the work.
5. Derivation of the Law of Sines and the Law of Cosines.
6. Proofs of fundamental relations and formulas, such as the formulas for tan $(x + y)$ and cos $2x$, and of simple identities derivable from them.
7. Solution of simple trigonometric equations.
8. Theory and use of logarithms. Solution of exponential equations.

Solid Geometry

Inasmuch as the scope of this part of Examination Gamma is necessarily much more restricted than that of the former examination in solid geometry, some indication is given in the following paragraphs of what this scope is to be.

1. The relations of lines and planes in space.
2. The properties and measurement of prisms, pyramids, cylinders, and cones. There will be little emphasis upon the logical aspect of the mensuration of the solids mentioned above.

3. The sphere and the spherical triangle.
4. The solution of locus problems, to the extent of visualizing, describing, and representing the figure on paper, is explicitly included; formal proofs will ordinarily not be required.

Advanced Algebra

Inasmuch as the scope of this part of Examination Gamma is necessarily much more restricted than that of the former examination in advanced algebra, some indication is given in the following paragraphs of what this scope is to be.

1. Elementary aspects of the theory of equations with emphasis on the following ideas: The theorem that an equation of the *n*th degree has *n* roots, if every such equation has one root. The remainder theorem and its applications. The coefficients as a symmetric function of the roots. Condition for a rational root. Conjugate complex roots of equations with real coefficients. Descrates's Rule of Signs. Simple transformations of equations. Preliminary location of the roots by the graph. Approximate solution of numerical equations. Determination of the roots in two or three significant figures, preferably by some less elaborate method than Horner's Method.
2. A numerical and geometric treatment of complex numbers, including the sum, difference, product, and quotient of two complex numbers.
3. Permutations and combinations, restricted to the case of *n* objects all of which are different. Probability restricted to problems of moderate difficulty.

The examination will contain no questions on determinants, simultaneous quadratics, scales of notation, or mathematical induction.

Analytic Geometry and the Calculus

1. The analytic geometry of straight line and circle. The distance between two points, the slope of a line, parallel and perpendicular lines; the equation of the circle and its relation to center and radius. Other simple loci.
2. The differentiation of algebraic functions with applications to slopes, maxima and minima, rates of change, velocity, acceleration. With respect to fundamental notions concerning the derivative, the candidate should distinguish clearly between definition, assumption, and proof and should have an accurate knowledge of the rôle of the limiting process.

EXAMINATION QUESTIONS

Alpha Examination

Part I—(Directions): Write the answers to these questions as quickly as you can without sacrificing accuracy. After you have worked on Part I for one hour and fifteen minutes, you will be told: "Even if you have not finished Part I, it is strongly advised that you begin the next part of the examination."

1. The length of a rectangle is 11.2, and its width is 8.43. Its area is ....

4. Fill in the parenthesis: $2\frac{3}{4}$ pounds = ( ) ounces.

7. In Figure 1, $AB = 6$, $BC = 8$, ABC is a right triangle. $AC =$ ....

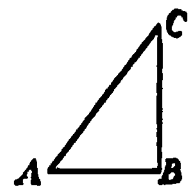


Figure 1

16. Arrange three equilateral triangles so that they form a trapezoid.

25. If $a = 2$, $b = 1$, $\frac{a^2 - ab}{3b} =$ ....

(These questions will be attempted only by candidates taking Mathematics Alpha.)

Part II—(Supplementary Directions): In questions in which you are asked to prove something it will be sufficient to write the important statements and reasons, omitting the formal statement of what is given and what is to be proved, and suppressing the most obvious details.

3. The difference of the areas of two circles is 7 square inches, and the sum of their areas is 25 square inches. Find the radii of the two circles.

4. In Figure 30, $ABCD$ is a square, $\angle ABE = \angle FBC = 15°$. Show that the $\triangle EBF$ is equilateral.

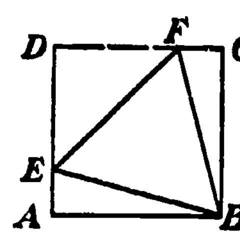


Figure 30

7. A motorist made a trip of 140 miles. At the end of 80 miles he was forced to reduce his speed. He finished the entire trip in 5 hours. If he had been able to keep his original speed for 120 miles before slowing down to the reduced speed, he would have finished the entire trip in 4 hours. Find his original speed and his reduced speed.

Beta Examination

Part I—(Beta candidates begin working at question 31.)

31. If $\sqrt{\frac{36}{b}} = 6$, $b =$.

37. In Figure 10, MN is parallel to BC, $AM = 7$, $MB = 4$, $AN = 9$. $NC =$ ______.

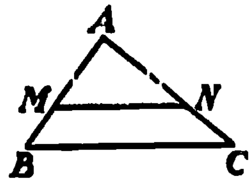


Figure 10

43. Reduce to simplest form:

$$\frac{a^2 - 5a + 4}{a + 3} \cdot \frac{a^2 + 2a - 3}{a - 4} = \qquad .$$

49. Fill in the parenthesis: $(\sqrt{3})(\sqrt{5}) = (\quad)$.

58. If $2^x = (-2)^6$, $x =$ ______.

(These questions will be attempted by candidates taking Mathematics Alpha and Mathematics Beta.)

Part III—(Supplementary Directions): In questions in which you are asked to prove something, it will be sufficient to write the important statements and reasons, omitting the formal statement of what is given and what is to be proved, and suppressing the most obvious details.

4. A child's bank contains \$2 in dimes and nickels. If 6 nickels and 7 dimes are withdrawn, 5 more of one kind of coin than of the other are left in the bank. Show that the question, "How many coins of each kind were there in the bank originally" can have only one solution.

5. In Figure 32, AB is a diameter of a circle, and CD is any other chord. At C and D perpendiculars are erected on CD, meeting AB at E and F respectively. Prove that $AE = BF$.

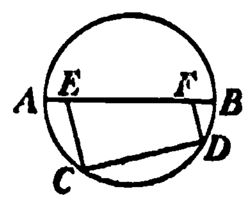


Figure 32

8. If $y = x^2 - 5x + 7$, is there a value of x for which y is negative? Give a reason for your answer.

9. Derive a formula for the value of an insurance policy n years after January 1, 1930, from the following data: (Consider n to be greater than 10)
   (a) On January 1, 1930, its value is \$1,000.
   (b) At the end of each year after January 1, 1930, the value is increased by \$25.
   (c) At the end of each year after January 1, 1935, it is increased by an additional \$20.
   (d) At the end of each year after January 1, 1940, it is increased by an additional \$15.
   (e) At the end of every year after January 1, 1930, the value is diminished by an amount equal to the number of years elapsed.

Gamma Examination

Part I—(Gamma candidates begin working on question 61.)

61. In Figure 21, draw the graph of the equation $y = 4x - x^2$ from $x = -2$ to $x = 6$.

82. What is the sum of the three consecutive odd numbers of which the middle one is $2n + 1$?

103. What is the equation of the axis of symmetry of the curve $y = 6x - x^2$?

124. Draw a diagram you would use in deriving the formula for $\sin (x + y)$.

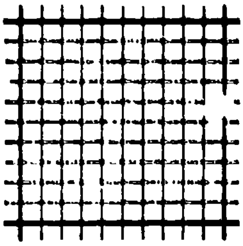


Figure 21

136. The sum of the sides of a spherical triangle is 240°. The spherical excess of its polar triangle is ____.

160. What is the relation between the roots of equations (a) and (b)?

(a) $2x^3 - \frac{3}{2}x - 2 = 0$

(b) $2x^3 - 3x - 1 = 0$.

Part IV—(Supplementary Directions): In questions in which you are asked to prove something it will be sufficient to write the important statements and reasons, omitting the formal statement of what is given and what is to be proved, and suppressing the most obvious details.

2. A light is at the center of a transparent spherical globe 10 inches in radius; the globe rests in a supporting ring two inches above its lowest point. What is the radius of the shadow of the ring cast on the pavement 6 feet below the light?

4. Find all the values of X between 0° and 360° which satisfy the equation, $9 \tan (45° - X) = \tan (45° + X)$.

7. Find the equation of the circle whose center is at the point (3, 4) and which passes through the point (2, 0). Prove that this circle also passes through the point (4, 0).

10. Show that any number which has 4 digits and which does not change when the order of the digits is reversed is exactly divisible by 11.

The University of the State of New York, Regents Examination. The Board of the Regents of the University of the State of New York conduct examinations twice a year for high school students attending schools in New York state. These are known as the Regents examinations. In mathematics, as Table V shows, six mathematical subjects are examined. The subject matter to be studied and the standard to be attained are indicated in the following extracts from syllabi and recent examination papers. The practice of setting a paper which combines a number of short-answer questions with a smaller number of essay type questions has been adopted in the Regents Examinations as well as in the College Entrance Examination Board's examinations. Table V also shows that intermediate algebra and plane geometry are taken by a majority of the candidates, that a considerable number take plane trigonometry, but that few take advanced algebra and solid geometry. The frequency with which these last two subjects are listed as mathematical courses in the catalogues of colleges, universities and teachers' colleges indicates that few candidates who gain admission to colleges have studied them previously. Since the Regents examinations are given twice yearly it is a common practice for students to be examined in only one or two mathematical subjects at the one examination.

An outline of the syllabus in each subject listed and representative questions from examination papers in them are given herewith:

Intermediate Algebra

Syllabus (1931 revision)

The language of algebra.

Formula: Types, interpretation, transformation, graphs, evaluation, construction.

Equations: Quadratic in one unknown.

Solution of the complete quadratic — relations between the coefficients and roots of the equation, nature of the roots.

Sets of two equations in two unknowns: first degree equations in three unknowns:

Radical equation, exponential equations

Checking.

Graphs: $y = mx + b$, $y = ax^2 + bx + c$, graphs of the simple conic sections.

Algebraic techniques: Factors, fractions, exponents and radicals, logarithms.

Series: AP and GP : binomial theorem.

Problems.

Functionality: Tables, formulas, graphs, equations, problems, variations, trigonometrical functions.

Examination Questions (1939)

Group I

Directions (Questions 1–8)—Indicate the correct answer to each of the following questions by writing on the dotted line at the right the letter *a*, *b*, or *c*.

2. The sum of the roots of the equation
$2x^2 - 5x + q = 0$ is (a) 5, (b) $\frac{-5}{2}$ or (c) $\frac{5}{2}$ 2. ....

5. When drawn on the same set of axes, the graphs of $y = 2x - 1$ and $y = -x + 3$
(a) do not intersect, (b) intersect in one point, or (c) intersect in more than one point. 5. ....

7. Log $\frac{a}{b}$ is equal to (a) $\frac{\log^a}{b}$, (b) $\frac{\log a}{\log b}$, or (c) $\log a - \log b$. 7. ....

Directions (Questions 9–25)—Write the correct answer in the space at the right.

12. The value of the discriminant of the equation
$x^2 - 4x + 1 = 0$ is.... 12. ....

15. The equation expressing the relation between x and y shown in the table

x	0	3	6	9
y	1	7	13	19

is .... 15. ..

18. The formula $F = \frac{Wv^2}{gr}$, when solved for W, is $W =$ .... 18. ....

22. In right triangle ABC, angle $C = 90°$, angle $A = 32°$, $AB = 10$; the length of AC correct to the nearest tenth is .... 22. ....

Group II

Answer three questions out of this group (out of five).

28. Using logarithms find, correct to the nearest hundredth, the value of

$$\sqrt[3]{\frac{18.2 \times \sin 42°}{.316}}$$

30. (a) Draw the graph of the equation $y = x^2 - 3x - 2$ from $x = -2$ to $x = 5$ inclusive
(b) On the same set of axes used in answer to (a), draw the graph of the equation $x + y = 1$.

(c) From the graphs drawn in answer to (a) and (b), determine the values of x and y common to the two equations.

Group III

Answer one question from this group.

31. Write the equations that would be used in solving each of the following problems. In each case state what the unknown letter or letters represent. (Solution of the equations is not required.)
   (a) A 5% solution of salt and water, weighing 100 pounds, is to be reduced to a 3% solution. How many pounds of water must be added?
   (b) The sum of the digits of a two-digit number is 11. If the digits are reversed, the resulting number is 20 less than twice the original number. Find the original number.

Group IV

Answer one question from this group.

35. An article costs a merchant $72. He wishes to find out at what price he should mark the article so that he can allow a discount of 10% of the marked price and still make a profit of 20% on the selling price. Answer each of the following, letting x represent the marked price:
   (a) What is the selling price in terms of x?
   (b) What is the profit in terms of x?
   (c) Write an equation in x which expresses the relation between the cost, the profit, and the selling price.
   (d) Find the value of x.

PLANE GEOMETRY

Syllabus (1931)

The syllabus for this course covers the work in the following groups of topics:

Rectilinear figures: Triangles, perpendicular and parallels, parallelograms, angle sum theorem, inequalities, loci, concurrent lines.
Circles: Arcs and chords, tangents, measures of angles, inequalities.
Similar figures: Similar triangles, applications of similar triangles, similar polygons.
Areas of polygons.
Measurement of a circle.
Constructions.

Examination questions (1939)

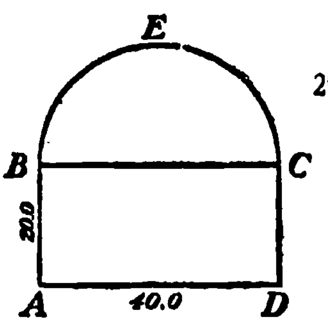
Group I

Directions (Questions 1–17)—Indicate the correct answer to each of the following questions by writing on the dotted line at the right the letter a, b or c.

4. If two parallel lines are cut by a transversal, the corresponding angles are always (a) supplementary or (b) equal or (c) acute 4. . . . .
8. If the corresponding sides of two similar triangles are in the ratio 1:2, the areas of the two triangles are in the ratio (a) 1:4, (b) 1:2 or (c) $1:\sqrt{2}$ 8. . . . .
11. If in the right triangle ABC, AB is the hypotenuse and CD is the altitude upon the hypotenuse, then (a) $(CD)^2 = AD \times DB$, (b) $(CD)^2 = AB \times AD$ or (c) $(CD)^2 = AC \times CB$ 11. . . . .
20. The formula for the circumference c of a circle in terms of its radius r is $c =$ 20. . . . .

Group II

28. Triangle ABC is inscribed in a circle. The bisector of angle C intersects side AB at D and arc AB at E. Prove: $AC \times BC = CD \times CE$ 28. . . . .



Group III

29. The accompanying figure $ABECD$ represents the cross section of an underground tunnel. $ABCD$ is a rectangle 40.0 feet by 20.0 feet, surmounted by the semicircle BEC. Find, correct to the nearest square foot, the area of the cross section. (Use $\pi = 3.14$)

Group IV

32. Consider each of the following statements and tell whether it is always true, sometimes true, or never true. Give reasons for your answers.
    (a) The area of a rectangle is equal to one half the product of its diagonals.
    (b) If two triangles have a side and any two angles of one equal to the corresponding parts of the other, the triangles are congruent.
    (c) If the radius of a circle is increased by x, then the circumference of the circle is increased by $2\pi x$.

(d) If the legs of a right triangle are represented by a and b and the hypotenuse by c, then $c^2 = (a + b)^2$.

PLANE TRIGONOMETRY

Syllabus (1934)

Functions of angles.
Radian measure.
Proofs and applications of the principal trigonometrical formulas:

$\sin (A \pm B)$;	$\cos (A \pm B)$,	$\tan (A \pm B)$
$\sin 2A$	$\cos 2A$	$\tan 2A$

Proofs of identities.
Graphs of the six trigonometrical functions.
Derivation of the general properties of the triangle.
Inverse trigonometrical functions.
Theory and use of logarithms.
Solution of triangles—four cases.
Problems.
Complex numbers—graphical representation of complex numbers.
Polar coordinates, de Moivres' theorem.

Examination Questions (1939)

Group I

Directions (questions 1–5)—Write the number that represents the value of each.

3. Sin $(csc^{-1} 2)$ 3. . . . .

7. If $\sin A = -\frac{4}{5}$ and A is in the fourth quadrant, what is the value of sin 2A? 7. . . . .

13. Express $\tan^2 \left(\frac{x}{2}\right)$ in terms of $\cos x$. 13. . . . .

19. Find the value of x between 90° to 180° for which $2 \sin^2 x - 1 = 0$. 19. . . . .

Group II

Answer at least two questions from this group.

21. (a) Find, correct to the nearest degree, the positive acute angle x that satisfies the equation $3 \cos^2 x + 8 \sin x - 7 = 0$.

(b) Prove the identity: $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

*25. Express in polar form:
   (a) The product of $1 + i\sqrt{3}$ and $1 - i$
   (b) The real root of $x^3 - 1 = 0$

* This question is based on one of the optional topics in the syllabus.

SOLID GEOMETRY

Syllabus (1936—revised in September, 1934)

Lines and planes in space.
Dihedral angles, polyhedral angles.
Loci and projections.
Prisms and pyramids.
Cylinders and cones.
Sphere—spherical triangles and polygons, polar triangles, measurement of a sphere

Examination Questions

Group I

Directions (questions 1–14)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

4. The lateral areas of two similar cones are 16 and 25. If the altitude of the smaller cone is h, the altitude of the larger cone in terms of h is .... 4. ....
9. The ratio of the diagonal of a cube to the diagonal of one of its faces is ....
   (Answer may be left in radical form.) 9. ....
14. The locus of points at a given distance from a given line is (a) two parallel lines, (b) two parallel planes, or (c) a cylindric surface. The correct answer is 14. ....
   (Answer a, b or c.)
19. The lateral faces of a frustum of a pyramid are trapezoids.
   (Answer "always, sometimes, or never.") 19. ....

Group II

21. Prove that if the first of two spheric triangles is the polar triangle of the second, then the second is the polar triangle of the first.
24. Derive the formula for the total area T of a frustum of a regular quadrangular pyramid whose base edges are a and b and whose slant height is 1.

Group III

28. The volume of a sphere is equal to that of a right circular cylinder whose altitude is 8.75 and the radius of whose base is 3.22. Find, correct to the nearest tenth, the radius of the sphere.

ADVANCED ALGEBRA

Syllabus (1932)

Number systems of algebra—complex numbers, fundamental operations with complex numbers, polar form, roots of equations $x^n = 1$.

Linear functions: $y = mx + b$, $y = mx$, lines parallel and perpendicular to given lines.

Quadratic functions: $y = ax^2 + bx + c$.

Solution of quadratic equations by factors, formulas, and graphs; relation between the roots of a quadratic equation.

Average rate of change of a quadratic function $(\Delta y \div \Delta x)$.

Instantaneous rate of change $\frac{(dy)}{(dx)}$.

Problems involving quadratics.

Graphic and algebraic solution of simultaneous equations.

Rational integral functions of the nth degree.

Solution of equations of degree higher than 2; graphs of cubic functions.

Approximate methods of solution: differentiation of rational integral functions.

Exponential and logarithmic functions.

Permutations and combinations, probability.

Differentiation: Notion of a limit: derivatives of rational integral polynomials, function of a function, fractions, irrational functions. Application to problems—maxima and minima, rates.

Examination Questions

Group I

4. If $f(x) = x^2 - x + 3$, what is the value of $f(-1)$? 4. ....

8. Write the equation whose roots are twice the roots of the equation

$$x^3 - \frac{x^2}{2} + \frac{3x}{4} - \frac{1}{4} = 0$$

8. ____

Directions (questions 10-14)—Answer yes or no.

12. Is the number 0.272727 .... rational? 12. ....

16. Find, correct to the nearest hundredth, the value of $\sqrt[3]{.0436}$ 16. . . . .
19. In a bus accident three out of the 20 passengers were reported injured. A family of three was in the bus. What is the probability that the three persons injured belonged to that family? 19. . . . .

Group II

22. Solve the equation

$$6x^4 + x^3 - 37x^2 - 6x + 6 = 0.$$

26. The breaking load P in tons which will crush a cast-iron strut with round ends is given by the formula

$$P = \frac{50d^{3.6}}{L^{1.7}}.$$

where d is the number of inches in the diameter and L is the number of feet in the length. Find P correct to the nearest ton when $d = 4$ and $L = 20$.

*29. Given $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2$
  (a) Find the coordinates of the maximum point.
  (b) Find the coordinates of the minimum point.
  (c) Find the coordinates of the point of inflection.
  (d) Sketch the curve.

* This question is based on one of the optional topics in the syllabus.

Ohio State Department of Education—Every Pupil Tests. The State Department of Education in the state of Ohio issues a Curriculum Guide for the direction of teachers in the high schools of the state, and prepares Every Pupil Tests in arithmetic, plane geometry, and elementary algebra. The Ohio Curriculum Guide for mathematics in grades nine and ten sets forth the general and specific objectives of the course, together with suggestions as to methods of teaching and an outline syllabus. Quotations from the syllabus and examination questions are given here.

NINTH GRADE. ELEMENTARY ALGEBRA

Syllabus

1. The language of algebra, including vocabulary of terms used in algebra.

2. The formula: evaluation, construction.
3. The equation: first degree equations in one unknown, sets of linear equations in two unknowns, graphic and algebraic solutions. Incomplete quadratic equations in one unknown.
   Complete quadratic equations in one unknown.
   Checking the solution.
4. Directed numbers—graphical representation, use.
5. Graphs—interpretations, construction, graphical solution of problems as picture of relation between two variables.
6. Algebraic technique—four fundamental operations, factoring, fractions, exponents and indices.
7. Problems.
8. Numerical trigonometry.
9. Relationship between variable quantities—in tables, graphs, of the form $y = mx + b$, $y = ax^2 + bx + c$, in trigonometrical functions.

Examination Questions from
Every Pupil Test in *Elementary Algebra* (1938)

Part I

Directions: Perform the indicated operations and place your answer on the line to the right of the problem. Simplify each answer by combining all like terms and reducing each fraction to lowest terms.

3. $4m - 6 = 2m - 9$, $m =$ ....
7. $6a - 2(3a - 2b) =$ ....
11. A binomial has how many terms? ....
17. If a bar graph 4 inches long represents 16000, how long a bar should be used to represent 10000? .... inches.
20. $3ab(3a^2b - 6ab^2) =$ ....

TENTH YEAR. PLANE GEOMETRY.

Syllabus

Emphasis is placed on problem solving and the nature of deductive proof as general objectives.

Specific objectives are:

I. The ability to make, to prove, and to use the following constructions:
   (Then follow some fourteen constructions based on circles, angles, line segment, triangles, regular polygons, areas of equivalent rectilinear figures, locus problems.)

II. Ability to understand and to use the following theorems:
   A. Geometry of One or Two Dimensions. (Then follows a list of some eighteen groups of theorems covering the usual range of theorems in plane geometry.)
   B. Geometry of One, Two, or Three Dimensions.[23] (Includes theorems on lines and planes, lateral area, and volume of pyramids, prisms, cylinders, cones: area and volume of a sphere.)

III. The ability to use understandingly the technical vocabulary of geometry . . .

This statement of objectives is followed by a short statement of general principles of method in teaching geometry. In this statement the various methods of teaching are discussed and illustrated.

Examination Questions, from
Every Pupil Test in *Plane Geometry* (1938)

Part I. General Information

Directions: Place the letter of the part which best completes the statement on the line provided.

. . . . 1. Each of two equal supplementary angles contains (a) 60° (b) 90° (c) 45° (d) 100°.

. . . . 4. The greatest number of acute angles which a parallelogram may have is (a) four (b) two (c) three (d) one.

. . . . 7. In an obtuse triangle, two of the exterior angles, formed by extending the sides in succession, are (a) obtuse (b) right (c) acute (d) equal.

Part II. Supplying Reasons

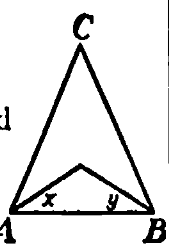
Directions: Study the following *figures*, the *given facts*, and *what is to be proved*. In the list of reasons may be found *one* of the reasons which is needed in the proof of an exercise. Select the *proper reason* in each case. Then place the letter corresponding to that reason on the blank provided.

[23] The following comment is added as a footnote to this part of the syllabus: "Not until both plane and solid geometry are more widely combined into a year course in geometry can much of this material be included in the tests. It is included here in order to stimulate the elimination of some less useful plane geometry and the inclusion of some solid geometry in the year course. Since most students can elect only one year of geometry it seems unfortunate for all of it to be two-dimensional."

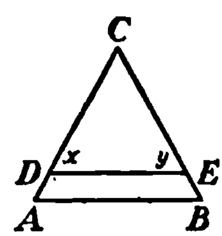
Reasons

A. If equals are added to equals, the sums are equal.
B. Complements of the same angle or of equal angles are equal.
C. An exterior angle of a triangle equals the sum of the two nonadjacent interior angles.
D. If two parallels are cut by a transversal, the corresponding angles are equal.
E. Halves of equals are equal.
F. If two angles of a triangle are equal, the sides opposite these angles are equal.
G. Side, angle, side = side, angle, side.
H. Angle, side, angle = angle, side, angle.
I. Side, side, side = side, side, side.

.... 12.
Given:
$AC = BC$
Angles A and B disected
Prove:
$\angle x = \angle y$



.... 15.
Given:
$CA = CB$
$DE // AB$
Prove:
$\angle x = \angle y$

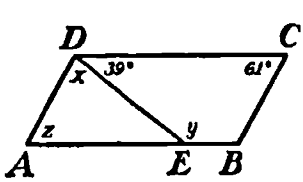


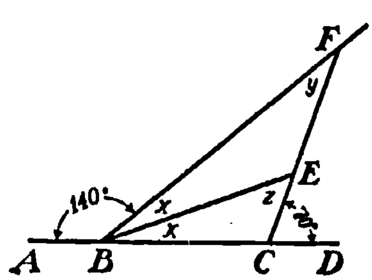
Part III. Numerical Problems

Directions: After studying the following figures and the given facts, compute the correct answers.

Given: $AB \parallel DC$ and $AD \parallel BC$.
$\angle CDE = 39°$ $\angle C = 61°$
.... °25. $\angle x$ = ???
.... °26. $\angle y$ = ???
.... °27. $\angle z$ = ???

Given: BE bisects CBF.
$ABF = 140°$
$FCD = 70°$
.... °19. $\angle x$ = ???
.... °20. $\angle y$ = ???
.... °21. $\angle z$ = ???

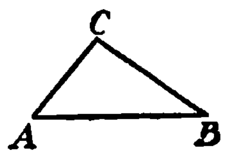




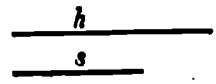
Part IV. Constructions

Directions: Perform the following constructions, using compasses and straight edge.

34. Given the triangle ABC, construct the median from C.



37. Given the hypotenuse h and the side s, construct a right triangle.



Textbooks. It has been stated above that the more modern school textbooks in mathematics in the United States have been written to meet the recommendations of the College Entrance Examination Board and the National Committee on Mathematical Requirements, and the requirements of state examinations. In some recent geometry textbooks informal geometry is dealt with in considerable detail in the early chapters and then follows the formal development of theorems in accordance with some sequence. Since plane geometry is usually studied for one year only, the tendency is to reduce the number of theorems to be proved, and to build up a consistent unit embracing this smaller number of theorems. Another characteristic of these textbooks is that, following recent investigations in the teaching of geometry, an effort has been made to apply the modes of thinking used in geometry, such as, for an example, the If-then formula, to non-geometrical data.

In elementary algebra the textbooks follow a plan of development which has become well established since Nunn's books on algebra appeared. The formula, graph, equation, directed numbers, graphs of functions, and quadratic equations are developed usually in that order. There is a strong tendency to delete from algebra textbooks the harder mechanical operations not necessary for use in the most difficult formulas taught, and to include in them problems whose solution requires the use of the simpler forms of these operations. Emphasis is also placed on the application of the technique of algebra to problems of everyday life as well as to problems which help to correlate the various branches of elementary mathematics.

From time to time textbooks have been written which depart from the organization of one subject in one book. The names given to the content of these texts are correlated mathematics, parallel mathematics, integrated mathematics, cooperative mathematics, and general mathematics. The particular use that is made of the subject matter of arithmetic, algebra, geometry, and trigonometry by the authors of these textbooks depends largely on their own personal points of view. For example, the term general mathematics has been used to describe that organization of subject matter which places "emphasis upon correlation only to the extent that the correlation represents the natural relationships of the subjects."[24] Again

In cooperative mathematics, each branch, arithmetic, algebra, geometry, etc., keeps its own individuality and organization, but at the same time it transfers from other branches and welcomes to itself any detail which may be logically useful.[25]

Textbooks in the more advanced school subjects follow a more logical development. They cover the range of topics required by the syllabi of the examinations boards and other bodies which prepare syllabi.

Commissions on the Teaching of Mathematics. Because of their importance in molding the type of mathematics which prospective mathematics teachers may have to teach, and because their recommendations will to some extent influence the discussions in methods courses in teacher training institutions in the future, a brief account is here added of the work of the Joint Commission on the Place of Mathematics in Secondary Education and of the Progressive Education Association on Mathematics in General Education.

Report of the Joint Commission. This commission has prepared a tentative report on the *Place of Mathematics in Secondary Educa-*

[24] C. McCormick, *The Teaching of General Mathematics in Secondary Schools of the United States*, p. 6.

[25] *Ibid.*, p. 7.

tion. This report contains nine chapters, the first four of which deal with the following problems: general aims and general objectives of secondary education, the rôle of mathematics in civilization, and the place of mathematics in education. The more mathematical part of the report is considered in the remaining five chapters, which deal respectively with the mathematical curriculum, the distribution and organization of materials, modified curriculum plans, mathematics in the junior college, evaluation in mathematics. In addition there is a section on the transfer of training.

Perhaps the most significant part of the report for present purposes is that dealing with the distribution and organization of materials. To focus attention on the problem of what should be done in the way of organizing materials, the Commission prepared a Grade Placement Chart on which was shown the suggested placement of mathematical topics in grades seven through twelve. An important feature of this chart is its implied advocacy of the teaching of several mathematical topics in each grade. In the words of the report:

> Continuity and flexibility are two of the primary requisites of a satisfactory mathematical program for our secondary schools. The accompanying Grade Placement Chart suggests a feasible plan to meet these requirements. The plan is intended for pupils of normal ability who have had good training; many of its features have already been successfully adopted in representative American schools. . . .
>
> In organizing the curriculum in terms of the major mathematical fields, care was taken to insure recurrence and thus give a sense of continuance. At the same time the program allows flexibility, for it is possible to assign parts of the fields to other school years than those shown. The work to be done in the different fields—number, form, and so on—extends through each year of the program, each year being expected to deepen and extend the pupil's experience in all the categories listed.[26]

This recommendation would tend to break down the present organization of subjects in separate units as arithmetic, algebra, geometry, and trigonometry. The suggestion is that two or three subjects should be emphasized in each school year and that certain

[26] *The Place of Mathematics in Secondary Education*, p. 108.

topics should be carried forward from year to year to preserve continuity of ideas.

Thus in grades seven and eight it is suggested that the principal mathematical subjects to be studied are arithmetic and informal geometry, with emphasis on mathematical modes of thinking. Aspects of the work in arithmetic and geometry, such as graphic representation and scale drawing, are introduced to prepare the way for the introduction of algebra and trigonometry later. Similarly in the tenth grade the major emphasis is to be on plane geometry as heretofore, but pupils are also taught how to apply the type of deductive thinking they learn in geometry to other situations. In addition they maintain contact with arithmetic, algebra, graphical work, and trigonometry which have been taught in previous years.

In the twelfth grade the central theme changes to advanced algebra, and opportunity is taken to introduce the calculus here. The suggested plan then makes provision for each of the following topics in each grade, the topic to be emphasized varying with the grade: arithmetic, geometry, graphic representation, algebra, trigonometry, mathematical modes of thinking, history of mathematics, correlated mathematical projects or activities.

Progressive Education Association. A point of view regarding the type of mathematics to be taught in schools which is somewhat different from that adopted by the Joint Commission is represented in a tentative report entitled *Mathematics in General Education*, prepared by a committee of the Progressive Education Association. Both reports pay tribute to the National Committee's 1923 report, and both reports point out the defects of parts of that report for present conditions in secondary schools. The report on *Mathematics in General Education* takes as its starting point the needs of the individual. In a chapter on "reflective thinking" it is pointed out that mathematics teachers have no monopoly of subject matter which promotes this type of thinking, so that, provided the situation dealt with by the pupils help their reflective thinking, "reference to mathematical material may well remain secondary or even casual."[27]

[27] *Mathematics in General Education*, A Report of the Progressive Education Association, 1938, p. 9.

The second part of the study deals with

> Major understandings growing out of mathematical experience, ranging from the semi-mathematical to the specifically mathematical. We discuss there, under suitable chapter headings, such concepts as measurements, tables, functions, and a brief exposition of the sort of understanding a well trained teacher should possess. This is followed by a list of objectives which summarize the behaviors pupils should ultimately develop. . . .
>
> The material presented is not intended to be a definitive body of subject matter for widespread verbatim textbook adoption. We hope rather to have it serve as an indication of the spirit in which the teaching may well be done.[28]

In this part chapters are devoted to analyzing problem situations, formulation and solution, data, approximation, functions, mathematical operations, symbolism and communication, proof and generalization, historical evolution.

In order further to illustrate the point of view adopted in this report the chapter on mathematical operations will be analyzed. The Commission seek to put the operational processes in mathematics in their proper place, which is, in elementary mathematics, as tools for the solution of problems. Emphasis is placed on understanding what the operations mean, what they will do and how they can be used, in addition to the importance of memorizing certain of them. These principles apply equally well in arithmetic, algebra, geometry, and calculus. The criteria of understanding, and of use in problems lead the Commission to question the desirability of continuing to teach in schools certain operations which are now being taught in mathematical classes. On this point the Commission say:

> But what the point of view of this Report really means is that practice in many techniques should be postponed until they are required in the study of some real problems. While this means that many pupils will never learn these techniques at all, it does not mean that they may not profit by rich mathematical experiences which are far more important than acquiring a technical proficiency which is rarely useful.[29]

[28] *Mathematics in General Education*, A Report of the Progressive Education Association, 1938, Introduction, p. 10.
[29] *Ibid.*, Chap. VII, p. 9.

To illustrate the general discussion given in the first two parts the Commission added a third part on *A Problem of Adolescents: Am I Normal?* The mathematical possibilities of this subject are explored and are shown to include (1) collection of data on factors chosen for study, (2) tabulation of data, (3) analysis of tabulated data leading to the introduction of the essential ideas and techniques of elementary statistics.

Summary. In this section mention was made of the various forces which contribute to form the present complex situation in mathematics in the secondary schools of the United States.

1. It was shown that the principal contribution of these agencies was represented to a considerable extent by the contribution of the external examinations boards.

2. An outline of the mathematics syllabi prepared by the two principal external examinations boards and by one state department was given, and selected examination questions were cited to illustrate the standard in mathematics which the schools were expected to reach in the subjects mentioned. The syllabi were in the nature of outline suggestions of topics to be taught. The analysis of examination papers revealed the importance attached to a type of mathematical test paper consisting of many short-answer questions and a smaller number of essay type questions

3. The practice of allowing high school pupils to elect subjects results in the choice of elementary algebra and plane geometry by the majority of pupils taking mathematics. Pupils expecting to study mathematics at a university, college, or teachers college may take one or more of the following subjects in addition: intermediate algebra, trigonometry, solid geometry. In a few schools analytic geometry and calculus are offered in addition to the above subjects.

4. Reference was made to the method of organizing the subject matter of mathematics in units. The discontinuity in teaching a particular subject which results from the grade placement of these units of subject matter was remarked upon.

5. Brief reference was made to the leadership directed toward improving mathematical instruction in schools, which has been offered by the various commissions on the teaching of mathematics.

6. The influence of textbooks on mathematical teaching was also discussed. It was shown that most modern textbooks claim to follow the recommendations of the National Committee in its 1923 report on *Reorganization of Mathematics in Secondary Education* and the requirements of the College Entrance Examination Board.

COMPARISONS

In this section of the study it has been shown that mathematics occupies a more prominent place in secondary schools of England and Wales than in the corresponding schools in the United States. This fact has been illustrated by stating the percentage of secondary school pupils studying mathematics, and by showing the range of mathematical subjects taught in secondary schools in the two countries. The lower percentage of secondary school pupils in the United States studying mathematics is probably due to the heterogeneous nature of the secondary school population in that country, and to the system of electives whereby pupils have a certain freedom of choice in the subjects which they study in grades ten through twelve.

The range of mathematical subjects taught in all grades in secondary schools in the United States includes elementary algebra, intermediate algebra, plane geometry, trigonometry, advanced algebra, solid geometry, elementary plane analytics, and elementary calculus. While these same subjects are taught in secondary schools in England and Wales, it has been shown, by quoting from syllabi and examination papers used in the two countries, that the standard of work in each of these subjects at the higher certificate examination stage in English and Welsh schools is higher than that reached in any grade in secondary schools in the United States. One direct consequence is that the abler pupils in schools in the United States who early exhibit a leaning toward and ability in mathematics do not have opportunities to develop their interest and power in mathematics comparable to those provided for students of equivalent ability in England and Wales. This is further shown by the fact that pupils entering universities in England and Wales to specialize in mathematics have studied such an amount of mathematics as to place them in this subject, according to

Cairns,[30] about two years in advance of pupils of corresponding age in schools in the United States.

While the syllabi and examination papers reveal that the same kind of subject matter in mathematics is examined in the external examinations in the two countries, an examination of widely used textbooks in the two countries revealed a difference in emphasis. For example, in English and Welsh textbooks in geometry the subject is presented in a rather formal manner, while in textbooks used in the United States there is a good deal of departure from the more formal treatment of the subject in the direction of illustrating the applications both of geometrical facts and of geometrical modes of thought in life situations. The application of mathematics to life situations is made explicit by writers of American textbooks but in English and Welsh textbooks these applications are implicit rather than explicit.

The associations of mathematics teachers in the two countries are active in the interests of mathematics teaching in their respective countries. To this end they prepare reports which both summarize for teachers the best current practice in teaching the various mathematical subjects, and indicate the trend of thought in the teaching of mathematics.

[30] W. D. Cairns, "Advanced Preparatory Mathematics in England, France and Italy," *American Mathematical Monthly*, XLII (January, 1935), p. 25.

Chapter V

ACADEMIC PREPARATION OF MATHEMATICS TEACHERS

ENGLAND AND WALES

Universities in England and Wales. In England and Wales there are two types of universities—residential and day. The Universities of Oxford and Cambridge belong to the first type and the University of London, its associated university colleges in the provinces, and provincial universities belong to the second type. The distinction between a university and a university college in England and Wales is roughly that the former is a degree-granting institution while the latter is not. Students in the university colleges associated with the University of London are awarded degrees by the University of London. The University of Wales is composed of four university colleges, each of which prepares students for degrees which are awarded by the University of Wales.

The academic year in universities in England and Wales is organized in three terms of ten weeks each and covers the period September to June. No regular summer terms are organized in these universities.

The universities in England and Wales offer curricula leading to degrees and diplomas. They offer degrees in Arts, Science, and other faculties, and a number of diplomas, including a Diploma in Education. The Bachelor's degrees are of two kinds, a degree in honors and a degree variously known as an Ordinary or General or pass degree. The courses required for the ordinary degree usually represent two or three years of work beyond the secondary school. In the University of London and in provincial universities there is a qualifying stage on the way toward a Bachelor's degree represented by the intermediate degree examination which is taken at the end of one year's work in the university (except by those who receive exemption at the higher certificate examination). The work for the Bachelor's degree (ordinary or general) occupies two years beyond the completion of the intermediate degree examination and thus may be accomplished in two years by those students who receive exemption from the intermediate degree examination. Bachelor's degrees in honors (B.A. Honors, B.Sc. Special) are granted at the end of three years' work after the intermediate degree examination. The distinction between a pass or ordinary or general degree and an honors degree rests solely in the amount of specialization which is implied in the latter. Separate courses of lectures are usually given to students reading for the two types of degrees. Whereas for the pass degree three or four subjects are studied less intensively, for the honors degree one or two subjects are studied in a more comprehensive, and at the same time, more detailed manner. Separate examinations for pass and honors Bachelor's degrees are conducted annually.

The quality of the work done by students in the honors examinations in each subject is recognized by the award of first, second, or third class honors in that subject. In the University of London honors may also be awarded on the work done in the courses leading to the general degree in science. The universities also award the M.A. and M.Sc. degrees as well as doctoral degrees. Regular courses leading to these degrees are sometimes offered, and the student reading for one or other of these higher degrees is encouraged to consult with the head of the department in which he wishes to work. In the University of London regular lecture courses are offered covering part of the requirements for the M.A. degree in education.

Certification Requirements. Prospective teachers in secondary schools in England and Wales are not required by law to have any specific qualifications, academic or professional. They need not hold a university degree nor need they hold a certificate of any kind to prove professional competence. In practice a degree in first or second class honors in the subject they propose to teach is demanded of them. Professional training is less frequently demanded by prospective employers, who in many cases act on the principle that experience, personality, and scholarship are the most important qualifications for teaching. Nevertheless the percentage of trained teachers in secondary schools is increasing, as will be seen from the data in Table VI, compiled from tables in the Board of Education's Annual Reports[1] on education for 1927, 1932, and 1937.

TABLE VI

FULL-TIME TEACHERS IN SECONDARY SCHOOLS ON THE GRANT LIST IN ENGLAND AND WALES

Assistant Teachers in Secondary Schools	1937 Men	1937 Women	1937 Total	1932 Total	1927 Total
Graduates					
Trained	6071	5271	11,342 (64%)	9102 (58.6%)	6336 (51.6%)
Not trained	4015	2357	6,372 (36%)	6422 (41.4%)	5941 (48.4%)
Non-graduates					
Trained and certificated	340	656	996	1058	1161
Certificated and not trained	36	52	88	130	171

Table VI shows that the number of graduates teaching in secondary schools in England and Wales greatly exceeds the number of non-graduates. Of the graduates, some 36 per cent (6372 out of 17,714) of those employed in secondary schools in 1937 had received no formal professional training. The corresponding percentages for 1932 and 1927 are respectively 41.4 per cent and 48.4 per cent, showing that the trend is toward the gradual elimination of the *not trained* graduate teacher in the secondary schools. Of the few non-graduate assistant teachers employed in the secondary schools all are certificated but some have not been trained. The distinction to be kept in mind here is that *trained* means that the teachers have completed a course of training in a training institution, and *certificated* means that the teachers have not attended a training institution but have passed a certification examination. The certification examinations were formerly conducted by the Board of Education, but since 1929 certificates to teach are awarded by the Board only to graduates of a training institution. The table also shows that the number of non-graduate teachers in secondary schools has steadily decreased during the ten-year period

[1] *Education in England and Wales*, 1927, p. 156, Table 41; 1932, p. 141, Table 41; 1937, p. 140, Table 46.

1927–1937, and that untrained non-graduate teachers have practically been eliminated from the secondary schools.

Since the overwhelming majority of teachers in the secondary schools in England and Wales is composed of university graduates, it will be necessary, for the purposes of this study, to investigate only the academic preparation in mathematics which prospective mathematics teachers receive at the universities. No separate discussion of the academic training of mathematics teachers in senior and central schools is necessary, for two reasons: In the first place, no specialist teachers in mathematics are employed in these schools; and, in the second place, an increasing number of graduates are receiving appointments to teach in them, and these graduates take much the same professional courses as do those preparing to teach in secondary schools.

The academic qualifications of mathematics teachers in secondary schools in England and Wales vary considerably. While a majority of new appointees are graduates in honors Class I or Class II in mathematics, teachers with lower academic qualifications are sometimes appointed. Table VII, compiled from data supplied by four selected educational authorities in England, shows the distribution of mathematics teachers in the service of these authorities according to their academic qualifications in mathematics.

TABLE VII

MATHEMATICS TEACHERS IN SECONDARY SCHOOLS UNDER FOUR SELECTED EDUCATIONAL AUTHORITIES IN ENGLAND

Authority	Mathematics Teachers in Secondary Schools						
	Honors Graduates			Pass Degree	Non-Graduates	Untrained	Total Mathematics Teachers
	Class I	Class II	Class III				
1	24 (10)*	27 (9)	10	16 (9)	3 (1)	29	80
2	1	3	2	2	—	1	8
3	4	—	—	3	2	5	9
4	17 (5)	30 (7)	7 (4)	50 (38)	9 (9)	63	113
Total	46	60	19	71	14	98	210
Per cent	21.9	28.6	9.1	33.8	6.6	46.7	

* Figures in parentheses indicate number of untrained teachers.

Table VII also shows that slightly over 50 per cent of the mathematics teachers in the service of these four educational authorities are graduates in honors Class I or Class II in mathematics, and that nearly 50 per cent of them are untrained. The data in Table VII give a sample for mathematics teachers of the information presented in Table VI for all assistant teachers in secondary schools.

Academic Preparation of Mathematics Teachers. The amount of mathematical knowledge that prospective teachers may have acquired during their school course has been set forth in some detail in Chapter IV. A teacher who aspires to teach mathematics in a secondary school will take a course in his secondary school leading to a Higher School Certificate in mathematics, with this subject counting as either one or two principal subjects.

In both cases he will be eligible to enter the honors courses in mathematics in a university. In provincial English universities, as in the University of Wales, mathematics may be studied in courses leading to an Arts or to a Science degree. A non-specialist in mathematics who has matriculated for the university normally reads for an ordinary or general degree. In the intermediate degree examination at the end of his first year at the university the student takes four subjects, those other than mathematics being selected from a language group or from a science group according as a degree in Arts or in Science is sought.

The course of study in mathematics for the intermediate degree examination in provincial universities in England and Wales usually consists of the following subjects:

Pure Mathematics
- Algebra
- Geometry
- Elementary analysis
- (Trigonometry and calculus)

Applied Mathematics
- Statics
- Dynamics
- Hydrostatics

It is examined by means of four three-hour papers two in pure mathematics and two in applied mathematics.

After successfully negotiating the intermediate degree examination the student selects three subjects from the following list and studies them for the B.A. (Ordinary) or B.Sc. (General) degree.[2]

B.Sc. (General) Degree
- Pure mathematics
- Applied mathematics
- Physics
- Chemistry
- Botany
- Zoology
- Geology
- Geography
- Psychology
- Physiology
- Military studies

B.A. (Ordinary) Degree
- Pure mathematics
- Applied mathematics
- Languages (classical and modern)
- History
- Geography
- Theology
- Botany
- Zoology
- Military studies

The same courses of study in mathematics usually serve for both of these degrees. The mathematical subjects usually studied in the first year for these two degrees are shown in the following list.

Pure Mathematics
- Higher algebra
- Trigonometry
- Geometry (pure and/or analytic)
- Elementary analysis
- Differential and integral calculus

[2] Data quoted, by permission, for the University of London. The same conditions usually apply in other provincial universities in England and Wales.

Applied Mathematics
Mechanics
(Statics, dynamics, hydrostatics)

During the second and third years, the following subjects are ordinarily studied:

Pure Mathematics
Higher algebra
Analytic geometry (2 and 3 dimensions)
Pure geometry
Advanced differential and integral calculus
Ordinary and partial differential equations

Applied Mathematics
Statics (harder)
Dynamics of rigid bodies

To show more clearly the range of work done in these courses and to facilitate comparisons which may be made later, a syllabus of work is given below. While this particular syllabus refers to the work for the Ordinary or General degree in the University of London it corresponds in its main outlines with the work done in other provincial universities in England and Wales.

Syllabi in Mathematics for the Ordinary Degree. The syllabi are as follows:[3]

PURE MATHEMATICS

Pure Mathematics of the Intermediate Course together with:—

Section A.—Summation of easy series. Determinants of the second and third order. Use of Binomial, Exponential, Logarithmic, Sine and Cosine Series; and their Convergence. Hyperbolic functions. Exponential values of sines and cosines. Complex numbers with their representation in a plane. De Moivre's theorem and easy applications.

Section B.—The simpler properties of plane figures involving straight lines and conic sections treated by the methods of coordinate geometry (excluding the use of homogeneous co-ordinates).

Section C.—Simple differentiation and integration of functions of one variable. Maxima and minima of functions of one independent vari-

[3] Quoted from the Regulations of the University of London by permission of the University.

able. Use of Taylor's and Maclaurin's theorems. The principle of superposition of small errors. Tangents, normals, and curvature of plane curves. Areas and lengths of curves. Volumes of solids of revolution.

Section D.—(Alternative with E). The simple properties of plane figures involving straight lines and conic sections treated by Euclidean or projective methods. Geometrical drawing illustrating the theory of such methods.

Section E.—(Alternative with D). The simple properties of plane figures involving straight lines and conic sections treated by Euclidean methods. The straight line, the plane, the sphere, the ellipsoid referred to its principal axes. Ordinary differential equations of the first order. Linear equations of the first and second orders with constant coefficients. Simple cases of the integration of differential equations of the first and second orders with two independent variables.

Candidates are required to take Sections A, B, and C, and either Section D or Section E.

Paper I will consist of Algebra and Trigonometry and Analytic Geometry; Paper II of Analytic Geometry and Calculus; Paper III of the alternatives under Sections D and E of the syllabus.

Applied Mathematics

The Applied Mathematics of the Intermediate Course treated more fully, a knowledge of Elementary Differential and Integral Calculus being assumed, together with:

Simple parts of uniplanar kinematics.

General dynamical notions.

Motion of a particle under a constant force. Motion of a particle under a central force varying inversely as the square of the distance or as the distance.

Rotation of a rigid body about a fixed axis, and simple cases of uniplanar motion of a rigid body.

Elementary theory of potential. (See Note below.)

Common catenary and suspension bridge.

Pressures of liquids on plane areas, pressure of liquids on solid bodies wholly or partially immersed, and conditions of equilibrium of such bodies, specific gravities and densities of substances, solid and liquid, and methods of determining them. Mata-centre in simple cases.

The elementary methods of graphical statics.

Note.—Candidates who are not offering Astronomy as one of their subjects for the B.Sc. (General) Examination may offer the following as an alternative to Elementary Theory of Potential:—

The celestial sphere and systems of co-ordinates. Diurnal and annual phenomena. Measurement of time. Phenomena depending on the observer's position. The principal astronomical instruments (omitting considerations of their errors.) General effects of the precession of the equinoxes. Simple treatment of phenomena depending on the motions of the moon and planets. Principles underlying the methods of estimating the distances of celestial bodies. Candidates will be expected to have an elementary knowledge of the historical development of Astronomy as far as the middle of the 17th century.

Examinations. The mathematical work described below is examined by means of five or six three-hour examination papers—three papers in pure mathematics and two or three papers in applied mathematics. The questions set are nearly all of the problem type, though sometimes the reproduction of bookwork is required in answer to part of a question. Twelve or more questions are usually set in each paper and full credit is received for seven or eight questions completely answered.

Honors Degrees. Students reading for a B.A. (Honors) or B.Sc. (Special) degree in mathematics take a course of study entirely different from that described above. To begin with they will normally be excused from the intermediate degree examination, because a condition of admission to the honors classes in mathematics is the holding of a Higher School Certificate in mathematics, and this, together with certain other requirements, is regarded by the universities as equivalent to work at the intermediate degree level.

Once admitted to the courses leading to an honors degree in mathematics prospective teachers may study mathematics alone for an honors degree. In some English universities a subsidiary subject is taken along with mathematics (pure and applied) and this subsidiary subject is selected from a group of science subjects in the case of the B.Sc. (Special) degree, and from a group of language and literary subjects in the case of the B.A. (Honors) degree.

The honors courses may extend for two, three, or four years beyond the school stage, but a student does not formally receive a degree till after the end of three years of post-school study. If the honors examination in mathematics is taken at the end of the second year, the third year is usually devoted to a study of advanced mathematics, on which the student is also examined. The content of the honors courses in mathematics covers a wide range of subjects. In addition to the mathematical subjects required for an ordinary or general degree the candidates for honors have the chance of specializing in pure mathematics or in applied mathematics. But this specialization is additional to a basic requirement of both pure and applied mathematics which all candidates for honors must take. The basic courses additional to those prescribed for the general degree are as follows:

Pure Mathematics
- Theory of functions of a real variable
- Differential equations (including partial differential equations and solution in series)
- Advanced calculus
- Differential geometry

Applied Mathematics
- Hydrodynamics
- Higher mechanics

Specialization for honors in pure mathematics or in applied mathematics requires the study of several of the following subjects in the field of specialization in addition to the above-mentioned courses:

Pure Mathematics
- Theory of functions of a complex variable
- Elliptic functions
- Projective geometry
- Special functions (Bessel, Legendre)
- Calculus of variations

Applied Mathematics
- Theory of potential
- Electricity and magnetism (mathematical theory)
- Astronomy

Theory of elasticity and sound
Relativity
Wave mechanics

Examinations for the Honors Degree. The examinations for the honors degree in mathematics are more comprehensive than those for the ordinary degree. For the honors degree from six to eight three-hour papers are set, and, in some universities, an essay on a mathematical subject is also required. Each paper contains eight or nine questions. In the pure mathematics papers the questions frequently require the reproduction of a difficult piece of bookwork, but in the applied mathematics papers the questions consist largely of problems with comparatively few pieces of bookwork to be reproduced.

The Universities of Oxford and Cambridge. No account of the academic training in mathematics of prospective teachers of mathematics in England is complete without reference to the special contribution of the Universities of Oxford and Cambridge to this training. In the days when the public and grammar schools were the only types of schools offering education at the secondary level, and when mathematical teaching in these schools had already graduated out of the control of itinerant spelling and arithmetic masters who taught these subjects in odd periods and on half holidays, the Universities of Oxford and Cambridge were the only source of mathematics teachers. They enjoyed this monopoly for many years, and during these years they exerted a dominating or even controlling influence over the mathematics taught in the schools. The advent of institutions for professional training of teachers and the slow recognition of the value of professional training in the preparation of teachers have supplemented rather than lessened their influence. For their influence has been both direct and indirect. Direct, because a great many of their graduates are engaged in teaching mathematics in the secondary and public schools in England; and indirect, because others of their graduates hold professorships and lectureships in mathematics in provincial universities in England, and these in their turn are responsible for the academic training in mathematics of students who gain employment in secondary schools without professional training as well as of those who complete a course of professional training in the local University Training Departments before gaining employment in secondary schools. The kind of mathematical training offered in these two universities may be illustrated by an examination of their courses of study in mathematics.

Cambridge University. Students reading mathematics for the B.A. degree in the University of Cambridge usually read for honors in mathematics. The honors examinations in mathematics are known as the Mathematical Tripos which consists of three parts. Part I is usually taken at the end of the first year, Part II at the end of the second or third year, and Part III at the end of the third or fourth year. In the years when a candidate is not being examined in one of the Parts of the Tripos he may take a Preliminary Examination. The subjects of the Preliminary Examination in mathematics are algebra, geometry, analysis, statics, dynamics, astronomy, mathematical methods, electricity, hydrodynamics.

The subjects examined in Part I of the Tripos are algebra, analysis, geometry, mechanics (including hydrostatics), electricity, and magnetism. On these subjects six three-hour examination papers are set, and for honors in Part I a candidate must do satisfactory work in both pure mathematics and applied mathematics. For a B.A. degree in honors in mathematics alone a candidate must also obtain honors in Part II of the Tripos. The subjects for Part II are: algebra, analysis, differential equations, geometry (including differential geometry), mechanics of solids and fluids, wave theory, astronomy, electricity, and magnetism. Those subjects are also examined by six three-hour problem papers. Part III of the Tripos does not contribute toward a candidate's honors in mathematics in Cambridge, but an additional recognition is given to the candidate who satisfies the examiners in this part of the Tripos. The subjects of examination for this Part III cover a wide range of mathematical subjects and the practice is for candidates to inform the examiners, through the proper channels, of the range of subjects in which they wish to be examined. Usually two or three such subjects are designated by a candidate who has to stand six three-hour papers in this part of the Tripos examination also. The range of subjects offered by students for this part of the Tripos is very wide, and during the current academic year some forty-nine separate advanced courses in pure and applied mathematics are being offered.

To cover the work required for the two parts of the Tripos leading to an honors degree, a student would normally attend about twelve lectures a week, though it must be understood that attendance at lectures is not compulsory in Cambridge.

Oxford University.[4] Students reading mathematics for the B.A. degree in Oxford University may do so either for a pass degree or for an honors degree. The course for the former extends over three years and for the latter over four years. In order to obtain a B.A. degree in Oxford a student must pass three examinations: I. Responsions or its equivalent for entrance to the University; II. First Public Examination (or Moderations, pass or honors) at the end of the first or second year; III. Second Public Examination (Final School, pass or honors) at the end of the third or fourth year.

In both Moderations and Final Schools the student must choose between pass and honors. Having taken honors in Moderations in a subject, a student is not obliged to take honors in that same subject in the Final School but may change to another subject if he wishes.

For Pass Moderations a student takes three subjects of which two must be languages; the third may, for example, be mathematics. The honors course in Moderations in mathematics is known as Mathematical Moderations. The examination in Mathematical Moderations consists of two parts, Part I being devoted to mechanics and pure mathematics and Part II to more advanced pure mathematics.

Students who take this course will find it a satisfactory preparation for the teaching of mathematics in schools. It covers ample technical material for this purpose, except for the highest teaching of specialists in schools where mathematics is most advanced.[5]

Mathematics is also examined in the Final Schools either at the pass or the honors standard. For the Final Pass School mathe-

[4] Account based on *Oxford University Handbook*, 1932, pp. 133-166.
[5] *Oxford University Handbook*, 1932, p. 140.

matics may be taken as one of three subjects of which one is a language subject. The Final Honor School in Mathematics is examined in two parts, Part I consisting of six papers of a straightforward character in both pure and applied mathematics, and Part II of a special examination on advanced mathematical subjects which a candidate chooses from a prescribed list and submits to the examiners. A candidate aiming at first class honors in mathematics must take Part II of the examination.

In Oxford University the B.Sc. degree is a research degree and not an undergraduate degree, as in other universities in England and Wales.

While the naming of courses taught indicates to some extent the range of mathematical work done in the university, the standard of work so taught cannot properly be gauged without reference to the examinations which the students take in each subject. Because of the difficulty of selecting what might be called typical examination questions on any topic, the task will not be attempted here. It will be assumed that an adequate range of work of a more formal or orthodox kind is done during the two or three years' course leading to one or other of the types of degrees mentioned. A comparison of examination papers for the ordinary or general degree in two English universities with the University Entrance scholarship papers and the distinction papers of the Higher Certificate examination would suggest that while a considerably wider range of mathematical topics is covered in the degree examinations the standard of work reached in comparable items (integration, differential equations) is not appreciably higher than that in the school examinations. This remark would suggest that mathematics teachers who have taken only an ordinary or general degree in mathematics may not be sufficiently well equipped mathematically to teach those pupils who aspire to distinction in the Higher Certificate examination or to a University Entrance scholarship. It may be for this reason that the practice has developed in English schools of appointing honors graduates to vacancies as teachers of mathematics, especially where there are advanced students to be taught.

Academic Training in a Second Teaching Subject. The list of subjects from which students reading for a Bachelor's degree in Arts or in Science select their courses was given earlier in this chapter. Students reading for a Science degree usually combine with mathematics the study of the physical sciences. Those reading for an Arts degree combine various other subjects - usually a language—with mathematics. Students reading for honors in mathematics need not take any other subject; frequently, however, students elect a second subject, usually one of the physical sciences.

The curriculum in the second teaching subject may be assumed to reach a comparable standard to that in mathematics. Since the second subject chosen varies to some extent, no detailed account of the curriculum in these subjects will be described here.

Summary. The academic training in mathematics of prospective mathematics teachers in England and Wales has been reviewed in this section. The principal points arising out of the discussion are as follows:

1. Prospective mathematics teachers for secondary schools in England and Wales receive their academic training in mathematics in a university.

2. At the university they may read mathematics at two different standards according as they read for an ordinary (or general) degree or for a degree in honors.

3. The mathematical courses at the university cover a wide range of subjects in pure and applied mathematics, the latter being usually understood to include mechanics, statics, dynamics, hydrostatics, and, in more advanced courses, applications of mathematics in electricity and magnetism, elasticity, theory of potential.

4. The mathematical courses in universities are examined by means of written problem-type examination papers. Owing to the difficulty of selecting representative examination questions from these papers, no selection of such questions was made for this study. A subjective opinion is offered, based on considerable experience with the subject matter of advanced mathematics, that the examination papers analyzed in connection with this part of the study reveal that mathematical work at a high standard is offered in the universities in England and Wales.

5. An account was given of the academic preparation offered in the Universities of Oxford and Cambridge. It was pointed out that the range of mathematical subjects offered in these universities was considerably wider than that offered in provincial universities.

THE UNITED STATES

Introduction. The pre-service training of secondary mathematics teachers in the United States is given principally in three types of institutions, namely, in state and private universities and colleges, and in teachers colleges.[6] Universities are institutions composed of a number of schools, or departments, or colleges, such as schools of education, of medicine, of law, of dentistry. In addition to these professional schools most universities maintain a college which students enter from high school and where they may continue their general education for a period of four years. At the end of the four-year period students may graduate with the degree of A.B. or B.S. Besides the liberal arts colleges associated with the larger universities there are also many other independent liberal arts colleges (or colleges) which grant a Bachelor's degree at the end of a four-year course. The remaining four-year institutions granting degrees are the teachers colleges, in many of which a student may either pursue a program of general education or of education directed more specifically to his future profession of teaching. Customarily, the four college years in universities, colleges, and teachers colleges starting with first year are called respectively the freshman, sophomore, junior, and senior years. The courses in freshman and sophomore years are usually taken by all students, and specialization along the lines of a student's major and minor interests begins in the junior year.

The courses of study offered in the three types of institutions mentioned above are divided into units each lasting for one semester of about eighteen weeks and covering two or three hours a week. On successfully completing courses or units of this length students are awarded credits (2, 3, or 4 credits depending on the particular

[6] In addition to four-year colleges there are in the United States junior colleges and general colleges offering a two-year course of study, and special schools such as schools of journalism, offering one-year courses of study. Students may graduate from these colleges to the four-year colleges or even to professional schools.

institution). Such courses may be taken each year as will, in the total, give from 120 to 130 earned credits for graduation with a Bachelor's degree.[7] In choosing courses which will contribute to the required number of credits in each college year students are required to take a certain number of courses and are free to elect others, either from prescribed groups of subjects or from the whole range of subjects offered by the institution. In this way some courses may be considered as constants, being required for a given curriculum, others as variables with limited choice, and still others as free electives offering unlimited choice.

A minimum of one year's further study, involving the writing of a thesis and acquisition of further credits varying from 24 to 32 in number, leads to the degree of Master of Arts in those institutions which award degrees beyond the Bachelor's degree.

Although the National Survey of the Education of Teachers[8] revealed that about 75 per cent of senior high school teachers and 66 per cent of junior high school teachers receive their training in universities and colleges each year, the training of teachers is by no means the principal function of these institutions.

Admission. The standards of admission to these three types of institutions vary greatly. On the one hand admission to most state-controlled institutions of higher learning in the United States is unrestricted, and, in accordance with state law, any student who completes a course of secondary education is eligible for admission to the state college no matter in what subjects he specialized in the secondary school. On the other hand admission to many private and a few state universities, colleges, and teachers colleges is selective, and the trend is toward requiring candidates to satisfy one or more of the following conditions of admission:[9]

1. To pass an extramural examination, such as the College Entrance Examination Board's examination, or the New York State Regents examinations.

[7] Earle R. Rugg (et al.), *Teacher Education Curricula*, National Survey of the Education of Teachers, Vol. III, Appendix B, Table B, p. 517.

[8] E. S. Evenden (et al.), *Teacher Personnel in the United States*, National Survey of the Education of Teachers, Vol. II, p. 233.

[9] P. Roy Brammell, *Articulation of High School and College*, National Survey of Secondary Education, Monograph No. 10, p. 22, Table 4.

2. To complete 15 units of work in high school with such grades as will insure placement, academically, in the upper quarter of the school population.
3. A personal interview.
4. An intelligence test.

Admission to college under these conditions does not automatically carry with it the implication that a student has had training in mathematical subjects.[10] In some states mathematics is not required at all for entrance to college; in other states some mathematics is required, usually not more than would yield $2\frac{1}{2}$ or 3 credits, say $1\frac{1}{2}$ credits in algebra, 1 credit in geometry, and perhaps $\frac{1}{2}$ credit in trigonometry. Since mathematics is an elective subject in the upper grades of the secondary school many pupils will take more mathematics than that prescribed, though the percentage of pupils doing so is small.

Mathematics as a Major and Minor Subject of Study. In the three types of institutions mentioned mathematics may be studied as a major subject or as a minor subject, i.e., as a subject in which a student's principal interest lies, or as a subject which he ranks second or third in point of interest. The number of credits to be earned by students studying mathematics as a major subject and the type of work for which these credits are awarded vary considerably both within the one type of institution and as between the three types of institutions. The principal reasons for this seem to be the unequal training of the faculties, the number of members of the faculty available for teaching the mathematical courses, and the quality of students admitted to the institutions.

Whether a student is regarded as having studied a subject as a major or a minor seems to be determined largely by the device of counting credits, for credit in a number of courses rather than evidence of mastery seems to be the main criterion. This does not necessarily exclude the qualitative aspects, but from interviews it appears that little, if any, emphasis is on anything but the quantitative side.

Table VIII, compiled from data given in the National Survey of

[10] *Ibid.*, p. 36.

the Education of Teachers,[11] will serve to summarize the distribution of credits taken by prospective teachers whose major interest was in mathematics.

Table VIII shows that in universities and colleges a median number of 28 credits in mathematics, with a range of 17 to 40, was prescribed for majors in mathematics; and that in teachers colleges

TABLE VIII

PRESCRIBED AND ELECTIVE COURSES FOR MATHEMATICS STUDENTS IN UNIVERSITIES AND COLLEGES, AND IN TEACHERS COLLEGES

Institution	Credits for Majors in Mathematics		Credits in First Minor Subject			Credits in Second Minor Subject			Special Method in Mathematics			Median Number of Electives
	M*	R*	O*	M	R	O	M	R	O	M	R	
Universities and Colleges (Median credits for graduation 125)	28	17-40	53	15	9-25	19	12	9-20	79	6	2-13	42
Teachers Colleges (Median credits for graduation, 129)	26	13-36	—	18	6-31	—	12	6-13	—	4	2-14	

* M—Median. R—Range. O—Percentage of institutions prescribing subject.

the corresponding figures were a median of 26 and a range of 13 to 36 credits respectively. Thus slightly more credits in mathematics on the average were prescribed by universities and colleges than by teachers colleges. The actual number of mathematics courses taken by students whose major subject is mathematics is greater than the number prescribed, because these students will certainly choose courses in mathematics among the electives which they are allowed, and this accounts for the fact, revealed also in the National Survey,[12] that the median number of semester-hour credits in mathematics actually taken by students in the universities and colleges referred to was 29 with a range from 21 to 60.

In institutions of higher learning in the United States mathematics is taken as a minor subject with a wide variety of other

[11] Earle R. Rugg, (et al.), *Teacher Education Curricula*, National Survey of the Education of Teachers, Vol. III, p. 517, Appendix B; pp. 530-531, Appendix C.

[12] *Ibid.*, Vol. III, p. 198.

subjects, but chiefly with physical sciences and English. The National Survey showed that these two subjects, together with social science, are chosen in that order as the chief minor subjects when mathematics is a major subject. This finding has been confirmed by the replies to the questionnaire sent out in connection with this study. Table VIII shows that considerably less work is done in the minor fields than in the major field of study. It shows further that only about 50 per cent of the universities and colleges prescribed one minor subject, and only 19 per cent prescribed two, to be studied with mathematics as a major subject.

These facts, incidentally, have a bearing on a second teaching subject and they will be considered in this light in a later section of this study. Information was sought on the relative standards of the work in the major and minor courses in mathematics. Replies from universities, colleges, and teachers colleges revealed that the mathematical courses studied by minors are also studied by the majors in mathematics during the undergraduate years, and that the same standard of work is expected from both groups. The data furnished by the questionnaire showed this to be true for eleven out of fourteen colleges, twelve out of thirteen universities, and twenty-one out of twenty-three teachers colleges who replied.

Certification Requirements. As a condition of employment in most states of the United States teachers are required to be certified. This means that the state or local education authority prescribes a minimum standard in academic and professional subjects which teachers must satisfy. Usually the regular academic and professional courses offered by teacher training institutions are accepted toward certification by the certifying authorities. This is certainly true of those institutions which have been accredited by any of the five large accrediting associations. The average number of credits to be earned for certification in academic subjects is about eighteen, though the range varies considerably. The requirements for certification in professional subjects will be considered in the appropriate section later in this study. It may be mentioned here that about one fifth (one fourth in the teachers colleges) of the credits earned for graduation in teacher training institutions are earned in professional subjects.

Mathematics Studied in Universities, Colleges, and Teachers Colleges. An examination of the catalogues and calendars of representative[13] institutions of the three types mentioned reveals that a common group of mathematical subjects is offered in all of them. This is shown in Table IX, where data for 106 institutions are tabulated.

TABLE IX

PRINCIPAL MATHEMATICS SUBJECTS OFFERED IN THREE TYPES OF INSTITUTIONS DURING UNDERGRADUATE YEARS

Subject	Universities		Colleges		Teachers Colleges	
	Number (36)	Per cent	Number (50)	Per cent	Number (20)	Per cent
Plane analytic geometry	34	94	48	96	18	90
Algebra	33	92	46	92	20	100
Trigonometry	33	92	45	90	20	100
Differential calculus	36	100	50	100	20	100
Theory of equations	31	86	42	84	12	90
Integral calculus	36	100	50	100	20	100
Theory of equations	28	78	27	54	9	45
Projective geometry	30	83	20	40	5	25
Advanced calculus	26	72	28	56	6	30
Statistics	25	70	26	52	14	70
College algebra	25	70	24	48	11	55
Mathematics of finance	21	58	33	66	9	45

Table IX shows that the subjects most frequently offered in undergraduate courses in mathematics in the three types of institutions are as follows:

Algebra
Trigonometry
Analytic geometry (of 2 dimensions)
Differential calculus
Integral calculus
Analytic geometry (of 3 dimensions)
Differential equations (ordinary and easy partial)
Statistics

[13] The institutions were chosen from lists of institutions accredited by the various accrediting agencies in the United States. The number of each chosen for study is in proportion to the number of institutions associated with each of the accrediting agencies.

This list agrees with that obtained by Sueltz[14] in 1933, when he tabulated the replies of 4000 teachers to questions about the academic courses in mathematics which they had taken prior to entering teaching. These subjects are spread over a four-year course and it is customary to state which courses are prerequisite to any given course. The table shows further that universities and colleges offer a wider range of mathematical subjects during the undergraduate years than do teachers colleges. For example, 70 per cent or more of universities offer projective geometry, advanced calculus, theory of equations, in addition to the above-mentioned subjects, and more than half of the liberal arts colleges offer theory of equations, advanced calculus, and mathematics of finance. It is to be noted, however, that some teachers colleges offer all these additional subjects.

It will be observed that mechanics is missing from this list of subjects. The reason for this is that in the United States mechanics is usually taught as part of physics and not as part of mathematics. It is also worthy of note that such a high percentage of institutions has to offer courses in algebra, trigonometry, and, to a less extent, solid geometry which students have either not taken at school or completed so long before entering a university, college, or teachers college that they are in need of additional work in these more elementary subjects. The practice regarding the award of credit for these last mentioned courses varies.

The above list of subjects does not account for all of the mathematical subjects taught in universities, colleges, and teachers colleges. Few students, however, who study mathematics as a major subject fail to take all of these subjects during their undergraduate course. The additional mathematical subjects which are available to advanced undergraduate students or to students studying for a higher degree in mathematics cover a very considerable range of work, especially in the various branches of modern pure mathematics. Mechanics is again usually omitted from the advanced mathematical courses and is taught as part of the physics or mathematical physics courses. The subjects listed immediately below are those offered fairly frequently to more advanced students

[14] B. A. Sueltz, *The Status of Teachers of Secondary Mathematics in the United States*, p. 57, Table 27.

in the universities, colleges, and teachers colleges investigated:

Projective geometry
Statistics
Mathematics of finance
Advanced calculus
History of mathematics
College geometry
Analytical mechanics
Introduction to complex numbers

The larger universities offer a still wider range of courses in mathematics in addition to those already mentioned. Such courses are normally taken by students studying for the degree of Master of Arts or of Doctor of Philosophy in mathematics. Among these additional courses (to name a few only) which these larger universities offer are:

Partial differential equations
Theory of groups
Calculus of finite differences
Calculus of variations
Theory of sets of points
Higher geometry
Elliptic functions
Integral equations
Theory of numbers
Theory of functions of a real variable
Theory of functions of a complex variable
Vector analysis

The evidence of Sueltz'[15] study would suggest that at the present time few prospective secondary school teachers have included these last subjects in their preparation. Of interest for the present study is the fact, revealed in the foregoing lists of subjects, that in universities, colleges, and teachers colleges in the United States some attempt is made to broaden the course of mathematical study by the inclusion of such topics as statistics and mathematics of finance. Moreover, the history of elementary mathematics finds a place in

[15] *Op. cit.*, p. 57, Table 27.

the mathematical curricula of a number of institutions (50 per cent universities, 26 per cent colleges, 55 per cent teachers colleges), as does also a subject called foundations of mathematics. From the catalogue accounts of the courses in foundations as well as from discussions with members of faculties of certain institutions in which they are given, it would appear that they do not penetrate sufficiently deeply into any mathematical topic to warrant recognition in terms of the principles[16] outlined in the introduction to this study.

Professionalized Subject Matter in Mathematics. An account of the academic training of mathematics teachers in the United States must include an account of a development which is related to both the academic and the professional training of teachers, and which therefore has been given the name professionalized subject matter. This professionalized subject matter, or the professional treatment of subject matter as Bagley calls it, is:

> . . . an intensive study of subject matter in intimate connection with the problem of teaching appropriate parts of it to others. The specific selection of those parts and the specific methods of teaching them to heterogeneous groups of learners are primarily bound up in the subject matter itself. They must be derived from the subject matter rather than imposed upon it from without. . . . The advocate of the professional treatment of subject matter maintains that it is a simpler matter for certain specialists in the various subject-matter fields to acquire a working knowledge of the educational theory and a working mastery of what educational investigation has discovered than it would be for the educationist to acquire a knowledge and a mastery of all the specialized fields of scholarship with which mass education must be concerned.[17]

From this account by an outstanding proponent of the idea, it is seen that this subject has a place both here and in the succeeding chapter in the section on the professional training of mathematics teachers in the United States. This method of dealing with subject matter has been adopted to some extent in teachers colleges, but to no considerable extent in universities and colleges. A fairly

[16] Refer to page 8 of this study, Principle III.

[17] W. C. Bagley, "The Teacher's Professional Study of Subject Matter in Mathematics," *Mathematics Teacher*, Vol. XXXI, (October, 1938), p. 275.

detailed account of the arguments for and against it has been given by E. S. Evenden in the final volume of the recent survey of teacher training in the United States.[18]

Arguing for the professionalization of subject matter Evenden points out that

1. The plan does not prevent periods of concentrated work in subject matter.
2. The functional integration of subject matter and method is made as the related materials are presented.
3. It makes possible—in fact it makes necessary—a closer relationship between subject matter instructors and the demonstration and practice schools.
4. Supervision of practice teaching is distributed among all the subject matter teachers.

On the other hand he realizes that

1. Professionalization tends to limit the orderly way in which either the subject matter or the methods may be presented.
2. It is exceedingly difficult to find teachers who can do this work properly [for, as he claims, it requires teachers who, while conceding nothing in respect of scholarship to teachers of more orthodox academic mathematics, have had such experience of teaching and have such ability in teaching as will make possible a real fusion between the appropriate professional and the academic parts of the subject].

Syllabi. Universities, colleges, and teachers colleges in the United States do not publish detailed syllabi for their mathematics courses. It is therefore difficult to estimate from the brief outlines of courses given in the catalogues how much of any one subject is taught. Examples of syllabi in several mathematical subjects are included here to serve as a rough indication of the work done in mathematics in each of the four years of the normal undergraduate course.

Freshman Year

1. Introduction to Mathematics 1 (required course). This course helps the student to see that the branches of mathematics form one great

[18] E. S. Evenden, *Summary and Interpretation*, National Survey of the Education of Teachers, Vol. VI, pp. 109–117.

system. Historical facts are used, the relation of arithmetic to algebra is made clear, the universality of mathematical laws becomes evident and the processes of algebraic manipulation take on significance. The course includes a study of the concept of number, fundamental operations with numbers, formulas and equations with numerous applications, graphical analysis, series and probability, and some of the elementary concepts of statistics.

(Teachers college)

2. Theory of Investments. Logarithms, simple and compound interest, annuities, amortization, valuation of bonds, sinking funds, depreciation.

(College)

3. College Algebra. Binomial theorem, complex numbers, arithmetic, geometric and harmonic progressions, theory of equations, limits, infinite series, partial fractions, permutations, combinations, probability, logarithms, determinants.

(University)

4. Trigonometry. (3 semester hours credit). Trigonometric functions, logarithms, derivation of formulas, solution of right and oblique triangles, and miscellaneous problems.

(College)

SOPHOMORE YEAR

1. Differential calculus. Differentiation, maxima and minima, rates, integration, applications, infinite series, partial differentiation.

(College)

2. Integral Calculus. The standard formulas of integration. Applications to geometry, mechanics and physics.

(Teachers college)

3. Advanced Statistics. More complete discussion of index numbers and linear correlation, nonlinear correlations and time series, multiple correlation, theory of moments and the probability curve.

(College)

4. Analytic Geometry. Elementary course—projections, loci, straight lines, conic sections, tangents and normals, together with a brief study of solid analytic geometry.

(Teachers college)

Junior Year

1. Advanced Calculus. A study of curvature, motion, centroids, moments of inertia, indeterminate forms, curve tracing, infinite series, partial differentiation and multiple integration.

(Teachers college)

2. Advanced Calculus. Multiple integrals, implicit functions, transformations of variables, Taylor's theorem, applications to geometry and physics, differential equations.

(College)

3. Synthetic Projective Geometry. Alignments, principle of duality, projection, section, perspectivity, projectivities of geometrical forms in one, two and three dimensions, harmonic constructions, conic sections, algebra of points and one, two, and three dimensional coordinate systems and families of lines.

(College)

4. Advanced Analytic Geometry. Principle of duality, double ratio, point and line coordinates in ordinary space, curves and surfaces of second order and of second class, poles and polars, linear transformations.

(Teachers college)

5. Projective Geometry. Harmonic forms, curves and pencils of rays of the second order, ruled surfaces of the second order, theory of poles and polars, theory of involutes.

(University)

6. Elementary Theory of Numbers. Divisibility, congruences, roots, quadratic forms, laws of reciprocity of Legendre-Gauss.

(University)

7. Theory of Equations. Complex numbers, De Moivre's theorem, theory of determinants.

(College)

Senior Year

1. Differential Equations. Definition and solution of first, second and higher order differential equations. Applications to geometry, chemistry, and physics.

(Teachers college)

2. Theory of Functions of a Complex Variable. Rational and integral functions, analytic functions, conformal mapping, contour integrals, infinite series.

(University)

Examinations. The syllabi listed in the previous paragraphs do not afford a definite means of judging the standard reached in each subject. Nor can these standards be estimated by making an analysis of examination papers. It is not the practice in universities, colleges, and teachers colleges in the United States to publish examination papers, and it was not possible to obtain a sufficiently large sample to make a discussion of them profitable.

Analysis of Textbooks. In order to give some kind of answer to this question of standards, an examination was made of the more widely used textbooks in the different mathematical subjects. On request heads of departments in a number of institutions supplied the names of textbooks used in the different courses which they offer and also indicated the amount of work usually covered in each of these texts. As a result of an examination of those texts which seem to be widely used, it is possible to give the following more detailed information about topics taught in the various mathematical subjects which constitute the basic courses of the undergraduate curriculum in mathematics.

Calculus. The book which for many years has been the standard textbook in the calculus in the United States is Granville's *Calculus*, recently revised by Smith and Longley. This book is divided into two parts dealing respectively with differential and integral calculus. The range of work covered in the first part is to some extent indicated by the following chapter headings: variables, functions and limits, differentiation (meaning of the process), rules for differentiating algebraic forms (includes the usual rules for differentiation), various applications of the derivative (geometry, maxima and minima, acceleration), differentiation of transcendental functions (exponential, logarithmic, trigonometric and inverse trigonometric functions), applications to parametric equations, polar equations; differentials (applications to small errors) curvature, radius and circle of curvature (involutes and evolutes), theorem of mean value and applications (to indeterminate forms, maxima and minima). The material in this part is organized as follows: A portion of the theory is developed, problems are worked to illustrate the application of the theory, and other problems are left for the student to do.

The second part of the book, on the integral calculus, also follows the traditional pattern in covering such topics as integration (standard elementary forms); constant of integration (its geometrical significance and use in solving problems); the definite integral (application to areas and approximate integration); integration as a process of summation (applications to area, volumes of solids of revolution, length of curve, surface of revolution; formal integration (rational fractions by substitution), reduction formulas, use of tables of integrals; centroids, fluid pressure and other applications; series (power series, convergence) expansion of functions (Taylor's and Maclaurin's series).

Then follows a section on differential equations and the applications of some of the earlier ideas to functions of two variables. The chapter on ordinary differential equations includes the usual development as far as linear differential equations with constant coefficients (with applications in kinematics). A chapter on partial differentiation is introduced to prepare for applications to envelopes: tangents and normals to skew curves and to surfaces; maxima and minima of functions of several variables, extension of Taylor's Theorem: then follows a chapter on multiple integrals in which double and triple integrals are applied to finding areas, surfaces, volumes, moments of inertia, and the like.

Analytic Geometry. Among the books on analytic geometry mentioned by a large number of persons responding to the questionnaire was the revised edition of a textbook by Love.[19] The first nine chapters of this textbook are devoted to plane analytic

[19] Clyde E. Love, *Analytic Geometry.*

geometry, and the next six to solid analytic geometry. The section on plane analytic geometry covers the usual topics of coordinates, curves, loci, the straight line, the circle, polar coordinates, conic sections, transformation of coordinates, the general equation of the second degree, central conics. The treatment of these topics is straightforward and essential bookwork is illustrated by worked examples. The bookwork seems to have been chosen to illustrate the important ideas rather than to present a detailed account of the subject matter of plane analytic geometry.

The last six chapters on solid analytic geometry deal respectively with coordinates in space, loci, the plane, the straight line, surfaces, curves (in space). Here again the aim seems to be to present the important ideas of the subject by means of simple bookwork and problems rather than to present a systematic account of it.

In order to compare the type of work in plane and solid analytic geometry and in calculus contained in these two representative American textbooks with that required in the same subjects in England and Wales, selected questions from the two textbooks and from examination papers for the B.Sc. (General) degree in the University of London[20] are quoted below.

Plane Analytic Geometry

1. Show that

$$x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0,$$

where λ is an arbitrary constant, represents a family of coaxial circles. Find the limiting points of the system given by

$$x^2 + y^2 - 2(x + y - 4) - 6 = 0.$$

Determine the equations to the circles which pass through these points and have radius 2 units.

(London, 1938)

2. Prove analytically that the radical axis of three circles taken in pairs, meet in a point.

(Love, p. 82, No. 14)[21]

[20] *Examination Papers, 1938*, B.Sc. (General), The University of London. (All questions quoted by permission of the University of London.)

[21] Extracts from Love's *Analytic Geometry* are made by kind permission of The Macmillan Co., Publishers, New York.

3. If θ is the angle between the lines given by $ax^2 + 2hxy + by^2 = 0$, show that

$$(a + b)^2 \tan^2 \theta = 4(h^2 - ab).$$

Prove that

$$y^2 - 4xy + x^2 - 10y + 8x + 13 = 0$$

represents a pair of straight lines; find their point of intersection and the angle between them.

(London, 1938)

4. Determine the kind of conic represented by each of the following equations; remove the xy term by rotation of axes; reduce the resulting equation to the standard form; trace the curve.

(i) $11x^2 - 24xy + 4y^2 + 6x + 8y - 10 = 0.$

(Love, p. 120, No. 12)

5. Show that the polar equation to the chord joining the points of the conic $\frac{1}{r} = 1 + e \cos \theta$ whose vectorial angles are $(\alpha \pm B)$ is

$$\frac{1}{r} = e \cos \theta + \sec B \cos (\theta - \alpha)$$

A chord of this conic subtends a right angle at the pole. Show that the locus of the foot of the perpendicular from the pole on the chord is in general a circle, but reduces to a straight line when the conic is a rectangular hyperbola.

(London, 1938)

6. Derive the polar equation of a parabola with vertex at the pole and focus at $(a, 0)$, (*a*) directly, (*b*) by transforming the equation $y^2 = 4ax$.

(Love, p. 125, No. 13)

Solid Analytic Geometry

1. The straight line joining the point $P(a, b, c)$ to a point Q is bisected perpendicularly by the line $2x = y = z$. Prove that the coordinates of Q are

$$\frac{-7a + 4b + 4c}{9}, \quad \frac{4a - b + 8c}{9}, \quad \frac{4a + 8b - c}{9}$$

If P moves along the line $x = -y = z$, find the equations of the locus of Q.

(London, 1938)

2. By various methods, find the equations of a line through $(1, -5, -3)$ parallel to each of the planes $3x + y = z$, $x + 4y + 3z = 6$.
(Love, p. 222, No. 25)

3. Prove that all chords of the surface
$$ax^2 + by^2 + cz^2 = 1$$
which are bisected by the point (α, B, γ) lie in the plane
$$a\alpha(x - \alpha) + bB(y - B) + c\gamma(z - \gamma) = 0.$$
Prove also that the center of the conic in which the above surface is cut by any tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$ lies on the surface
$$(ax^2 + by^2 + cz^2)^2 = r^2(a^2x^2 + b^2y^2 + c^2z^2)$$
(London, 1938)

4. Find the equation of the surface generated by a variable circle parallel to the yx-plane, intersecting the x-axis, and having its center on the circle $x^2 + y^2 = a^2$, $z = 0$.
(Love, p. 248, No. 30)

Calculus

1. Show how to solve the differential equation
$$\frac{dy}{dx} + Py + Qy^n = 0,$$
where P and Q are functions of x.
Solve the equation
$$\frac{dy}{dx} = y \tan x + y^3 \tan^3 x.$$
(London, 1933)

2. Solve the equation,
$$\frac{d^2s}{dt^2} - 9\frac{ds}{dt} + 20s = t^2e^{3t}$$
(Granville, p. 406, No. 17)[22]

3. With the usual rotation prove the formula
$$P = r\frac{dr}{dp}$$

[22] Questions from Granville's *Differential and Integral Calculus* are made by permission of Ginn & Co., Publishers, Boston.

Find the radius of curvature at any point of the curve $r^n = a^n \cos n\theta$, and show that the length of the intercept made by the circle of curvature on the radius vector is proportional to the length of the radius vector.

(London, 1938)

4. Calculate the radius of curvature at any point (P_1, θ_1) on the following curve,

$$P = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

(Granville, p. 155, No. 5)

The questions quoted above provide one answer to the question of relative standards of mathematical work in universities in England and Wales, and in universities, colleges, and teachers colleges in the United States. The answer is that the mathematical attainment of students reading for the Ordinary or General degree in universities in England and Wales (assuming the standard set by the University of London to be representative) is, at the end of three years of study, about the same as that of students in institutions of equivalent rank in the United States after they have completed a four-year course of study for the Bachelor's degree.

In-service Training. Teachers in the United States have many opportunities for in-service training. It is the practice in many universities, colleges, and teachers colleges to offer courses during the summer session as well as during the regular academic year. These summer sessions are used by students for three main purposes:

1. To continue progress toward a degree, usually a Bachelor's or a Master's degree.

2. To continue the regular year's program of study, and thereby to shorten the length of the course leading to a degree.

3. To raise the grade of the teachers' certificate held.

According to the findings of the National Survey of the Education of Teachers[23] teachers used the earliest summer sessions to prepare to meet certification requirements in the various states. More recently they have been used for in-service training leading to a

[23] Earle R. Rugg (et al.), *Teacher Education Curricula*, National Survey of the Education of Teachers, Vol. III, p. 403.

degree. Sueltz found, for a group of mathematics teachers whose records he investigated, that "approximately 60 per cent of them had done some studying since they entered their present teaching position."[24] He reaches the conclusion that "the teachers who are extending their education are doing so in the field of educational theory rather than in the field of mathematics."[25] The field of educational theory may be taken to include professional courses in the teaching of mathematics. There is no way of finding out just what academic mathematical courses are taken by teachers during the summer sessions, but it may be assumed that they will take those courses which follow on most naturally from the courses for the Bachelor's degree which they have already taken, and which have been listed earlier in this section. The point here made is that many opportunities exist for in-service training, and that mathematics teachers as a group make considerable use of these opportunities for the improvement at least of their professional qualifications.

Inspection of a small sample of university, college, and teachers college catalogues for the current summer session seems to indicate both that there is a restricted offering of subject matter courses in mathematics during the summer as compared with the regular academic year, and that the subjects which are offered during the summer sessions usually are confined to trigonometry, algebra, calculus, and analytic geometry.

Summary. The purpose of this section has been to give an account of the academic training in mathematics received by prospective mathematics teachers in universities, colleges, and teachers colleges in the United States. The following are the principal findings:

1. In these three types of institutions mathematics, along with other subjects, is organized in units, each of which is studied for two or three hours a week for one semester (18 weeks). Two, three, or four credits or points are allowed for each unit of work successfully completed.

2. Mathematics may be taken as a major or minor subject of

[24] B. A. Sueltz, *op. cit.*, p. 85.
[25] *Ibid.*, p. 89.

study; specialization in mathematics usually begins in the junior year for those who wish to take mathematics as a major subject.

3. On the average slightly more credits in mathematics are earned by students who study mathematics as a major subject in universities and colleges than in teachers colleges.

4. Students who study mathematics as a minor subject usually do so with physical sciences, English, and social sciences as major subjects. When mathematics is studied as a major subject one of these same three subjects is usually taken as a minor subject.

5. The mathematical courses taken in these institutions by mathematics majors and minors are adequate to cover the certifications requirements in the home state and frequently in several adjacent states.

6. The mathematical subjects most frequently offered in these institutions are as follows:

Freshman year: algebra, trigonometry, analytic geometry, elements of calculus.

Sophomore year: differential and integral calculus, analytic geometry.

Junior year: analytic geometry (3 dimensions), calculus, differential equations, statistics.

Senior year: advanced calculus, theory of equations, projective geometry, statistics, history of mathematics.

It will be noted that the freshman year is devoted largely to work which is offered in secondary schools, but which pupils do not take at school.

7. Mechanics is rarely taught as a part of a course of study in mathematics; it is included in courses in physics, and on this account many prospective teachers may miss studying it.

8. With the exception that mechanics is omitted, the mathematical courses are broad in the sense of including a variety of subjects, such as statistics, history of mathematics, mathematics of finance, in addition to the more orthodox subjects.

9. Really advanced mathematics is taught chiefly in graduate courses; consequently graduates from teachers colleges and liberal arts colleges do not normally include in their mathematical training any study of the more advanced branches of modern pure mathematics.

10. In teachers colleges there is a trend toward organizing the mathematics courses so as to relate the academic and professional aspects of the subject in what is called professionalized subject matter. This trend has not been followed by universities and colleges.

11. Since no mathematical examination papers are published by these three types of institutions in the United States an attempt was made to estimate the standard of work done in mathematical courses by examining syllabi and widely used textbooks.

12. It was pointed out that opportunities for in-service training in mathematics is available through the many institutions which offer summer courses in mathematics, but it was also pointed out that teachers make less use of summer courses for improving their academic qualifications in mathematics than for improving their professional qualifications.

COMPARISONS

The academic training of mathematics teachers in England and Wales and in the United States has been discussed in this section of the study. It was stated that the academic training given in the former does not have to conform to any standards, while that given in the latter has to conform to minimum standards set up by certifying agencies. In both countries the academic training is given in degree-granting institutions, in which the courses leading to the Bachelor's degree extend for three or four years beyond the secondary school in the United States. The mathematical training received in courses leading to a Bachelor's degree in universities in England and Wales is of two kinds, according as a student studies for a pass or an honors degree in mathematics. The range and difficulty of mathematical work required for an honors degree greatly exceeds that required for a pass or ordinary degree in universities in England and Wales, and the honors work is taken in separate courses from that required for a pass degree. In universities, colleges, and teachers colleges in the United States a student may study mathematics as a major or as a minor subject. The mathematical courses prescribed for mathematics as a major subject include the courses studied when it is a minor subject, so that the standard in the common courses is the same, but for a major in mathematics many additional mathematical courses are taken.

There is a common core of mathematical subjects studied by prospective mathematics teachers in the two countries. The core of mathematical subjects leading to a pass degree in English and Welsh universities corresponds roughly to that for which a Bachelor's degree is given in universities, colleges, and the better teachers colleges in the United States. The mathematical subjects included in this core are algebra, trigonometry, modern geometry, analytic geometry (2 and 3 dimensions), differential and integral calculus, advanced calculus, and differential equations. In addition to these core subjects students in universities in England and Wales study mechanics for the pass degree in mathematics.

While the core curriculum in mathematics is the typical one taken by prospective mathematics teachers in secondary schools in the United States, the mathematical curriculum taken by typical secondary school mathematics teachers in England and Wales contains many more mathematical subjects. The reason for this is that most of the mathematics teachers in England and Wales find it necessary to have an honors degree in mathematics before they are appointed to secondary schools. An account was given of the range of additional mathematical work covered in the honors courses in universities in England and Wales, and this range of work was shown to be approximately the same as is covered in the larger universities in the United States. It was shown, however, that mathematics teachers in the United States rarely take these additional courses, consequently the typical mathematics teacher in secondary schools in England and Wales has received a more advanced and more specialized training in mathematics than his counterpart in the United States.

Since the mathematical examination papers given by universities, colleges, and teachers colleges in the United States are not published it was not possible to compare by this means the standard of work taught in mathematical courses in the United States with that taught in universities in England and Wales. An attempt was made to compare the standard of work by examining textbooks in specific subjects. It was shown by this means that while the work done in calculus by a pass degree student in England and Wales and by a student in the United States taking mathematics as a major subject is approximately the same, that done by the former in analytic geometry of three dimensions covers a somewhat more extensive course.

The development of what is known as professionalized subject matter courses, particularly in teachers colleges in the United States, whereby an attempt is made to coordinate advanced mathematics with secondary school mathematics, has no counterpart in English and Welsh universities.

Opportunities for in-service training in mathematics exist both in England and Wales and in the United States. In the former countries, however, special courses are not usually organized and it is left to the individual student to take the initiative if he wishes to pursue the study of mathematics further. When such initiative is shown by the student help may be received from the mathematics department of the university. In the United States, on the other hand, regular courses for in-service academic training are organized during the summer sessions by some universities, colleges, and teachers colleges. In practice it is found that only a minority of mathematics teachers who seek in-service training avail themselves of the opportunities thus provided. The majority, as will be shown later, attend courses in professional subjects during the summer sessions.

Chapter VI

THE PROFESSIONAL TRAINING OF MATHEMATICS TEACHERS

ENGLAND AND WALES

DURING THE PRESENT CENTURY three agencies have been active in contributing to the professional training of mathematics teachers for secondary schools in England. These are the university training departments, the vacation and summer courses organized by universities, local education authorities and the Board of Education, and the professional organizations for teachers, of which the Mathematical Association is by far the most important for the purposes of the present study. While these agencies contribute a great deal to the efficient training of mathematics teachers, it must be understood that a prospective secondary school teacher in England is not required by law to have any specific academic or professional qualifications as a condition of appointment to a secondary school. It is true, however, that the percentage of secondary school teachers who add to their qualifications a course of professional training prior to receiving an appointment in a secondary school is increasing. This is illustrated in Table X, compiled from data given in a report on the training of mathematics teachers in England.[1]

TABLE X

GRADUATES IN SECONDARY SCHOOLS IN ENGLAND

Year	Sex	Percentage of Graduates	Percentage of Trained Teachers	Percentage of Teachers Both Graduates and Trained
1913	Men	71.6	37.5	27.9
	Women	52.3	47.4	29.7
1931	Men	83.6	49.0	44.0
	Women	65.5	46.0	39.0

[1] "The Training of Mathematics Teachers in England," *Mathematical Gazette*, XVI (December, 1932), p. 332.

University Training Departments. While the Board of Education in England endeavors to encourage a variety of institutions to train teachers for secondary schools, in practice the chief agencies of pre-service professional training are the university training departments. There are twenty-two training departments in England attached either to a university or to a university college. They train for teaching two types of students Four-Year students and One-Year students. The Four-Year students are students who are admitted to the university training department after matriculation from a secondary school, usually with a Higher School Certificate. While officially attached to the university training department these students are allowed to pursue a three-year course of study leading to a degree in one of the other faculties of the university Arts, Science, Economics and they then spend a fourth year in the training department, devoting their time wholly to professional preparation for teaching. One-Year students are those who have completed their degree course in some university before they enter the One-Year course of professional training in the university training department. The professional course of study in the training department leads either to the award of the university Diploma in Education or to completion of the requirements for certification by the Board of Education.

The course of professional training described below may be taken by students who wish to obtain the university Diploma in Education, but who do not wish to obtain the Board's Teachers' Certificate, as well as by students interested only in the Teachers' Certificate. The prescribed courses of study and particularly the prescribed period of practice teaching are taken by Diploma students. The chief difference between the two groups of students lies in the fact that the award of the Diploma is in the control of the university and that of the Teachers' Certificate is in the control of, or is subject to, the final approval of the Board of Education. Consequently different standards of assessment, as, for example, in practice teaching, may be put into operation by the two controlling bodies, though this is not usually done. Certification is necessary for all teachers who seek employment in public elementary schools (including senior and central schools). Since teachers in secondary schools need not satisfy any such requirements, further consideration of certification requirements is not necessary for the purposes of this study.

Courses of Study. While the arrangement of professional courses differs to some extent from one university to another, all training departments offer the following:

Principles of education	2 hours per week for $1\frac{1}{2}$ terms
Psychology	3 hours per week for $1\frac{1}{2}$ terms
English system of education	1 hour per week for 1 term
Two special methods courses	1 hour per week (each) for 2 terms
School hygiene	1 hour per week for 1 term
Physical training	1 hour per week for 1 term
Speech training	1 hour per week for 1 term

One or more of the following subjects is also taken in addition to the above-mentioned subjects, but no one training department offers all of them as alternatives:

Comparative education
History of education
Music
Art
Craft
Scripture (short course of six lectures)

In addition to these professional courses which are taken during the fourth year, the Four-Year students will have attended certain courses at the university training department during the three years when they are reading for their degree in another faculty of the university. The subjects thus studied are intended to be recreational in character and include physical training, and one of art, music, or handicraft.

Students are also required to teach for sixty days during their year of professional training. This is equivalent to spending one term out of three teaching in the schools.

It is necessary now to examine the kind of training which prospective mathematics teachers receive in the university training departments. In the previous chapter it was stated that a candidate who applies with any hope of success for a teaching position in mathematics in a secondary school in England is usually required to hold an honors degree in first or second class honors in mathematics. While a few student teachers with no degree in mathematics, or with a pass or third class honors degree in mathematics are appointed to secondary schools, students with these latter qualifications usually receive appointments to the higher divisions of the elementary schools, i.e., to senior and central schools. All students who have graduated in mathematics, whether in honors or with a pass degree, take the same course in methods of teaching mathematics at the training department, and their training differs only in the type of school in which they do their practice teaching.

Methods Courses. Of what do these methods courses consist? Information obtained from the university training departments in England and Wales shows that the number of hours devoted to methods of teaching mathematics during the professional year range from 10 to 60, the average total number of hours being about 15. These hours are devoted to lectures and discussions, the average distribution of time as between the various subjects being as follows: arithmetic (5), algebra (5), geometry (3-5), mechanics (5, in a few institutions only), advanced mathematics (2 or 3, in a few institutions only). In so few lectures little more is attempted than to state the principles underlying the teaching of the various subjects mentioned above, and to illustrate these principles by using one or two specific topics. It is left to the student to increase his familiarity with the theory of the teaching of mathematics through the more important literature on the subject. In this connection the Reports of the Mathematical Association (on the *Teaching of Arithmetic*, *Teaching of Algebra*, *Teaching of Geometry*, and *Teaching of Mechanics*), and *The Teaching of Algebra* and *Exercises in Algebra* by Sir Percy Nunn are regarded as of fundamental importance.

Evaluation. The work done in the course of lectures and in the prescribed reading on the teaching of mathematics is examined in various ways. In one training department an essay on some subject connected with the teaching of mathematics is the only form of examination set. In the remainder, students are examined by means of a composite "methods" paper. This paper contains questions on all the methods courses given in the training department, and it is customary to set four or five questions of the essay type in each of the methods subjects. Each student is expected to answer questions on at least two methods subjects.

The following questions, quoted by permission of the examining authorities, are typical of the questions set in the above-mentioned type of examination.

1. "The matter of geometry is *mental* matter; Mathematics is an abstract science."
   "The teaching of geometry should begin with concrete examples of shape and size."
   How do you reconcile these apparently conflicting statements? Show the educational implications of your argument.
   (Cardiff, 1936)

2. Show how your pupils' knowledge of the decimal system of numbers may be turned to account in the teaching of algebraic indices.
   How would you develop your exposition, so as to lead them to an understanding of $x^{\frac{1}{2}}$, x^0, x^{-3}?
   (Cardiff, 1937)

3. "The study of mathematics has contributions of great, even of unique, importance to make towards training in the use of English; it is impossible to teach mathematics properly unless these contributions are made."
   Discuss this view, illustrating by example what provision for the teaching of English can be made in the mathematical course.
   (London, 1937)

4. Discuss your treatment of one of the following:
   (a) The 'function of a function' formula in differentiation.
   (b) Composition and resolution of forces.
   (c) Summation of series whose n-th term is on the type
   $$a_0 + a_1n + a_2n^2 + \ldots\ldots + a_kn^k \, .$$
   (d) Loci in elementary geometry, a locus problem in analytic geometry.
   (e) Trigonometric ratios of compound angles.
   At what stage should each topic be introduced and how would you prepare the way for it in the earlier parts of the course?
   (London, 1937)

5. How would you develop for a class the idea of negative numbers and derive the laws to which operations with such numbers conform?
(Liverpool, 1936)
6. What methods would you advocate for teaching the first principles of trigonometry? Give reasons for your answer.
(Liverpool, 1936)

Demonstration Lessons. The short course of lectures and discussions on methods of teaching mathematics is supplemented in most of the training departments by demonstration lessons in mathematics. These lessons are used to illustrate the lectures and to acquaint the students with the level of mathematical development in the various classes of the upper elementary and secondary schools. The lessons may be given by the class teacher or by the mathematics lecturer in the training department.

Practice Teaching. A more important factor in the training of mathematics teachers than the short course of lectures and demonstrations mentioned above is the period of sixty days' practice teaching which is required by the Board of Education's Regulations of all students attending a training department as recognized students, that is, as students who receive grants from the Board of Education. The practice teaching period is organized in different ways by different training departments. Acting on the principle that theory and practice should not be divorced, some of the departments require their students to teach for two days each week for two terms, the remaining three days being spent in regular course work. The more usual plan, however, is to require the teaching to be done in blocks either a single block of twelve weeks, or two blocks of eight weeks and four weeks, or of seven and five weeks, during the first and second terms respectively. While most of the training departments do not have special practice schools as part of their organization, an attempt is made to give students experience in different types of schools and with pupils of different abilities in mathematics. For this purpose use is made of conveniently situated higher elementary and secondary schools. Where the practice teaching period is arranged in two blocks, a student may do his teaching first in the senior division of the elementary school and then in a secondary school. Graduates in first and second class honors in mathematics are allowed to practice in a secondary school, but this concession cannot always be made to graduates in third class honors or with a pass degree, owing to lack of facilities for practice.

On the average students teach three periods a day and their teaching is supervised both by the lecturer in mathematics in the training department and by the mathematical master in the school. The students are expected to take full responsibility for the preparation and testing of their work, and also to make full use of the opportunities they may have for observing the work of other mathematics teachers. Considerable effort is made to place students in schools where teachers are sympathetic and willing to guide and assist them.

In all of the training departments the quality of the students' teaching ability is assessed. Marks are awarded on a literal scale A to E, of which E represents a failure in teaching. A student must earn a mark of C or above before a university will award him its Diploma in Education.

In whatever way the practice teaching is organized, it is the opinion of the staff of the departments who responded to the question that such contact with real teaching as the practice period offers is a far more valuable factor in the training of mathematics teachers than is the taking of additional methods courses in the teaching of mathematics.

The Institute of Education, University of London. The Institute of Education is the largest training center for secondary school teachers in England and Wales. It provides courses for graduates who are preparing for the university Diploma in Education and/or the Teachers Certificate of the Board of Education, as well as courses leading to higher degrees in education. Students accepted for the courses in secondary school teaching are required to have graduated with first- or second-class honors.

The Institute provides a full-time lecturer in the department of mathematics and the courses for prospective mathematics teachers are better organized and are more specialized there than are corresponding courses in other university training departments in England and Wales. Whereas the latter provide for less than twenty lectures a year on the teaching of mathematics in secondary schools, the Institute offers a course in this subject consisting of two hours a week for three terms (thirty weeks), making a total of some sixty hours in all. Of the two weekly lectures one is usually devoted to the systematic discussion of problems in the teaching of arithmetic, algebra, and geometry in secondary schools; the second is reserved for a seminar in which a variety of problems is discussed. In some of these seminars an opportunity is afforded to consider problems arising out of demonstration lessons seen; in others an effort is made to bridge the gap between certain topics in school mathematics and the corresponding advanced topics studied in the university course in mathematics; in still others prominent mathematics teachers and inspectors are invited to discuss certain aspects of the teaching of mathematics in schools. This theoretical part of the training in the teaching of mathematics of prospective mathematics teachers is examined at the end of the year by examination questions of the type already quoted in this chapter.

A feature of the training which prospective mathematics teachers receive at the Institute of Education is the opportunity to see demonstration lessons in mathematics once a week during the first term.

The Institute of Education arranges for its students to teach two days a week for two terms, thus fulfilling the requirements of the Board of Education which demand sixty days of practice teaching for the Teachers' Certificate. Students who have obtained first- or second-class honors in mathematics are given the opportunity to teach in secondary schools in and near London. They are required to teach mathematics for three forty-five minute periods each teaching day and to observe for some of the remaining periods. The teaching experience affords variety both in mathematical subjects taught and in types of pupils taught.

The Mathematics Staff of University Training Departments. The members of the mathematics staff (or faculty) in universities in England and Wales are honors graduates in mathematics. Most probably all of them have received their mathematical training at either Oxford or Cambridge, since these are the two most famous mathematical schools in the country.

The qualifications of the persons in charge of mathematics in university training departments vary considerably. In only one training department is there a full-time lecturer in mathematics. The only qualification found to be common to all of these members of staff is experience in teaching secondary school mathematics. Their academic qualifications in mathematics varied from the holding of an honors degree in mathematics from Oxford or Cambridge to a pass degree in mathematics from a provincial university.

Another Type of Professional Training. The importance which seems to be attached to the idea of learning to teach by actually teaching may perhaps explain why many headmasters in England appoint to their staff teachers who, while they have had no teaching experience, possess a good personality and high qualifications in their subject. These newly appointed, inexperienced teachers are placed under the guidance of the senior mathematics masters in the school, and in their care they may well receive, in the course of one or two years, a thorough training in the methods of teaching mathematics a training, moreover, which a student in a training department cannot hope to match in thoroughness since he is obliged to devote a considerable part of his professional year to the discussion of general educational questions.

This training of selected persons to teach mathematics under the guidance of an experienced mathematics teacher is regarded in this study as professional training, even though it may lack some of the characteristics usually thought of as accompanying professional training. As an example of this type of professional training the following procedure adopted by Mr. A. W. Siddons, of the Harrow School, England, and explained to the writer in an interview, is given: Mr. Siddons adopted the plan of having a new teacher teach in a room adjacent to his own classroom. He visited the new teacher with or without notice, and always discussed each lesson seen. When necessary, and frequently at the request of the new teacher himself, Mr. Siddons gave lessons for him, these lessons being followed by discussions. The whole program of work for his class was from time to time discussed with the new teacher, as also was the teaching of any particular lesson. The aim of this training was to enable the teacher to develop power in the pupils and not merely to impart information. In other observation lessons Mr. Siddons showed the teacher how to follow sidetracks profitably, particularly those using a knowledge of the history of mathematics. This participation in observing, teaching, and examining very quickly led to confidence and competence in teaching.

Second Teaching Subject. A second teaching subject is required of all students preparing to teach in secondary schools in England. The subject most frequently chosen as a second teaching subject by prospective teachers of mathematics is one of the physical sciences, usually physics. Chemistry is quite frequently chosen for a second subject, and other subjects, such as botany, general science, geography, are also occasionally found in combination with mathematics. The same conditions as to number of lectures, and the general manner of conducting the methods courses hold for the second teaching subject. Students are given some experience in teaching and observing their second subject during the practice teaching periods.

The Mathematical Association. Among the "other agencies" which contribute in some measure to the training of mathematics teachers in England is the Mathematical Association, an association of teachers interested in the teaching of mathematics whether in the elementary school or in the secondary school. The Association[2] enjoys at the present time (1938) a membership of some seventeen hundred persons, most of whom live in England and Wales, though geographically the members are distributed in many countries. The contribution of this body to the training of mathematics teachers does not appear to have been given adequate notice in the literature, passing references to its services in improving mathematics teaching have been made from time to time.

A forerunner of the Mathematical Association, the Association for the Improvement of Geometrical Teaching or the A. I. G. T. as it is called was organized in 1871 by mathematics teachers in public schools in England, in order to correct the state of affairs in the teaching of Euclid that was revealed by the Public Schools Commission Report of 1868. The sixty-one original members, of whom fifty-two were schoolmasters, met for the first time in Janu-

[2] From data furnished by the Secretary of the Mathematical Association of England.

ary, 1871, with the avowed purpose of delivering the schools from the tyranny of Euclid.

What did the A. I. G. T. actually do? First of all it stirred up much discussion and circulated its reports, which gave much admirable advice and thereby did much to improve the geometrical teaching in the country. In 1888 Professor Chrystal wrote: "I think your Association has already done much good to the teaching of geometry. I begin to find intelligent teachers of geometry in a good many places now, where formerly no such thing could be found." Apart from the stirring up of the teachers of the country, the more tangible work of the A. I. G. T. consisted of the production of an admirable syllabus and textbook of geometry. The trouble taken over these was enormous. . . . Eventually the syllabus was published in 1875, after more than four years' work. . . . Meanwhile the committee was perpetually trying to influence examining bodies but without great success. . . . But "the questions in geometry proposed at the Matriculation Examination of the University of London in June 1876 deviated from the old Euclidian type so far as to provoke a spirited controversy in the columns of the *Times*". . . . In 1887 after sixteen years work they are still bombarding the universities. At last, in 1887 and 1888, Oxford and Cambridge both passed regulations that allowed proofs other than Euclid's provided that Euclid's order was not violated.[3]

The above extracts from the Presidential Address to the Mathematical Association by A. W. Siddons indicates the work done by the A. I. G. T. over a period of seventeen or eighteen years. Although it did not meet with the success it had expected or the efforts and enthusiasm of its members warranted, the A. I. G. T. laid the foundations early in the present century for the final overthrow of Euclid as a textbook in geometry. The A. I. G. T. became moribund about 1890, apparently having failed to make much impression on the existing extramural examining bodies. However, its work was revived by the organization of the Mathematical Association in 1894. With the formation of this body, its objectives began to be realized and its influence widened.

The Mathematical Association, like the A. I. G. T., was an organization composed of schoolmasters. After a few years of effective work and steady progress it was stirred to considerable

[3] "Progress," Presidential Address to the Mathematical Association, *Mathematical Gazette*, XX (February, 1936), p. 16.

activity by the address of John Perry at the meeting of the British Association for the Advancement of Science in Glasgow in September, 1901, in which he attacked the current teaching of mathematics in schools and made a strong plea for reform in the teaching of mathematics. From 1902 onward the volumes of the *Mathematical Gazette*, the official publication of the Association from its rebirth in 1894, has contained articles and discussions in which the charges against mathematics made by Perry are examined carefully, and in which the movement for reform in the teaching of mathematics receives whole-hearted support.

The Association formed a Teaching Committee in 1902 to draw up a report on the teaching of geometry. This committee has, since its inception, been a most important influence toward the improvement of mathematical instruction in the schools of England, because its members have usually been active teachers and their deliberations have always reflected the best current practice in mathematical teaching in the schools, as well as an awareness of the important trends in mathematical teaching.

It may be asked why the Mathematical Association is thought of as contributing to the training of the mathematics teachers in England. The foregoing outline of its development shows the close contact of the Association with, and its influence on, problems relating to the mathematics taught in the secondary schools and to the methods of teaching mathematics. In addition to its influence on the teachers already in the schools, the Mathematical Association has also influenced the teaching of mathematics in the training colleges and university training departments through the discussions of the Association's Reports which form so considerable a part of the methods courses in these institutions. The influence of the Association has been further widened because writers of mathematics textbooks have adopted the recommendations of the Association's Reports as they were released.

In elaboration of the above an account will now be given of the influence of the Mathematical Association on the present professional training of students-in-training and on the in-service training of active teachers of mathematics.

Influence of the Association on Pre-service Training. Undoubtedly the most important influence exerted by the Mathematical Association on the pre-service training of prospective mathematics teachers is by means of the reports which the Association has issued from time to time. As has been stated, these reports form the basis of the lectures and discussions on the teaching of the various mathematical subjects which are given in training colleges and university training departments.

Since 1902 some seventeen reports have been prepared by the Teaching Committee of the Association and have been printed in the *Mathematical Gazette* or have been published as separate reports. The most widely used are those on *The Teaching of Mechanics* (1918 and 1930), *The Teaching of Algebra* (1933), *The Teaching of Geometry* (1923, revised and amplified 1938), and *The Teaching of Arithmetic* (1931). Approximate figures supplied by the Mathematical Association show that the number of copies issued varies from about 2000 (*The Teaching of Mechanics*, 1930) to 3500 (*The Teaching of Geometry*, 1923). It may be assumed, therefore, that these reports have been widely distributed to teachers of mathematics and to libraries, and that their influence has been correspondingly widespread.

The reports most widely used to date have been those on *The Teaching of Geometry* (1923, 1938). An analysis of these reports will serve to illustrate the nature of the others, and to show how carefully practical considerations are kept in mind by the framers of the reports, who are active teachers of mathematics.

The 1923 report on *The Teaching of Geometry* deals in separate chapters with the following topics: General Principles, Elementary Practical Considerations, Discussion on Some Disputed Points, The Question of an Agreed Sequence, Notes on Some Minor Points, A Word with the Examiner and a Note on Relativity.

In the section on General Principles the Report argues for the organization of propositions into those which may be assumed (at any rate in the early stages of a school course) and those which should be proved. Many of the latter may well be known as a result of work with riders (originals) based on the assumed theorems.

This idea of dividing propositions into these two main groups is developed further in the next chapter where, following an earlier suggestion contained in a Circular on Geometry issued by the Board of Education, geometry is divided for teaching purposes into three stages—Stage A, an informal and experimental stage, Stage B, a deductive stage, Stage C, a systematizing stage. Some suggestions as to the content of each of these stages were made. A syllabus of classroom work and field work was suggested for Stage A. It was assumed that the content of Stage B was fairly well established, though the Committee did discuss a number of disputed points which normally arise in connection with the work of Stage B, as, for example, superposition as a method of proof, the parallel postulate and its implications and alternatives. Since some of these ideas were not generally known to teachers of geometry at the time a section of considerable length was devoted to these disputed questions. In particular, an alternative to the parallel postulate in the form of a "principle of similarity" was discussed in the hope that some schools would experiment with it. At that stage the question of an agreed sequence of theorems was still a vexed question and the Report offers the opinion that no fixed sequence of theorems is necessary. Nevertheless it realizes the difficulties the taking of such a position raises, and it makes specific suggestions about overcoming the more obvious of these difficulties.

Certain minor points (though many of them of great practical interest to teachers) are then discussed, such as the arrangement of a group of theorems, the double aspect of a locus, etc.

While this report thus covered the important issues in the teaching of geometry and left no uncertainty as to the position of the Committee on certain controversial matters, it necessarily gave too little information on the practical details for carrying into effect its recommendations. Particularly did this apply to work of Stage A, the informal stage in the teaching of geometry which the report recommended. These defects or omissions which have in the meantime given rise to discussions on the teaching of geometry, have received attention in the most recent (1938) Report of of the Association,[1] also devoted to the teaching of geometry. In the

introduction to this Report its connection with the 1923 Report is stated. It is to serve as a companion to the earlier report. The brief account of the content of the introductory stage in teaching geometry called

[1] *A Second Report on the Teaching of Geometry in Schools*, prepared by the Mathematical Association of England, 1938.

Stage A—is here developed into a syllabus for a course on informal geometry based on class room and field work. A whole section of some 28 pages is devoted to a discussion of "details of teaching in Stage A." This discussion on teaching is characterized by such statements as "This work can be driven home further by questions such as this"; "A good problem would be as follows"; "One effective way of doing this, one no doubt of many, is to say . . . "; each followed by the appropriate problem or explanation which can be used to advantage by a novice in teaching. Here then is a thoroughly practical section on the informal stage, Stage A, of geometry teaching. What of the other stages? The deductive stage, Stage B of the earlier Report, is treated in like manner in four chapters as follows:

(iv) Stage B.
 The Relation between the Stages.
 Main Aims of Stage B.
 Freedom of Order.
(v) The Groups of Theorems.
(vi) Riders (Original), Constructions and Standard Theorems.
(vii) Further Details of Teaching in Stage B.

Chapter vii above again runs to 22 pages, which are full of useful suggestions for teaching the groups of theorems of Chapter v, for dealing with solid geometry, for correlating geometry with geography, and so on.

Again Stage C, the systematizing stage in teaching geometry, is elaborated from the earlier Stage C into two chapters, one dealing with the logical structure of geometry (for school purposes), the other dealing with more philosophical questions relating to the theory of parallels. Appendices of some 50 pages deal with miscellaneous topics of interest and of value to teachers of geometry, including, in Appendix 1, a list of suitable class room apparatus for teaching geometry.

In 1902 a committee of the Association prepared a report which contained recommendations on the teaching of geometry in the light of the discussions which followed Perry's address and also in the light of the consideration which had been given by the Mathematical Association to the replacement of Euclid as a text-book in geometry. It is interesting to compare these recommendations with those of the later reports (1923, 1938): In the earlier report the following recommendations were made: (i) that geometry should be introduced informally, (ii) that the working of riders (originals) is important (iii) that the formal subject matter be divided into constructions and theorems, (iv) that the proof by superposition be retained, and (v) that theorems be presented in a given order. In the later report the Committee, while approving the emphasis on riders and on the informal introduction to geometry, allowed the further liberty of not requiring a fixed sequence of theorems and of not requiring the use of proof by superposition. This comparison affords one illustration of many which might be given to show how the Mathematical Association through the years has altered its point of view as emphases in the teaching of a particular subject have changed. At the time each report was produced, however, it represented a balanced account of good current teaching practices.

A list of the subjects about which reports have been prepared will illustrate the range of the Association's activities.

1. Report on Arithmetic and Algebra (1902)
2. Teaching of Mechanics (1904)
3. Advanced School Mathematics (1904) (Introduction of differential and integral calculus recommended)
4. Mathematics in Preparatory Schools (1907)
5. Course of Mathematics Required for Entrance Scholarship Examinations (1908)
6. Teaching of Mathematics in English Public Schools (for International Commission) (1908)
7. Reform of Mathematical Teaching in Germany (1913) (Informs teachers of reform movements in another important country)
8. Mathematics in Secondary Schools (1914) (Protests against the effects of external examinations on the teaching of mathematics in Secondary Schools)
9. Teaching of Calculus in Public Schools (1913) (Prepared for International Commission)
10. General Mathematics Syllabus for Non-specialists in Public Schools (1913)
11. The Teaching of Arithmetic (1916) (Recommends reducing the subject matter then being taught in Arithmetic in the schools, and stresses importance of setting problems "such as occur in practice")
12. The Teaching of Mathematics in Secondary Schools (1919) (Urges importance of trigonometry, calculus, mechanics as school subjects gives definite opinions on the training of teachers of mathematics)

13. The Teaching of Mechanics (1918) (mentioned above)
14. The Teaching of Geometry (1923)
15. Mathematics for Girls (1926)
16. Teaching of Mechanics (1930)
17. The Training of Mathematics Teachers (1902)

Of course not all of these reports have made equally valuable contributions to the pre-service training of a teacher. However, the Association's more recent reports on the teaching of arithmetic, algebra, geometry, and mechanics form the basis of discussions and reading which the prospective mathematics teachers do during their course of professional training, and, because of this, they have a very definite influence in helping to make prospective teachers aware both of the problems confronting mathematics teachers and of some of the methods of dealing with these problems which long and thoughtful experience have devised.

Influence of the Association on In-service Training. It must not be thought that the effect of the Mathematical Association on the training of mathematics teachers has been confined to its contributions to the pre-service training of prospective mathematics teachers. Perhaps its greatest influence has been to serve as an agency for the in-service training of teachers. In addition to the reports, the Association publishes the *Mathematical Gazette* five times a year. Each issue of the *Gazette* runs to about 2200 copies which are distributed first to its 1700 members and then to libraries or to general sales. Important features of each issue of the *Mathematical Gazette* are articles, mathematical notes, reviews and notices, and gleanings.

Articles. The articles appearing in the *Gazette* may be classified into (i) addresses by the President of the Association or other visiting lecturers, (ii) articles dealing with the teaching of some topic or subject in mathematics, (iii) discussions on selected subjects pertaining to the teaching of mathematics which perhaps are engaging the attention of a considerable body of teachers.

According to the record provided by the *Gazette* it seems the yearly custom for the President of the Mathematical Association to deliver a presidential address. In the past the respective presidents have used the opportunity provided by the annual meetings to give an account either of recent developments in advanced branches of mathematics, physics, and other related sciences, or of some branch of knowledge unfamiliar to the layman but having mathematical implications, or to give authoritative statements on issues more directly relating to the teaching of mathematics. In addition to the presidential addresses those by other eminent lecturers also fall into the given categories. Such addresses, given as they are by eminent authorities in each field, are of great importance for the "further education of teachers." Perusal of these addresses leads to the surmise that they may well open up to teachers new realms of ideas because of comprehensive introductions to novel aspects of science by acknowledged experts; other addresses again provide an important background of ideas and principles for many of the topics which teachers will have to teach (at a much lower standard) in the secondary schools. Of course it is obvious that all teachers will not read all of these articles, but the evidence obtained from interviews with many teachers in different parts of England and Wales indicates that they are being used to considerable advantage by them. The following are titles of addresses given before the Association and published in the *Mathematical Gazette*. They indicate how important is the influence thus brought to bear on the thinking of mathematics teachers.

1. A. Andrade, Problems of Atomic Structure
   S. Brodetsky, Aeroplane Mathematics
   A. Eddington, Relativity
   Decline of Determinism
   E. Picard, Development of Mathematical Analysis and Its Relation to the Other Sciences
   E. T. Whittaker, What Is Energy?
   A. Sommerfeld, The Scientific Results and Aims of Modern Applied Mechanics
2. C. Kuczynski, Population Trends
   H. T. H. Piaggio, Psychology and Mathematics
   F. P. Ramsey, Mathematical Logic
   A. N. Whitehead, Aims of Education
3. G. StC. Carson, Intuition
   G. Darboux, The Development of Geometrical Methods
   G. H. Hardy, What Is Geometry?

R. L. Heath, Greek Mathematics and Science
E. W. Hobson, Democratization of Mathematics
C. S. Jackson, The Elementary Arithmetic of the Theory of Numbers
P. E. B. Jourdain, Introduction to Irrational Numbers
E. H. Neville, The Food of the Gods (An appeal for reorganization of scholarship mathematics in secondary and public schools.)
A. W. Siddons, Progress. (An account of the activities of the A. I. G. T. and Mathematical Association.)
W. H. Young, Introduction to the Mathematical Ideas of Infinity

While it seems apparent that stimulus is provided by lectures of the type listed, the publication of such lectures by no means constitutes the sum total of the Mathematical Association's contribution to the in-service training of the mathematics teacher. The members of the Association share in the discussions which follow the publication of a report. These discussions are frequently held in the various centers where there are branches of the Association, and members are encouraged to examine and challenge the findings listed in the report. Reports of such discussions are also frequently published. In addition to the discussions which follow the publication of each report, other discussions on topics of considerable interest to a group of mathematics teachers are arranged by the Association, usually at its annual meetings. The practice is for three or four speakers to open a discussion and then for members to contribute as they wish. The record of the subjects so discussed during the present century already makes an impressive list. No attempt will be made to summarize all of these discussions, but attention must be drawn to certain of the more significant among them.

Naturally there first come to mind the lengthy discussions on the Perry Movement for the reform of mathematics. It is true to say that for the first few years of the present century the greater part of the Association's attention was devoted to discussions on the teaching of geometry. But as early as 1911 12 the attention had shifted to the teaching of algebra and, although no report on algebra was prepared at that early date, the *Gazette* had already published articles and addresses on the teaching of algebra by Nunn, which, elaborated in his important books on the subject, have contributed so greatly to the reform of the teaching of algebra in England. From its inception it has been the good fortune of the Mathematical Association to name among its members leaders in the field of mathematics as well as of education who have given of their knowledge, experience, wisdom, and energy in promoting the cause of mathematics teaching in England. The contribution of these leaders to the types of discussions listed below constitute in themselves no inconsiderable part of the value for teachers of the ideas recorded in the volumes of the *Mathematical Gazette*.

Topics on which valuable discussions have been arranged from time to time are:

The "New" Geometry
An Arithmetic Syllabus for Secondary Schools
Aims and Methods of School Algebra
On Some Unrealized Possibilities in Mathematical Education
A Plea for Earlier Introduction of the Calculus
Teaching Mechanics to Beginners
Sequence of Theorems in School Geometry (1922) (See 1923 Report)
Plea for Probability in Schools
Syllabus in Additional Mathematics
The History of Mathematics: its Relation to Pupil and Teacher
Methods of Learning Geometrical Theorems
Rider Work in Geometry
Teaching of Loci
Bearing of Higher Geometry on the School Course
Teaching of Elementary Astronomy
Reform of Mathematics
First Encounter with a Limit
Study of Statistics in a School Course

It is evident that these discussions cover a very wide range of topics of interest to mathematics teachers, and their content reveals the useful contribution which they may make both to the more effective teaching of a particular topic, and also as a stimulus to experiment with teaching of new topics in mathematics classes.

To some extent such discussions may well fill the gaps left in the mathematics teacher's equipment after he has attended the more formal course work in mathematics in a university. If sufficiently disseminated among teachers, they serve to provide bases for the kind of experience which is of the utmost importance in the training of mathematics teachers, according to the principles which are central to this thesis. The value of these addresses and discussions both for prospective and active mathematics teachers may be judged by the following review of the discussion on "Methods of Learning Geometrical Theorems."

The discussion[5] was introduced by the senior mathematical master at Harrow School who began by considering the organization of the theorems of geometry into groups for teaching purposes. Of special interest was what he called a digression on method, in which he outlined his own method of teaching geometry. The steps suggested were as follows: Teacher requires each boy to have a piece of scrap paper beside him (in every lesson) on which he attempts to draw the figure for the new proposition; teacher draws figure on the blackboard and letters it; pupils check their figures, discuss differences in drawings and letter figure as on blackboard (for convenience of reference); teacher allows a few minutes in which each boy attempts to prove the proposition; cross-questions the class to draw from them what is given, what is required, and what they have found out; marks this information on the figure on the blackboard (in colored chalk); in this way builds up the complete proof of the proposition; does not necessarily write out the proof at this lesson, but waits till allied theorems have been proved in the same way, and later devotes a period to the writing out of two or three theorems. Again, the writing out of a theorem depends on knowing the theorems which can be assumed and this illustrates the importance of dealing with chains of theorems "to drive home the perspective of the theorems as a whole or of a group." A child knows a new theorem when he is convinced of its truth, he knows its enunciation, he can apply it to numerical cases and he can use it, if need be, in a logical proof.

After further explanation along these lines the subject was discussed by other teachers who raised questions and contributed information on the most suitable method of naming triangles, methods of teaching pupils how to remember which construction

[5] Review of discussion reported in *Mathematical Gazette*, XVII (May, 1933), pp. 76–83.

lines to draw when proving a difficult theorem such as the Pythagoras theorem, in what order theorems and riders (originals) should be taught (riders first), the importance or otherwise of the formal method of writing out the proof of a theorem.

It will be noted that the discussion was kept on a thoroughly practical level, and that it contains information which would be of great value both to experienced teachers who cannot frequently meet with other teachers to discuss difficulties and to exchange experiences, and also to students or young teachers in the early days of their teaching career. The discussions on the topics listed above are all on the same practical level.

The evidence here presented concerning the contribution of the Mathematical Association to the in-service training of the mathematics teachers in England and Wales through reports, articles, and discussions is impressive. That it has been effective in achieving the result here claimed for it may be seen from an examination of the modifications made in mathematical textbooks following the publication of the various reports of the Association.

Reviews and Notices. Another service performed by the Association which seems to be of value to mathematics teachers, if it is not so directly valuable as a contributory factor in in-service training, is the publication of reviews of mathematical books. Considerable space is allocated to competent reviews of school textbooks in mathematics and by this means teachers are able to judge the merits of the various books reviewed. A more important service, however, is provided by the careful and scholarly reviews of works on advanced mathematics. Books on advanced mathematics whether written in English or in other languages are so reviewed by experts in the particular subject as to provide teachers who wish to continue the study of some branch of mathematics with expert guidance in the selection of suitable treatises for study.

Mathematical Notes. In each issue of the *Gazette* there are a number of notes on topics relating to elementary mathematics, which are regarded as of interest to teachers. Teachers themselves contribute many of these notes.

The Training of Mathematics Teachers. From time to time the Mathematical Association has published statements and reports on the training of mathematics teachers. Some of the principles outlined in the introduction to this study have the support of the Association. It has expressed its recognition of the need for the professional training as well as for the academic training of mathematics teachers. While various numbers of the *Gazette* contain articles devoted to this subject, the Association has not pressed for reform in this direction with the zeal with which it has attacked the problem of improving mathematical instruction in the schools. Speakers at the meetings of the Association have from time to time called attention to the inadequate training of teachers. A committee of the Association prepared a report on the *Training of Mathematics Teachers in England* in 1932, and use has already been made in this study of the findings of this report. While the report states important principles governing the training of mathematics teachers, the point is made in this study that the Association has contributed more definitely in this respect to teachers in England in form of in-service training than through any direct influence it has had on the institutions, organized for that purpose, considerable though that undoubtedly is.

Vacation Courses. A question of some interest is whether the untrained teachers in secondary schools attend courses of professional training during their active teaching career. Unfortunately there is no reliable source from which such information can be obtained. However, some mathematics teachers, including a few untrained teachers, do take advantage of the facilities for further training provided annually by a number of university training departments, by the Board of Education, and by certain local education authorities. These facilities for further professional training take the form of vacation courses or summer courses. The avowed purpose of these courses is to provide an opportunity for a limited number of mathematics teachers to meet together to discuss with experienced teachers and inspectors their teaching problems and difficulties, and in this way to receive help and inspiration for their own teaching.

The number of vacation courses in mathematics offered each year in England and Wales is small, and varies to some extent from year to year, as will be seen from Table XI.

Table XI shows that the number of persons attending the courses is small, but this does not mean that the response of the teachers is poor. Rather it means that attendance on the courses, especially those courses conducted by the Board of Education, is limited.

TABLE XI*

OTHER COURSES FOR MATHEMATICS TEACHERS

Subject	Year	Short Full-Time Courses		Part-Time Courses	
		Number of Courses	Number Attending	Number of Courses	Number Attending
Mathematics	1934	2	92	15	490
	1936	3	160	4	261
	1937	3	141	7	262

* Compiled from tables in *Education in 1934*, p. [illegible]; *Education in 1936*, p [illegible]; *Education in 1937*, p 203

For example, out of some 250 applicants for the Board of Education's summer course in mathematics it is customary to select between 60 and 70 persons; and no attempt is made by the Board to provide additional classes to absorb the unsuccessful applicants. The reason for this practice is that the Board release only a small number of their officers to conduct each course, and it is felt that the discussion groups into which the classes are divided should be sufficiently small to allow free participation in discussions by all members of each discussion group.

To illustrate the kind of work done in these summer courses and to indicate their value for the in-service professional training of teachers attending them, the following summary of the work done in a recent summer school[6] conducted by four Inspectors of the Board of Education and two visiting teachers is given.

From some 250 applicants 60 men and women were chosen to attend this summer course. In making the selection of persons for summer courses each year, the selectors are usually guided by:

1. The influence which a person is likely to have on the improvement of mathematical teaching by virtue of his position and therefore senior mathematical masters are welcomed.

[6] Permission to quote this information has been kindly granted by the Board of Education. It relates to the Summer School in Mathematics, held in Oxford, August, 1938.

2. The need of the individual teacher very inexperienced teachers or teachers who are too old are rejected in favor of persons who have taught mathematics for perhaps seven to ten years.

It is felt that the latter have had sufficient experience both to be aware of the difficulties in teaching mathematics and to have given considerable thought to their solution; that therefore they are in a position to make valuable contributions to the discussions.

The course consisted of twenty-eight lectures, three lectures a day for ten days and, in addition, discussions in the evening. These evening discussions are usually devoted to topics outside the range of work covered in the lectures.

The subjects discussed during the course and the number of lectures devoted to each were as follows:

Inaugural address	1
Arithmetic	2
Algebra	4
Geometry	5
Mechanics	2
Mathematics and English	1
Astronomy for schools	2
Trigonometry	1
Statistics (introductory)	2
History of mathematics	2
Calculus	3
Graphs	1
School textbooks in mathematics choice and quality	1
Concluding meeting	1
Total	28

While no topic could be discussed in any great detail in the time allowed for it, the practice is to discuss issues of major importance and to exchange opinions and experiences on these issues. Certain subjects, as, for example, history of mathematics, statistics, not usually taught in schools were discussed in order both to serve as an introduction to them for the teachers to whom they were unfamiliar, and to illustrate how they might be taught in schools.

Vacation courses for secondary school mathematics teachers have occasionally been organized by the mathematics department of universities in England. A report of a two weeks' summer course appeared in the *Mathematical Gazette*, No. 149, p. 161. The course was given at the University of Leeds and was attended by thirty-five teachers from the following types of schools: public schools, grammar schools, country secondary schools, city secondary schools, girls' high schools, and pupil teacher centers.

The course consisted of (i) lectures and discussions on the teaching of school mathematics (geometry, algebra, calculus), (ii) lectures on nomography and computation, and (iii) a series of biographical lectures dealing with the work of Carnot, Maclaurin, Kelvin, Rayleigh, Cayley, Sylvester, Salmon, Babbage, Whitworth, Clerk, Rankine, Descartes (and graphs), Perry, D'Ocagne, Argand, Abel, Riemann, Neumann, Maxwell, and Faraday.

It will be seen from this account that the vacation courses do not attempt to be systematic courses on the methods of teaching mathematics. The subjects chosen for lectures and discussion are such, nevertheless, as to focus attention on the more important problems in the teaching of mathematics, and, in addition, to provide an introduction to new topics which will serve to stimulate the thinking of the mathematics teachers along new lines. The biographical lectures in the Leeds course and the lectures on non-school mathematics of the Board of Education's course definitely belong to the latter category.

Because of the small numbers of teachers who attend these vacation courses, it cannot be and is not claimed that the courses provide as effective a method of in-service training as do the other agencies. However, for those mathematics teachers privileged to attend them the courses may be of great practical value and may serve as a source of real inspiration for their own teaching. For this reason they merit consideration in a study of the training of mathematics teachers in England.

Higher Degrees in Education. In-service professional training of another kind is provided in some of the universities in England and Wales in the form of courses leading to higher degrees in education. The degree most commonly awarded in education is the Master of

Arts, though the Master of Education and Doctor of Philosophy degrees also are awarded in education by some of the universities.

In the University of London lecture courses are provided for students taking the M.A. degree in education, and these are arranged in the late afternoon or early evening for the convenience of teachers who wish to attend. In addition to presenting satisfactory work at the written examinations in education, educational psychology, comparative education, history of education candidates for the M.A. degree in education must prepare a thesis on an approved subject.

Candidates for the Doctor's degree in education in universities in England and Wales are required to prepare and to defend a thesis on some approved subject.

Summary. This section dealt with the contributions of three agencies to the professional preparation of mathematics teachers: namely, the university training departments, the Mathematical Association, and vacation courses for teachers. The following are the principal findings.

1. The contribution of the university training departments was shown to be entirely professional in character. It consists of courses of lectures on the theory of education, psychology, special methods, hygiene, physical training, and, in addition, of the requirement of a period of twelve weeks' supervised practice teaching in the schools. The special methods courses in mathematics which are of chief interest in this study consist of comparatively few lectures (average number about twenty) on various aspects of the teaching of mathematics. An attempt was made to show the nature of the work covered in the special methods lectures by citing representative examination questions from examination papers. It was pointed out, however, that the examinations on special methods do not occupy a very important place in the scheme of training.

2. Great importance is attached to the period of practice teaching. In many institutions this period, together with the demonstration and observation lessons which accompany the methods lectures, is regarded as the most valuable part of the whole period of training.

3. Some training is given in the method of teaching another subject besides mathematics. For a majority of prospective

mathematics teachers this second subject is one of the physical sciences.

4. An account was given of the professional training in mathematics offered in the I stitute of Education to prospective mathematics teachers in secondary schools in England.

5. The important contribution of the Mathematical Association to the professional training of mathematics teachers was illustrated in some detail by referring to the influence of the Association's reports on the teaching of mathematical subjects, and to the articles in its journal, the *Mathematical Gazette*.

6. It was shown that though the facilities provided in England and Wales for professional in-service training through vacation courses are inadequate, yet those few vacation courses which are conducted in the teaching of mathematics provide valuable in-service training for those teachers who attend them.

7. Most universities in England and Wales award higher degrees in education, principally the Master's degree. Definite courses are organized leading to this degree in education, and a thesis is an integral part of the requirements.

THE UNITED STATES

Certification Requirements. In a previous section it has been shown that the minimum amount of training, both academic and professional, which prospective mathematics teachers must receive before being eligible for appointments to high schools in many states in the United States is determined by the state certification requirements. These certification requirements, so far as they concern the professional aspects of training, are that students should have completed an average of about fifteen credits in the theory and practice of education, though this number of credits varies in different states from zero to thirty or more semester-hours. Without significant exceptions secondary school teachers in the United States are certified. A certificate valid in one state is not necessarily valid in another state. Indeed, it is common to find more than one type of certificate in a state. Owing to these conditions a wide network of certifying bodies has grown up throughout the nation, some of them local, others organized on a state-wide basis, but each with its own requirements, and this accounts for the present wide range in the professional and academic standards to be met for certification.

An important influence is exerted on the stabilization of certification requirements by the large accrediting associations which were mentioned earlier. While some educational authorities, as, for example, New York City, conduct their own examinations for certification, in general courses which are adequate to cover certification requirements in the various states are provided for in the universities, colleges, and teachers colleges. In the schools of education in the universities and in teachers colleges, ample provision for the study of education is made. In the colleges, however, it seems to be the common practice to provide no more education courses than are necessary to meet the requirements of state certification.

A summary of the practices regarding certification of teachers was prepared by Sueltz from data collected for the National Survey of the Education of Teachers. The conclusions relevant for this study are as follows:

> Certificates are now most commonly issued on the basis of educational training. . . .
>
> In general, certificates for teaching in high schools are based upon four years of college training, which includes between fifteen and eighteen semester hours of credit in the field of educational theory. . . .
>
> There is a trend toward requiring teachers of mathematics to have special preparation in this field (education) before they are certified to teach the subject (mathematics). . . .
>
> Thirty states have a definite requirement of some form of apprentice or practice teaching. A tendency toward requiring training agencies to provide adequate facilities for this work is indicated.
>
> There is a tendency to specify some of the separate courses in educational theory that are accepted to meet the requirement in this field, (as for example, a course on methods of teaching high school mathematics, for prospective mathematics teachers).[7]

[7] B. A. Sueltz, *The Status of Teachers of Secondary Mathematics in the United States*, pp. 114-116.

Professional Training in Universities and Colleges. The National Survey showed that schools or colleges of education associated with universities or colleges are of recent growth, their number having increased from twelve in 1905 to one hundred and five in 1938. The important findings of the Survey concerning these schools of education which are relevant for this study are as follows:

> In more than half of them differentiated under-graduate curricula for teachers begin with the freshman year; in 28 per cent with the junior year; in 8 per cent at the graduate level. . . . Schools and colleges of education include the largest teacher education institutions in the United States. . . .
>
> In only 24 per cent of the institutions was preparation for teaching restricted to the school or college of education: in 76 per cent prospective teachers were enrolled in other colleges as well. . . . However, a median of 71 per cent of all undergraduate students who receive degrees with complete qualifications for certification were reported as enrolled in the schools of education. . . .
>
> The extent of control exercised by schools of education is shown by some of their policies and practices. In four-fifths of the cases the schools of education reported that they designate the specific fields that are acceptable as major or minor fields of concentration for teaching.[8]

In view of the importance of these institutions for the professional training of teachers it is of interest to examine the professional courses which they prescribe for prospective teachers. Information on this point is given in Table XII which is adapted from the report of the National Survey.

Table XII shows that purely professional subjects account for about one-fifth of the total number of credits required for graduation from universities and colleges, and for about one-fourth of the number for graduation from teachers colleges. This difference arises because in the universities and colleges practice teaching is included as part of education and educational psychology, whereas in the teachers colleges it is given separate recognition in terms of points. The educational subjects taken by a considerable majority

[8] Earle R. Rugg (et al.), *Teacher Education Curricula*, National Survey of the Education of Teachers, Vol. III, pp. 164-167.

of students in these institutions seem to be education, educational psychology, general psychology, special methods, student-teaching, and general methods in that order.[9]

Another factor influencing the number of credits in education and related subjects taken by students in universities and colleges is that the accrediting associations have designated minimum requirements in these subjects. For example, two of these associations specify a minimum of 15 credits, two others a minimum of 12 credits; the remaining association has a requirement of professional preparation or successful teaching without prescribing a minimum.

TABLE XII*

PRESCRIPTIONS IN MAJOR ACADEMIC AND PROFESSIONAL COURSES CATALOGUE DATA FOR 57 UNIVERSITIES AND COLLEGES AND 29 TEACHERS COLLEGES

Institutions	Median Credit for Graduation	Major Subject Mathematics		Education† and Educational Psychology			Special Methods			Student Teaching		
		R‡	M‡	P‡	R	M	P	R	M	P	R	M
Universities and Colleges	125	17-40	28	97	6-32	[illegible]	78	2-13	6			
Teachers Colleges	130	12-45	25	100	5-41	18	81	1-24	7	93	2-16	7

* Earle R. Rugg (et al.), *Teacher Education Curricula*, National Survey of the Education of Teachers, Vol. III, p. 530, Tables B and C.
† Includes student teaching and observation.
‡ P = per cent of institutions prescribing.
R = range of semester hours.
M = median number of semester hours.

Special Methods. Of particular concern for this study is the kind of preparation which prospective mathematics teachers receive in these institutions in the special methods courses and in practice teaching. According to data collected by the National Survey, special methods courses, regarded as agencies of professional training, rank high in the estimation of teachers. In mathematics these special methods courses deal with the teaching of mathematics in junior and or senior high schools.

[9] Earle R. Rugg (et al.), *Teacher Education Curricula*, National Survey of the Education of Teachers, Vol. III, p. 260, Table 18.

Table XIII was compiled from data supplied in answer to a questionnaire. While the number of responses does not give the data great significance, the fact that the institutions responding were widely distributed geographically will permit their being regarded as representing trends in methods courses. The table shows that the teachers colleges more consistently offer special methods courses in mathematics than do the other two types of institution, and that the university mathematics departments tend to offer courses in the teaching of senior high school mathematics rather than in junior high school mathematics. It should be noted that

TABLE XIII

SPECIAL METHODS COURSES IN MATHEMATICS IN THREE TYPES OF TRAINING INSTITUTIONS

Institutions	No. of Institutions	Number Offering Courses in Teaching of: Senior High School Mathematics	Junior High School Mathematics
Universities	13	8	3
Liberal Arts Colleges	14	6	7
Teachers Colleges	23	17	15

these data do not refer to the schools of education in universities, but only to the mathematics departments in universities.

While the catalogues of colleges and universities do not give very detailed information about the content of these courses, the following statements taken from current catalogues of a graduate school of education and a liberal arts college will illustrate the type and range of work done in these special methods courses.

The Teaching of Algebra.

A study of modern methods of teaching high school algebra. Topics discussed: the cultural values of algebra; reasonable outcomes from the study of this subject; the place of algebra in the curriculum; fundamental laws and principles of algebra; the function concept as the unifying element of the course; the treatment of formulas and equations; directed numbers; problem solving; the real problem movement; obsolete topics; numerical trigonometry; recent changes in American and European syllabi in algebra studies and experiments relating to the teaching of this subject.

Emphasis on the possibilities of making algebra a thinking rather than a manipulative subject, with educational values corresponding to those of geometry.

(Graduate School of Education)

The Teaching of Mathematics (Three credits).

A study is made of the secondary school curricular offerings in mathematics. The curricula of secondary schools are examined and evaluated. Instruction in organizing subject matter into suitable teaching units is given. Lesson plans and appropriate test materials are prepared and evaluated. The reference material, judged and tabulated, is arranged for possible future uses.

(College)

Materials and Methods in the Teaching of Secondary Mathematics.

3 semester-hours credit: 3 recitations per week. (Elective for juniors and seniors.) Brief history of mathematics, relation of mathematics to civilization; value of mathematics; reorganization of the curriculum; modern tendencies; objectives of the teaching of mathematics and their validity; special methods: suggestions on teaching critical parts; marks; tests, their construction, validity and reliability.

(College)

In each type of institution the special methods courses count for credit. Usually 3 hours a week for one semester, in all, about 50 hours of lectures, are devoted to their study, though a wide range of such courses is offered in some of the larger teacher training institutions.

Practice Teaching. Practice teaching or student teaching is coming to be regarded in the United States as of great importance in the professional preparation of teachers. In virtually all schools of education in universities in the United States provisions are made for teaching in schools variously called laboratory schools and university high schools. The advantages regarding practice teaching enjoyed by students in universities and colleges situated in large cities are not available to students attending liberal arts colleges in smaller communities which do not train for teaching a sufficient number of students each year to make it worth while for the college to have its own training school. For instance, in a recent study Baugher[10] showed that liberal arts colleges in the United States have on the average only twenty-nine students engaged in practice teaching each year. He showed also that in these colleges practice teaching is most frequently done during the senior year. The National Survey[11] showed that in 1930 31 all teachers colleges and about 85 per cent of colleges and universities in the United States, required their students to have experience in practice teaching. Thus in some institutions in the United States practice teaching is regarded as a privilege to be granted only to suitable persons. However, as the data show, the trend is toward requiring all students to engage in practice teaching during their course of professional training.

Under a widely accepted plan the experience afforded by practice teaching is divided into three main stages observation, participation, and teaching. These stages, however, are not rigidly separated from one another. Practice teaching usually occupies the equivalent of one hour a day for one semester (18 weeks) during the senior year; this conforms to the 90 hour standard for student teaching, established by the American Association of Teachers Colleges and adopted widely throughout the country. Credit to the amount of from 3 to 6 points toward a degree is allowed for the work in practice teaching.

The first stage, observation, begins when the student has been assigned to a class. He is under the immediate direction and supervision of the supervising instructor[12] and he observes the work of the class and of the instructor for some time. Gradually, he is allowed to participate, at first in the non-teaching activities of the classroom, such as constructing and correcting exercises, and later in part-teaching activities, such as assisting individual pupils with their difficulties. After participating for some time the student is allowed to develop a unit of work with the class and as time goes on his teaching responsibility in the class increases. Baugher showed in his investigation that the distribution of time as between

[10] J. H. Baugher, *Practice Teaching in Liberal Arts Colleges*, p. 10, Table 2.

[11] *Special Survey Studies*, National Survey of the Education of Teachers, Vol. V, p. 120.

[12] Supervising instructor is the name given to a classroom teacher in a special laboratory or practice school.

these three stages of practice teaching in liberal arts colleges was: observation 38 per cent, participation 13 per cent, teaching 49 per cent.[13]

A limitation of this plan for organizing practice teaching seems to be that a student may observe teaching in only one class, and may have experience in teaching one or at most two units of work in mathematics to one group of pupils during his pre-service professional training.

Supervision of Practice Teaching. An analysis of the replies to the question of whether the mathematics faculty of teacher training institutions were responsible for supervising the practice teaching is shown in Table XIV. These data, while possibly not significant

TABLE XIV

SUPERVISION OF PRACTICE TEACHING BY MATHEMATICS FACULTY

Institutions	Supervision by Mathematics Faculty		Number of Institutions
	Yes	No	
Universities	3	9	12
Colleges	0	14	14
Teachers Colleges	8	16	24
Totals	11	39	50

for the whole country, have some significance as indicating trends, since the institutions which supplied data were widely distributed geographically. The data point to the fact that in colleges and to a considerable extent in universities the mathematics faculty of these institutions do not supervise the student teaching of prospective mathematics teachers. To some extent this is true also in teachers colleges. Supervision is quite commonly done by special supervisors on the faculty of schools of education and teachers colleges. The fact that the mathematics faculty in about one-third of the teachers colleges supervises student teaching is probably connected with the movement toward professionalization of subject matter in these institutions, since professionalization necessitates close coöperation between the work in the schools and in the training institutions.

[13] Baugher, *op. cit.*, pp. 58-59.

The Professional Preparation of Mathematics Teachers in Teachers Colleges. Teachers colleges in the United States at the present time educate only about one-fifth of the teachers for senior high schools and about three-tenths of the teachers for junior high schools. These proportions are significant, however, and they gain in significance when it is remembered that the number of teachers colleges is only about one-fifth of the total number of institutions training teachers for secondary schools.

There are fundamental differences between the organization, as training institutions, of teachers colleges on the one hand and universities and colleges on the other. In teachers colleges the whole four years' program of a student's professional and academic education is carried forward in the one institution. In universities a separation is usually made between the academic training received in the liberal arts, or other colleges, and the professional training received in the school of education. While in liberal arts colleges the whole program of work is carried on in the one institution, little attempt is made to give a professional bias to the work until the junior and/or senior years. The continuous four-year curricula of teachers colleges tend to direct the whole of the student's college experience toward his future work. One important result of this difference in organization has been illustrated earlier in the account of professionalized subject matter which is taught in teachers colleges. An important possibility in professionalizing subject matter in mathematics is that the relations between the more advanced mathematics and school mathematics can be kept in mind by the instructor who teaches both the advanced and the special methods courses in mathematics to students who have frequent opportunities to see the mathematics lessons taught in the laboratory school attached to the college.

Another result of this organization is that the student's practical contact with the schools may begin earlier in his course perhaps even in his freshman year. An example of the way subject matter is professionalized is given later in this section in connection with the account of the training of mathematics teachers at New Jersey State Teachers College, at Montclair. Apart from the fundamental difference in organization mentioned above, the content of the professional courses in education, educational psychology, and general psychology in teachers colleges parallel closely those already described for schools of education in universities and for liberal arts colleges.

Special Methods and Practice Teaching. The following extract from the bulletins of two of the larger teachers colleges in the United States will serve to illustrate the purpose and nature of the work done in teachers colleges in the special methods courses in mathematics and during the period of practice teaching. Their similarity to the corresponding courses in schools of education in universities will be noted. That rather more credit is allowed on the average for both of these courses in teachers colleges than in universities and colleges is shown in Table XII above.

The Teaching of Secondary School Mathematics.

> The courses in the Teaching of Mathematics in Secondary Schools coordinates and brings to a focus all of the professionalization of his previous courses. Here his attention is concentrated solely on a careful study of the teaching of mathematics in secondary schools. He becomes acquainted with the literature of the teaching of mathematics and with discussions by leading teachers in mathematical periodicals. In supervised student-teaching the student puts into practice, under expert direction and supervision, in high school classes, the theories and methods he has studied. Thus we have the combination of a sound scholarship in mathematics and an apprenticeship under successful high school teachers. . . . He participates, under the direction of high school instructors, in organizing material, in making, administering, and marking tests, and in assisting in experimental work. A study of recent trends in the teaching of mathematics, of noteworthy research, and of modern texts and tests is included.
>
> (3 credits—State Teachers College)

Teaching Practices in High School Mathematics.

> Modern points of view toward instruction in the high school; dominating ideas in arithmetic, algebra and geometry; values in mathematics and reasonable outcomes to be expected from the study of mathematics; recent experiments to improve the teaching of mathematics and problems demanding study; necessary equipment for school mathematics; organization and presentation of the materials of instruction. (3 credits—State Teachers College)

The Program of Professional Education of Mathematics Teachers in Three Institutions in the United States. In the following section a summary is given of the methods employed in training mathematics teachers in three institutions representing respectively the graduate schools of education, the schools of education in universities, and the teachers colleges. The program of training offered in these three institutions represents that of the better training institutions in the United States.[14]

Teachers College, Columbia University, New York.[15] Teachers College, Columbia University, is a postgraduate professional school for the training of teachers and others engaged in the various branches of the profession of education. In general, students entering Teachers College are expected to have the A.B. degree or its equivalent. Its courses are organized in five main divisions; namely, foundations of education, organization and administration of education, guidance, instruction, and nursing education. The department of the teaching of mathematics forms a part of the division of instruction. The courses in the teaching of mathematics offered in Teachers College

> are designed to meet the needs of students who are engaged in, or wish to prepare for, the teaching or supervision of mathematics in elementary schools, secondary schools, normal schools, or teachers colleges. Provision is also made for students majoring in other subjects who wish to take some work in mathematics.
>
> Programs may be arranged to lead to the degrees of Master of Arts, Doctor of Philosophy, and Doctor of Education.[16]

For the degree of Master of Arts, for which most of the students in Teachers College study, 32 tuition points, or 30 points plus a thesis, must be taken, distributed over a period of not less than one academic year beyond the Bachelor's degree. Of the required points, 8 must be taken in the foundations of education, and from 12 to 24 in the major field. Thus more than half the points necessary for a Master's degree in Teachers College may be earned in special

[14] Not sufficient difference was found in the training program offered in institutions in other states to warrant special mention of more than these three institutions.

[15] Based on information given in the *Teachers College Bulletin*, Winter and Spring Sessions, 1938 1939.

[16] *Ibid.*, pp. 141.

methods courses in the teaching of mathematics and in courses in professionalized subject matter in mathematics.

The College also offers professional diplomas of which three are available in the department of mathematics; namely, (i) Teacher of Mathematics, (ii) Supervisor of Mathematics, (iii) Teacher of Junior High School Mathematics. The course for a professional diploma extends over two years and includes, as part of its requirement, the usual program for the Master's degree,

> and additional work . . . comprising courses in mathematics, science, statistics, education or related fields, chosen from the offerings of the College or the University. An orientation course in educational administration and a course in guidance are recommended for all diploma programs.[17]

The following professional courses[18] suitable for secondary school teachers of mathematics are at present offered in the department of mathematics in Teachers College:

	No. of Points
1. Teaching and supervision of mathematics (junior and senior high school)	3*
2. The reorganization of secondary school mathematics	3
3. Teaching of algebra	3
4. Teaching of geometry	3
5. Student observation and teaching in mathematics	3
6. Professionalized subject matter in junior high school mathematics	3*
7. Field work in mathematics	2 or 3
8. Foundations of mathematics Professionalized subject matter for teachers in secondary schools	3*
9. Elementary mechanics for teachers in secondary schools	3
10. Professionalized subject matter in senior high school mathematics	3*

* Each semester.

[17] *Teachers College Bulletin*, Winter and Spring Sessions, 1938-1939, p. 141.

[18] Each course includes two lectures a week for eighteen weeks. In cases where a course may be taken for different numbers of points additional reading is required to earn the larger number of points.

Since academic courses in mathematics as such are not given in Teachers College, and since in any case Teachers College students come from all parts of the country, it is expected that students taking mathematics as a major teaching subject will have had courses in mathematics at least through the calculus before entering the College. An analysis of the previous mathematical training of students at present registered as offering mathematics as a major subject in Teachers College showed that more than one-half the students had taken the following mathematical courses before entering the College:

	Per Cent
Freshman college mathematics	69*
Solid analytic geometry	62
Differential calculus	89
Higher college algebra	60
Plane analytic geometry	93
Plane trigonometry	73
Integral calculus	82

* Percentage of the 1938 class who had previously taken the subject.

A third or more of these students had also taken courses in the history of mathematics, differential equations, theory of equations, and spherical trigonometry.

Examinations. The examinations on the courses in the teaching of mathematics are frequently of the short-answer objective type, though essay type questions are also given from time to time. Examples of examination questions are listed below:

Education 268P Final Test (1937)

Place a plus sign before those statements which you believe to be entirely true and a minus sign before those with which you are in complete or partial disagreement.

.... 4. Trigonometric ratios should be taught as a department of the doctrine of direct proportion.

.... 7. The concept of the doubly-endless sequence is of little importance in algebra.

.... 10. Nunn objects to the use of x and y for the unknown at the beginning of algebra.

.... 18. The fundamental idea over a large part of mathematics, particularly in the field of signed number, is not magnitude but order.

.... 19. A graphic record is useless unless its object is clearly stated and the scales properly labelled.

.... 32. $a^0 = 1$ cannot be proved.

.... 37. The graph is superior to the formula for expression of the relationship between two or more variables.

.... 45. The dominant notion underlying the scheme of directed numbers is the notion of magnitude.

Final Examination in The Teaching of Geometry

Place a plus sign at the left of each statement which is always true, a minus sign at the left of each statement which is always false, and a D at the left of each statement which is debatable.

.... 1. The study of geometry should be primarily a course in the solution of originals and general methods of attack.

.... 6. Every theorem should prove some new fact which could not have been perceived by direct intuition, symmetry, or superposition.

.... 10. No part of Euclid's scheme can be said to be valuable for its own sake or necessary in other parts of mathematics.

.... 13. If a good preliminary course in geometry has been taught, pupils will be unwilling to accept postulates without argument.

.... 20. If a pupil can perceive the truth of certain theorems, there can be few processes more destructive than the elaboration of deductions from other intuitions with the sense of imparting a notion of proof.

.... 25. The "wonder motive" should be the controlling factor in teaching "informal geometry."

.... 36. Ratio, proportion, and variation are so interwoven that a proper understanding of them demands their simultaneous teaching.

.... 60. The presence of schemes of deduction based on statements which find universal acceptance as descriptions of our space impressions must make for good in the child's development.

Practice Teaching. Owing to the fact that Teachers College is a postgraduate professional school of education, a large percentage of its students has had previous training and experience. Consequently a small percentage of its students is concerned with practice teaching experience.

Teachers College offers facilities for practice teaching to those students who satisfy certain conditions of health, of scholarship, and of credit in the field of specialization. Two types of practice teaching are offered, namely practice teaching and observation, and interneship, and facilities in both these types of practice teaching are available for students in the mathematics department. Under the plan of practice teaching and observation a student may spend an hour a day in a school for as long as one semester observing and teaching under guidance, while under the interneship plan the student serves as a teaching assistant in a school, and devotes more of his time to teaching than under the first plan.

University of Chicago,[19] *Chicago, Illinois*. As a result of a recent reorganization there is no longer a College of Education in the University of Chicago.

> General responsibility for the organization and direction of programs for teachers is now assumed by the university as a whole, and responsibility for the preparation of teachers in special subjects or fields is vested in designated divisions and departments.[20]

The policy underlying the plan of preparation of teachers for secondary schools is stated in the Announcements as follows:

> In view of the number and variety of experiences involved in the development of such teachers, university programs planned to this end are based on the assumption that study will be continued by prospective secondary teachers at least until the Master's degree has been attained. . . .
>
> To students who pursue appropriate programs and who give evidence of satisfactory attainments as defined by the particular Division or department involved, the University will grant a Master's degree and, in addition, a secondary school certificate. . . The certificates granted by the University are not substitutes for those issued the teachers by states or other legal certifying agencies. They are documents which attest evidence of professional competence and serve as official recommendations from the University to those who employ teachers.[21]

[19] Announcements, The University of Chicago, *The Preparation of Teachers*, 1938–39.

[20] *Ibid.*, p. 6.

[21] *Ibid.*, pp. 7–8.

The general requirements for the secondary school certificate are (1) a good general education, (2) academic competence in the field of specialization, (3) professional competence as shown by attainments in examinations in education, and special methods.

Training of Mathematics Teachers—The Secondary School Certificate. The Announcements state clearly the requirements for earning the secondary school certificate in mathematics. A minimum program of mathematical courses leading to the secondary school certificate is outlined which students must take in the mathematics department of the University. The general plan recommends (1) that prospective mathematics teachers spend the first two years in the College studying the fundamental branches of elementary mathematics, namely, plane trigonometry, college algebra, plane analytic geometry, calculus, and elementary mathematical analysis I to III; (2) that on entering the Division (of the physical sciences), a further two years is spent in studying additional mathematics courses and certain fundamental courses in education, as follows:

Mathematics. (*a*) Elementary theory of equations, synthetic projective geometry or solid analytic geometry.

(*b*) *Two courses from:* differential equations, infinite series and definite integrals, introductory theory of functions, introduction to higher algebra, theory of functions of a complex variable, metric differential geometry.

Teaching of Mathematics.[22] Introduction to the teaching of mathematics, apprentice teaching in mathematics in secondary schools.

Education. An introduction to the study of the American school system: an introduction to educational psychology.

(3) In the third year of the Division, leading to the Master's degree in mathematics, the prospective teacher is recommended further to take:

(*a*) Eight courses acceptable to the Department of Mathematics.
(*b*) A thesis acceptable to the Department as fulfilling part of the requirement for the Master's degree.

[22] The courses in the Teaching of Mathematics cover the material of the following books by E. R. Breslich: *The Technique of Teaching Mathematics in Secondary Schools*, *Problems in Teaching Secondary School Mathematics*, *Administration of Mathematics in Secondary Schools.*

(c) Advanced problems in teaching mathematics in secondary school and junior college: or a course in a second teaching field.

Requirements when mathematics is the second teaching subject are also stated and include Part 1 and a little of Part 2 of the above program.

New Jersey State Teachers College at Montclair. New Jersey State Teachers College is a four-year teachers college preparing teachers for secondary schools. It offers graduate courses and awards the degree of Master of Arts in specific subjects. In the field of mathematics this college has adopted the idea of the professionalization of subject matter in the sense in which the sponsors of that idea would have it regarded. The point of view regarding mathematics teaching in the College is presented in the *College Bulletin* as follows:

The training in mathematics in a teachers college should differ from that given in a liberal arts college or in an engineering college in a number of important respects. Throughout the entire course it should be remembered that the student is preparing to teach mathematics to secondary school pupils. Thus he must not only thoroughly understand the uses and limitations of formulas, but also be able to derive them from simpler ideas; he must not only thoroughly understand fundamental principles, but also acquire the facility of making them clear to others, of searching out the obstacles that hinder another's understanding. He must not only have a mastery of the particular topics he is to teach, but also see them as an integral part of the entire body of subject matter of mathematics, and know the place of mathematics in the history of civilization and its uses in practical life.

The courses offered by the Department of Mathematics have been selected with the following objectives (among others) in view:

. . . .

2. To professionalize the course so that the student will be conscious of teaching problems and will have abundant practice in logical reasoning and in making lucid explanations.

. . . .

4. To supply a cultural background and an awareness of the specific contributions which mathematics has made to civilization.

. . . .

6. To integrate the work with other courses, particularly science, social

studies, and economics, so that the student will realize the effectiveness of mathematics as a tool in solving scientific and sociological problems.[23]

Courses. Required courses[24] at State Teachers College, Montclair, for students whose major subject is mathematics are as follows:

First Year. Mathematical analysis, 1, 11. (3 each)
Second Year. Differential and integral calculus. (3 each)
Third Year. Modern college geometry (3), higher algebra (3).
Fourth Year. The teaching of mathematics in secondary schools (3), solid analytic geometry and analysis, supervised student teaching, reading and lectures in mathematics.

In addition, a large number of mathematics subjects are offered as electives. Information supplied by the College indicates that students studying mathematics as a major subject choose a large number of these mathematics courses as electives:

Third Year. Solid geometry.
Fourth Year. History of mathematics, educational statistics.
Graduate Courses: (Open by permission to seniors).

Second Teaching Subject. Since it has been shown that no single subject is widely adopted by teachers as a second teaching subject, with mathematics as the principal teaching subject, no attempt will be made here to describe in detail the type of training offered in the three subjects which are most frequently studied with mathematics. It will be sufficient to note that the three types of institutions considered offer courses both in the academic and professional aspects of these subjects which correspond to those offered in the academic and professional aspects of mathematics.

The Mathematics Faculty of Training Institutions. The qualifications of members of the mathematics faculty of certain teacher training institutions in the United States are shown in Table XV. The table shows that a higher percentage of mathematics teachers in colleges and universities than in teachers colleges have studied

[23] *Catalogue of Courses, 1938–1940,* Bulletin of the New Jersey State Teachers College at Montclair, p. 82.
[24] The numbers in parentheses refer to credits for the course.

TABLE XV

QUALIFICATIONS OF MATHEMATICS FACULTY MEMBERS

Institutions	Number of—		Per Cent of Faculty Who Have—				
	Institutions	Mathematics Faculty	Taught in Secondary School	Done Graduate Work in Mathematics	M.A. Degree	Ph.D. Degree	Other Degrees
Colleges	12	46	28	67	100		
Universities	11	100	37	94	22	76	2
Teachers Colleges	27	78	42	59	61	30	9

mathematics beyond the undergraduate stage. While the great majority of the faculties of all three types of institutions has earned higher degrees, the fact just mentioned would suggest that a larger percentage of mathematics teachers in teachers colleges than in colleges and universities have done their postgraduate work in subjects other than mathematics, presumably in professional subjects. A larger percentage of the mathematics faculties of teachers colleges, however, have had experience in teaching mathematics in secondary schools.

In-service Professional Training. The contribution of summer schools to the in-service academic training of mathematics teachers has already been considered. It was pointed out that, according to Sueltz, teachers tend to take courses in professional subjects rather than in academic subjects during the summer sessions. The professional subjects which they may offer include among them the courses in the teaching of mathematics. Inspection of a small sample of catalogues of institutions offering summer courses shows that, except in the larger graduate schools of education, such as Teachers College, Columbia University, only two or three courses in the teaching of mathematics are offered. These include the teaching of arithmetic, the teaching and supervision of mathematics, and the teaching of mathematics.

The fact that so few courses in the teaching of mathematics and, as was shown earlier, in mathematics seem to be offered during summer sessions, makes almost inevitable Sueltz's[25] conclusion

[25] *Op. cit.*, p. 89.

cited earlier, that mathematics teachers take professional subjects in preference to academic subjects during the summer sessions. The courses offered during the summer sessions cover the same material as do the corresponding courses offered during the regular year, but it has to be given in more concentrated form.

Associations of Mathematics Teachers. The associations of mathematics teachers in the United States have made important contributions to the pre-service and in-service training of mathematics teachers. The two most active associations in this field are the Mathematical Association of America, an association with a membership drawn largely from college teachers of mathematics, and the National Council of Teachers of Mathematics which draws its membership largely from high school teachers.

Mathematical Association of America. The Mathematical Association of America has exercised real leadership in the campaign for the improvement of mathematics teaching in, and of mathematics teachers for, the high schools of the United States. At the instigation of this Association the National Committee on Mathematical Requirements was organized in 1916 "for the purpose of giving national expression to the movement for reform in the teaching of mathematics."[26] For several reasons the National Committee did not really begin to function adequately and continuously till 1919 when J. W. Young and J. A. Foberg were appointed to devote their whole time to the Committee's work. As a result of this more concentrated attention to the subject of their report, the Committee was able to publish in 1923 a report entitled *The Reorganization of Mathematics in Secondary Education.* In this report the Committee dealt principally with the improvement in mathematical instruction in schools in the United States, and an account of this work has been given earlier in this study.

The Committee's recommendations on school mathematics have colored the thinking of all writers on mathematical education who contribute to the two important professional journals, *The Mathematics Teacher* and *School Science and Mathematics.* They have also influenced textbook writers who in their turn have played a major part in gradually disseminating the main ideas of the report.

[26] *The Reorganization of Mathematics in Secondary Education*, p. vii.

This contribution of writers of textbooks to the reform of mathematics in the schools of the United States gains especial significance from the practice in many states of having state-wide adoption of textbooks. It will be assumed, then, that the Committee's principal contribution is to the improvement of mathematical teaching in secondary schools. While this contribution to school mathematics is important for this study, a fact of greater importance is that the Committee devoted some attention to the training of teachers. They deplore the fact that (in 1922) so many teachers who had no special training in mathematics were allowed to teach the subject.

The Report of the Committee[27] devotes a long chapter to describing methods of training in operation in some of the states, cities, and teacher-training institutions in various parts of the United States. This description revealed features which exist at the present time, namely, the heterogeneity in the requirements for certification, the unequal preparation in subject matter required of prospective teachers of mathematics, the variety of offerings in the courses on the teaching of mathematics, and the unequal requirements in courses in educational subjects.

Having in mind the diversity which actually existed in methods of training, the Committee thought it desirable

> to formulate a standard for certification which would be generally desirable if it could be put into effect - a formulation suggestive to colleges and universities throughout the land in raising standards to a reasonable status, even though this be far below that of several European countries.[28]

Briefly the standard proposed for mathematics teachers in senior high schools was as follows:

(1) Graduation from a four-year college, university or teachers college with credit for at least the following mathematical courses: plane and spherical trigonometry, analytic geometry (two and three dimensions), college algebra, differential and integral calculus, synthetic projective geometry, scientific training in geometry, scientific training in algebra.

[27] *The Reorganization of Mathematics in Secondary Education*, pp. 429-508.
[28] *Ibid.*, p. 507.

(2) Credit for theoretical and practical physics, chemistry.
(3) Credit of history of education, principles of education, methods of teaching, educational psychology, organization and function of secondary education.
(4) Satisfactory performance of the duties of a teacher of mathematics in a secondary school for a period of not less than 10 year, or 20 semester hours.
(5) The desirability of college semester courses in solid geometry, descriptive geometry, analytic projective geometry, theory of statistics, mathematics of investment, surveying, practical and descriptive astronomy.
(6) The desirability of college semester courses in history, economics, sociology, political science, general psychology, philosophy and ethics.[29]

These recommendations have been summarized in some detail because they reveal at once the source from which has come the organization of training courses for mathematics teachers which have been described up to the present in this study. While this standard has not been uniformly met there is evidence that the mathematics courses have been broadened to include some of the subjects in paragraph (5) above, in addition to most of those mentioned in paragraph (1).

The Mathematical Association again considered the subject of the training of mathematics teachers in 1935 in the report which has been cited frequently. While this report was concerned chiefly with the training of mathematics teachers in colleges, it devoted a section to the training of secondary school teachers of mathematics. Its recommendations for secondary school teachers of mathematics are quoted below:

(1) Minimum training in mathematics
   a. Courses in mathematics including complete treatments of trigonometry, college algebra, analytic geometry, and six semester-hours of calculus.
   b. A college treatment of synthetic Euclidean geometry (or, possibly, descriptive geometry) (three semester hours).
   c. Advanced algebra, such as the theory of equations (three semester-hours).

[29] *The Reorganization of Mathematics in Secondary Education*, pp. 507–508.

d. Either directed reading or a formal course in the history of mathematics and the fundamental concepts of mathematics.

(2) Minimum college training in fields related to mathematics.

a. Introductory courses in physics and in another science (twelve semester-hours).

b. A course in the mathematics of investment (three semester-hours).

c. An introductory course in economics.

d. A first course in statistics, with a mathematical viewpoint (three semester-hours).

(3) Desirable additional training in mathematics and related fields.

a. Advanced calculus and differential equations or mechanics (six semester-hours).

b. Additional work in geometry, such as projective geometry, solid analytic geometry, etc. (three semester-hours).

c. Additional study of algebra (three semester-hours).

d. Introduction to astronomy.

e. Additional study of physics and other sciences to complete a background in three or more sciences (nine semester-hours).

(4) Adequate training in English composition and cultural training outside of mathematics and related fields. Work in languages, literature, fine arts, and the social sciences in preference to increased specialization in mathematics and related fields, and in preference to elective work in the theory of education beyond the legal requirements.

(5) Training in the theory of education and practice teaching.

a. A one-year course in methods of teaching and practice teaching in secondary mathematics, together with any distinctly pertinent material concerning educational measurements and other content from educational theory (ten semester-hours). It is our belief that this essential part of the student's training should, if possible, be under the direction of professors who have taught mathematics at the secondary level, and who have maintained contacts with the secondary field.

b. Study of methods of teaching in the principal minor field selected by the student and any additional material relating to the history, psychology, or administration of education which can be objectively justified in the training of a teacher (not more than five semester-hours).[30]

[30] "Report on the Training of Teachers of Mathematics," *American Mathematical Monthly*, XLII (May, 1935), pp. 275–76.

Judging from the evidence of the two reports just cited it would seem that mathematics teachers in the United States have definite ideas on the kind of training which prospective mathematics teachers should receive. They seem to regard knowledge of subject matter and specific professional training in the teaching of mathematics as the two most important elements in the training of prospective mathematics teachers.

The reports cited above provide evidence of the direct influence of the Mathematical Association of America on the training of mathematics teachers in the United States. Through its monthly journal, *The American Mathematical Monthly*, the Association provides opportunities for in-service training for teachers. While the general run of articles in this journal are technical in character and are likely to appeal chiefly to teachers who devote some time to the advanced study of mathematics, there are articles which deal with modern trends in various fields of mathematics. Many of these articles represent accounts of addresses on special subjects given at the invitation of the Association by eminent scholars in the subject. Such articles may stimulate teachers to pursue the study of newer branches of mathematics, but at least they will keep them informed of trends in some branches of modern mathematics. The titles of some such articles taken from recent volumes of *The American Mathematical Monthly* are as follows:

1. R. D. Carmichael, Recent Researches in the Theory of Numbers
   W. Weaver, Conformal Representation, with Applications to Problems of Applied Mathematics
   E. V. Huntington, The Postulational Method in Mathematics
   J. L. Coolidge, The Rise and Fall of Projective Geometry
   R. Wavre, Is There a Crisis in Mathematics?
2. R. E. Langer, René Descartes
   W. H. Bussey, Geometrical Constructions without the Classical Restriction
   W. B. Carver, Thinking versus Manipulation

Joint Commission of the Mathematical Association of America and The National Council of Teachers of Mathematics. As a result of the extensive experiments that have been made with the suggestions of the National Committee, it has recently become clear that in certain respects their recommendations needed reconsideration. It should be understood, however, that for the most part the report contains much material that is still relevant and valuable, if not, indeed, still in advance of the best current practice. In 1936 a Joint Commission of the Mathematical Association and the National Council of Teachers of Mathematics undertook to review the situation and they have published a tentative report called *The Place of Mathematics in Secondary Education.* While this Joint Commission intends to discuss the problem of teacher training, this section of their report is not yet available. Mention of the report of this Commission is relevant for this study because it has stimulated considerable discussion among mathematics teachers and thus has made them aware of the trends of thought in mathematical education, and of the possible direction in which their teaching may develop in the future.

The National Council of Teachers of Mathematics. Of the professional associations of mathematics teachers in the United States, this organization most nearly corresponds in character and in influence to the Mathematical Association of England. The N. C. T. M. (as it will be designated hereafter) publishes *The Mathematics Teacher,* a magazine devoted entirely to the interest of mathematical teaching in schools, and also a yearbook. Of these two publications, the yearbooks are extremely important and influential, although, on account of its more frequent publication, its wide geographical circulation, and the practical value of many of its articles *The Mathematics Teacher* exerts a not inconsiderable influence on mathematical teaching. Owing, however, to the size of the country and the number of teachers of mathematics in the United States, it is not possible accurately to estimate the influence of the publications of the National Council.

Yearbooks. The yearbooks of the N. C. T. M., of which thirteen have been published to date, deserve a high place in the regard of mathematics teachers in the United States, and they hold that place, if opinion expressed to the writer by mathematics teachers from many parts of the United States can be considered as a criterion.

The subjects to which yearbooks have been devoted since the publication of the first yearbook in 1926 are as follows:

1. A Survey of Progress in the Past Twenty-Five Years
2. Curriculum Problems in Teaching Mathematics
3. Selected Topics in the Teaching of Mathematics
4. Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910
5. The Teaching of Geometry
6. Mathematics in Modern Life
7. The Teaching of Algebra
8. The Teaching of Mathematics in the Secondary Schools
9. Relational and Functional Thinking in Mathematics
10. The Teaching of Arithmetic
11. The Place of Mathematics in Modern Education
12. Approximate Computation
13. The Nature of Proof

An important characteristic of the yearbooks of the N. C. T. M. is that they represent individual points of view on mathematical subjects in secondary schools rather than the collective view of a committee of teachers. This is not intended to imply that the contributions of individuals are in any way less valuable than are those of committees. Indeed, in the case of three of the yearbooks the whole of the yearbook is devoted to the work of one person, namely, the ninth, twelfth, and thirteenth yearbooks. To illustrate the kind of contribution which the yearbooks make to the growth of mathematics teachers, a discussion is given here of the content of the fifth and ninth yearbooks. The fifth yearbook on *The Teaching of Geometry* is selected so that the discussion of this subject may be compared with that given on the *Report on the Teaching of Geometry* prepared by the Mathematical Association of England.

The treatment of this subject in the fifth yearbook does not represent the work of a committee, but it is composed of the individual contributions of a number of leaders in the field of the teaching of mathematics in the United States. The articles in the yearbook center round four main problems: the place of geometry as a school subject and the current ideas about teaching of geometry; the development of teaching units in geometry of a more or less experimental kind, and a new approach to the teaching of geometry; special methods used in teaching geometry; the modern views of transfer of training as applied to geometry.

These articles will now be discussed briefly.

1. The articles in the first group are by W. D. Reeve[31] and W. R. Longley.[32] These writers are concerned not so much with details of classroom procedure as with expounding the modern ideas on the teaching of geometry. For example, Reeve's article on "The Teaching of Geometry" discusses informal and demonstrative geometry, the purposes of studying geometry, the content of the course and conditions controlling its selection, the place of principles and definitions in teaching geometry, methods of attacking original exercises and some points on the conduct of a recitation. The aim of the article seems to be rather to make mathematics teachers aware of the modern problems in teaching geometry than to give them any systematic help in resolving particular difficulties.

   Longley's article does the same kind of service for solid geometry as Reeve's article does for plane geometry. In addition he also makes out a case for the study of geometry in schools. Because geometry as a school subject is not studied by a large percentage of pupils articles of this type are necessary in American publications for their propaganda value as well as for their inherent professional value.

2. In the second group of articles one by Vera Sanford[33] gives an account of an experiment with the teaching of informal geometry in the seventh grade, i.e., to pupils eleven years old. The National Committee had recommended that an introduction to geometry might be given in the ninth grade before the formal study of the subject is normally begun in the tenth grade. Following this recommendation several attempts have been made to organize such units of demonstrative geometry. Two of these, by Orleans[34] and by Seidlin,[35] are

[31] W. D. Reeve, "The Teaching of Geometry," *National Council of Teachers of Mathematics, Fifth Yearbook*, pp. 1-28.

[32] W. R. Longley, "What Shall We Teach in Geometry?," *ibid.*, pp. 29-38.

[33] Vera Sanford, "Demonstrative Geometry in the Seventh and Eighth Years," *National Council of Teachers of Mathematics, Fifth Yearbook*, pp. 39-43.

[34] Joseph B. Orleans, "A Unit of Demonstrative Geometry for the Ninth Year," *ibid.*, pp. 44-53.

[35] Joseph Seidlin, "A Unit of Demonstrative Geometry for the Ninth Year," *ibid.*, pp. 54-63.

published in this yearbook. Orleans' unit is developed around angles at a point, congruence, parallels, angle-sum theorem, similar triangles. He gave sufficient illustrative material to make it easy for other teachers to teach this unit should they so wish. Seidlin's unit develops along different lines in that he assumes a number of theorems (e.g., vertical angles, first congruence theorem), parallel lines (corresponding angle theorem). The unit develops from these assumptions and other definitions by suggesting theorems which might be proved, but it does not contain very detailed accounts of exercises or originals which might be used in connection with the development of the unit.

A third article in this group, by Gertrude E. Allen,[36] gives an account of an experiment on the redistribution of material for high school geometry which was carried out in Oakland, California. The subject matter content is characterized by an increased number of postulates and a decreased number of theorems to be proved, a close correlation between plane and solid geometry, the incorporation of some algebra, trigonometry and arithmetic, and the application of general principles in the solution of practical problems.

3. The two articles on Methods of teaching, namely, the one by Upton[37] on "The Use of Indirect Proof in Geometry and in Life" and the one by Schlauch,[38] "The Analytic Method in the Teaching of Geometry," give thorough and well illustrated accounts of the uses of these methods, and give sufficient applications from the field of geometry and (in the case of Upton's article) from life, to make them valuable sources of reference for teachers of geometry.

4. In the final group which has been chosen for discussion here there is a rather elaborate article on "The Transfer of Training, with Particular Reference to Geometry" by W. Betz.[39] After discussing relevant psychological data Betz presents a clear account of the possibilities of geometry as a field of transfer. In summary he points out that the problem of transfer of training is fundamentally one of good teaching: that there must be training for transfer.

[36] Gertrude E. Allen, "An Experiment in Redistribution of Material for High School Geometry," *National Council of Teachers of Mathematics, Fifth Yearbook*, p. 79.

[37] C. B. Upton, "The Use of Indirect Proof in Geometry and in Life," *ibid.*, pp. 102-133.

[38] W. S. Schlauch, "The Analytic Method in the Teaching of Geometry," *ibid.*, pp. 134-44.

[39] W. Betz, "The Transfer of Training, with Particular Reference to Geometry," *ibid.*, pp. 149-98.

As an example of a yearbook which contains an individual contribution a review will be given of the *Relational and Functional Thinking in Mathematics*, by H. R. Hamley.

In this important monograph Hamley begins by examining from the purely mathematical point of view the terms function and variable and the meaning of functionality. He then examines the function concept from the psychological and historical point of view using this examination to prepare for the study of the function concept in its relation to secondary school mathematics. The influence of this important concept is traced, using current textbooks, in several of the more important European countries and in the United States. This influence of course starts with the work of Klein in Germany and its development is traced in England, Austria, France, and America. The most valuable part of this study from a practical point of view is reserved for the last chapters, "The Function Concept in Practice," and "A Course of Study Based on the Function Concept." In the former the foundation is laid for the course of study which follows by means of a discussion of the following topics: relations between elements of a single class, correspondence between the elements of two classes, some debatable topics in geometry, the graph, the formula. In the last chapter some two hundred graded questions forming typical problem material are listed to illustrate the progression of ideas developed during the study. The material listed here is practical in character. It has the great merit that it provides a way which has been taught successfully in schools of organizing the course of study in mathematics around the function concept. The material introduced into this course provides a good example of what can be made of school mathematics if there is kept clearly in mind some definite coordinating principle.

Central Association of Mathematics and Science Teachers. This Association, through its publication *School Science and Mathematics*, exerts an influence, principally in midwestern states in the United States, corresponding to that exerted by the National Council of Teachers of Mathematics through its journal, *The Mathematics Teacher*. The mathematical articles appearing in *School Science and Mathematics* deal chiefly with problems in the teaching of mathematics, although, as the following titles chosen from recent issues of the journal show, some of them deal with wider aspects of the subject:

H. C. Christofferson, Geometry as a Way of Thinking
Edith L. Mossman, The Purpose of Mathematics
P. A. Constantinides, Mathematical Time
C. A. Ellis, Mathematics and Engineering Education
F. W. Bubb, The Role of Mathematics in Science
R. F. Forbes, Mathematics Functioning in Industry

Summary. This section of the study has revealed the following features of the professional training of mathematics teachers in the United States.

1. The minimum requirements in professional subjects are prescribed by certifying agencies, and these minimum requirements are realized in the three types of training institutions in the United States.

2. There is a difference in the emphasis placed on the professional aspects of training as between the liberal arts colleges of universities and colleges on the one hand, and the schools of education and teachers colleges on the other hand. The former seem to favor the fairly definite separation of the academic from the professional aspects of training, while the latter appear to favor the course of training which duly emphasizes both aspects of training and which aims to professionalize subject matter.

3. The college group favor a modicum of theory and considerable experience of teaching as elements in the professional training, while the teachers colleges and schools of education in universities lean more heavily to theory without reducing the requirements of practice teaching.

4. Special methods courses in the teaching of mathematics are offered in most of the teacher training institutions, and usually about fifty lecture hours are devoted to them.

5. Practice teaching is advocated as a necessary part of the preparation of mathematics teachers, but it is regarded as a privilege to be granted to those specially fitted to undertake it.

6. On the whole the experience in teaching gained by prospective mathematics teachers during practice teaching periods tends to lack variety, being too often confined to a single class and to a single unit of work.

7. In a few graduate schools of education a wide variety of special methods courses in mathematics is offered.

8. An account was given of the courses for the professional training of mathematics teachers provided in three of the leading teacher training institutions in the United States, namely, in Teachers College, Columbia University, the University of Chicago, New Jersey State Teachers College at Montclair.

9. In-service professional training is provided in the sessions which many training institutions hold during the summer months. Except in the larger institutions the professional courses for mathematics teachers offered during summer sessions consist of courses in the teaching of arithmetic and of mathematics.

10. Opportunities for in-service training are also provided by three associations of mathematics teachers through their publications. Of these associations, The National Council of Teachers of Mathematics have through their journal, *The Mathematics Teacher*, and through their yearbooks, the most direct influence of a professional character. The Central Association of Mathematics and Science Teachers also exerts an influence affecting the professional in-service training of mathematics teachers through their publication, *School Science and Mathematics*.

In so far as it has sponsored the National Committee on Mathematical Requirements and has undertaken from time to time to prepare reports on the Training of Mathematics Teachers, the Mathematical Association of America has had considerable influence both on the subject matter taught in secondary schools and on the type of training which mathematics teachers receive. In addition, through its journal, *The American Mathematical Monthly*, this Association continues to provide opportunities for the in-service academic growth of mathematics teachers.

COMPARISONS

The account given in this chapter has revealed that the English theory and practice regarding the professional training of mathematics teachers corresponds very closely to that approved by liberal arts colleges, mathematics departments of universities, and associations of mathematics teachers in the United States. The theory and practice of professional training which they advocate is somewhat in contrast to that advocated by teachers colleges and colleges of education in universities in the United States. The former prefer to separate the period of professional training from that of academic training, while the latter prefer to combine the two.

Whereas in England and Wales professional training is not obligatory, in the United States all teachers must be certified and certification implies that some type of professional training be received. One important part of this professional training, so far as mathematics teachers are concerned, is the training given in methods of teaching mathematics. Except in one training institution in England and Wales, special methods do not receive adequate treatment there. More time is allowed for the study of special methods in all three types of training institutions in the United States, and in some of the larger institutions as many as ten special methods courses are provided.

Practice teaching also received emphasis in both countries. Whereas in England and Wales a long period of continuous practice teaching is regarded as necessary for all students in training institutions, in the United States practice teaching is regarded as a privilege. In the former countries practice teaching is marked by continuity of teaching and diversity of experience with pupils of different ages, while in the latter the experience is restricted frequently to a particular group of pupils and to one mathematical topic.

The courses in education and allied subjects given in training institutions in both countries correspond to some extent. The chief difference is to be found in the increased number of such courses offered in teachers colleges and especially in graduate schools of education in the United States.

A comparison of the qualifications of members of the faculty who prepare prospective mathematics teachers shows that in training institutions in England and Wales and in teachers colleges in the United States the faculty members combine scholarship with experience in teaching mathematics in secondary schools, and that, while they are adequately prepared in scholarship, the members of the mathematics faculties in universities and colleges in the United States have had little experience in secondary school teaching.

The opportunities for in-service professional training provided for mathematics teachers in England and Wales are very meager when compared with those offered in summer schools in the United States. In the former countries not sufficient courses are provided to take care of all who wish to attend, while in the latter country large summer schools are organized and a wide variety of courses is offered.

The associations of mathematics teachers in England and Wales and in the United States make similar contributions to the in-service professional training of mathematics teachers. In their respective journals they publish reports and articles which are designed to keep mathematics teachers informed of trends of thought and of current practices in the teaching of mathematics.

Chapter VII

SUMMARY AND COMPARISONS

IN THE INTRODUCTION to this study seven questions were proposed. The answers to these questions have been reported at appropriate places throughout the study and for convenience they will be summarized here.

1. ***How much mathematics is studied by secondary school pupils in the two countries, and what standard is achieved?***

ENGLAND AND WALES

The data presented in this study show that at present all pupils in secondary schools in England and Wales are required to study mathematics for four or five years, at the end of which time they take the first certificate examination in Elementary Mathematics. The mathematical subjects examined in Elementary Mathematics are arithmetic, algebra, geometry, and trigonometry. Those pupils who are especially interested in mathematics also take Additional Mathematics in the first certificate examination. The additional mathematical subjects studied for Additional Mathematics are analytic geometry, elementary differential and integral calculus, modern euclidean geometry, and certain topics in harder algebra. Mechanics is an optional subject for both Elementary and Additional Mathematics. Pupils who intend to study mathematics at a university remain at school for two years after the first certificate examination and prepare for the higher certificate examination. Mathematics is taken either as one or two principal subjects in this examination. The mathematical subjects cover both pure and applied mathematics and include higher algebra, pure geometry, analytic geometry (two dimensions), differential and integral calculus, differential equations, statics, dynamics, hydrostatics.

THE UNITED STATES

The secondary school period in schools in the United States extends either for four years or for six years beyond the elementary school. During the last three of these school years mathematics is an elective subject in most states of the union. The mathematical subjects studied by a majority of secondary school pupils are elementary algebra and plane geometry. Pupils interested in mathematics have the option of electing one or more of the following additional subjects — intermediate algebra, advanced algebra, solid geometry, trigonometry, plane analytic geometry, elementary differential and integral calculus. The proportion of pupils electing these additional mathematical subjects, however, is very small. There are two external examinations in mathematics — the College Entrance Examination Board examinations and the New York Regents examinations.

COMPARISONS

It will be observed that secondary school pupils in England and Wales are required to study considerably more mathematics than pupils of a corresponding age in the United States. The elective system in the United States tends toward discontinuity of treatment of school mathematics as compared with the continuous study of mathematics for four years in secondary schools in England and Wales to the first certificate stage. Consequently pupils in English schools have a wider knowledge of the various mathematical subjects at the end of four years than do pupils in American schools.

Judging from the examination papers in mathematics set by the external examinations boards in the two countries, it would seem fair to say that the highest standard of mathematical work required in secondary schools in the United States corresponds, subject for subject, to that required for Additional Mathematics at the first certificate examination in England and Wales. Thus at about the age of sixteen pupils in England and Wales have reached the same standard in mathematics as have pupils in the United States at about the age of eighteen.

Whereas in the United States this standard of mathematical work is the maximum required — the minimum being considerably lower —

for entrance to universities, colleges, and teachers colleges, in England and Wales students are required to study mathematics for a further two years before gaining admission to a university to study mathematics. During these two years mathematical specialists study pure mathematics of the kind normally offered in the freshman and sophomore years in the better universities, colleges, and teachers colleges, and, in addition, they study mechanics. W. D. Cairns, when speaking of and exhibiting mathematical examination papers taken by pupils in English secondary schools, says:

> It will be seen at once in a very concrete way that this portion of the public schools covers the mathematics of at least the first two college years in the United States.[1]

It should be said, however, that many educators and all of the mathematical associations in the United States frequently protest against the small range of mathematical work taught in the secondary schools and against the conditions which have produced the decrease in percentage enrollment in mathematics in the secondary schools during recent years. From a study of the data it would seem that this small range of mathematical work is tied to the heterogeneous nature of the school population. It is certainly true that pupils with greater ability in mathematics are not doing work in this subject to a standard to which they are capable.

This comparison was made to show the standard of mathematical work which teachers in the two countries should be prepared to teach.

2. *Is the academic training of the mathematics teachers adequate to guarantee reasonably good teaching of school mathematics at the standard required in schools?*

ENGLAND AND WALES

The answer to this question so far as it concerns teachers in England and Wales is that prospective teachers who have received the mathematical training required of students in the honors courses

[1] W. D. Cairns, "Advanced Preparatory Mathematics in England, France and Italy," *American Mathematical Monthly*, XLII (January, 1935), p. 25.

in mathematics in the universities are certainly well equipped on the academic side to teach mathematics to all classes of pupils in secondary schools. This may not be true without qualification of those prospective teachers who have studied mathematics to the standard required in the Ordinary or General (or pass) degree courses. While these teachers are academically well equipped to teach all mathematical subjects required at the first certificate stage, their training may not cover adequately the specialized branches of mathematics studied by pupils preparing for the higher certificate examination or for university entrance scholarships. It was pointed out that this is probably the reason why, in practice, the qualification of an honors degree in mathematics is required of prospective mathematics teachers in England and Wales.

THE UNITED STATES

The academic training of mathematics teachers in the United States is given in universities, colleges, and teachers colleges. The amount of mathematical training received in these institutions varies considerably from state to state, and in the main the minimum requirements are determined by agencies of certification. Most of these agencies prescribe for certification an average of between fifteen and eighteen points in mathematics, i.e., the equivalent of six three-point courses in mathematics at the college level. Those teachers who teach mathematics after receiving this minimum training in mathematics may not be well qualified to teach all subjects included in school mathematics. The reasons for this are that the minimum number of points required for certification may be earned in the freshman and sophomore years in college, and that the mathematical courses selected toward satisfying point requirements may include courses in such parts of elementary mathematics as solid geometry, analytic geometry, trigonometry, and algebra. Those prospective teachers who take considerably more than the minimum number of points in mathematics and who definitely include in their academic training courses in differential and integral calculus, differential equations, theory of equations, college geometry will be well qualified on the academic side to teach all of the mathematics taught in secondary schools in the United

States. In teachers colleges there is a movement toward professionalization of subject matter whereby the attempt is made definitely to relate the advanced mathematics to the appropriate parts of school mathematics and to keep together the professional and the academic aspects of the subject.

COMPARISONS

The evidence adduced in answer to the second question would seem to suggest that a higher percentage of mathematics teachers in secondary schools in England and Wales are qualified to teach all of the mathematics offered in those schools than is the case with mathematics teachers in the United States. Whereas in England and Wales no legal requirements are set up concerning qualifications for teaching in secondary schools, in practice it happens that the majority of mathematics teachers appointed to secondary schools must have the qualification represented by an honors degree in mathematics. It has been shown that the mathematical courses leading to these degrees guarantee that prospective teachers who pass through them are adequately prepared to teach secondary school mathematics in so far as the subject matter aspect of teaching is concerned. In England and Wales no attempt is made during the period of academic training in mathematics to take account of the fact that the learner is preparing to teach the subject. This means that there is no movement toward what is called in the United States professionalization of subject matter in any of the mathematical courses taken by prospective teachers during their training period.

It has also been shown in the preceding chapters that the academic qualifications in mathematics of mathematics teachers in the United States are not so high as are those of mathematics teachers in England and Wales. This fact has caused concern to some educators and particularly to associations of mathematics teachers in the United States. The minimum program of academic training in mathematics recommended by the Mathematical Association of America falls considerably short of that required of mathematics teachers in England and Wales, although representing a definite advance on the minimum demands at present made by the certifying bodies in the United States. In view of the evidence presented it cannot be said that prospective mathematics teachers in the United States who study mathematics as a minor subject, or those who satisfy only the minimum requirements for certification, are adequately prepared to teach all branches of school mathematics. If and when the recommendations of the Mathematical Association as to minimum requirements in subject matter are put into effect, teachers trained in accordance with these recommendations will be more suitably prepared to teach the mathematics at present taught in secondary schools in the United States. Prospective teachers who study mathematics as a major subject and who elect a considerable number of mathematical courses in addition to the prescribed courses might be regarded as sufficiently well prepared to teach the amount of mathematics now offered in secondary schools in the United States. Should the range and difficulty of secondary school mathematics be increased, even these teachers will need to undergo a more intensive academic training in mathematics to be regarded as adequately prepared in subject matter.

Some account has already been given of a method of organizing subject matter in which the more advanced mathematics, as it is being taught, is related to the corresponding parts of school mathematics. This method of organizing subject matter is in operation in teachers colleges mainly. There is reason to believe that it can be made a most effective instrument for the proper training of mathematics teachers, provided teachers can be found who possess scholarship as well as experience and ability in teaching.

3. *What is the nature of and how effective are the present methods of professional training of mathematics teachers in the two countries?*

ENGLAND AND WALES

It has been shown that professional training is not a necessary qualification for those seeking to teach in secondary schools in England and Wales. However, an increasing percentage of appointees to secondary schools has received training before accepting appointments. It has also been shown that three agencies contribute to the professional education of mathematics teachers in these two countries. Of these the university training departments represent the more formal agency of training. At the university training departments prospective teachers devote two thirds of their time to lectures in education and allied subjects, and in special methods, the remaining third of their time (sixty days) being devoted to practice teaching. The special methods courses in mathematics in most training institutions are so short as to allow time only to discuss the main principles underlying the teaching of mathematics. The period of practice teaching is regarded as the most important single factor in preparing prospective teachers for their work, and on this account every effort is made to insure that the experience gained during this period is significant and varied. Neighboring schools are used for this purpose since the university training departments do not have special training schools associated with them.

THE UNITED STATES

In the United States universities, colleges, and teachers colleges offer courses for the professional training of teachers. As in the case of the academic subjects the minimum number of courses in professional subjects is determined by the requirements of certification agencies. These agencies prescribe an average minimum number of between fifteen and eighteen credits in education and allied subjects. Frequently prospective and active teachers of mathematics accumulate many more than the minimum number of credits in professional subjects. Special methods courses in mathematics are given, usually during the junior or senior year, in the three types of training institutions. In teachers colleges students are given an opportunity for contact with schools during the freshman and sophomore years also. The minimum course in special methods of teaching mathematics in all types of institutions provides for about fifty lecture hours. In many institutions, especially institutions for graduate study in education, additional courses in methods are available.

Practice teaching is an integral part of the professional training offered in most institutions, and it is usually done during the senior year. Facilities for practice teaching are unequal in the various parts of the country. They vary from fairly adequate in the institutions in large cities to unsatisfactory in the colleges remote from a sufficient number of public schools.

COMPARISONS

Whereas in England and Wales professional training is not obligatory, in the United States minimum standards for certification have been set up. A more important place seems to be given to practice teaching in the preparation of teachers in England and Wales than is the case in the United States, but in the latter when it is given much more attention is paid both to methods courses and to courses in education and allied subjects. In England and Wales a more definite separation of the professional from the academic aspects of training is made than is the case in the United States where, in universities, colleges, and teachers colleges, academic and professional subjects are studied concurrently in the junior and/or senior year, and frequently also in the freshman and/or sophomore years.

4. *What facilities are provided for in-service training of mathematics teachers? How effective are these facilities?*

ENGLAND AND WALES

Additional facilities in the form of in-service training for teachers are being provided to an increasing extent in England and Wales. These facilities relate entirely to in-service professional training. It is true, of course, that teachers in service may study independently for higher degrees in mathematics at a university and may even receive some help from the university mathematics department to this end. But universities do not normally offer courses in mathematics leading to higher degrees either during the academic year or during vacations. Spasmodic attempts have been made to organize colloquia in mathematics during vacations, but these are not by any means a regular offering, though the need for them has been expressed from time to time. On the other hand, opportunities for in-service professional training are provided regularly but not on a scale sufficiently large to meet the demand. Of interest to mathematics teachers are the short vacation courses on the teaching of mathematics provided by the Board of Education, by local education authorities, and occasionally by universities.

These courses aim to cover a wide range of topics in the teaching of mathematics and, through discussions and lectures, to assist teachers with their teaching problems, rather than to offer a systematic account of any one aspect of the teaching of mathematics.

The University of London and provincial universities in England and Wales also offer higher degrees in education and courses of lectures are offered in the evenings and/or on Saturday mornings for students reading for these higher degrees.

THE UNITED STATES

In the United States opportunities for in-service training both in academic and in professional subjects are provided in summer schools which are organized in the three types of training institutions. In summer schools, mathematics teachers tend to patronize the professional courses in preference to the academic courses in mathematics, though the latter are provided in many institutions. In general the academic courses in mathematics organized during the summer sessions are restricted in number and range of subjects offered as compared with those offered during the regular academic year. Very great use is made of the summer courses in professional subjects. Each course covers a definite unit of work which may be taken for credit. Credits or points earned during the summer sessions count toward the total number of credits for graduation and for graduate degrees.

COMPARISONS

The facilities for in-service academic and professional training in England and Wales are meager when compared with those offered in the regular summer sessions in training institutions in the United States. In the former countries practically no facilities for in-service academic training are available. Those available for in-service professional training are restricted in extent and are therefore open only to a selected few. In the United States, however, all who wish may attend summer courses. Here mathematics teachers tend to use the facilities provided to improve their professional qualifications rather than to improve their academic qualifications in mathematics.

There is perhaps more incentive for American teachers than for English and Welsh teachers to make use of the summer school courses because the credits which they earn for these courses count toward the number of credits required for graduation, and because they can in this manner meet the higher requirements for certification which are from time to time imposed by certifying authorities. Use is made of summer courses both to complete the requirements for a Bachelor's degree, and also to obtain a Master's degree, all of the courses for which may be taken in professional subjects. In England and Wales no special vacation courses are offered toward the higher degrees in education. Except in the case of the vacation courses held biennially in Oxford University, attendance at which carries eligibility for the University Diploma in Education, no vacation course in England and Wales may be used to count for credit toward a degree or diploma.

5. *What agencies other than the regular teacher training institutions contribute to the training of mathematics teachers, and what is the nature and importance of their contribution?*

ENGLAND AND WALES

It was shown in this study that the Mathematical Association of England, through its reports on the teaching of the various school mathematical subjects and through discussions at its annual meetings which are afterwards published in the *Mathematical Gazette*, has made and still does make a significant contribution to the in-service professional training of mathematics teachers. It also provides the opportunity for a type of in-service academic training through the articles and book reviews which are a special feature of the *Gazette*.

THE UNITED STATES

So far as secondary mathematics teachers are concerned, the principal agencies of this kind in the United States are three professional associations of teachers, the Mathematical Association of America, the National Council of Teachers of Mathematics, and the Central Association of Science and Mathematics Teachers. The last two associations are concerned chiefly with school mathematics and their contribution to the improvement of teaching is made through the journals which they publish, namely, *The Mathematics Teacher*, *School Science and Mathematics*, and the Yearbooks of the National Council of Teachers of Mathematics. The Mathematical Association of America is really an association of college teachers of mathematics, consequently, their journal *The American Mathematical Monthly* is devoted to the interests of college mathematics. On this account therefore the Association provides facilities for in-service academic training in mathematics of those teachers who are able to read the more advanced subjects.

COMPARISONS

It is thus seen that the work of the associations of mathematics teachers in the two countries is quite similar in character. These associations offer leadership in the field of the teaching of mathematics in initiating, and interpreting to teachers, new movements in mathematical education. They also perform, when necessary, a valuable function as defenders of the interests of mathematics as a school subject: in England and Wales, against the too rigid requirements of external examinations; in the United States, against the tendency to omit so much mathematics from the curriculum of secondary schools.

6. *What assumptions are implicit in the methods of training adopted in the two countries?*

ENGLAND AND WALES

From the account given in this study of the training of mathematics teachers for secondary schools in England and Wales it follows that this training is based on the assumption that academic preparation in subject matter is the most important single element in training. Further, the committees of the Mathematical Association in their reports bearing on teacher training have emphasized this point, and have stated as a generally accepted fact that the academic and professional aspects of training should be conducted separately, the latter aspect after the former has been completed.

In keeping with the above it is generally assumed that persons of ability must be enlisted in the profession of teaching and, once recruited, they must be given an appropriate training. A consequence of this assumption is that a fairly rigid process of selection of prospective teachers in secondary schools is put into operation, beginning with entrance to the secondary school and continuing through successive stages of their academic development. Until recently it was generally assumed, and in some quarters it is still assumed, that there is no place for so-called professional subjects in the training of mathematics teachers. The only qualifications needed were assumed to be scholarship, personality, and a broad cultural education. It was assumed that the art of teaching could best be acquired by actually teaching and perhaps only in that way. These assumptions about professional preparation have now been fairly generally discarded, and in their place it is assumed that a period of professional training is necessary, and that this can best be given after prospective teachers have completed their academic training. It is on these terms that professional training is now admitted as part of the training of mathematics teachers, and in this form it has received the support of the Mathematical Association of England. The opinion seems to be fairly generally held that into this year of professional training there should be fitted a great deal of teaching experience and actual contact with pupils at the expense, if necessary, of some of the theoretical aspects.

THE UNITED STATES

It would appear from the discussion given in this study that the training institutions in the United States which train mathematics teachers for secondary schools operate under somewhat different sets of assumptions. On the one hand the mathematical departments of universities (as distinct from their schools of education) and of colleges, supported by the associations of mathematics teachers, seem to assume that the most important elements in training center round a thorough knowledge of the subject and active contact with the problems of teaching through practice teaching and observation. On the other hand, the teachers colleges and schools of education in universities organize their courses apparently on the assumption that the important elements in teacher training center round a knowledge of pupils. Therefore they tend to regard a knowledge of a specific subject or subjects as only one element in the picture, and to emphasize those studies, education, educational psychology, guidance, methods, and the like, which are normally regarded as belonging to professional as distinct from academic training.

To state the position thus baldly may imply, what indeed is not true, that the adherents of the subject matter or academic point of view will have none of professional subjects, and that the adherents of the professional point of view will have nothing to do with subject matter. Nevertheless there is a very real contrast in emphases here, as may be seen from Valentine's article in a recent issue of the *Educational Record.* In this article Valentine presents forcibly and with an admission of partisanship the case for a general return to the academic type of training for teachers:

> Those colleges [the liberal arts colleges] must regain their leadership in education by re-establishing the principle that a school teacher must above all else have a sound general education.[2]

The real need would seem to be, as Kandel points out,

> to provide for an integration of the two. . . . As contrasted with medicine, law, engineering or other professions, the difficulty in the teaching is that academic subjects . . . are the tools of the profession: these the teacher must learn how to apply to the other datum of education—the pupil. There is no guarantee whatever that a series of uncoordinated professional subjects tacked on to the academic subjects will produce a good teacher. Certificating authorities seem to have taken their cue from those teachers colleges or schools of education which have already committed the error of confusing professional preparation with the study of education with the result that constant proliferation of the latter has infected the former, with the further confusion between teaching as an art and education as a science.[3]

In the United States the assumption is implicit that the problem

[2] Alan Valentine, "Teacher Training versus Teacher Education," *Educational Record*, (July, 1938).

[3] I. L. Kandel, "The State's Right to Certify Teachers," *New York State Education*, (October, 1938), p. 11.

of selection may be handled largely through the certifying agencies, rather than through the schools. In the United States the secondary school population is more heterogeneous than in England and Wales, and a greater proportion of this population goes on to colleges and universities. Further, the nature of the promotion of pupils through secondary schools, colleges, and universities has the effect that relatively unselected groups offer themselves for teaching.

COMPARISONS

The above account would seem to suggest that persons belonging to liberal arts colleges, mathematical departments of universities and mathematical associations in the United States, and educators and mathematicians in England and Wales make the basic assumptions in regard to teacher training that the most important single element in such training is through preparation in subject matter. They assume also that a measure of professional training is desirable, to be undertaken after the completion of a period of academic training. The emphasis in professional training is to be on observation, practice teaching, and special methods rather than on the theoretical study of education and allied subjects.

Somewhat in contrast to the above assumptions are those made by persons associated with teachers colleges and departments of education in universities in the United States. While recognizing that subject matter must occupy a position of great importance in the training of teachers, they do not give it an all-important place. Rather do they base the type of training they offer and recommend on the assumption that the pupil is the most important element in the picture and that academic subjects, professional subjects, and experience in teaching should be given to the prospective teacher in such proportions as will fit him to make his optimum contribution to the whole development of the pupil.

It should be added that the contrast in method of selection of prospective teachers is in harmony with the social philosophies in the two countries. In England and Wales opportunities tend to narrow rapidly, not only in teaching but in all preferred professions, whereas in the United States a conscious effort is made to keep the approaches open to a wide range of occupational opportunities.

Chapter VIII

EVALUATION IN TERMS OF PRINCIPLES

THE SEVENTH QUESTION proposed for answer was: On what principles should the methods of training be based? In the light of these principles what are the strengths and weaknesses of the training of mathematics teachers in England and Wales and in the United States?

The principles which were developed and expounded in the second chapter of this study may be briefly stated as follows:

1. Prospective mathematics teachers should receive a thorough course of training in mathematics.

2. This training should be given in a university or an institution of equivalent rank by teachers who are themselves mathematicians of outstanding competence and who appreciate and understand the difficulties inherent in mathematics, whether it be regarded as a subject of learning or of teaching.

3. Mathematics teachers should study the important branches of pure mathematics, mechanics, the history of mathematics, applications of mathematics, as, for example, study of its logical foundations, and particularly the essential connection between the various branches of advanced mathematics and their counterpart at the more elementary stage.

4. Mathematics teachers should make a less intensive study of some subject, preferably one closely related to mathematics.

5. The teacher, during his period of active service, should strive to progress in his acquaintance with and mastery of many aspects of mathematical knowledge.

6. A period of professional training is a necessary part of pre-service training of mathematics teachers.

7. The content of this course of professional training should be organized principally for the purpose of training teachers for teaching mathematics.

8. Mathematics teachers should be equipped to teach at least a second (and preferably an allied) subject, and they should therefore undertake a course of professional training in this second subject.

9. This period of professional preparation of mathematics teachers should include some courses in the theory and practice of education, and in psychology.

ENGLAND AND WALES

That part of the first principle which related to the thoroughness of the course of training in mathematics is certainly satisfied by the training received by honors graduates in mathematics from universities in England and Wales. Those teachers whose mathematical training was received in courses leading to an ordinary degree are sufficiently well prepared academically to teach all but the more difficult sections of school mathematics. The university teachers who give these students their academic training in mathematics qualify as suitable teachers of future teachers in respect to their scholarship. Without visiting their classes there would be no way of discovering to what extent these university teachers of mathematics are aware of the difficulties of teaching and of learning mathematics. However, it may be mentioned that many of the leading mathematicians in England and Wales have served as presidents or as members of the council of the Mathematical Association, and have devoted considerable time to the problems of secondary school mathematics in connection with their tenure of one or other of these offices.

When the third principle is considered, however, it is found that the kind of mathematics taught in universities in England and Wales lacks elements which are of the greatest importance for future teachers of mathematics. During their academic training in mathematics, it is unusual for prospective teachers to receive any instruction in the history of mathematics, in statistics, in applications of mathematics to other fields of learning, in the foundations of mathematics, and in the bearing of higher mathematics on the mathematics they will teach in schools. The lack of these elements in the training of mathematics teachers is here regarded as a weakness in the method of training.

Practically all mathematics students in English and Welsh universities are required to study at least one other subject besides mathematics and in the great majority of cases the second subject is one of the physical sciences. Exceptions to this practice occur at the Universities of Oxford and Cambridge, where a student may study mathematics only for a degree.

It has been shown that the responsibility of studying additional mathematics or of investigating the various fields of application of mathematics after the completion of the work for a Bachelor's degree in mathematics rests with each individual. Higher degrees are awarded at the universities, but students who wish to earn them must do so largely by independent study. Up to the present the universities have made no effort to offer courses, during vacations or at week ends, which are designed to present to teachers the development of ideas in the various branches of modern mathematics. In failing to provide courses of this kind the universities are restricting the opportunities for the in-service academic growth of mathematics teachers. The Mathematical Association of England, however, offers one means for the type of in-service growth which is contemplated here.

The methods of training mathematics teachers in England and Wales conform in part to the principle that a period of professional training is a necessary adjunct of the pre-service training of the teacher. It is still true that no such requirement is generally made, and many teachers are appointed to secondary schools without having received any professional training. In so far as this means, in practice, that an untrained teacher receives no assistance and direction, it is regarded as undesirable in terms of the present principle. However, some evidence was adduced to show that such a practice may be defensible under special circumstances, as in the case where a teacher is appointed to a school where the Senior Mathematical master assumes responsibility for his training.

Strict separation of the academic and professional training of secondary school mathematics teachers is practiced in England and Wales on the ground that the greatest advantage can be obtained if each type of training is treated separately from the other

In so far as considerable emphasis is placed on inclusion of a long period of practice teaching as part of the professional training of mathematics teachers, and on provision of opportunities to teach mathematics to a variety of classes or grades, the professional training of mathematics teachers in England and Wales fulfills in part the seventh principle enunciated. It fails to satisfy this principle completely, however, since so little attention is given during the professional year either to an exposition of methods of teaching mathematics or to a consideration of the bearing of the academic mathematics which prospective teachers have studied on the school mathematics which they will be required to teach.

Since it is a regular practice in most universities in England and Wales to require the study of at least one other subject besides mathematics for a degree, the eighth principle is satisfied so far as the academic preparation in a second teaching subject is concerned. The same defects as were noted in the last paragraph, however, also apply to the professional training in this second subject. For, while similar arrangements are provided for methods courses in the second teaching subject, prospective teachers do not get an opportunity to have any considerable experience with practice teaching in that subject.

Since the study of education and allied subjects occupies the greater portion of the time not devoted to practice teaching during the year of professional training, it is true to say that the ninth principle is satisfied by the type of training offered to teachers in England.

The strengths noted in the teacher training program in England and Wales are:

1. The emphasis placed on mastery of subject matter, albeit subject matter of an orthodox kind.
2. The high qualifications of the teachers who teach prospective teachers both in universities and in university training departments.
3. The provision of a long period of practice teaching during the year of professional training, with opportunities for teaching mathematics to pupils at various stages in their mathematical education and to pupils of varying ability in mathematics.

In terms of the principles adopted in this study the program of training for mathematics teachers in England and Wales showed certain weaknesses:

1. The academic training in mathematics lacks breadth in that certain subjects of fundamental importance in the training of mathematics teachers are omitted.

2. The special methods courses offered in most of the training institutions are too short to be of real value, for, except at the Institute of Education, London, little more is done in this regard than to devote a few lectures to a discussion of principles.

3. Too little provision is made for in-service academic and professional training of mathematics teachers. Although there is considerable demand for such training, especially professional training, no attempt is made to meet the demand by forming additional courses.

THE UNITED STATES

It cannot be said without reservation that the academic preparation in mathematics which prospective mathematics teachers of secondary schools in the United States receive meets the requirements of the first principle. It has been shown that many such teachers receive adequate preparation in subject matter. There are, however, many who are inadequately prepared to teach mathematics in all grades of the secondary school because their own academic preparation in mathematics does not exceed that demanded either of students studying mathematics as a minor subject or of those meeting the minimum certification requirements in subject matter.

The qualifications of the teachers of prospective mathematics teachers vary greatly with the type of institution. In general universities and colleges command the services of teachers with the higher scholarship in mathematics, while teachers colleges attract teachers who combine scholarship with experience in and familiarity with the problems of secondary school teaching.

In respect of the range of mathematics taught to prospective teachers the training institutions in the United States go some distance toward satisfying the demands of the third principle. Indeed, the types of mathematical courses offered in these institutions would seem to suggest that in selecting them emphasis was placed on their suitability for prospective teachers, for they comprise a wide range of mathematical subjects rather than a smaller number to be studied more intensively.

Although the minor subject of study selected when mathematics is the major subject may be one of a large number, there is a trend toward the choice of one of three subjects, namely, English, physical sciences, and social sciences.

The idea that prospective mathematics teachers should prepare to teach a second subject is gaining in favor in the United States and many certifying authorities and state boards of education demand a second teaching subject. However, it cannot be said at the present time that the requirement of a second teaching subject is at all general. To the extent that this is not required the training of mathematics teachers in the United States fails to meet the demands of the fourth principle.

It has been shown that, although academic courses in mathematics are offered by institutions which conduct summer sessions, mathematics teachers in the United States as a group tend to choose subjects other than these academic mathematical courses. To the extent that they do this they do not satisfy the fifth principle, which asserts the desirability of academic training in-service. Other agencies in the form of meetings of the associations of mathematics teachers and their publications also offer opportunities for such academic in-service training, but no estimate could be made of the extent to which teachers avail themselves of these additional opportunities.

The training of mathematics teachers in the United States satisfies the sixth principle, for considerable importance is attached to professional training, particularly in the schools of education in universities and in teachers colleges. This is not so true of the professional training offered in liberal arts colleges, for they tend to provide courses in professional subjects which will meet the minimum demands of certifying bodies.

In order to examine the content of the professional training in the light of the seventh principle, it will be necessary to think of it in two parts. So far as courses in methods of teaching of mathematics are concerned it would seem that ample time is allowed in most training institutions for a careful treatment of the more important aspects of mathematical teaching in secondary schools. In certain of the graduate schools of education many detailed courses on the teaching of mathematics are offered. The seventh principle is not so well satisfied, however, in respect of the arrangements for practice teaching. The arrangements would seem to fall short of what is desirable (i) because the practice is too often restricted, throughout its entire period, to a single grade or even to a single topic, and (ii) because of the lack of opportunity for continuous class teaching in the sense of being on the job all day and every day for a number of weeks. One reason for this would seem to be that many teachers are unwilling to give up their classes to students.

To satisfy the eighth principle the professional training of mathematics teachers in the United States would need to be improved in both of the aspects referred to in the last paragraph. Opportunities for taking courses in the teaching of the second subject are provided more frequently than arrangements are made for students to have practice in teaching that subject.

Professional courses in all teacher training institutions satisfy the ninth principle in so far as these institutions provide sufficient courses in education and allied subjects. There is probably a greater range of separate courses in education and allied subjects than in any other subject.

So far as they are revealed by the above principles the principal strengths in the methods adopted for the training of mathematics teachers in the United States are:

1. The variety and range of mathematical courses offered to students during the period of their academic training, such, for example, as courses in statistics, mathematics of finance, history of mathematics.

2. The variety and range of courses in education and allied subjects offered; where, however, these courses are studied too greatly to the exclusion of subject matter courses they may represent a weakness rather than a strength in the system of training.

3. The opportunities provided for in-service training both in academic and in professional aspects of training. While the facili-

ties so provided must be regarded as a strength, it has been shown that too little use is made of them for in-service academic training.

Aspects of the training of mathematics teachers which must be regarded as presenting weaknesses in terms of the principles of this study are:

1. The low standards in subject matter at present accepted as the minimum requirement both for purposes of certification and for mathematics when studied as a minor subject in training institutions. (These standards are regarded as low compared with what teachers need to know to teach effectively all of the mathematical subjects offered in secondary schools.)

2. The omission of mechanics from the academic courses in mathematics. This is regarded as a weakness because mechanics offers unique opportunities for the application of important mathematical ideas and techniques.

3. The fact that so little use is made of the opportunities for in-service academic training in mathematics represents a weakness in terms of the fifth principle.

4. The frequent limitation of practice teaching experience in mathematics to teaching and observing pupils in only one grade during a whole semester does not provide an amount of variety and experience in this aspect of training sufficient to satisfy completely the principles herein stated. Lack of facilities for gaining practice in teaching the second subject also represents a weakness of the present system.

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